A Unified Model of Derivative Securities

Keith A. Lewis

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Abstract

Market instruments can be bought or sold at a price and entail cash flows. Every arbitrage-free model of prices and cash flows is parameterized by a vector-valued martingale whose components are indexed by market instruments and a positive, adapted process. This can be used to value, hedge, and manage the risk of derivative securities.

A *derivative security* is a contract between two parties: I will give you this on these dates if you will give me that on those dates. Derivatives must have existed since before recorded history. The Nobel prize winning breakthrough of Black, Scholes, and Merton was to show how to synthesize derivatives by dynamically trading market instruments based on the borrowing cost used to fund the hedge instead of trying to estimate the the actual growth rate of the underlying securities.

This short note provides a unified model for valuing, hedging, and managing the risk of any derivative security. It shows how they can be synthesized by trading market instruments and turns the spotlight on what may be the next Nobel prize winning problem: how should you hedge if you can't do it continuously?

Market Model

We assume the usual setup, $\langle\Omega, P, (\mathcal{A}_t)_{t\in T}\rangle$, of a sample space Ω, a probability measure P, and an increasing filtration of algebras $(\mathcal{A}_t)$ over the set of trading times T. If you are not familiar with this see the Notation section below.

Every *instrument* has a *price*, X_t, and a *cash flow*, C_t, at any trading time, $t \in T$. Instruments are assumed to be perfectly liquid: they can be bought or sold at the given price in any amount. Cash flows are associated with owning an instrument and are almost always 0, e.g., stocks have dividends, bonds have coupons, European options have exactly one cash flow at expiration. Futures always have price 0.

A *market model* specifies *prices* $X_t\colon \mathcal{A}_t \to \mathbf{R}^I$, and *cash flows* $C_t\colon \mathcal{A}_t \to \mathbf{R}^I$, where I are the available market instruments.

We use the notation $X\colon \mathcal{A} \to \mathbf{R}$ to indicate $X\colon \Omega \to \mathbf{R}$ is $\mathcal{A}$-measurable.

Trading

A *trading strategy* is a finite collection of strictly increasing stopping times, τ_j, and trades, $\Gamma_j : \mathcal{A}_{\tau_j} \to \mathbf{R}^I$ indicating the number of shares to trade in each instrument. Trades accumulate to a *position*, $\Delta_t = \sum_{\tau_j < t} \Gamma_j = \sum_{s<t} \Gamma_s$ where $\Gamma_s = \Gamma_j$ when $s = \tau_j$. Note the trade at time t is not included in the position at time t: it takes some time for trades to settle.

The *value* (or *mark-to-market*) of a position at time t is $V_t = (\Delta_t + \Gamma_t)\cdot X_t$: what you would get from liquidating your existing position and the trades just executed. The *amount* generated by the trading strategy at time t is $A_t = \Delta_t \cdot C_t - \Gamma_t \cdot X_t$: you receive the cash flows associated with your existing position and pay for the trades you just executed.

Arbitrage

Arbitrage is a trading strategy with $A_{\tau_0} > 0$, $A_t \geq 0$ for $t > \tau_0$ and $\sum_j \Gamma_j = 0$: you make money on the first trade and never lose until the strategy is closed out.

The Fundamental Theorem of Asset Pricing states there is no arbitrage if and only if there exists a *deflator*, $D_t : \mathcal{A}_t \to (0, \infty)$, with

$$X_t D_t = E_t[X_v D_v + \textstyle\sum_{t<u\leq v} C_u D_u].$$

We can assume $D_0 = 1$. If (D_t) is a deflator then so is (D_t/D_0).

Note that if there are no cash flows, $C_t = 0$ for all $t \in T$, this says $X_t D_t$ is a martingale. For an infinite time horizon where the price times the deflator goes to 0, this says the current price is the expected price of discounted future cash flows, just as in Graham and Dodd valuation.

A consequence of the above and the definition of value and amount is

$$V_t D_t = E_t[V_v D_v + \sum_{t<u\leq v} A_u D_u].$$

Note the similarity to the previous displayed equation. Value corresponds to price and amount corresponds to cash flow. This equation is the skeleton key for valuing derivative securities. It shows how dynamic trading can create synthetic market instruments.

