

# Integration and Integration

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# Introduction

One problem of the Riemann integral is that some functions are not Riemann integratable.

**Example 0.0.1** (Dirichlet function). For  $[a, b] \subset \mathbb{R}$ , define the Dirichlet function as

$$g : [a, b] \rightarrow \mathbb{R}, x \mapsto g(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \quad (1)$$

What are the properties a generalized concept of volumina should have?

1. positive valued
2. null empty set
3. monotonous
4. translationinvariance
5. normalization

**Definition 0.1.** Let  $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow \overline{\mathbb{R}}_0^+$ .

- $\mu$  is monotonous.
- $\mu$  is translationinvariant.
- $\mu$  is  $\sigma$ -additive.

**Theorem 0.1.1** (Vitali's Theorem).



## Part I

# $\sigma$ -algebra and measures

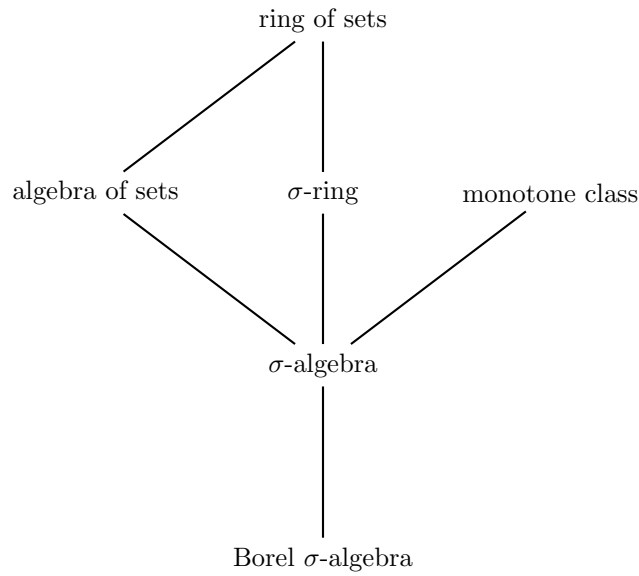




# Chapter 1

## Family of Sets

We have the following tree of inclusion. NOTATION GUIDE:



1.  $X$  as the superset
2.  $\mathcal{P}(X)$  is the power set of  $X$ .
3.  $A, B \in \mathcal{P}(X)$  as subsets
4.  $\mathcal{R}, \mathcal{A} \subset \mathcal{P}(X)$  system of subsets

### 1.1 Symmetric Difference

**Definition 1.1** (Symmetric difference). Let  $A, B$  be sets. The binary set operation symmetric difference is defined as

$$A \triangle B := (A \setminus B) \cup (B \setminus A). \quad (1.1)$$

In other words,  $x \in A \triangle B$  implies  $x$  is either in  $A$  or  $B$ , but not in both.

**Proposition 1.1.1** (Properties of Symmetric Difference). Let  $A, B, C, X$  and  $Y$  be sets. Moreover, let  $A_i$  and  $X_i$  be sets with an arbitrary non-empty index set  $i \in I$ . Then, the following identities hold.

1.  $A \triangle B = (A \cup B) \setminus (A \cap B)$ .
2.  $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ . (Symmetric difference is **associative**.)
3.  $A \triangle B = B \triangle A$ . (Symmetric difference is **commutative**.)
4.  $A \triangle \emptyset = A$  and  $A \triangle A = \emptyset$
5.  $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$ .
6.  $A \cap B = \emptyset \Rightarrow A \triangle B = A \cup B$ .
7.  $B \subset A \Rightarrow A \triangle B = A \setminus B$ .
8.  $X \cap Y = \emptyset \Rightarrow A \cap B \subset (X \triangle A) \cup (Y \triangle B)$ .
9.  $(\bigcup_{i \in I} X_i) \triangle (\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} (X_i \triangle A_i)$

*Proof.* Elementary. □

## 1.2 Ring of Sets

**Definition 1.2** (Ring of sets). There are two equivalent definitions. Let  $X$  be a set and  $\mathcal{R} \subset \mathcal{P}(X)$  a system of subsets. Then  $\mathcal{R}$  is a **ring of sets over  $X$** , if

1. the following axioms are met.
  - (a)  $\mathcal{R} \neq \emptyset$  ( $\mathcal{R}$  is **nonempty**.)
  - (b)  $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$  ( $\mathcal{R}$  is **closed under relative complement**.)
  - (c)  $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$  ( $\mathcal{R}$  is **closed under finite unions**.)
2.  $(\mathcal{R}, \triangle, \cap)$  is a ring in the algebraic sense, with  $\triangle$  as addition and  $\cap$  as multiplication.

*Proof.* We show that the two definitions above are indeed equivalent.

(1  $\Rightarrow$  2) Let  $\mathcal{R}$  be nonempty, closed under the relative complement, and closed under finite unions. First, consider  $(\mathcal{R}, \triangle)$ . Let  $A, B \in \mathcal{R}$ . It is

1. (Closure under addition)  $A \cup B \in \mathcal{R}$  because  $\mathcal{R}$  is closed under finite unions. We also have  $A \cap B = A \setminus (A \setminus B) \in \mathcal{R}$  as  $\mathcal{R}$  is closed under the relative complement. From these it follows that  $A \triangle B = (A \cup B) \setminus (A \cap B) \in \mathcal{R}$  by using the closure under the relative complement again.
2. (Associativity)
3. (Commutativity)
4. (Neutral element)  $\emptyset$
5. (Inverse element)  $A$

Therefore,  $(\mathcal{R}, \triangle)$  is an abelian group. Secondly, consider  $(\mathcal{R}, \cap)$ .  $\cap$  is associative and commutative. The identity element is the union of all sets (does this exist??). □

**Remark.** Since we have the identity  $A \cap B = A \setminus (A \setminus B)$ , the condition that  $\mathcal{R}$  is closed under the relative complement, i.e.

$$A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R} \tag{1.2}$$

can be replaced with closure under finite intersection, therefore

$$A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}. \tag{1.3}$$

**Example 1.2.1.** Let  $X$  be a set.

1.  $\mathcal{P}(X)$  and  $\{\emptyset, X\}$  are ring of sets.
2.  $\{\emptyset\}$  is a ring of sets.

