A Unified Model of Derivative Securities

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Abstract

Market instruments can be bought or sold at a price and entail cash flows. Every arbitrage-free model of prices and cash flows is parameterized by a positive, adapted process and a vector-valued martingale whose components are indexed by market instruments. This can be used to value, hedge, and manage the risk of derivative securities.

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A *derivative security* is a contract between two parties: I will give you this on these dates if you will give me that on those dates. Derivatives must have existed since before recorded history. The Nobel prize winning breakthrough of Black, Scholes, and Merton was to show how to synthesize derivatives by dynamically trading market instruments based on the borrowing cost used to fund the hedge instead of trying to estimate the the actual growth rate of the underlying securities.

This short note provides a unified model for valuing, hedging, and managing the risk of any derivative security. It shows how they can be synthesized by trading market instruments and turns the spotlight on what may be the next Nobel prize winning problem: how should you hedge if you can't do it continuously?

Notation

If $\mathcal{A}$ is an *algebra* on the set Ω we write $X\colon \mathcal{A} \to \mathbf{R}$ to indicate $X\colon \Omega \to \mathbf{R}$ is $\mathcal{A}$-*measurable*. If $\mathcal{A}$ is finite then the *atoms* of $\mathcal{A}$ form a *partition* of Ω and being measurable is equivalent to being constant on atoms. In this case X is indeed a function on the atoms.

If $\mathcal{A}$ is an algebra of sets, the *conditional expectation* of X given $\mathcal{A}$ is defined by $Y = E[X|\mathcal{A}]$ if and only if Y is $\mathcal{A}$ measurable and $\int_A Y\,dP = \int_A X\,dP$ for all $A \in \mathcal{A}$. This is equivalent to $Y(P|_{\mathcal{A}}) = (XP)|_{\mathcal{A}}$ where the vertical bar indicates restriction of a measure.

A *filtration* indexed by $T \subseteq [0, \infty)$ is an increasing collection of algebras, $(\mathcal{A}_t)_{t\in T}$. A process $M_t\colon \mathcal{A}_t \to \mathbf{R}$, $t \in T$, is a *martingale* if $M_t = E[M_u|\mathcal{A}_t] = E_t[M_u]$ for $t \leq u$.

A *stopping time* is a function $\tau\colon \Omega \to T$ such that $\{\omega \in \Omega \mid \tau(\omega) \leq t\}$ belongs to $\mathcal{A}_t$, $t \in T$.

Market Model

Every *instrument* has a *price*, X_t, and a *cash flow*, C_t, at any trading time, $t \in T$. Instruments are assumed to be perfectly liquid: they can be bought or sold at the given price in any amount. Cash flows are associated with owning an instrument and are almost always 0, e.g., stocks have dividends, bonds have coupons, European options have exactly one cash flow at expiration, futures always have price 0.

A *market model* specifies *prices* $X_t : \mathcal{A}_t \to \mathbf{R}^I$, and *cash flows* $C_t : \mathcal{A}_t \to \mathbf{R}^I$, where I are the available market instruments.

Trading

A *trading strategy* is a finite collection of strictly increasing stopping times, τ_j, and trades, $\Gamma_j : \mathcal{A}_{\tau_j} \to \mathbf{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, $\Delta_t = \sum_{\tau_j < t} \Gamma_j = \sum_{s<t} \Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$. Note the trade at time t is not included in the position at time t: it takes some time for trades to settle.

The *value* (or *mark-to-market*) of a position at time t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: what you would get from liquidating your existing position and the trades just executed. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades you just executed.

Arbitrage

Arbitrage is a trading strategy with $A_{\tau_0} > 0$, $A_t \geq 0$ for $t > \tau_0$ and $\sum_j \Gamma_j = 0$: you make money on the first trade and never lose until the strategy is closed out.

The Fundamental Theorem of Asset Pricing states there is no arbitrage if and only if there exists a *deflator*, $D_t : \mathcal{A}_t \to (0, \infty)$, with

$$X_t D_t = E_t[X_v D_v + \textstyle\sum_{t<u\leq v} C_u D_u].$$

We can assume $D_0 = 1$. If (D'_t) is a deflator then so is (D'_t/D'_0).

