Bootstrapping Curves

Keith A. Lewis

October 17, 2019

Abstract

Given a collection of fixed income instruments and corresponding prices the bootstrap algorithm constructs a unique curve having piecewise constant forward rates that reprices each instrument.

The *discount* curve, $D(u)$, is the price of a zero coupon bond paying 1 unit at maturity u. The *forward rate*, $f(u)$, and *spot rate*, $r(u)$, are defined by $D(u) = \exp(-\int_0^u f(t)\,dt) = \exp(-ur(u))$. Any one of discount, forward, or spot determine the other two.

Although discounts are easier to use in formulas and software, traders prefer to think in terms of rates.

Note the spot $r(u) = (1/u)\int_0^u f(t)\,dt$ is the average of the forward and $f(u) = r(u) + ur'(u)$. Since averaging smooths things out and taking derivatives can amplify small variations it is numerically more stable to work with forwards.

A *fixed income instrument* is specified by a set of cash flows, (c_j), at times, (u_j). The *present value* is $pv = \sum_j c_j D(u_j)$. When the instrument *price* equals its present value we say the curve "(re)prices" the instrument.

Fitting a Curve

Given a set of fixed income instruments and their prices, we would like to find a discount curve that reprices all the instruments. This problem is highly underdetermined: there are an uncountably infinite number of curves that will do the job.

Bootstrap

The *bootstrap* algorithm is a method for fitting a curve. Suppose we have a curve defined on the interval $[0, u]$ and a fixed income instrument with price p and cash flows $c_1, \ldots, c_m$ at times $u_1, \ldots, u_m$ where $u_m > u$. We can find a unique forward, f, on the interval $[u, u_m]$ that prices the instrument by solving

$$p = \sum_{u_j \le u} c_j D(u_j) + \sum_{u_j > u} c_j D(u) \exp(-f(u_j - u))$$

If there is exactly one cash flow past the end of the curve then there is an explicit solution. We have $p = pv + c_m D(u) \exp(-f(u_m - u))$ where pv is the first sum, so $f = [\log c_m D(u)/(p - pv)]/(u_m - u)$.

There is also an explicit solution when extending the curve using an instrument having exactly two cash flows and a price of 0, e.g., an at-the-money forward rate agreement. We have $0 = c_0 D(u_0) + c_1 D(u_1)$. If $u_0 \le u$ then there is exactly one cash flow past the end of the curve and the previous case holds. If $u_0 > u$ then $0 = c_0 D(u) \exp(-f(u_0 - u)) + c_1 D(u) \exp(-f(u_1 - u))$ so $f = \log(-c_1/c_0)/(u_1 - u_0)$.

The bootstrap method orders the collection of instruments used to build the curve in order of increasing maturity. It then constructs the unique piecewise constant forward using the above method.

Remarks

It is possible that there is no solution to the bootstrap algorithm but this never happens when using accurate market data.

If the interval from the end of the curve to the maturity of the instrument being used to extend it is small then the computation may become unstable.

It is common to see implementations use various splining techniques to smooth the forward curve. It is better to add synthetic instruments with interpolated prices to do "smoothing".