Probability

Keith A. Lewis

August 31, 2019

Abstract

This note collects salient facts about probability theory.

Contents

Probability is an extension of logic. Instead of propositions being either true or false a degree of belief can be specified for events occurring. All probabilities are conditional on models of available information.

Probability Model

A *probability model* specifies a *sample space* and a *probability measure*.

Sample Space

A sample space is what can happen: heads or tails as the outcome of a coin toss, the integers from 1 to 6 as the outcomes of rolling a single die, the set of all sequences of not more than 280 characters as a model of possible Twitter tweets.

An *event* is a subset of a sample space.

People seem to be surprised probabilities are modeled using sets. Sets have no structure, they are just a bag of things (*elements*).

Probability Measure

A *probability measure* assigns a number between 0 and 1 to events. If Ω is a sample space and P is a probability measure then the measure of the union of sets is the sum of the measure of each set minus the measure of the intersection: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ for events E and F. This is the mathematical way to say measures do not double count.

A probability measure must also satisfy $P(\emptyset) = 0$ and $P(\Omega) = 1$.

Exercise. If Q is a measure with $Q(\emptyset) = a$ and $Q(\Omega) = b$, show $(Q - a)/(b - a)$ is a probability measure.

Expected Value

The *indicator* (or *characteristic*) function $1_A(\omega)$ is 1 if $\omega \in A$ and 0 if $\omega \notin A$. If $X = \sum a_i 1_{A_i}$ where $a_i \in \mathbf{R}$ and A_i are events, Define the *expected value* of X by $EX = \sum_i a_i P(A_i)$.

Exercise. Show that if $\sum_i a_i 1_{A_i} = 0$ then $\sum_i a_i P(A_i) = 0$.

Hint: Replace the A_i by disjoint B_j so $b_j = 0$ for all j.

This shows expected value is well-defined.

Exercise. Show $P(\cup_i A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \cdots$.

Hint: Use $(1_A - 1_{A_1}) \cdots (1_A - 1_{A_n}) = 0$, where $A = \cup_{k=1}^n A_k$.

Algebra

An *algebra of sets*, or *algebra*, on Ω is a collection of subsets (events), $\mathcal{A}$, that is closed under complement and union. This lets us talk about and event not happening and whether event A or B occured.

We also assume the empty set belongs to $\mathcal{A}$, hence also Ω. By De Morgan's Laws an algebra is also closed under intersection. The *power set* of Ω, $2^\Omega = \{E : E \subseteq \Omega\}$, clearly satisfies these conditions.

An *atom* of an algebra is a member, A, of the algebra such that if $B \subseteq A$ and B is in the algebra, then either $B = A$ or B is the empty set.

Partition

A *partition* of a set is a collection of pairwise disjoint subsets who's union is equal to the set.

Exercise. If an algebra on Ω is finite its atoms form a partition of Ω.

Hint: Show $A_\omega = \cap\{B \in \mathcal{A} : \omega \in B\}$, $\omega \in \Omega$, is an atom

This shows there is a one-to-one correspondence between finite partitions and finite algebras of sets. A partition is the mathematical way of specifying partial information. Knowing the outcome, $\omega \in \Omega$, corresponds to complete knowledge. Knowing which atom the outcome belongs to corresponds to partial knowledge. For example, the partition $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ corresponds to knowing whether the roll of a die is odd or even.

The coarsest partition $\{\Omega\}$ corresponds to no knowledge while the finest partition $\{\{\omega\} : \omega \in \Omega\}$ corresponds to complete knowledge.

Measurable

A function $X\colon \Omega \to \mathbf{R}$ is $\mathcal{A}$-*measureable* if the sets $X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\}$ belong to $\mathcal{A}$ for $x \in \mathbf{R}$.

Exercise: If $\mathcal{A}$ is finite, show that a function is measurable if and only if it is constant on atoms of $\mathcal{A}$.

In this case $X\colon \mathcal{A} \to \mathbf{R}$ is indeed a function on the atoms.

Cumulative Distribution

A *random variable* is a variable, a symbol that can be used in place of a number, with additional information: the probability of the values it can take on. The *cumulative distribution function* is $F(x) = F^X(x) = P(X \le x)$. It tells you everything there is to know about X. For example, $P(a < X \le b) = F(b) - F(a)$.

Exercise. Show $P(a \le X \le b) = \lim_{x\uparrow a} F(b) - F(x)$.

Hint: $[a, b] = \cap_n (a - 1/n, b]$.

In general, $P(X \in A) = E1_A = \int 1_A(x)\, dF(x)$ for sufficiently nice $A \subset \mathbf{R}$ where we are using Riemann–Stieltjes integration.

Exercise: Show for any cumulative distribution function, F, that F is non-decreasing, $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ and F is right continuous with left limits.

Every such function defines a random variable.

