Chapter 1

Noetherian Rings

Cheat Sheet

Definition 1. A ring R is called Noetherian if it fulfills one of the following equivalent conditions.

1. Each ideal $\mathfrak{a}$ of R is finitely generated.
2. Any infinite increasing sequence of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$ in R eventually stabilized, i.e. $\mathfrak{a}_k = \mathfrak{a}_{k+1}$ for k large enough.
3. Every nonempty collection S of ideals of R contains a maximal element with respect to inclusion.

Theorem 2. Let R be a Noetherian ring. Each surjective ring homomorphism $\varphi : R \longrightarrow R$ is injective, and thus an isomorphism.

Theorem 3. If R is a Noetherian integral domain, then every nonzero nonunit in R can be factored into irreducibles.

Theorem 4.

1. R/I is Noetherian.
2. R Noetherian $\iff$ polynomial ring with finite variables is Noetherian $\iff$ formal power series with finite variables is Noetherian
3. localization is Noetherian

Theorem 5. In a nonzero Noetherian ring, each proper ideal is an intersection of finitely many primary ideals.

1.1 Introduction

Definition 6. A ring R is called Noetherian if it fulfills one of the following equivalent conditions.

1. Each ideal $\mathfrak{a}$ of R is finitely generated.
2. Any infinite increasing sequence of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$ in R eventually stabilized, i.e. $\mathfrak{a}_k = \mathfrak{a}_{k+1}$ for k large enough.
3. Every nonempty collection S of ideals of R contains a maximal element with respect to inclusion.

Example 6.1. PIDs and thus fields are Noetherian rings.

Intuition. Noetherian rings can be regarded as a good generalization of PIDs: the property of all ideals being singly generated is often not preserved under common ring-theoretic constructions ($\mathbb{Z}$ is a PID but $\mathbb{Z}[X]$ is not), but the property of all ideals being finitely generated does remain valid under many constructions of new rings from old rings. For example, every quadratic ring $\mathbb{Z}[\sqrt{d}]$ is Noetherian; many of these rings are not PIDs.

Example 6.2. The polynomial ring $K[X_1, X_2, \ldots]$ in infinitely many variables over a field K is not Noetherian.

Remark. The third condition shows a Noetherian ring R other than the zero ring has a maximal ideal and every proper ideal $\mathfrak{a}$ in a Noetherian ring R is contained in a maximal ideal. This does not need Zorn's lemma, which is used to show maximal ideals exist in arbitary rings.

Theorem 7. Let R be a Noetherian ring. Each surjective ring homomorphism $\varphi : R \longrightarrow R$ is injective, and thus an isomorphism.

Proof. Let $\varphi : R \longrightarrow R$ be surjective.

The trick is to see that kernel of φ is an ideal and that a n-fold composition of φ give exactly the ideals we want to study.

We have the following sequence of increasing ideals

$$\ker(\varphi) \subset \ker(\varphi^2) \subset \ker(\varphi^3) \subset \cdots$$

This is valid because: Let $x \in \ker(\varphi^n)$, then $\varphi^n(x) = 0$ and that means $\varphi^{n+1}(x) = \varphi^n(\varphi(x)) = \varphi^n(0) = 0$, thus $x \in \ker(\varphi^{n+1})$.

Since R is Noetherian, this sequence of ideals becomes stationary, hence $\ker(\varphi^n) = \ker(\varphi^{n+1})$ for n large enough.

Let $x \in \ker(\varphi)$, then $\varphi(x) = 0$. Since φ is surjective, φ^n is surjective, so $x = \varphi^n(x')$ for some $x' \in R$.

That means $0 = \varphi(x) = \varphi(\varphi^n(x')) = \varphi^{n+1}(x')$.

So $x' \in \ker(\varphi^{n+1}) = \ker(\varphi^n)$. Thus $x = \varphi^n(x') = 0$. So $\ker(\varphi) = \{0\}$. $\square$

Example 7.1. $\mathbb{R}[X]$ is a Noetherian ring and the map $f : \mathbb{R}[X] \longrightarrow \mathbb{R}[X], f(X) \mapsto f(X^2)$ is an injective ring homomorphism that is not surjective.

