NE 806: Neutronics
Homework 1

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August 31, 2020

Slide Set 1

Exercise 1.

1. Add the computation of sigma_n (for elastic scattering) and add it to sigma_p to form sigma_e, the total elastic scattering cross section.

2. Plot sigma_g (as computed above) together with sigma_e and the corresponding values extracted directly from BNL's Sigma plot (when you plot a cross section, you can view the actual as-plotted values as a text file).

Solution. For the exercises contained in this slide set the Breit-Wigner single-level resonance formula will be used to model the energy-dependent cross sections of U238, U235, and H1. For a more in depth description of the Breit-Wigner single see Duderstadt and Hamilton 1976 pg. 26. Table 1 is provided to describe the many variables necessary to formulate the Breit-Wigner equation.

Symbol	Description
σ_γ	Radiative Capture Cross section
σ_0	Total Cross Section at Resonance Energies
σ_s	Scattering Cross Section
Γ_γ	Radiative Line Width
Γ_n	Neutron Line Width
Γ	Total Line Width
E_0	Energy at which Resonance Occurs
E_c	Center of Mass Energy
$\lambda\!\!\!^-$	Reduced Neutron Wavelength
g	Statistical Spin Factor
J	Total Spin
I	Nuclear Spin
A	Mass Number
R	Nuclear Radius

Table 1: Descriptions of variables used in Eq. 1.

With all the necessary variables listed we will now begin by considering the Breit-Wigner single-level resonance formula for radiative capture.

$$\sigma_\gamma(E_c) = \sigma_0 \frac{\Gamma_\gamma}{\Gamma} \left(\frac{E_0}{E_c}\right)^{1/2} \frac{1}{1+y^2}, \quad y = \frac{2}{\Gamma}(E_c - E_0) \tag{1}$$

Where σ_0 may be calculated by,

$$\sigma_0 = 4\pi\lambda\!\!\!^-_0{}^2 \frac{\Gamma_n}{\Gamma} g = 2.608 \times 10^6 \frac{(A+1)^2}{A^2 E_0(eV)} \frac{\Gamma_n}{\Gamma} g, \tag{2}$$

and g with

$$g = \frac{2J+1}{2(2I+1)}. \tag{3}$$

Additional terms are needed to calculate σ_s using Breit-Wigner, these two additional terms account for interference scattering and potential scattering.

$$\sigma_s(E_c) = \sigma_0 \frac{\Gamma_n}{\Gamma} \left(\frac{E_0}{E_c}\right)^{1/2} \frac{1}{1+y^2} + \sigma_0 \frac{2R}{\lambda\!\!\!^-} \frac{1}{y+y^2} + 4\pi R^2 \tag{4}$$

The nuclear radius of the target nuclide may be calculated using,

$$R = 1.25 \times 10^{-13} \cdot A^{1/3} (cm). \tag{5}$$

The form that the reduced wavelength takes for a neutron is given by

$$\lambdabar = \frac{4.55 \times 10^{-10}}{\sqrt{E_c}}(cm). \tag{6}$$

Cross section data can be constructed by plugging resonance parameter data, (which may be obtained on the Brookhaven National Laboratory website) into Eqs. 1 and 4.

