Exercise Sheet 1

Exercise 1

A polynomial $f(X) \in \mathbb{Z}[X]$ is primitive if the greatest common divisor of its coefficients is 1. Show the following:

1. If $f(X), g(X) \in \mathbb{Z}[X]$ are primitive, then the product $f(X)g(X)$ is also primitive.

Solution

1.

Denote the coefficients of f and g with a_i and b_j for $1 \leq i \leq \deg f$ and $1 \leq j \leq \deg g$ such that

$$f(X) = \sum_{i=0}^{\deg f} a_i X^i \qquad\qquad g(X) = \sum_{j=0}^{\deg g} b_j X^j \tag{1}$$

Assume there is a prime $x \in \mathbb{Z}$ that divides all coefficients of fg and let a_n and b_m be the first coefficients in f and g respectively that are not divisble by x.
Consider X^{n+m} in the polynomial fg. The coefficient for this term is the sum of products of a_i and b_j for which $i + j = n + m$, i.e.

$$a_n b_m + a_{n-1} b_{m+1} + a_{n+1} b_{m-1} + a_{n-2} b_{m+2} + \ldots. \tag{2}$$

This coefficient is however not divisible by x as x divides all but the first term. Hence we have a contradiction.

2.