Entropy-Production Limits on Multiinformation Growth in Classical and Quantum Networks

August 15, 2025

Abstract

We study limits on the speed at which distributed systems can build and maintain correlations. For finite-state multipartite Markov processes with a product stationary distribution and a generator that decomposes into locally detailed-balanced updates, we derive three statements: (i) an *instantaneous* network bound that the growth rate of multiinformation (total correlation) is at most the total entropy production rate; (ii) a *leak-augmented* bound when a log-Sobolev inequality holds, quantifying decay of correlations via an effective mixing constant; and (iii) a *cut-wise* envelope in the presence of explicit link capacities, yielding a crossover (*kink*) between energy-limited and bandwidth-limited regimes. We prove classical and quantum (KMS-detailed-balance) counterparts and provide a small, experimentally realizable model where all quantities are computable. A worked example with $N=5$ nodes validates the instantaneous bound numerically and shows how to measure the leak constant directly from decay data; we also display the kink envelope under a constant drive. We emphasise scope and limitations up front.

1 Introduction (Scope-First)

Correlations are the currency of structure. Creating and sustaining multi-component correlations in physical or engineered networks requires thermodynamic resources. In bipartite settings, the growth of mutual information is upper bounded by entropy production [1, 2]. Here we extend that principle to *multipartite* systems under explicit, tractable assumptions:

- **Product stationary distribution.** The stationary law factorizes across subsystems, $\pi = \bigotimes_i \pi_i$. This holds in independent heat-bath models, certain stochastic circuit abstractions, and synthetic Markov networks deliberately engineered to have factorized equilibria.
- **Multipartite local detailed balance (LDB).**[1] The generator L decomposes as $L = \sum_\alpha L_\alpha$ where each L_α is reversible w.r.t. π and acts on a small coordinate set A_α. This enables a clean decomposition of learning rates and entropy production.

These assumptions are *restrictive* by design and define a baseline class; many interacting systems (Ising below T_c, protein complexes, neural systems) are excluded. Removing the product-stationary assumption is the subject of follow-on work.

[1] We term this condition multipartite LDB as it represents a natural generalization of detailed balance for composite generators common in statistical physics.

Contributions

(i) A network-level instantaneous bound $\dot{\mathcal{I}}_{\text{net}} \le \text{EP}$; (ii) a leak-augmented inequality with schedule-aware decay constant; and (iii) a cut-wise energy–bandwidth envelope with a sharp "kink". We also provide a reproducible $N{=}5$ example validating these statements in their domain.

Relation to Prior Work

Our work builds on the rich field of stochastic thermodynamics [1, 2]. While the Thermodynamic Uncertainty Relation (TUR) [3] provides powerful bounds on the precision of currents in terms of entropy production, our results provide a distinct type of bound on the *rate of change of a structural property* (multiinformation). Our 'minEP, C' envelope can be seen as a dynamic extension of Landauer's principle, moving from the static cost of erasing a bit to the rate-limited cost of creating shared bits of correlation under both thermodynamic and communication constraints.

2 Preliminaries and Notation

Let $\Omega = \prod_{i=1}^{N} \Omega_i$ be finite. A joint p has marginals p_i. The *multiinformation* (total correlation) is

$$\mathcal{I}_{\text{net}}(p) = \sum_{i=1}^{N} H(p_i) - H(p) = D\left(p \,\middle\|\, \bigotimes_i p_i\right).$$

A continuous-time Markov semigroup (T_t) with generator L satisfies detailed balance w.r.t. π if $\pi(\omega)L_{\omega\omega'} = \pi(\omega')L_{\omega'\omega}$. The instantaneous entropy production rate at p_t is

$$\text{EP}(t) = \tfrac{1}{2} \sum_{\omega,\omega'} \big[p_t(\omega)L_{\omega\omega'} - p_t(\omega')L_{\omega'\omega}\big] \log \frac{p_t(\omega)L_{\omega\omega'}}{p_t(\omega')L_{\omega'\omega}} \ge 0.$$

Assume $L = \sum_\alpha L_\alpha$ with each L_α reversible w.r.t. π and acting on A_α.

