Notes on Algebraic Geometry

Kei Thoma

August 21, 2024

Contents

TODO

Part I

Pre: Commutative Algebra

1. Prove R int domain, then $R[X]$ is int domain.

Proposition 0.0.1. *If R is an integral domain, then the polynomial ring $R[X]$ is again an integral domain.*

Proof.

1. Since $1 \in R \subset R[X]$, the polynomial ring $R[X]$ is nonempty.
2. Let $f, g \in R[X]$ be two nonzero polynomials with
$$f = \sum_{i=0}^{m} a_i X^i \quad \text{and} \quad g = \sum_{j=0}^{n} b_j X^j.$$
Consider its product
$$f \cdot g = \sum_{k=0}^{m+n} c_j X^k$$
and suppose $f \cdot g = 0$.
3. Since the leading coefficient of the product c_{m+n} is obtained by multiplying the leading coefficients of f and g, we have $c_{m+n} = a_m \cdot b_n$.
4. We had $f \cdot g = 0$, thus $c_{m+n} = a_m \cdot b_n = 0$.
5. R is an integral domain, therefore $a_m \cdot b_n = 0$ means $a_m = 0$ or $b_n = 0$.
6. This contradicts that f and g were nonzero polynomials.

$\square$

Part II

Topology

Definition 0.0.2 (Product Topology)**.** $X = \prod_{i \in I} X_i$

$$\left\{ p_i^{-1}(U_i) \mid i \in I \text{ and } U_i \subset X_i \text{ is open in } X_i \right\}$$

Part III

Algebraic Geometry

Chapter 1

Affine Varieties

Cheat Sheet

Definition 1.0.1.
1. The affine n-space $\mathbb{A}^n$ over an algebraically closed field K is the set of all n-tuples of elements of K.
2. For a subset $S \subset K[X_1, \ldots, X_n1]$, we define the zero-locus as

$$Z(S) = \{\, x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \,\}.$$

3. A subset $V \subset \mathbb{A}^n$ is an algebraic set if there exists a subset $S \subset K[X_1, \ldots, X_n]$ such that $V = Z(S)$.

Definition 1.0.2.
1. Zariski topology.
2. Irreducible

Full Text

Algebraic Sets

Definition 1.0.3. Let K be an algebraically closed field and let $n \in \mathbb{N}_0$ be a natural number.

1. The affine n-space over K is the set of all n-tuples of elements of K.
2. An element p in $\mathbb{A}^n$ is called a point.
3. If $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is a point, then a_i is called the coordinate for each $1 \le i \le n$.

Intuition 1.0.4. It's just space with points. But not vectors, because we don't add points.

Definition 1.0.5. For each subset S of polynomials in $K[X_1, \ldots, X_n]$, we define the zero-locus $Z(S)$ to be the set of points in the affine n-space $\mathbb{A}^n$ on which the functions in S simultaneously vanish, i.e.

$$Z(S) = \{ x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \} .$$

Intuition 1.0.6. These are just curves.

Remark 1.0.7. 1. If $\mathfrak{a}$ is generated by T, then $Z(T) = Z(\mathfrak{a})$.

2. $Z(T)$ can be written in finitely many generators.

Definition 1.0.8 (Algebraic Set)**.** A subset V of $\mathbb{A}^n$ is an algebraic set if there exists a subset $S \subset k[X_1, \ldots, X_n]$ such that $V = Z(S)$.

Intuition 1.0.9. So if the points on the space is a curve, then it's an algebraic set.

Zariski Topology

Definition 1.0.10. Zariski topology on $\mathbb{A}^n$. Closed sets are algebraic sets.

Example 1.0.11. Consider $\mathbb{A}^2$.

1. Any point is a closed subset.

2. Any curve described by a polynomials is a closed subset.

Definition 1.0.12. Irreducible subsets

Affine Algebraic Variety

Definition 1.0.13. An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a quasi-affine variety.

Proposition 1.0.14 (1.2.)**.**

Theorem 1.0.15 (Hilbert's Nullstellensatz)**.**

Corollary 1.0.16. *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

Example 1.0.17. $\mathbb{A}^n$ is irreducible, since it corresponds to the zero ideal in $K[X_1, \ldots, X_n]$, which is prime.

Example 1.0.18. Let f be an irreducible polynomial in $K[X, Y]$. Then f generates a prime ideal in $K[X, Y]$, since $K[X, Y]$ is a UFD, so the zero set $V = Z(f)$ is irreducible.