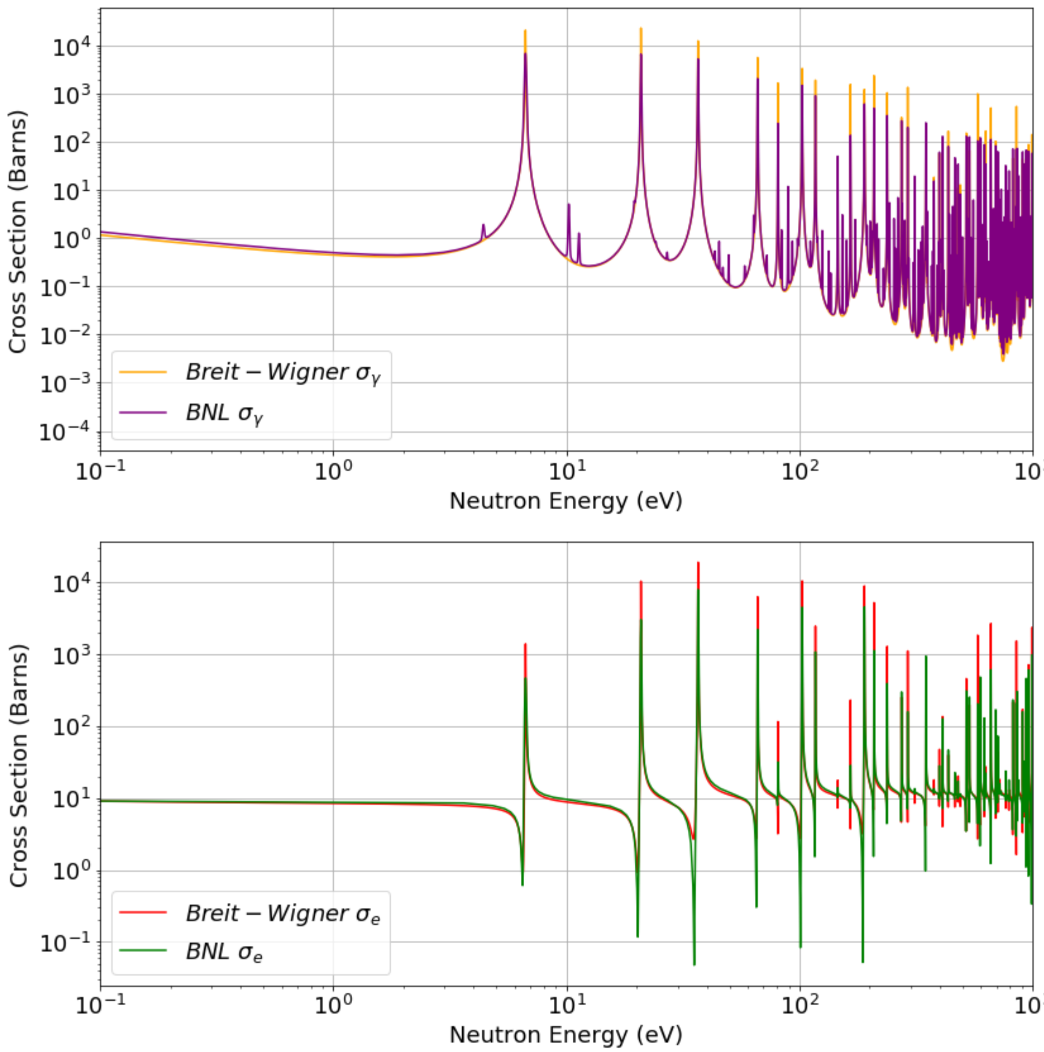

Figure 1: Comparison of cross section data for U238 obtained using Breit-Wigner and directly from BNL cross section files.

Slide Set 2

Exercise 1.

Prove $n(\boldsymbol{r}, \hat{\boldsymbol{\Omega}}, E, t) = \frac{v}{m} n(\boldsymbol{r}, \boldsymbol{v}, t)$

Solution 1.

To prove the above equation it is sufficient to show that

$$\int_0^\infty \int_{4\pi} n(\boldsymbol{r}, \hat{\boldsymbol{\Omega}}, E, t) d\hat{\boldsymbol{\Omega}} dE = \frac{v}{m} \int n(\boldsymbol{r}, \boldsymbol{v}, t) d^3 v. \tag{7}$$

We may neglect volume and time when integrating over the phase space as these terms are equivalent in both forms of the equation. Let us begin by writing a variation of the equality found on pg. 36 of Duderstadt and Hamilton

$$\int d^3 v n(\boldsymbol{r}, \boldsymbol{v}, t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z n(\boldsymbol{r}, \boldsymbol{v}, t) = \int_0^\infty v^2 dv \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta) n(\boldsymbol{r}, v, t) \tag{8}$$

