OPTIMAL MICROSTRUCTURE TRADING WITH A LONG-TERM UTILITY FUNCTION

JEROME BENVENISTE AND GORDON RITTER*

Courant Institute of Mathematical Sciences
New York University
251 Mercer St., New York, NY 10012

ABSTRACT. We combine Almgren–Chriss optimal execution with market microstructure in a framework where passive (joining the queue in a limit order book) or aggressive (willing to cross the bid-offer spread) modes of execution are allowed. To achieve this, we represent the Almgren–Chriss strategy within the framework of Hamiltonian dynamics. We then show that if a risk-neutral agent has expected returns equal to Hamilton's generalized momenta, then such agent repeatedly solving a myopic wealth-maximization problem reproduces the Almgren and Chriss solution. Hence the vector of generalized momenta p represents effective microstructure alphas, and also is the gradient of the Bellman value function. We demonstrate that our formulation is computationally efficient and provide a practical algorithm, accompanied by a numerical example which illustrates what can go wrong in the naive approach.

1. INTRODUCTION

The paper by Almgren and Chriss (2001) is one of the most influential works on the optimal execution of portfolio transitions, and provided the groundwork for many subsequent studies. Their work exhibits optimal execution as a special case of the celebrated expected utility theory of Arrow (1963) and Pratt (1964), as the latter framework is general enough to include any situation in which wealth is a random variable. An execution desk with a suitable utility of wealth $u(w)$ should optimally trade to maximize $E[w] - (\kappa/2)V[w]$ where $\kappa > 0$ is the risk-aversion constant. In the execution context, $E[w]$ is typically negative and describes the total trading cost

Date: December 2, 2017.
Key words and phrases. Finance; Investment analysis; Bayesian statistics; Portfolio optimization.
* Corresponding author. *E-mail address:* ritter@post.harvard.edu.

incurred during the execution, while $V[w]$ is the total variance of wealth during the execution.

The work of Almgren and Chriss (2001) is both mathematically elegant and practically useful, but the authors chose to maximize utility over a certain restricted set of execution strategies. Specifically, they abstract away microstructure details by assuming that a parent order is split into a sequence of child orders, which are then passed to an *executor* (or "microtrader" in the language of Gatheral (2012)). In the Almgren-Chriss model, the microtrader doesn't know the overall utility function, and it has a mandate to always completely fill the child order it is given. Almgren and Chriss (2001) concern themselves entirely with the actions of the "macrotrader" whose job it is to optimally slice a large order into a sequence of child orders, or in the multiple-security case, a wave of larger orders into a sequence of child waves composed of smaller orders.

In other words, the "microtrader" is effectively a list of constraints: it *cannot* know the parent utility function, it *must* completely fill the child order. As in any constrained problem, the value of the utility function at optimality may be less than its value in a larger unconstrained problem. Moreover, the distinction between microtrader and macrotrader is arbitrary and imposed for convenience, rather than necessity. In the present work, we lift these constraints. In our framework, there is no macrotrader or microtrader; there is simply a *trader* who interacts with a continuous limit order book in such a way as to optimize a longer-term utility function. In that sense our trader is somewhat analogous to the dealer in Stoll (1978), but is under no pressure to make two-sided markets and may trade aggressively when it is optimal to do so.

Typical *modus operandi* for trading in a limit order book market include "aggressive" and "passive" execution, each of which really describes a style of execution, or a whole range of loosely-related systems, and not a specific algorithm or order type.

"Aggressive" execution refers to trading primarily by means of market orders (or marketable limit orders) and ensures that a child order will be filled completely and with some immediacy, but also pays a real (and perhaps very high) cost for the convenience of immediacy. This cost is typically modeled as a linear term dependent on the bid-offer spread, plus a nonlinear term capturing market impact. Such a cost structure is assumed by many authors; for example, Alfonsi, Fruth, and Schied (2010) assume that executions cross the spread and eat into the order book when calculating costs. Aggressive execution is analogous to writing one or more in-the-money options, which are then exercised immediately by other market participants.

"Passive" execution corresponds roughly to trading via limit order placement strategies which aim to constantly remain in the queue on the near side of the spread until the order is filled, presumably by an adversary who chooses to aggress. Passive execution avoids both spread pay and, arguably, avoids market impact as well. There is still impact in the sense that some liquidity on the other side is removed from the market, but the price at which passive orders are filled – if they are – is guaranteed to be better than any level on the other side of the book. However, passive execution provides no guarantees regarding immediacy or completeness of the fill. Thus passive execution is analogous to writing a near-the-money option – it's always possible that such an option may never be exercised.

At the microstructure level, the space of possible decision variables is large. In order to move forward, we introduce a simplified three-state model of the decision space which captures the spirit, if not the full detail, of the decision variable involved in interacting with the limit order book. In our simplified model of the decision space, a trader may choose, for each asset and at each time instant, one of three *states of aggression*, defined momentarily.

Definition 1. By the term *states of aggression* we mean:

- ($\mathfrak{w}$) *Wait* – cancel all orders in this security
- ($\mathfrak{p}$) *Participate passively* – join the queue on the near side, resulting in the possibility of a fill with some probability per unit time, or
- ($\mathfrak{a}$) *Participate aggressively* – place a marketable limit order for one lot at the other side of the inside market, typically resulting in an immediate fill.

There is another way to think of our simple three-state model, which clarifies the duality with market-making strategies. Suppose one is a market maker, but also has a utility function of wealth and, hence, a desired target portfolio. Stoll (1978) considers just such a model, in which "by appropriate changes in his bid-ask quotations the dealer encourages transactions by the public that will rebalance his portfolio. In other words, the dealer acts passively setting prices and letting the public choose which stocks it will purchase from him and which stocks it will sell to him. Portfolio adjustments arising from his dealer activity are therefore restricted to those stocks in which he makes a market." Our three states of aggression $\{\mathfrak{w}, \mathfrak{p}, \mathfrak{a}\}$ can be thought of as prescriptions for how to set bid and offer prices. The "wait" state $\mathfrak{w}$ corresponds to setting bid and offer prices much wider than the current inside market, while the "passive" state $\mathfrak{p}$ corresponds to making a market in which the near-side quote is right at the market, and the far side quote is very far away from the current inside market. Finally, the "aggressive" state $\mathfrak{a}$ corresponds to placing both sides of our market beyond the far side of the current inside market.

One must also choose the quantity to be traded with each microstructure decision. Such quantities are typically multiples of the relevant security's lot size. Multiple-lot aggressions can be viewed as a sequence of aggressive one-lot executions. This is not true for passive trading, since limit orders must join a queue; here we ignore this complication for simplicity. Hence within the framework of these assumptions, we lose little generality in viewing each aggressive or passive order as being an order for one lot (typically 100 shares). This allows us to model aggression as a tri-nary decision variable. This decision variable enlarges the decision space relative to the Almgren-Chriss model. Moreover, introducing aggression transforms the original dynamic programming problem into a stochastic one, since the fill of any passive order is a random variable.

