

Topology

K

May 12, 2022

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Chapter 1

Topological Space

1.1 Definitions and Theorems

Definition 1 (Topological Space). A **topological space** is an **ordered pair** (X, τ) , where X is a **set** and τ is a **collection of subsets** that satisfies the following **axioms**.

1. The **empty set** \emptyset and the **entire set** X belongs to τ .
2. Any **arbitrary union** of members of τ belongs to τ .
3. The **intersection** of **finite number** of members of τ belongs to τ .

The **collection** τ is called a **topology** on X and the **elements** of τ are called **open sets**. A **subset** $A \subset X$ is said to be **closed** if its **complement** $X \setminus A$ is **open**.

Definition 2 (Continuous Maps). Let (X, τ_X) and (Y, τ_Y) be **topological spaces**. A **map** $f : X \rightarrow Y$ is said to be **continuous** if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \quad (1.1)$$

Lemma 3. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \rightarrow Y$ is ϵ - δ -continuous if and only if f is continuous.

Definition 4 (Homeomorphism). Let X and Y be **topological spaces**.

1. A **map** $f : X \rightarrow Y$ is a **homeomorphism** if it has the following properties.
 - (a) f is **bijective**.
 - (b) f and the **inverse map** f^{-1} is **continuous**.
2. Two topological spaces X and Y are said to be **homeomorphic** if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. If $Y = X$ we also write $\text{Homeo}(X)$.

Definition 5 (Homeomorphism). Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a **basis** of the topology, if any member of \mathcal{O} is the **union of subsets** from \mathcal{B} .
2. $\mathcal{S} \subset \mathcal{O}$ is a **subbasis** of the topology, if any member of \mathcal{O} is the **union of finite intersections of subsets** from \mathcal{S} .

We say that \mathcal{B} and \mathcal{S} **generates** \mathcal{O} and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 6. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a **collection of subsets**, then there **exists exactly one** topology $\tau \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \tau$
2. If $\tau' \subset \mathcal{P}(X)$ a topology with $\mathcal{S} \subset \tau'$, then $\tau \subset \tau'$.

Definition 7. 1. Given (X, τ) be a **topological space**, $S \subset X$ a subset, the **subspace topology** (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \}.$$

2. Let (X, τ_X) and (Y, τ_Y) be two **topological spaces**. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

3. Let (X, τ_X) and (Y, τ_Y) be two **topological spaces**. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

Definition 8. Let (X, τ) be a topological space.

1. Given a **point** $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in \tau$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U .
2. Let $S \subset X$ be a subset. The interior of S , denoted by $\overset{\circ}{S}$ or $\text{int}(S)$, is the **set** of all interior points of S .
3. Let $S \subset X$ be a subset. The closure of S , denoted by \overline{S} or $\text{cl}(S)$, is defined by

$$\text{cl}(S) := X \setminus \text{int}(X \setminus S).$$

1.2 Proofs, Remarks, and Examples

Example 8.1. Let X be a [set](#).

1. $\tau = \mathcal{P}(X)$ is called the [discrete topology](#). In this case, (X, τ) is called the [discrete space](#). It is the [finest topology](#) that can be defined on a set. (The set of all possible topologies on a given set forms a partially ordered set.)
2. $\tau = \{\emptyset, \mathcal{P}(X)\}$ is called the [trivial topology](#).
3. Let (X, d) be a [metric space](#). Set

$$\tau_d := \{U \in \mathcal{P}(X) \mid U \text{ is an open subset in the metric space } (X, d)\}. \quad (1.2)$$

Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U .

Here, τ is a topology. In other words, a metric induces a topology.

(Proof as homework.)

4. The Zariski-topology.

Example 8.2. List of natural topologies.

1. On \mathbb{R}^n the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of \mathbb{R}^n are arbitrary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{x \in \mathbb{R}^n \mid d(p, x) < r\}.$$

This definition agrees with the topology endowed on arbitrary metric spaces.

Remark. The set of all homeomorphisms of X to itself $\text{Homeo}(X)$ is a group with composition as its operation.

Remark. This lemma does not hold for basis.