## 1.3 Algebra of Sets

**Definition 1.3** (Algebra of sets). There are two equivalent definitions. Let  $X$  be a set and  $\mathcal{R} \subset \mathcal{P}(X)$  a system of subsets. Then  $\mathcal{A}$  is an algebra of sets over  $X$ ,

1. if  $\mathcal{A}$  is a ring of sets that contains  $X$ , or
2. if the following axioms are met
  - (a)  $\mathcal{A} \neq \emptyset$  ( $\mathcal{A}$  is nonempty.)
  - (b)  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$  ( $\mathcal{R}$  is closed under the absolute complement.)
  - (c)  $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$  ( $\mathcal{R}$  is closed under finite unions.)

## 1.4 $\sigma$ -Ring

**Definition 1.4** ( $\sigma$ -Ring). Let  $X$  be set and  $\mathcal{R} \subset \mathcal{P}(X)$  a system of subsets.  $\mathcal{R}$  is a  $\sigma$ -ring over  $X$ , if

1.  $\mathcal{R} \neq \emptyset$ . ( $\mathcal{A}$  is nonempty.)
2.  $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$  (closed under the relative complement.)
3.  $A_1, A_2, A_3, \dots \in \mathcal{R} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$  (Closed under countable unions.)

## 1.5 Monotone Class

**Definition 1.5** (Notation for Monotonous Sequence of Sets).

**Definition 1.6** (Monotone class). Let  $\mathcal{M} \subset \mathcal{P}(\Omega)$  a system of sets and  $k \in \mathbb{N}^*$ . Then,  $\mathcal{M}$  is a monotone class, if

1. Let  $X_k \in \mathcal{M}$  with  $X_k \uparrow X$ , then  $X \in \mathcal{M}$ .
2. Let  $Y_k \in \mathcal{M}$  with  $Y_k \downarrow X$ , then  $X \in \mathcal{M}$ .

Intersection of arbitrary many monotonous class is again a monotonous class. Therefore, for all  $\mathcal{E} \subset \mathcal{P}(\Omega)$  with  $\mathcal{E} \neq \emptyset$  there exists the smallest monotonous class around  $\mathcal{E}$

$$\mathcal{M}_{\mathcal{E}} := \bigcap_{\mathcal{M} \text{ is monotonous class, } \mathcal{E} \subset \mathcal{M}} \mathcal{M} \quad (1.4)$$

**Remark.** All  $\sigma$ -algebras are monotone class.

**Theorem 1.6.1.** Let  $\mathcal{A} \subset \mathcal{P}(\Omega)$  an algebra of sets. Then, the following are equivalent

- $\mathcal{A}$  is a  $\sigma$ -algebra.
- For  $A_k \uparrow A$ ,  $A \in \mathcal{A}$ .

## 1.6 $\sigma$ -Algebra

### CHEAT SHEET

1. closed under complementation (absolute and relative)
2. closed under countable unions
3. closed under countable intersections
4. closed under symmetric differences

**Definition 1.7** ( $\sigma$ -algebra). Let  $X$  be a set and  $\mathcal{A} \subset \mathcal{P}(X)$  a system of subsets.  $\mathcal{A}$  is a  $\sigma$ -algebra over  $X$ , if

1.  $\mathcal{A} \neq \emptyset$ .
2.  $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3.  $A_1, A_2, A_3, \dots \in \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

**Example 1.7.1.** Trivial examples for the above structures.

**Definition 1.8.** Let  $\mathcal{E} \subset \mathcal{P}(\Omega)$  be a system of sets. Define

$$\mathcal{F}(\mathcal{E}) := \{\mathcal{A} \subset \mathcal{P}(\Omega) \mid \mathcal{E} \subset \mathcal{A}, \mathcal{A} \text{ } \sigma\text{-Algebra}\} \quad (1.5)$$

$$\langle \mathcal{E} \rangle^\sigma := \sigma(\mathcal{E}) := \bigcap_{\mathcal{A} \in \mathcal{F}(\mathcal{E})} \mathcal{A} \quad (1.6)$$

The first is the family of all  $\sigma$ -algebras that contain  $\mathcal{E}$ . The second is the smallest  $\sigma$ -algebra that contains  $\mathcal{E}$ .

## 1.7 Product Algebra??

**Definition 1.9.** Let  $\Omega_1$  and  $\Omega_2$  be sets; let  $\mathcal{R}_1 \subset \mathcal{P}(\Omega_1)$  and  $\mathcal{R}_2 \subset \mathcal{P}(\Omega_2)$  be ring of sets, and  $\Omega := \Omega_1 \times \Omega_2$ . Define

$$\mathcal{R} := \mathcal{R}_1 \boxtimes \mathcal{R}_2 := \left\{ \bigcup_{i=1}^m A_i \times B_i \mid A_i \in \mathcal{R}_1, B_i \in \mathcal{R}_2, m \in \mathbb{N} \right\} \quad (1.7)$$

$\mathcal{R}$  is a ring of sets over  $\Omega$ .

**Theorem 1.9.1.** In above definition, if  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are algebra of sets, then  $\mathcal{R}$  is a algebra of set.

**Theorem 1.9.2.**

$$\mathfrak{Q}(\mathbb{R}^n) \quad (1.8)$$

is a ring of sets.

**Remark.** From  $\mathfrak{Q}(\mathbb{R}^n)$  we can construct one very important  $\sigma$ -algebra, the Borel-Algebra of  $\mathbb{R}^n$ .

**Definition 1.10** (Products of  $\sigma$ -algebras). Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be  $\sigma$ -algebras on  $\Omega_1, \Omega_2$ . Then, let

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \quad (1.9)$$

**Example 1.10.1.**

$$\mathcal{B}(\mathbb{R}^{n+m}) = \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \quad (1.10)$$

**Definition 1.11.** Let  $(X_k)_{k \in \mathbb{N}^*}$  be a sequence of sets with  $X_1 \subset X_2 \subset X_3 \subset \dots$  and  $X := \lim_{k \rightarrow \infty} X_k := \bigcup_{k \in \mathbb{N}^*} X_k$ . Similar for monotonously decreasing.

