Barrier Options

Keith A. Lewis

November 1, 2019

Abstract

A short description of the taxonomy of barrier options.

Let X_t be the price of a stock at time t.

Define the running maximum of the price $\overline{X}_t = \max_{s\leq t} X_s$ and the running minimum of the price $\underline{X}_t = \min_{s\leq t} X_s$.

The *draw up* is $\overline{X}_t - X_t$. The *draw down* is $X_t - \underline{X}_t$.

A put option with strike k expiring at time t has payoff $(k - X_t)^+$ at t.

A call option with strike k expiring at time t has payoff $(X_t - k)^+$ at t.

Barrier options with strike k and barrier h expiring at t come in eight forms: put/call, in/out, up/down. A put or call option can knock in or out if the running max or min hits the barrier as it goes up or down.

The terminology goes in reverse order.

An *up-in-put* pays $(k - X_t)^+ 1(\overline{X}_t > h)$ at t. You get the put payoff if the running max hits the barrier.

A *down-in-put* pays $(k - X_t)^+ 1(\underline{X}_t < h)$ at t. You get the put payoff if the running min hits the barrier.

An *up-out-put* pays $(k - X_t)^+ 1(\overline{X}_t < h)$ at t. You get the put payoff if the running max does not hit the barrier.

A *down-out-put* pays $(k - X_t)^+ 1(\underline{X}_t > h)$ at t. You get the put payoff if the running min does not hit the barrier.

An *up-in-call* pays $(X_t - k)^+ 1(\overline{X}_t > h)$ at t. You get the call payoff if the running max hits the barrier.

A *down-in-call* pays $(X_t - k)^+ 1(\underline{X}_t < h)$ at t. You get the call payoff if the running min hits the barrier.

An *up-out-call* pays $(X_t - k)^+ 1(\overline{X}_t < h)$ at t. You get the call payoff if the running max does not hit the barrier.

A *down-out-call* pays $(X_t - k)^+ 1(\underline{X}_t > h)$ at t. You get the call payoff if the running min does not hit the barrier.

These can be combined into one formula.

If $c = 1$ in the case of a call option and $c = -1$ in case of a put option, then $(c(X_t - k))^+$ is the payoff of the corresponding call or put option.

If $d = 1$ in the case of the running max and $d = -1$ in the case of the running min, $X_t^d = d \max_{s \le t} dX_s$ is the corresponding running max or min.

If $e = 1$ in the case of an in option and $e = -1$ in the case of an out option, then $eX_t^d > eh$ is the corresponding in or out criterion.

The general barrier option payoff formula is $c(X_t - k)^+ 1(eX_t^d > eh)$.

Exercise. How does the initial price, X_0, affect these formulas?

Exercise. What are the corresponding formulas for early ending barrier options?

Notes

These can be priced using the *reflection principal* for Brownian motion and *Girsanov's Theorem.*

One form of the reflectiona principal for Brownian motion is

$$E[f(B_t)1(\overline{B}_t > a)] = E[f(B_t)1(B_t > a)] + E[f(2a - B_t)1(B_t > a)].$$

Note the right hand side depends only on Brownian motion at time t. If $f(x) = 1$ this formula says $P(\overline{B}_t > a) = 2P(B_t > a)$.

The proof involves *reflected Brownain motion.* Define $B'_t = B_t$ if $\overline{B}_t < a$ and $B'_t = 2a - B_t$ if $\overline{B}_t \ge a$. Both B'_t and B_t are standard Brownian motions.

$$\begin{aligned}
E[f(B_t)1(\overline{B}_t > a)] &= E[f(B_t)1(\overline{B}_t > a, B_t > a)] + E[f(B_t)1(\overline{B}_t > a, B_t \le a)] \\
E[f(B_t)1(\overline{B}_t > a)] &= E[f(B_t)1(\overline{B}_t > a, B_t > a)] + E[f(B'_t)1(\overline{B'}_t > a, B'_t \le a)] \\
E[f(B_t)1(\overline{B}_t > a)] &= E[f(B_t)1(B_t > a)] + E[f(2a - B_t)1(\overline{B'}_t > a, 2a - B_t \le a) \\
E[f(B_t)1(\overline{B}_t > a)] &= E[f(B_t)1(B_t > a)] + E[f(2a - B_t)1(\overline{B'}_t > a, B_t \ge a)] \\
E[f(B_t)1(\overline{B}_t > a)] &= E[f(B_t)1(B_t > a)] + E[f(2a - B_t)1(B_t \ge a)]
\end{aligned}$$

Girsanov's theorem states that for any number α, $B_t - \alpha t$ is Brownian motion under the measure $dP^\alpha/dP = e^{\alpha B_t - \alpha^2 t/2}$, where P is Wiener measure. This and the above can be used to price barrier options.