

Topology

K

December 20, 2022

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Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring). A ring is a set A equipped with two binary operations $+$ (addition) and \cdot (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. (A, \cdot) is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
 - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
 - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Definition 2 (Ideal).

Definition 3 (Ideal Operation). Let \mathfrak{a} and \mathfrak{b} be ideals of a ring A .

1. The sum of two ideals \mathfrak{a} and \mathfrak{b} is defined by

$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \}$$

which is again an ideal. It is the smallest ideal in A that contains \mathfrak{a} and \mathfrak{b} .

2. The product of an ideal
3. The intersection of
4. The radical of an ideal \mathfrak{a} is defined by

$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$

which is again an ideal.

5. The transporter

Proposition 4. Let \mathfrak{a} be an ideal.

1. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.

Definition 5 (Nilpotent Element and Nilradical). An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.

The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\text{Nil}(A)$.

1.2 Exercises and Notes

Example 5.1. Let K be a field and $A = K[X, Y]/(X - XY^2, Y^3)$.

1. Compute the nilradical $\text{Nil}(A)$.

Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

$$\begin{aligned}
 X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\
 &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\
 &= XY \cdot Y^3 + \mathfrak{a} \\
 &= XY \cdot 0 + \mathfrak{a} \\
 &= 0 + \mathfrak{a}.
 \end{aligned}$$

Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism $K[X, Y]/(X - XY^2, Y^3) \simeq K[Y]/(Y^3)$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

Clearly, $Y \in \text{Nil}(A)$ or in other words $(Y) \subset \text{Nil}(A)$. But we also have that $K[Y]/(Y) = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \text{Nil}(A)$ conclude $\text{Nil}(A) = (Y)$. \square