## 1.8 Rectangles

**Example 1.11.1.** Let

$$\mathfrak{Q}(\mathbb{R}) := \left\{ \bigcup_{i=1}^m [a_i, b_i) \mid m \in \mathbb{N}; a_i, b_i \in \mathbb{R} \right\} \quad (1.11)$$

be the set of all unions of finitely many right half open intervals on  $\mathbb{R}$ . Then,  $\mathfrak{Q}(\mathbb{R})$  is a set of rings. Similar for the left half open sets, but not for open or closed intervals!  $\mathfrak{Q}(\mathbb{R})$  is neither  $\sigma$ -ring,  $\sigma$ -algebra nor an algebra of sets. One can generalize this to higher dimensions.

## 1.9 Borel $\sigma$ -algebra

**Definition 1.12** (Topology). Let  $X$  be a set. A collection  $\mathcal{U} \subset \mathcal{P}(\Omega)$  of subsets of  $X$  is called a topology on  $X$  if it satisfies the following axioms.

1. The empty set and  $X$  itself belong to the topology, i.e.  $\emptyset \in \mathcal{U}$  and  $X \in \mathcal{U}$ .
2. The topology is stable under finite intersections, i.e. if  $n \in \mathbb{N}$  and  $U_1, \dots, U_n \in \mathcal{U}$  then  $\bigcap_{i=1}^n U_i \in \mathcal{U}$ .
3. The topology is stable under arbitrary unions, i.e. if  $I$  is any index set and  $U_i \in \mathcal{U}$  for  $i \in I$  then  $\bigcup_{i \in I} U_i \in \mathcal{U}$ .

A topological space is a pair  $(X, \mathcal{U})$  consisting of a set  $X$  and a topology  $\mathcal{U} \in \mathcal{P}(X)$ .

**Example 1.12.1** (Standard Topology on  $\mathbb{R}$ ). The set of open subsets  $\mathcal{T}$  of  $\mathbb{R}$  is the standard topology on  $\mathbb{R}$ . Concretely,  $\mathcal{T}$  contains countable union of open intervals in  $\mathbb{R}$  and sets of the form  $(a, \infty]$  or  $[-\infty, b)$  for  $a, b \in \mathbb{R}$ .

**Definition 1.13** (Borel algebra). Let  $(\Omega, \mathcal{T})$  be a topological space, then  $\mathcal{B}(\Omega) := \sigma(\mathcal{T})$  is the Borel  $\sigma$ -algebra of  $\Omega$ . The elements of  $\mathcal{B}$  are called Borel (measurable) sets.

**Theorem 1.13.1.** Let  $(\Omega, \mathcal{T})$  be a topological space. Then the following holds.

1. Every closed subset  $F \subset \Omega$  is a Borel set.
2. Every countable union  $\bigcup_{i=1}^{\infty} F_i$  of closed subsets  $F_i \subset \Omega$  is a Borel set.
3. Every countable intersection  $\bigcap_{i=1}^{\infty} F_i$  of open subsets  $F_i \subset \Omega$  is a Borel set.

**Theorem 1.13.2.** It is

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{Q}(\mathbb{R}^n)) \quad (1.12)$$

Moreover, define

$$\mathcal{Q}_{\mathbb{Q}}(\mathbb{R}^n) := \left\{ \bigcup_{i=1}^m [a_{1,i}, b_{1,i}) \times \dots \times [a_{n,i}, b_{n,i}) \mid m \in \mathbb{N}; a_{\nu,i}, b_{\nu,i} \in \mathbb{Q}; \nu = 1, \dots, n \right\} \quad (1.13)$$

the ring of sets of finite unions of quadern with rational edge points. Then, we even have

$$\mathcal{R}(\mathbb{R}^n) = \sigma(\mathcal{Q}_{\mathbb{Q}}(\mathbb{R}^n)) \quad (1.14)$$

**Lemma 1.13.1.** Open subsets  $U \subset \mathbb{R}^n$  are disjoint union of countably many right half open dices with edge points in  $\mathbb{Q}^n$

## 1.10 Exercises

### Exercise 1.1

Let  $X$  be a nonempty set and for all  $1 \leq i \leq m$  with  $m \in \mathbb{N}$  let  $A_i \subset X$  be a finite amount of subsets. Set

$$S := A_1 \triangle A_2 \triangle \dots \triangle A_m. \quad (1.15)$$

Because of the associative property of the symmetric difference,  $S$  is uniquely defined regardless of the order of the operations.

Show that  $x \in X$  belongs to  $S$  if and only if  $x$  belongs to an odd number of sets  $A_k$ , i.e. when the number of indices  $k \in \{1, 2, \dots, m\}$  with  $x \in A_k$  is odd.

**Solution 1.1****Exercise 1.2**

Let  $X$  be a nonempty set and  $R := \{f : X \rightarrow \mathbb{F}_2\}$  where  $\mathbb{F}_2 = \{0, 1\}$  is a field of two elements equipped with the common addition and the common multiplication. Moreover, define the operations

$$(f \oplus g)(x) := f(x) + g(x) \quad (1.16)$$

$$(f \otimes g)(x) := f(x) \cdot g(x). \quad (1.17)$$

Show the following statements.

1.  $(R, \oplus, \otimes)$  is a commutative ring with the identity element.
2. The map  $\mathcal{P}(X) \rightarrow R, A \mapsto \chi_A$  that maps a subset  $A \subset X$  to its characteristic function is bijective.
3. For all  $A, B \in \mathcal{P}(X)$  we have

$$\chi_{A \triangle B} = \chi_A \oplus \chi_B \quad \chi_{A \cap B} = \chi_A \otimes \chi_B. \quad (1.18)$$

4. Conclude from the statements above that  $\mathcal{P}(X)$  is isomorphic to  $R$  as a ring with  $\triangle$  as its addition and with  $\cap$  as its multiplication.
5. A subset  $\mathcal{R} \subset \mathcal{P}(X)$  is a ring of sets if and only if  $\mathcal{R}$  is a subring of  $\mathcal{P}(X)$  with respects to the ring structure defined above.

