

Series 2

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Exercise 2 c)

Solution. 1. \mathcal{B} is a subbasis for the discrete topology. Take an arbitrary subset $\mathcal{U} \subset \mathbb{R}$. If $\mathcal{U} = \mathbb{R}$, then we simply have

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x, x+1\}$$

as $\{x, x+1\}$ are members of the subbasis \mathcal{B} . Similarly, if $\mathcal{U} = \mathbb{R} \setminus \{y\}$ for some $y \in \mathbb{R}$, then we have

$$\mathbb{R} \setminus \{y\} = \left(\bigcup_{\substack{x \in \mathbb{R} \\ x+1 \neq y}} \{x, x+1\} \right) \cup \{y-1, y+1\}$$

because again $\{y-1, y+1\}$ lies in \mathcal{B} . For any other cases, notice that there are two distinct points $y \neq z$ with $y, z \notin \mathcal{U}$, thus the two sets $\{x, y\}$ and $\{x, z\}$ are members of \mathcal{B} . Therefore, we have

$$\begin{aligned} \mathcal{U} &= \bigcup_{x \in \mathcal{U}} \{x\} \\ &= \bigcup_{x \in \mathcal{U}} \{x, y\} \cap \{x, z\}. \end{aligned}$$

In other words, every subset of \mathbb{R} is a union of finite intersections of members in \mathcal{B} , thus \mathcal{B} as a subbasis generates the discrete topology.

2. However, \mathcal{B} is not a basis of the discrete topology. Plainly, a singleton set cannot be generated from a union of elements of \mathcal{B} .

□

Exercise 3 b)

Suppose \mathcal{B} is a subbasis for a topology \mathcal{T} on a set X . Given another topological space Y , show that a map $f : Y \rightarrow X$ is continuous if and only if for every $\mathcal{U} \in \mathcal{B}$, $f^{-1}(\mathcal{U})$ is open in Y .

Lemma. The preimage of a map is stable under arbitrary unions and finite intersections.

Proof. Let $f : X \rightarrow Y$ be a map, $\{A_i\}_{i \in I}$ be a family of subsets in Y , and A and B subsets in Y .

1. It is plainly

$$\begin{aligned}
x \in f^{-1}\left(\bigcup_{i \in I} A_i\right) &\iff f(x) \in \bigcup_{i \in I} A_i \\
&\iff \text{there is a } i \in I \text{ such that } f(x) \in A_i \\
&\iff \text{there is a } i \in I \text{ such that } x \in f^{-1}(A_i) \\
&\iff x \in \bigcup_{i \in I} f^{-1}(A_i).
\end{aligned}$$

2. We simply have

$$\begin{aligned}
x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\
&\iff f(x) \in A \text{ and } f(x) \in B \\
&\iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \\
&\iff x \in f^{-1}(A) \cap f^{-1}(B).
\end{aligned}$$

□

Solution. Denote the topology of Y by \mathcal{S} .

“ \Rightarrow ”: Let $f : Y \rightarrow X$ be continuous and fix an $\mathcal{U} \in \mathcal{B}$. Since \mathcal{B} is subbasis, all its elements are open subsets, thus \mathcal{U} is open. Then by definition of continuous maps, the preimage $f^{-1}(\mathcal{U})$ is also open in Y . As we have fixed an arbitrary $\mathcal{U} \in \mathcal{B}$, we may conclude the desired result.

“ \Leftarrow ”: On the other hand, let for every $\mathcal{U} \in \mathcal{B}$ the preimage $f^{-1}(\mathcal{U})$ be open in Y . Consider an arbitrary open subset $\mathcal{V} \in \mathcal{T}$. By the definition of a subbasis, \mathcal{V} is a union of finite intersection of members of \mathcal{B} , i.e.

$$\mathcal{V} = \bigcup_{\alpha \in I} (\mathcal{U}_1^\alpha \cap \dots \cap \mathcal{U}_{n_\alpha}^\alpha)$$

with I being an arbitrary index set, and $n_\alpha \in \mathbb{N}$ for each $\alpha \in I$. The preimage of \mathcal{V} therefore is

$$\begin{aligned}
f^{-1}(\mathcal{V}) &= f^{-1}\left(\bigcup_{\alpha \in I} (\mathcal{U}_1^\alpha \cap \dots \cap \mathcal{U}_{n_\alpha}^\alpha)\right) \\
&= \bigcup_{\alpha \in I} (f^{-1}(\mathcal{U}_1^\alpha) \cap \dots \cap f^{-1}(\mathcal{U}_{n_\alpha}^\alpha))
\end{aligned}$$

where we applied the aforementioned lemma on the last step. Now, $f^{-1}(\mathcal{U}_i)$ are open subsets for all $i \in \mathbb{N}$. By the definition of topological spaces, unions of finite intersections of open subsets are also open, hence $f^{-1}(\mathcal{V})$ is open. Thus, f is continuous. □

Exercise 3 c)

Now suppose $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ is a collection of topological spaces, (X, \mathcal{T}) is $\prod_{\alpha \in I} X_\alpha$ with the product topology, and the subbase $\mathcal{B} \subset \mathcal{T}$ is taken to consist of all sets of the form

$$\{x_{\alpha \in I} \mid x_\beta \in \mathcal{U}_\beta\} \subset \prod_{\alpha} X_\alpha$$

for arbitrary $\beta \in I$ and $\mathcal{U}_\beta \in \mathcal{T}_\beta$.

