The dynamics of a Mobile Inverted Pendulum (MIP)

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1 Introduction

In this document, a Mobile Inverted Pendulum (MIP) is a robotic platform that resembles an autonomous segway-like vehicle with two independently-actuated wheels and enough motor torque to not only balance while driving aggressively in in a segway-like mode, but also to perform a self-uprighting maneuver from a horizontal configuration. Several MIP prototype designs are currently being developed at the UCSD Coordinated Robotics Lab; myMIP is one of the simplest of these designs, and is built from inexpensive, commercial-off-the-shelf (COTS) parts together with a few custom pieces easily fabricated on an inexpensive 3D printer. Other prototypes that climb stairs and pick up and throw balls are also being developed. In this document, the equations of motion of a MIP, moving in a straight line on a level surface, are developed from first principles.

2 Equations of Motion

2.1 Free Body Diagrams

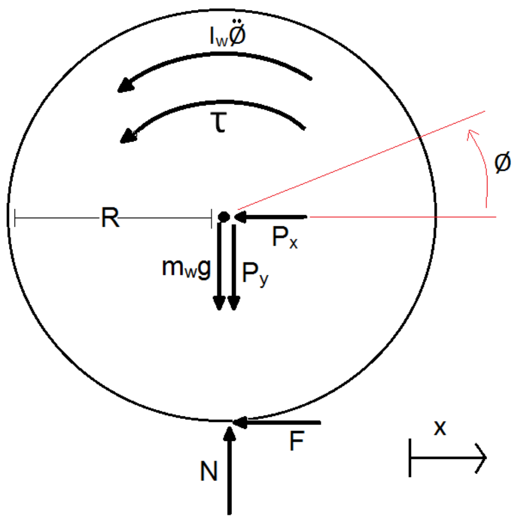

Figure 1: Wheel

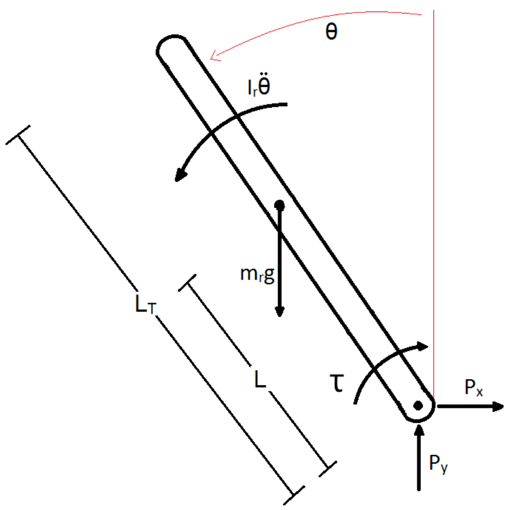

Figure 2: Rod

Where:
m_w= mass of wheel
m_r = mass of rod
I_w= Inertia of wheel
I_r = Inertia of rod
τ = torque from motors (input)
R = radius of wheel
L = length from end to center of mass of the rod
L_T= length of rod
P_x and P_y = reaction force between wheel and rod in the x and y direction, respectively
a_t= tangential force
a_n = normal force
g = gravity (9.81 m/s)

2.2 Kinematics

Position (r) of rod center of mass (CM):

$$r = x\hat{i} - L\sin(\theta)\hat{i} + L\cos(\theta)\hat{j} \tag{2.2.1}$$

Velocity ($\dot{r}$) of rod CM:

$$\dot{r} = \dot{x}\hat{i} - \dot{\theta}L\cos(\theta)\hat{i} - \dot{\theta}L\sin(\theta)\hat{j} \tag{2.2.2}$$

Acceleration ($\ddot{r}$) of rod CM:

$$\ddot{r} = (\ddot{x} - \ddot{\theta}L\cos\theta + \dot{\theta}^2 L\sin\theta)\hat{i} - (\ddot{\theta}L\sin\theta + \dot{\theta}^2 L\cos\theta)\hat{j} \tag{2.2.3a}$$

$$= (\ddot{x}\cos\theta - \ddot{\theta}L)(\cos\theta\hat{i} + \sin\theta\hat{j}) - (\ddot{x}\sin\theta + \dot{\theta}^2 L)(\cos\theta\hat{j} - \sin\theta\hat{i}) \tag{2.2.3b}$$