Additional complications arise with passive execution in the multi-asset case. Almgren and Chriss (2001) show in an appendix that much of their analysis generalizes to multiple-security portfolios in a fairly straightforward manner, but their assumption is that child orders will be filled completely and at the specified time. As in the single-asset case, this amounts to working in a restricted strategy space, in which one doesn't control the additional decision variable representing the decision to aggress or join the near-side queue.

However, executing certain kinds of portfolio transitions passively could have catastrophic effects.

Example 1. Consider a market-neutral portfolio which submits a market-neutral wave to an execution desk. In other words, if h_0 is the current portfolio, and h is the target portfolio, then h_0, h, and $h - h_0$ all have zero net exposure and zero exposure to market beta. The portfolio transition (or *wave*) is given to the desk with one hour allowed to complete the execution. We proxy the market via a broad index such as the S&P 500. Let m_0, m denote the index value at the start and end of the trading interval. Suppose that the market falls by 5% over the trading interval, so $m/m_0 \approx 0.95$. Then most of the passive buys will be filled relatively quickly, while many of the passive sells will not be filled at all. For most of the trading interval, the investor's portfolio will have a positive exposure to the market as the market is falling, leading to a loss of wealth. This argument is *symmetric* with respect to the sign of the market move; a 5% rise in the market leads to a similar loss.

The loss due to slippage which occurs in Example 1 would be mostly avoided if the portfolio transition were executed in such a way that the overall market beta remained low throughout the execution. Intuitively, as the portfolio evolved out of hedge due to passive execution, the executor could become more aggressive in those orders which would most effectively get the portfolio back to being hedged.

Further, many authors have documented significant short-term predictability in the limit order books of the more liquid cash equities and futures markets. Gould and Bonart (2015) find that queue imbalance provides significant predictive power for the direction of the next mid-price movement, while Cont, Stoikov, and Talreja (2010) present a stylized model of the dynamics of the limit order book that permits the computation of the conditional probability of a passive fill.

Very short-term predictability in the limit order book dynamics does not necessarily represent an arbitrage opportunity, as the size of the effect may not be sufficient to justify trading based on the signal alone. Nonetheless, such signals may still add value to an execution desk who will be trading regardless. For example, the short-term signals could determine the optimal current state of aggressiveness in the sense of Def. 1.

Almgren and Chriss (2001) do not specify how to use microstructure signals, possibly because such signals are of limited use when all child orders are traded aggressively. The present work provides a way to blend the risk-minimization properties of the Almgren-Chriss framework with any arbitrary set of short-term execution alphas.

In the rest of the paper, we show first that the Almgren-Chriss strategy can be considered as a special case of Hamiltonian dynamics. The associated dynamics has an intuitive trading interpretation, in which the generalized momenta $p := \delta L/\delta \dot{q}$ play the role of microstructure alphas, and the trader at each instant must solve a very simple expected-wealth-maximization problem with respect to these "alphas." This expected-wealth-maximization is simply the optimization involved in computing the Legendre-Fenchel conjugate.

We then show that the Hamiltonian representation allows us to generalize the Almgren-Chriss strategy to cases in which the trader may:

(1) use passive execution to reduce costs when it would be beneficial to do so, and more generally,
(2) choose among the three states of aggressiveness detailed above at each time instant, and
(3) make use of short-term microstructure alphas as implied by Cont, Stoikov, and Talreja (2010), Gould and Bonart (2015), or any other model.

The use of Hamiltonian methods in optimal control problems and more specifically in optimal investment and trading problems is of course not new. Pontryagin introduced "adjoint equations" as an approach to deterministic optimal control problems in the 1950s; these methods were extended to stochastic control theory through the "stochastic maximum principle" of Bismut (Bismut (1976); see also Bismut (1978)); this work has been influential in the optimal portfolio selection literature. Much more recently, Naujokat and Westray (2011) recognized the importance of using passive orders in the context of tracking a continuous-time trading trajectory in illiquid markets. However, like much of the existing literature, Naujokat and Westray (2011) treat the single-asset case; the advantage of our approach is that it gives a strategy for execution of multiple assets (which scales to portfolios of many thousands of assets) that's implementable in practice.

Notation. For a matrix A, the transpose will be denoted by A' or $A^\top$. We apply the same notation to vectors, hence both $v'w$ and $v \cdot w$ denote the dot product, or scalar product. A dot over a function of time, such as $\dot{q}(t)$, means the time derivative.

2. Almgren-Chriss and Hamiltonian Dynamics

We first re-derive the Almgren-Chriss path using the calculus of variations. This is necessary both to prove that our framework is a generalization of Almgren and Chriss (2001), but also because our algorithm requires as inputs the generalized momenta, p, familiar from Hamiltonian dynamics. Below we show how to compute p in semi-closed form, ie. in terms of an eigen-decomposition that need only be computed once for the entire trading path. This derivation entails working in the continuous-time limit, even though real-world trading paths are discrete sequences of trades. We shall then subsequently show explicitly how the continuous-time formulation informs the discrete formulation. Along these lines, Gatheral and Schied (2011) solve the HJB equation explicitly in the Almgren–Chriss framework.

Almgren and Chriss (2001) consider two kinds of impact: permanent and temporary, where temporary is really *instantaneous* as it is assumed to have no impact at all on the price process in any period other than the one in which the trade occurs. Furthermore, Almgren and Chriss (2001) consider only the linear form of permanent impact, and ultimately show that linear permanent impact plays no role in the continuous-time limit. Throughout this paper, we follow Almgren and Chriss (2001) in treating all temporary impact as instantaneous, and treating permanent impact as linear, hence irrelevant to the continuous-time limit.

These impact assumptions are an approximation which is only valid within a certain regime. For example, if we repeatedly aggress with medium to large order sizes within a short timeframe, it is unrealistic to assume that the impact will revert instantly. We must never push ourselves, by result of our own actions, into a regime where we know that our model no longer holds.

Let $q(t)$ denote a hypothetical continuous-time trading path, so $q(t) \in \mathbb{R}^n$ denotes the portfolio positions (or holdings) at time t, denominated in dollars or any other convenient numeraire. Following the notation of Gatheral (2010), we let $x(t)$ denote the holdings in shares, rather than in numeraire units.

Much of the literature on market impact (see Gatheral (2010) for example) represents impact as a function of the *trading rate* or the time-derivative of the positions. It is conventional in physics, and increasingly in finance as well, to denote the time-derivative of a path by a dot over the letter. Hence in our notation the trading rate (in dollars per unit time) is $\dot{q}(t)$, defined to be $dq(t)/dt$.

The *instantaneous trading cost function* is defined as the continuous-time limit of any ordinary trading cost function. In the single-security case, if we trade δq dollars in some small time interval of length δt, and this costs $\lambda \cdot \delta q/\delta t$ times traded notional for some $\lambda > 0$, then the total cost in dollars per unit time is

$$\lambda(\delta q/\delta t)^2 \equiv c(\delta q/\delta t)$$

where $c(v) = \lambda v^2$. However, we stress that $c()$ need not be quadratic, merely convex. In the multiple-security case, $c : \mathbb{R}^n \to \mathbb{R}_+$.