**Solution 1.2****Exercise 1.3****Exercise 1.4****Exercise 1.5**

Show explicitly that the following subsets generate the same  $\sigma$ -algebra on  $\mathbb{R}$ .

$$\mathcal{E}_1 := \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\} \quad \mathcal{E}_2 := \{[a, b) \mid a, b \in \mathbb{Q}, a \leq b\} \quad (1.19)$$

$$\mathcal{E}_3 := \{[a, b) \mid a, b \in \mathbb{R}, a \leq b\} \quad \mathcal{E}_4 := \{(-\infty, b) \mid a, b \in \mathbb{Q}, a \leq b\} \quad (1.20)$$

**Solution 1.5**

We want to proof

$$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_3) = \sigma(\mathcal{E}_4). \quad (1.21)$$

We will do this by showing four inclusions. In each step, our goal is to show that an arbitrary interval from the generator of the superset is included in the  $\sigma$ -algebra of the subset.

1. First we show  $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$ . Fix  $a, b \in \mathbb{Q}$  with  $a \leq b$  and consider the interval  $[a, b)$ . If  $a = b$ , then the interval is empty and  $[a, b) = \emptyset \in \sigma(\mathcal{E}_1)$  immediately. Now let  $x, y \in \mathbb{R}$  with  $x < y < a$ . The set  $(x, a)^c \cap (y, b)$  is included in  $\sigma(\mathcal{E}_1)$  as  $\sigma$ -algebras are closed under absolute complements and intersections. We also have

$$(x, a)^c \cap (y, b) = ((-\infty, x] \cup [a, \infty)) \cap (y, b) = [a, b). \quad (1.22)$$

Therefore, it follows that  $[a, b) \in \sigma(\mathcal{E}_1)$  and hence  $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$ .

2. Next, we show  $\sigma(\mathcal{E}_2) \subset \sigma(\mathcal{E}_3)$ . As before, fix  $a, b \in \mathbb{R}$  with  $a \leq b$  and consider the interval  $[a, b]$ . If this interval is empty, it is included in  $\sigma(\mathcal{E}_2)$ , so assume  $a < b$ . Let  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  sequences in  $\mathbb{Q}$  with  $a < a_k$  and  $b_k < b$  and with  $a$  and  $b$  as their limits respectively. Since a  $\sigma$ -algebra is closed under countable unions,  $\bigcup_{k=1}^{\infty} [a_k, b_k]$  is included in  $\sigma(\mathcal{E}_2)$ , but we also have

$$\bigcup_{k=1}^{\infty} [a_k, b_k] = \lim_{k \rightarrow \infty} [a_k, b_k] = [a, b] \quad (1.23)$$

We conclude that  $[a, b] \in \sigma(\mathcal{E}_2)$  and therefore,  $\sigma(\mathcal{E}_2) \subset \sigma(\mathcal{E}_3)$ .

3. Now we will show  $\sigma(\mathcal{E}_3) \subset \sigma(\mathcal{E}_4)$ . Again, fix  $b \in \mathbb{Q}$  and consider  $(-\infty, b)$ . Let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{Q}$  with  $x_k < b$  for each  $k \in \mathbb{N}$  and diverging to negative infinity. As  $\sigma$ -algebras are closed under countable unions, we have  $\bigcup_{k=1}^{\infty} (x_k, b) \in \sigma(\mathcal{E}_3)$ . On the other hand, it is

$$\bigcup_{k=1}^{\infty} (x_k, b) = \lim_{k \rightarrow \infty} (x_k, b) = (-\infty, b). \quad (1.24)$$

This means that  $(-\infty, b) \in \sigma(\mathcal{E}_3)$  and from this we have  $\sigma(\mathcal{E}_3) \subset \sigma(\mathcal{E}_4)$ .

4. Lastly, we want to show  $\mathcal{E}_4 \subset \mathcal{E}_1$ . Fix  $a, b \in \mathbb{R}$  with  $a \leq b$  and consider  $(a, b)$ . Again, if  $a = b$ , then the interval is empty and included in  $\sigma(\mathcal{E}_4)$ . So assume  $a < b$ .

Let  $[x, y]$  with  $x, y \in \mathbb{Q}$  and  $x < y$  be a half left open interval. This is included in  $\sigma(\mathcal{E}_1)$  because  $(-\infty, x)^c \cap (-\infty, y) = [x, y]$  and  $\sigma$ -algebras are closed under absolute complements and intersections.

Now let  $(x_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{Q}$  with  $x_k < y$  for all  $k \in \mathbb{N}$  and converging to  $x$ , then  $\bigcap_{k=1}^{\infty} [x, x_k] = \lim_{k \rightarrow \infty} [x, x_k] = \{x\}$  is also in  $\sigma(\mathcal{E}_1)$ .





## Chapter 2

# Measure

**Definition 2.1.** Let  $\mathcal{R} \subset \mathcal{P}(\Omega)$  a ring of sets, and let  $\mu : \mathcal{R} \rightarrow [0, \infty]$  be an application.  $\mu$  is called a content, if

1.  $\mu(\emptyset) = 0$ .
2.  $\mu(A \dot{\cup} B) = \mu(A) + \mu(B)$

An  $\sigma$ -additive content is called premeasure.

A premeasure  $\mu : \mathcal{A} \rightarrow [0, \infty]$  on  $\sigma$ -algebra  $\mathcal{A}$  is called a measure.

$\mu$  is finite if for all  $A \in \mathcal{R} : \mu(A) < \infty$ .

$\mu$  is  $\sigma$ -finite if there exists a sequence  $(A_m)_{m \in \mathbb{N}^*}$  in  $\mathcal{R}$  with  $\mu(A_m) < \infty$  and  $\bigcup_{m \in \mathbb{N}^*} A_m = \Omega$ .

**Lemma 2.1.1.** If  $\mu(A \cap B) < \infty$ , then

$$\mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cup B) \quad (2.1)$$

**Theorem 2.1.1** (Properties of premeasure).

**Example 2.1.1** (Dirac-measure). Let  $\Omega \neq \emptyset$ . Let  $\mathcal{A} \subset \mathcal{P}(\Omega)$  a  $\sigma$ -algebra. Define for all  $x \in \Omega$  a  $\delta_x : \mathcal{A} \rightarrow \mathbb{R}_0^+$  with

$$\delta_x(A) := \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{else.} \end{cases} \quad (2.2)$$

$\delta_x$  is a finite measure, called the Dirac-measure.