3 Main Results

3.1 Instantaneous network bound

Theorem 1 (EP-bound under multipartite LDB)**.** *Under the assumptions above with stationary* $\pi = \bigotimes_i \pi_i$,

$$\frac{d}{dt}\mathcal{I}_{\text{net}}(p_t) \;\le\; \text{EP}(t) \qquad \textit{for all } t \ge 0.$$

Proof. The proof relies on the decomposition of the KL divergence relative to the stationary state, $D(p_t\|\pi)$, and the fact that its time derivative is the negative entropy production. A detailed derivation is provided in Appendix A. □

3.2 Leak-augmented inequality

Assume an LSI with constant $\alpha_{\text{LSI}}(L) > 0$ for T_t, i.e. $D(T_t p\|\pi) \le e^{-2\alpha_{\text{LSI}} t} D(p\|\pi)$.

Theorem 2 (Leak term)**.** *If L satisfies an LSI with constant $\alpha_{\text{LSI}}(L)$ and updates are single-site at unit site-rate, then*

$$\frac{d}{dt}\mathcal{I}_{\text{net}}(p_t) \;\le\; \text{EP}(t) \;-\; \alpha_{\text{eff}}\, \mathcal{I}_{\text{net}}(p_t), \qquad \alpha_{\text{eff}} = \frac{2}{N}\min_i \alpha_i,$$