Example 1.0.19.

1. Consider $Z((X+1)(Y+1)) = Z(XY + X + Y + 1)$ in $\mathbb{A}^2$. Clearly, $X^2 - 1$ is not prime, hence it is not irreducible. We must have $Z(XY + X + Y + 1) = Z(X+1) \cup Z(Y+1)$. Indeed, the above example is just two lines crossing.
2. Consider $Z((X^2 - Y)(Y + 1)) = Z(X^2Y + X^2 - Y^2 - Y)$. This is a parabola and a line.

Definition 1.0.20. If f is an irreducible polynomial in $K[X_1, \ldots, X_n]$ then the affine variety $H = Z(f)$

Affine Coordinate Ring

Definition 1.0.21. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of Y, to be $A/I(Y)$.

Dimension

Definition 1.0.22 (Noetherian Topological Space)**.** A topological space X is called Noetherian if it satisfies the descending chain condition for closed subsets, i.e. for any sequence $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots$ becomes stationary.

Proposition 1.0.23 (1.5.)**.** *In a Noetherian topological space X, every nonempty closed subset $\mathcal{V}$ can be expressed as finite union of irreducible, closed subsets.*

Corollary 1.0.24. *Every algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of varieties, no one containing another.*

Definition 1.0.25 (Dimension of a Topological Space)**.**

Definition 1.0.26 (Height of a Prime Ideal)**.**

Definition 1.0.27 (Dimension of a Ring)**.**

Theorem 1.0.28. *Let K be a field, and let B be an integral domain which is a finitely generated K-algebra. Then:*

1. *the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over K*

2. *For any prime ideal $\mathfrak{p}$ in B, we have*

$$height \mathfrak{p} + dim B/\mathfrak{p} = dim B$$

BOOKMARK

Definition 1.0.29 (Affine Algebraic Variety)**.** For an algebraically closed field K and a natural number $n \in \mathbb{N}_+$, let $\mathbb{A}^n$, be an affine n-space over K. The polynomials in $K[X_1, \ldots, X_n]$ can be viewed as K-valued functions on $\mathbb{A}^n$.

1. For each subset S of polynomials in $K[X_1, \ldots, X_n]$, define the zero-locus $Z(S)$ to be the set of points in $\mathbb{A}^n$ on which the functions in S simultaneously vanish, i.e.

$$Z(S) = \{ \, x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \, \} \, .$$

2. A subset V of $\mathbb{A}^n$ is called affine agebraic set if $V = Z(S)$ for some $S \subset K[X_1, \ldots, X_n]$.

3. A nonempty affine algebraic set is called irreducible if it is not the union of two proper algebraic subsets. An irreducible affine algebraic set is also called an affine variety.

Definition 1.0.30. If X is a topological space, we define the dimension of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

1.1 Exercises

Exercise 1.1.1 (1.1. (a))**.** *Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.*

Solution. By definition 1.0.21 of a coordinate ring, we simply have $A(Y) = k[X, Y]/(Y - X^2)$. The isomorphism follows from the isomorphism theorem and the map $f : k[X, Y] \to k[X]$ where we set $f(Y) = X^2$.

Exercise 1.1.2 (1.1. (b))**.** *Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.*

Solution. $A(Z) = k[X, Y]/(XY - 1)$

We know $A(Z)$ is an k-algebra (see remark). Consider $f : k[X, Y] \longrightarrow k[T]$. We must have $\ker f = (XY - 1)$, thus $f(XY - 1) = 0$, so $f(X) = 1/f(Y)$

I'll think about the rigorous details later, but basically $A(Z) \cong k[X, X^{-1}]$

Exercise 1.1.3 (1.1. (c))**.** *Let f be any irreducible quadratic polynomial in $k[X, Y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?*

Solution. Let f be irreducible.

$A(W) = k[X, Y]/(f)$

isn't this kinda clear ...? I'll come back to write it down rigorously, but in general ...

Exercise 1.1.4. *Let $V \subset \mathbb{A}^3$ be the set $Y = \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}$.*

1. *Show that V is an affine variety of dimension* 1.
2. *Find generators for the ideal $I(V)$.*

Solution. An affine variety is an irreducible, closed subset of $\mathbb{A}^n$ with regard to the Zariski topology.