Proof. If $u > t$ is sufficiently small then $X_t D_t = E_t[(X_u + C_u)D_u]$ and $V_t D_t = (\Delta_t + \Gamma_t) \cdot X_t D_t = \Delta_u \cdot E_t[(X_u + C_u)D_u]$. Since $\Delta_u \cdot C_u = \Gamma_u \cdot X_u + A_u$ we have $V_t D_t = E_t[(\Delta_u \cdot X_u + \Gamma_u \cdot X_u + A_u)D_u] = E_t[(V_u + A_u)D_u]$. The formula above follows by induction.

For a trading strategy that closes out, $V_{\tau_0} D_{\tau_0} = E_{\tau_0}[\sum_{t>\tau_0} A_t D_t] \geq 0$. Since $V_{\tau_0} = \Gamma_{\tau_0} \cdot X_{\tau_0}$, $A_{\tau_0} = -\Gamma_{\tau_0} \cdot X_{\tau_0}$, and $D_{\tau_0} > 0$ we have $A_{\tau_0} \leq 0$. This proves the "easy" direction of the FTAP.

There is no need to prove the "hard" direction since we have a large supply of arbitrage free models: every arbitrage-free model has the form $X_t D_t = M_t - \sum_{s\leq t} C_s D_s$ where $M_t : \mathcal{A}_t \to \mathbf{R}^I$ is a martingale and $D_t : \mathcal{A}_t \to (0, \infty)$. This is immediate by substituting $X_v D_v = M_v - \sum_{s\leq v} C_s D_s$ in the first displayed equation.

Valuing

If a derivative security pays amounts $\bar{A}_j$ at times $\bar{\tau}_j$ and there is a hedge, $(\Gamma_t)_{t\in T}$, that replicates these amounts, the value of the derivative is the cost of setting up the initial hedge: $V_0 = \Gamma_0 \cdot X_0$. The hedge must satisfy $A_t = 0$ if $t \neq \bar{\tau}_j$ for all j (*self financing*) and $A_t = \bar{A}_j$ if $t = \bar{\tau}_j$ for some j.

The formula $V_0 = E[\sum_j \bar{A}_j D_{\bar{\tau}_j}]$ is the value of the initial hedge, $\Gamma_0 \cdot X_0$. It can be computed using the derivative security payments, $\bar{A}_j$, and the deflator.

An European option has a single payment, $\bar{A}_T$, at a fixed time T and has value $V_0 = E\bar{A}_T D_T$. Sometimes it is useful to compute this as $E\bar{A}_T D_T = E^* \bar{A}_T E D_T$, where E^* is the expected value under the probability measure defined by $dP^*/dP = D_T/ED_T$. P^* is called the *forward measure* at time T.

Hedging

The trades at time t are similarly determined by $\Delta_t + \Gamma_t = dV_t/dX_t$, where the last term is the Fréchet derivative. Since we know the position, Δ_t, at time t this determines the trades: $\Gamma_t = dV_t/dX_t - \Delta_t$.

In the continuous time case where stocks are modelled by geometric Brownian motion, this becomes classical Black-Scholes/Merton delta hedging where delta is Δ and gamma is Γ. Under their mathematical assumptions, the hedge perfectly replicates the derivative.

In the real world, it is not possible to perfectly replicate the derivative security. There is still research to be done on when to hedge and how to mangage the associated risk of imperfect replication.

Black-Scholes/Merton

The Black-Scholes/Merton model of a stock with no dividents is specified by the martingale $M_t = (r, s\exp(\sigma B_t - \sigma^2 t/2))$ and deflator $D_t = \exp(-\rho t)$. No need for Ito's lemma, partial differential equations, or the Hahn-Banach theorem.

Canonical Deflator

There is a canonical choice for a deflator if repurchase agreements are available.

Repurchase Agreements

Assume trades occur at discrete times, like they actually do, so $T = \{t_j\}$ where $t_i < t_j$ if $i < j$.

A *repurchase agreement* at time t_j, R_j, has price $X_{t_j}^{R_j} = 1$ and cash flow $C_{t_{j+1}}^{R_j} = R_j$ so for any arbitrage free model $D_{t_j} = E_{t_j}[R_j D_{t_{j+1}}]$. We assume $D_{t_{j+1}}$ is $\mathcal{A}_{t_j}$ measurable so $D_{t_j} = R_j D_{t_{j+1}}$ and $D_{t_j} = \Pi_{j<n} R_j^{-1}$.

