Useful Properties in Chapters 1-6

Online S520 Supplemental Material

1 Properties of Expected Values

1.1 Linear transformation

1. Adding or subtracting a constant from the values of a random variable shifts the mean/expected value.
$$E(X \pm c) = E(X) \pm c$$

2. Adding or subtracting a constant from the values of a random variable doesnt change the variance or standard deviation.
$$Var(X \pm c) = Var(X)$$
$$SD(X \pm c) = SD(X)$$

3. Multiplying each value of a random variable by a constant multiplies the mean/expected value by the constant.
$$E(aX) = aE(X)$$

4. Multiplying each value of a random variable by a constant multiplies the variance by the square of the constant.
$$Var(aX) = a^2 Var(X)$$

5. Multiplying each value of a random variable by a constant multiplies the standard deviation by the absolute value of the constant.
$$SD(aX) = |a| SD(X)$$

Summary (Linear transformation 1-5)

$$\begin{aligned} E(aX + c) &= aE(X) + c \\ Var(aX + c) &= a^2 Var(X) \\ SD(aX + c) &= |a| SD(X) \end{aligned}$$

Examples

1. Suppose that $X_1 \sim N(-2, 4)$ and $X_2 \sim N(5, 9)$ are independent. Determine the mean and variance of $-3X_1 + 8$ and $2X_2 - 5$.

Solution: $E(X_1) = -2, Var(X_1) = 4,\ E(X_2) = 5, Var(X_2) = 9.$

$$\begin{aligned} E(-3X_1 + 8) &= -3E(X_1) + 8 = -3 \cdot (-2) + 8 = 14 \quad (\text{let } a = -3, c = 8) \\ Var(-3X_1 + 8) &= (-3)^2 Var(X_1) = 9 \cdot 4 = 36 \quad (\text{let } a = -3, c = 8) \end{aligned}$$

$$\begin{aligned} E(2X_2 - 5) &= 2E(X_2) - 5 = 2 \cdot 5 - 5 = 5 \quad (\text{let } a = 2, c = -5) \\ Var(2X_2 - 5) &= 2^2 Var(X_2) = 4 \cdot 9 = 36 \quad (\text{let } a = 2, c = -5) \end{aligned}$$

2. Similar exercises: Trosset 5.6.8 (b) and (d)

1.2 Sum of Random Variables

1. The expected value of the sum (or difference) of two random variables is the sum (or difference) of their expected values.
$$E(X \pm Y) = E(X) \pm E(Y)$$

2. The variance of the sum (or difference) of two **independent** random variables is the sum of their individual variances.
$$Var(X \pm Y) = Var(X) + Var(Y)$$

Summary (Linear transformation and Sum)

$$\begin{aligned} E(aX + bY) &= aE(X) + bE(Y) \\ Var(aX + bY) &= a^2 Var(X) + b^2 Var(Y) \quad (\text{if } X \text{ and } Y \text{ are independent}) \end{aligned}$$

Examples

1. Suppose that $X_1 \sim N(-2, 4)$ and $X_2 \sim N(5, 9)$ are independent. Determine the mean and variance of $-3X_1 + 2X_2$.
Solution: $E(X_1) = -2, Var(X_1) = 4,\ E(X_2) = 5, Var(X_2) = 9.$ $-3X_2$ and $2X_2$ are also independent.
$$\begin{aligned} E(-3X_1 + 2X_2) &= -3E(X_1) + 2E(X_2) = 6 + 10 = 16 \quad (\text{let } a = -3, b = 2) \\ Var(-3X_1 + 2X_2) &= (-3)^2 Var(X_1) + 2^2 Var(X_2) = 36 + 36 = 72 \quad (\text{let } a = -3, b = 2) \end{aligned}$$

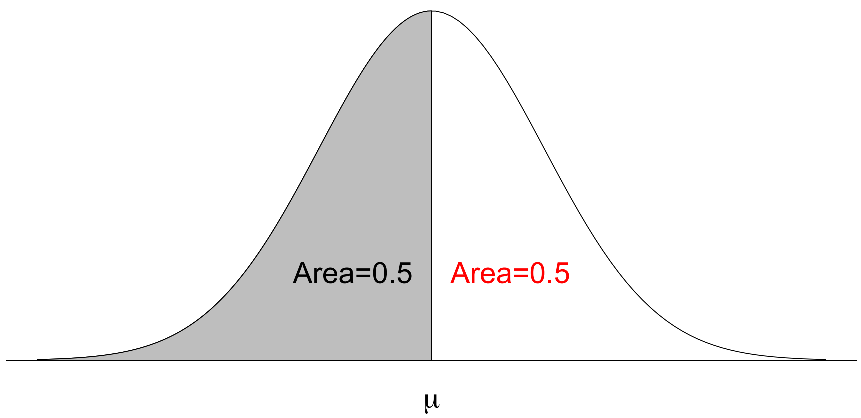
2. Similar exercises: Trosset 5.6.8 (a), (c) and (e).

