Normal Short Rate model

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If f_t is the short rate process, 1 invested at t returns $1 + f_t\,dt = e^{f\,dt}$ at $t + dt$, then the price at time t of a zero coupon bond maturing at time u is $D_t(u) = E_t \exp(-\int_t^u f_s\,ds)$, where E_t denotes conditional expectation at time t.

Let $(B_t)_{t\geq 0}$ be standard brownian motion. The *normal interest rate model* specifies the short rate process $f_t = \phi(t) + \sigma(t)B_t$.

If $\Sigma(t)$ is an integral of $\sigma(t)$, we have $d(\Sigma(t)B_t) = \Sigma(t)\,dB_t + B_t\sigma(t)\,dt$ so

$$
\begin{aligned}
\int_t^u \sigma(s)B_s\,ds &= \Sigma(u)B_u - \Sigma(t)B_t - \int_t^u \Sigma(s)\,dB_s \\
&= \Sigma(u)B_u - \Sigma(u)B_t + \Sigma(u)B_t - \Sigma(t)B_t - \int_t^u \Sigma(s)\,dB_s \\
&= (\Sigma(u) - \Sigma(t))B_t + \int_t^u (\Sigma(u) - \Sigma(s))\,dB_s
\end{aligned}
$$

Using $\exp(-\int_0^t \Lambda^2(s)\,ds/2 + \int_0^t \Lambda(s)\,dB_s)$ is a martingale we have $\exp(\int_t^u \Lambda(s)\,dB(s))|_t = \exp(\int_t^u \Lambda(s)^2\,ds/2)$ and so

$$
\exp(\int_t^u (\Sigma(u) - \Sigma(s))\,dB_s)|_t = \exp(\int_t^u (\Sigma(u) - \Sigma(s))^2\,ds/2)
$$

Putting the facts together yields

$$
D_t(u) = e^{-\int_t^u [\phi(s) - (\Sigma(u) - \Sigma(s))^2/2]\,ds - (\Sigma(u) - \Sigma(t))B_t}
$$

If σ is constant then $\Sigma(t) = \sigma t$ and

$$
-\log D_t(u) = \int_t^u [\phi(s) - \sigma^2(u-s)^2/2]\,ds + \sigma(u-t)B_t
$$

Since $D_0(u) = D(u) = \exp(-\int_0^u f(s)\,ds)$ is today's discount curve we have $\int_0^u f(s)\,ds = \int_0^u \phi(s)\,ds - \sigma^2 u^3/6$ and so $f(t) = \phi(t) - \sigma^2 u^2/2$. In terms of the forward the equation above becomes

$$-\log D_t(u) = \int_t^u f(s)\,ds + \sigma^2 ut(u-t)/2 + \sigma(u-t)B_t$$