Let $\Sigma \in S^n_{++}$ denote the asset-level covariance matrix, which we assume has strictly positive spectrum. Practically, if n is large the estimation of Σ should be approached via a stable procedure such as (17) below.

Consider the problem of starting from a portfolio q_0, and executing a wave of orders over a fixed, pre-determined time interval $[0, T]$, leading to target portfolio q_{opt}. This can be viewed as a problem in the calculus of variations. As in classical mechanics, it has both a Lagrangian and a Hamiltonian formulation, which are convex duals to each other. The Hamiltonian is related to the Lagrangian by the Legendre-Fenchel transform.

The optimal-trading problem is then

$$\min_q \int_0^T L(q(t), \dot{q}(t))\, dt \quad \text{s.t.} \quad q(0) = q_0 \;\text{ and }\; q(T) = q_{\text{opt}} \tag{1}$$

where L is the *Lagrangian*

$$L(q, v) = c(v) + \frac{1}{2}\kappa(q - q_{\text{opt}})'\Sigma(q - q_{\text{opt}})\,. \tag{2}$$

where $\kappa > 0$ is the risk-aversion constant.

At a stationary point, the Euler-Lagrange equation is satisfied:

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q} = 0.$$

A first-order system can be obtained if we introduce a new variable p, whose components are called *generalized momenta* and defined as

$$p := \frac{\delta L}{\delta v}(q, \dot{q}) \tag{3}$$

If the conditions of the implicit function theorem are satisfied, we could in principle algebraically solve (3) for $\dot{q}$, obtaining

$$\dot{q} = \phi(q, p)$$

for some function ϕ defined implicitly by (3). The Euler equation then takes the form

$$\dot{p} = \frac{\delta L}{\delta q}(q, \dot{q}) = \frac{\delta L}{\delta q}(q, \phi(q,p)) \equiv \psi(q,p)$$

As the functions ϕ, ψ are algebraic (not involving derivatives), we have a system of $2n$ first-order ODE given by

$$\dot{q} = \phi(q,p), \quad \dot{p} = \psi(q,p) \tag{4}$$

These equations can be expressed more symmetrically by introducing the *Hamiltonian*

$$H(q,p) = p\phi(q,p) - L(q, \phi(q,p))$$

Eqns. (4) are equivalently written in a form known as *Hamilton's equations*:

$$\dot{q} = \frac{\delta H}{\delta p}, \quad \dot{p} = -\frac{\delta H}{\delta q} \tag{5}$$

We can now solve the variational problem (1), as it is sufficient to solve (5) subject to the indicated boundary conditions. If we do this with the same cost function as Almgren and Chriss (2001), we must obtain the same solution.

Suppose $c(v) = \frac{1}{2} v' \Lambda v$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix. We assume that no trading is free of cost, so $\lambda_i > 0$ for all i. From (2) and (3), we see that the generalized momenta are $p = \Lambda \dot{q}$ and hence algebraically solving, one has $\phi(q,p) = \Lambda^{-1} p$. The Hamiltonian is then

$$H(q,p) = \frac{1}{2} p \Lambda^{-1} p - \frac{1}{2} \kappa (q - q_{\text{opt}})' \Sigma (q - q_{\text{opt}}) \, .$$

Hamilton's equations then become:

$$\dot{q} = \Lambda^{-1} p, \qquad \dot{p} = \kappa \Sigma (q - q_{\text{opt}}) \tag{6}$$

or more simply

$$\frac{dx}{dt} = Ax + b, \tag{7}$$

where $x : \mathbb{R} \to \mathbb{R}^{2n}$ and $A \in M(2n; \mathbb{R})$, $b \in \mathbb{R}^{2n}$ are given by

$$x = \begin{pmatrix} q \\ p \end{pmatrix}, \ A = \begin{pmatrix} 0 & \Lambda^{-1} \\ \kappa\Sigma & 0 \end{pmatrix}, \ \text{ and } \ b = \begin{pmatrix} 0 \\ -\kappa \Sigma q_{\text{opt}} \end{pmatrix}. \tag{8}$$

Duhamel's formula yields

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} b \, ds \tag{9}$$

To make use of (9) one must compute the one-parameter group e^{tA}. The usual way to do this involves the Jordan canonical form of A. In fact, the special structure of this problem implies that A is diagonalizable, as we show below.

Before proceeding to the multi-asset case, it is instructive to consider the single-asset case, in which case $\Lambda = (\lambda)$ and $\Sigma = (\sigma^2)$ are 1×1 matrices and

$$e^{tA} = \begin{pmatrix} \cosh(\omega t) & (\lambda\omega)^{-1}\sinh(\omega t) \\ \lambda\omega \sinh(\omega t) & \cosh(\omega t) \end{pmatrix}, \qquad \omega := \sqrt{\frac{\kappa\sigma^2}{\lambda}} \tag{10}$$

Applying (10) to (9), the full solutions to Hamilton's equations (6) in the single-asset case are then

$$q(t) = q_{\mathrm{opt}} + (q_0 - q_{\mathrm{opt}})\cosh(t\omega) + (\lambda\omega)^{-1} p_0 \sinh(t\omega) \tag{11}$$

$$p(t) = \lambda\omega\,(q_0 - q_{\mathrm{opt}})\sinh(t\omega) + p_0 \cosh(t\omega) \tag{12}$$

The path given by (11) necessarily satisfies the initial condition $q(0) = q_0$, and imposing the final condition $q(T) = q_{\mathrm{opt}}$ gives

$$p_0 = -\lambda\omega\,(q_0 - q_{\mathrm{opt}})\coth(T\omega). \tag{13}$$

Eqns. (11) and (12) verify the claim made above that, if we use the same (quadratic) cost function as Almgren and Chriss (2001), then the Almgren and Chriss (2001) trading paths can be derived from Hamiltonian dynamics. Amusingly, this provides a rather direct analogy to classical mechanics, in which the variance term plays the role of a "potential energy" and the trading cost term plays the role of a "kinetic energy."

Theorem 1. In the general, multi-asset case, A defined by (8) is always diagonalizable. The eigenvalues of A are the positive and negative square roots of the eigenvalues of the positive-definite symmetric matrix $\kappa\Lambda^{-1/2}\Sigma\Lambda^{-1/2}$.

