Exercise 0.1.

Definition 0.2 — Saturation.
Let A be a ring and S be a multiplicative closed subset. The complement in A of the union of prime ideals that do not meet S is denoted by $\overline{S}$ and is called the saturation of S. It is the smallest and unique multiplicativly closed subset that contains S.

Exercise 0.3 (3.8).

- i) $\Rightarrow$ ii): Let $\phi : S^{-1}A \longrightarrow T^{-1}A$ be the ring homomorphism that maps $a/s \in S^{-1}A$ to a/s as an element of $T^{-1}A$. We show that if ϕ is bijective, then for each $t \in T$, $t/1$ is a unit in $S^{-1}A$.
- ii) $\Rightarrow$ iii): For each $t \in T$, $t/1$ is a unit in $S^{-1}A$. We show that for each $t \in T$ there exists $x \in A$ such that $xt \in S$.
  1. Let $t \in T$, then $t/1$ is a unit in $S^{-1}A$, so there exists a $a/s \in S^{-1}A$ such that $t/1 \cdot a/s = 1$.
  2. We have
  $$1 = \frac{t}{1} \cdot \frac{a}{s} = \frac{ta}{s} \tag{1}$$
  $$\iff 1 \cdot s = \frac{ta}{s} s \tag{2}$$
  $$\iff s = ta. \tag{3}$$
  3. If we set $x := a$, then there exists a $x \in A$ such that $xt \in S$.
- iii) $\Rightarrow$ iv): For each $t \in T$ there exists $x \in A$ such that $xt \in S$. We show that T is contained in the saturation of S.
  1. Let $t \in T$, $x \in A$ such that $xt \in S$, and $\mathfrak{p}$ be a prime ideal that contains t.
  2. Then $xt \in \mathfrak{p}$, so $\mathfrak{p} \cap S \neq \emptyset$, or in other words, prime ideals that contain t do not meet S.
  3. Hence, t is also not contained in the union of prime ideals that do not meet S.
  4. But it is contained in the complement of the union of prime ideals that do not meet S.
  5. So t is in the saturation of S.
- iv) $\Rightarrow$ v): Let T be contained in the saturation of S. We show that every prime ideal that meets T also meets S.
  1. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \cap T \neq \emptyset$.
  2. Then there is a $x \in \mathfrak{p}$ with $x \in T$.
  3. x is also in $\overline{S}$ because T is contained in the saturation of S.
  4. This means that x is not in the union of prime ideals that do not meet S.
  5. So if $x \in \mathfrak{p}$, then $\mathfrak{p}$ must meet S.
- v) $\Rightarrow$ i): Every prime ideal that meets T also meets S. Let $\phi : S^{-1}A \longrightarrow T^{-1}A$ be the ring homomorphism which maps $a/s \in S^{-1}A$ to a/s as an element of $T^{-1}A$. We show ϕ is bijective.
  1.