Probability Refresher

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Abstract

This note collects salient facts about probability theory.

Contents

Probability is an extension of logic. Instead of propositions being either true or false a degree of belief can be specified for events occurring. All probabilities are conditional on models of available information.

Probability Model

A *probability model* specifies a *sample space* and a *probability measure*.

Sample Space

A sample space is what can happen: heads or tails as the outcome of a coin toss, the integers from 1 to 6 as the outcomes of rolling a single die, the set of all sequences of not more than 280 characters as a model of possible Twitter tweets.

An *event* is a subset of a sample space.

People seem to be surprised probabilities are modeled using sets. Sets have no structure, they are just a bag of things (*elements*).

Probability Measure

A *probability measure* assigns a number between 0 and 1 to events. If Ω is a sample space and P is a probability measure then the measure of the union of sets is the sum of the measure of each set minus the measure of the intersection: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ for events E and F. This is the mathematical way to say measures do not double count.

A probability measure must also satisfy $P(\emptyset) = 0$ and $P(\Omega) = 1$.

Exercise. If Q is a measure with $Q(\emptyset) = a$ and $Q(\Omega) = b$, show $(Q - a)/(b - a)$ is a probability measure.

Algebra

An *algebra of sets*, or *algebra*, on Ω is a collection of subsets (events), $\mathcal{A}$, that is closed under complement and union. This lets us talk about and event not happening and whether event A or B occured.

We also assume the empty set belongs to $\mathcal{A}$, hence also Ω. By De Morgan's Laws an algebra is also closed under intersection. The *power set* of Ω, $2^\Omega = \{E : E \subseteq \Omega\}$, clearly satisfies these conditions.

An *atom* of an algebra is a member, A, of the algebra such that if $B \subseteq A$ and B is in the algebra, then either $B = A$ or B is the empty set.

Partition

A *partition* of a set is a collection of pairwise disjoint subsets who's union is equal to the set.

Exercise. If an algebra on Ω is finite its atoms form a partition of Ω.

Hint: Show $A_\omega = \cap\{B \in \mathcal{A} : \omega \in B\}$, $\omega \in \Omega$, is an atom

This shows there is a one-to-one correspondence between finite partitions and finite algebras of sets. A partition is the mathematical way of specifying partial information. Knowing the outcome, $\omega \in \Omega$, corresponds to complete knowledge. Knowing which atom the outcome belongs to corresponds to partial knowledge. For example, the partition $\{\{1,3,5\},\{2,4,6\}\}$ corresponds to knowing whether the roll of a die is odd or even.

The coarsest partition $\{\Omega\}$ corresponds to no knowledge while the finest partition $\{\{\omega\} : \omega \in \Omega\}$ corresponds to complete knowledge.

Measurable

A function $X\colon \Omega \to \mathbf{R}$ is $\mathcal{A}$-*measureable* if the sets $X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \le x\}$ belong to $\mathcal{A}$ for $x \in \mathbf{R}$.

Exercise: If $\mathcal{A}$ is finite, show that a function is measurable if and only if it is constant on atoms of $\mathcal{A}$.

In this case $X\colon \mathcal{A} \to \mathbf{R}$ is indeed a function on the atoms.

Random Variable

A *random variable* is a variable, a symbol that can be used in place of a number, with additional information: the probability of the values it can take on. The *cumulative distribution function* is $F(x) = F^X(x) = P(X \le x)$. It tells you everything there is to know about X. For example, $P(a < X \le b) = F(b) - F(a)$.

Exercise. Show $P(a \le X \le b) = \lim_{x\uparrow a} F(b) - F(x)$.

Hint: $[a,b] = \cap_n (a - 1/n, b]$.

In general, $P(X \in A) = E1_A = \int 1_A(x)\,dF(x)$ for sufficiently nice $A \subset \mathbf{R}$ where we are using Riemann–Stieltjes integration.

Exercise: Show for any cumulative distribution function, F, that F is non-decreasing, $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ and F is right continuous with left limits.

Every such function defines a random variable.

The cdf $F(x) = \max\{0, \min\{1, x\}\}$ defines the uniformly distributed random variable U. For $0 \le a < b \le 1$, $P(a < U < b) = b - a$.

Two random variables, X and Y, have the same *law* if they have the same cdf.

Exercise. If X has cdf F, then X and $F^{-1}(U)$ have the same law.

Exercise. If X has cdf F, then $F(X)$ and U have the same law.

This shows a uniformly distributed random variable has sufficient randomness to generate any random variable. There are no random, random variables.

The mathematician's definition of a random variable is that it is a measurable function $X\colon \Omega \to \mathbf{R}$. Its cumulative distribution function is $F(x) = P(X \le x) = P(\{\omega \in \Omega \mid X(\omega) \le x\})$. Given a cdf F we can define $X\colon \mathbf{R} \to \mathbf{R}$ to be the identity function and let P be the probability measure defined by F: $P(A) = \int 1_A(x)\,dF(x)$.