Proof. Write

$$A = \begin{pmatrix} 0 & D \\ S & 0 \end{pmatrix}$$

where $D = \Lambda^{-1}$ is diagonal and $S = \kappa\Sigma$ is symmetric. The eigenvalue equation for A then reads $Av = \gamma v$, and if $v = (x \quad y)^\top$ then this becomes $Dy = \gamma x, \quad Sx = \gamma y$ which gives for x and y:

$$\gamma^2 x = DSx, \quad \gamma^2 y = SDy.$$

The eigenvectors of A must therefore be of the form $(x \;\; y)^\top$ where $x \in \mathbb{R}^n$ is an eigenvector of DS and $y \in \mathbb{R}^n$ is an eigenvector of SD, and the eigenvalues of A are square roots of the associated eigenvalues of DS or SD. Note that DS or SD have the same eigenvalues, and each has the same eigenvalues as the symmetric, positive-definite matrix

$$M := D^{1/2} S D^{1/2}. \tag{14}$$

Let $(v_i)_{1 \le i \le n}$ be a basis of eigenvectors for M, so that $Mv_i = m_i v_i$ with $m_i > 0$. Let $x_i = D^{1/2} v_i$ and let $y_i = m_i^{-1/2} S x_i$. Then

$$A \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} D y_i \\ S x_i \end{pmatrix} = \begin{pmatrix} m_i^{-1/2} D S x_i \\ m_i^{1/2} y_i \end{pmatrix} = \sqrt{m_i} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Similarly, if we take $y_i = -m_i^{-1/2} S x_i$ we construct the eigenvector with eigenvalue $-\sqrt{m_i}$. We have thus constructed a basis for $\mathbb{R}^{2n}$ consisting of eigenvectors of A. Hence A is diagonalizable, completing the proof. □

By Theorem 1 we can write $A = P \Gamma P^{-1}$ for some invertible matrix P whose columns are the eigenvectors of A, and some diagonal matrix $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_{2n})$, where $\gamma_\alpha, \alpha = 1 \ldots 2n$ are the eigenvalues of A. This is also the Jordan canonical form of A, and gives the usual way of computing the matrix exponential:

$$\exp(tA) = P \exp(t\Gamma) P^{-1} \tag{15}$$

Moreover, as the proof of Theorem 1 actually constructs the eigenvectors, we can see that the columns of P are of the form $(x_i \;\; y_i)^\top$ where $x_i = \Lambda^{-1/2} v_i$ and $y_i = \pm m_i^{-1/2} \kappa \Sigma \Lambda^{-1/2} v_i$ for some eigenvector v_i of M with $M v_i = m_i v_i$. As an aside, we have also shown that A is *traceless*, $\operatorname{tr}(A) = 0$, and hence that $\det(e^{tA}) = \exp(\operatorname{tr}(tA)) = 1$ which checks out for (10).

To complete the solution, we need to impose the final condition $q(T) = q_{\text{opt}}$ as before. In so doing we find a multi-asset analogue of (13). For simplicity, take $q_{\text{opt}} = 0$ (ie. liquidation as opposed to trading towards a fixed target), and introduce notation for the $n \times n$ blocks within $\exp(tA)$ as follows:

$$U(t) = \exp(tA) = P \exp(t\Gamma) P^{-1} = \left(\begin{array}{c|c} U_{11}(t) & U_{12}(t) \\ \hline U_{21}(t) & U_{22}(t) \end{array} \right)$$

Then $p_0 \in \mathbb{R}^n$ is determined by solving the n equations $[U(T)(q_0 \;\; p_0)^\top]_i = 0$ for all $i = 1, \ldots, n$. These equations determine p_0, implicitly, as a function of q_0 and T. Explicitly,

$$p_0 = \Xi(q_0, T) = -U_{12}(T)^{-1} U_{11}(T) q_0 \tag{16}$$

where this equation serves to define $\Xi(q_0, T)$. Note (16) is the n-dimensional analogue of (13).

This formulation allows a computationally efficient implementation. Computing $U(t) = P \exp(t\Gamma) P^{-1}$ for various values of t is fast once the diagonalization of A (yielding P and Γ) has been computed. Furthermore, the matrix $-U_{12}(t)^{-1} U_{11}(t)$ could be computed once at the beginning of the day for all t in a sufficiently fine grid, ie. it does not need to be recomputed each time one has a new portfolio q_0 and wishes to find the associated generalized momenta from (16). Hence an execution desk could effect the liquidation of hundreds of different portfolios using the same set of $U(t)$, computed once before the trading day begins.

Eq. (15) together with the observations just made thus provide a complete prescription for computing the paths $q(t)$ and generalized momenta $p(t)$ from the Duhamel formula (9). The only expensive step is obtaining the eigenvalue decomposition of M, defined above in (14), but this need only be done once at the beginning of the execution path (not at every step of the algorithm to be presented below). To build intuition in a simple special case, note that when all assets have the same cost function, then Λ is proportional to the identity matrix, and v_i and x_i are proportional to the principal component directions.

Finally, to implement the above procedure in practice, the asset-level covariance $\Sigma \in S^n_{++}$ must be estimated. We adopt the standard factor model approach, in which we assume that a stock's returns over a given period is the sum of components arising from (unobserved) returns to certain common factors and an idiosyncratic component. This implies a reduction of Σ to the form

$$\Sigma := \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2) + XFX' \tag{17}$$

where X is an $n \times k$ vector matrix of factor loadings, F is the covariance matrix of the factors, and $\sigma_i^2 > 0$ for all $i = 1, \ldots, n$.

If n is large, we choose $k \ll n$ and use a model of the form (17) to estimate Σ. Practically, we take the matrix X to consist of several "style factors" including beta, size, momentum, value, and volatility inspired by Fama and French (1993) and Carhart (1997), augmented with an industry classification based on GICS. We estimate σ_i^2 and F empirically. Bayesian estimation of σ_i^2 ensures that σ_i^2 are bounded away from zero; the inverse-gamma posterior has vanishingly small probability mass around the origin. Practically, this is important: with the diagonal elements of D bounded away from zero, Σ is stably and efficiently invertible because it is diagonal plus low-rank.

3. Generalized Momenta and the Value Function

In this section, we wish to connect the generalized momenta discussed above with optimization theory and dynamic programming. In the notation from the previous section, define the *value function*

(18)
$$V(t,x) = -\min_q \int_t^T L(q(s), \dot{q}(s))\, ds \quad \text{s.t.} \quad q(t) = x \ \text{ and } \ q(T) = q_{\text{opt}}.$$

Thus $V(t,x)$ is the remaining utility gain from time t obtained from following the best policy, when the current state at time t is x. (We use the negative of the integral so that a higher value is better.)

