The Unified Model

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Abstract

There is a unified way to value derivative securities using models of market prices.

Notation

If $\mathcal{A}$ is an algebra on the set Ω we write $X\colon \mathcal{A} \to \mathbb{R}$ to indicate $X\colon \Omega \to \mathbb{R}$ is $\mathcal{A}$-*measurable*. If $\mathcal{A}$ is finite then the atoms of $\mathcal{A}$ form a partition of Ω and being measurable is equivalent to being constant on atoms. In this case X is indeed a function on the atoms.

If $\mathcal{A}$ is an algebra of sets, the *conditional expectation* of X given $\mathcal{A}$ is defined by $Y = E\left[X \mid \mathcal{A}\right]$ if and only if Y is $\mathcal{A}$ measurable and $\int_A Y\,dP = \int_A X\,dP$ for all $A \in \mathcal{A}$. This is equivalent to $Y(P|_\mathcal{A}) = (XP)|_\mathcal{A}$ where the vertical bar indicates restriction of a measure.

A *filtration* indexed by $T \subseteq [0, \infty)$ is an increasing collection of algebras, $(\mathcal{A}_t)_{t\in T}$. A process $M_t\colon \mathcal{A}_t \to \mathbb{R}$, $t \in T$, is a *martingale* if $M_t = E[M_u \mid \mathcal{A}_t] = E_t[M_u]$ for $t \le u$. A *stopping time* is a function $\tau\colon \Omega \to T$ such that $\{\omega \in \Omega \mid \tau(\omega) \le t\}$ belongs to $\mathcal{A}_t$, $t \in T$.

Instruments, Prices, Cash Flows

Every *instrument* has a *price*, X_t, and a *cash flow*, C_t, at any trading time, $t \in T$, although cash flows are almost always 0. Instruments are assumed to be perfectly liquid: they can be bought or sold at the given price in any amount. Cash flows are associated with owning an instrument: stocks have dividends, bonds have coupons, European options have exactly one cash flow at expiration.

The *unified model* specifies *prices* $X_t : \mathcal{A}_t \to \mathbb{R}^I$, and *cash flows* $C_t : \mathcal{A}_t \to \mathbb{R}^I$, where I are the available market instruments.

Trades, Values, Amounts

A *trading strategy* is a finite collection of strictly increasing stopping times, τ_j, and trades, $\Gamma_j : \mathcal{A}_{\tau_j} \to \mathbb{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, $\Delta_t = \sum_{\tau_j < t} \Gamma_j = \sum_{s<t} \Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$. Note the trade at time t is not included in the position at time t: it takes some time for trades to settle.

The *value* (or *mark-to-market*) of a position at time t is $V_t = (\Delta_t + \Gamma_t) \cdot X_t$: what you would get from liquidating your existing position and the trades just executed. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades you just executed.

Arbitrage

A model is *arbitrage-free* if there is no trading strategy with $\sum_j \Gamma_j = 0$, $A_{\tau_0} > 0$ and $A_t \geq 0$ for $t > \tau_0$: it is impossible to make money on the first trade and never lose until the strategy is closed out.

The Fundamental Theorem of Asset Pricing states this is the case if and only if there exists a *deflator*, $D_t : \mathcal{A}_t \to (0, \infty)$, with

$$X_t D_t = E_t[X_v D_v + \textstyle\sum_{t<u\leq v} C_u D_u].$$

Note that if $C_t = 0$ for all $t \in T$, this says $X_t D_t$ is a martingale. If the prices are eventually 0, this says the current price is the expected price of discounted future cash flows. A consequence of the above and the definition of value and amount is

$$V_t D_t = E_t[V_v D_v + \textstyle\sum_{t<u\leq v} A_u D_u].$$

If $u > t$ is sufficiently small then $X_t D_t = E_t[(X_u + C_u)D_u]$ and $V_t D_t = (\Delta_t + \Gamma_t) \cdot X_t D_t = \Delta_u \cdot E_t[(X_u + C_u)D_u]$. Since $\Delta_u \cdot C_u = \Gamma_u \cdot X_u + A_u$ we have $V_t D_t = E_t[(\Delta_u \cdot X_u + \Gamma_u \cdot X_u + A_u)D_u] = E_t[(V_u + A_u)D_u]$. The formula above follows by induction.

For a trading strategy that closes out, $V_{\tau_0} D_{\tau_0} = E_{\tau_0}[\sum_{t>\tau_0} A_t D_t] \geq 0$. Since $V_0 = \Gamma_0 \cdot X_0$, $A_0 = -\Gamma_0 \cdot X_0$, and $D_0 > 0$ we have $A_0 \leq 0$, where the 0 subscript denotes time τ_0. This proves the "easy" direction of the theorem.

There is no need to prove the "hard" direction since we have a large supply of arbitrage free models: every model of the form $X_t D_t = M_t - \sum_{s\leq t} C_s D_s$ where $M_t : \mathcal{A}_t \to \mathbb{R}^I$ is a martingale and $D_t : \mathcal{A}_t \to (0, \infty)$ is arbitrage-free. This is immediate by substituting $X_v D_v$ in the first displayed equation.

Canonical Deflator

There is a canonical choice for a deflator if repurchase agreements are available.

A *repurchase agreement* at time t, R_t, has price $X_t^{R_t} = 1$ and cash flow $C_{t+dt}^{R_t} = R_t$ so for any arbitrage free model $D_t = E_t[R_t D_{t+dt}]$. Define the *forward repo rate*, f_t, by $R_t = \exp(f_t\, dt)$ and the *canonical deflator* to be $D_t = \exp(-\int_0^t f_s\, ds)$. The repos are arbitrage free for any forward repo rate process.

Hedging

Given a derivative contract paying amounts A_j at times τ_j, how does one value this? One way is to find a trading strategy that replicates the payments. The value is just the cost to set up the initial trading strategy, $V_0 = \Gamma_0 \cdot X_0$. The initial hedge is $\Gamma_0 = dV_0/dX_0$. From $V_0 = E_0 \sum_{\tau_j} A_j D_{\tau_j}$ we can compute this in terms of the contract amounts and deflators.

The trades at time t are determined by $\Delta_t + \Gamma_t = dV_t/dX_t$, where the last term is the Fréchet derivative. Since we know the position, Δ_t, at time t this determines the trades, $\Gamma_t = dV_t/dX_t - \Delta_t$.

In the continuous time case, this becomes classical Black-Scholes/Merton delta hedging where delta is Δ and gamma is Γ. In discrete time the "best" hedge needs to be defined and can be found using dynamic programming or other optimization techniques.