

# Integration and Integration

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# Introduction



## Part I

# $\sigma$ -algebra and measures





## Chapter 1

# Family of Sets



# Chapter 2

## Measure

### 2.1 Content, Premeasure, and Measure

**Definition 2.1.** Let  $\mathcal{R} \subset \mathcal{P}(X)$  be a ring of sets. A set function  $\mu \rightarrow [0, \infty]$  is called

- finitely additive if for all disjoint  $A, B \in \mathcal{R}$  it is  $\mu(A \sqcup B) = \mu(A) + \mu(B)$ .
- $\sigma$ -additive if for all disjoint  $A_k \in \mathcal{R}$  with  $k \in \mathbb{N}$  and  $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$  it is

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \quad (2.1)$$

- subadditive if for all  $A, B \in \mathcal{R}$  it is  $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- $\sigma$ -subadditive if for all  $A_k \in \mathcal{R}$  with  $k \in \mathbb{N}$  and  $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$  it is

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k). \quad (2.2)$$

- finite if for all  $A \in \mathcal{R}$  it is  $\mu(A) < \infty$ .
- $\sigma$ -finite if there exists a collection of subsets  $\{A_k\}_{k \in \mathbb{N}}$  in  $\mathcal{R}$  with  $\mu(A_k) < \infty$  for all  $k \in \mathbb{N}$  such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \quad (2.3)$$

- monotonous if for all  $A, B \in \mathcal{R}$  with  $A \subset B$  it is  $\mu(A) \leq \mu(B)$ .

**Remark.** In the definition of  $\sigma$ -additivity, checking whether  $\bigsqcup_{k=1}^{\infty} A_k$  is included in  $\mathcal{R}$  is required. For  $\sigma$ -rings and therefore  $\sigma$ -algebras, it is guaranteed that a countable union of disjoint sets are included.

In general, not all finite set functions  $\mu \rightarrow [0, \infty]$  are  $\sigma$ -finite as  $X$  need not be included in a ring of sets.

**Definition 2.2** (Content). Let  $\mathcal{R} \subset \mathcal{P}(X)$  be a ring of sets. A set function  $\mu \rightarrow [0, \infty]$  is called a content if

1.  $\mu(\emptyset) = 0$ .
2.  $\mu$  is finitely additive.

**Definition 2.3** (Premeasure). Let  $\mathcal{R} \subset \mathcal{P}(X)$  be a ring of sets. A  $\sigma$ -additive content  $\mu \rightarrow [0, \infty]$  is called a premeasure.

**Definition 2.4** (Measure). Let  $\mathcal{A} \subset \mathcal{P}(X)$  a  $\sigma$ -algebra. A  $\sigma$ -additive content  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is called a measure.

## 2.2 Lebesgue Content

**Definition 2.5** (Lebesgue Content). Let  $\mathcal{Q}(\mathbb{R}^n)$  be the ring of sets over  $\mathbb{R}^n$ .

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigcup_{k=1}^m [a_{1,k}, b_{1,k}) \times \cdots \times [a_{n,k}, b_{n,k}) \mid m \in \mathbb{N}; a_{i,k}, b_{i,k} \in \mathbb{R}; 1 \leq k \leq n \right\} \quad (2.4)$$

Set  $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$  as

$$\lambda^n(A) := \sum_{k=1}^m \prod_{i=1}^n (b_{i,k} - a_{i,k}) \quad (2.5)$$

$\lambda^n$  is the Lebesgue content.

**Theorem 2.5.1.**  $\lambda^n$  is a well-defined finite content.

**Theorem 2.5.2.**  $\lambda^n$  is a premeasure.

## 2.3 Lebesgue Measure

### CHEET SHEET

1. Content  $\mu : \mathcal{R} \rightarrow [0, \infty]$  is empty set 0 and finitely additive.
2. Premeasure  $\mu : \mathcal{R} \rightarrow [0, \infty]$  is  $\sigma$ -additive content.
3. First extension  $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$
4. Outer measure  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$

$$\mathcal{A} \subset \mathcal{A}^\uparrow \subset \sigma(\mathcal{A}) \subset \hat{\mathcal{A}} \quad (2.6)$$

**Definition 2.6.** Let  $\mathcal{R} \subset \mathcal{P}(X)$  a set of rings. Set

$$\mathcal{R}^\uparrow := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k \in \mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \quad (2.7)$$

**Remark.**  $\mathcal{R}^\uparrow$  is the set of all  $A \in \mathcal{P}(X)$  that can be expressed as a countable many unions of sets in  $\mathcal{R}$ .