Note that if $C_t = 0$ for all $t \in T$, this says $X_t D_t$ is a martingale. If the prices are eventually 0, this says the current price is the expected price of discounted future cash flows. A consequence of the above and the definition of value and amount is

$$V_t D_t = E_t[V_v D_v + \sum_{t<u\leq v} A_u D_u].$$

Note the similarity to the previous displayed equation. Value corresponds to price and amount corresponds to cash flow. This equation is the skeleton key for valuing derivative securities.

If $u > t$ is sufficiently small then $X_t D_t = E_t[(X_u + C_u) D_u]$ and $V_t D_t = (\Delta_t + \Gamma_t) \cdot X_t D_t = \Delta_u \cdot E_t[(X_u + C_u) D_u]$. Since $\Delta_u \cdot C_u = \Gamma_u \cdot X_u + A_u$ we have $V_t D_t = E_t[(\Delta_u \cdot X_u + \Gamma_u \cdot X_u + A_u) D_u] = E_t[(V_u + A_u) D_u]$. The formula above follows by induction.

For a trading strategy that closes out, $V_{\tau_0} D_{\tau_0} = E_{\tau_0}[\sum_{t>\tau_0} A_t D_t] \geq 0$. Since $V_{\tau_0} = \Gamma_{\tau_0} \cdot X_{\tau_0}$, $A_{\tau_0} = -\Gamma_{\tau_0} \cdot X_{\tau_0}$, and $D_{\tau_0} > 0$ we have $A_{\tau_0} \leq 0$. This proves the "easy" direction of the FTAP.

There is no need to prove the "hard" direction since we have a large supply of arbitrage free models: every arbitrage-free model has the form $X_t D_t = M_t - \sum_{s\leq t} C_s D_s$ where $M_t : \mathcal{A}_t \to \mathbf{R}^I$ is a martingale and $D_t : \mathcal{A}_t \to (0, \infty)$. This is immediate by substituting $X_v D_v = M_v - \sum_{s\leq v} C_s D_s$ in the first displayed equation.

Valuing

If a derivative security pays amounts B_j at times v_j and there is a hedge, $(\Gamma_t)_{t\in T}$, that replicates these amounts, the value of the derivative is the cost of setting up the initial hedge: $\Gamma_0 \cdot X_0$. The hedge must satisfy $A_t = 0$ if $t \neq \upsilon_j$ for all j (*self financing*) and $A_t = B_j$ if $t = \upsilon_j$ for some j.

The formula $V_0 = E[\sum_j B_j D_{\upsilon_j}]$ is the value of the initial hedge, $\Gamma_0 \cdot X_0$. It can be computed using the derivative security payments and the deflator.

An European option has a single payment, B_T, at a fixed time T and has value $V_0 = EB_T D_T$. Sometimes it is useful to compute this as $EB_T D_T = E^* B_T ED_T$, where E^* is the expected value under the Esscher transform of the probability measure defined by $dP^*/dP = D_T/ED_T$. P^* is called the *forward measure*.

The only problem with this is ... Not only is it impossible to trade in continuous time, it leads to absured results.

Hedging

The trades at time t are similarly determined by $\Delta_t + \Gamma_t = dV_t/dX_t$, where the last term is the Fréchet derivative. Since we know the position, Δ_t, at time t this determines the trades, $\Gamma_t = dV_t/dX_t - \Delta_t$.

In the continuous time case where stocks are modelled by geometric Brownian motion, this becomes classical Black-Scholes/Merton delta hedging where delta is Δ and gamma is Γ. Under their mathematical assumptions, the hedge perfectly replicates the derivative.

In the real world, it is not possible to perfectly replicate the derivative security. There is still research to be done on when to hedge and how to mangage this risk.

Examples

The Black-Scholes/Merton model is specified by $M_t = (r, s\exp(\sigma B_t - \sigma^2 t/2))$ and $D_t = \exp(-\rho t)$. No need for Ito's lemma, self-financing conditions, or PDE's, much less the Hahn-Banach theorem.