2.3 Dynamics

Angular acceleration of wheel:

$$I_w\ddot{\phi} = \tau - FR \tag{2.3.1}$$

Linear acceleration of wheel CM in $\hat{i}$ direction:

$$m_w\ddot{x} = -P_x - F \tag{2.3.2}$$

Linear acceleration of wheel CM in $\hat{j}$ direction:

$$0 = N - P_y - m_w g \tag{2.3.3}$$

Angular acceleration of rod:

$$I_r\ddot{\theta} = -\tau + P_y L\sin\theta + P_x L\cos\theta \tag{2.3.4}$$

Linear acceleration of rod CM in $\hat{i}$ direction:

$$m_r(\ddot{x} - \ddot{\theta}Lcos\theta + \dot{\theta}^2 L\sin\theta) = P_x \tag{2.3.5}$$

Linear acceleration of rod CM in $\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j}$ direction:

$$m_r(\ddot{x}\cos\theta - \ddot{\theta}L) = -m_r g\sin\theta + P_y\sin\theta + P_x\cos\theta \tag{2.3.6}$$

2.4 Derivation of Equations of Motion

Using the dynamic equations we can derive the equations of motion:

Starting with (**??**), rearrange to get

$$P_y \sin\theta + P_x \cos\theta = \frac{I_r\ddot{\theta} + \tau}{L} \tag{2.4.1}$$

plug the above equation into (**??**)

$$m_r(\ddot{x}\cos\theta - \ddot{\theta}L) = -m_r g \sin\theta + \frac{I_r\ddot{\theta} + \tau}{L} \tag{2.4.2}$$

multiplying everything by L and distributing m_r

$$m_r L\ddot{x}\cos\theta - m_r\ddot{\theta}L^2 = -m_r g L \sin\theta + I_r\ddot{\theta} + \tau \tag{2.4.3}$$

and rearrange to get **Equation of Motion #1**

$$-(m_r L\cos\theta)\ddot{x} + (I_r + m_r L^2)\ddot{\theta} = m_r g L\sin\theta - \tau \tag{2.4.4}$$

Now using the (**??**) and (**??**)

$$P_x = -m_w\ddot{x} - F \tag{2.4.5}$$

$$F = -\frac{I_w\ddot{\phi} - \tau}{R} \tag{2.4.6}$$

Plugging both into (**??**)

$$m_r(\ddot{x} - \ddot{\theta}L\cos\theta + \dot{\theta}^2 L\sin\theta) = -m_w\ddot{x} - F \tag{2.4.7}$$

$$m_r(\ddot{x} - \ddot{\theta}L\cos\theta + \dot{\theta}^2 L\sin\theta) = -m_w\ddot{x} + \frac{I_w\ddot{\phi} - \tau}{R} \tag{2.4.8}$$

multiplying everything by R and distrubuting m_r

$$m_r R\ddot{x} - m_r R\ddot{\theta}L\cos\theta + m_r R\dot{\theta}^2 L\sin\theta = -m_w R\ddot{x} + I_w\ddot{\phi} - \tau \tag{2.4.9}$$

and rearrange to get to get **Equation of Motion #2**

$$I_w\ddot{\phi} - (m_r R + m_w R)\ddot{x} + (m_r R L\cos\theta)\ddot{\theta} = m_r R\dot{\theta}^2 L\sin\theta + \tau \tag{2.4.10}$$

Now applying no slip condition, $\ddot{x} = -R\ddot{\phi}$, to (**??**) and (**??**)

Final Equations of Motion for the MIP are:

$$\boxed{(m_r R L\cos\theta)\ddot{\phi} + (I_r + m_r L^2)\ddot{\theta} = m_r g L\sin\theta - \tau} \tag{2.4.11}$$

$$\boxed{(I_w + (m_r + m_w)R^2)\ddot{\phi} + (m_r R L\cos\theta)\ddot{\theta} = m_r R L\dot{\theta}^2\sin\theta + \tau} \tag{2.4.12}$$