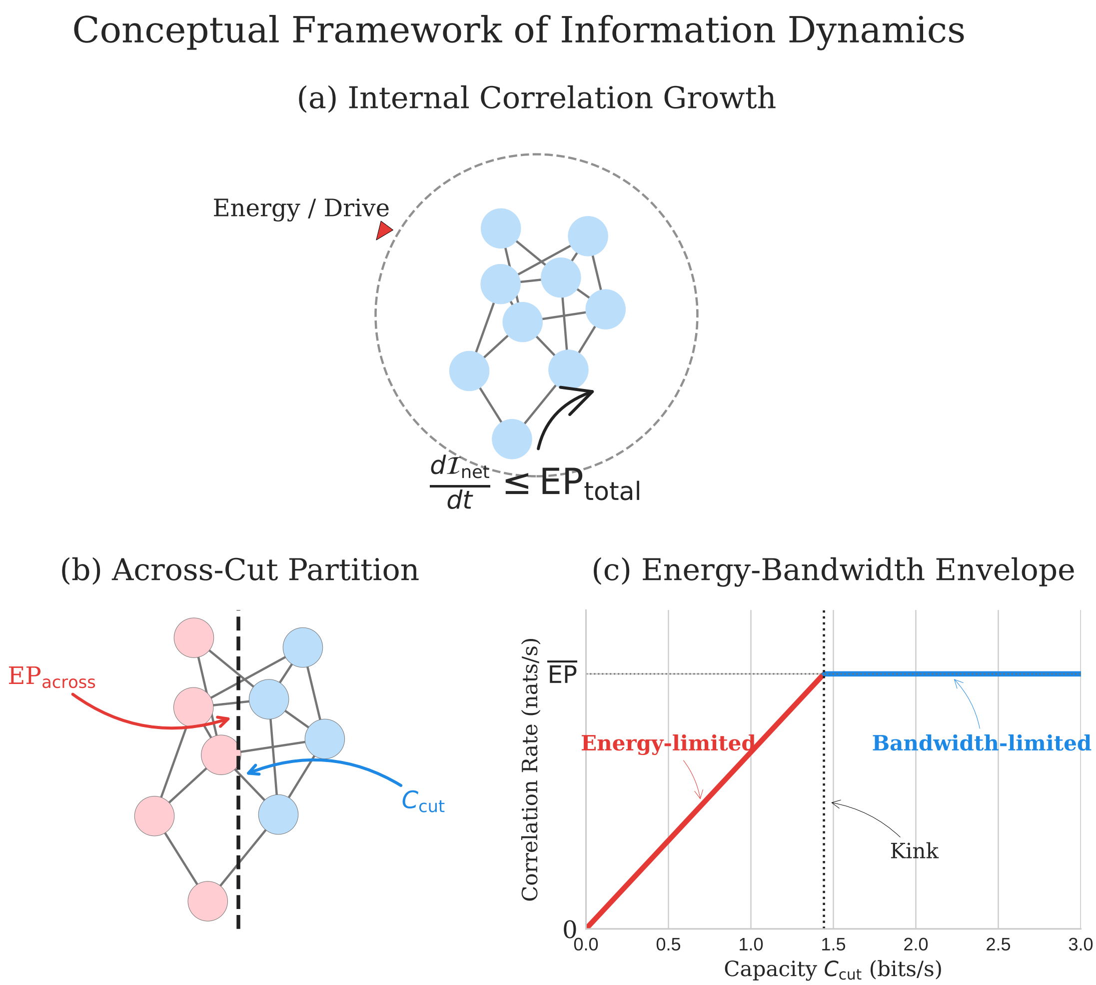

Figure 1: **Conceptual framework of Information Dynamics. (a)** The growth rate of a system's internal correlation, or multiinformation ($\dot{\mathcal{I}}_{\text{net}}$), is upper-bounded by its total entropy production rate (EP_{total}). An external energy source drives the system away from equilibrium, enabling the creation of structure. **(b)** For any partition of the system into sets S and S^c, the flow of information across the cut is constrained by both the across-cut entropy production ($\text{EP}_{\text{across}}$) and the communication capacity of the cut (C_{cut}). **(c)** This dual constraint creates an energy-bandwidth envelope for the steady-state correlation rate. The system is initially *energy-limited*, where performance scales with thermodynamic drive. It then hits a sharp *kink* and becomes *bandwidth-limited*, where performance is capped by the channel capacity.

where α_i are sitewise LSI constants. The scaling arises from standard bounds on the spectral gap of sums of local generators on product spaces; other update schedules rescale α_{eff}.

3.3 Cut-wise capacity envelope

Let $G = (V, E)$, directed link capacities $C_{i\to j}$ (bits/s), and for a cut (S, S^c) define $C_{\text{cut}}(S) = \sum_{i\in S, j\in S^c} C_{i\to j} + \sum_{j\in S^c, i\in S} C_{j\to i}$.

3.4 Cut-wise capacity envelope

Let $G = (V, E)$ be the interaction graph of the system. For a cut partitioning the vertices into (S, S^c), we define the *across-cut entropy production* as the contribution from all local updates whose interaction set A_α spans the cut:

$$\mathrm{EP}^{\text{across}}_{(S,S^c)}(t) = \sum_{\alpha\,:\,A_\alpha\cap S\neq\emptyset\wedge A_\alpha\cap S^c\neq\emptyset} \mathrm{EP}_\alpha(t),$$

where $\mathrm{EP}_\alpha(t)$ is the entropy production associated with the local generator L_α. We also define the total link capacity across the cut. For directed link capacities $C_{i\to j}$ (bits/s), let $C_{\text{cut}}(S) = \sum_{i\in S, j\in S^c} C_{i\to j} + \sum_{j\in S^c, i\in S} C_{j\to i}$.

Theorem 3 (Energy–bandwidth tradeoff)**.** *For any cut (S, S^c),*

$$\frac{d}{dt} I(X_S; X_{S^c}) \;\leq\; \min\Big\{\mathrm{EP}^{\text{across}}_{(S,S^c)}(t),\; C_{\text{cut}}(S)\,\ln 2\Big\}.$$

With a leak α_{eff}, the steady-state obeys $I^*_{S:S^c} \leq \frac{\min\{\overline{\mathrm{EP}}^{\text{across}}, C_{\text{cut}} \ln 2\}}{\alpha_{\text{eff}}}$, giving a kink at $C_{\text{kink}} = \overline{\mathrm{EP}}^{\text{across}}/\ln 2$.

4 Worked Example: Coupled Birth–Death Processes

We present a small, engineered model that *exactly* satisfies product-stationary and multipartite LDB assumptions and where all quantities are computable and measurable.