**Definition 2.2.** Let

$$\mathfrak{Q}(\mathbb{R}^n) := \left\{ \bigcup_{i=1}^m [a_{1,i}, b_{1,i}) \times \dots \times [a_{n,i}, b_{n,i}) \mid m \in \mathbb{N}; a_{\nu,i}, b_{\nu,i} \in \mathbb{R}; \nu = 1, \dots, n \right\} \quad (2.3)$$

define

$$\lambda^n : \mathfrak{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+, A \mapsto \lambda^n(A) := \sum_{i=1}^m \prod_{\nu=1}^n (b_{\nu,i} - a_{\nu,i}) \quad (2.4)$$

is a premeasure.

**Definition 2.3.**

$$\mathcal{R}^\uparrow := \{A \in \mathcal{P}(\Omega) \mid \exists (A_k)_{k \in \mathbb{N}^*} \subset \mathcal{R} \text{ with } A_k \uparrow A\} \quad (2.5)$$

$\mathcal{R}^\uparrow$  is the set of all  $A \in \mathcal{P}(\Omega)$  that can be expressed as countably many sets from  $\mathcal{R}$ .  $\mathcal{R}^\uparrow$  is not a ring of sets.

**Definition 2.4.** Let  $\mu : \mathcal{R} \rightarrow [0, \infty]$  be a premeasure on  $\mathcal{R}$ , and  $A_k \uparrow A$ . Then,

$$\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty], A \mapsto \tilde{\mu}(A) = \lim_{k \rightarrow \infty} \mu(A_k) \quad (2.6)$$

is an extension of  $\mu$  on  $\mathcal{R}^\uparrow$ . This is not in general a premeasure.

**Theorem 2.4.1** (Properties of the first extension).

**Definition 2.5.** Let  $\mathcal{R} \subset \mathcal{P}(\Omega)$  a set of rings,  $\mu : \mathcal{R} \rightarrow [0, \infty]$  a  $\sigma$ -finite premeasure on  $\mathcal{R}$ , and  $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$  the first extension on  $\mathcal{R}^\uparrow$ . Moreover, let  $X \subset \Omega$  a subset of  $\Omega$ . Then,

$$\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty], X \mapsto \mu^*(X) := \inf \{ \tilde{\mu}(A) \mid A \in \mathcal{R}^\uparrow, X \subset A \} \quad (2.7)$$

is the outer measure.

**Theorem 2.5.1** (Properties of the second extension).

Bla Bla bla

**Definition 2.6** (Lebesgue measure).

Part II

Lebesgue Integral



## Chapter 3

# Measurable Functions

There is measurable, Borel measurable and Lebesgue measurable.

**Definition 3.1** (Measurable Function). Let  $(X, \mathcal{A}_X)$  and  $(Y, \mathcal{A}_Y)$  be measurable spaces. A map  $f : X \rightarrow Y$  is called measurable if the pre-image of every measurable subset of  $Y$  under  $f$  is measurable subset of  $X$ , i.e.

$$B \in \mathcal{A}_Y \Rightarrow f^{-1}(B) \in \mathcal{A}_X. \quad (3.1)$$

**Definition 3.2.** Let  $(X, \mathcal{A}_X)$  be a measurable space. A function  $f : X \rightarrow \overline{\mathbb{R}}$  is called measurable if it is measurable with respect to the Borel  $\sigma$ -algebra on  $\overline{\mathbb{R}}$

**Definition 3.3** (Borel Measurable Maps).

**Theorem 3.3.1.** Let  $(\Omega, \mathcal{A})$  be a measurable space, and  $\mathcal{B} = \sigma(\mathcal{E})$  for a generator  $\mathcal{E} \subset \mathcal{P}(\Omega)$ . If for all  $E \in \mathcal{E}$  it is  $f^{-1}(E) \in \mathcal{A}$ , then  $f$  is measurable.

**Example 3.3.1.** Let  $f : (\mathbb{R}, \mathcal{B}) \rightarrow (\mathbb{R}, \mathcal{B})$  defined as

$$f(x) := \begin{cases} 1 & x \in Q \\ -1 & x \notin Q \end{cases} \quad (3.2)$$

for a  $Q \notin \mathcal{B}(\mathbb{R})$ . Then,  $f^{-1}(1) = Q \notin \mathcal{B}$  and therefore,  $f$  is not measurable even though  $|f| = 1$  is measurable.



## Chapter 4

# Convergence Theorems

**Theorem 4.0.1** (Beppo Levi). Let  $(\Omega, \mathcal{A}, \mu)$  a measure space, and for  $k \in \mathbb{N}^*$ , let  $f_k : \Omega \rightarrow \mathbb{R}$  be a sequence of integrable functions such that

$$\forall x \in \Omega, \forall n \in \mathbb{N} : f_n(x) \leq f_{n+1}(x). \quad (4.1)$$

Moreover, if there exists  $M \in \mathbb{R}$  with  $\forall k : \int f_k d\mu \leq M$ , then

$$f := \lim_{k \rightarrow \infty} f_k : \Omega \rightarrow \overline{\mathbb{R}} \quad (4.2)$$

integrable with

$$\int f d\mu = \lim_{k \rightarrow \infty} \int f_k d\mu \quad (4.3)$$

**Theorem 4.0.2.** If the Riemann integral exists, it matches the Lebesgue integral.

**Theorem 4.0.3.** Let  $(\Omega, \mathcal{A}, \mu)$  a measure space, let  $g : X \rightarrow [0, \infty)$  be an integrable function, and let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of integrable functions satisfying

$$|f_n(x)| \leq g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N} \quad (4.4)$$

and converging pointwise to  $f : X \rightarrow \mathbb{R}$ . Then  $f$  is integrable and, for every  $E \in \mathcal{A}$

$$\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu \quad (4.5)$$





# Part III

## Applications



## Chapter 5

# Cavalieri's Principle

**Definition 5.1** (Cross-section). Let  $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$  with  $n, k, l \in \mathbb{N}^*$ , and  $A \subset \mathbb{R}^n$ . Then for a  $y \in \mathbb{R}^l$

$$A_y := \{x \in \mathbb{R}^k \mid (x, y) \in A\} \quad (5.1)$$

is the  $l$ -dimensional cross-sections of  $A$ .

**Remark.** Immediately from the definition above, we have

$$A = \bigcup_{y \in \mathbb{R}^l} (A_y, y). \quad (5.2)$$

In other words,  $\{(A_y, y)\}_{y \in \mathbb{R}^l}$  is a patition of  $A$ .