Remark. 1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. ¹ Let (X, \mathcal{O}) be a [topological space](#). A [subset](#) that is **both** [open](#) and [closed](#) is called [clopen](#). Moreover, a subset is [clopen](#) if and only if its [boundary](#) is [empty](#).

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\text{cl}(A) = A$, but on the other hand, A is open, so we also have $\text{int}(A) = A$. Then, the boundary of A is $\partial A = \text{cl}(A) \setminus \text{int}(A) = A \setminus A = \emptyset$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. \square

¹The following is a definition and a small proposition.

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 9. A topological space (X, \mathcal{O}) is said to be **connected**, if one of the following **equivalent** conditions is met.

1. X is **not** a **union** of two **nonempty**, **disjoint**, and **open subsets**, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$.
2. The **only subsets** of X that are **both open** and **closed** (**clopen**) are the empty set \emptyset and the entire set X , i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \emptyset$ or $A = X$.
3. The **only subsets** of X with empty **boundary** are the emptyset \emptyset and the entire set X .
4. All **continuous** maps from X to the two point space $\{0, 1\}$ endowed with the **discrete** topology is **constant**.

A **subset** of X is **connected** if it is a **connected space** when viewed as a **subspace** of X .

Lemma 10. Any **interval** $I \subset \mathbb{R}$ is **connected**.

Lemma 11. Let X and Y be **topological spaces** and $f : X \rightarrow Y$ a **continuous function**. If X is **connected**, then $f(X) \subset Y$ is **connected**.

Definition 12. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 13. Connected components are closed subsets.

Lemma 14 (Lemma 11). Let X be connected and $f : X \rightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$, then f is constant.

Definition 15. X is said to be **path connected**, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 16. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 17. A topological space (X, \mathcal{O}) is said to be **connected**, if one of the following **equivalent** conditions is met.

1. X is **not** a **union** of two **nonempty**, **disjoint**, and **open** subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** **open** and **closed** (**clopen**) are the empty set \emptyset and the entire set X , i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \emptyset$ or $A = X$.
3. The **only** subsets of X with empty **boundary** are the emptyset \emptyset and the entire set X .
4. All **continuous** maps from X to the two point space $\{0, 1\}$ endowed with the **discrete** topology is **constant**.

A **subset** of X is **connected** if it is a **connected** space when viewed as a **subspace** of X .

Proof. We verify the equivalence of the different definitions. So, let (X, \mathcal{O}) be a topological space.

- “1. \Rightarrow 2.”: Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X , then $X \setminus A$ is also not the empty set nor X . Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \emptyset$ or $A = X$.
- “2. \Rightarrow 1.”: Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.
- “2. \iff 3.”: This is one of the properties of clopen subsets and was proven in remark XXX.
- “1. \Rightarrow 4.”: Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \rightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.
- “4. \Rightarrow 1.”: Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \rightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

□

Lemma 18. Any **interval** $I \subset \mathbb{R}$ is **connected**.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \quad (2.1)$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \text{cl}(B)$ and since the complement of B is an open subset A , so $B = \text{cl}(B)$. It is therefore $s \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. \square

Lemma 19. Let X and Y be **topological spaces** and $f : X \rightarrow Y$ a **continuous function**. If X is **connected**, then $f(X) \subset Y$ is **connected**.

Example 19.1. The general linear group $\text{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Remark. Let $f : X \rightarrow Y$ be continuous and X be connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$ so $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$, so $A = \emptyset$ or $B = \emptyset$ and we are done. \square

Proof. \square

Example 19.2. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proof. Locally constant implies continuous with regards to the discrete topology on Y . Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \emptyset$. Conclude f is identical to $f(x)$. \square

Application: $f : X \rightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \rightarrow X$ be continuous path with $\gamma(0) = a$ and $\gamma(1) = b$. We have that γ^{-1} \square

Remark. The converse statement is not true in general.

Example 19.3. $X = \{ (x, \sin(\frac{1}{x})) \mid x > 0 \} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected but not path connected.

Proof. Homework \square

Remark. missing

2.3 Exercises and Notes

Remark. If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are two connected topological spaces, then their product $X \times Y$ with the product topology $\mathcal{O}_{X \times Y}$ is also connected.

