European Option Pricing

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Abstract

We demonstrate an explicit formula for pricing any European option in terms of the standard normal distribution and cumulants that uses Bell and Hermite polynomials.

An European option pays some function of the underlying instrument value at expiration. Let F be the (random) value of an instrument at option expiration and g be the payoff function. The forward value of the option is $E[g(F)]$. Any piecewise linear, continuous function can be approximated by a cash position, a forward contract and a linear combination of puts and calls. We demonstrate an explicit formula for puts and calls in terms of the standard normal distribution and cumulants that uses Bell and Hermite polynomials.

Puts and Calls

A put has payoff $p(x) = \max\{k - x, 0\} = (k - x)^+$ and a call has payoff $c(x) = (x - k)^+$. Since $c(x) - p(x) = x - k$ we can calculate the call value using the put value and a forward contract having the same strike.

If F is lognormal then the standard Black-Scholes/Merton theory can be used to value puts and calls. If F is a perturbation of a lognormal random variable there is an explicit formula for computing the option value using the cumulants of the perturbation.

Valuation formulas depend on the cumulative distribution function of F. The forward value of a put option is the expected value of the payoff

$$
\begin{aligned}
E[(k-F)^+] &= E[(k-F)1(k-F \geq 0)] \\
&= E[(k-F)1(F \leq k)] \\
&= kP(F \leq k) - E[F1(F \leq k)] \\
&= kP(F \leq k) - E[F]E[1(F \leq k)F/E[F]] \\
&= kP(F \leq k) - E[F]P^*(F \leq k)
\end{aligned}
$$

where P^* is the measure defined by $dP^*/dP = F/E[F]$. I.e., $E^*[X] = E[XF/E[F]]$. Similarly, the value of a call is $E[(F-k)^+] = E[F]P^*(F \geq k) - kP(F \geq k)$.

Exercise. Derive this formula.

The Black model uses $F = f\exp(\sigma B_t - \sigma^2 t/2)$ where B_t is standard Brownian motion. Note $F \leq k$ is equivalent to $B_t/\sqrt{t} \leq (\sigma^2 t/2 + \log k/f)/\sigma\sqrt{t}$. The right-hand side of the last inequality is often called $-d_2$ and the probability that the inequality holds is $\Phi(d_2)$ where Φ is the standard normal cumulative density function. For any normally distributed random variable, N, we have $E[\exp(N)f(N)] = E[\exp(N)]E[f(N + \operatorname{Var}(N))]$ and $E[\exp(N)] = \exp(E[N] + \operatorname{Var}(N)/2)$.

Exercise. Show $P^*(F \leq k) = E[e^{\sigma B_t - \sigma^2 t/2} 1(F \leq k)] = P(Fe^{\sigma^2 t} \leq k)$.

Exercise. Show $P^*(F \leq k) = \Psi(-d_1)$ where $d_1 = d_2 + \sigma\sqrt{t}$.

Cumulant

The *cumulant* of a random variable X is $\kappa^X(s) = \log E[\exp(sX)]$ and the *cumulants* are the coefficients in the power series expansion $\kappa^X(s) = \kappa(s) = \sum_{n>0} \kappa_n s^n/n!$.

Note $\kappa(0) = 0$, $\kappa'(0) = \kappa_1 = E[X]$, and $\kappa''(0) = \kappa_2 = \operatorname{Var} X$.

Exercise. Prove this.

Exercise. If c is constant show $\kappa_n^{cX} = c^n \kappa_n$.

Exercise. If X and Y are independent show $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$.

Define $X = z(F) = (\kappa(s) + \log F/f)/s$, so $F = f\exp(sX - \kappa(s))$. Note $E[F] = f$ and $\operatorname{Var}(\log F) = s^2 \operatorname{Var}(X)$.

Exercise. Prove this.

The Black model has X standard normal and $s = \sigma\sqrt{t}$. In this case the cumulant of X is $s^2/2$ so all cumulants are zero except the second $\kappa_2 = 1$.

Exercise. Use $E\exp(N) = \exp(E[N] + \operatorname{Var}(N)/2)$ for any normally distributed random variable N to prove this.

We can and do assume X has mean 0 and variance 1 as in the Black model.

Exercise. Show $f\exp(s(\sigma X + \mu) - \kappa^{\sigma X + \mu}(s)) = f\exp(s^* X - \kappa^X(s^*))$ where $s^* = \sigma s$.

Exercise. Show if X is Poisson with parameter λ, i.e., $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \ldots$, then $\kappa_n = \lambda$ for all n.

Note $F/E[F] = e^{sX - \kappa(s)}$ and define P^* as above by $dP^*/dP = e^{sX - \kappa(s)}$.

Exercise. If X^* is the random variable defined by $P(X^* \leq x) = P^*(X \leq x)$ show the cumulant of X^* is $\kappa^*(u) = \kappa(u+s) - \kappa(s)$.

Exercise. Show $(\kappa^*)^{(n)}(0) = \kappa^n(s)$ for derivatives $n > 0$.

Exercise. Show the cumulants of X^* are $\kappa_n^* = \sum_{k=0}^\infty \kappa_{n+k} s^k/k!$.

In particular, $E[X^*] = \kappa'(s)$ and $\mathrm{Var}(X^*) = \kappa''(s)$.

