1. We want to find the decomposition of $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$.
2. It is $\sqrt{\mathfrak{q}_i} = \sqrt{(I : x)}$ for some $x \in R$.
3. What we know though is if $x \notin \sqrt{I}$ then $(I : x) = I$ which doesn't help.
4.

Example 0.1. $I = (X^2Y, XY^2)$.

1. $\sqrt{(X^2Y, XY^2)} = (XY, X^2, Y^2)$
2. $X^2Y^2 - XY^2 = (X^2 - X)Y^2$

Theorem 1. In a Noetherian ring, each ideal has a minimal primary decomposition.

Every irreducible ideal is primary.

1.

Theorem 2. Let

$$I = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

Then $\sqrt{\mathfrak{q}_i} = (I : x)$ for some x

$I = (X^2Y, XY^2)$

1. $(X^2Y, XY^2) = (X^2, XY^2) \cap (Y, XY^2) = (X^2, XY^2) \cap (Y)$
2. $(X^2, XY^2) \cap (Y) = (X^2, X) \cap (X^2, Y^2) \cap (Y) = (X) \cap (X^2, Y^2) \cap (Y)$

Theorem 3.

Proof.
1. Let $\{u_1, \dots, u_r\}$ generate I.
2. If u_1 is not a pure power, we can write $u_1 = vw$ where v and w are coprime monomials.
3. We claim: $I = (v, u_2, \dots, u_r) \cap (w, u_2, \dots, u_r)$.

□

The associated primes are $\{(X), (Y)\}$. The embedded primes are $\{(X, Y)\}$

1. $(X^2, XY, XZ) = (X^2, XY, XZ) \cap (X^2, XY, Z) = (X) \cap (X^2, X, Z) \cap (X^2, Y, Z)$

$(X^3Y, XY^4) = (X^3, XY^4) \cap (Y, XY^4) = (X) \cap (X^3, Y^4) \cap (Y, X) \cap (Y) = (X) \cap (X^3, Y^4) \cap (Y)$
$(X^2Z, YZ, Z - XY)$

Exercise 3.1. If an ideal $\mathfrak{a}$ has a primary decomposition, then $\operatorname{Spec}(A/\mathfrak{a})$ has only finitely many irreducible components.

Solution. Let $\mathfrak{a}$ be a decomposable ideal in a ring A. Denote the set of prime ideals that contain $\mathfrak{a}$ by

$$V(\mathfrak{a}) := \{\, \mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subset \mathfrak{p} \,\} \subset \operatorname{Spec}(A)$$

and denote the canonical projection to the quotient ring by $\pi : A \longrightarrow A/\mathfrak{a}$.

1. Since $\mathfrak{a}$ is decomposable, by proposition 4.6., the isolated primes of $\mathfrak{a}$ are precisely the minimal elemenets of $S_{\mathfrak{a}}$.
2. The prime ideals of the quotient ring $A/\mathfrak{a}$ are precisely the images of prime ideals that contain the ideal $\mathfrak{a}$ of A under the canonical projection, i.e. if $\overline{\mathfrak{p}} \subset A/\mathfrak{a}$ is prime, then $\mathfrak{p} = \pi^{-1}(\overline{\mathfrak{p}})$ is prime in A and contains $\mathfrak{a}$, and if $\mathfrak{p} \subset A$ is a prime ideal that contains $\mathfrak{a}$, then $\overline{\mathfrak{p}} = \pi(\mathfrak{p})$ is prime in $A/\mathfrak{a}$.

   Or state it differently, there is a one-to-one correspodence between the sets

   $$V(\mathfrak{a}) = \{\, \mathfrak{p} \subset \operatorname{Spec}(A) \mid \mathfrak{a} \subset \mathfrak{p} \,\} \longleftrightarrow \operatorname{Spec}(A/\mathfrak{a}).$$