Expected Value

The *expected value* of a random variable is defined by $EX = \int_{-\infty}^{\infty} x\,dF(x)$. The expected value of any function of a random variable is $Ef(X) = \int_{-\infty}^{\infty} f(x)\,dF(x)$.

The *indicator* (or *characteristic*) function $1_A(\omega)$ is 1 if $\omega \in A$ and 0 if $\omega \notin A$. If $X = \sum a_i 1_{A_i}$ where $a_i \in \mathbf{R}$ and A_i are events, The *expected value* of X by $EX = \sum_i a_i P(A_i)$.

Exercise. Show that if $\sum_i a_i 1_{A_i} = 0$ then $\sum_i a_i P(A_i) = 0$.

Hint: Replace the A_i by disjoint B_j so $b_j = 0$ for all j.

This shows expected value is well-defined.

Exercise. Show $P(\cup_i A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \cdots$.

Hint: Use $(1_A - 1_{A_1}) \cdots (1_A - 1_{A_n}) = 0$, where $A = \cup_{k=1}^n A_k$.

Moments

The *moments* of a random variable, X, are $m_n = E[X^n]$, $n = 0, 1, 2, \ldots$. They don't necessarily exist for all n, except for $n = 0$. They also cannot be an arbitrary sequence of values.

Suppose all moments of X exist, then for any complex numbers, (c_i), $0 \le E|\sum_i c_i X^i|^2 = E \sum_{j,k} c_j \bar{c_k} X^{j+k} = \sum_{j,k} c_j \bar{c_k} m_{j+k}$. This says the Hankel matrix, $M = [m_{j+k}]_{j,k}$, is positive definite. The converse is also true: if the Hankel matrix is positive definite there exists a random variable with the corresponding moments. This is not a trivial result and the random variable might not be unique.

Cumulants

The *cumulant* of a random variable, X, is $\kappa(s) = \kappa^X(s) = \log E \exp(sX)$. The *cumulants*, κ_n, are defined by $\kappa(s) = \sum_{n>0} \kappa_n s^n/n!$.

It is easy to see $\kappa_1 = EX$ and $\kappa_2 = \operatorname{Var} X$. The third and fourth cumulants are related to skew and kurtosis. We will see the exact relationship below.

If c is a constant then $\kappa^{cX}(s) = \kappa^X(cs)$ so $\kappa_n^{cX} = c^n\kappa_n^X$. If X and Y are independent then $\kappa^{X+Y}(s) = \kappa^X(s) + \kappa^Y(s)$ so $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$\$

Bell Polynomial

The relationship between moments and cumulants is given by *Bell polynomials*. They are defined by $\exp(\sum_1^i nftya_n s^n/n!) = \sum_0^\infty B_n(a_1,\ldots,a_n)s^n/n!$. Taking the derivative with respect to s and equating powers of s shows $B_0 = 1$ and $B_{n+1}(a_1,\ldots,a_{n+1} = \sum_{k=0}^n \binom{n}{k} B_{n-k}(a_1,\ldots,a_{n-k})a_{k+1}$.

Bell polynomials show the connection between the moments and the cumulants of a random variable since $E\exp(sX) = \sum_0^\infty EX^n s^n/n! = \sum_0^\infty m_n s^n/n!$ where m_n is the n-th moment and $E\exp(sX) = \exp(\kappa(s)) = \exp(\sum_{n=1}^\infty \kappa_n s^n/n!)$.

Excercise: Show $m_n = \sum_{k=1}^n B_k(\kappa_1,\ldots,\kappa_n)$.

Exercise: Find the first five Bell polynomials.

In particular $m_1 = \kappa_1$ and $m_2 = \kappa_1^2 + \kappa_2$ so κ_1 is the mean and κ_2 is the variance. If the mean is 0 and the variance is 1, then κ_3 is the skew and κ_4 is the excess kurtosis.

Conditional Expectation

The *conditional expectation* of an event B given an event A is $P(B|A) = P(B\cap A)/P(A)$. In some sense, this reduces the sample space to A. In particular, $P(A|A) = 1$. Since $P(A|B) = P(A\cap B)/P(B)$ we have $P(A|B) = P(B|A)P(A)/P(B)$. This is the simplest form of Bayes Theorem. It shows how to update your degree of belief based on new information. Every probability is conditional on given information.

Define $E[X|A] = E[X1_A]/P(A)$ for any random variable X. If $X = 1_B$ then this coincides with the definition of conditional expectation above.

This is how we define $E[X|\mathcal{A}]\colon \mathcal{A} \to \mathbf{R}$ on atoms of $\mathcal{A}$.

Joint Distribution

Two random variables, X and Y, are defined by their *joint distribution*, $H(x,y) = P(X\le x, Y\le y)$. For example, the point (X,Y) is in the square $(a,b]\times(c,d]$ with probability $P(a < X \le b, c < Y \le d) = P(X\le b, Y\le d) - P(X \le a) - P(Y\le c) + P(X\le a, Y\le c)$.

