Answers (Problem set 5)

Online S520

1. (a) We want to find the point q_2 such that $F(q_2) = 0.5$. By geometry, the CDF is

$$F(y) = \begin{cases} 0 & y < 0 \\ 0.3y & 0 \leq y < 1 \\ 0.3 + 0.7(y-1) & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$

and hence $q_2 = 9/7$.

Or use geometry to find the value q_2 on interval [1, 2) to the left of which the area under $f(x)$ is 0.5. Since the area under $f(x)$ on interval [0, 1) is 0.3, the area of the rectangular with base $q_2 - 1$ and height 0.7 should be 0.2, i.e. $0.7 \times (q_2 - 1) = 0.2$. So $q_2 = 9/7$.

(b) There's a 0.3 chance that X is between 0 and 1 (uniformly), and a 0.7 chance that X is between 1 and 2 (again, uniformly.) So the expected value is $0.3 \times 0.5 + 0.7 \times 1.5 = 1.2$.

The above is equivalent to considering the random variable X as a weighted sum of two uniform random variables, X_1 on [0, 1) and X_2 on [1, 2), with weights 0.3 and 0.7. So $E(X) = E(0.3X_1 + 0.7X_2) = 0.3E(X_1) + 0.7E(X_2) = 0.3 \times 0.5 + 0.7 \times 1.5 = 1.2$.

Or calculate the integral using calculas:

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 0.3x\, dx + \int_1^2 0.7x\, dx \\ &= 0.3 \times \left.\frac{x^2}{2}\right|_0^1 + 0.7 \times \left.\frac{x^2}{2}\right|_1^2 \\ &= 0.3 \times 0.5 + 0.7 \times 1.5 \\ &= 1.2 \end{aligned}$$

2. *Trosset exercise 6.4.1*

The graph of the pdf is:

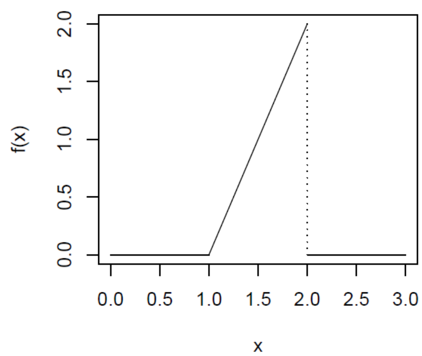

Figure 1: pdf for Exercise 5.6.2.

(a) The triangle to the left of q_2 ($1 < q_2 < 2$) has area 0.5. This triangle has base $(q_2 - 1)$ and height $f(q_2)$. So:

$$\begin{aligned}\frac{1}{2}(q_2 - 1) \times 2(q_2 - 1) &= 0.5 \\ (q_2 - 1)^2 &= 0.5 \\ q_2 - 1 &= \sqrt{0.5} \\ q_2 &= 1 + \sqrt{0.5}\end{aligned}$$

So the population median is $1 + \sqrt{0.5}$.

(b) The triangle to the left of q_1 ($1 < q_1 < q_2$) has area 0.25. This triangle has base $(q_1 - 1)$ and height $f(q_1)$. So:

$$\begin{aligned}\frac{1}{2}(q_1 - 1) \times 2(q_1 - 1) &= 0.25 \\ (q_1 - 1)^2 &= 0.25 \\ q_1 - 1 &= \sqrt{0.25} = 0.5 \\ q_1 &= 1 + 0.5 = 1.5\end{aligned}$$

The triangle to the left of q_3 ($1 < q_2 < q_3$) has area 0.75. This triangle has base $(q_3 - 1)$ and height $f(q_3)$. So:

$$\begin{aligned}\frac{1}{2}(q_3 - 1) \times 2(q_3 - 1) &= 0.75 \\ (q_3 - 1)^2 &= 0.75 \\ q_3 - 1 &= \sqrt{0.75} = 0.5\sqrt{3} \\ q_3 &= 1 + 0.5\sqrt{3}\end{aligned}$$

So the IQR is $q_3 - q_1 = 1 + 0.5\sqrt{3} - 1.5 \approx 0.366$

3. *Trosset exercise 6.4.7*

   (a) True. For a symmetric random variable X, the average of the first and third quartiles of X is the second quartiles or median of X.

   (b) False. Interquartile range is rather insensitive to the influence of extreme values that occur with small probability, and standard deviation is sensitive to the influence of extreme values. See example 6.6 (Trosset pg 147), $q_1 = q_3 = 0$, so $IQP = q_3 - q_1 = 0$. But $\sigma \geq 0$, and $\sigma_1 = \sqrt{2.34}, \sigma_2 = \sqrt{196.38}, \sigma_3 = \sqrt{19600.38}$. (Trosset pg 150)

   (c) False. Expected value is sensitive to the influence of extreme values. See example 6.6, the first and third quartile of X are both 0, but the expected value is $2 \times 10^{k-2}$ where $k = 1, 2, 3, 4$

   (d) True. When $\sigma = 0$, it indicates that there is no variation in the random variable, i.e. the random variable can only assume one value. Therefore, the first and third quartiles are also 0.

   (e) False. A counter example: a random variable takes the following values $\{-2, -1, 0, 3\}$ with probability $(0.2, 0.2, 0.4, 0.2)$. The mean and median are both 0 and it is obviously not symmetric.

4. Let X be the weekly demand for granola. Then $X \sim N(\mu, \sigma^2)$ where $\mu = 85$ and $\sigma = 5$. We want to find the 2.5% quantile q such that $P(X < q) = 2.5\% = 0.025$. Use R function:

$$q = \texttt{qnorm(0.025, mean = 85, sd = 5)} = 75.2\text{pounds}.$$

5. *Trosset exercise 6.4.5*
   $CCQ \sim N(\mu = 100, \sigma^2 = 20^2)$. We want to find the quantile q such that $P(CCQ > q) = 10^{-6}$. So $P(CCQ < q) = 1 - 10^{-6}$. Use R function:

$$q = \texttt{qnorm(1-10\^{}\{-6\}, mean = 100, sd = 20)} = 195.1.$$

6. (5 points) *Let X be a standard normal random variable. Let $Y = X^2$. Find the 0.9-quantile of Y. (Use R and give code.)*

   We want to find the quantile q such that $P(Y < q) = 0.9$, which is equivalent to $P(X^2 < q) = 0.9$ or $P(-\sqrt{q} < X < \sqrt{q}) = 0.9$.

   Since X is a standard normal random variable, its distribution is symmetric around 0, or $P(-\sqrt{q} < X < 0) = P(0 < X < \sqrt{q}) = 0.9/2 = 0.45$. Hence,

$$P(X < \sqrt{q}) = P(X < 0) + P(0 < X < \sqrt{q}) = 0.5 + 0.45 = 0.95,$$

   implying that $\sqrt{q}$ is the 0.95-quantile of X.

   $\sqrt{q} = \texttt{qnorm(.95)} = 1.644854.$

   $q = \texttt{qnorm(.95)}^2 = 2.71.$