Given two random realized returns on an investment, which is to be preferred? This is a fundamental problem in finance that has no definitive answer except in the case one investment always returns more than the other, in which case arbitrage exists. In 1952 Markowitz[@Mar1952] and Roy[@Roy1952] introduced a criterion for risk vs. return in portfolio selection: if two portfolios have the same expected realized return then prefer the one with smaller variance. An efficient portfolio has the least variance among all portfolios having the same expected realized return. This was developed into the Capital Asset Pricing Model by Treynor[@Tre1961], Sharpe[@Sha1964], Lintner[@Lin1965], and many others.

The Capital Asset Pricing Model marked the transition from the due diligence required for Graham-Todd security analysis to using the wisdom of the markets to inform investing. The “market portfolio” was assumed to be in an efficient “equilibrium” resulting from the cadre of investment professionals performing “market clearing” trades. This short note is agnostic to the quoted terms and proves a simple mathematical result about efficient portfolios.

There are well-founded criticisms of the CAPM, but it has value as an easily understood model. Portfolio managers use Sharpe ratios to tailor returns for an investment strategy while accounting for risk. The CAPM demonstrates a constraint on expected returns and covariance of efficient portfolios. We show a much stronger constraint: efficient portfolios satisfy an equality of realized returns as random variables. This allows the value-at-risk, or any risk measure, of efficient portfolios to be calculated, something not possible using the classical result that only holds for expected values.

This result follows directly from writing down a mathematical model for one period investments. The only thing remarkable is that this has not heretofore been noted in the literature. Prior work fails to explicitly specify a sample space and probability measure, the first step in any model involving probability since Kolomogorov legitimized probability as a branch of measure theory [@Kol1956].

## CAPM

The CAPM places a constraint on the excess expected realized return of efficient portfolios.

where is the realized return of an efficient portfolio, is the realized return of a risk-less portfolio, is the realized return of the “market portfolio”, and .

This short note shows the random realized return of any efficient portfolio satisfies

where and are the random realized returns of any two independent efficient portfolios. This implies . Taking expected values of both sides when has zero variance and is the “market portfolio” yields the classical CAPM formula

## One-Period Model

Let be the set of *market instruments* and be the set of possible *market outcomes* over the period. The *one-period model* specifies the initial instrument prices and the final instrument prices depending on the outcome that occurs. The one period model also specifies a probability measure on the space of outcomes. It is common to assume is , is the identity function, and is multivariate normal. We allow arbitrary distributions to be specified for final prices.

A *portfolio* is the number of shares initially purchased in each instrument. It costs to acquire the portfolio at the beginning of the period and returns when liquidated at the end of the period if occurs. The *realized return* of is when .

## Efficient Portfolio

A portfolio is *efficient* if its variance is less than or equal to the variance of any portfolio having the same expected realized return. Note for any non-zero so there is no loss in assuming . In this case is the realized return of the portfolio. If then the variance of the realized return is where .

For a given expected realized return we can use Lagrange multipliers to minimize over , . As is well-known, the first order condition on is . See the [Appendix](#appendix) for a proof.

If is invertable then

This shows every efficient portfolio is in the span of and .

The only novel result in this paper is the observation that if and are any two independent efficient portfolios then for some scalars and so . This shows

as random variables where . Taking the covariance with on both sides gives

If is not invertable then there exists with . The first order condition gives . The first order conditions , and show so . This is a special case of the condition for a one-period model to be [arbitrage-free](#ftap).

There may be two independent portfolios having variance zero. If they have different returns then arbitrage exists. If they have the same return then the model has redundant assets.

## Appendix

We use the notation for what is usually denoted by the transpose or . It is simpler and more illuminating to work with abstract vector spaces and linear operators between them than with and matrices. Matrix multiplication is just composition of linear operators.

Recall is the vector space of all functions from the set to with scalar multiplication and vector addition defined point-wise: and for , , and .

For define by if is finite. Note is linear.

Let be the set of all linear operators from the vector space to . Note is also a vector space with scalar multiplication and addition defined point-wise. The dual of a vector space is . For we have and allows us to identify with . If its adjoint is defined by where , , . If then and .

Let be the set of bounded linear operators from the normed linear spaces to . A linear operator is bounded if there exists with for all . The least upper bound of such constants is the norm of . This makes a normed vector space.

### Fréchet derivative

If is a function between normed vector spaces its Fréchet derivative is defined by

where means as . If the Fréchet derivative exists at then can be approximated by a linear operator near .

Given define by . We have so since .

Given define by . We have

This shows so if .

### Lagrange Multiplier

To find the minimum value of given we use Lagrange multipliers and solve

for , . If then and where .

Since , the first order conditions for an extremum are

Assuming is left invertable . Note every extremum lies in the (at most) two dimensional subspace spanned by and .

The constraints and can be written

with , , and . Inverting gives

where . The solution is , , and

A straightforward calculation shows the variance is

### FTAP

Arbitrage exists in the one-period model if there is a with and for . The cost of putting on a position is so you make money entering the position and never lose money unwinding it.

Note if where and are non-negative scalars then if . In this case there is no arbitrage.

The one-period Fundamental Theorem of Asset Pricing states there is no model arbitrage if and only if belongs to the smallest closed cone containing the range of . Note this statement does not involve any measures. The FTAP is a geometric result, not a probabilistic result.

Recall that a *cone* is a subset of a vector space closed under addition and multiplication by a positive scalar, that is, and for . For example, the set of arbitrage portfolios is a cone.

The above proves the “easy” direction. The contra-positive follows from the

**Lemma.** *If and is a closed cone in with then there exists with and for .*

*Proof.* Since is closed and convex there exists nearest with for all . Let . For any and we have so . Simplifying gives . Dividing by and letting decrease to 0 shows . Take then for . By similar reasoning, letting increase to 0 shows so . Because we have .

The proof also shows is an arbitrage when one exists.

If is bounded, as it is in the real world, then there exists a positive finitely-additive measure [@DunSch1958] with . Since is a positive measure with mass 1 we have under this “probability” measure.

We say is a zero coupon bond if . Since the realized return on is is the constant . The *discount* of the zero coupon bond is . In this case is the discounted “expected value” of .

## References