1. We show that V is a closed subset with regard to the Zariski topology.
   (a) Since any algebraic set is immediately a closed subset, it is enough to show that V is the zero-locus of some subset of polynomials in $K[X_1, \dots, X_n]$.
   (b) Consider the ideal $(Y - X^2, Z - X^3) \subset K[X, Y]$ and it's zero set $Z(Y - X^2, Z - X^3)$.
   (c) Writing out the definition of the zero set gives

$$\begin{aligned} Z(Y - X^2, Z - X^3) &= \{ (x, y, z) \in \mathbb{A}^3 \mid y - x^2 = 0,\ z - x^3 = 0 \} \\ &= \{ (x, y, z) \in \mathbb{A}^3 \mid y = x^2,\ z = x^3 \} \\ &= \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}. \end{aligned}$$

   Thus, V is the zero set of the ideal $(Y - X^2, Z - X^3)$.

(d) Hence, by definition, $V = Z(Y - X^2, Z - X^3)$ is an algebraic set.

2. Here, we prove that V is irreducible.

(a) Consider the quotient $K[X,Y,Z]/(Y - X^2, Z - X^3)$.

(b) By substitution, we get the isomorphism

$$K[X,Y,Z]/(Y - X^2, Z - X^3) \cong K[X, X^2, X^3] = K[X].$$

(c) Since K is a field it in particular an integral domain and so is $K[X]$.

(d) Thus, $(Y - X^2, Z - X^3)$ is prime in $K[X,Y,Z]$.

(e) With corollary 1.0.16 we may conclude the variety $V = Z(Y - X^2, Z - X^3)$ is irreducible.

3. We show that V is of dimension 1.

(a) By proposition 1.7, the dimension of V corresponds with the dimension of its affine coordinate ring $A(V)$.

(b) It is

$$A(V) = K[X,Y,Z]/(Y - X^2, Z - X^3) \cong K[X].$$

(c) $K[X]$ is a PID. All prime ideals are generated by irreducible elements. Thus dimension is 1.

Exercise 1.1.5 (1.3.). *Let V be the algebraic set in $\mathbb{A}^3$ defined by two polynomials $X^2 - YZ$ and $XZ - X$. Show that V is a union of three irreducible components. Describe them and find their prime ideals.*

Solution. $V = Z(X^2 - YZ, XZ - X)$

If $z = 0$, then $x = 0$ and y can be any thing, so one irreducible component is the y-axis. This is described by $V_1 = Z(x, z)$. V_1 is irreducible because its ideal (x, z) is prime because $K[X,Y,Z]/(X,Z) \cong K[Y]$ is an integral domain.

If $x = 0$, then $yz = 0$. If $z = 0$, then see above. $y = 0$ gives the z-axis $V_2 = Z(x, y)$

If $Z = 1$, then $X^2 - Y = 0$, so $X^2 = Y$. We have $V_3 = Z(X^2 - Y, Z - 1)$. This is also irreducible because $K[X,Y,Z]/(X^2 - Y, Z - 1) \cong K[X,Y]/(X^2 - Y) \cong K[X, X^2] = K[X]$.

If $Z \neq 1$, then $X(Z - 1) = 0$ gives $X = 0$.

We will find the irreducible components by investigating cases.

1. Consider the subset $V_1 = \{\, (x,y,z) \in V \mid z = 1 \,\}$.

(a) If $Z = 1$, then the polynomials reduce to

$$X^2 - YZ \Rightarrow X^2 - Y \quad \text{and} \quad XZ - X \Rightarrow X - X \Rightarrow 0.$$

(b) Thus, V_1 is the zero-locus of the ideal $(X^2 - Y, Z - 1)$.

(c) This ideal $(X^2 - Y, Z - 1)$ is prime because

$$K[X, Y, Z]/(X^2 - Y, Z - 1) \cong K[X, Y]/(X^2 - Y) \cong K[X, X^2] = K[X]$$

is an integral domain.

(d) Hence V_1 is irreducible.

2. Consider the subset $V_1 = \{ (x, y, z) \in V \mid z \neq 1 \}$.

(a) If $Z \neq 1$, then we have for the second polynomial

$$XZ - X = 0 \Rightarrow X(Z - 1) = 0 \Rightarrow X = 0$$

(b) The first polynomial gives

$$YZ = 0$$

Exercise 1.1.6. *If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.*

Solution. Consider $V = Z(X^2 - Y)$.