2 Properties of Normal Distributions

2.1 Symmetry

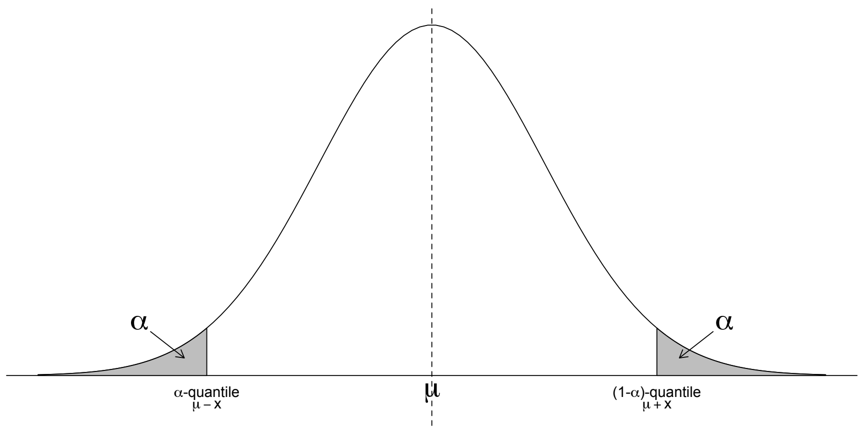
1. A normal distribution with mean μ and standard deviation σ is symmetric about the mean μ. So the area below μ is the same as the area above μ.

$$P(X < \mu) = P(X > \mu) = 0.5$$



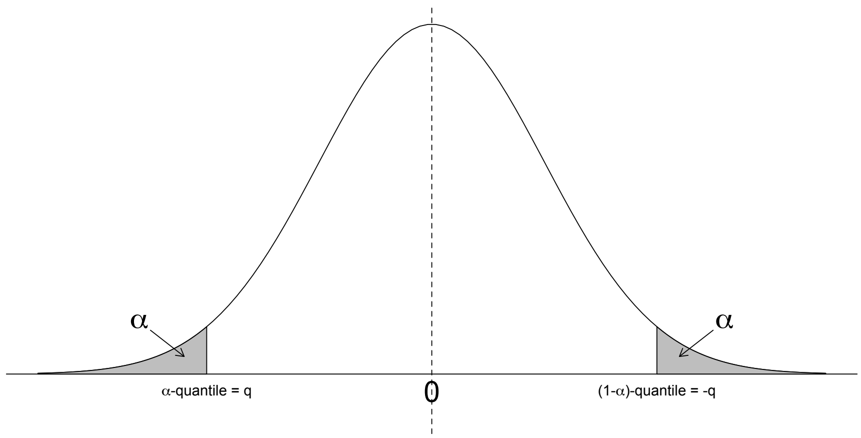
2. The α quantile and $1-\alpha$ quantile of a normal distribution are symmetric about the mean μ.

$$P(X < \mu - x) = P(X > \mu + x) = \alpha$$



3. The α quantile and $1-\alpha$ quantile of a **standard** normal distribution are symmetric about the mean 0, and therefore are opposite numbers.

$$P(X < q) = P(X > -q) = \alpha$$



Examples

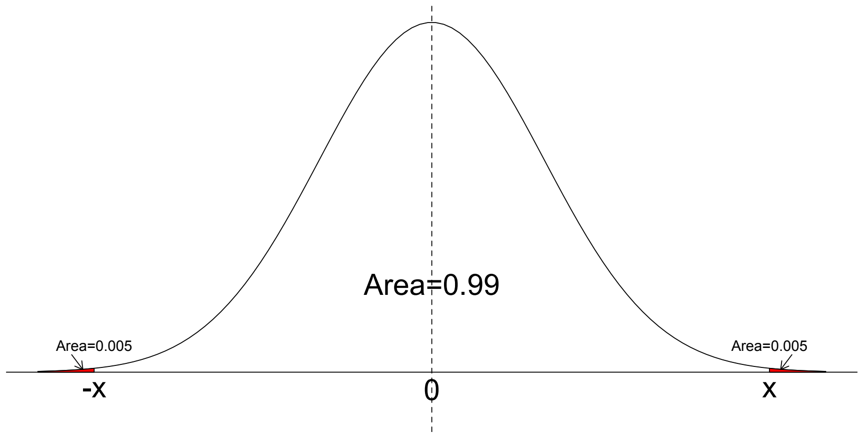
1. The third quartile of Z, the standard normal random variable, is 0.674. What is the first quartile of Z?
   Solution: Since the standard normal distribution is symmetric about 0, the third quartile (0.75 quantile), q_3, and the first quartile (0.25 quantile), q_1, are symmetric about 0. Therefore, $q_1 = q_3 = -0.674$.