Bell polynomials

The (complete) Bell polynomials $B_n(\kappa_1, \ldots, \kappa_n)$ are defined by $B_0 = 1$ and

$$B_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) = \sum_{k=0}^n \binom{n}{k} B_{n-k}(\kappa_1, \ldots, \kappa_{n-k})\kappa_{k+1}$$

for $n > 0$. They satisfy

$$\exp\bigl(\sum_{n>0} \kappa_n s^n/n!\bigr) = \sum_{n\geq 0} B_n(\kappa_1, \ldots, \kappa_n) s^n/n!$$

Differentiation both sides with respect to s and and equating terms of equal power gives the recursive definition.

Note $B_1(\kappa_1) = \kappa_1$ and $B_2(\kappa_1, \kappa_2) = \kappa_1^2 + \kappa_2$.

The *reduced Bell polynomials*, $b_n(\kappa_1, \ldots, \kappa_n) = B_n(\kappa_1, \ldots, \kappa_n)/n!$ satisfy the recursion

Excercise. Prove this.

$$b_{n+1}(\kappa_1, \ldots, \kappa_{n+1}) = \frac{1}{n+1} \sum_{k=0}^n b_{n-k}(\kappa_1, \ldots, \kappa_{n-k})\kappa_{k+1}/k!$$

Hermite polynomials

The (probabalists') Hermite polynomials are defined by $H_0(x) = 1$, $H_1(x) = x$, and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$ for $n \geq 1$. They satisfy

$$\frac{d^n}{dx^n} e^{-x^2/2}(x) = (-1)^n H_n(x) e^{-x^2/2}$$

Fourier transform

The Fourier transform of ψ is $\hat{\psi}(u) = E[\exp(-iuX)]$. It is a general property of Fourier transforms that $\widehat{\psi'}(u) = iu\hat{\psi}(u)$.

Exercise. Prove this.

Let $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ be the standard normal density function and let $\psi(x)$ be the probability density function of X. Note $\hat{\phi}(u) = \phi(u)$.

Cumulative distribuiton

This section shows how to compute the cumulative distribution of a perturbation of a standard normal random variable.

$$\begin{aligned}
\hat{\psi}(u) &= E[\exp(-iuX)] \\
&= \exp(\kappa(-iu)) \\
&= \exp(\sum_{n>0} \kappa_n(-iu)^n/n!) \\
&= \exp(-u^2/2)\exp(\sum_{n\geq 3} \kappa_n(-iu)^n/n!) \\
&= \hat{\phi}(u)\exp(\sum_{n\geq 3} \kappa_n(-iu)^n/n!) \\
&= \hat{\phi}(u)(\sum_{n\geq 0} B_n(0,0,\kappa_3,\ldots,\kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u)(1 + \sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n)(-iu)^n/n!) \\
&= \hat{\phi}(u) + \sum_{n\geq 3} B_n(\kappa_1,\ldots,\kappa_n)(-1)^n\widehat{\phi^{(n)}}(u)/n!
\end{aligned}$$

Taking inverse Fourier transforms yields

$$\psi(x) = \phi(x) + \sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n)(-1)^n\phi^{(n)}(x)/n!.$$

Integrating gives

$$
\begin{aligned}
\Psi(x) &= \Phi(x) + \sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n)(-1)^n\phi^{(n-1)}(x)/n! \\
&= \Phi(x) - \phi(x)\sum_{n\geq 3} B_n(0,0,\kappa_3,\ldots,\kappa_n)H_{n-1}(x)/n!
\end{aligned}
$$

The value of a put is $E[(k-F)^+] = k\Psi(z) - f\Psi^*(z)$ where $z = (\kappa(s)+\log(f/F))/s$ and Φ^* is the formula above with κ_n replaced with κ_n^*.

Remarks

The Hermite polynomials can be expressed using Bell polynomials, $H_n(x) = B_n(x,1,0,\ldots,0)$.

The first seven Bell polynomials with $\kappa_1 = \kappa_2 = 0$ are

$$
\begin{aligned}
B_1(0) &= 0 \\
B_2(0,0) &= 0 \\
B_3(0,0,\kappa_3) &= \kappa_3 \\
B_4(0,0,\kappa_3,\kappa_4) &= \kappa_4 \\
B_5(0,0,\kappa_3,\kappa_4,\kappa_5) &= \kappa_5 \\
B_6(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6) &= 10\kappa_3^2 + \kappa_6 \\
B_7(0,0,\kappa_3,\kappa_4,\kappa_5,\kappa_6,\kappa_7) &= 35\kappa_3\kappa_4 + \kappa_7
\end{aligned}
$$

If $f\colon \mathbf{R}\to\mathbf{R}$ has a piecewise continuous second derivatve, then

$$
f(x) = f(a) + f'(a)(x-a) + \int_{-\infty}^{a} (k-x)^+ f''(k)\,dk + \int_a^{\infty} (x-k)^+ f''(k)\,dk.
$$

Note this formula holds for $x = a$. Taking a derivative with respect to x yields

$$
\begin{aligned}
f'(x) &= f'(a) + \int_{-\infty}^{a} -1(x\leq k) f''(k)\,dk + \int_a^{\infty} 1(x\geq k) f''(k)\,dk \\
&= f'(a) - \int_{\min\{x,a\}}^{a} f''(k)\,dk + \int_a^{\max\{x,a\}} f''(k)\,dk
\end{aligned}
$$

Note this formula holds for $x = a$. Taking a derivative with respect to x yields

$$f''(x) = f''(x)1(x < a) + f''(x)1(x > a)(k)$$

for $x \neq a$. Note the left and right limits as $x \to a$ equal $f''(a)$. This proves the original formula is valid.

Remarks

If the payoff function has jumps, digital options can be used to replicate it.