Unified Model

How to value, hedge, and manage the risk of **any** portfolio

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4 Feb 2021

What Quants Do

- Model prices and cash flows.
- Fit model parameters to market data.
- Compute expected values and derivatives.
- Specify hedges/trading strategies.
- Provide measures of how good a hedge is.

What Quants Don’t

– Provide realistic models.
– Specify *when* to hedge.
– Use parameters traders understand.
– Manage risk.

Holdings

Holdings are the atoms of finance.

- A *holding* (i, a, e) is an *instrument*, *amount*, and *entity*.
- Instrument (*I*): stocks, bonds, futures, commodities, currencies, …
- Amount (*A*): shares, notional, contracts, physical quantity, units,…
- Entity (*E*): individual or corporation owning the holding

Trades

Holdings interact through *trades*.

- A trade $(t; i, a, e; i', a', e')$ is an exchange of holdings at a time.
- At time $t \in T$ buyer e exchanges (i, a) for (i', a') with seller e'.
- *Price* is $X = a/a'$ (after the trade).
- Buyers decide what and how much to exchange.
- Sellers quote a price for that.
- Buyer instrument is usually currency.

Price

Prices depend on time, the instruments, the entities, and the amount the buyer wants to acquire, among other things.

– The trade $(t; i, a'X, e; i', a', e')$ is available to e from e', where $X = X(t; i, e; i', a', e')$.

– "Among other things" is denoted Ω, the sample space.

Cash Flow

Stocks have dividends, bonds have coupons, futures have margin adjustments, commodities have holding costs, …

- Holding (i, a) at time t results in a cash flow $aC(t; i; i')$ of i' at time t.
- Usually i' is a currency, but payment-in-kind is possible.
- Cash flows are zero except at discrete times.
- Cash flows do not depend on entities, only on the issuer of i.

Market (Model)

A *market model* specifies possible future prices and cash flows of instruments.

- Price: $X\colon T \times I \times E \times I' \times A' \times E' (\times\, \Omega) \to \mathbf{R}$.
- Cash Flow: $C\colon T \times I \times I' (\times\, \Omega) \to \mathbf{R}$.
- The trade $(t, i, a'X(t; i, e; i', a', e'), e; i', a', e')$ is available to the buyer.
- Assuming price does not depend on amount and e, e' are understood…
- … $X(t; i, e; i', a', e')$ is a vector-valued stochastic process $X_t\colon \Omega \to \mathbf{R}^I$ indexed by instruments

Trading

A *trading strategy* specifies when, what, and how much to trade.

- Trading times $\tau_0 < \tau_1 < \cdots$…
- .. and amounts $\Gamma_j\colon \Omega \to A^I$.
- Position: $\Delta_t = \sum_{\tau_j < t} \Gamma_j = \sum_{s<t} \Gamma_s$.
- Value: $V_t = (\Delta_t + \Gamma_t) \cdot X_t$ is the mark-to-market.
- Amount: $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$ shows up in the trade blotter.

Arbitrage

Arbitrage exists (for a model) if there exists a trading strategy that eventually closes out with $A_{\tau_0} > 0$ and $A_t \geq 0, t > \tau_0$.

- Make money up front and never lose money until strategy is closed.
- *Closed out* means the position is zero after some finite time.
- This definition does not depend on a measure.
- Traders consider RoI: $A_0/|\Gamma_0| \cdot |X_0|$.

FTAP

The Fundamental Theorem of Asset Pricing. No arbitrage if and only if there exist *deflators* $D_t\colon \Omega \to (0,\infty)$ such that

$$X_t D_t = E_t[X_u D_u + \sum_{t<s\leq u} C_s\, D_s],$$

where E_t is conditional expectation at time t.

– If $C_t = 0$, deflated prices are a martingale.

– As $u \to \infty$, value is discounted future cash flows.