   (a) By proposition 1.1., the ideals in $A/\mathfrak{a}$ corresponds to an ideal in A that contains $\mathfrak{a}$, i.e. if $\overline{\mathfrak{b}} \in A/\mathfrak{a}$ is an ideal, then $\mathfrak{b} = \pi^{-1}(\overline{\mathfrak{b}})$ is an ideal in A that contains $\mathfrak{a}$, and if $\mathfrak{b} \in A$ is an ideal that contains $\mathfrak{a}$, then $\overline{\mathfrak{b}} = \pi(\mathfrak{b})$ is an ideal in $A/\mathfrak{a}$.
   (b) If $\overline{\mathfrak{p}} \in A/\mathfrak{a}$ is a prime ideal, then $\mathfrak{p} = \pi^{-1}(\overline{\mathfrak{p}})$ is a prime ideal in A because preimages preserves prime ideals.
   (c) If $\mathfrak{p} \subset A$ is a prime ideal, then $\overline{\mathfrak{p}} = \pi(\mathfrak{p})$ is a prime ideal in $A/\mathfrak{a}$.
      i. Let $\overline{x} \cdot \overline{y} \in \overline{\mathfrak{p}}$.
      ii. Since π is surjective, there are x and y in A such that $\pi(x) = \overline{x}$ and $\pi(y) = \overline{y}$.
      iii. Moreover, since π is a ring homomorphism, we have $\overline{x} \cdot \overline{y} = \pi(x) \cdot \pi(y) = \pi(xy)$.
      iv. Since $\overline{x} \cdot \overline{y} \in \overline{\mathfrak{p}}$ and since
3. Thus, by 1. and 2., the isolated primes of $\mathfrak{a}$ corresponds to minimal elements of $\operatorname{Spec}(A/\mathfrak{a})$. Since the number of associated primes is finite, the number of isolated primes and hence the number of minimal elements of $\operatorname{Spec}(A/\mathfrak{a})$ must be finite as well.
4. Bla bla bla, finite minimal elements, finite irreducible components.

□

Exercise 3.2 (Atiyah & MacDonald 4.2., Bosch 2.1.)**.** If $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then $\mathfrak{a}$ has no embedded prime ideals.

Solution. Let $\mathfrak{a}$ be an ideal in a ring A.

1. If $\mathfrak{a}$ is not decomposable, then the statement is trivially[1] true. Thus, consider an ideal $\mathfrak{a}$ that is decomposable and denote one of its minimal primary decomposition by

   $$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

2. Taking the radical on both sides yields

   $$\begin{aligned} \sqrt{\mathfrak{a}} &= \sqrt{\bigcap_{i=1}^{n} \mathfrak{q}_i} \\ &= \bigcap_{i=1}^{n} \sqrt{\mathfrak{q}_i} \qquad \text{bla bla} \end{aligned}$$

   Since $\mathfrak{a} = \sqrt{\mathfrak{a}}$, the last expression gives

   $$\mathfrak{a} = \bigcap_{i=1}^{n} \sqrt{\mathfrak{q}_i}$$

   which is a primary decomposition of $\mathfrak{a}$.

3. Assume $\sqrt{\mathfrak{q}_i} \subset \sqrt{\mathfrak{q}_j}$ for some $j \neq i$. In that case, we have a primary decomposition

   $$\mathfrak{a} = \bigcap_{\substack{i=1 \\ i \neq j}}^{n} \sqrt{\mathfrak{q}_i}$$

   which has less primary components than the thing above which is a contradiction.

   I think I can write this much better.

□

Additional Bosch: Is the converse true?

Exercise 3.3 (Atiyah & MacDonald 4.3)**.** If A is absolutely flat, every prime ideal is maximal.

Hints.
- Exercise 2.28 is crucial.

□

Solution. Let A be an absolutely flat ring and fix a prime ideal $\mathfrak{p}$ in A. Our goal is to show that $A/\mathfrak{p}$ is a field. For that endeavour, fix an element $\overline{x} \in A/\mathfrak{p}$ with $\overline{x} \neq \overline{0}$. We will show $\overline{x}$ is invertible.

1. By exercise 2.28., if A is absolutely flat, then so is $A/\mathfrak{p}$. Furthermore, exercise 2.28. says in any absolutely flat ring, all non-units are zero-divisors.

---

[1] While it is always aluring to use the word "vacuously", I don't think this is a case of a vacuous truth. If $\mathfrak{a}$ is not decomposable, then the set of associated ideals of $\mathfrak{a}$ is empty and thus embedded primes which a subset of associated ideals is also empty. It fulfills the statement by definition and not because there is nothing to check.

2. Suppose there exists a non-zero non-unit $\overline{x}$ in $A/\mathfrak{p}$. By 1., $\overline{x} \in A/\mathfrak{p}$ must be a non-zero zero-divisor. Thus, there is some non-zero $\overline{y} \in A/\mathfrak{p}$ such that $\overline{x} \cdot \overline{y} = \overline{0}$.
3. Now, $\overline{x} \cdot \overline{y} = \overline{0} \in A/\mathfrak{p}$ implies $x \cdot y \in \mathfrak{p}$. By the definition of prime ideals, we hence have $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, but this is equivalent to saying $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$ which were both excluded. We arrived at a contradiction. There are no non-zero non-units in $A/\mathfrak{p}$.