**Theorem 5.1.1** (Generalized Cavalieri's principle). Let  $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$  with  $n, k, l \in \mathbb{N}^*$ . Let  $A \subset \mathbb{R}^k \times \mathbb{R}^l$  a Borel subset of  $\mathbb{R}^n$ , and let  $\{(A_y, y)\}_{y \in \mathbb{R}^l}$  be a patition of  $A$  via cross-sections. Then we have the following

1. For all  $y \in \mathbb{R}^l$ ,  $A_y$  is Borel subset of  $\mathbb{R}^k$ .
2. Let  $F_A : \mathbb{R}^l \rightarrow [0, \infty], y \mapsto F_A(y) := \text{Vol}_K(A_y) = \lambda^k(A_y)$  be the  $k$ -dimensional volume of  $A_y$ . Then  $F_A$  is Borel measurable on  $\mathbb{R}^l$ .
3.  $\text{Vol}_n(A) := \int_{\mathbb{R}^l} \text{Vol}_k(A_y)$

*Proof.* 1. Fix  $y \in \mathbb{R}^l$

**Corollary 1.** For  $K \subset \mathbb{R}^n$  compact, we have

$$\text{Vol}_n(K) = \int_{\mathbb{R}} \text{Vol}_{n-1}(K_t) \quad (5.3)$$

**Example 5.1.1** (Volume of a Cylinder). Let  $B \subset \mathbb{R}^{n-1}$  be compact and  $Z := B \times [0, h] \subset \mathbb{R}^n$ .  $Z$  is called the generalized  $n$ -dimensional cylinder with a basis  $B$  and height  $h$  where  $h \geq 0$ . For the cross-sections we have

$$Z_t = \begin{cases} B & \text{if } t \in [0, h] \\ \emptyset & \text{else.} \end{cases} \quad (5.4)$$

and for the Volume of the Cylinder, we have

$$\text{Vol}_n(Z) = \int_0^h \text{Vol}_{n-1}(B) dt \quad (5.5)$$

$$= \text{Vol}_{n-1}(B) \int_0^h dt \quad (5.6)$$

$$= \text{Vol}_{n-1}(B) \cdot h \quad (5.7)$$

as  $\text{Vol}_{n-1}(\emptyset) = 0$ .

**Exercise 5.1**

Let  $B \subset \mathbb{R}^n$  be a Borel set and  $f : B \rightarrow \mathbb{R}_0^+$  a non-negative measurable function. Show the following.

1. The set

$$V := \{(x, y) \in B \times \mathbb{R} \mid 0 \leq y \leq f(x)\} \quad (5.8)$$

is a measurable subset of  $\mathbb{R}^{n+1}$ .

2.  $f$  is integratable over  $B$  if and only if the  $(n+1)$ -dimensional Lebesgue measure of  $V$  is finite and it is

$$\lambda^{n+1}(V) = \int_B f(x) \, d\lambda^n(x) \quad (5.9)$$

**Solution 5.1**

2. Fix a  $x \in \mathbb{R}^n$  and define

$$V_x := \{y \in \mathbb{R} \mid (x, y) \in V\} \quad (5.10)$$

$$= \{y \in \mathbb{R} \mid (x, y) \in B \times \mathbb{R}, 0 \leq y \leq f(x)\} \quad (5.11)$$

$$\text{Vol}(V_x) = f(x) - 0 = f(x) \quad (5.12)$$

$$\lambda^{n+1}(V) = \text{Vol}_n(V) = \int_B \text{Vol}_1(A_y) \, d^n y \quad (5.13)$$

$$\text{Vol}_1(A_y) \quad (5.14)$$

## Chapter 6

# Fubini's Theorem

### Exercise 6.1

1. In case of existence, compute the following Riemann and Lebesgue integrals for each permutation of the integration order.

$$\int_0^2 \int_1^2 y \sin(\pi xy) \, dx \, dy \quad (6.1)$$

$$\int_{[0,2]} \int_{[1,2]} y \sin(\pi xy) \, d\lambda(x) \, d\lambda(y) \quad (6.2)$$

2. In case of existence, compute the following Riemann and Lebesgue integrals for each permutation of the integration order.

$$\int_0^1 \int_0^\infty (2 - xy)xy \exp(-xy) \, dx \, dy \quad (6.3)$$

### Solution 6.1

First, we solve the inner integral. First, apply linearity.

$$\int (2 - xy)xy \exp(-xy) \, dx = -y \int x(yx - 2) \exp(-yx) \, dx \quad (6.4)$$

Now define

$$f(x) = x(yx - 2) \Rightarrow f'(x) = 2yx - 2 \quad (6.5)$$

$$g'(x) = \exp(-yx) \Rightarrow g(x) = -y^{-1} \exp(-yx) \quad (6.6)$$

Rewrite and apply integration by parts.

$$-y \int x(yx - 2) \exp(-yx) \, dx = -y \int f(x)g'(x) \, dx \quad (6.7)$$

$$= -y \left( f(x)g(x) - \int f'(x)g(x) \, dx \right) \quad (6.8)$$

$$= -y \left( x(yx - 2) \cdot y^{-1} \exp(-yx) - \int (2yx - 2)(-y^{-1} \exp(-yx)) \, dx \right) \quad (6.9)$$

We will solve this integral

$$\int (2yx - 2) - y^{-1} \exp(-yx) \, dx = \quad (6.10)$$



## Chapter 7

# Finding Volume by Rotation

**Definition 7.1.**  $F : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is rotationally symmetric in  $\mathbb{R}^n$  if there exists a  $f : [0, \infty) \rightarrow \overline{\mathbb{R}}$  such that for all  $x \in \mathbb{R}^n$  it is  $F(x) = f(\|x\|)$ .

**Theorem 7.1.1.** The volume of the unit sphere is

$$\tau_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \quad (7.1)$$

**Theorem 7.1.2.** Let  $B \subset [0, \infty)$  a Borel subset and  $A := \{x \in \mathbb{R}^n \mid \|x\| \in B\}$ . Then the Lebesgue measure of  $A$  is

$$\lambda^n(A) = n\tau_n \int_B r^{n-1} dr \quad (7.2)$$

where  $\tau_n$  is the volume of the unit sphere.

**Theorem 7.1.3.** Let  $f : [0, \infty) \rightarrow \overline{\mathbb{R}}$  is Borel measurable. Then the following are equivalent.