The cdf $F(x) = \max\{0, \min\{1, x\}\}$ defines the uniformly distributed random variable U. For $0 \le a < b \le 1$, $P(a < U < b) = b - a$.

Two random variables, X and Y, have the same *law* if they have the same cdf.

Exercise. If X has cdf F, then X and $F^{-1}(U)$ have the same law.

Exercise. If X has cdf F, then $F(X)$ and U have the same law.

This shows a uniformly distributed random variable has sufficient randomness to generate any random variable. There are no random, random variables.

The mathematician's definition of a random variable is that it is a measurable function $X\colon \Omega \to \mathbf{R}$. Its cumulative distribution function is $F(x) = P(X \le x) = P(\{\omega \in \Omega \mid X(\omega) \le x\})$. Given a cdf F we can define $X\colon \mathbf{R} \to \mathbf{R}$ to be the identity function and let P be the probability measure defined by F: $P(A) = \int 1_A(x)\, dF(x)$.

Expected Value

The *expected value* of a random variable is defined by the $EX = \int_{-\infty}^{\infty} x\, dF(x)$. The expected value of any function of a random variable is $Ef(X) = \int_{-\infty}^{\infty} f(x)\, dF(x)$.

Joint Distribution

Two random variables, X and Y, are defined by their *joint distribution*, $F(x,y) = F^{X,Y}(x,y) = P(X \le x, Y \le y)$. For example (X,Y) is in the square $(a,b] \times (c,d]$ with probability $P(a < X \le b, c < Y \le d) = P(X \le b, Y \le d) - P(X \le a) - P(Y \le c) + P(X \le a, Y \le c)$.

The *marginal distbutions* are $F^X(x) = F^{X,Y}(x,\infty)$ and $F^Y(y) = F^{X,Y}(\infty,y)$

Independent

The random variables X and Y are *independent* if $F^{X,Y}(x,y) = F^X(x)F^Y(y)$ for all x and y. This is equivalent to $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B.

We also have that $Ef(X)g(Y) = Ef(X)Eg(Y)$ for and functions f and g whenever all expected values exist.

Exercise: Prove this for the case $f = \sum_i a_i 1_{A_i}$ and $g = \sum_j b_j 1_{B_j}$.

Conditional Expectation

The *conditional expectation* of an event B given an event A is $P(B|A) = P(B \cap A)/P(A)$. In some sense, this reduces the sample space to A. In particular, $P(A|A) = 1$. Since $P(A|B) = P(A \cap B)/P(B)$ we have $P(A|B) = P(B|A)P(A)/P(B)$. This is the simplest form of Bayes Theorem. It shows how to update your degree of belief based on new information. Every probability is conditional on given information.

Define $E[X|A] = E[X1_A]/P(A)$ for any random variable X. If $X = 1_B$ then this coincides with the definition of conditional expectation above.

If we write this as $E[X|A]P(A) = E[X1_A]$ then defining $E[X|\mathcal{A}]$ by $E[X|\mathcal{A}]P|_\mathcal{A} = (XP)_\mathcal{A}$ agree on atoms of $\mathcal{A}$.

Moments

The *moments* of a random variable, X, are $m_n = E[X^n]$, $n = 0, 1, 2, \ldots$. They don't necessarily exist for all n, except for $n = 0$. They also cannot be an arbitrary sequence of values.

Suppose all moments of X exist, then for any complex numbers, (c_i), $0 \le E|\sum_i c_i X^i|^2 = E\sum_{j,k} c_j \bar{c_k} X^{j+k} = \sum_{j,k} c_j \bar{c_k} m_{j+k}$. This says the Hankel matrix, $M = [m_{j+k}]_{j,k}$, is positive definite. The converse is also true: if the Hankel matrix is positive definite there exists a random variable with the corresponding moments. This is not a trivial result and the random variable might not be unique.

Cumulants

The *cumulant* of a random variable, X, is $\kappa(s) = \kappa^X(s) = \log E\exp(sX)$. The *cumulants*, κ_n, are defined by $\kappa(s) = \sum_{n>0} \kappa_n s^n/n!$.

It is easy to see $\kappa_1 = EX$ and $\kappa_2 = \operatorname{Var} X$. The third and fourth cumulants are related to skew and kurtosis. We will see the exact relationship below.

If c is a constant then $\kappa^{cX}(s) = \kappa^X(cs)$ so $\kappa^{cX}_n = c^n\kappa^X_n$. If X and Y are independent then $\kappa^{X+Y}(s) = \kappa^X(s) + \kappa^Y(s)$ so $\kappa^{X+Y}_n = \kappa^X_n + \kappa^Y_n$\$

Characteristic Function

The *characteristic function* of a random variable, X, is $\xi(t) = \kappa(it)$.

Fourier Transform

The *Fourier transform* is $\psi(t) = \xi(-t) = \kappa(-it)$. Clearly $\psi(t) = \xi(-t)$.

