Chapter 1

Commutative Rings

Exercise 1.1. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals in A, and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of B. Prove the following statements.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

   Proof. We show $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$. Let $x \in (\mathfrak{a}_1 + \mathfrak{b}_2)^e$, then we have for some index set I

$$x = \sum_{i \in I} \lambda_i x_i, \tag{1.1}$$

   where $\lambda_i \in B$ and $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$ for all $i \in I$. For each $i \in I$ it is $x_i = \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2})$, hence

$$\begin{aligned}
x &= \sum_{i \in I} \lambda_i \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2}) && (1.2)\\
&= \sum_{i \in I} \lambda_i \left(\varphi(\mu_{i,1} a_{i,1}) + \varphi(\mu_{i,2} a_{i,2}) \right) && \text{(by linearity)} \quad (1.3)\\
&= \sum_{i \in I} \lambda_i \left(\mu_{i,1} \varphi(a_{i,1}) + \mu_{i,2} \varphi(a_{i,2}) \right) && \text{(by linearity)} \quad (1.4)\\
&= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \lambda_i \mu_{i,2} \varphi(a_{i,2}) && \text{(by distributivity)} \quad (1.5)\\
&= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) && \text{(reordering the sum).} \quad (1.6)\\
& && (1.7)
\end{aligned}$$

   The last term is exactly the elements expressed by $\mathfrak{a}_1^e + \mathfrak{a}_2^e$, therefore, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

   I think the above proof should work into both directions. □