By definition

$$\int_{4\pi} d\hat{\boldsymbol{\Omega}} \equiv \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta) \tag{9}$$

Plugging Eq. 9 into 8 we obtain,

$$\int d^3 v n(\boldsymbol{r}, \boldsymbol{v}, t) = \int_0^\infty dv v^2 \int_{4\pi} d\hat{\boldsymbol{\Omega}} n(\boldsymbol{r}, v, \hat{\boldsymbol{\Omega}}, t) \tag{10}$$

Now all we need to do is make a change of variables to energy from velocity. To do so we will use

$$E = \frac{1}{2} m v^2 \quad \text{and} \quad dv = \frac{dE}{mv}, \tag{11}$$

which allows us to rewrite the Eq. 10 as

$$\int d^3 v n(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{v}{m} \int_0^\infty dE \int_{4\pi} d\hat{\boldsymbol{\Omega}} n(\boldsymbol{r}, v, \hat{\boldsymbol{\Omega}}, t) \tag{12}$$

Well ****.

Slide Set 2

Exercise 2.

Derive the streaming operator (i.e., $\hat{\mathbf{\Omega}} \cdot \nabla\psi(\boldsymbol{r}, \hat{\mathbf{\Omega}}, E, t)$) for 1-D cylindrical and spherical coordinates, for brevity, omit time and energy dependence in your derivation.

Solution 2:

A detailed derivation of this problem for spherical coordinates in 1-D may be found in Lewis and Miller pg. 29. Using that text as a reference let's begin.

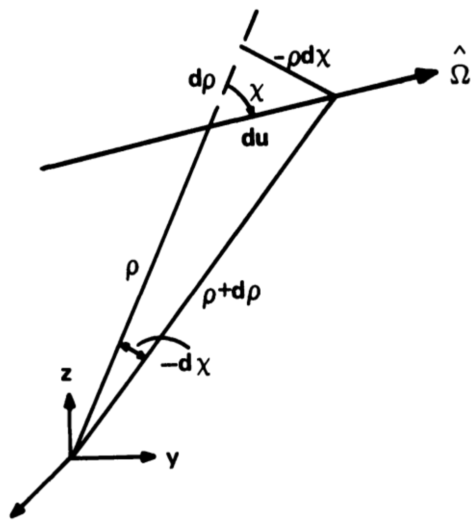

Figure 2: Coordinates for change of variables for one-dimensional spherical geometry, Lewis and Miller pg. 30.

The angular flux depends only on ρ and the angular variable $\mu \equiv cos(\chi)$, Hence the direction derivative in polar coordinates becomes

$$\frac{d\psi}{du} = \frac{\partial\psi}{\partial\rho}\frac{d\rho}{du} + \frac{\partial\psi}{\partial\mu}\frac{d\mu}{du}. \tag{13}$$

Now let us derive equations using Fig. 2.

$$\frac{d\rho}{du} = \mu = \cos(\chi) \quad \text{and} \quad -\rho\frac{d\chi}{du} = \sin(\chi) \tag{14}$$

$$\frac{d\mu}{du} = \frac{d(\cos(\chi))}{d\chi}\frac{d\chi}{du} = (\sin(\chi))\left(-\frac{\sin(\chi)}{\rho}\right) \tag{15}$$

$$\frac{d\mu}{du} = \frac{1-\mu^2}{\rho} \tag{16}$$

Plugging the above geometric identities into the directional derivative we obtain

$$\boxed{\hat{\mathbf{\Omega}} \cdot \nabla\psi(\boldsymbol{r}, \hat{\mathbf{\Omega}}, E, t) = \frac{d\psi}{du} = \mu\frac{\partial\psi}{\partial\rho} + \frac{(1-\mu^2)}{\rho}\frac{\partial\psi}{\partial\mu}} \tag{17}$$

