Chapter 1

Interpolation

1.1 Lagrange Interpolation

Definition 1. For $n \in \mathbb{N}_0$, $n+1$ distinct $x_0, \dots, x_n \in \mathbb{K}$ and the index $j \in \{0, \dots, n\}$ the j-th Lagrange polynomials respective to the nodes $x_0, \dots, x_n$ are defined as

$$L_j = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k} \in \mathbb{K}[x]. \tag{1.1}$$

If $f(x_0), \dots, f(x_n)$ are now the data respective to the nodes $x_0, \dots, x_n$ then the Lagrange interpolation p is given by

$$p(x) = \sum_{j=0}^{n} f(x_j) L_j(x). \tag{1.2}$$

Example 1.1. Consider the domain $[2, 10]$ partitioned into 5 points, i.e. $\{2, 4, 6, 8, 10\}$ and a function $f : [0, 10] \to \mathbb{R}$, $x \mapsto f(x) = \ln(x)$. The y-values then are

$$\ln(2) \approx 0.6931 \quad \ln(4) \approx 1.3862 \quad \ln(6) \approx 1.7917 \quad \ln(8) \approx 2.0794 \quad \ln(10) \approx 2.3025. \tag{1.3}$$

Computing the Lagrange polynomials gives

$$\begin{aligned}
L_1(x) &= \ln(2) \cdot \frac{x-4}{2-4} \cdot \frac{x-6}{2-6} \cdot \frac{x-8}{2-8} \cdot \frac{x-10}{2-10} && (1.4)\\
&= 5\ln(2) - \frac{77}{24} x \ln(2) + \frac{71}{96} x^2 \ln(2) - \frac{7}{96} x^3 \ln(2) + \frac{1}{384} x^4 \ln(2) && (1.5)\\
L_2(x) &= \ln(2) \cdot \frac{x-2}{4-2} \cdot \frac{x-6}{4-6} \cdot \frac{x-8}{4-8} \cdot \frac{x-10}{4-10} && (1.6)\\
&= -10\ln(2) + \frac{107}{12} x \ln(2) - \frac{59}{24} x^2 \ln(2) + \frac{13}{48} x^3 \ln(2) - \frac{1}{96} x^4 \ln(2) && (1.7)
\end{aligned}$$

Example 1.2. Let $f(x) = x^8$. We want to interpolate f on the grid points $\{-3, -2, -1, 0, 1, 2, 3\}$. The Lagrange polynomials are

$$L_1(x) = -6561 \cdot \frac{x+2}{-3+2} \cdot \frac{x+1}{-3+1} \cdot \frac{x+0}{-3+0} \cdot \frac{x-1}{-3-1} \tag{1.8}$$

Example 1.3. We interpolate $\log_2(x)$ on the points $\{16, 32, 64\}$. It is

$$\begin{aligned}
\log_2(16)L_1(x) &= \log_2(16) \cdot \frac{x-32}{16-32} \cdot \frac{x-64}{16-64} && (1.9)\\
&= \frac{1}{192}x^2 - \frac{1}{2}x + \frac{32}{3} && (1.10)\\
\log_2(32)L_2(x) &= \log_2(32) \cdot \frac{x-16}{32-16} \cdot \frac{x-64}{32-64} && (1.11)\\
&= -\frac{5}{512}x^2 + \frac{25}{32}x - 10 && (1.12)\\
\log_2(64)L_3(x) &= \log_2(64) \cdot \frac{x-16}{64-16} \cdot \frac{x-32}{64-32} && (1.13)\\
&= \frac{1}{256}x^2 - \frac{3}{16}x + 2. && (1.14)
\end{aligned}$$

Summing up yields

$$p(x) = -\frac{1}{1536}x^2 + \frac{3}{32}x + \frac{8}{3}. \tag{1.15}$$

1.2 Spline Interpolation

Definition 2. A function $s \in C^q([a,b],\mathbb{K})$ is called a spline of degree p and the smoothness q respective to the partition $\triangle$ if s is a polynomial on the subinterval $[x_{j-1}, x_j]$ with a degree equal or less than q. We denote $s \in \mathcal{S}_q^p(\triangle)$.

