Exercise 0.1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. 1. Let $S^{-1}M = 0$, then for all $s \in S$ and for all $m \in M$ we have that

$$\frac{m}{s} = 0, \tag{1}$$

or in other words, $(m, s) \equiv (0, s')$ for some $s' \in S$. By definition, there exists a $t \in S$ such that

$$\begin{aligned} t(s \cdot 0 - s'm) = 0 &\iff t(0 - s'm) = 0 && (2)\\ &\iff ts'm = 0. && (3) \end{aligned}$$

Choose ts' to be the factor and we get $sM = 0$.

2. If there is an $s \in S$ such that $sM = 0$, then we can write for all $m \in M$ that $s \cdot m = 0$. We have

$$0 = s \cdot m = s(1 \cdot m - 1 \cdot 0) \tag{4}$$

which means again $(m, 1) \equiv (0, 1)$, hence $S^{-1}M = 0$.

□

Exercise 0.2. 3.2

Hints

1. To show that $x \in S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, use Proposition 1.9., i.e.

$$x \in \operatorname{Jac}(A) \iff 1 - xy \in A^\times \text{ for all } y \in A. \tag{5}$$

2. If $s \in S$, then $f(s)$ is a unit in $S^{-1}A$.

Proof. We want to show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Let $x \in S^{-1}\mathfrak{a}$. From Proposition 1.9. we have that $x \in \operatorname{Jac}(S^{-1}A)$ if and only if $1 - xy \in (S^{-1}A)^\times$ for all $y \in S^{-1}A$. Because $S = 1 + \mathfrak{a}$, we can write

$$x = \frac{a_1}{1 + a_2} \tag{6}$$

for some $a_1, a_2 \in \mathfrak{a}$. With this, it is

$$1 - xy = 1 - \frac{a_1}{1 + a_2} y = \frac{1 + a_2}{1 + a_2} - \frac{a_1 y}{1 + a_2} = \frac{1 + a_2 - a_1 y}{1 + a_2}. \tag{7}$$

Now, $a_2 - a_1 y$ is contained in $\mathfrak{a}$, hence the whole numerator $1 + a_2 - a_1 y$ is contained in S, and in turn, this means the whole expression is a unit in $S^{-1}A$.

We want to give an alternative proof to Corollary 2.5. If $\mathfrak{a}M = M$, then $(S^{-1}\mathfrak{a})(S^{-1}M) = (S^{-1}M)$ because modules of fractions are determined uniquely up to isomorphisms. From above, we have that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. By Nakayama's lemma, we get $S^{-1}M = 0$. Now applying Exercise 3.1 yields that there is a $s \in S$ such that $sM = 0$. Since $S = 1 + \mathfrak{a}$, it is clearly $s \equiv 1 \mod \mathfrak{a}$. □