We will prove the following theorem from classical mechanics:

Theorem 2. The function V above satisfies the Hamilton-Jacobi differential equation:

$$\frac{\partial V}{\partial t} + H(x, \nabla V) = 0 \tag{19}$$

with the singular final condition:

$$V(T,x) = \begin{cases} 0, & \text{if } x = q_{\text{opt}} \\ \infty, & \text{if } x \neq q_{\text{opt}}. \end{cases}$$

Proof: Let q^* be the path solving the problem (18), and let $h > 0$ be a small number. By the Bellman's principle, we have

$$V(t,x) = -\int_t^{t+h} L(q^*(s), \dot{q}^*(s)) ds + V(t+h, q^*(t+h)) \tag{20}$$

$$\approx -L(x, \dot{q}^*(t))h + V(t,x) + \frac{\partial V}{\partial t} h + \nabla V \cdot \dot{q}^*(t) h. \tag{21}$$

where we have dropped terms of order greater than 1 in h. Then the optimality of q^* implies that the path $q^*|_{[t,t+h]}$ must maximize the right hand side of (20) among all paths on the interval $[t, t+h]$ such that $q^*(t) = x$; since h is small, such paths can be well approximated by linear paths; thus

$\dot{q}^*$ must solve

$$\sup_v \{-L(x,v) + \nabla V \cdot v\}. \tag{22}$$

Inserting this expression into (21), collecting terms gives

$$\begin{aligned} 0 &\approx \frac{\partial V}{\partial t} h + \sup_v \{-L(x,v) + \nabla V \cdot v\} h \\ &= \frac{\partial V}{\partial t} h + H(x, \nabla V) h; \end{aligned}$$

dividing through by h and letting $h \to 0$ gives the Hamilton-Jacobi equation.

The above proof also gives an interpretation of the generalized momenta p in terms of the value function. Indeed, the first-order conditions for (22) imply that along an optimal trajectory q, the equation

$$\nabla V = \frac{\partial L}{\partial v}(q, \dot{q})$$

holds. The right-hand side is precisely the definition of p from (3). Thus p is the direction in which the "value of the position" increases most rapidly.

4. Passive Execution and Fill Probability

We now show how to include passive execution and fill probability into the formulation. Let $q(t)$ be an optimal path and $\dot{q}(t)$ be its velocity vector, or vector of trading rates. Using the natural convex duality between the Lagrangian and Hamiltonian, we note that the optimal instantaneous trade $\dot{q}(t)$ at each time t is the argument v which solves

$$\operatorname*{argmax}_v \,[p \cdot v - c(v)]. \tag{23}$$

We interpret (23) as an optimization in a risk-neutral world. Indeed, if a risk-neutral agent had a vector of expected returns which happened to exactly equal the vector of generalized momenta, p, and sought only to maximize net profit, irrespective of risk, then (23) is the problem faced by this agent. This agent can be considered "myopic" because any information concerning more than one period ahead is available to the agent only indirectly, insofar as p depends on the rest of the trading path.

This intuition is related to the interpretation of the generalized momenta as the gradient of the value function, established in the previous section. Indeed, for a myopic risk-neutral investor who does not face transaction costs, the "value of a position" x is simply the expected profit over the next period, i.e.

$$V(x) = r \cdot x$$

where r is the vector of expected returns. But then

$$\nabla V = r.$$

One could ask the question: "do there exist microstructure alphas which somehow encode the informational content of the value function?" The answer is yes, and the generalized momenta p are such alphas. A myopic agent needs only to know these "microstructure alphas" in order to make the right decisions for long-term utility. In other words, we consider a hypothetical agent whose views are

$$\mathbb{E}[r] = p \tag{24}$$

where r denotes a very short-term return associated to the mid price. Eq. (23) is then trivially equivalent to

$$\operatorname*{argmax}_v \, \mathbb{E}[r \cdot v - c(v)]\,. \tag{25}$$

The generalized momenta p are *effective microstructure alphas* because the agent behaves as a rational agent with expectations (24) would behave, even though the components of p do not literally represent forecasts of price return.

Now consider the same agent optimizing expected utility, but the agent must choose not only the trade size v, but also one of the three states of aggressiveness as per Definition 1. Let a_i be the aggression state in the i-th security. The fill probability is best represented conditional on the aggression state.

$$p_f \,|\, a_i = \begin{cases} 0 & a_i = \mathfrak{w} \\ f_i & a_i = \mathfrak{p} \\ 1 & a_i = \mathfrak{a} \end{cases} \tag{26}$$

In this generalized situation, we assume that (25) remains valid, but the agent must choose one of the three states of aggression, denoted a, and the expectation in (25) is computed over a larger probability space which includes the randomness of getting filled if the agent should happen to choose $a = \mathfrak{p}$, the passive state. Let us write

$$R_{i,t}(v, a)$$

for the (random) P&L from an order of quantity v on stock i using action a over a short interval $[t, t+\epsilon]$. Then

$$R_{i,t}(v,a) = \mathbf{1}_{\text{fill}}(r_i v_i - \text{cost}(v_i, a))$$

where $r_i = \text{mid}_i(t+\epsilon)/\text{mid}_i(t) - 1$ is the midpoint return over the interval, $\mathbf{1}_{\text{fill}}$ is the indicator variable for the event of the order being filled (for simplicity, we assume there are no partial fills), and $\text{cost}(v_i, a)$ is the cost of filling the quantity v_i by using the action a. This cost function will in general be different from the cost function $c(v)$ appearing in our objective, which represents an average cost of trading, without reference to particular execution choices. For example, for $a = \mathfrak{a}$, $\text{cost}(v, a)$ will include the spread; for $a = \mathfrak{p}$, it may include an adverse selection cost. In our setting, the myopic risk-neutral agent's beliefs are such that $\mathbb{E}[r_i] = p_i$.

For the sake of deriving a concrete trading rule, assume that r_i and $\mathbf{1}_{\text{fill}}$ are independent, that passive executions incur the average cost, and that aggressive ones, in addition, incur a spread cost. Then we have

$$\mathbb{E}[R(v_i, a_i) \,|\, a_i] = \begin{cases} 0 & a_i = \mathfrak{w} \\ f_i(p_i v_i - c(v_i)) & a_i = \mathfrak{p} \\ p_i v_i - c(v_i) - s_i v_i & a_i = \mathfrak{a} \end{cases} \tag{27}$$

where s_i denotes one-half the bid-offer spread. We assume that, since the aggressive state involves executing at the far side of the limit order book, the agent incurs a cost of one half-spread relative to the mid price if the agent chooses $a_i = \mathfrak{a}$, ie. chooses to aggress.

Theorem 3. Under the assumptions used to derive (27), the state a which optimizes

$$\mathbb{E}[R(v_i, a_i) \,|\, a]$$

is determined as follows. If

$$p_i v_i - c(v_i) - s_i v_i > f_i(p_i v_i - c(v_i)) > 0$$

then the agent chooses $a = \mathfrak{a}$, to aggress. If $p_i v_i > c(v_i)$, the agent chooses the passive mode, $a = \mathfrak{p}$, and finally if $p_i v_i < c(v_i)$ the agent chooses to wait, $a = \mathfrak{w}$ (intuitively because further trading would go against the gradient of the value function).

The proof of Theorem 3 is simply by direct computation, but the implications are interesting. Theorems 1 and 3, taken together, essentially provide an algorithm for how to implement the execution optimally.

Algorithm 1. Let $q_0 \in \mathbb{R}^n$ be the initial portfolio, and T is the total time allowed for the execution.

