

Problem 01.2

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and (M, \mathcal{F}) a measurable space. Moreover, let $X : \Omega \rightarrow M$ a $(\mathcal{A}, \mathcal{F})$ -measurable random variable. Show that

$$\mathbb{P}^X(B) := \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B)), \quad B \in \mathcal{F} \quad (1)$$

defines a probability measure on (M, \mathcal{F}) .

Solution

1. We have

$$\mathbb{P}^X(M) \stackrel{\text{def.}}{=} \mathbb{P}(X \in M) \quad (2)$$

$$\stackrel{\text{def.}}{=} \mathbb{P}(\{\omega \in M \mid X(\omega) \in M\}) \quad (3)$$

$$= \mathbb{P}(\{\omega \in M\}) \quad (4)$$

$$= \mathbb{P}(M) \quad (5)$$

$$\stackrel{\text{def.}}{=} 1. \quad (6)$$

In (4), we used that the codomain of X is M and in the last step, we used the normed property of the probability measure \mathbb{P} .

2. Let $A_i \in \mathcal{F}$ with $i \in \mathbb{N}$ disjoint subsets. We have

$$\mathbb{P}^X\left(\bigcup_{i=1}^{\infty} A_i\right) \stackrel{\text{def.}}{=} \mathbb{P}\left(X \in \bigcup_{i=1}^{\infty} A_i\right) \quad (7)$$

$$\stackrel{\text{def.}}{=} \mathbb{P}\left(\left\{\omega \in M \mid X(\omega) \in \bigcup_{i=1}^{\infty} A_i\right\}\right). \quad (8)$$

As A_i are disjoint, $X(\omega)$ is included in one and only one A_i . Therefore with the σ -additivity of \mathbb{P} , we have

$$= \mathbb{P}\left(\bigcup_{i=1}^{\infty} \{\omega \in M \mid X(\omega) \in A_i\}\right) \quad (9)$$

$$\stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}\{\omega \in M \mid X(\omega) \in A_i\} \quad (10)$$

$$\stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}(X \in A_i) \quad (11)$$

$$\stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}^X(A_i). \quad (12)$$

In short, \mathbb{P}^X is σ -additive.

From above, it follows that \mathbb{P}^X is a probability measure.

Problem 01.3 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $n \in \mathbb{N}$ and $A_k \in \mathcal{A}$ for all $k \in \{1, \dots, n\}$. Prove the following formula.

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n \left((-1)^{k-1} \sum_{I \subset \{1, \dots, n\}, |I|=k} \mathbb{P}\left(\bigcap_{i \in I} A_i\right) \right) \quad (13)$$

Solution Let $n = 1$, then we simply have

$$\mathbb{P}\left(\bigcup_{k=1}^1 A_k\right) = \mathbb{P}(A_1) = \sum_{k=1}^1 \left((-1)^{k-1} \sum_{I \subset \{1\}, |I|=k} \mathbb{P}\left(\bigcap_{i \in I} A_i\right) \right). \quad (14)$$

Now consider the statement for the case $n + 1$.

$$\mathbb{P}\left(\bigcup_{k=1}^{n+1} A_k\right) = \mathbb{P}\left(A_{n+1} \cup \left(\bigcup_{k=1}^n A_k\right)\right) \quad (15)$$

$$= \mathbb{P}(A_{n+1}) + \mathbb{P}\left(\bigcup_{k=1}^n A_k\right) - \mathbb{P}\left(A_{n+1} \cap \left(\bigcup_{k=1}^n A_k\right)\right) \quad (16)$$

$$(17)$$

The following must hold

$$\mathbb{P}(A_n) - \mathbb{P}\left(A_{n+1} \cap \left(\bigcup_{k=1}^n A_k\right)\right) = (-1)^n \sum_{I \subset \{1, \dots, n+1\}, |I|=n+1} \mathbb{P}\left(\bigcap_{i \in I} A_i\right) \quad (18)$$

$$= (-1)^n \mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right) \quad (19)$$