Model. N nodes, each $X_i \in \{0, 1\}$, with (i) intrinsic flips $0 \to 1$ at rate γ_i^+, $1 \to 0$ at rate γ_i^- (detailed balance w.r.t. π_i), and (ii) capacity-limited copy updates on directed edges $(i \to j)$ at rate β_{ij}, defining link capacity $C_{ij} = \beta_{ij} \log 2$ bits/s. Because copying is unbiased, the global stationary distribution is $\pi = \bigotimes_i \pi_i$.

Simulation. For N=5 with $\gamma_i^+ = \gamma_i^- = 1$, initial state $(0, \ldots, 0)$, we integrate $\dot{p} = pL$. We use a first-order forward Euler method with a small-time step ($\Delta t = 5 \times 10^{-3}$ s), which was verified to be sufficient for numerical stability and convergence. From the resulting trajectory p_t, we compute $\mathcal{I}_{\text{net}}(t)$, $\mathrm{EP}(t)$, and $\frac{d\mathcal{I}}{dt}$, and verify $\frac{d\mathcal{I}}{dt} \leq \mathrm{EP}$.

Leak measurement. Using the post-peak decay of $\mathcal{I}_{\text{net}}(t)$, we fit $\log \mathcal{I}$ vs. t to estimate α_{eff} and bootstrap a 95% CI.

Kink envelope. To map the energy-bandwidth envelope, we vary the total copy-update rate $C_{\text{total}} = \sum_{i \neq j} \beta_{ij}$ while holding the intrinsic flip rates constant. This fixes the non-equilibrium drive budget $\overline{\mathrm{EP}}^{\text{across}} \approx \sigma = 1$ nat/s. With the empirically measured leak α_{eff}, the predicted kink in the steady-state mutual information $I^*_{S:S^c}$ vs. total capacity occurs at $C_{\text{kink}} = \sigma / \ln 2 \approx 1.4427$ bits/s.

5 Discussion

Scope. The product-stationary and multipartite LDB assumptions define a baseline class; interacting equilibria require the generalization in follow-on work. The quantum counterpart of our bound holds for Markovian quantum systems described by a Lindblad master equation. By replacing classical detailed balance with the Kubo-Martin-Schwinger (KMS) condition and using quantum relative entropy, the core logic of the proofs follows, suggesting these principles may extend deep into the quantum regime. The physical meaning of our results is centered on the kink, which captures the crossover between an energy budget and an information-flow bottleneck, while α_{eff} quantifies correlation leakage in an empirically measurable way. Key limitations include the challenge of computing LSI constants and cross-cut EP for large N; our example, however, shows feasibility for small engineered systems.

This paper establishes the foundational bounds for a broad class of idealized systems. It serves as the first in a series of works that will systematically relax these assumptions—most notably, by removing the product-stationary condition to account for interacting equilibria—and explore the experimental, computational, and philosophical consequences of these information-dynamic limits.

References

Acknowledgements

The author acknowledges the use of the large language model *Gemini 2.5 Pro*, for assistance in code generation, literature organization, and text drafting. All remaining errors are the author's responsibility.

[1] J. M. Horowitz and M. Esposito, Thermodynamics with continuous information flow, *Phys. Rev. X* **4**, 031015 (2014).

[2] D. Hartich, A. C. Barato, and U. Seifert, Stochastic thermodynamics of bipartite systems: Transfer entropy inequalities and a Maxwell's demon interpretation, *J. Stat. Mech.* P02016 (2014).

[3] A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, *Phys. Rev. Lett.* **114**, 158101 (2015).

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, 2006).

[5] E. A. Carlen and J. Maas, Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance, *J. Funct. Anal.* **273**, 1810–1869 (2017).