(1) On initialization, set $q = q_0$ and compute $p = \Xi(q, T)$.
(2) For $i = 1, \dots, n$, do the following:
    (a) Using the closed-form solution in Theorem 3 with the current value of p_i, find the aggression level a_i^* which optimizes
$$a_i^* = \operatorname*{argmax}_a \mathbb{E}[R(v_i, a_i) \,|\, a]$$
    for v_i equal to a single round lot.
    (b) If $a_i^* = \mathfrak{p}$ then remain in the queue, or join the queue if not already. If $a_i^* \in \{\mathfrak{a}, \mathfrak{w}\}$ then cancel any existing orders in the i-th security. If $a_i^* = \mathfrak{a}$ then aggress, and record the associated fill.
    (c) Update q to include any fills received since the last update. If q was updated, then also update $p = \Xi(q, T - t)$ where $T - t$ is the time left remaining.
(3) If $q \neq 0$, i.e. there is any remaining quantity to be filled, return to step 2.

Note in step 2(b) that new fills could arise either due to aggression in step 2(a), or due to passive limit orders being filled in the market. The computation required in step 2(c) is just a matrix multiplication.

5. A Detailed Numerical Example

Mathematical elegance and simplicity are to be prized of course, but an execution model cannot pass the test of practicality until it actually helps us execute portfolio transitions. In this section we establish this in a particular example using real-world market data.

One of the most important features of our framework, as compared with a plain-vanilla Almgren–Chriss executor, is that it allows the executor to fill many of its trades passively (hence avoiding certain types of market impact and spread costs). The main point we wish to make is that our method potentially avoids the pitfalls of a purely-passive execution model, because it can consider the utility gradient (and its multi-period analog, the gradient of the value function) in the formation of aggression levels. With this specific aim in mind, we consider the liquidation of a market-neutral portfolio with our method, and contrast this with comparable results for a purely-passive method.

The specific example we choose is the liquidation of a market-neutral portfolio on October 15, 2008. The portfolio to be liquidated is long IBM and short AAPL. We choose the long position in IBM arbitrarily to be 1000 shares. We estimate the CAPM beta of each security, denoted $\hat{\beta}_i$ where $i = 1, 2$, using three years of daily data, and size the short position so that the beta exposure of the portfolio $\sum_i h_i \hat{\beta}_i$ is as near zero as possible.

We approach this problem using the method of the previous sections. The model has several parts, which we explain in turn. One part is the risk model, which manifests itself via the asset-level covariance matrix Σ. Covariance matrix estimation for a large universe of assets proceeds by estimating a covariance matrix of the form (17). The example of this section is much simpler: $n = 2$ so the covariance matrix is determined by the asset betas and the market volatility. Another part is the high-level transaction cost model appearing in the value function, denoted by $c(v)$. The third part entails a simulation of the model's interaction with the market microstructure and, in particular the simulation of passive fills.

We take $\kappa = 10^{-3}$. Note that κ represents the aversion to risk in dollar terms; larger κ simply means that the theoretical Almgren-Chriss path will liquidate more quickly because the system is more averse to the risk associated to long-lived inventory. Our parameter κ directly corresponds to what was called λ by Almgren and Chriss (2001), but we prefer to reserve λ for the parameters of the cost model, to be discussed presently.

5.1. **Transaction Cost Model.** For continuity with previous sections, we continue to let $c(v_i)$ denote the cost due to market impact of trading v_i dollars per day in the i-th security, and let $v = (v_1, \ldots, v_n)$ denote the vector of all trades and $c(v)$ the total cost of all trades. In order to use the computationally tractable results derived previously, we continue to assume $c(v) = v'\Lambda v$ for some diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. This reduces the transaction cost modeling problem to one of estimating appropriate values for each λ_i.

As an aside, we remark that with non-quadratic market impact costs, our method should still be more tractable than the HJB approach, because in our method, one would obtain the generalized momenta directly from solving a first-order ODE system. By contrast, in the HJB approach one must find the action (by solving the Hamilton-Jacobi PDE).

Let advp_i denote our prediction of the daily dollar volume in the i-th security. The notation "advp" comes from the fact that it is computed as the average daily volume "adv" in shares, times the price "p." For simplicity we assume trading one percent of advp_i will cause 20 basis points of market impact, with extension by linearity, so trading 2 percent of advp_i will cause 40 basis points of impact, etc. This means that

$$\lambda_i = 20 \times 10^{-4} \times \frac{1}{0.01\mathrm{advp}_i} \tag{28}$$

For very large trades (say, more than 0.05 advp_i), simple models such as (28) break down. For this reason, we restrict our attention in this example to trades that are relatively small with respect to the anticipated volume.

5.2. **Microstructure Simulation.** One of the most challenging aspects of this study is simulating passive execution, which we defined previously as a process of continually joining the queue on the near side of the limit order book until the order is filled, but never crossing the spread.

We are limited to the academic data sets available via the Wharton Research Data Services (WRDS). For this exercise, we used the New York Stock Exchange Trade and Quote (TAQ) database, which contains intraday transactions data (trades and quotes) for all securities listed on the New York Stock Exchange (NYSE) and American Stock Exchange (AMEX), as well as Nasdaq National Market System (NMS) and SmallCap issues.

The TAQ database represents the aggregate inside quote for each exchange. Therefore, it includes both specialists and the public limit order book. In fact, subtle distinctions of this sort are not relevant for this study; *we simply need a data set which has the same statistical distribution as the quote streams one is likely to encounter in practice.*

Only having access to the consolidated feed, we construct a conservative simulation of when passive fills occur. Specifically, if we have a "buy" limit order (the entire process is similar for limit "sell" orders with "bid" replaced by "ask") which is simulated as existing in the queue on the bid side of the order book, when can we assume such an order was filled? Conservatively, if the order book changes and the new ask price is less or equal to the existing limit order price, we assume that in the process of this change, markets would have cleared and our limit order would have been filled, at least partially. We limit the amount of fill to the posted quantity at the new ask price. If this quantity is simulated to have been taken out, then no further fills are allowed to occur in the simulation until the price level changes. We assume that when the price level of the national best bid or offer has changed, the liquidity is also replenished to the reported value at the new price level. This is a fairly conservative set of conventions; in reality, a larger number of passive fills could occur than merely the ones we simulate. This is because if there are multiple limit orders in the queue, one limit order can of course be filled without either bid or ask price levels changing.

Any simulation of microstructure based entirely on the consolidated feed cannot be perfect. However, given that our results are very well grounded theoretically, we believe that this simulation is sufficiently realistic to be indicative of the procedure's merit.

Predicting the probability of a passive fill, denoted f_i above, is tantamount to predicting the next transition of the limit order book and hence requires a model of limit order book dynamics. Indeed, such fill probabilities are one of the possible outputs of the very detailed model of Cont, Stoikov, and Talreja (2010). As the discussion and implementation of such models are outside of our current scope, and our dataset is only the consolidated feed, we simply take $f_i = 0.1$ as the passive fill probability. If this could be improved, then we expect an associated improvement in the performance of our model over the vanilla model.