The *marginal distbutions* are $F(x) = H(x,\infty)$ and $G(y) = H(\infty,y)$, where F and G are the cumulative distributions of X and Y respectively.

In general, the joint distribution of $X_1, \ldots, X_n$ is $F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$.

Independent

The random variables X and Y are *independent* if $H(x, y) = F(x)G(y)$ for all x and y. This is equivalent to $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B.

We also have that $Ef(X)g(Y) = Ef(X)Eg(Y)$ for and functions f and g whenever all expected values exist.

Exercise: Prove this for the case $f = \sum_i a_i 1_{A_i}$ and $g = \sum_j b_j 1_{B_j}$.

In general, $X_1, \ldots, X_n$ are independent if $F(x_1, \ldots, x_n) = F_1(x_1) \cdots F_n(x_n)$, where F_j is the law of X_j.

Copulas

A *copula* is the joint distribution of uniformly distributed random variables on the unit interval. Let U and V be two uniformly distributed random variables. The copula of X and Y is the joint distribution of $F^{-1}(X)$ and $G^{-1}(Y)$ where F and G are the cumulative distributions of X and Y respectively: $C^{X,Y=}(u, v) = P(F^{-1}(X) \leq u, G^{-1}(Y) \leq v)$.

Exercise: Show $C^{X,Y}(u, v) = H(F(u), G(v))$ where $C^{X,Y}$ is the copula of X and Y, and H is the joint distribution of X and Y.

Exercise: Show $H(x, y) = C(F^{-1}(x), G^{-1}(y))$

This shows how to use the copula and marginal distributions to get the joint distribution.

An equivalent definition is a copula is a probability measure on $[0, 1]^2$ with uniform marginals.

Exercise: Prove this.

If U and V are independent, uniformly distributed random variables on the unit interval then $C(u, v) = uv$.

If $V = U$ then their joint distribution is $C(u, v) = P(U \leq u, V \leq v) = P(U \leq u, U \leq v) = P(U \leq \min\{u, v\}) = \min\{u, v\} = M(u, v)$.

If $V = 1-U$ then their joint distribution is $C(u, v) = P(U \leq u, V \leq v) = P(U \leq u, 1-U \leq v) = P(1-v \leq U \leq u) = \max\{u-(1-v), 0\} = \max\{u+v-1, 0\} = W(u, v)$

Exercise: (Fr'echet-Hoeffding) For every (2-dimensional) copula, $W \leq C \leq M$.

Hint: For the upper bound use $H(x, y) \le F(x)$ and $H(x, y) \le G(y)$. For the lower bound note $0 \le C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2)$ for $u_1 \ge u_2$ and $v_1 \ge v_2$.

Examples

Move!!! These can be used to prove the *central limit theorem*: if X_j are independent, identically distributed random variables with mean zero and variance one, then $(X_1 + \cdots X_n)/sqrtn$ converges to a standard normal random variable.

If X is normal then $E\exp(X) = \exp(EX + \mathrm{Var}(X)/2)$ so the cumulants satisfy $\kappa_n = 0$ for $n > 2$.

If X is Poisson with parameter λ then

$$
\begin{aligned}
Ee^{sX} &= \sum_{k=0}^{\infty} e^{sk} e^{-\lambda} \lambda^k / k! \\
&= \sum_{k=0}^{\infty} (e^s \lambda)^k e^{-\lambda} / k! \\
&= \exp(\lambda(e^s - 1))
\end{aligned}
$$

so $\kappa(s) = \lambda(e^s - 1)$ and $\kappa_n = \lambda$ for all n.

Infinitely Divisible

A random variable, X, is *infinitely divisible* if for any positive integer, n, there exist independent, identically distributed random variables X_1,…,X_n such that $X_1 + \cdots + X_n$ has the same law as X.

Unsorted

Characteristic Function

The *characteristic function* of a random variable, X, is $\xi(t) = \kappa(it)$.

Fourier Transform

The *Fourier transform* is $\psi(t) = \xi(-t) = \kappa(-it)$. Clearly $\psi(t) = \xi(-t)$.

Remarks

Cheval de Mere

Pascal

Bernoulli(s)

Kolmogorov

Willy Feller

Examples

Discrete

A *discrete* random variable is defined by $x_i \in \mathbf{R}$ and $p_i > 0$ with $\sum p_i = 1$. The probability the random variable takes on value x_i is p_i.

If a discrete random variable takes on a finite number of values, n, then if $p_i = 1/n$ for all i the variable is called *discrete uniform*.

Bernoulli

A *Bernoulli* random variable is a discrete random variable with $P(X = 1) = p$, $P(X = 0) = 1 - p$.

Binomial

A *Binomial* random variable is a discrete random variable with $P(X = k) = \binom{n}{k}/2^n$, $k = 0, \ldots, n$.

Uniform

A *continuous uniform* random variable on the interval $[a, b]$ has density $f(x) = 1_{[a,b]}/(b - a)$.

Normal

The *standard normal* random variable, Z, has density function $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$.