Canonical Deflator

There is a canonical choice for a deflator if repurchase agreements are available.

A *repurchase agreement* at time t, R_t, has price $X_t^{R_t} = 1$ and cash flow $C_{t+dt}^{R_t} = R_t$ so for any arbitrage free model $D_t = E_t[R_t D_{t+dt}]$. Define the *forward repo rate*, f_t, by $R_t = \exp(f_t\, dt)$. The *canonical deflator* is $D_t = \exp(-\int_0^t f_s\, ds)$. As we will see below, the prices of all (non-risky) fixed income securites are determined by the deflator.

Forward

A *forward* contract on underlying S pays $A_t = S_t - f$ at t. The *par forward* is defined so that $V_0 = 0$ so $0 = E(S_t - f)D_t$ and $S_0 = fED_t$. This formula is called the *cost of carry*.

Put-Call Parity

The first thing every trader checks when using a new model is *put-call parity*. A (European) *put option* pays $A_t^p = \max\{k - S_t, 0\}$ at t and a *call option* pays $A_t^c = \max\{S_t - k, 0\}$ at t. Since $A_t^c - A_t^p = S_t - k$ we have $V_0^c - V_0^p = c - p = S_0 - kED_t$, where c and p are the value of the call and put at time 0.

Zero Coupon Bond

A *zero coupon bond* pays one unit at maturity u so it has a cash flow of 1 unit at time u. An arbitrage free model requires the price at time t, $Z_t(u)$, to satisfy $Z_t(u)D_t = E_tD_u$, so $Z_t(u) = E_t \exp(-\int_t^u f_s\,ds)$.

Forward Rate Agreement

A *forward rate agreement* pays -1 unit at the *effective date* u, and $1 + f\delta(u,v)$ at the *termination date* v, where f is the *coupon* and $\delta(u,v)$ is the *day count fraction* for the interval $[u,v]$. The day count fraction is approximately equal to the time in years from u to v for any *day count basis*.

The *par coupon* at time t, $F_t(u,v;\delta)$ is the coupon that makes the price at time $t \leq u$ equal to 0, $0 = E_t - D_u + (1 + F_t\delta(u,v))D_v$. Hence $F_t(u,v;\delta) = (Z_t(u)/Z_t(v) - 1)/\delta(u,v)$ is determined by zero coupon bond prices. Note $F_t\delta E_tD_v = E_t[D_u - D_v]$.

There are also forward rate agreements not involving the exchange of notional. A (fixed rate) *payer* has the single cash flow $(f - F_u(u,v;\delta))\delta(u,v)$ at time v. A *receiver* has the negative of this cash flow. The value at any time $t \leq u$ is determined by

$$
\begin{aligned}
X_tD_t &= E_t(f - F_u(u,v;\delta))\delta(u,v)D_v \\
&= E_tf\delta(u,v)D_v - E_u[D_u - D_v] \\
&= E_tf\delta(u,v)D_v - D_u + D_v \\
&= E_t - D_u + (1 + f\delta(u,v))D_v
\end{aligned}
$$

which is the same as for a forward rate agreement that does exchange notional. These two types of FRAS's have very different risk characteristics.

Remarks

The price of an instrument is not a number. Not only does it depend on whether you are buying or selling, the amount being purchased, and the counterparties involved, determine the price.

The atoms of finance are *exchanges*: $(t; a, i, c; a', i', c')$, where t is the time of the exchange, a is the amount of instrument i the *buyer*, c, decides to obtain for the amount a' in instrument i' the *seller*, c', charges.

Price is a function $X\colon T \times A \times I \times C \times I \times C \to \mathbf{R}$, where T is the set of trading times, A the set of amounts that can be traded, I is the set of market instruments, and C is the set of legal trading entities.

There is no need for a probability measure. The *dual* of the space of bounded functions on a set Ω, $B(\Omega)$, is the space of finitely additive measures on Ω, $ba(\Omega)$.