

0.1 Integral Dependence

Definition 0.1. Let $A \subset B$ be an extension of rings.

1. An element $x \in B$ is integral over A if there is a monic polynomial $f(T) \in A[T]$ such that $f(x) = 0 \in B$. That is, if for certain $n \in \mathbb{N}_0$ there are elements $a_1, \dots, a_n \in A$ such that

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0. \quad (1)$$

Such a polynomial $f(T) = T^n + a_1T^{n-1} + \dots + a_n$ is called an equation of integral dependence of x over A .

Lemma 0.1.1. All UFDs are integrally closed.

Definition 0.2 (A -Module).

Remark. 1. The submodules of the ring A itself are precisely the ideals of A .

2. In an extension of rings $A \subset B$, B is an A -module.

Proposition 0.2.1. Let $A \subset B$ be an extension of rings and $x \in B$. Then the following are equivalent.

- x is integral over A .
- The subring $A[x] := \{f(x) \mid f(T) \in A[T]\} \subset B$ is a finitely generated A -module.
- There is a finitely generated A -submodule $M \subset B$ such that $1 \in M$ and $x \cdot M \subset M$.