In general,  $\mathcal{R}^\uparrow$  is not a set of rings.

**Definition 2.7.** Let  $\mathcal{R} \subset \mathcal{P}(X)$  be a ring of sets and  $\mu : \mathcal{R} \rightarrow [0, \infty]$  a premeasure. For  $A_k \uparrow A$  with  $A_k \in \mathcal{R}$  for  $k \in \mathbb{N}$  define

$$\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty], A \mapsto \tilde{\mu}(A) := \lim_{k \rightarrow \infty} \mu(A_k). \quad (2.8)$$

$\tilde{\mu}$  is called the first extension of the premeasure  $\mu$ .

**Remark.** In general,  $\tilde{\mu}$  is not a premeasure as  $\mathcal{R}^\uparrow$  need not be a ring of sets.

$\tilde{\mu}$  restricted on  $\mathcal{R}$  is identical with  $\mu$ , i.e.  $\tilde{\mu}|_{\mathcal{R}} \equiv \mu$ .

**Lemma 2.7.1.** The first extension  $\tilde{\mu}$  is well-defined.

**Proposition 2.7.1** (Properties of  $\mathcal{R}^\uparrow$ ).

**Proposition 2.7.2** (Properties of the First Extension).

**Definition 2.8** (Second Extension or the Outer Measure). Let  $\mathcal{R} \subset \mathcal{P}(X)$  be a ring of sets,  $\mu : \mathcal{R} \rightarrow [0, \infty]$  a  $\sigma$ -finite premeasure on  $\mathcal{R}$ , and  $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$  the first extension of  $\mu$  on  $\mathcal{R}^\uparrow$ . Moreover, let  $B \subset X$  be a subset of  $X$ . Then, the map

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty], B \mapsto \mu^* := \inf \{ \tilde{\mu}(A) \mid A \in \mathcal{R}^\uparrow, A \supset B \} \quad (2.9)$$

is called the outer measure induced by  $\tilde{\mu}$  on  $\mathcal{P}(X)$ .

**Proposition 2.8.1** (Properties of the Second Extension).

**Proposition 2.8.2** (Properties of the Outer Measure).

**Definition 2.9** (Lebesgue Outer Measure). Let  $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$  the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty], B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^\uparrow, A \supset B \right\} \quad (2.10)$$

is called the Lebesgue outer measure induced by  $\tilde{\lambda}^n$ .

**Definition 2.10** (Pseudo Metric). Let  $X$  be a set. A map  $d : X \times X \rightarrow \overline{\mathbb{R}}$ ,  $(x, y) \mapsto d(x, y)$  is called pseudo metric on  $X$  if for all  $x, y, z \in X$  it is the following three axioms are met.

1.  $x = y \Rightarrow d(x, y) = 0$ .
2.  $d(x, y) = d(y, x)$ . (Symmetry.)
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

**Proposition 2.10.1.** The outer measure induces a pseudo metric, i.e.

$$d_{\mu^*} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty], (A, B) \mapsto d_{\mu^*}(A, B) := d_{\mu^*}(A \triangle B) \quad (2.11)$$

is a pseudo metric.

**Proposition 2.10.2.** The outer measure is continuous.

**Definition 2.11** (Approximation through elements of Rings). Let  $\mathcal{R} \subset \mathcal{P}(X)$  a set of rings,  $\mu : \mathcal{R} \rightarrow [0, \infty]$  a premeasure on  $\mathcal{R}$ , and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  the outer measure induced by  $\mu$ . Then,  $A \in \mathcal{P}(X)$  is called  $\mathcal{R}$ -approximatable in respect to  $\mu^*$  if for all  $\epsilon > 0$  there exists an  $B \in \mathcal{R}$  such that  $d_{\mu^*}(A, B) = \mu^*(A \triangle B) < \epsilon$ .

**Theorem 2.11.1.** Let  $\mathcal{A} \subset \mathcal{P}(X)$  be a  $\sigma$ -algebra and  $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$  a finite premeasure. Let the first extension  $\tilde{\mu} : \mathcal{A}^\uparrow \rightarrow \mathbb{R}_0^+$  also be finite and  $\mu^* : \mathcal{P}(X) \rightarrow \mathbb{R}_0^+$  the outer measure. Then,

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \quad (2.12)$$

is a  $\sigma$ -algebra on  $X$ .

