Chapter 1

Commutative Rings

Definition 1.1 — .

$$A^\times := \{\, a \in A \mid \exists b \in A : a \cdot b = 1 \,\} \tag{1.1}$$

is the group of units in A.

Example 1.2. Consider the ring $(\mathbb{Z}, +, \cdot)$.

1. $\mathbb{Z}^\times = \{\, -1, 1 \,\}$.
2. $\mathrm{ZD}(\mathbb{Z}) = \{\, 0 \,\}$.
3. $\mathrm{Nil}(\mathbb{Z}) = \{\, 0 \,\}$.

$\mathbb{Z}$ is therefore an integral domain and is reduced.

Example 1.3. Consider an arbitary field $\mathbb{K}$ as a ring.

1. $\mathbb{K}^\times = \{\, -1, 1 \,\}$.
2. $\mathrm{ZD}(\mathbb{K}) = \{\, 0 \,\}$.
3. $\mathrm{Nil}(\mathbb{Z}) = \{\, 0 \,\}$.

All fields are therefore an integral domain and are reduced.

Example 1.4. Consider the set of all continuous real-valued functions defined on the real numbers $C(\mathbb{R})$ with the operations of addition and multiplication.

1. $(C(\mathbb{R}))^\times =???$

Example 1.5. Consider $\mathbb{Z}[c]$ with $c \in \mathbb{C}$.

1. $(\mathbb{Z}[c])^\times = \{\, -1, 1 \,\}$.
2. $\mathrm{ZD}(\mathbb{Z}[c]) = \{\, 0 \,\}$.
3. $\mathrm{Nil}(\mathbb{Z}[c]) = \{\, 0 \,\}$.

Lemma 1. *1. $(A \setminus ZD(A), \cdot)$ is a semigroup containing $A^\times$.*

2. For $a \in A \setminus ZD(A)$ and $b_1, b_2 \in A$ with $a \cdot b_1 = a \cdot b_2$ one can clear, so we have $b_1 = b_2$.

3. $Nil(A)$ is an ideal in A.

4. The set $A_{red} := A \big/ Nil(A)$ is a reduced ring.

5. $\{\, 0 \,\} \subseteq Nil(A) \subseteq ZD(A) \subseteq A \setminus A^\times$

Proof. 1. Consider $(A \setminus \mathrm{ZD}(A), \cdot)$. The associativity of the multiplication is inherited from A, hence the only thing to show is that the operation is well-defined. Let $a, b \in \mathrm{ZD}(A)$, then there is a $x \in \mathrm{ZD}(A)$ such that $x \cdot a = 0$. We have

$$x \cdot a = 0 \iff x \cdot a \cdot b = 0. \tag{1.2}$$

This means that $a \cdot b \in \mathrm{ZD}(A)$ or in other words the set is closed under multiplication.

Now let $u \in A^\times$ be an unit, hence we have $u \cdot u^{-1} = 1$ for some $u^{-1} \in A^\times$. Assume $u \in \mathrm{ZD}(A)$. Then, there is a $x \in \mathrm{ZD}(A)$ with $x \neq 0$ such that $x \cdot u = 0$. We have

$$\begin{aligned} u \cdot u^{-1} = 1 &\iff x \cdot u \cdot u^{-1} = x & (1.3)\\ &\iff 0 \cdot u^{-1} = x & (1.4)\\ &\iff 0 = x. & (1.5) \end{aligned}$$

This is a contradiction with the assumption $x \neq 0$.

2.

□

Definition 1.6 — .
Two integers a and b are coprime if the only positive integer that is a divisor of both of them is 1. Equivalently, their greatest common divisor is 1.

Two ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in A are called coprime if $\mathfrak{a}_1 + \mathfrak{a}_2 = A$.

Proposition 1.

Exercise 1.7. 1. $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.

2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.

Proof. 1. Fix an $a \in \mathfrak{a}$. We want to show that $a \in (\mathfrak{a} : \mathfrak{b}) = \{ x \in A \mid x\mathfrak{b} \subseteq \mathfrak{a} \}$ or in other words $a\mathfrak{b} \subseteq \mathfrak{a}$. For all $b \in \mathfrak{b}$ we have $a \cdot b \in \mathfrak{a}$, therefore, $a\mathfrak{b} \subseteq \mathfrak{b}$. Conclude $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.

□

Exercise 1.8. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and unit is a unit.

Proof. Let $x \in \mathrm{Nil}(A)$. Since the nilradical is the intersection of all prime ideals, we have $x \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. All prime ideals are contained in some maximal ideal $\mathfrak{m}$, therefore, we have $x \in \mathrm{Jac}(A)$. It follows that $1 - xy \in A^\times$ for all $y \in A$. Choose $y = -1$ and conclude $1 + x$ is a unit. □

Exercise 1.9. Let A be a ring and let $A[X]$ be the ring of polynomials in an indeterminate X with coefficients in A. Let

$$f = a_0 + a_1 X + a_2 X^2 + \ldots + a_n x^n \in A[X]. \tag{1.6}$$

Prove that

1. f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

Exercise 1.10 (Robert's Lemma)**.** Let $a \in A$ and $b \in \mathrm{ZD}(A) \cup \{ 0 \}$ and $a + b \in A^\times$, then $a \in A^\times$.

Chapter 2

Modules

Definition 2.1 — A-module.
Modules are the generazation of vector spaces. While vector spaces are defined over a field, modules are over a ring. Modules is also a generalization of abelian groups.

An A-module is an abelian group $(M,+)$, together with a map $A \times M \longrightarrow M$, given by $(a,m) \mapsto a \cdot m$, satisfying for all $m, m_1, m_2 \in M$ and all $a, a_1, a_2 \in A$ the following:

Definition 2.2 — A-submodules.

Definition 2.3 — A-module homomorphism.
A-module homomorphism or A-linear maps are structure perserving maps between two A-modules.

Proposition 2 (Hom(M,N) is an A-module)**.**

Proposition 3 (Quotient with submodule forms an module)**.** *Let M be an A-module an N its submodule. Then M/N is an A-module.*

Proposition 4. *Let M be an A-module and $\mathfrak{a} \subseteq A$ an ideal with $am = 0$ for all $a \in \mathfrak{a}$, then M is a an $A/\mathfrak{a}$-module by means of $(a+\mathfrak{a})m := am$ (for $m \in M$, $a \in A$).*

Theorem 2.4 (Isomorphism Theorems)**.** *Let M and N be A-modules, S and T submodules of M, and $\varphi : M \longrightarrow N$ be a module homomorphism. Then:*

1. $M \big/ ker(\varphi) \cong im(\varphi)$.
2. $(S+T)/S \cong S \big/ (S \cap T)$.
3. *If* $T \subseteq S \subseteq M$, *then* $M/T \big/ S/T \cong M/S$.

Definition 2.5 — .
For a ring A and an index set I put

$$A^{(I)} := \{\, f : I \longrightarrow A \mid f(i) = 0 \text{ for almost all } i \in I \,\}. \tag{2.1}$$

Theorem 2.6 (Nakayama's Lemma)**.** *Let A be a ring, $\mathfrak{a}$ be an ideal in A, and M a finitely-generated module over A. If $\mathfrak{a}M = M$, then there exists an $x \in A$ with $x \equiv 1 \mod \mathfrak{a}$, such that $xM = 0$.*