Proof. We will use the definition that all continuous maps from $X \times Y$ to $\{0, 1\}$ endowed with the discrete topology must be constant. Fix a continuous $f : X \times Y \rightarrow \{0, 1\}$.

First, consider the image $f(\{x\} \times Y)$ with $x \in X$. Assume f is not constant on $\{x\} \times Y$, then $f(\{x\} \times Y) = \{0, 1\}$. So we have the preimages $f^{-1}(\{0\}) = \{x\} \times U$ and $f^{-1}(\{1\}) = \{x\} \times V$ with $U, V \subset Y$, $U, V \neq \emptyset$, and $U \cap V = \emptyset$. Because f is continuous, U and V must also be open. This would however mean that $U \sqcup V = Y$ and Y would not be connected, therefore, f is constant on $\{x\} \times Y$. Similarly, we get that f is constant on $X \times \{y\}$ for all $y \in Y$.

Let $(x, y) \in X \times Y$ and $(x', y') \in X \times Y$ be two arbitrary points. We have $f(x, y) = f(x, y')$ because f is constant on $\{x\} \times Y$ and similarly $f(x, y') = f(x', y')$ because f is constant on $X \times \{y'\}$. Putting everything together, it is $f(x, y) = f(x', y')$, therefore all continuous $f : X \times Y \rightarrow \{0, 1\}$ are constant. \square

Example 19.4. Clearly, the union of two connected sets need not be connected. Take for example $[0, 1] \subset \mathbb{R}$ and $[2, 3] \subset \mathbb{R}$. Their union $[0, 1] \cup [2, 3]$ is not connected.

Set difference of connected sets are also not necessarily connected, e.g. $[0, 2] \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ are connected, but $[0, 2] \setminus \{1\} = [0, 1] \cup (1, 2]$ is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and another unit circle around $(1, 0)$ $A := \{(x, y) \mid (x - 1)^2 + y^2 = 1\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3} \right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3} \right) \right\}$$

which is not connected.

Example 19.5. Connectedness does not imply path-connectedness. Let \mathbb{R}^2 be endowed with the Euclidean topology and consider

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \cup (\{0\} \times [-1, 1]) \subset \mathbb{R}^2.$$

and see figure XXX. X is connected, but it is not path-connected.

Proof. Denote

$$A := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \quad B := \{0\} \times [-1, 1],$$

then $X = A \sqcup B$.

1. First, define $f : \mathbb{R}^+ \rightarrow \mathbb{R}^2$ as

$$f(x) := \left(x, \sin\left(\frac{1}{x}\right) \right).$$

f is continuous, \mathbb{R}^+ is an interval, therefore connected, so $f(\mathbb{R}^+) = A$ is connected. On the other hand, $\{0\}$ and $[-1, 1]$ are connected and so is their product B .

Assume there is a clopen subset $S \subset X$ that is not empty. Without loss of generality, we have that $(0, 0) \in U$ (otherwise, consider the complement of U which also must be clopen). Since A is clopen in A , the intersection $A \cap U$ must also be clopen in A , but A is connected, so A is contained in U .

Moreover, the closure of A is also contained in U . So there is an $\epsilon > 0$ such that the ball $B(p, \epsilon)$ that contains $(0, 0)$ is in U . I got lazy to go into the details, but this ball contains a point of B . Follow the same reason as above.

- 2.

\square

Chapter 3

Separation Axioms

Literature: Groessere Liste in Sten, Seibeck

Definition 20 (T_1 Space). Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 21. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X , i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 22 (T_2 Space). Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \emptyset$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 23. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x .
3. The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed as a subset of the product space $X \times X$.

Proposition 24. T_2 spaces are also T_1 spaces.

Chapter 4

Compact Spaces

Definition 25. 1. A topological space X is called compact if each of its open cover has a finite subcover.

2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X .

Theorem 26. Satz 17

Theorem 27. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 28. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 29. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \rightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Lemma 30. $[0, 1] \subset \mathbb{R}$ is compact.