Theorem 8. If R is a Noetherian integral domain, then every nonzero nonunit in R can be factored into irreducibles.

Remark.

1. If R is a field, then there are no nonzero units and the theorem is vacuously true.
2. This theorem is not saying that a Noetherian integral domain has a unique factorization. Many Noetherian integral domains do not have unique factorization.
3. If an integral domain R contains a nonzero nonunit x that has no irreducible factorization, then R can't be Noetherian.

1.2 Ring Operations

Which operations keep Noetherian?

1. subring
2. quotient
3. polynomial
4. fractions
5. direct product the ideals in $A \times B$ corresponds to ideals in A and B if the ring is infinite
6.

Example 8.1. *A subring of a Noetherian ring need not be Noetherian.* Take any non-Noetherian ring. It is a subring of its field of fractions which is trivially Noetherian.

Example 8.2. 1. The polynomial ring $K[X_1, X_2, \ldots]$ with infinitely many variables is not Noetherian, but $\mathrm{Frac}(K[X_1, X_2, \ldots])$ is Noetherian.

2. $K[X, Y]$ is Noetherian, but the subring $\{ XY^k \mid k \in \mathbb{N}^+ \}$ is not.

Theorem 9. If R is a Noetherian ring, then so is R/I for any ideal I in R.

Proof. There are at least two ways.
First is direct by using the fact that $\pi^{-1}(\mathfrak{a})$ is an ideal in R that is finitely generated.
The other is using exact sequences. □

Theorem 10. R is a Noetherian ring if and only if $R[X]$ is Noetherian if and only if $R[X_1, X_2]$ is Noetherian.

Example 10.1. $\mathbb{R}[X_1, X_2, \ldots]$ is not Noetherian.

Theorem 11. For a ring A its polynomial ring $A[X_1, X_2, \ldots]$ is not Noetherian.

Proof. The chain of ideals $(X_1) \subset (X_1, X_2) \subset \cdots$ never becomes stationary. □

Theorem 12. If R is Noetherian, then so is the formal power series $R[[X]]$ in (finitely many variables)

What about the converse? What about infinite number of variables?

Proof. □

Theorem 13. If A is Noetherian, so is its ring of fractions $S^{-1}A$.

Proof. 1. Let A be a Noetherian ring.

2. Let S be a multiplicative set in A.
3. Let $S^{-1}A$ be the localization of A at S with $\tau : A \longrightarrow S^{-1}A$ the canonical homomorphism.
4. Fix an ideal $\mathfrak{a}$ in $S^{-1}A$. *(We will show $\mathfrak{a}$ is finitely generated.)*
5. By ideal correspodence, $\tau^{-1}(\mathfrak{a})$ is an ideal in A.
6. Since A is Noetherian, $\tau^{-1}(\mathfrak{a})$ is finitely generated.
7. Let $x_1, \ldots x_n \in A$ generate $\tau^{-1}(\mathfrak{a})$.
8. Let $\frac{x}{s} \in \mathfrak{a}$.

□

What about the converse?

Example 13.1. The quadratic ring $\mathbb{Z}[\sqrt{d}]$ for a nonsquare integer $d \in \mathbb{Z}$ is Noetherian.

Proof. 1. It is $\mathbb{Z}[\sqrt{d}] \cong \mathbb{Z}[X]/(X^2 - d)$.

2. $\mathbb{Z}[X]$ is Noetherian.
3. Thus $\mathbb{Z}[X]/(X^2 - d)$ is Noetherian.
4. By the isomorphism, $\mathbb{Z}[\sqrt{d}]$ is Noetherian.

□

1.2.1 Corollaries

Theorem 14. Let $\varphi : R \longrightarrow R'$ be a ring homomorphism. If R is Noetherian, then so is $\mathrm{im}(\varphi)$.

Proof.

1. Let R be a Noetherian ring.
2. Let $\varphi : R \longrightarrow R'$ be a ring homomorphism.
3. By the isomorphism theorem, we have the isomorphism $R/\ker(\varphi) \cong \mathrm{im}(\varphi)$.
4. Since R is Noetherian, $R/\ker(\varphi)$ is as well.
5. By the isomorphism given in (3), $\mathrm{im}(\varphi)$ is Noetherian.

