European Option Pricing

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September 21, 2019

Abstract

There is a simple formula for pricing European options using cumulants.

An European option pays some function of the underlying instrument value at expiration. Let F be the (random) value of an instrument at option expiration and g be the payoff function. The forward value of the option is $E[g(F)]$.

If F is lognormal then the standard Black-Scholes/Merton theory can be used to value this. If F is a perturbation of a lognormal random variable there is an explicit formula for computing the option value using the cumulants of the perturbation.

Valuation formulas depend on the cumulative distribution function of F. The (forward) value of a digital put paying $1(F \leq k)$ is $P(F \leq k)$.

The forward value of a put option is the expected value of the payoff

$$\begin{aligned}
E[\max\{k-F,0\}] &= E[(k-F)1(k-F \geq 0)] \\
&= E[(k-F)1(F \leq k)] \\
&= kP(F \leq k) - E[F1(F \leq k)] \\
&= kP(F \leq k) - E[F]E[1(F \leq k)F/E[F]] \\
&= kP(F \leq k) - E[F]P^F(F \leq k)
\end{aligned}$$

where P^F is the measure defined by $dP^F/dP = F/E[F]$. I.e., $E^F[X] = E[XF/E[F]]$.

The *cumulant* of a random variable X is $\kappa^X(s) = \log E[\exp(sX)]$ and the *cumulants* are the coefficients in the power series expansion $\kappa(s) = \sum_{n>0} \kappa_n s^n/n!$, where we write $\kappa(s)$ instead of $\kappa^X(s)$ when the random variable obvious.

Note $\kappa(0) = 0$, $\kappa'(0) = \kappa_1 = E[X]$, and $\kappa''(0) = \kappa_2 = \operatorname{Var} X$.

Define $X = z(F) = (\kappa(s) + \log F/f)/s$, so $F = f\exp(sX - \kappa(s))$. Note $E[F] = f$. The Black model has X standard normal and $s = \sigma\sqrt{t}$. In this case the cumulant of X is $s^2/2$ so all cumulants are zero except the second $\kappa_2 = 1$. We can and do assume X has mean 0 and variance 1 as in the Black model.

Exercise. Show $f\exp(s(\sigma X + \mu) - \kappa^{\sigma X+\mu}(s)) = f\exp(s^* X - \kappa^X(s^*))$ where $s^* = \sigma s$.

The (complete) Bell polynomials $B_n(\kappa_1, \ldots, \kappa_n)$ are defined by $B_0 = 1$ and

$$B_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(\kappa_1, \ldots, \kappa_{n-k})\kappa_{k+1}$$

for $n > 0$. They satisfy

$$e^{\sum_{n>0} \kappa_n s^n/n!} = \sum_{n\geq 0} B_n(\kappa_1, \ldots, \kappa_n)s^n/n!$$

Differentiation both sides with respect to s and and equating terms of equal power gives the recursive definition.

Note $B_1(\kappa_1) = \kappa_1$ and $B_2(\kappa_1, \kappa_2) = \kappa_1^2 + \kappa_2$.

The (probabalists') Hermite polynomials are defined by $H_0(x) = 1$, $H_1(x) = x$, and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ for $n \geq 1$. They satisfy

$$fracd^n dx^n e^{-x^2/2}(x) = (-1)^n H_n(x)e^{x^2/2}$$

Let $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ be the standard normal density function and let $\psi(x)$ be the probability density function of X. The Fourier transform of ψ is $\hat{\psi}(u) = E[\exp(-iuX)]$. Note $\hat{\phi}(u) = \phi(u)$. It is a general property of Fourier transforms that $\widehat{\psi'}(u) = iu\hat{\psi}(u)$.

$$
\begin{aligned}
\hat{\psi}(u) &= E[\exp(-iuX)] \\
&= \exp(\kappa(-iu)) \\
&= \exp(\sum_{n>0} \kappa_n (-iu)^n/n!) \\
&= \exp(-u^2/2) \exp(\sum_{n\geq 3} \kappa_n (-iu)^n/n!) \\
&= \hat{\phi}(u) \exp(\sum_{n\geq 3} \kappa_n (-iu)^n/n!) \\
&= \hat{\phi}(u)(\sum_{n\geq 0} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u)(1 + \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u) + \sum_{n\geq 3} B_n(\kappa_1, \ldots, \kappa_n)(-1)^n \widehat{\phi^{(n)}}(u)/n!
\end{aligned}
$$

Taking inverse Fourier transforms yields

$$
\psi(x) = \phi(x) + \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-1)^n \phi^{(n)}(x)/n!.
$$

Integrating gives

$$
\begin{aligned}
\Psi(x) &= \Phi(x) + \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n)(-1)^n \phi^{(n-1)}(x)/n! \\
&= \Phi(x) - \phi(x) \sum_{n\geq 3} B_n(0, 0, \kappa_3, \ldots, \kappa_n) H_{n-1}(x)/n!
\end{aligned}
$$

Remarks

The Hermite polynomials can be expressed using Bell polynomials, $H_n(x) = B_n(x, 1, 0, \ldots, 0)$.

$$
\begin{aligned}
B_1(0) &= 0 \\
B_2(0,0) &= 0 \\
B_3(0,0,\kappa_3) &= \kappa_3 \\
B_4(0,0,\kappa_3,\kappa_4) &= \kappa_4 \\
B_5(0,0,\kappa_3,\kappa_4,\kappa_5) &= \kappa_5 \\
B_6(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6) &= 10\kappa_3^2 + \kappa_6 \\
B_7(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6,\kappa_7) &= 35\kappa_3\kappa_4 + \kappa_7
\end{aligned}
$$

If $f\colon \mathbf{R} \to \mathbf{R}$ has a piecewise continuous second derivatve, then

$$
f(x) = f(a) + f'(a)(x-a) + \int_{-\infty}^{a} (k-x)^+ f''(k)\,dk + \int_{a}^{\infty} (x-k)^+ f''(k)\,dk.
$$