Define the *forward repo rate*, f_j, by $R_j = \exp(f_j\,\Delta t_j)$ where $\Delta t_j = t_{j+1} - t_j$, so $D_{t_j} = \exp(-\sum_{j<n} f_j \Delta_{t_j})$. The continuous time version of the *canonical deflator* is $D_t = \exp(-\int_0^t f_s\,ds)$.

As we will see below, the prices of all (non-risky) fixed income securites are determined by the deflator.

Futures

A *futures* on underlying S expiring at t_n has price $X_{t_j} = 0$ for all j, cash flows $C_{t_j} = \Phi_{t_j} - \Phi_{t_{j-1}}$ for $0 < j \le n$, where Φ_{t_j} is the futures *quote* at time t_j, and $\Phi_{t_n} = S_{t_n}$ at expiration. Since $0 = E_{t_j}(\Phi_{t_{j+1}} - \Phi_{t_j}) D_{t_j}$, we have $\Phi_{t_j} = E_{t_j}\Phi_{t_{j+1}}$ since $D_{t_j} > 0$ is t_j measurable.

This shows futures quotes are martingales.

Forward

A *forward* contract on underlying S with stike k expiring at time t pays $A_t = S_t - k$ at t. It has initial value $V_0 = E[(S_t - k)D_t] = S_0 - kED_t$. The *par forward*, f, is the strike that makes the initial value equal to zero: $0 = V_0 = E(S_t - f)D_t$ so $S_0 = fED_t$. This formula is called the *cost of carry*.

Put-Call Parity

The first thing every trader checks when using a new model is *put-call parity*. A (European) *put option* on underlying S with strike k expiring at time t pays $A_t^p = \max\{k - S_t, 0\}$ at t. A *call option* on underlying S with strike k expiring at time t pays $A_t^c = \max\{S_t - k, 0\}$ at t. Since $A_t^c - A_t^p = S_t - k$ we have $V_0^c - V_0^p = c - p = S_0 - kED_t$, where c and p are the value of the call and put at time 0. This result is independent of any (arbitrage-free) model.

An American option with strike k and expiration t pays A_τ at a stopping time $\tau \le t$ at the discretion of the option holder. In the unified model this is represented by extending the sample space Ω to $\Omega \times (0, t]$. The point (ω, τ) represents exercising at time τ when ω occurs. Note that the model does not assume the option is exercised at the "optimal" time. In practice, not every market paricipant does this.

American options do not satisfy put-call parity in general. The exercise time of the put and the call are usually different.

Zero Coupon Bond

A *zero coupon bond* pays one unit at maturity u. An arbitrage free model requires the price at time t, $Z_t(u)$, to satisfy $Z_t(u)D_t = E_t D_u$, so $Z_t(u) = E_t D_u / D_t = E_t \exp(-\int_t^u f_s\,ds)$.

Forward Rate Agreement

A *forward rate agreement* over the period $[u, v]$ with coupon f and day count basis δ pays -1 unit at the *effective date* u, and $1 + f\delta(u, v)$ at the *termination date* v, where $\delta(u, v)$ is the *day count fraction* for the interval. The day count fraction is approximately equal to the time in years from u to v for any day count basis.

The *forward par coupon* at time t, $F_t(u, v; \delta)$ is the coupon that makes the price at time $t \le u$ equal to 0: $0 = E_t[-D_u + (1 + F_t(u, v; \delta)\delta(u, v))D_v]$. Hence the par coupon

$$F_t(u, v; \delta) = (Z_t(u)/Z_t(v) - 1)/\delta(u, v)$$

is determined by zero coupon bond prices. Note, writing $F_t = F_t(u, v; \delta)$ and $\delta = \delta(u, v)$,

$$E_t F_t \delta D_v = F_t \delta E_t D_v = E_t[D_u - D_v] = Z_t(u) - Z_t(v)$$

There are also forward rate agreements not involving the exchange of notional. A (fixed rate) *payer* has the single cash flow $(f - F_u(u, v; \delta))\delta(u, v)$ at time v.