2.5 Re-derivation of Equations of Motion using Lagrangian Dynamics

Kinetic Energy of Wheel:

$$T_w = \frac{1}{2}m_w\dot{x}^2 + \frac{1}{2}I_w\dot{\phi}^2 \tag{2.5.1}$$

Kinetic Energy of Rod:

$$T_r = \frac{1}{2}m_r(\dot{x} - L\dot{\theta}\cos\theta)^2 + \frac{1}{2}m_r(L\dot{\theta}\sin\theta)^2 + \frac{1}{2}I_r\dot{\theta}^2 \tag{2.5.2}$$

Potential Energy:

$$V = m_r g L\cos\theta \tag{2.5.3}$$

Now using the Euler-Lagrange Equation:

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}) - \frac{\partial\mathcal{L}}{\partial q} \tag{2.5.4}$$

where the lagrangian is written as

$$\mathcal{L} = T_w + T_r - V \tag{2.5.5}$$

and the generalized coordinate is

$$q = \begin{bmatrix} x \\ \phi \\ \theta \end{bmatrix} \tag{2.5.6}$$

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{x}}) - \frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}(m_w\dot{x} + m_r(\dot{x} - L\dot{\theta}\cos\theta)) + 0 \tag{2.5.7}$$

$$= m_w\ddot{x} + m_r\ddot{x} - m_r L\ddot{\theta}\cos\theta + m_r L\dot{\theta}^2\sin\theta \tag{2.5.8}$$

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}) - \frac{\partial\mathcal{L}}{\partial\phi} = \frac{d}{dt}(I_w\dot{\phi}) - 0 \tag{2.5.9}$$

$$= I_w\ddot{\phi} \tag{2.5.10}$$

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}) = \frac{d}{dt}(-m_r L(\dot{x} - L\dot{\theta}\cos\theta)\cos\theta + m_r L(L\dot{\theta}\sin\theta)\sin\theta + I_r\dot{\theta}) \tag{2.5.11}$$

$$= \frac{d}{dt}(-m_r L\dot{x}\cos\theta + m_r L^2\dot{\theta} + I_r\dot{\theta}) \tag{2.5.12}$$

$$= -m_r L(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) + m_r L^2\ddot{\theta} + I_r\ddot{\theta} \tag{2.5.13}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = m_r L\dot{\theta}\sin\theta(\dot{x} - L\dot{\theta}\cos\theta) + m_r L\dot{\theta}\cos\theta(L\dot{\theta}\sin\theta) + m_r g L\sin\theta \tag{2.5.14}$$

Putting everything together, you get:

$$(m_w + m_r)\ddot{x} - (m_r L\cos\theta)\ddot{\theta} + m_r L\dot{\theta}^2 \sin\theta = Q_x \tag{2.5.15}$$

$$I_w\ddot{\phi} = Q_\phi \tag{2.5.16}$$

$$(-m_r L\cos\theta)\ddot{x} + (I_r + m_r L^2)\ddot{\theta} - m_r g L\sin\theta = Q_\theta \tag{2.5.17}$$

Now by solving the Euler-Lagrange equations, the equations of motion can be written in the form:

$$S^T M(q)S\dot{\nu} + S^T F(q,\dot{q}) = S^T B(q)u \tag{2.5.18}$$

where u is the torque from the motors.

Now using the equations found above and putting them into the form above, you get:

$$M(q) = \begin{bmatrix} (m_w + m_r) & 0 & -(m_r L\cos\theta) \\ 0 & I_w & 0 \\ (-m_r L\cos\theta) & 0 & (I_r + m_r L^2) \end{bmatrix} \tag{2.5.19}$$

$$q = \begin{bmatrix} x \\ \phi \\ \theta \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix}, \quad \dot{\nu} = \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix}, \quad F(q,\dot{q}) = \begin{bmatrix} m_r L\dot{\theta}^2 \sin\theta \\ 0 \\ -m_r g L\sin\theta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Now defining the no slip equation $\dot{x} + R\dot{\phi} = 0$ and applying it:

$$A = \begin{bmatrix} 1 & R & 0 \end{bmatrix} \tag{2.5.20}$$

$$S = \begin{bmatrix} -R & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.5.21}$$