Show that a sequence $\{x_\alpha^n\}_{\alpha \in I} \in X$ converges to $\{x_\alpha\}_{\alpha \in I} \in X$ as $n \rightarrow \infty$ if and only if $x_\alpha^n \rightarrow x_\alpha$ for every $\alpha \in I$.

Solution. “ \Rightarrow ”: Let the sequence $\{x_\alpha^n\}_{\alpha \in I} \in X$ converge to $\{x_\alpha\}_{\alpha \in I} \in X$. By exercise 3 a), we have that for all subbase $\mathcal{U} \in \mathcal{B}$ with $\{x_\alpha\}_{\alpha \in I} \in \mathcal{U}$ it is $\{x_\alpha^n\}_{\alpha \in I} \in \mathcal{U}$ for sufficiently large n . The members of the subbase was in the form

$$\{\{x_\alpha\}_{\alpha \in I} \mid x_\beta \in \mathcal{U}_\beta\} \subset \prod_{\alpha \in I} X_\alpha.$$

Thus, for each $\alpha \in I$, we have $x_\alpha^n \rightarrow x_\alpha$.

“ \Leftarrow ”: On the other hand, let x_α^n converge to x_α for every $\alpha \in I$. By definition of convergence, we have that for every neighbourhood \mathcal{V}_α of x_α , it is $x_\alpha^n \in \mathcal{V}_\alpha$ for all sufficiently large n . Denote the open subsets of these neighbourhoods by \mathcal{U}_α . Then, $\prod_{\alpha \in I} \mathcal{U}_\alpha$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in I}$ in the product topology and also contains all $\{x_\alpha^n\}_{\alpha \in I}$ if n is sufficiently large enough. Thus, $\{x_\alpha^n\}_{\alpha \in I}$ converges to $\{x_\alpha\}_{\alpha \in I}$. \square

Exercise 5 b)

Proof. A sequence of functions $f_n : I \rightarrow X$ converges to $f : I \rightarrow X$ if and only if f_n converges pointwise to f on finitely many terms and for any other terms f_n lies in every neighbourhood of f with sufficiently large n .

Proof. “ \Rightarrow ”: Let f_n converge to f in the box topology. If I is finite, then the box topology is just the product topology and we get the desired result. Thus, let I be infinite. Assume the statement to prove is false, i.e. there is a neighbourhood \mathcal{U}_α of $f(\alpha)$ (α is a term that does not converge pointwise) such that $f_n(\alpha) \notin \mathcal{U}_\alpha$.

Construct a sequence $\alpha_n \in I$ such that the neighbourhoods \mathcal{U}_{α_n} of $f(\alpha_n)$ and a subsequence f_{k_n} of f_n such that $f_{k_n}(\alpha_n) \notin \mathcal{U}_{\alpha_n}$ for all $n \in \mathbb{N}$. With this sequence, $\{g \in X^I \mid g(\alpha_n) \in \mathcal{U}_{\alpha_n}\}$ is an open subset. It contains f , but not f_{k_n} which would mean that f_n does not converge to f .

“ \Leftarrow ”: On the other hand, let the converse condition hold. Then, f_n lies in every neighbourhood of f for sufficiently large n , thus f_n converges to f . \square

Exercise 7

Solution. 1. The error is in the following part.

“which means that x_n **cannot enter** arbitrary neighbourhoods of $x \in X$ for arbitrary large values of n , i.e. there exists $N_x \in \mathbb{N}$ and an open neighbourhood of $\mathcal{U}_x \subset X$ of x such that $x_n \notin \mathcal{U}_x$ **for every** $n \geq N_x$ ”

The definition of convergence of a sequence was that for every neighbourhood $\mathcal{U} \subset X$ of x it is $x_n \in \mathcal{U}$ for all $n \in \mathbb{N}$ sufficiently large. Thus, if we negate the definition, we have that for every neighbourhood $\mathcal{U} \subset X$ of x it is $x_n \notin \mathcal{U}$ **for some** $n \in \mathbb{N}$ sufficiently large.

This mistake makes the last conclusion false. The proof says “each of these finitely many subsets contains at most finitely many terms of x_n ”, but in actuality the subsets may contain infinitely many terms of x_n .

2. It suffices to require X to be a first-countable space. Then, going back to the proof, we have for every $x \in X$, no sequence of x_n converges to x . Let $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$ be a neighbourhood base (we may choose such a base without loss of generality as it was done in the proof of Lemma 4.16). Then, each of these neighbourhood bases would only have finitely many terms of x_n and the given proof would work. \square