Introduction to Mathematical Logic

FOURTH EDITION

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To Arlene

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Preface

This is a compact introduction to some of the principal topics of mathematical logic. In the belief that beginners should be exposed to the easiest and most natural proofs, I have used free-swinging set-theoretic methods. The significance of a demand for constructive proofs can be evaluated only after a certain amount of experience with mathematical logic has been obtained. If we are to be expelled from 'Cantor's paradise' (as non-constructive set theory was called by Hilbert), at least we should know what we are missing.

The major changes in this new edition are the following.

1. In Chapter 2, a section has been added on logic with empty domains, that is, on what happens when we allow interpretations with an empty domain.
2. In Chapter 4, Section 4.6 has been extended to include an outline of an axiomatic set theory with urelements.
3. The subjects of register machines and random access machines have been dropped from Section 5.5 Chapter 5.
4. An appendix on second-order logic will give the reader an idea of the advantages and limitations of the systems of first-order logic used in Chapters 2–4, and will provide an introduction to an area of much current interest.
5. The exposition has been further streamlined, more exercises have been added, and the bibliography has been revised and brought up to date.

The material of the book can be covered in two semesters, but, for a one-semester course, Chapters 1–3 are quite adequate (omitting, if hurried, Sections 1.5, 1.6 and 2.10–2.16). I have adopted the convention of prefixing a D to any section or exercise that will probably be difficult for a beginner, and an A to any section or exercise that presupposes familiarity with a topic that has not been carefully explained in the text. Bibliographic references are given to the best source of information, which is not always the earliest paper; hence these references give no indication as to priority.

I believe that the essential parts of the book can be read with ease by anyone with some experience in abstract mathematical thinking. There is, however, no specific prerequisite.

This book owes an obvious debt to the standard works of Hilbert and

Bernays (1934; 1939), Kleene (1952), Rosser (1953) and Church (1956). I am grateful to many people for their help and would especially like to thank the following people for their valuable suggestions and criticism: Richard Butrick, James Buxton, Frank Cannonito, John Corcoran, Newton C.A. da Costa, Robert Cowen, Anil Gupta, Eric Hammer, Bill Hart, Stephen Hechler, Arnold Koslow, Byeong-deok Lee, Alex Orenstein, Dev K. Roy, Atsumi Shimojima and Frank Vlach.

Elliott Mendelson
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Introduction

One of the popular definitions of logic is that it is the analysis of methods of reasoning. In studying these methods, logic is interested in the form rather than the content of the argument. For example, consider the two arguments:

1. All men are mortal. Socrates is a man. Hence, Socrates is mortal.
2. All cats like fish. Silvy is a cat. Hence, Silvy likes fish.

Both have the same form: All A are B. S is an A. Hence, S is a B. The truth or falsity of the particular premisses and conclusions is of no concern to logicians. They want to know only whether the premisses imply the conclusion. The systematic formalization and cataloguing of valid methods of reasoning are a main task of logicians. If the work uses mathematical techniques or if it is primarily devoted to the study of mathematical reasoning, then it may be called *mathematical logic*. We can narrow the domain of mathematical logic if we define its principal aim to be a precise and adequate understanding of the notion of *mathematical proof*.

Impeccable definitions have little value at the beginning of the study of a subject. The best way to find out what mathematical logic is about is to start doing it, and students are advised to begin reading the book even though (or especially if) they have qualms about the meaning and purpose of the subject.

Although logic is basic to all other studies, its fundamental and apparently self-evident character discouraged any deep logical investigations until the late 19th century. Then, under the impetus of the discovery of non-Euclidean geometry and the desire to provide a rigorous foundation for calculus and higher analysis, interest in logic revived. This new interest, however, was still rather unenthusiastic until, around the turn of the century, the mathematical world was shocked by the discovery of the paradoxes – that is, arguments that lead to contradictions. The most important paradoxes are described here.

1. *Russell's paradox* (1902). By a set, we mean any collection of objects – for example, the set of all even integers or the set of all saxophone players in Brooklyn. The objects that make up a set are called its members or

elements. Sets may themselves be members of sets; for example, the set of all sets of integers has sets as its members. Most sets are not members of themselves; the set of cats, for example, is not a member of itself because the set of cats is not a cat. However, there may be sets that do belong to themselves – for example, the set of all sets. Now, consider the set A of all those sets X such that X is not a member of X. Clearly, by definition, A is a member of A if and only if A is not a member of A. So, if A is a member of A, then A is also not a member of A; and if A is not a member of A, then A is a member of A. In any case, A is a member of A and A is not a member of A.

2. *Cantor's paradox* (1899). This paradox involves the theory of cardinal numbers and may be skipped by those readers having no previous acquaintance with that theory. The cardinal number $\overline{\overline{Y}}$ of a set Y is a measure of the size of the set; $\overline{\overline{Y}} = \overline{\overline{Z}}$ if and only if Y is equinumerous with Z (that is, there is a one–one correspondence between Y and Z). We define $\overline{\overline{Y}} \leqslant \overline{\overline{Z}}$ to mean that Y is equinumerous with a subset of Z; by $\overline{\overline{Y}} < \overline{\overline{Z}}$ we mean $\overline{\overline{Y}} \leqslant \overline{\overline{Z}}$ and $\overline{\overline{Y}} \neq \overline{\overline{Z}}$. Cantor proved that, if $\mathscr{P}(Y)$ is the set of all subsets of Y, then $\overline{\overline{Y}} < \overline{\overline{\mathscr{P}(Y)}}$. Let V be the universal set – that is, the set of all sets. Now, $\mathscr{P}(V)$ is a subset of V; so it follows easily that $\overline{\overline{\mathscr{P}(V)}} \leqslant \overline{\overline{V}}$. On the other hand, by Cantor's theorem, $\overline{\overline{V}} < \overline{\overline{\mathscr{P}(V)}}$. Bernstein's theorem asserts that, if $\overline{\overline{Y}} \leqslant \overline{\overline{Z}}$ and $\overline{\overline{Z}} \leqslant \overline{\overline{Y}}$, then $\overline{\overline{Y}} = \overline{\overline{Z}}$. Hence, $\overline{\overline{V}} = \overline{\overline{\mathscr{P}(V)}}$, contradicting $\overline{\overline{V}} < \overline{\overline{\mathscr{P}(V)}}$.
3. *Burali-Forti's paradox* (1897). This paradox is the analogue in the theory of ordinal numbers of Cantor's paradox and requires familiarity with ordinal number theory. Given any ordinal number, there is a still larger ordinal number. But the ordinal number determined by the set of all ordinal numbers is the largest ordinal number.
4. *The liar paradox*. A man says, 'I am lying', If he is lying, then what he says is true and so he is not lying. If he is not lying, then what he says is true, and so he is lying. In any case, he is lying and he is not lying.†
5. *Richard's paradox* (1905). Some phrases of the English language denote real numbers; for example, 'the ratio between the circumference and diameter of a circle' denotes the number π. All the phrases of the English language can be enumerated in a standard way: order all phrases that have k letters lexicographically (as in a dictionary) and then place all phrases with k letters before all phrases with a larger number of letters. Hence, all phrases of the English language that denote real numbers can

†The Cretan 'paradox', known in antiquity, is similar to the liar paradox. The Cretan philosopher Epimenides said, 'All Cretans are liars'. If what he said is true, then, since Epimenides is a Cretan, it must be false. Hence, what he said is false. Thus, there must be some Cretan who is not a liar. This is not logically impossible; so we do not have a genuine paradox. However, the fact that the utterance by Epimenides of that false sentence could imply the existence of some Cretan who is not a liar is rather unsettling.