Now let us consider the case of polar coordinates, the geometry of which is provided below.
This time our streaming term is given by the equation

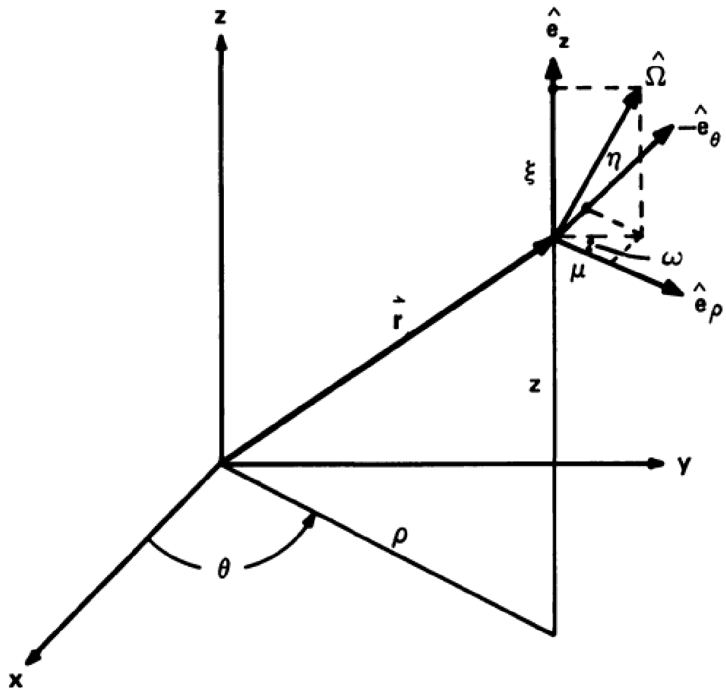

Figure 3: Cylindrical space-angle coordinate system in three dimensions, Lewis and Miller pg. 33.

$$\hat{\mathbf{\Omega}} \cdot \nabla\psi(\boldsymbol{r}, \hat{\mathbf{\Omega}}, E, t) = \frac{d\psi}{du} = \frac{\partial\psi}{\partial\rho}\frac{d\rho}{du} + \frac{\partial\psi}{\partial\mu}\frac{d\omega}{du} \tag{18}$$

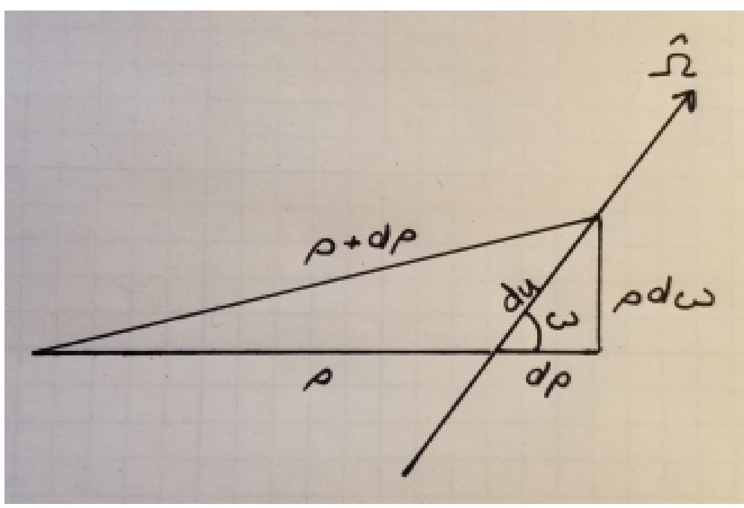

Figure 4: Coordinates for change of variables for one-dimensional cylindrical geometry

$$sin(\omega) = \frac{\rho d\omega}{du} \quad cos(\omega) = \frac{d\rho}{du} \tag{19}$$

Where $\sin(\omega) = \eta$ and $cos(\omega) = \mu$, which gets us

$$\frac{\eta}{\rho} = \frac{d\omega}{du} \quad \mu = \frac{d\rho}{du} \tag{20}$$

Allowing us to write

$$\boxed{\hat{\boldsymbol{\Omega}} \cdot \nabla \psi(\boldsymbol{r}, \hat{\boldsymbol{\Omega}}, E, t) = \frac{d\psi}{du} = \mu \frac{\partial \psi}{\partial \rho} + \frac{\eta}{\rho} \frac{\partial \psi}{\partial \mu}} \tag{21}$$

Slide Set 2

Exercise 3.