1.  $F : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, x \mapsto F(x) := f(\|x\|)$  is Lebesgue integrable over  $\mathbb{R}^n$ .
2.  $r^{n-1}f : [0, \infty) \rightarrow \overline{\mathbb{R}}, r \mapsto r^{n-1}f(r)$  is Lebesgue integrable over  $[0, \infty)$ .

Moreover, if one of the above is true, then we have the formula

$$\int_{\mathbb{R}^n} f(\|x\|) d^n x = n\tau_n \int_{[0, \infty)} r^{n-1} f(r) dr \quad (7.3)$$

**Example 7.1.1.** For a  $R \in \mathbb{R}^+$  and  $1 \leq i \leq n$  let

$$I_i := \int_{\|x\| \leq R} x_i^2 d^n x. \quad (7.4)$$

We Immediately have  $I_i = I_j =: I$  for all  $i, j$ .

$$I = \frac{1}{n} \sum_{i=1}^n I_i \quad (7.5)$$

$$= \frac{1}{n} \sum_{i=1}^n \int_{\|x\| \leq R} x_i^2 d^n x \quad (7.6)$$

$$= \frac{1}{n} \int_{\|x\| \leq R} \sum_{i=1}^n x_i^2 d^n x \quad (7.7)$$

$$= \frac{1}{n} \int_{\|x\| \leq R} \|x\|^2 d^n x \quad (7.8)$$

$$(7.9)$$

Now with the formula above, we have

$$I = \frac{1}{n} \cdot n \cdot \tau_n \int_0^R r^{n-1} r^2 dr \quad (7.10)$$

$$= \tau_n \int_0^R r^{n+1} dr \quad (7.11)$$

$$= \tau_n \frac{R^{n+2}}{n+2} \quad (7.12)$$

**Example 7.1.2.**

$$\int_0^\infty \exp(-x^2) = \frac{\sqrt{\pi}}{2} \quad (7.13)$$

*Proof.* Define

$$I = \int_{-\infty}^\infty \exp(-x^2) dx \quad (7.14)$$

Consider

$$I^2 = \left( \int_{-\infty}^\infty \exp(-x^2) dx \right) \left( \int_{-\infty}^\infty \exp(-y^2) dy \right) \quad (7.15)$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \exp(-x^2) \exp(-y^2) dx dy \quad (7.16)$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \exp(-(x^2 + y^2)) dx dy \quad (7.17)$$

$$= \int_{\mathbb{R}^2} e^{-\|x\|^2} d^2 \lambda \quad (7.18)$$

$$= \int_0^\infty r e^{-r^2} dr \quad (7.19)$$

**Example 7.1.3.** Let  $B_1 := \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$  be the open unit disk. Find the integral

$$\int_{B_1} \frac{1}{\sqrt{1 - \|x\|^2}} d\lambda^2(x) \quad (7.20)$$

*Proof.* Define  $f : [0, \infty) \rightarrow \overline{\mathbb{R}}$  as

$$f(x) = \frac{1}{\sqrt{1 - x^2}} \chi_{[0,1)}(x). \quad (7.21)$$

As  $[0, 1)$  is a Borel set of  $\mathbb{R}$ ,  $\chi_{[0,1)}$  is Borel measurable. On the other hand,  $\frac{1}{\sqrt{1-x^2}}$  is continuous for all  $x \in [0, 1)$ , so the composition of these two functions  $f$  is again Borel measurable.

Now consider,  $rf(r)$ . We have

$$\int |rf(r)| dr = \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr \quad (7.22)$$

$$= -\sqrt{1 - r^2} \quad (7.23)$$

$$= 0 + 1 \quad (7.24)$$

$$= 1 \quad (7.25)$$

**Example 7.1.4.** Compute the following integral

$$f(\xi, \eta) := \int_{B_1} \frac{\exp(i(x\xi + y\eta))}{\sqrt{1 - x^2 - y^2}} dx dy \quad (7.26)$$



## Chapter 8

# Transformation Formula

**Theorem 8.0.1.** Suppose  $\phi : U \rightarrow V$  is a  $C^1$ -diffeomorphism between open subsets of  $\mathbb{R}^n$ . If  $f : V \rightarrow \mathbb{R}$  is Lebesgue integrable OR continuous with a compact support, then

$$\int_U (f \circ \phi) |\det(d\phi)| dm = \int_V f dm. \quad (8.1)$$

**Example 8.0.1.** (2D) From polar coordinates to cartesian coordinates.

$$\phi : \mathbb{R}_0^+ \times [0, 2\pi) \rightarrow \mathbb{R}^2, (r, \varphi) \mapsto \phi(r, \varphi) := (r \cos \varphi, r \sin \varphi) \quad (8.2)$$

$$D\phi(r, \varphi) = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix} \quad (8.3)$$

$$\det D\phi(r, \varphi) = r \quad (8.4)$$

(3D) From spherical coordinates to cartesian coordinates.

$$\phi : \mathbb{R}_0^+ \times [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{R}^3 \quad (8.5)$$

$$(r, \theta, \varphi) \mapsto \phi(r, \theta, \varphi) := (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) \quad (8.6)$$

$$D\phi(r, \theta, \varphi) := \begin{pmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix} \quad (8.7)$$

$$\det D\phi(r, \theta, \varphi) = r^2 \sin \theta \quad (8.8)$$

(3D) From cylindrical coordinates to cartesian coordinates.

$$\phi : \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \rightarrow \mathbb{R}^3 \quad (8.9)$$

$$x = r \cos \theta \quad (8.10)$$

$$y = r \sin \theta \quad (8.11)$$

$$z = z \quad (8.12)$$

$$D\phi(r, \theta, z) = \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (8.13)$$

$$\det D\phi(r, \theta, z) = r \quad (8.14)$$



**Part IV**

**More Theory**



## Chapter 9

# Lebesgue Space

**Definition 9.1** ( $L^p$ -Norm). Let  $X, \mathcal{A}, \mu$  a measure space, and  $f : X \rightarrow \overline{\mathbb{R}}$  measurable. Then for  $p \in [1, \infty)$  the  $L^p$ -norm is defined as

$$\|f\|_p := \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}. \quad (9.1)$$