Plugging in everything into equation **??**, you get:

$$\begin{bmatrix} (m_w + m_r)R^2 + I_w & m_r RL\cos\theta) \\ (m_r RL\cos\theta) & (I_r + m_r L^2) \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -m_r RL\dot{\theta}^2 \sin\theta \\ -m_r g L\sin\theta \end{bmatrix} = \begin{bmatrix} \tau \\ -\tau \end{bmatrix} \tag{2.5.22}$$

Finally, multiplying through all the matrices you get the below equations of motion for the MIP which are the same as those computed previously.

$$\boxed{(I_w + (m_w + m_r)R^2)\ddot{\phi} + (m_r RL\cos\theta)\ddot{\theta} = m_r RL\dot{\theta}^2 \sin\theta + \tau} \tag{2.5.23}$$

$$\boxed{(m_r RL\cos\theta)\ddot{\phi} + (I_r + m_r L^2)\ddot{\theta} = m_r g L\sin\theta - \tau} \tag{2.5.24}$$

3 Control of MIP

There are a number of ways that we control iFling or a segway as it is a SIMO system. We will develop both a controller using classical controls and state space control

3.1 Classical Controls

Typically classical controls are used to develop a controller for a SISO system. In the case of iFling/Segway we have a SIMO system. As this system has two modes of interest (controlling θ and ϕ) that have natural frequencies that are significantly different we can use a method called Successive Loop Closure (SLC). SLC allows us to apply classical control techniques to control SIMO systems. The basic idea behind SLC is that we can create a feedback loop within a second feedback loop as depicted in figure **??**. The inner loop contains the control of the fast dynamics (θ) while the outer feedback loop contains the controls of the slower dynamics (ϕ).

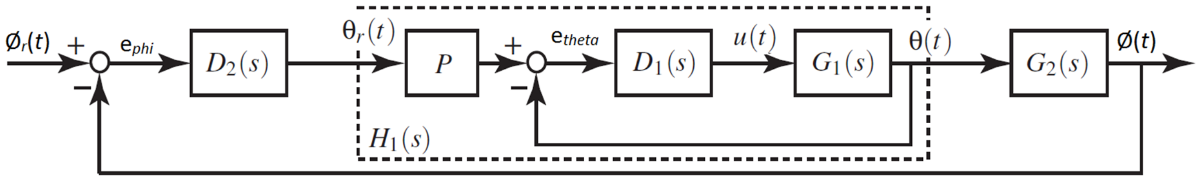

Figure 3: Successive loop closure of a SIMO plant G(s) designed with "fast" θ dynamics and "slow" ϕ dynamics. Note this is the idealization of the system for the purpose of control design.

Starting with the Equations of motion for iFling found in previous sections:
Linearize (using small angle approximation) the equations of motion:

$$(I_w + (m_w + m_r)R^2)\ddot{\phi} + (m_r RL)\ddot{\theta} = \tau \tag{3.1.1}$$

$$(m_r RL)\ddot{\phi} + (I_r + m_r L^2)\ddot{\theta} = m_r gL\theta - \tau \tag{3.1.2}$$

Now take the Laplace transform of the linearized equations of motion:

$$(I_w + (m_w + m_r)R^2)s^2\Phi + (m_r RL)s^2\Theta = \tau \tag{3.1.3}$$

$$(m_r RL)s^2\Phi + (I_r + m_r L^2)s^2\Theta = m_r gL\Theta - \tau \tag{3.1.4}$$

Now to get the transfer function of the fast dynamics we want $\frac{\Theta}{\tau}$. So solve for phi and then plug it into the next other equation

$$\Phi = \frac{\tau - m_r RLs^2\Theta}{(I_w + (m_w + m_r)R^2)s^2} \tag{3.1.5}$$

$$m_r RL\tau - m_r^2 R^2 L^2 s^2\Theta + (I_w + (m_w + m_r)R^2)(I_r + m_r L^2)s^2\Theta = (m_r gL\Theta - \tau)(I_w + (m_w + m_r)R^2) \tag{3.1.6}$$

Solving for $\frac{\Theta}{\tau}$ and simplifying, the transfer function for fast dynamics of iFling is:

$$\boxed{G_1 = \frac{\Theta}{\tau} = \frac{m_r RL + I_w + (m_w + m_r)R^2}{(-(I_w + (m_w + m_r)R^2)(I_r + m_r L^2) + m_r^2 R^2 L^2)s^2 + m_r gL(I_w + (m_w + m_r)R^2)}} \tag{3.1.7}$$