1. We get the two projections

$$p_1(V) = \mathbb{A}^1 \quad \text{and} \quad p_2(V) = [0, \infty).$$

2. In $\mathbb{A}^1$, the only closed sets are finite sets and $\mathbb{A}^1$ itself.

3. Thus $p_2(V) = [0, \infty)$ is not closed.

Exercise 1.1.7 (1.5.)**.** *Show that k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.*

Solution. • B isomorphic to some coordinate ring

• B finitely generated with no nilpotent elements

Let B be a K-algebra that is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$ for some n. We show that B is finitely generated with no nilpotent elements.

1. I mean, any coordinate ring is finitely generated by $1, X_1, X_2, \ldots, X_n$.

2. Isn't it just if an element is nilpotent, it is in the nilradical, thus in any prime ideal and this prime ideal lies in the ideal?

3. So let $A/I(V)$ be a coordinate ring and assume $x \in A/I(V)$ be nilpotent.

4. So $x^n \in I(V)$.

Let B be a finitely generated K-algebra with no nilpotent elements. We show that B is isomorphic tp the affine coordinate ring of some algebraic set in $\mathbb{A}^n$.

1. Firstly, let $1, b_1, \ldots, b_n$ generate B.

2. Consider the map

$$\begin{aligned} \varphi : K[X_1, \ldots, X_n] &\to B \\ b_i &\mapsto X_i \end{aligned}$$

3. We now need the kernel.

$$\ker(\varphi) = \{\, f \in K[X_1, \ldots, X_n] \mid \varphi(f) = 0 \,\}$$

Exercise 1.1.8 (1.6.). *Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure cl(Y) is also irreducible.*

Solution. Let X be a irreducible topological space and let $\mathcal{U}$ be an open subset of X. We show $\mathcal{U}$ is dense.

1. Assume $\mathcal{U}$ does not lie dense in X, i.e. $\mathrm{cl}(\mathcal{U}) \neq X$. In other words, there is a point $p \in X$ with $p \notin \mathrm{cl}(\mathcal{U})$.

2. If all open neighborhoods $\mathcal{U}_p$ of p intersects $\mathcal{U}$, i.e. $\mathcal{U} \cap \mathcal{V}_p \neq \varnothing$, then by definition, p must lie in the closure $\mathrm{cl}(\mathcal{U})$. Hence let $\mathcal{V}_p$ be an open neighborhood of p with $\mathcal{U} \cap \mathcal{V}_p = \varnothing$.

3. This contradicts the irreduciblility of X hence $\mathcal{U}$ lies dense in X.

We show $\mathcal{U}$ is irreducible.

1. Assume $\mathcal{U}$ is not irreducible, i.e. we find two open sets $\mathcal{V}_1$ and $\mathcal{V}_2$ with $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{U}$.

2. I mean this is obvious.

Exercise 1.1.9 (1.7.). *Show that the following conditions are equivalent for a topological space X.*

1. *X is Noetherian.*

2. *Every nonempty family of closed subset has a minimal element.*

3. *X satisfies the ascending chain condition for open subset.*

4. *Every nonempty family of open subsets has a maximal element.*

Solution. This is simple.

Exercise 1.1.10 (1.7. (b))**.** *A Noetherian topological space is quasi-compact, i.e. every open cover has a finite subcover.*

Solution. This is also just simple topology.

Exercise 1.1.11 (1.8.)**.** *Let V be an affine variety of dimension d in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume $V \not\subset H$. Then every irreducible component of $V \cap H$ has dimension $d-1$.*

Solution. I need to understand hypersurfaces first.

1. Consider any chain of $Z_0 \subset Z_1 \subset \cdots \subset Z_k$ distinct irreducible closed subsets in V. By definition, we have $k \leq d$ for all chains.

2. Consider now the chain

$$Z_0 \cap H \subset Z_1 \cap H \subset \cdots \subset Z_k \cap H$$

Exercise 1.1.12. *Let $\mathfrak{a} \subset A = K[X_1, \ldots, X_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n-r$.*

Solution. 1. Let $\mathcal{U}$ be an irreducible component of $Z(\mathfrak{a})$.

2. $\mathcal{U}$ is closed subset.

3. $\mathcal{U}$ is thus an algebraic set.

4. Since is irreducible, $\mathcal{U}$ has a prime ideal.

5. We have

$$height \mathfrak{p} + dim K[X_1, \ldots, X_n]/\mathfrak{p} = n$$

6. $dim K[X_1, \ldots, X_n]/\mathfrak{p}$ this dimension of the coordinate ring corresponds to the dimension of the irreducible component.

