The Unified Model

There is a unified way of pricing every market instrument and derivative security using a single model.

Notation

If $\mathcal{A}$ is an algebra on the set Ω we write $X\colon \mathcal{A} \to \mathbb{R}$ to indicate $X\colon \Omega \to \mathbb{R}$ is $\mathcal{A}$-*measurable*. If $\mathcal{A}$ is finite then the atoms of $\mathcal{A}$ form a partition of Ω and being measurable is equivalent to being constant on atoms. In this case X **is** a function on the atoms.

If X is $\mathcal{A}$-measurable and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ then the *conditional expectation* of X given $\mathcal{B}$ is defined by $Y = E[X|\mathcal{A}]$ if and only if Y is $\mathcal{B}$ measurable and $\int_B Y\,dP = \int_B X\,dP$ for all $B \in \mathcal{B}$. This is equivalent $Y(P|_\mathcal{B}) = (XP)|_\mathcal{B}$ where the vertical bar indicates restriction.

A *filtration* is a be a totally ordered set, T, and an increasing collection of algebras, $(\mathcal{A}_t)_{t\in T}$. A process $M_t\colon \mathcal{A}_t \to \mathbb{R}$, $t \in T$, is a *martingale* if $M_t P|_{\mathcal{A}_t} = M_u P|_{\mathcal{A}_t}$. If P is understood we write this as $M_t = M_u|_{\mathcal{A}_t}$. The usual notation is $M_t = E[M_u|\mathcal{A}_t] = E_t[M_u]$.

Model

The *unified model* specifies *prices* $X_t : \mathcal{A}_t \to \mathbb{R}^I$, and *cash flows* $C_t : \mathcal{A}_t \to \mathbb{R}^I$, where I are the available market instruments. Instruments are assumed to be perfectly liquid: they can be bought and sold at the given price in any amount. Cash flows are associated with owning an instrument: stocks have dividends, bonds have coupons, European options have exactly one cash flow at expiration.

A *trading strategy* is a finite collection of strictly increasing stopping times, τ_j, and trades, $\Gamma_j : \mathcal{A}_{\tau_j} \to \mathbb{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, $\Delta_t = \sum_{\tau_j < t} \Gamma_j = \sum_{s<t} \Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$.

The *value* (or *mark-to-market*) of a position at time t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: what you would get from liquidating your existing position and the trades just executed. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades you just executed.

A model is *arbitrage-free* if there is no trading strategy with $\sum_j \Gamma_j = 0$, $A_{\tau_0} > 0$ and $A_t \geq 0$ for $t > \tau_0$: it is impossible to make money on the first trade and never lose until the strategy is closed out.

The Fundamental Theorem of Asset Pricing states this is the case if and only if there exists a positive adapted process, $D_t : \mathcal{A}_t \to (0, \infty)$, with

$$X_t D_t = (X_u D_u + \sum_{t<s\leq u} C_s D_s)|_{\mathcal{A}_t}$$

Note that if $C_t = 0$ for all $t \in T$ this says $X_t D_t$ is a martingale. A consequence of the above and the definition of value and amount is

$$V_t D_t = (V_u D_u + \sum_{t<s\leq u} A_s D_s)|_{\mathcal{A}_t}$$

If $u > t$ is sufficiently small then $X_t D_t = (X_u + C_u) D_u|_{\mathcal{A}_t}$ and $V_t D_t = (\Delta_t + \Gamma_t) \cdot X_t D_t = \Delta_u \cdot (X_u + C_u) D_u|_{\mathcal{A}_t}$. Since $\Delta_u \cdot C_u = \Gamma_u \cdot X_u + A_u$ we have $V_t D_t = (\Delta_u \cdot X_u + \Gamma_u \cdot X_u + A_u) D_u\_\mathcal{A}_t = (V_u + A_u) D_u|_{\mathcal{A}_t}$. The formula above follows by induction.

For a strategy as above, $V_{\tau_0} D_{\tau_0} = (\sum_{t>\tau_0} A_t D_t)|_{\mathcal{A}_{\tau_0}} \geq 0$. Since $V_0 = \Gamma_0 \cdot X_0$, $A_0 = -\Gamma_0 \cdot X_0$, and $D_0 > 0$ we have $A_0 \leq 0$, where the 0 subscript denotes time τ_0. This proves the "easy" direction of the theorem.

There is no need to prove the "hard" direction since we have a large supply of arbitrage free models: every model of the form $X_t D_t = M_t - \sum_{s\leq t} C_s D_s$ where $M_t : \mathcal{A}_t \to \mathbb{R}^I$ is a martingale and $D_t : \mathcal{A}_t \to (0, \infty)$ is a positive adapted process is arbitrage-free. This is immediate by substituting $X_u D_u$ in the first displayed equation.

Examples

A *zero coupon bond* maturing at time u, $D(u)$, has a single cash flow $C_u^{D(u)} = 1$ so its price at time t satisfies $X_t^{D(u)} D_t = E_t 1 D_u$. We write $D_t(u) = E_t D_u / D_t$ for its price at time t. In particular $D_0(u) = E D_u$.

A *repurchase agreement* quotes a rate, f_t, at time t. It has price $X_t^{f_t} = 1$ and cash flow $C_{t+dt}^{f_t} = e^{f_t\,dt}$. For any arbitrage free model $D_t = E_t[e^{f_t\,dt} D_{t+dt}$. Define the *canonnical deflator* to be $D_t = \exp(-\int_0^t f_s\,ds)$.

Let $D_t = e^{-\rho t}$ and $M_t = (r, se^{\sigma B_t - \sigma^2 t/2})$. This is the Black-Merton/Scholes model. No need for self-financing portfolios, Ito's Lemma, or partial differential equations when using the Unified Model!