Linear Algebra

Linear algebra is the study of linear structures and functions that preserve this structure. Vector spaces are a mathematical sweet spot. They are completely characterized by their dimension.

Vector Space

A *vector space* over the real numbers $\boldsymbol{R}$ is set V with a binary operation $V \times V \to V$, $(v, w) \mapsto v + w$, and a scalar product $\boldsymbol{R} \times V \to V$, $(a, v) \mapsto av$, satisfying the distributive law. The binary addition is *commutative* $(v + w = w + v)$, *associative* $((u + v) + w = u + (w + v))$, has an *identity element* $\mathbf{0}$ $(v + \mathbf{0} = v)$, and each element has an inverse $((-v) + v = \mathbf{0})$. The scalar product satisfies the *distributive laws* $a(v + w) = av + aw$, $(a + b)v = av + bv$, $(ab)v = a(bv)$ and $1v = v$, $a, b \in \boldsymbol{R}$, $v, w \in V$.

Exercise. *Show* $a\mathbf{0} = \mathbf{0}$, $0v = \mathbf{0}$, *and* $(-1)v = -v$, $a \in \boldsymbol{R}$, $v \in V$.

If S is any set then the set of all functions from S to the real numbers, $\boldsymbol{R}^S = \{v \colon S \to \boldsymbol{R}\}$, is a vector space. The addition is defined pointwise

$$(v + w)(s) = v(s) + w(s) \text{ for } v, w \in \boldsymbol{R}^S$$

as is the scalar product

$$(av)(s) = av(s) \text{ for } a \in \boldsymbol{R}, v \in \boldsymbol{R}^S.$$

The additive identity, $\mathbf{0}$, is the function $\mathbf{0}(s) = 0$ for all $s \in S$.

Exercise. *Show* $\boldsymbol{R}^S$ *is a vector space.*

You are probably already familiar with the vector space $\boldsymbol{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \boldsymbol{R}, 1 \le i \le n\}$. If $S = \{1, \ldots, n\}$ and $x \in \boldsymbol{R}^S$ then $x(i) = x_i$ provides a correspondance between $\boldsymbol{R}^{\{1,\ldots,n\}}$ and $\boldsymbol{R}^n$.

Subspace

A subset $W \subseteq V$ is a *subspace* of V if W is also a vector space. Clearly $\{0\}$ is the smallest subspace and V is the largest subspace of V.

Exercise. *If* $v \in V$ *then* $\boldsymbol{R}v = \{av \mid a \in \boldsymbol{R}\}$ *is a subspace.*

Given any set $S \subset V$ define $\operatorname{span} S$ to be the smallest subspace containing S.

Exercise. *If* $S \subseteq V$ *then* $\operatorname{span} S = \{\sum_{s\in S_0} a_s s \mid a_s \in \boldsymbol{R}, S_0 \subseteq S$ *finite* $\}$.

We need S_0 to be finite for the sum to be defined.

Solution

Taking $S_0 = \{s\}$ and $a_s = 1$ we see S is a subset of the right hand side. Every term of the form $\sum_{s\in S_0} a_s s$ must belong to $\operatorname{span} S$. Since the right hand side is a subspace it must be equal to the span of S.

A set of vectors $\{v_i\}_{i\in I}$ are *independent* if every finite sum $\sum_i a_i v_i = 0$ implies $a_i = 0$ for all i. A *basis* of a vector space is a set of independent vectors that span V.

Linear Operators

A *linear operator* from the vector space V to the vector space W, $T\colon V \to W$, is a function that preserves the vector space structure: $T(u + v) = Tu + Tv$ and $T(au) = aTu$ for $u, v \in V$, and $a \in \boldsymbol{R}$.

Exercise. *Show* $T(au + v) = aTu + v$, $a \in \boldsymbol{R}$, $u, v \in V$ *implies* T *is a linear operator.*

If $T\colon V \to V$ is a linear operator and U is a subspace of V then it is *invariant* under T if $TW \subseteq W$.

The set of all linear operators from a vector space V to a vector space W, $\mathcal{L}(V, W)$, is also a vector space. The addition is defined by $(S + T)v = Sv + Tv$, $S, T \in \mathcal{L}(V, W)$, $v \in V$ and scalar multiplication by $(aT)v = a(Tv)$, $a \in \boldsymbol{R}$.

Exercise. *Show* $\mathcal{L}(V, W)$ *is a vector space.*

The *kernel* of a linear transformation $T\colon V \to W$ is $\ker T = \{v \in V \mid Tv = 0\}$ and the *range* is $\operatorname{ran} T = \{Tv \mid v \in V\} \subseteq W$.

Exercise. *If* $T\colon V \to W$ *is a linear operator show the kernel is a subspace of* V *and the range is a subspace of* W.

Exercise. *Show* $\ker T = \{0\}$ *implies* T *is one-to-one.*

Hint: Show $Tu = Tv$ imples $u = v$, $u, v \in V$.

If $T\colon V \to W$ is one-to-one we can define the inverse $T^{-1}\colon \operatorname{ran} T \to V$ by $T^{-1}w = v$ if and only if $w = Tv$.

Exercise. *If* $T\colon V \to V$ *is a linear operator then the kernel and range are invariant under* T.

Solution

We have $T(\ker T) = \{\mathbf{0}\} \subseteq \ker T$ and $T(\operatorname{ran} T) = T(TV) \subseteq TV = \operatorname{ran} T$.

Eigenvectors/values

If $T\colon V \to V$ is a linear operator and $\boldsymbol{R}v$ is invariant under T then v is an *eigenvector* of T. The number $\lambda \in \boldsymbol{R}$ with $Tv = \lambda v$ is the *eigenvalue* corresponding to v. Note if $Tv = \lambda v$ then $v \in \ker T - \lambda I \neq \{0\}$ and so $T - \lambda I$ is not invertible.

The *spectrum* of an operator is the set of all eigenvalues: $\sigma(T) = \{\lambda \in \boldsymbol{R} \mid \ker(T - \lambda I) \neq \{\mathbf{0}\}\}$. For $\lambda \in \sigma(T)$ let $V_\lambda = \ker(T - \lambda I)$.

Exercise. *Show* $V_\lambda \cap V_\mu = \{\mathbf{0}\}$ *if* $\lambda \neq \mu$.

For $v \in V$ let $V_v = \operatorname{span}\{T^j v \mid j \geq 0\}$. Clearly V_v is invariant for T.

Spectral mapping theorem If p is a polynomial then $p(\sigma(T)) = \sigma(p(T))$.

Shift Operator

Given $v_1, \dots, v_n \in V$ define the *shift operator* $J\colon V \to V$ by $Jv_i = v_{i+1}$, $1 \leq j < n$ and $Jv_n = \mathbf{0}$.