Consider a slab of width W cm. Solve the time-independent, monoenergetic, transport equation for this slab assuming it consists of a purely absorbing medium having an absorption cross section Σ_a 1/cm. Assume further that the slab is filled with an isotropic source $s(x,\mu) = \frac{s}{2}$ for $x \in [0, W/2]$ (with no source in the other half). Determine ψ, ϕ, and $\mathbf{J}$. Plot ϕ and $\mathbf{J}$ for $\Sigma_a = 5$ and $s = 2$.

Solution 2:

Let us begin by considering the following equation which may be found in lecture slide 2.

$$\mu\frac{d\psi}{dx} + \Sigma_t\psi(x,\mu) = s(x,u) \tag{22}$$

Because the slab is purely absorbing we may make the substitution $\Sigma_t = \Sigma_a$. Note that the above equation can be written in the conventional form of a linear first-order ODE, a special case of inexact ODEs,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We may rewrite Eq. 22 in the form of a linear first order ODE where $y = \psi$ and $x = x$.

$$\frac{d\psi}{dx} + \frac{\Sigma_a}{\mu}\psi(x,\mu) = \frac{s(x,u)}{\mu}. \tag{23}$$

However, this equation is not exact, an integrating factor $\lambda(x)$, can be multiplied through to make this equation exact. The integrating factor is found using

$$\lambda(x) = e^{\int P(x)}dx$$

Solving for our case we find $\lambda(x) = e^{\frac{\Sigma_a}{\mu}x}$. Now let us solve for the angular flux moving in the positive x direction $\mu > 0$ in the domain $x \in [0, W/2]$.

$$\psi_+ = e^{-\frac{\Sigma_a}{\mu}x}\int_0^x \frac{s}{2\mu}e^{\frac{\Sigma_a}{\mu}x}dx$$

$$\boxed{\psi_+ = \frac{s}{2\Sigma_a}\left(1 - e^{-\frac{\Sigma_a}{\mu}x}\right), x \in [0, W/2] \text{ and } \mu > 0} \tag{24}$$

Likewise the angular flux in the negative x direction, $\mu < 0$, can be found. Note that the substitution $-|\mu| = \mu$ was used in this case.

$$\psi_- = e^{-\frac{\Sigma_a}{|\mu|}x}\int_{W/2}^x \frac{s}{2|\mu|}e^{\frac{\Sigma_a}{|\mu|}x}dx$$

$$\boxed{\psi_- = \frac{s}{2\Sigma_a}\left(1 - e^{-\frac{\Sigma_a}{|\mu|}[W/2-x]}\right), x \in [0, W/2] \text{ and } \mu < 0} \tag{25}$$

To solve for angular flux in the domain $x \in [W/2, W]$ let us reconsider Eq. 23, this time setting $s(x,\mu) = 0$.

$$\frac{d\psi}{dx} + \frac{\Sigma_a}{\mu}\psi(x,\mu) = 0 \tag{26}$$

This time we have a separable differential equation, the solution of which is

$$\psi(x,\mu) = ke^{-\frac{\Sigma_a}{\mu}x}, \tag{27}$$

where k is some constant which may be found by plugging in boundary conditions. Boundary conditions may be found by evaluating the previously solved for formulas for ψ_+ and ψ_- at the location $x = W/2$. These boundary conditions being,

$$k = \frac{s}{2\Sigma_a}\left(e^{-\frac{\Sigma_a}{\mu}\frac{W}{2}} - 1\right), \mu > 0 \quad \text{and} \quad k = 0, \mu < 0.$$