A *receiver* has the negative of this cash flow. The value at any time $t \leq u$ is determined by

$$
\begin{aligned}
V_t D_t &= E_t[(f - F_u(u, v; \delta))\delta(u, v) D_v] \\
&= E_t[f\delta(u, v) D_v - E_u[D_u - D_v]] \\
&= E_t[f\delta(u, v) D_v - D_u + D_v] \\
&= E_t[-D_u + (1 + f\delta(u, v)) D_v]
\end{aligned}
$$

which is the same as for a forward rate agreement that does exchange notional. These two types of FRAS's have very different risk characteristics. If either counterparty defaults during the time notionals are exchanged the loss can be much larger than when the payment is only the difference of the fixed and floating rate.

Interest Rate Swap

An *interest rate swap* with *calculation dates* $(t_j)_{j=0}^n$ with coupon c, and day count basis δ pays -1 unit at the effective date t_0, $c\delta(t_{j-1}, t_j)$ at t_j, $0 < j < n$, and $1 + c\delta(t_{n-1}, t_n)$ at *termination* t_n.

The *swap par coupon* at time t, $F_t(t_0, \dots, t_n; \delta)$, is the coupon that makes the price at time $t \leq t_0$ equal to 0:

$$
0 = E_t[-D_{t_0} + \sum_{0<j<n} F_t \delta(t_{j-1}, t_j) D_{t_j} + (1 + F_t \delta(t_{n-1}, t_n) D_{t_n}].
$$

Hence the par coupon, $F_t(t_0, \dots, t_n; \delta) = (Z_t(t_0) - Z_t(t_n)) / \sum_{0<j\leq n} \delta(t_{j-1}, t_j) Z_t(t_j)$, is determined by zero coupon bond prices.

Note that if $n = 1$ this is identical to a forward rate agreement.

There are also interest rate swaps not involving the exchange of notional. A (fixed rate) *payer* has the cash flows $(c - F_{t_j}(t_{j-1}, t_j; \delta))\delta(t_{j-1}, t_j)$ at times t_j, $0 < j \leq n$. A *receiver* has the negative of these cash flow.

As with forward rate agreements, the coupon making the value at time t equal to zero is the swap par coupon.

Floorlet

A *floorlet* is a put option on an at-the-money *forward rate agreement*. It pays $\max\{k - F_u(u, v), 0\}\delta(u, v)$ at time v. Its value at time $t < u$ is determined by

$V_t D_t = E_t \max\{k - F_u(u,v), 0\}\delta(u,v) D_u$. Writing $F_u = F_u(u,v)$ and $\delta = \delta(u,v)$ we have

$$
\begin{aligned}
V_t D_t &= E_t[\max\{k - F_u, 0\}\delta D_v] \\
&= E_t[\max\{k\delta - (1/Z_u(v) - 1), 0\} D_v] \\
&= E_t[\max\{1 + k\delta - 1/Z_u(v), 0\} D_v] \\
&= E_t^*[\max\{1 + k\delta - 1/Z_u(v), 0\}] E_t D_v \\
&= E_t^*[\max\{1 + k\delta - 1/Z_u(v), 0\}] Z_t(v) D_t
\end{aligned}
$$

where E_t^* is the expectation under the forward measure P^* defined by $dP_t^*/dP_t = D_v/E_t D_v$. This shows the value at t of a floorlet is $V_t = E_t^*[\max\{1 + k\delta - 1/Z_u(v), 0\}] Z_t(v)$.

Caplet

A *caplet* is a call option on an at-the-money *forward rate agreement*. It pays $\max\{F_u(u,v) - k, 0\}\delta(u,v)$ at time v. Its value at time $t < u$ is determined by $V_t D_t = E_t \max\{F_u(u,v) - k, 0\}\delta(u,v) D_u$. Similar to floorlets, the value at t of a caplet is $V_t = E_t^*[\max\{1/Z_u(v) - (1 + k\delta), 0\}] Z_t(v)$

Floor, Cap

A *floor* and a *cap* are just a sequence of back-to-back floorlets or caplets.