5.3. **Results.** As indicated above, we construct a market-neutral portfolio of $n = 2$ securities in which the long side is initially 1000 shares of IBM. Security $i = 1$ is IBM and $i = 2$ is AAPL. We estimate the security betas to the S&P 500 (via regression on several years of daily data) as

$$\hat{\beta}_1 = 0.705, \quad \hat{\beta}_2 = 1.276\,. \tag{29}$$

We begin the simulation at 10:00am on October 15, 2008, rather than immediately at the open since there are often outlier quotes, wide spreads and other effects around the open. The most recent midpoint price of IBM at 10:00am was $p_1 = 93.06$ and for AAPL, $p_2 = 105.985$.

For convenience we keep track of a cash balance for each position. The $n_1 = 1000$ shares of IBM are financed by borrowing $n_1 p_1 =$ USD 93,060 in cash and purchasing a position initially worth USD 93,060, so the net value (cash + stock) of that position is initially zero. Similarly, the short position in AAPL is obtained by borrowing $n_2 = -485$ shares and immediately selling them for USD 51,403 and this position also initially has net (cash + stock) value zero. Note that with these holdings, (29) implies that the portfolio's beta is

$$n_1 p_1 \hat{\beta}_1 + n_2 p_2 \hat{\beta}_2 \approx 0\,.$$

Any cash generated from further stock sales or cash used for further purchases of the same security is considered part of the separate cash balance allocated to that position. As prices change and as orders are filled, the values of each position will fluctuate.

Let $n_{i,t}$ denote the number of shares held in the i-th security at time t, and $p_{i,t}$ the latest midpoint price as of time t. Also let $c_{i,t}$ denote the amount of cash (which can be positive or negative) attributed to the i-th security at time t, according to the accounting conventions outlined above. These variables change throughout the lifetime of the execution.

Definition 2. The *value of a position* is the number of shares held times the most recent midpoint price, plus the total amount of cash associated to the position, ie. $n_{i,t}p_{i,t} + c_{i,t}$. The *value of a portfolio* is the sum of the values of all its positions, ie.

$$\text{value}_t := \sum_{i=1}^{n} (n_{i,t}p_{i,t} + c_{i,t}) \tag{30}$$

The value process, eq. (30), and especially its drift, is one measure of the quality of the execution. If the value tends to drift downward, as in the AlwaysPassive model detailed below, then the execution desk is losing money due to slippage. This is perhaps the typical situation – one expects execution to have associated costs. A particularly pleasant situation arises when the drift of the portfolio value process (30) is zero, as in Figure 2, and it's possible that with very good microstructure alphas added to the generalized momenta, the drift could even become positive.

All monetary values are reported in USD. The predicted daily volumes are estimated as

$$\text{advp}_1 \approx 1.16 \times 10^9, \quad \text{advp}_2 \approx 6.13 \times 10^9\,.$$

The covariance matrix predicted by the APT model as per (17) is

$$\Sigma = 10^{-4} \times \begin{pmatrix} 15.5728 & 17.7558 \\ 17.7558 & 28.6519 \end{pmatrix}$$

which implies a correlation of 0.84 among the two assets, and daily volatilities of approximately 3.9% and 5.4%.

The output of our algorithm is the instantaneous aggression level: aggressive, passive, or wait. There isn't a unique benchmark to gauge such an algorithm's performance, but it is sensible to compare a complicated method of choosing the aggression level to a simple method for choosing the aggression level, to see if the additional complexity is justified. Hence one could compare it to a constant aggression level – always passive.

Figure 1 reveals that, as the market was falling, the passive "buy" orders in AAPL were all filled very quickly, while unsurprisingly the "sell" orders in IBM were filled very slowly, and indeed were not even finished by the end of the trading day. This drove the Gross Market Value (gmv) down while pushing the net and beta higher, where we define

$$\beta_t := \sum_i n_{i,t} p_{i,t} \hat{\beta}_i, \quad \mathrm{net}_t := \sum_i n_{i,t} p_{i,t}, \quad \mathrm{gmv}_t := \sum_i |n_{i,t} p_{i,t}|, \tag{31}$$

with $\hat{\beta}_i$ given by (29). Thus the portfolio had $\beta_t > 0$ in a falling market. Note that the losses incurred in this manner do not become gains if the sign of the market move is reversed; they remain *losses* irrespective of the market's direction. In a rising market, the "always passive" model would have the same problem: the "sell" orders would be filled quickly, the "buy" orders would linger, and the portfolio would build up negative beta in a rising market.

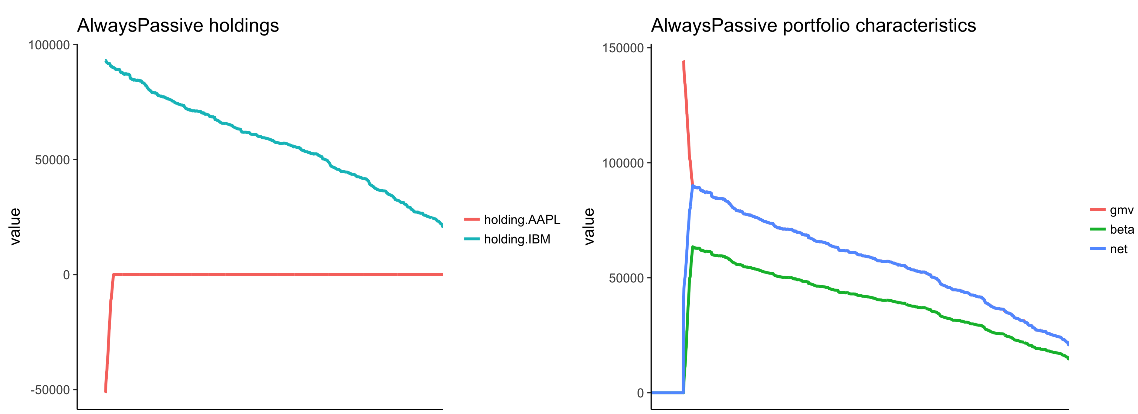

FIGURE 1. Portfolio holdings in the "always passive" model, and portfolio characteristics: gross market value (gmv), net, and β_t given by (31).

We now show the analogous graphs for our more sophisticated execution model developed in previous sections. Note that the model retains a fairly small beta exposure throughout the lifetime of the execution. This is because CAPM beta is also a factor in the APT risk model, and the generalized momenta point along the gradient of the Hamilton-Jacobi-Bellman value function and hence drive trading towards the optimal value of multiperiod utility (including the risk term). This is the key advantage of our model over simpler execution algos.