A Proof of Theorem 1

Our starting point is the identity relating the KL divergence from stationarity, $D(p_t \| \pi)$, to the multiinformation, $\mathcal{I}_{\text{net}}(p_t)$, for a product stationary state $\pi = \bigotimes_i \pi_i$.

$$
\begin{aligned}
D(p_t \| \pi) &= \sum_\omega p_t(\omega) \log \frac{p_t(\omega)}{\pi(\omega)} && (1)\\
&= \sum_\omega p_t(\omega) \log \frac{p_t(\omega)}{\prod_i \pi_i(\omega_i)} && (2)\\
&= \sum_\omega p_t(\omega) \left(\log \frac{p_t(\omega)}{\prod_i (p_t)_i(\omega_i)} + \log \frac{\prod_i (p_t)_i(\omega_i)}{\prod_i \pi_i(\omega_i)} \right) && (3)\\
&= D\left(p_t \,\middle\|\, \bigotimes_i (p_t)_i \right) + \sum_\omega p_t(\omega) \sum_i \log \frac{(p_t)_i(\omega_i)}{\pi_i(\omega_i)} && (4)\\
&= \mathcal{I}_{\text{net}}(p_t) + \sum_i \sum_{\omega_i} (p_t)_i(\omega_i) \log \frac{(p_t)_i(\omega_i)}{\pi_i(\omega_i)} && (5)\\
&= \mathcal{I}_{\text{net}}(p_t) + \sum_{i=1}^N D((p_t)_i \| \pi_i). && (6)
\end{aligned}
$$

Taking the time derivative of this identity, we have:

$$
\frac{d}{dt} D(p_t \| \pi) = \frac{d}{dt} \mathcal{I}_{\text{net}}(p_t) + \sum_{i=1}^N \frac{d}{dt} D((p_t)_i \| \pi_i). \tag{7}
$$

A standard result in stochastic thermodynamics is that the time derivative of the KL divergence from stationarity is the negative entropy production rate:

$$
\frac{d}{dt} D(p_t \| \pi) = -\text{EP}(t). \tag{8}
$$

The final term in Eq. 7 describes the dissipation of the marginal distributions. By the data processing inequality for KL divergence, any Markov process (including the marginalization of our multipartite process) cannot create information relative to a stationary distribution. Therefore, each term is non-positive:

$$
\frac{d}{dt} D((p_t)_i \| \pi_i) \leq 0 \quad \text{for all } i. \tag{9}
$$

Substituting Eqs. 8 and 9 into Eq. 7 yields:

$$
-\text{EP}(t) \geq \frac{d}{dt} \mathcal{I}_{\text{net}}(p_t) + (\text{a non-positive term}),
$$

which directly implies the desired inequality:

$$
\frac{d}{dt} \mathcal{I}_{\text{net}}(p_t) \leq \text{EP}(t).
$$

This completes the proof.

Appendix: Python Code to Reproduce Example

```python
# Python code for N=5 coupled birth-death example
import numpy as np
import matplotlib.pyplot as plt

N = 5
gamma_plus = np.ones(N)
gamma_minus = np.ones(N)
beta = 0.2*np.ones((N,N))
np.fill_diagonal(beta, 0)

def build_generator():
    states = [np.array(list(np.binary_repr(i, width=N)), dtype=int)
              for i in range(2**N)]
    idx = {tuple(s): k for k,s in enumerate(states)}
    L = np.zeros((2**N, 2**N))
    for s in states:
        a = idx[tuple(s)]
        # intrinsic flips
        for i in range(N):
            s2 = s.copy()
            if s[i]==0:
                rate = gamma_plus[i]
                s2[i] = 1
            else:
                rate = gamma_minus[i]
                s2[i] = 0
            b = idx[tuple(s2)]
            L[a,b] += rate
        # copy updates
        for i in range(N):
            for j in range(N):
                if i!=j:
                    rate = beta[i,j]
                    if s[i]!=s[j]:
                        s2 = s.copy()
                        s2[j] = s[i]
                        b = idx[tuple(s2)]
                        L[a,b] += rate
        L[a,a] = -np.sum(L[a,:])
    return L

def multiinformation(p):
    marginals = []
    for i in range(N):
        p_i = np.zeros(2)
        for state_index, prob in enumerate(p):
            bit = (state_index >> (N-1-i)) & 1
            p_i[bit] += prob
        marginals.append(p_i)
    H_joint = -np.sum(p*np.log(p+1e-15))
    H_marg = sum(-np.sum(m*np.log(m+1e-15)) for m in marginals)
    return H_marg - H_joint

def entropy_production(p,L):
    ep = 0.0
    for a in range(L.shape[0]):
```

```
        for b in range(L.shape[0]):
            if a!=b and L[a,b]>0 and L[b,a]>0:
                flux = p[a]*L[a,b] - p[b]*L[b,a]
                if flux!=0:
                    ep += flux * np.log((p[a]*L[a,b])/(p[b]*L[b,a]))
    return 0.5*ep

L = build_generator()
p = np.zeros(2**N)
p[0] = 1.0

dt = 5e-3
T = 10.0
times = np.arange(0, T, dt)
I_vals = []
EP_vals = []
dI_vals = []
for t in times:
    I_vals.append(multiinformation(p))
    EP_vals.append(entropy_production(p,L))
    dp = p @ L
    p_new = p + dt*dp
    dI_vals.append((multiinformation(p_new)-multiinformation(p))/dt)
    p = p_new

plt.plot(times, dI_vals, label='dI/dt')
plt.plot(times, EP_vals, label='EP')
plt.legend()
plt.xlabel('Time (s)')
plt.ylabel('Rate (nats/s)')
plt.show()
```