Bell Polynomial

The relationship between moments and cumulants is given by *Bell polynomials*. They are defined by $\exp(\sum_1^i nftya_n s^n/n!) = \sum_0^\infty B_n(a_1,\dots,a_n)s^n/n!$. Taking the derivative with respect to s and equating powers of s shows $B_0 = 1$ and $B_{n+1}(a_1,\dots,a_{n+1} = \sum_{k=0}^n \binom{n}{k} B_{n-k}(a_1,\dots,a_{n-k})a_{k+1}$.

Bell polynomials show the connection between the moments and the cumulants of a random variable since $E\exp(sX) = \sum_0^\infty EX^n s^n/n! = \sum_0^\infty m_n s^n/n!$ where m_n is the n-th moment and $E\exp(sX) = \exp(\kappa(s)) = \exp(\sum_{n=1}^\infty \kappa_n s^n/n!)$.

Excercise: Show $m_n = \sum_{k=1}^n B_k(\kappa_1,\dots,\kappa_n)$.

Exercise: Find the first five Bell polynomials.

In particular $m_1 = \kappa_1$ and $m_2 = \kappa_1^2 + \kappa_2$ so κ_1 is the mean and κ_2 is the variance. If the mean is 0 and the variance is 1, then κ_3 is the skew and κ_4 is the excess kurtosis.

Copulas

A *copula* is the joint distribution of uniformly distributed random variables on the unit interval. Let U and V be two uniformly distributed random variables.

The copula of X and Y is the joint distribution of $F^{-1}(X)$ and $G^{-1}(Y)$ where F and G are the cumulative distributions of X and Y respectively: $C^{X,Y=}(u,v) = P(F^{-1}(X) \leq u, G^{-1}(Y) \leq v)$.

Exercise: Show $C^{X,Y}(u,v) = H(F(u), G(v))$ where $C^{X,Y}$ is the copula of X and Y, and H is the joint distribution of X and Y.

Exercise: Show $H(x,y) = C(F^{-1}(x), G^{-1}(y))$

This shows how to use the copula and marginal distributions to get the joint distribution.

If $V = U$ then their joint distribution is $C(u,v) = P(U \leq u, V \leq v) = P(U \leq u, U \leq v) = P(U \leq \min\{u,v\}) = \min\{u,v\} = M(u,v)$.

If $V = 1-U$ then their joint distribution is $C(u,v) = P(U \leq u, V \leq v) = P(U \leq u, 1-U \leq v) = P(1-v \leq U \leq u) = \max\{u-(1-v), 0\} = \max\{u+v-1, 0\} = W(u,v)$

Exercise: For every copula, $W \leq C \leq M$.

Examples

Move!!! These can be used to prove the *central limit theorem*: if X_j are independent, identically distributed random variables with mean zero and variance one, then $(X_1 + \cdots X_n)/sqrtn$ converges to a standard normal random variable.

If X is normal then $E\exp(X) = \exp(EX + \mathrm{Var}(X)/2)$ so the cumulants satisfy $\kappa_n = 0$ for $n > 2$.

If X is Poisson with parameter λ then

$$
\begin{aligned}
Ee^{sX} &= \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \lambda^k / k! \\
&= \sum_{k=0}^{\infty} (e^s \lambda)^k e^{-\lambda} / k! \\
&= \exp(\lambda(e^s - 1))
\end{aligned}
$$

so $\kappa(s) = \lambda(e^s - 1)$ and $\kappa_n = \lambda$ for all n. ### Normal

Poisson

Infinitely Divisible

A random variable, X, is *infinitely divisible* if for any positive integer, n, there exist independent, identically distributed random variables X_1,…,X_n such that

$X_1 + \cdots + X_n$ has the same law as X.

Characteristic function ...

moments, Hamburger moment problem.

cumulants, Bell polynomials

Normal

Poisson

Infinitely Divisible

Stochastic Processes

A *stochastic process* is ...

Brownian Motion

reflection

L'evy Processes

Remarks

Cheval de Mere

Pascal

Bernoulli(s)

Kolmogorov

Willy Feller

Examples

Discrete

A *discrete* random variable is defined by $x_i \in \mathbf{R}$ and $p_i > 0$ with $\sum p_i = 1$. The probability the random variable takes on value x_i is p_i.

Bernoulli

A *Bernoulli* random variable has $P(X = 1) = p$, $P(X = 0) = 1 - p$.

Binomial

A *Binomial* random variable ... $P(X = k) = \binom{n}{k}/2^n$.

Uniform

The random variable, U, that is *uniformly distributed* on the *unit interval*, $[0, 1]$, has cdf $F(x) = x$ if $0 \leq x \leq 1$, $= 0$ if $x < 0$, and $= 1$ if $x > 1$.

Normal