Exercise 6.1

Consider $K := \mathbb{Q}(\sqrt{-10})$.

1. Show that $(2) = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ and find the generators of $\mathfrak{p}$ explicitly.

   Proof. Because $-10 \equiv 2 \mod 4$, the ring of integer of K is $\mathcal{O}_K = \mathbb{Z}[\sqrt{-10}]$. The minimal polynomial of $\sqrt{-10}$ is $X^2 + 10$ and we have

   $$X^2 + 10 \equiv X^2 \mod 2.$$

   Thus, $(2) = (2, \sqrt{-10})^2$. $\square$

2. Prove that the ideal $\mathfrak{p}$ you just found is prime, but not principal. Deduce the order of $[\mathfrak{p}] \in \mathrm{Cl}(K)$ is 2.

   Proof. $(2, \sqrt{-10})$ being prime arises from the theorem that gives the method of computation. Now assume $(2, \sqrt{-10})$ is principal, then there is an $\alpha \in \mathcal{O}_K$ that divides 2. Using the multiplicativity of the norm gives $N(\alpha)$ divides $N(2) = 4$, so $N(\alpha) = 2$, but this is impossible. So 2 is irreducible in $\mathcal{O}_K$ and clearly $\sqrt{-10}$ is not a multiple of 2. Hence the generators do not share a divisor and $(2, \sqrt{-10})$ is principal. Moreover, because $(2, \sqrt{-10})^2 = (2)$ is principal (and all principal ideals are equivalent to (1)), the order of $(2, \sqrt{-10})$ is 2. $\square$

3. Prove that $(3) \subset \mathcal{O}_K$ is prime. Using Minkowski's bound, deduce that $Cl(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

   Proof. Similarly as in 1., we have

   $$X^2 + 10 \equiv X^2 + 1 \mod 3$$

   which is irreducible in $\mathbb{Z}/3\mathbb{Z}$, so (3) is prime. The Minkowski's bound for K is

   $$M_K = \sqrt{|D_K|}\left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} = \sqrt{40}\frac{4}{\pi}\frac{2}{4} = \frac{4\sqrt{10}}{\pi} = 4.03$$

   $\square$