$\square$

1.3 Primary Decomposition

1.3.1 Primary Ideals

Definition 15. A proper ideal $\mathfrak{q}$ of a ring A is called primary if one of the following equivalent conditions are fulfilled.

1. For all $ab \in \mathfrak{q}$, it is either $a \in \mathfrak{q}$ or $b^n \in \mathfrak{q}$ for some $\mathbb{N}^+$.
2. The zero divisors in $A/\mathfrak{q}$ is nilpotent.

Example 15.1. In the ring of integers $\mathbb{Z}$, the primary ideals are 0 and $(p)^k$ for prime numbers p and $k \in \mathbb{N}^+$.
More concretely, consider (6). This is not a primary ideal because $6 = 2 \cdot 3$, but 6 neither contains 2, 3 nor the powers of them.
On the other hand, $(2)^3 = (8)$ is primary. For example, the decomposition $8 = 2 \cdot 4$
Concretely, consider $(2)^3 = (8)$.
On the other hand, (6) is not primary because $6 = 2 \cdot 3$, but

Theorem 16. Let $\mathfrak{q}$ be an ideal in $\mathbb{Z}$. Then $\mathfrak{q}$ is primary if and only if $\mathfrak{q} = (0)$ or $\mathfrak{q} = (p^n)$ for some prime number p and some $\mathbb{N}^+$.

Proof. "$\Rightarrow$":

1. Let $\mathfrak{q}$ be a primary ideal.
2. Fix $x \in \mathfrak{q}$.
3. $x = p_1^{k_1} \cdots p_n^{k_n}$.
4. By induction, we must find $p_i^{k_i} \in \mathfrak{q}$
5. Thus $\mathfrak{q} = (p_i^n)$

"$\Leftarrow$": trivial $\square$

Example 16.1. Let K be a field and $A = K[X, Y, Z]/(XY - Z^2)$. $\mathfrak{a} = (X, Z)$.
$(X, Z)^2$ is not primary in A
$(X, Z)^2/(XY - Z^2) \cong (X^2, XZ, Z^2)/(XY - Z^2)$
$XZ \in (X, Z)^2$

Theorem 17. Let $\mathfrak{q}$ be a primary ideal. Then $\sqrt{\mathfrak{q}}$ is a prime ideal in R. It is the unique smallest prime ideal containing $\mathfrak{q}$.

Theorem 18. All prime ideals are primary.

Example 18.1. Clearly, the converse is false.

Intuition. We can think of primary ideals as generalization of prime powers.

Example 18.2. Power of prime ideal need not be a primary ideal.

Example 18.3. *$\mathfrak{m}$-primary ideals need not be powers of $\mathfrak{m}$.*
Consider the ring $A = K[X, Y]$ and the ideal $\mathfrak{q} = (X, Y^2)$.
X, Y^2 is primary.

1.3.2 Primary Decomposition

Definition 19. An ideal $\mathfrak{a}$ is called reducible if $\mathfrak{a} = \mathfrak{b}_1 \cap \mathfrak{b}_2$ for ideals $\mathfrak{b}$ and $\mathfrak{b}'$ strictly containing $\mathfrak{a}$. An irreducible ideal is proper and not reducible.

Example 19.1. $(6) \cap (9) = (18)$

Theorem 20. In a nonzero Noetherian ring, each proper ideal is an intersection of finitely many irreducible ideals.

Theorem 21. In a nonzero Noetherian ring, each irreducible ideal is a primary ideal.

What about the converse?

Theorem 22. In a nonzero Noetherian ring, each proper ideal is an intersection of finitely many primary ideals.

What about the converse?
Steps to prove the result above: 1.

Example 22.1. Consider $K[X, Y, Z]$. Ideals (X, Y), (X, Z), and (X, Y, Z).

$$(X, Y) \cdot (X, Z) = (X, Y) \cap (X, Z) \cap (X, Y, Z)^2 \tag{1.1}$$

Theorem 23. Let $\mathfrak{q}$ be an primary ideal and $\mathfrak{p} = \sqrt{\mathfrak{q}}$ be the corresponding prime ideal. Then for any $x \in R$

1. $(\mathfrak{q} : x) = R$ if $x \in \mathfrak{q}$. I think this is true for any $\mathfrak{q}$.