2. If $P(-x < Z < x) = 0.99$, what is $P(Z < x)$?

$$P(Z < -x) + P(Z > x) = 1 - P(-x < Z < x) = 0.01 \quad \text{(total area under the curve is 1)}$$

$$P(Z < -x) = P(Z > x) = \frac{0.01}{2} = 0.005 \quad \text{(symmetry)}$$

$$P(Z < x) = P(Z < -x) + P(-x < Z < x) = 0.005 + 0.99 = 0.995$$



2.2 Empirical Rule

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.683$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.954$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997$$

2.3 Linear transformation of a normal random variable

1. A linear transformation of a normal random variable is a normal random variable.

$$\text{If } X \sim N(\mu, \sigma^2), \text{ then } aX + c \sim N(a\mu + c, a^2\sigma^2) \quad (a \neq 0)$$

2. Standard normal random variable has mean 0 and standard deviation 1.

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Note: If you compare this conversion to the linear transformation above, you will find

$$a = \frac{1}{\sigma}, \quad c = -\frac{\mu}{\sigma}$$

Examples

1. Suppose that $X_1 \sim N(-2, 4)$, what is the distribution of $2X_1 - 3$?

   Solution: $\mu = -2, \sigma^2 = 4$.

$$2X_1 - 3 \sim N(2 \cdot (-2) - 3, 2^2 \cdot 4) = N(-7, 16) \quad (\text{let } a = 2, c = -3)$$

2. A specialty foods company mails out "gourmet steaks" to customers willing to pay a gourmet price. The steaks vary in size, and the uncooked weights follow a normal distribution with a mean of 1.2 pounds and standard deviation of 0.10 pound. The steaks are mailed out in a box which weighs 0.2 pound. What is the distribution of the total weight of the package?

   Solution: Let X denote the weight of uncooked steak. $X \sim N(1.2, 0.1^2)$. Then

$$X + 0.2 \sim N(1.2 + 0.2, 0.1^2) \quad (\text{let } a = 1, c = 0.2)$$

   So the total weight of the packageis also normally distributed with a mean of 1.4 pounds and standard deviation of 0.10 pound.

2.4 Sum of independent normal random variables

1. The sum of **independent** normal random variables is a normal random variable.

   If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Summary (2.3 + 2.4)

If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$$

Examples

1. Suppose that $X_1 \sim N(-2, 4)$ and $X_2 \sim N(5, 9)$ are independent. What is the distribution of $-3X_1 + 2X_2$?

   Solution: Let $a = -3, b = 2$, then

$$-3X_1 + 2X_2 \sim N((-3)(-2) + (2)(5), (-3^2)(4) + 2^2(9)) = N(16, 72).$$

2. A specialty foods company mails out "gourmet steaks" to customers willing to pay a gourmet price. The steaks vary in size, and the uncooked weights follow a normal distribution with a mean of 1.2 pounds and standard deviation of 0.10 pound. What is the distribution of the average weight of 4 independent steaks?

   Solution: Let X_i denote the weight of each of the 4 uncooked steaks. $X_i \sim N(1.2, 0.1^2), i = 1, 2, 3, 4$. Then

$$X_1+X_2+X_3+X_4 \sim N(1.2+1.2+1.2+1.2, 0.1^2+0.1^2+0.1^2+0.1^2) = N(4.8, 0.04) \quad (\text{independent})$$

$$\frac{X_1 + X_2 + X_3 + X_4}{4} \sim N\left(\frac{4.8}{4}, \frac{0.04}{4^2}\right) = N(1.2, 0.05^2) \quad (a = 1/4, c = 0)$$

   Hence, the average weight of 4 independent steaks is also normally distributed with a mean of 1.2 pounds and standard deviation of 0.05 pound.