□

Exercise 3.4 (Atiyah & MacDonald 4.4)**.** In the polynomial ring $\mathbb{Z}[X]$, the ideal $\mathfrak{m} = (2, X)$ is maximal and the ideal $\mathfrak{q} = (4, X)$ is $\mathfrak{m}$-primary, but is not the power of $\mathfrak{m}$.

Proof. "The ideal $\mathfrak{m} = (2, X)$ is maximal in the polynomial ring $\mathbb{Z}[X]$."

1. We have the isomorphism $\mathbb{Z}[X]/(2, X) \cong \mathbb{Z}/2\mathbb{Z}$.
   (a) Consider the map
   $$\begin{aligned} \varphi : \mathbb{Z}[X] &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ P(X) &\mapsto P(0) \mod 2\mathbb{Z}. \end{aligned}$$
   (b) The map φ is a homomorphism because it is the composition of substitution homomorphism $\mathbb{Z}[X] \longrightarrow \mathbb{Z}$ and the canonical projection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$.
   (c) Moreover, the map φ is surjective because it is the composition of two surjective maps. Thus, $\operatorname{im}(\varphi) = \mathbb{Z}/2\mathbb{Z}$.
   (d) We also have $\ker(\varphi) = (2, X)$.
      i. Let $P \in \ker(\varphi)$. Then, $\varphi(P) = P(0) \mod 2\mathbb{Z}$.
      ii. zzzzzzzzzzzzzzzzzzzzzzzzz
   (e) Thus, by the isomorphism theorem, we have $\mathbb{Z}[X]/(2, X) \cong \mathbb{Z}/2\mathbb{Z}$.
2. Since $\mathbb{Z}/2\mathbb{Z}$ is a field, $(2, X)$ is a maximal ideal.

"The ideal $\mathfrak{q} = (4, X)$ is $\mathfrak{m}$-primary."

1. We simply have $\sqrt{(4, X)} = \sqrt{\sqrt{4} + \sqrt{X}} = \sqrt{(2) + (X)} = \sqrt{(2, X)}$.
2. Thus, $(4, X)$ is $\mathfrak{m}$-primary.

"The ideal $\mathfrak{m} = (4, X)$ is not a power of $\mathfrak{m}$."

1. We have the chain of strict inclusion
   $$\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^k \supsetneq \cdots$$
2. $\mathfrak{m}^2 = (4, 2X, X^2)$.
3. So $(4, X) \subset (2, X)$ but $(4, X) \not\subset (4, 2X, X^2)$
4. Thus it cannot be a power of $\mathfrak{m}$.

□

Exercise 3.5 (Atiyah & MacDonald 4.5)**.** In the polynomial ring $K[X, Y, Z]$ where K is a field, let $\mathfrak{p}_1 = (X, Y)$, $\mathfrak{p}_2 = (X, Z)$, and $\mathfrak{m} = (X, Y, Z)$. Then, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime and $\mathfrak{m}$ is maximal. Let $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Show that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a reduced primary decomposition of $\mathfrak{a}$. Which components are isolated and which are embedded?

"reduced primary decomposition"

1. $\mathfrak{a} = (X, Y)(X, Z) = (X^2, XY, XZ, YZ)$.
   (a) $X^2 \in (X, Y) \cap (X, Z) \cap (X, Y, Z)^2$
   (b) $XY \in$ bla bla
2. The representation is $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ a primary decomposition because $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, hence primary and $\mathfrak{m}^2$ is $\mathfrak{m}$-primary.
3. Clearly, the radicals of the ideals are distinct.
4. as
   (a) $(X, Y) \not\supset (X, Z) \cap (X, Y, Z)^2 = (X, Z) \cap (X^2, XY, XZ, Y^2, YZ, Z^2) = (X^2, XY, XZ, YZ, Z^2)$
   (b) $(X, Y, Z)^2 \not\supset (X, Z) \cap (X, Y) = (X, YZ)$

Exercise 3.6 (Atiyah & MacDonald 4.7)**.** Let A be a ring and $A[X]$ be a polynomial ring. $\mathfrak{a}[X]$ is the extension of $\mathfrak{a}$ of $\mathfrak{a}$ to $A[X]$.

Proof. □

Exercise 3.7 (Bosch 2.2.)**.** Let $\mathfrak{p} \subset A$ be a prime ideal and assume that A is Noetherian. Show that the n-th power $\mathfrak{p}^n$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$. Can we expect that there is a smallest primary ideal containing $\mathfrak{p}^n$?