Lemma

If

$$X_t D_t = E_t[X_u D_u + \sum_{t<s\leq u} C_s\, D_s],$$

then

$$V_t D_t = E_t[V_u D_u + \sum_{t<s\leq u} A_s\, D_s].$$

- Price ↔ value, cash flow ↔ amount.
- Trading strategies create synthetic market instruments.

FTAP…

Easy direction: deflators imply no arbitrage.

If $A_t \geq 0, t > 0$, then

$$V_0 D_0 = E_0\left[\sum_{t>0} A_t\, D_t\right] \geq 0.$$

Since $V_0 = \Gamma_0 \cdot X_0 = -A_0$, $A_0 \leq 0$ so there is no arbitrage.

…FTAP

Hard direction: no need to prove!

Given any vector-valued martingale M_t and any deflator D_t

$$X_t D_t = M_t - \sum_{s \le t} C_s \, D_s$$

is an arbitrage free model.

Proof: More high school algebra.

Canonical Deflator

If the market has repos then $D_t = e^{-\int_0^t f_s \, ds}$ where f_s is the repo/short rate at time s.

- The repo/short rate determines the value of all fixed income instruments.

Zero

A *zero coupon bond* $D(u)$ has $C_u = 1$.

- $X_t D_t = E_t[D_u]$ so $X_t D_t = D_t(u) D_t = E_t[D_u]$,
- $X_t = D_t(u) = E_t[D_u]/D_t = E_t[e^{-\int_t^u f(s)\,ds}]$.
- Risky zero $D^{R,T}(u)$ has cash flow $C_u = 1(T > u)$ or $C_T = R1(T \leq u)$.

FRA

A *forward rate agreement* $F^{\delta}(t,u)$ has $C_t = -1$, $C_u = 1 + f\delta$, where δ is the dcf for the interval from t to u.

- $0 = X_0 = E[-D_t + (1 + f\delta)D_u]$, so
- $f = (D(t)/D(u) - 1)/\delta$.
- *Par forward* at s defined by
- $0 = X_s = E_s[-D_t + (1 + F_s(t,u)\delta(t,u))D_u]$,
- so $F_s(t,u) = (D_s(t)/D_s(u) - 1)/\delta(t,u)$

Arrear

A FRA $\overline{F}^{\delta}(t,u)$ paying in *arrears* has one cash flow $C_u = (f - F_t(t,u))\delta$.

- Model: $M_t = (E_t[D_u])_u, u \geq t$, all zeros indexed by maturity.
- Trades: $\tau_0 = t, \Gamma_t = +1_u, t_1 = u, \Gamma_u = -1_u$.
- FRA + trades = arrears.

B-M/S

The Black-Merton/Scholes model is

$$M_t = (r, se^{\sigma B_t - \sigma^2 t/2}), D_t = e^{-\rho t},$$

so $X_t = M_t / D_t = (e^{\rho t}, se^{\sigma B_t + (\rho - \sigma^2) t/2})$.

- Fixed dividends $C_{t_j} = d$.
- Proportional dividends $C_{t_j} = pS_{t_j}$.
- Equilibrium arguments, self-financing portfolios, Ito's formula not required.

Hedge

Given a set of amounts A_j at t_j how do we find a trading strategy that produces these?

- $V_0 = E[\sum_j A_j D_{t_j}] = \Gamma_0 \cdot X_0$ so $D_{X_0} V_0 = \Gamma_0$.
- $V_j = E_j[\sum_{k>j} A_k D_{t_k}] = (\Delta_j + \Gamma_j) \cdot X_j$ so $D_{X_j} V_j = \Delta_j + \Gamma_j$ (Frechet derivative.)
- Note $\Delta_{j+1} - \Delta_j = \Gamma_j$, so Δ is delta and Γ is gamma.
- This is *never* a perfect hedge.

Remarks

Trajectory of mathematical finance is increasingly accurate models covering all instruments.

Incorporate trading strategies when measuring risk.

Monte Carlo all the things! We have Moore's Law on our side and can put the ML headcount to good use.

Provide real-time valuation and risk reporting across all asset classes.

Links

– Unified Model
– Unified Finance
– Excel add-in libary