Option Pricing

Keith A. Lewis

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Abstract

There is a simple formula for pricing European puts and calls in terms of cumulants.

Let F be the (random) value of an instrument at option expiration. The (forward) value of a put option is the expected value of the payoff

$$\begin{aligned}
E[\max\{k - F, 0\}] &= E[(k - F)1(k - F \geq 0)] \\
&= E[(k - F)1(F \leq k)] \\
&= kP(F \leq k) - E[F1(F \leq k)] \\
&= kP(F \leq k) - E[F]E[1(F \leq k)F/E[F]] \\
&= kP(F \leq k) - E[F]P^*(F \leq k)
\end{aligned}$$

where P^* is the Esscher transform of the probability measure, i.e., its Radon-Nikodym derivative is $dP^*/dP = F/E[F]$.

The *cumulant* of a random variable X is $\kappa^X(s) = \log E[\exp(sX)]$ and the *cumulants* are the coefficients in the power series expansion $\kappa(s) = \sum_{n>0} \kappa_n s^n/n!$, where we write $\kappa(s)$ instead of $\kappa^X(s)$ when the random variable obvious.

Note $\kappa(0) = 0$, $\kappa'(0) = \kappa_1 = E[X]$, and $\kappa''(0) = \kappa_2 = Var\, X$.

Define $X = z(F) = (\kappa(s) + \log F/f)/s$, so $F = f\exp(sX - \kappa(s))$. Note $E[F] = f$. The Black model has X standard normal and $s = \sigma\sqrt{t}$. In this case the cumulant of X is $s^2/2$ so all cumulants are zero except the second $\kappa_2 = 1$. We can and do assume X has mean 0 and variance 1 as in the Black model.

Exercise. Show $f\exp(s(\sigma X + \mu) - \kappa^{\sigma X+\mu}(s)) = f\exp(s^* X - \kappa^X(s^*))$ where $s^* = \sigma s$.

The (complete) Bell polynomials $B_n(\kappa_1, \ldots, \kappa_n)$ are defined by $B_0 = 1$ and

$$B_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(\kappa_1, \ldots, \kappa_{n-k})\kappa_{k+1}$$

for $n > 0$. They satisfy

$$\exp(\sum_{n>0} \kappa_n s^n/n!) = \sum_{n\geq 0} B_n(\kappa_1, \ldots, \kappa_n) s^n/n!$$

Differentiation both sides with respect to s and and equating terms of equal power gives the recursive definition.

Note $B_1(\kappa_1) = \kappa_1$ and $B_2(\kappa_1, \kappa_2) = \kappa_1^2 + \kappa_2$.

The (probabalists') Hermite polynomials are defined by $H_0(x) = 1$, $H_1(x) = x$, and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ for $n \geq 1$. They satisfy

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}(x)$$

Let $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ be the standard normal density function and let $\psi(x)$ be the probability density function of X. The Fourier transform of ψ is $\hat{\psi}(u) = E[\exp(-iuX)]$. Note $\hat{\phi}(u) = \phi(u)$. It is a general property of Fourier transforms that $\widehat{\psi'}(u) = iu\hat{\psi}(u)$.

$$\begin{aligned}
\hat{\psi}(u) &= E[\exp(-iuX)] \\
&= \exp(\kappa(-iu)) \\
&= \exp(\sum_{n>0} \kappa_n (-iu)^n/n!) \\
&= \exp(-u^2/2) \exp(\sum_{n\geq 3} \kappa_n (-iu)^n/n!) \\
&= \hat{\phi}(u) \exp(\sum_{n\geq 3} \kappa_n (-iu)^n/n!) \\
&= \hat{\phi}(u)(\sum_{n\geq 0} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u)(1 + \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u) + \sum_{n\geq 3} B_n(\kappa_1, \ldots, \kappa_n)(-1)^n \widehat{\phi^{(n)}}(u)/n!
\end{aligned}$$

Taking inverse Fourier transforms yields

$$\psi(x) = \phi(x) + \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-1)^n \phi^{(n)}(x)/n!.$$

Integrating gives

$$\begin{aligned}\Psi(x) &= \Phi(x) + \sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n)(-1)^n \phi^{(n-1)}(x)/n! \\ &= \Phi(x) - \phi(x) \sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n) H_{n-1}(x)/n!\end{aligned}$$

Remarks

The Hermite polynomials can be expressed using Bell polynomials, $H_n(x) = B_n(x,1,0,\ldots,0)$.

$$\begin{aligned}B_1(0) &= 0 \\ B_2(0,0) &= 0 \\ B_3(0,0,\kappa_3) &= \kappa_3 \\ B_4(0,0,\kappa_3,\kappa_4) &= \kappa_4 \\ B_5(0,0,\kappa_3,\kappa_4,\kappa_5) &= \kappa_5 \\ B_6(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6) &= 10\kappa_3^2 + \kappa_6 \\ B_7(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6,\kappa_7) &= 35\kappa_3\kappa_4 + \kappa_7\end{aligned}$$