Swaption

A *swaption* is an option on a swap. It has a single cash flow $\max\{k - F_{t_0}(t_0, \ldots, t_n; \delta), 0\}$ at the effective date of the swap, t_0.

Remarks

The price of an instrument is not a number. Not only does it depend on whether you are buying or selling, the amount being purchased, and the counterparties involved, determine the price.

The atoms of finance are *exchanges*: $(t; a, i, c; a', i', c')$, where t is the time of the exchange, a is the amount of instrument i the *buyer*, c, decides to obtain for the amount a' in instrument i' from the *seller*, c'. a' in instrument i' from the *seller*, c'.

We can incorporate more realistic features by defining *price* to be a function $X\colon T\times A\times I\times C\times I\times C\to\mathbf{R}$, where T is the set of trading times, A the set of amounts that can be traded, I is the set of market instruments, and C is the set of legal trading entities. The exchanges available to the buyer at any time t are $(t;a,i,c;aX(t;a,i;i',c'),i',c')$

One Period Model

There is no need for probability measures: the Fundamental Theorem of Asset Pricing is a geometic result. In the one period case where $T=\{t_0,t_1\}$ the no arbitrage condition is there does not exist Γ_0 with $A_0=-\Gamma_0\cdot X_0>0$ and $A_1=\Gamma_0\cdot X_1\geq 0$.

In this case the FTAP states there is no arbitrage if and only if there exists a positive measure, Π, on Ω with $X_0=\int_\Omega X_1\,d\Pi$.

If X_0 belongs to the smallest closed cone containing the range of X_1, then there exists a positive measure, Π, with $X_0=\int_\Omega X_1\,d\Pi$. For any $\gamma\in\mathbf{R}^I$ such that $\gamma\cdot X_1\geq 0$ we have $\gamma\cdot X_0=\int_\Omega\gamma\cdot X_1\,d\Pi\geq 0$ hence there can be no arbitrage.

If X_0 does not belong to the smallest closed cone containing the range of X_1, then there exists a hyperplane through the origin separating X_0 from the cone. E.g., if x is the closest point in the cone to X_0 then $\gamma=x-X_0$ will do the job. Let γ be a vector in $\mathbf{R}$ normal to the hyperplane. We can choose γ such that $\gamma\cdot X_0<0$ and $\gamma\cdot X_1(\omega)\geq 0$ for all $\omega\in\Omega$. This shows arbitrage exists.

If a zero coupon bond exists, i.e., there is a $\zeta\in\mathbf{R}^I$ with $\zeta\cdot X_1(\omega)=1$ for all $\omega\in\Omega$, then $\zeta\cdot X_0=\int_\Omega\zeta\cdot X_1\,d\Pi=||\Pi||=z$ is the price of the zero coupon bond and $P=\Pi/z$ is a probability measure.

Notation

If $\mathcal{A}$ is an *algebra* on the set Ω we write $X\colon\mathcal{A}\to\mathbf{R}$ to indicate $X\colon\Omega\to\mathbf{R}$ is $\mathcal{A}$-*measurable*. If $\mathcal{A}$ is finite then the *atoms* of $\mathcal{A}$ form a *partition* of Ω and being measurable is equivalent to being constant on atoms. In this case X is indeed a function on the atoms.

If $\mathcal{A}$ is an algebra of sets, the *conditional expectation* of X given $\mathcal{A}$ is defined by $Y=E[X|\mathcal{A}]$ if and only if Y is $\mathcal{A}$ measurable and $\int_A Y\,dP=\int_A X\,dP$ for all $A\in\mathcal{A}$. This is equivalent to $Y(P|_\mathcal{A})=(XP)|_\mathcal{A}$ where the vertical bar indicates restriction of a measure.

A *filtration* indexed by $T\subseteq[0,\infty)$ is an increasing collection of algebras, $(\mathcal{A}_t)_{t\in T}$. A process $M_t\colon\mathcal{A}_t\to\mathbf{R}$, $t\in T$, is a *martingale* if $M_t=E[M_u|\mathcal{A}_t]=E_t[M_u]$ for $t\leq u$.

A *stopping time* is a function $\tau\colon\Omega\to T$ such that $\{\omega\in\Omega\mid\tau(\omega)\leq t\}$ belongs to $\mathcal{A}_t$, $t\in T$.