Chapter 1

Interpolation

Chapter 2

Finite Element

Definition 1. Let

1. $K \subset \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary (the element domain),
2. $\mathcal{P}$ be a finite-dimensional space of function on K (the space of shape functions), and
3. $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$ be a basis for $\mathcal{P}'$ (the set of nodal variables).

Then $(K, \mathcal{P}, \mathcal{L})$ is called a finite element.

Definition 2. Let $(K, \mathcal{P}, \mathcal{L})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{L}$ (i.e. $N_i(\phi_j) = \delta_{ij}$) is called the nodal basis of $\mathcal{P}$.

Lemma 2.1. Let $\mathcal{P}$ be a k-dimensional vector space and let $\{L_1, L_2, \ldots, L_k\}$ be a subset of the dual space $\mathcal{P}'$. Then the following two statements are equivalent.

1. $\{L_1, L_2, \ldots, L_k\}$ is a basis for $\mathcal{P}'$.
2. Given $p \in \mathcal{P}$ with $L_i(p) = 0$ for all $i \in \{1, 2, \ldots, k\}$, then $p = 0$.

Example 2.1. Let $K = [0, 1]$, $\mathcal{P}$ be the set of linear polynomials and $\mathcal{L} = \{L_1, L_2\}$ where $L_1(p) = p(0)$ and $L_2(p) = p(1)$ for all $p \in \mathcal{L}$. Then $(K, \mathcal{P}, \mathcal{L})$ is a finite element.

First proof by verifying linearity. Just check

$$\lambda_1 L_1 + \lambda_2 L_2 = 0$$

□

Second proof by construction of a nodal basis. We construct the nodal basis $\{\phi_1, \phi_2\}$ of $\mathcal{P}$ explicitly. Since ϕ_j must fulfill $L_i(\phi_j) = \delta_{ij}$, we have

$$\begin{aligned}
L_1(\phi_1) = 1 &\iff a_1 \cdot 0 + b_1 = 1 \\
&\iff b_1 = 1 \\
L_2(\phi_1) = 0 &\iff a_1 \cdot 1 + b_1 = 0 \\
&\iff a_1 = -1 \\
L_1(\phi_2) = 0 &\iff a_2 \cdot 0 + b_2 = 0 \\
&\iff b_2 = 0 \\
L_2(\phi_2) = 1 &\iff a_2 \cdot 1 + b_2 = 1 \\
&\iff a_2 = 1
\end{aligned}$$

Set $\phi_1(x) = -x + 1$ and $\phi_2(x) = x$, then $\{\phi_1, \phi_2\}$ is a nodal basis of $\mathcal{P}$ and $(K, \mathcal{P}, \mathcal{L})$ is a finite element. □

Third proof with lemma. Denote $p \in \mathcal{P}$ with $L_1(p) = L_2(p) = 0$ as $p(x) = ax + b$. We have $L_1(p) = b$ and $L_2(p) = a + b$, so $a = b = 0$ and $p = 0$. By lemma XXX $\{L_1, L_2\}$ is a basis for $\mathcal{P}'$. □

We can generalize the previous example.

Example 2.2. Let $K = [a, b]$, $\mathcal{P}_k$ be the set of all polynomials of degree less than or equal to k, and $\mathcal{L} = \{L_0, L_1, \ldots, L_k\}$ where

$$L_i(p) = p\left(a + \frac{(b-a)i}{k}\right) \qquad \text{for all } p \in \mathcal{P}_k \text{ and } i \in \{0, 1, \ldots, k\}. \tag{2.1}$$

Then $K, \mathcal{P}_k, \mathcal{N}_k$ is a finite element.

Proof. Let $p \in \mathcal{P}_k$ with $L_1(p) = L_2(p) = \cdots = L_k(p) = 0$. So p has $k+1$ roots but is only of degree k, therefore p must be the zero-polynomial. □

Example 2.3 (Counter Example to Nonconform P_2-FE)**.** Denote

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \qquad P_3 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \tag{2.2}$$

and write $p(x) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$. Then, for L_1 we have

$$\begin{aligned}
L_1(p) &= p(Q_1) && (2.3)\\
&= p(\mu P_2 + (1-\mu) P_3) && (2.4)\\
&= \mu p(P_2) + p(P_3) - \mu p(P_3) && (2.5)\\
&= \mu \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right) \\
&+ \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right) && (2.6)\\
&- \mu \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right)
\end{aligned}$$

$$\mu \tag{2.7}$$

Example 2.4. Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_k$ and $\mathcal{N}$ denote point evaluations at $\{(t_i, t_j) \mid 0 \leq i, j \leq k\}$ where $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element.