Chapter 1

Interpolation

1.1 Lagrange Interpolation

Example 0.1. Consider the domain $[2, 10]$ partitioned into 5 points, i.e. $\{2, 4, 6, 8, 10\}$ and a function $f : [0, 10] \to \mathbb{R}$, $x \mapsto f(x) = \ln(x)$. The y-values then are

$$\ln(2) \approx 0.6931 \quad \ln(4) \approx 1.3862 \quad \ln(6) \approx 1.7917 \quad \ln(8) \approx 2.0794 \quad \ln(10) \approx 2.3025. \tag{1.1}$$

Computing the Lagrange polynomials gives

$$\begin{aligned}
L_1(x) &= \ln(2) \cdot \frac{x-4}{2-4} \cdot \frac{x-6}{2-6} \cdot \frac{x-8}{2-8} \cdot \frac{x-10}{2-10} && (1.2)\\
&= 5\ln(2) - \frac{77}{24}x\ln(2) + \frac{71}{96}x^2\ln(2) - \frac{7}{96}x^3\ln(2) + \frac{1}{384}x^4\ln(2) && (1.3)\\
L_2(x) &= \ln(2) \cdot \frac{x-2}{4-2} \cdot \frac{x-6}{4-6} \cdot \frac{x-8}{4-8} \cdot \frac{x-10}{4-10} && (1.4)\\
&= -10\ln(2) + \frac{107}{12}x\ln(2) - \frac{59}{24}x^2\ln(2) + \frac{13}{48}x^3\ln(2) - \frac{1}{96}x^4\ln(2) && (1.5)
\end{aligned}$$

Example 0.2. Let $f(x) = x^8$. We want to interpolate f on the grid points $\{-3, -2, -1, 0, 1, 2, 3\}$. The Lagrange polynomials are

$$L_1(x) = -6561 \cdot \frac{x+2}{-3+2} \cdot \frac{x+1}{-3+1} \cdot \frac{x+0}{-3+0} \cdot \frac{x-1}{-3-1} \tag{1.6}$$

Example 0.3. We interpolate $\log_2(x)$ on the points $\{16, 32, 64\}$. It is

$$\begin{aligned}
\log_2(16)L_1(x) &= \log_2(16) \cdot \frac{x-32}{16-32} \cdot \frac{x-64}{16-64} && (1.7)\\
&= \frac{1}{192}x^2 - \frac{1}{2}x + \frac{32}{3} && (1.8)\\
\log_2(32)L_2(x) &= \log_2(32) \cdot \frac{x-16}{32-16} \cdot \frac{x-64}{32-64} && (1.9)\\
&= -\frac{5}{512}x^2 + \frac{25}{32}x - 10 && (1.10)\\
\log_2(64)L_3(x) &= \log_2(64) \cdot \frac{x-16}{64-16} \cdot \frac{x-32}{64-32} && (1.11)\\
&= \frac{1}{256}x^2 - \frac{3}{16}x + 2. && (1.12)
\end{aligned}$$

Summing up yields

$$p(x) = -\frac{1}{1536}x^2 + \frac{3}{32}x + \frac{8}{3}. \tag{1.13}$$

1.2 Spline Interpolation