Going through the same process of finding the fast dynamics transfer function using equations(??) we can find also find the slow dynamics transfer function, $\frac{\Phi}{\Theta}$

$$\boxed{G_2 = \frac{\Phi}{\Theta} = \frac{-(I_r + m_r RL)s^2 + m_r gL}{(I_w + (m_w + m_r)R^2 + m_r RL)s^2}} \tag{3.1.8}$$

Now that we have the transfer functions for the fast (G1) and slow (G2) system we can design the controllers that stabilize the fast (G1) and slow (G2) dynamics of the system. Note that though we will be finding the controllers necessary to control the system that these controllers will only work in simulations. The controllers will very likely fail to control the MIP in reality. The reason for this is that our current transfer functions do not not include the motor dynamics. This will be introduced in the next section. The reader is also encouraged to write a MatLab program to help in designing a controller as this is an iterative process. The reader is highly discourage from applying the random control design (RCD) strategies facilitated by root locus graphical user interfaces such as sisotool in Matlab.

3.1.1 Design of the Fast Controller Without Motor Dynamics

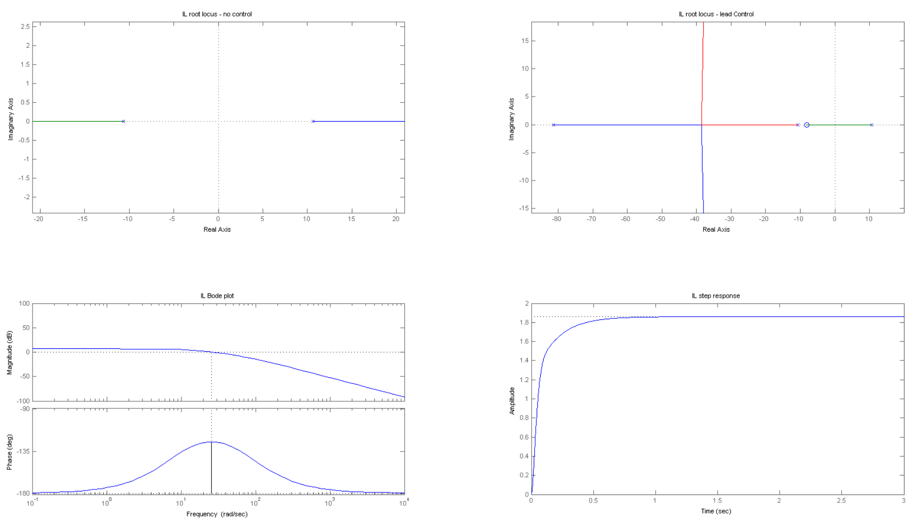

Figure 4: Root Locus, Bode Plot and Step Response for the Fast Inner Loop.

To start the design of the controller plot the root locus for G1. Looking at the uncontrolled root locus in the above figure it should be apparent that a lead controller with negative gain is needed to bring the root locus into the stable region (left hand plane). To determine the amount of lead required we will need to determine our desired crossover frequency (ω_c). To do this will need to first selected an appropriate rise time (t_r) to stabilize the fast system quickly. A good start for a MIP will be $t_r = 0.07 secs$

Now, knowing our desired t_r, the desired crossover frequency can be found using the following rule of thumb:

$$w_c = \frac{1.8}{t_r} = \frac{1.8}{0.07} = 25.714 \tag{3.1.9}$$

With a desired crossover frequency ($\omega_{cdesired}$) we will also want to selected a desired dampening (ζ) for the system. Let choose $\zeta = .55$. Using this information we get the following equation:

$$\phi_{max} = sin^{-1}\frac{p/z - 1}{p/z + 1} \longrightarrow \frac{p}{z} = \frac{1 + sin(100\zeta)}{1 - sin(100\zeta)} = 10 \tag{3.1.10}$$

Finally, the last equation needed for the design of a lead controller is:

$$\omega_{cdesired} = \sqrt[2]{zp} \longrightarrow zp = 25.714^2 \tag{3.1.11}$$

Now with the knowledge that $pz = 25.714^2$ and $p/z = 10$ we can determine the pole and zero of the lead controller. From here we can look at the bode plot to determine the amount of gain (K) needed to place our crossover frequency at the desired frequency of 25.714 rad/sec.