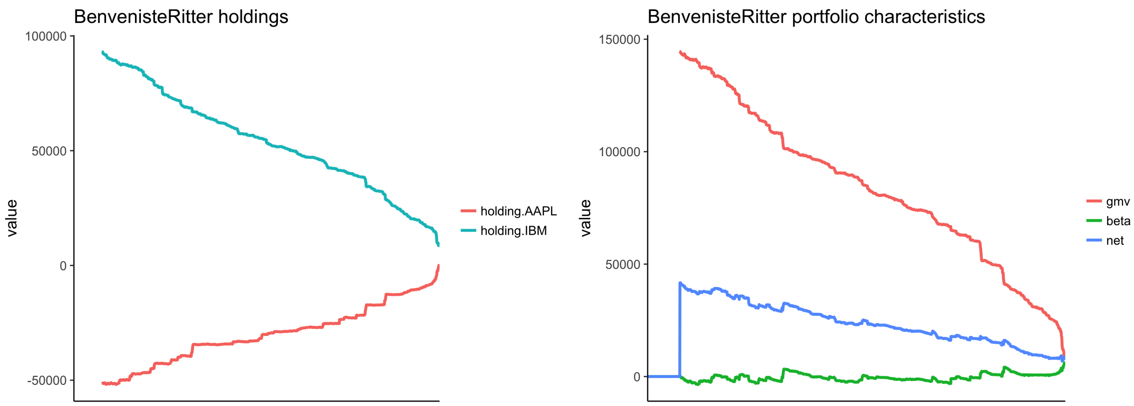

FIGURE 2. Portfolio holdings in the sophisticated model, and portfolio characteristics: gross market value (gmv_t), net_t, and β_t given by (31).

Finally, we consider the portfolio value over the lifetime of the execution. Note that in our model, the value process (30) is approximately driftless, which as explained above is a desirable property, and outperforms the "always passive" value process realization. In particular, in our model value_t is able to avoid negative drift in a falling market precisely because the portfolio remains approximately beta-neutral. In a portfolio with many assets (large n), our method would allow it to remain approximately neutral to all factors in the APT model.

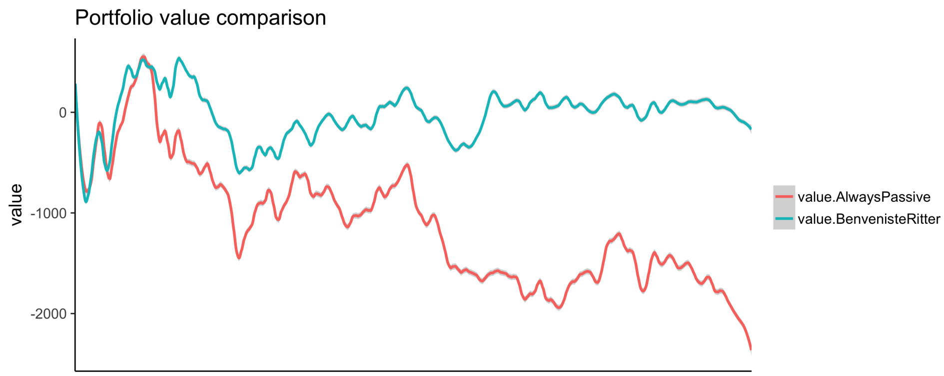

FIGURE 3. Portfolio value (30) over the lifetime of the execution, for both execution methods.

The difference in Figure 3 is both statistically and economically significant. The t-statistic for the difference is about 78, hence significant at the 99.999% level. Moreover, the dollar value of the difference between the two methods is about 1.5% of the initial gross market value to be liquidated.

6. Conclusions

Consider a hypothetical execution desk with orders to execute in n distinct assets, where n is large (many thousands). If the desk attempts to complete this task by only trading passively, they will encounter serious problems: namely, they will fall behind on orders which are moving in the direction of the trade, and they will build up unwanted factor exposure (and associated risk) at the portfolio level. This can be catastrophic, as some of our examples show.

Suppose the desk's portfolio has gotten "out of balance." Then one may ask: "is there a vector of microstructure alphas which point the way back to optimality?" We answer this in the affirmative: the necessary microstructure alphas are given by the generalized momenta, or equivalently the gradient of the value function.

What we mean by this, is as follows. A hypothetical myopic risk-neutral agent who simply trades to maximize expected profit of the next trade will actually exhibit fully optimal behavior as long as this agent uses the generalized momenta as microstructure alphas; the details are in Theorem 3 and the ensuing algorithm.

To be practical, the myopic expected-profit maximizations would need to be performed very quickly, because today's electronic limit order book markets can change rapidly, and experience high message throughput. Fortunately, this can be done; our method is computationally tractable. As we remark at the end of Sec. 2, all of the computationally-expensive aspects of the solution can be precomputed once per day (ideally, in the morning before trading begins).

Our method provides the user with other interesting bits of intuition which deserve further exploration, such as the connection to principal components also noted at the end of Sec. 2. Also, we wonder which other trading problems can be viewed within the unifying framework of a myopic risk-neutral wealth-maximizer whose microstructure alphas are aligned with the value function gradient.

References

Alfonsi, Aurélien, Antje Fruth, and Alexander Schied (2010). "Optimal execution strategies in limit order books with general shape functions". *Quantitative Finance* 10.2, pp. 143–157.

Almgren, Robert and Neil Chriss (2001). "Optimal execution of portfolio transactions". *Journal of Risk* 3, pp. 5–40.

Arrow, Kenneth J (1963). "Liquidity preference, Lecture VI in "Lecture Notes for Economics 285, The Economics of Uncertainty", pp 33-53".

Bismut, Jean-Michel (1976). "Linear quadratic optimal stochastic control with random coefficients". *SIAM Journal on Control and Optimization* 14.3, pp. 419–444.

– (1978). "An introductory approach to duality in optimal stochastic control". *SIAM review* 20.1, pp. 62–78.

Carhart, Mark M (1997). "On persistence in mutual fund performance". *The Journal of Finance* 52.1, pp. 57–82.

Cont, Rama, Sasha Stoikov, and Rishi Talreja (2010). "A stochastic model for order book dynamics". *Operations Research* 58.3, pp. 549–563.

Fama, Eugene F and Kenneth R French (1993). "Common risk factors in the returns on stocks and bonds". *Journal of financial economics* 33.1, pp. 3–56.

Gatheral, Jim (2010). "No-dynamic-arbitrage and market impact". *Quantitative finance* 10.7, pp. 749–759.

Gatheral, Jim (2012). “Optimal order execution”. *Imperial College Finance and Stochastics Seminar*.

Gatheral, Jim and Alexander Schied (2011). “Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework”. *International Journal of Theoretical and Applied Finance* 14.03, pp. 353–368.

Gould, Martin David and Julius Bonart (2015). “Queue imbalance as a one-tick-ahead price predictor in a limit order book”. *Available at SSRN*.

Naujokat, Felix and Nicholas Westray (2011). “Curve following in illiquid markets”. *Mathematics and Financial Economics* 4.4, pp. 299–335.

Pratt, John W (1964). “Risk aversion in the small and in the large”. *Econometrica: Journal of the Econometric Society*, pp. 122–136.

Stoll, Hans R (1978). “The supply of dealer services in securities markets”. *The Journal of Finance* 33.4, pp. 1133–1151.