**Theorem 2.11.2.** Let  $\mu, \tilde{\mu}, \mu^*$  and  $\mathcal{A}, \mathcal{A}^\uparrow, \hat{\mathcal{A}}$  be given. Then, a finite premeasure  $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$  can be uniquely extended to a finite measure  $\hat{\mu} : \hat{\mathcal{A}} \rightarrow \mathbb{R}_0^+$  where  $\hat{\mu} \equiv \mu^*|_{\hat{\mathcal{A}}}$ .

**Theorem 2.11.3.** Let  $\mathcal{R} \subset \mathcal{P}(X)$  a set of rings and  $\mu : \mathcal{R} \rightarrow [0, \infty]$  a  $\sigma$ -finite premeasure on  $\mathcal{R}$  and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  the outer measure induced by  $\mu$ . Then,  $\mu$  can be uniquely extended to a measure  $\hat{\mu} : \sigma(\mathcal{R}) \rightarrow [0, \infty]$  where  $\hat{\mu} \equiv \mu^*|_{\sigma(\mathcal{R})}$ .

**Definition 2.12.** Let  $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$  a  $\sigma$ -finite Lebesgue premeasure. In this chapter, we constructed a unique extension of  $\lambda^n$  on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathbb{R}^n)$ , the Lebesgue-Borel measure  $\hat{\lambda} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$ .

## 2.4 Measure Space

**Definition 2.13.** Let  $\mathcal{A} \subset \mathcal{P}(X)$  a  $\sigma$ -algebra. The tuple  $(X, \mathcal{A})$  is called measurable space and the sets in the  $\sigma$ -algebra  $A \in \mathcal{A}$  are called measurable sets.

Moreover, let  $\mu : \mathcal{A} \rightarrow [0, \infty]$  be a measure on  $\mathcal{P}(X)$ . Then,  $(X, \mathcal{A}, \mu)$  a measure space.

**Definition 2.14** (Null Sets). Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  the induced outer measure. Then  $N \subset X$  with  $\mu^*(N) = 0$  is called null set.

For  $X = \mathbb{R}^n$  with  $\lambda^n(N) = 0$  called Lebesgue null set.

$S = \emptyset$  is called the trivial null set.

**Definition 2.15** (Completion of a Measure Space). Let  $(X, \mathcal{A}, \mu)$  be a measure space. This measure space is called complete if all null sets are included in  $\mathcal{A}$ , i.e. for all  $N \in \mathcal{A}$

$$\mu^* N = 0 \Rightarrow N \in \mathcal{A}. \quad (2.13)$$

**Definition 2.16.** Let

$$\overline{\mathcal{A}}^\mu := \{A \cup N \mid A \in \mathcal{A}, N \subset X \text{ with } \mu^*(N) = 0\} \quad (2.14)$$

then  $\overline{\mathcal{A}}^\mu$  is called the completion of  $(X, \mathcal{A}, \mu)$ .

**Definition 2.17.** The completion of the Lebesgue-Borel measure space  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \hat{\lambda}^n)$  to  $(\mathbb{R}^n, \mathcal{B}^{\hat{\lambda}^n}(\mathbb{R}^n), \hat{\lambda}^n)$  or shorter  $(\mathbb{R}^n, \overline{\mathcal{B}}^\lambda(\mathbb{R}^n), \lambda^n)$  is called the (completed) Lebesgue measure space.

$B \in \overline{\mathcal{B}}^\lambda(\mathbb{R}^n)$  is called Lebesgue measurable to differentiate from  $B \in \mathcal{B}(\mathbb{R}^n)$  Borel measurable.

**Part II**

**Lebesgue Integral**





## **2.5 Measurable Maps**

## **2.6 Lebesgue Integral**



# Part III

## Applications



**Part IV**

**More Theory**



## Chapter 3

# Lebesgue Space

**Definition 3.1** ( $L^p$ -Norm). Let  $X, \mathcal{A}, \mu$  a measure space, and  $f : X \rightarrow \overline{\mathbb{R}}$  measurable. Then for  $p \in [1, \infty)$  the  $L^p$ -norm is defined as

$$\|f\|_p := \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}. \quad (3.1)$$

**Theorem 3.1.1** (Holder Inequality). Let  $p, q \in (1, \infty)$  such that  $p^{-1} + q^{-1} = 1$ . Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \quad (3.2)$$

**Theorem 3.1.2** (Minkowski Inequality). Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  measurable and  $f + g$  well defined on  $X$ . Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (3.3)$$

**Definition 3.2.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $p \in [1, \infty)$ . Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \rightarrow \overline{\mathbb{R}} \mid f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\}. \quad (3.4)$$