Proof. 1.

$\square$

1.4 Noetherian Modules

Example 23.1. Let $R = \mathbb{R}[X_1, X_2, \ldots]$. Let $I = (X_1, X_2, \ldots)$ be an ideal. I is the set of polynomials with constant term 0. I is not finitely generated.

Proof. Let $\mathfrak{a}$ □

Definition 24. An R-module M is called

1. Every ascending chain

$$M_1 \subset M_2 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M$$

of submodules in M becomes stationary, i.e. there is an index $i_0 \in \mathbb{N}$ such that $M_{i_0} = M_i$ for all $i \geq i_0$.

2. All submodules of M are finitely generated.

Theorem 25. If M is a Noetherian R-module then every submodule of M is Noetherian.

Proof. This follows immediately from the definition of Noetherian modules, because a submodule of a submodule is again a submodule.

1. Let M be a Noetherian R-module.
2. Fix a submodule N of M.
3. Fix a submodule L of N.
4. Since L is also a submodule of M and M is Noetherian, L is finitely generated.
5. Thus all submodules of N are finitely generated and N is Noetherian.

□

Theorem 26. If M is a Noetherian R-module then every quotient module M/N is Noetherian.

Theorem 27. Let M be an R-module and N be a submodule. Then M is Noetherian if and only if N and M/N are Noetherian.

Proof. "⇒": see above
"⇐":

1. Let M be an R-module.
2. Let N be a Noetherian submodule in M such that M/N is Noetherian.
3. Fix a submodule L of M. *(We will show that L is finitely generated.)*
4. The image $\pi(L)$ in M/N is finitely generated because $\pi(L)$ is a submodule in M/N and M/N is Noetherian.
5. Let $x_1, \ldots, x_n \in L$ such that $\pi(x_1), \ldots \pi(x_n)$ generate $\pi(L)$.
6. The intersection $L \cap N$ is finitely generated because $L \cap N$ is a submodule in N and N is Noetherian.
7. Let $y_1, \ldots, y_m \in L$ generate $L \cap N$.
8. Fix an element $x \in L$. *(We will show that x is a linear combination of geneators in L.)*
9. We have $\pi(x) = r_1\pi(x_1) + \cdots + r_n\pi(x_n) = \sum_{i=1}^n r_i\pi(x_i)$ for some $r_1, \ldots, r_n \in R$.
10. $\pi(x) - \sum_{i=1}^n r_i\pi(x_i) = 0$ thus $x - \sum_{i=1}^n r_i x_i$ lies in N.
11. Since $x \in L$ and $\sum_{i=1}^n r_i x_i \in L$, their difference $x - \sum_{i=1}^n r_i x_i$ lies also in L.
12. Hence $x - \sum_{i=1}^n r_i x_i \in L \cap N$.
13. Therefore $x - \sum_{i=1}^n r_i x_i = \sum_{i=1}^m s_i y_i$.
14. Thus $x = \sum_{i=1}^n r_i x_i + \sum_{i=1}^m s_i y_i$

□

1.5 More Noether

Theorem 28. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Then M is Noetherian if and only if M' and M'' are Noetherian.

Proof. "$\Rightarrow$":

1. Let M be Noetherian.
2. By exactness, $M \to M''$ is surjective.
3. Thus M'' is Noetherian.
4. By exactness, $M' \to M$ is injective.
5. Thus M' can be regarded as a submodule of M.
6. Hence M' is Noetherian.

"$\Leftarrow$":

1. Let M' be Noetherian.
2. Let M'' be Noetherian.
3. Fix a submodule N of M.
4. The image $g(L)$ is finitely generated.
5. The intersection $L \cap f(M')$ is finitely generated.
6. Fix an element $x \in L$.
7. $g(x) = r_1 g(x_1) + \cdots + r_n g(x_n)$
8.

□

Theorem 29. Let $M_1, \ldots, M_n$ be Noetherian A-modules. Then their direct sum $\bigoplus_{i=1}^n M_i$ is Noetherian.

Proof. It is enough to show $M_1 \oplus M_2$ is Noetherian. We have the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0 \tag{1.2}$$

□