Applying these boundary conditions to Eq. 27 yields the following two formulas,

$$\boxed{\psi_+(x,\mu) = \frac{s}{2\Sigma_a}\left(e^{-\frac{\Sigma_a}{\mu}\frac{W}{2}} - 1\right)e^{-\frac{\Sigma_a}{\mu}x}, x \in [W/2, W] \text{ and } \mu > 0} \tag{28}$$

and

$$\boxed{\psi_-(x,\mu) = 0, x \in [W/2, W] \text{ and } \mu < 0} \tag{29}$$

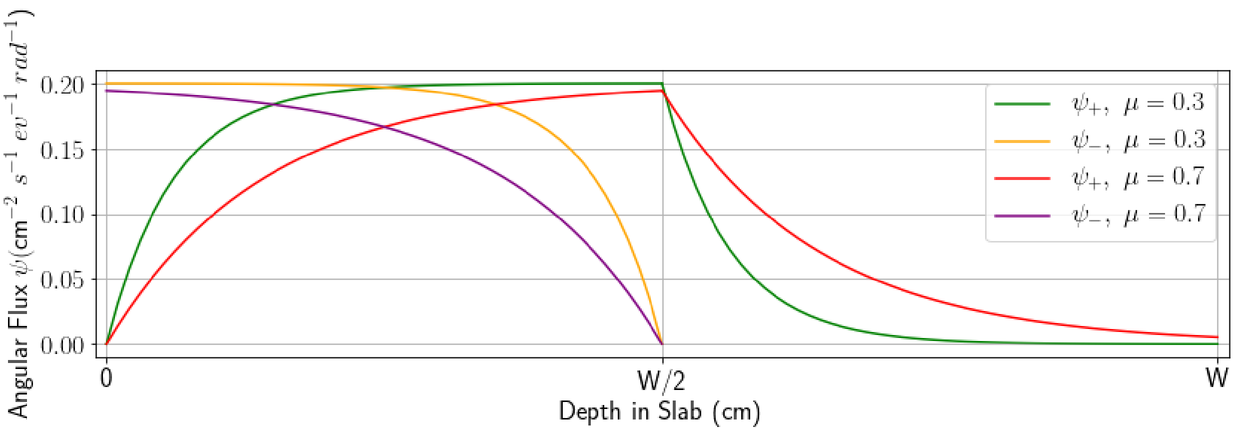

Figure 5: Angular flux at specific angles across the absorbing slab.

Scalar flux, $\phi(x)$, can be calculated wit the following,

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu) d\mu \tag{30}$$

which may be broken down to

$$\phi(x) = \int_{-1}^{0} \psi_-(x,\mu) d\mu + \int_{0}^{1} \psi_+(x,\mu) d\mu \tag{31}$$

Let us first evaluate Eq. 31 in the domain $x \in [0, W/2]$.

$$\phi(x) = \frac{S}{2\Sigma_a}\left(\int_{-1}^{1} d\mu - \int_{-1}^{0} e^{-\frac{\Sigma_a}{|\mu|}[W/2-x]} d\mu - \int_{0}^{1} e^{-\frac{\Sigma_a}{\mu}x} d\mu\right)$$

$$\phi(x) = \frac{S}{2\Sigma_a}\left(2 - \int_{0}^{1} e^{-\frac{\Sigma_a}{\mu'}[W/2-x]} d\mu' - \int_{0}^{1} e^{-\frac{\Sigma_a}{\mu}x} d\mu\right)$$

$$\boxed{\phi(x) = \frac{S}{2\Sigma_a}\left(2 - E_2(\Sigma_a(W/2 - x)) - E_2(\Sigma_a x)\right)} \tag{32}$$