Example 2.1.
- The space of linear spline functions is $\mathcal{S}_0^1(\triangle)$. We have $s \in C^0([a,b],\mathbb{K})$ so s is continuous but not necessarily differentiatable.
- The space of cubic spline functions is $\mathcal{S}_2^3(\triangle)$.

Theorem 2.1. $\mathcal{S}_{m-1}^m(\triangle)$ is a $\mathbb{R}$-vector space. In particular, it contains all polynomials of degree $\leq m$. Moreover, the dimension of $\mathcal{S}_{m-1}^m(x_0, \dots, x_n)$ is $m + n$.

Example 2.2. Let $[-1, 1]$ be a domain partitioned into $[-1, 0] \cup [0, 1]$. Consider the function

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \qquad x \in [-1, 1]. \tag{1.16}$$

1.2.1 Linear Splines

1.2.2 Cubic Splines

Definition 3.
1. hermitian
2. natural
3. periodic

Theorem 3.1. If $\triangle = (x_0, \dots, x_n)$ is a partition of the interval $[a, b]$ and $y_0, \dots, y_n$ the respective data then there is exactly one interpolating natural cubic spline $s \in \mathcal{S}_2^3(\triangle)$ with $s''(a) = s''(b) = 0$.

Example 3.1. Given the data set $(0, 0)$, $(1, 0.5)$, $(2, 2)$ and $(3, 1.5)$ we want to find the interpolating cubic spline $s(x)$ satisfying $s'(0) = 0.2$ and $s'(3) = -1$.

We start by finding s_i''. Denote $s_i''(x_i) = M_i$. With Lagrange interpolation we get the general formula

$$s_i''(x) = M_{i-1}\frac{x - x_i}{x_{i-1} - x_i} + M_i\frac{x - x_{x-1}}{x_i - x_{i-1}}. \tag{1.17}$$

Plugging in the values for x_i gives

$$
\begin{aligned}
s_1''(x) &= M_0 \frac{x-1}{0-1} + M_1 \frac{x-0}{1-0} && (1.18)\\
&= M_0 + (M_1 - M_0)x && (1.19)\\
s_2''(x) &= M_1 \frac{x-2}{1-2} + M_2 \frac{x-1}{2-1} && (1.20)\\
&= 2M_1 - M_2 + (M_2 - M_1)x && (1.21)\\
s_3''(x) &= M_2 \frac{x-3}{2-3} + M_3 \frac{x-2}{3-2} && (1.22)\\
&= 3M_2 - 2M_3 + (M_3 - M_2)x. && (1.23)
\end{aligned}
$$

Now we integrate.

$$
\begin{aligned}
\int s_1''(x)\,\mathrm{d}x &= s_1'(x) = C_1 + M_0 x + \frac{M_1 - M_0}{2} x^2 && (1.24)\\
\int s_2''(x)\,\mathrm{d}x &= s_2'(x) = C_2 + (2M_1 - M_2)x + \frac{M_2 - M_1}{2} x^2 && (1.25)\\
\int s_3''(x)\,\mathrm{d}x &= s_3'(x) = C_3 + (3M_2 - 2M_3)x + \frac{M_3 - M_2}{2} x^2. && (1.26)
\end{aligned}
$$

With the given values, we get

$$
\begin{aligned}
s_1'(0) &= 0.2 = C_1 && (1.27)\\
s_3'(3) &= -1 = C_3 + (3M_2 - 2M_3) \cdot 3 + \frac{M_3 - M_2}{2} \cdot 3^2 && (1.28)\\
&= C_3 + \frac{9}{2} M_2 - \frac{3}{2} M_3 && (1.29)
\end{aligned}
$$

We start by finding s_1. It is $s_1(x) = a + bx + cx^2 + dx^3$ and $s_1'(x) = b + cx + dx^2$ for some $a, b, c, d \in \mathbb{R}$. Plugging in the conditions gives

$$
\begin{aligned}
s_1(0) &= a = 0 && (1.30)\\
s_1(1) &= a + b + c + d = 0.5 && (1.31)\\
s_1'(0) &= b = 0.2 && (1.32)
\end{aligned}
$$