Risky Bonds

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Abstract

A simple model for the price of risky bonds.

The *unified model* specifies prices $X_t\colon \mathcal{A}_t \to \mathbf{R}^I$, and cash flows $C_t\colon \mathcal{A}_t \to \mathbf{R}^I$, where I are the available market instruments. It is arbitrage free if and only if there exists deflators $D_t\colon \mathcal{A}_t \to (0,\infty)$ with

$$X_t D_t = E_t[X_v D_v + \sum_{t<u\leq v} C_u D_u].$$

Since we can replace any deflators $(\hat{D}_t)$ by $(\hat{D}_t/\hat{D}_0)$ we assume $D_0 = 1$.

If there is a finite number of cash flows then the initial prices are

$$X_0 = E\sum_{t>0} C_t D_t.$$

A *trading strategy*, (τ_j, Γ_j), is a set of strictly increasing stopping times and trades $\Gamma_j\colon \mathcal{A}_{\tau_j} \to \mathbf{R}^I$. The *position*, $\Delta_t = \sum_{\tau_j<t} \Gamma_j = \sum s < t\Gamma_s$, is the accumulated trades to date.

The *value* of a trading stategy at t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: the amount of money you recieve for unwinding your existing position and the trade you just dit. The *amount* of a trading strategy at t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you get all cash flows proprtional to your position in each instrument and you pay the prevailing market price of every trade you execute.

As simple consequence of the definition of value and amount is

$$V_t D_t = E_t[V_v D_v + \sum_{t<u\leq v} A_u D_u].$$

If a trading strategy closes out, i.e., the position is eventually 0, then the initial value is

$$V_0 = E\sum_{t>0} A_t D_t.$$

Zero Coupon Bonds

Let $D(u)$ denote a zero coupon bond maturing at u. It has a unit cash flow at time u. Its price at time t, $D_t(u)$, is determined by $D_t(u)D_t = E_t[D_u]$ for $t \leq u$. Since there are no cash flows after time u we have $D_t(u) = 0$ for $t > u$.

In particular, $D_t(u) = E_t D_u / D_t$ and $D_0(u) = E_t D_t$. We write $D(u)$ for $D_0(u)$ in order to usefully confuse it with the name of the zero coupon bond.

Risky Zero Coupon Bonds

Let $D^{T,R}(u)$ denote a zero coupon bond maturing at u issued by a company that defaults at time T and pays (proportional) recovery R at default. It has a cash flow of 1 at u if $T > u$ and a cash flow of R at T if $T \leq u$, where T and R are random variables with $T \geq 0$ and $0 \leq R \leq 1$. A natural sample space for the random variables T and R is $\Omega = [0, \infty) \times [0, 1]$.

The value of this instrument at time t, $D_t^{T,R}(v)$, is determined by

$$D_t^{T,R}(v)D_t = E_t\big[\sum_{t<u\leq v} R1(T = u)D_u + 1(T > v)D_v\big],$$

where D_t is a deflator.

Assuming rates are 0, so $D_t = 1$, and R is constant, the only random variable is the default time and the sample space is $\Omega = [0, \infty)$. There is a natural filtration $(\mathcal{A}_t)_{t\geq 0}$ associated with any default time. The atoms of $\mathcal{A}_t$ are the singletons $\{s\}$ for $s < t$ and the set $[t, \infty)$: if default occurs prior to t we know exactly when it happened, otherwise all we know is default happened at or after t. Note we do not have instantaneous knowledge of default so $\{t\}$ is not an atom of $\mathcal{A}_t$.

Under these assumptions

$$D_t^{T,R}(v) = RE_t[\sum_{t<u\leq v} 1(T = u)] + E_t[1(T > v)].$$

Recall that for a random variable, X, on $\langle \Omega, P, \mathcal{A} \rangle$, where $\mathcal{A}$ is finite, we have $E_t[X](\omega) = \int_A X\,dP/P(A) = (XP)(A)/P(A)$ where A is the atom of $\mathcal{A}$ containing ω. The conditional expectation is constant on the atom and we define $0/0 = 0$.

Consider $E_t[\phi(T)]$. For $s < t$ we have $E_t[\phi(T)](s) = \phi(s)$ since $\{s\}$ is the atom containing s. For $s \geq t$ we have $E_t[\phi(T)](s) = \int_t^\infty \phi\,dP/P(T \geq t)$ since $[t, \infty)$ is the atom containing s.

If $\phi(t) = 1(T = t)$ then for $s < t$ $E_t[1(T = u)](s) = 0$ and for $s \geq t$ $E_t[1(T = u)](s) = P(T = u)/P(T \geq t)$.

If $\phi(t) = 1(T > u)$ then for $s < t$ $E_t[1(T > v)](s) = 0$ and for $s \geq t$ $E_t[1(T > v)](s) = P(T > v)/P(T \geq t)$.

This shows

$$D_t^{T,R}(v) = [RP(t < T \leq v) + P(T > v)]1(T \geq t)/P(T \geq t).$$

If R is random this becomes

$$D_t^{T,R}(v) = \{\int_{t+}^{v} E_t[R|T = u]\, dP(T \leq u) + P(T > v)\}1(T \geq t)/P(T \geq t).$$

For the case of stochastic short rate and the standard assumption rates are independent of default and recovery, we have

$$D_t^{T,R}(v) = \{\int_{t+}^{v} D(t,u)E_t[R|T = u]\, dP(T \leq u) + D(t,T)P(T > v)\}1(T \geq t)/P(T \geq t),$$

where $D(t,u) = E_t D_u/D_t$ is the price at time t of a zero coupon bond maturing at u.

It is convenient to factor the formula above into two components: a *fee leg* and a *protection leg*.

Fee Legs

A fee leg consists of a stream of fixed cash flows that stop at default. Assuming zero recovery, the present value of the stream c_j and times t_j is

$$E[\sum_j c_j D_{t_j} 1(T > t_j)] = \sum_j c_j D(t_j) P(T > t_j)$$

assuming (as above) rates and defaults are independent.

Protection Legs

A protection leg pay unit notional at time of default. The present value of the leg for protection over the period $[0,u]$ is

$$E[\int_0^u D_t P(T = t)\, dt] = \int_0^u D(t) f(t)\, dt$$

where f is the probability density function of the default time.

Possible Implementation

Given a *hazard rate*, $\lambda(t)$, defined by $P(t < T < T + h | T > t) = \lambda(t)h + o(h)$ we get $P(T > t) = \exp(-\int_0^t \lambda(s)\,ds)$ so $dP(T \leq t) = \lambda(t)P(T > t)$ is the density for the default time.

If we assume piecewise constant forward and hazard rates the integral for the protection leg has a closed form solution. Taking the union of the knots for the forward and hazard we will need to compute integrals of the form

$$\int_a^b \exp(-fs)\lambda\exp(-\lambda s)\,ds = \lambda\int_a^b \exp(-(f{+}\lambda)s)\,ds = -(\lambda/(f{+}\lambda))\exp(-(f{+}\lambda)s)|_a^b.$$