7. Yeah, obviously because the height of $\mathfrak{p}$ is r or smaller.

Chapter 2

Projective Varieties

2.1 Full Text

2.1.1 Projective Space

Definition 2.1.1. Projective Space, points, set of homogeneous coordinates

Definition 2.1.2 (Graded Ring)**.** 1. A graded ring is a ring R that is decomposed into a direct sum

$$R = \bigoplus_{k=0}^{\infty} R_k = R \oplus R_1 \oplus R_2 \oplus \cdots$$

of additive groups such that $R_m R_n \subset R_{m+n}$.

2. An element of R_k is called a homogenous element of degree k.

3. An ideal $\mathfrak{a} \subset R$ is a homogenous ideal if $\mathfrak{a} = \bigoplus_{k \geq 0}(\mathfrak{a} \cap R_k)$.

Example 2.1.3. Consider $\mathbb{P}^3$. Let $T = (X - 1)$.

2.2 To Affine Space

If $f \in R$ is a linear homogeneous polynomial, then the zero set of f is called a hyperplane. In particular, we denote the zero set of X_i by H_i for $0 \leq i \leq n$. Let U_i be the open set $\mathbb{P}^n - H_i$. Then $\mathbb{P}^n$ is covered by the open sets U_i, because if $p = (a_0, \ldots, a_n)$ is a point, then at least one $a_i \neq 0$, hence $p \in U_i$.

2.3 Exercises

Exercise 2.3.1 (2.1.)**.**

Exercise 2.3.2 (2.2.)**.** *Let $\mathfrak{a}$ be a homogeneous ideal in a graded ring R. Then the following are equivalent.*

1. $Z(\mathfrak{a}) = \varnothing$.
2. $\sqrt{\mathfrak{a}} = R$ *or*

Solution. Let $Z(\mathfrak{a}) = \varnothing$.

1. By definition,
$$Z(\mathfrak{a}) = \{\, p \in \mathbb{P}^n \mid f(p) = 0 \text{ for all } f \in \mathfrak{a} \,\}$$
2. Let $f \in \bigoplus_{k=1}^{\infty} R_k$.

Chapter 3

Morphisms

Definition 3.0.1. Let X be a quasi-affine variety in $\mathbb{A}^n_K$ and $f : X \to K$ a function.

1. f is regular at a point $p \in X$ if there is an open neighborhood $\mathcal{U} \subset X$ of p, and polynomials $g, h \in K[X_1, \ldots, X_n]$, such that $h(x) \neq 0$ for all $x \in \mathcal{U}$, and $f = g/h$ on $\mathcal{U}$.

2. f is regular on X if it is regular at every point on X.

Lemma 3.0.2. *A regular function is continuous, when K is identified with $\mathbb{A}^1_K$ in its Zariski topology.*

Definition 3.0.3 (Germ)**.** Given a point p of a topological space X, and two maps $f, g : X \to Y$ where Y is any set, then f and g define the same germ at p if there is a neighbourhood $\mathcal{U}$ of p such that restricted to $\mathcal{U}$, f and g are equal, i.e.

$$f(x) = g(x) \text{ for all } u \in \mathcal{U}.$$

Definition 3.0.4. Let X be a variety.

1. We denote the ring of all regular functions on X by $\mathcal{O}(X)$.

2. If p is a point on X, we define the local ring of p on X, $\mathcal{O}_p$ to be the ring of germs of regular functions on X near p. In other words, an element of $\mathcal{O}_p$ is a pair $(\mathcal{U}, f)$ where $\mathcal{U}$ is an open subset of X containing p, and f is a regular function on $\mathcal{U}$, and where we identify two such pairs $(\mathcal{U}, f)$ and $(\mathcal{V}, g)$ if $f = g$ on $\mathcal{U} \cap \mathcal{V}$.

Theorem 3.0.5. *Let $X \subset \mathbb{A}^n$ be an affine variety.*

1. *The ring of all regular functions on X is isomorphic to the coordinate ring of X, i.e.*

$$\mathcal{O}(X) \cong A(X).$$

2. *There is a one-to-one correspondence between the points of X and the maximal ideals of $A(Y)$.*

3. *The localization of the ring of all regular functions at $p \in X$*

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.