Exercise 0.1
Let

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - 2z^2 = 0,\ x + y + z - 1 = 0\}. \tag{1}$$

Show the following.

1. M is a onedimensional submanifold of $\mathbb{R}^3$.

Solution 0.1
Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$f(x, y, z) := (f_1(x, y, z),\ f_2(x, y, z)) := (x^2 + y^2 - 2z^2,\ x + y + z - 1). \tag{2}$$

For the partial derivatives of f, we have

$$\begin{aligned} \partial_x f_1(x, y, z) &= 2x & \partial_x f_2(x, y, z) &= 1 \qquad (3)\\ \partial_y f_1(x, y, z) &= 2y & \partial_y f_2(x, y, z) &= 1 \qquad (4)\\ \partial_z f_1(x, y, z) &= -4z & \partial_z f_2(x, y, z) &= 1 \qquad (5) \end{aligned}$$

hence we have

$$\mathrm{D}f(x, y, z) = \begin{pmatrix} 2x & 2y & -4z \\ 1 & 1 & 1 \end{pmatrix} \tag{6}$$

This matrix has the rank 1 if $x = y = -2z$. But if we look at the given equation, we have

$$\begin{aligned} x + y + z - 1 &= x + x - \frac{1}{2}x - 1 &(7)\\ &= \frac{3}{2}x - 1 &(8) \end{aligned}$$

As this equation should equal 0, we have $x = \frac{2}{3}$. With that, we also have $y = \frac{2}{3}$ and $z = -\frac{1}{3}$. But if we plug it into the second equation, we get

$$(\frac{2}{3})^2 + (\frac{2}{3})^2 - 2(-\frac{1}{3})^2 = \frac{8}{9} - \frac{2}{9} = \frac{6}{9} \neq 0 \tag{9}$$

So for all $(x, y, z) \in M$, the differential $\mathrm{D}f$ is surjective and by definition M is a onedimensional submanifold in $\mathbb{R}^3$.