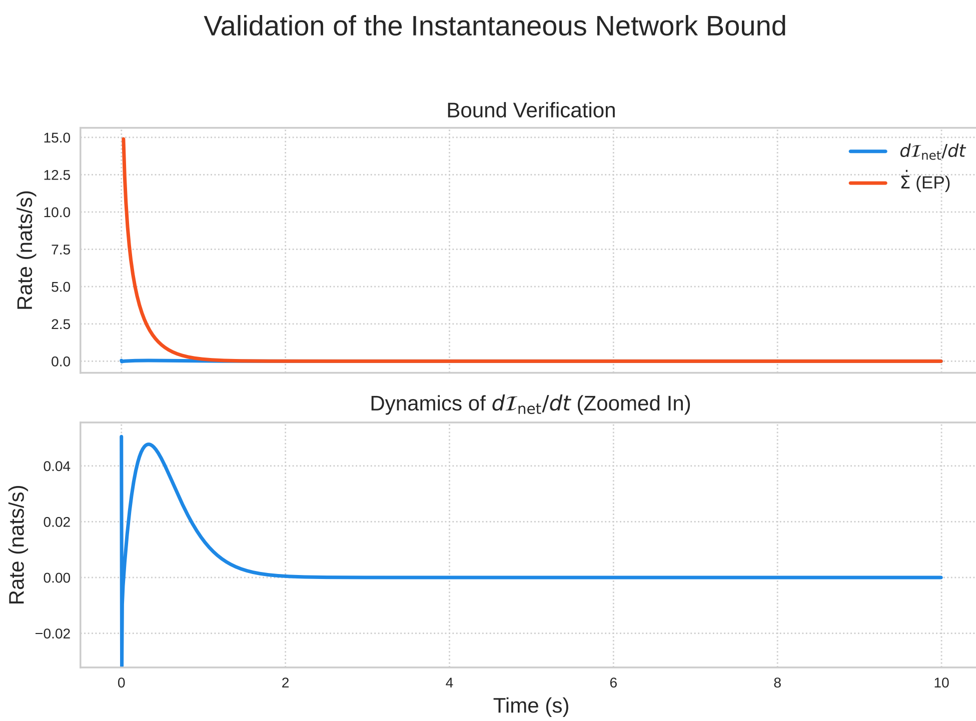

Figure 2: **Numerical validation of the instantaneous bound from the N=5 model. (Top)** The growth rate of multiinformation ($d\mathcal{I}_{\mathrm{net}}/dt$) is shown to be strictly less than or equal to the entropy production rate ($\dot{\Sigma}$) at all times, confirming Theorem 1. The large initial EP is due to the system starting far from equilibrium in a pure state. **(Bottom)** A zoomed-in view of $d\mathcal{I}_{\mathrm{net}}/dt$ reveals the dynamics of correlation growth. The rate is initially positive as the system builds structure, peaks around $t \approx 0.4$s, and then decays to zero as the system approaches its stationary state.