Where E_n in this case E_2 are exponential integrals and are characteristic of slab problems. They are defined as

$$E_n \equiv \int_{0}^{1} \mu^{n-2} e^{-x/\mu} d\mu. \tag{33}$$

The exponential integrals may be plotted in python using the function *expn* from the *scipy.special* module. Now let's calculate the scalar flux in the region $x \in [W/2, W]$.

$$\phi(x) = \frac{s}{2\Sigma_a}\int_0^1 e^{-\frac{\Sigma_a}{\mu}(W/2+x)}d\mu - \frac{s}{2\Sigma_a}\int_0^1 e^{-\frac{\Sigma_a}{\mu}x}d\mu$$

$$\boxed{\phi(x) = \frac{s}{2\Sigma_a}\left[E_2(\Sigma_a(W/2+x)) - E_2(\Sigma_a x)\right]} \tag{34}$$

Unfortunately the above seems to be invalid based on plotting :(.

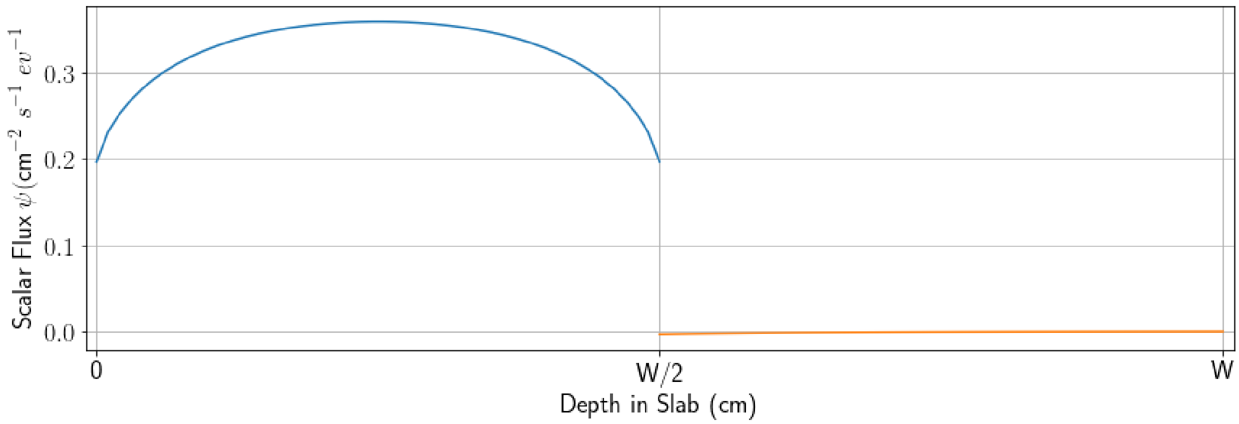

Figure 6: Scalar flux across the absorbing slab.

Finally we have the scalar current density $\boldsymbol{J}$ which may be calculated using

$$\boldsymbol{J}(x) = \int_{-1}^{1} \mu\psi(x,\mu)d\mu \tag{35}$$

Slide Set 3

Exercise 1.

Prove the following:

1. $\cos(\theta_L) = \frac{A\cos(\theta_c)+1}{\sqrt{A^2+2A\cos(\theta_c)+1}}$
2. $\frac{E'}{E} = \frac{1+\alpha+(1-\alpha)\cos(\theta_c)}{2}$
3. For isotropic scattering in CM, the average of the cosine of the LAB scattering angle is defined by $\bar{\mu} = \frac{2}{3A}$.

Solution 1.

