Definition 0.1 — .
An ideal q in A is primary if $q \neq A$ and if

$$xy \in q \Rightarrow \text{ either } x \in q \text{ or } y^n \in q \text{ for some } n > 0 \tag{1}$$

Proposition 1. *q is primary $\iff$ $A/q \neq 0$ and every zero-divisor in A/q is nilpotent*

Proof. Let q be a primary ideal.

1. Let $x \in A/q$ be a zero-divisor, then there is a $y \in A/q$ such that $(x+q)(y+q) = xy + q = \overline{0}$.
2. So $xy \in q$ and by definition, we either have $x \in q$ or $y^n \in q$ for some $n > 0$.
3. $x \in q$ and $y \in q$ cannot be, because we required x to be a zero-divisor in A/q.
4. The only other option is $x^n = 0$ for some $n > 0$.
5. Hence, x is nilpotent in A/q.

□

Proposition 2. *Every prime ideal is primary.*

Proposition 3. *Contraction of primary ideals are primary.*

Proposition 4. *Let q be a primary ideal in a ring A. Then $\sqrt{q}$ is the smallest prime ideal containing q.*

Proof. The nilradical of A is the intersection of all the prime ideals of A. □

Theorem 0.2 (First Uniqueness Theorem)**.** *Let $\mathfrak{a}$ be a decomposable ideal and let $\mathfrak{a} = \cap_{i=1}^{n} q_i$ be a minimal primary decomposition of $\mathfrak{a}$. Let $\mathfrak{p}_i = \sqrt{q_i}$. Then the $\mathfrak{p}_i$ are precisely the prime ideals which occur in the set of ideals $\sqrt{(\mathfrak{a} : x)}$ with $x \in R$, and hence are independent of the particular decomposition of $\mathfrak{a}$.*

Exercise 0.3. 4.2

Proof.

1. If $\mathfrak{a}$ is not decomposable, then the statement is vacuously true, so let $\mathfrak{a}$ be decomposable, i.e.

$$\mathfrak{a} = \bigcap_{i=1}^{n} q_i \tag{2}$$

We have

$$\mathfrak{a} = \sqrt{\mathfrak{a}} = \tag{3}$$

2.

□