Once we have our crossover frequency set we can look at the closed loop step response to double check that we have a stable system with the desired performance.

The resulting controller (D1) using the provide mass properties of MyMIP and the method outline above is:

$$D1 = -1.3700 * \frac{s + 8.1316}{s + 81.3157} \tag{3.1.12}$$

Note: Notice the negative gain!

3.1.2 Design of the Slow Controller Without Motor Dynamics

The process for designing the slow controller (D2) is exactly the same as designing the fast controller (D1), in this case, even though the root locus is different. A lead controller will need to be used to bring the root locus into the stable region. The difference between the slow controller and the fast controller as mentioned before is with the desired crossover frequency, which allows for the use of SLC. Lets use a desired $t_r = 0.7secs$ for the outer loop. Going through exactly the same steps as outlined in the design of the fast controller section the resulting controller for the slow system will be:

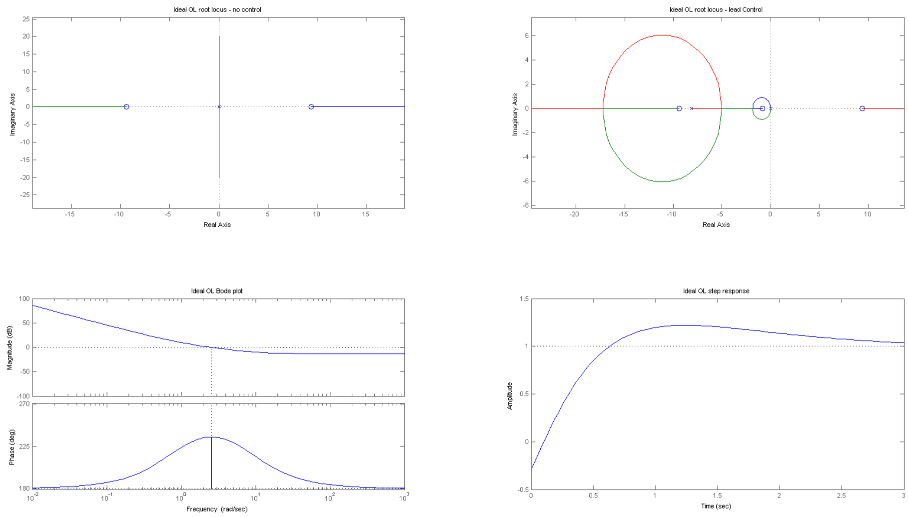

Figure 5: Root Locus, Bode Plot and Step Response for the Slow Outer Loop.

$$D1 = 0.1076 * \frac{s + 0.8132}{s + 8.1316} \tag{3.1.13}$$

Notice: No negative gain this time!

3.1.3 The Final SLC Controlled System Without Motor Dynamics

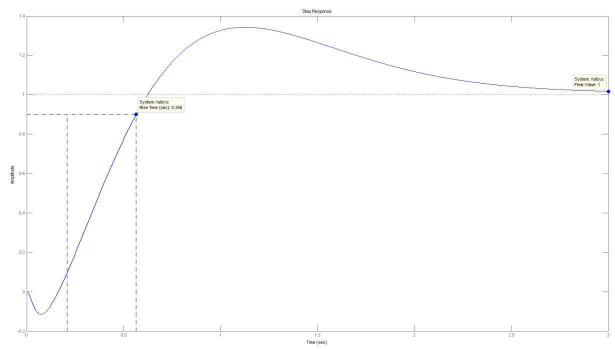

Figure 6: Root Locus, Bode Plot and Step Response for the Slow Outer Loop.

Note that the provide rise time and dampening values are provided as a good starting point for designing the controllers for MIPs. These controllers are not meant to be the best that can be designed. They were used only to show the reader the process involved in designing a controller for each system. It is strongly encouraged that the read play around with the rise time and dampening values to see their affect on the controllers and the control of the overall system. //