To begin we will write the following equality which is derived using the following substitutions: $v_c = A/(1+A) \cdot v_L$ and v

$$v'_L \cos(\theta_L) = v'_c \cos(\theta_c) + v_{cm} = \frac{A}{A+1} v_L \cos(\theta_c) + \frac{1}{A+1} v_L \tag{36}$$

$$v'_L = v_L \cdot \frac{(1 + A^2 + 2A\cos(\theta_c))^{1/2}}{A+1} \tag{37}$$

$$v_L \cdot \frac{(1 + A^2 + 2A\cos(\theta_c))^{1/2}}{A+1} \cos(\theta_c) = \frac{A}{A+1} v_L \cos(\theta_c) + \frac{1}{A+1} v_L \tag{38}$$

Becoming

$$\boxed{\cos(\theta_L) = \frac{A\cos(\theta_c)+1}{\sqrt{A^2+2A\cos(\theta_c)+1}}}$$

Solution 2.

From Duderstadt and Hamilton pg. 42 we know

$$\frac{v'_L}{v_L} = \frac{1 + A^2 + 2A\cos(\theta_c)}{(A+1)^2} \tag{39}$$

Which may be rewritten as a ratio of energies

$$\frac{E'}{E} = \frac{1/2 \cdot m \cdot v'_L}{1/2 \cdot m \cdot v_L} = \frac{1 + A^2 + 2A\cos(\theta_c)}{(1+A)^2} \tag{40}$$

Using the substitutions

$$\alpha = \left(\frac{A-1}{A+1}\right)^2 \tag{41}$$

and

$$1 + \alpha = 2\frac{A^2+1}{(A+1)^2} \quad \text{and} \quad 1 - \alpha = 2\frac{2A}{(A+1)^2} \tag{42}$$

Eq. 40 becomes

$$\boxed{\frac{E'}{E} = \frac{1 + \alpha + (1-\alpha)\cos(\theta_c)}{2}} \tag{43}$$

Solution 3.

The mean value of μ may be found using

$$\overline{\mu} = \overline{\cos(\theta_L)} = \frac{\int_0^{4\pi} \cos(\theta_L) d\Omega}{\int_0^{4\pi} d\Omega} \tag{44}$$

where

$$d\Omega = 2\pi \sin(\theta_c) d\theta_c$$

$$\overline{\mu} = \frac{1}{4\pi} \int_0^{\pi} \cos(\theta_L) \cdot 2\pi \sin(\theta_c) d\theta_c \tag{45}$$

$$\overline{\mu} = \frac{1}{2} \int_0^{\pi} \frac{A\cos(\theta_c) + 1}{\sqrt{A^2 + 2A\cos(\theta_c) + 1}} \cdot \sin(\theta_c) d\theta_c \tag{46}$$

$$y \equiv \cos(\theta_c) \quad \text{and} \quad dy = -\sin(\theta_c) d\theta_c$$

$$\overline{\mu} = \frac{1}{2} \int_{-1}^{1} \frac{Ay + 1}{\sqrt{A^2 + 2Ay + 1}} dy \tag{47}$$

$$\overline{\mu} = \frac{1}{2} \left[\int_{-1}^{1} \frac{Ay}{\sqrt{A^2 + 2Ay + 1}} + \int_{-1}^{1} \frac{1}{\sqrt{A^2 + 2Ay + 1}} dy \right] \tag{48}$$

Using the following solutions from integral tables

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b}, \quad \int \frac{x dx}{\sqrt{a + bx}} = -\frac{2(2a - bx)}{3b^2} \sqrt{a + bx} \tag{49}$$

Where for our particular case $a = A^2 + 1$ and $b = 2A$

$$\overline{\mu} = \frac{1}{2} \left[\frac{A(-2)[2(A^2 + 1) - 2Ay]}{12A^2} \sqrt{(A^2 + 1) + 2Ay} + \frac{2}{2A} \sqrt{(A^2 + 1) + 2Ay} \right]_{-1}^{1} \tag{50}$$

$$\overline{\mu} = \frac{1}{2A} \left[-\frac{1}{3} [(A^2 - A + 1)(A + 1) - (A^2 + A + 1)(A - 1)] + (A + 1) - (A - 1) \right] \tag{51}$$

$$\boxed{\overline{\mu} = \frac{2}{3A}} \tag{52}$$