**Theorem 9.1.1** (Holder Inequality). Let  $p, q \in (1, \infty)$  such that  $p^{-1} + q^{-1} = 1$ . Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \quad (9.2)$$

**Theorem 9.1.2** (Minkowski Inequality). Let  $f, g : X \rightarrow \overline{\mathbb{R}}$  measurable and  $f + g$  well defined on  $X$ . Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (9.3)$$

**Definition 9.2.** Let  $X, \mathcal{A}, \mu$  be a measure space and  $p \in [1, \infty)$ . Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = \left\{ f : X \rightarrow \overline{\mathbb{R}} \mid f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\} \quad (9.4)$$

aaa Also define

$$L^p(\mu) := \mathcal{L}^p(\mu) / \sim \mu \quad (9.5)$$



**Part V**

**Manifolds**





**Definition 9.3** (Homeomorphism). A function  $f : X \rightarrow Y$  between two topological spaces is a homeomorphism if it has the following properties:

1.  $f$  is a bijection.
2.  $f$  is continuous.
3. the inverse function  $f^{-1}$  is continuous.

**Definition 9.4** (Immersion). Let  $T \subset \mathbb{R}^k$  be an open subset.  $\varphi : T \rightarrow \mathbb{R}^n, t \mapsto \varphi(t)$  is called an immersion or embedding if  $\varphi$  is continuously partially differentiable and for all  $t \in T$  it is  $\text{rank}(D\varphi) = k$

**Remark.** If  $k = n$ , then  $\varphi$  is locally invertible.

**Theorem 9.4.1.** Let  $T \subset \mathbb{R}^k$  open and  $\varphi : T \rightarrow \mathbb{R}^n$  an immersion. Then, for all  $t \in T$  there exists an open neighbourhood  $V \in \mathcal{U}_t$ , such that  $\varphi|_V : V \rightarrow \varphi(V)$  is a homeomorphism.

**Definition 9.5.**  $M \subset \mathbb{R}^n$  is a  $k$ -dimensional submanifold, if

- For all  $a \in M$  there exists an open neighbourhood  $U$  around  $a$  in  $\mathbb{R}^n$  and there exists a  $n - k$  continuously partial differentiable  $f_j : U \rightarrow \mathbb{R}$  for  $j = 1, \dots, n - k$  such that

$$M \cap U = \{x \in U \mid f_1(x) = \dots = f_{n-k}(x) = 0\} \quad (9.6)$$

and for all  $x \in U$

$$\text{rank} \frac{\partial(f_1, \dots, f_{n-k})}{\partial(x_1, \dots, x_n)}(x) = n - k \quad (9.7)$$

**Example 9.5.1.** Let's construct the simplest submanifold. Let  $n = 2$  and  $k = 1$ .

$$M = \{x \in \mathbb{R}^2 \mid f(x, y) = c\} \quad (9.8)$$

**Theorem 9.5.1.** If  $M \subset \mathbb{R}^n$  is a  $k$ -dimensional submanifold then the following are equivalent.

1. For all points  $a \in M$  there exists a open neighbourhood  $U \in \mathcal{U}_a(\mathbb{R})$ , and there exists a function  $f_i : U \rightarrow \mathbb{R}$  with  $1 \leq i \leq n - k$  that is  $n - k$  continuously (partially) differentiable such that

$$M \cap U = \{x \in U \mid f_1(x) = \dots = f_{n-k}(x) = 0\} \quad (9.9)$$

and for all  $x \in U$   $Df(x) = n - k$ .

2. There exists a numbering of the coordinate system, such that

**Example 9.5.2.** The figure eight is described by  $f : \mathbb{R} \rightarrow \mathbb{R}^2, f(t) := (\cos t, \sin 2t)$ . Define

$$M := \{x \in \mathbb{R} \mid \cos x = 0, \sin 2x = 0\} \quad (9.10)$$

then

$$D\phi(x) = \begin{pmatrix} -\sin t \\ 2 \cos 2t \end{pmatrix} \quad (9.11)$$

**Definition 9.6.** A submanifold is  $k$ -dimensional of the class  $\mathcal{C}^\alpha$  if the  $n - k$  functions that describe the submanifold is  $\alpha$  times continuously differentiable.

**Theorem 9.6.1.** Let  $M \subset \mathbb{R}^n$  a  $k$ -dimensional submanifold of the class  $\mathcal{C}^\alpha$ . Let  $i = 1, 2$   $T_i \subset \mathbb{R}^k$  open and  $\varphi_i : T_i \rightarrow V_i \subset M$  KARTEN, i.e. in parameter form of the class  $\mathcal{C}^\alpha$  with  $V := V_1 \cap V_2 \neq \emptyset$ .

**Exercise 9.1**

Let  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined as

$$f(x, y, z) := x^2 + xy - y - z \qquad g(x, y, z) := 2x^2 + 3xy - 2y - 3z \quad (9.12)$$

Show that

$$C := \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = g(x, y, z) = 0\} \quad (9.13)$$

is a submanifold of  $\mathbb{R}^3$  and that

$$\phi : \mathbb{R} \rightarrow \mathbb{R}^3, \phi(t) := (t, t^2, t^3) \quad (9.14)$$

is a global parametrization of  $C$ .

**Solution 9.1**

Define  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  as  $F(x, y, z) = (f(x, y, z), g(x, y, z))$ , then  $C$  can be rewritten as

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}. \quad (9.15)$$

We have

$$\partial_x f(x, y, z) = 2x + y \qquad \partial_x g(x, y, z) = 4x + 3y \quad (9.16)$$

$$\partial_y f(x, y, z) = x - 1 \qquad \partial_y g(x, y, z) = 3x - 2 \quad (9.17)$$

$$\partial_z f(x, y, z) = -1 \qquad \partial_z g(x, y, z) = -3 \quad (9.18)$$

therefore

$$DF(x, y, z) = \begin{pmatrix} 2x + y & x - 1 & -1 \\ 4x + 3y & 3x - 2 & -3 \end{pmatrix} \quad (9.19)$$

To check if  $DF$  surjective, it is enough to show that

$$\begin{pmatrix} x - 1 \\ 3x - 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad (9.20)$$

are linearly independent. For that, we compute the determinant of the matrix created by the two vectors.

$$\det \begin{pmatrix} x - 1 & -1 \\ 3x - 2 & -3 \end{pmatrix} = -3x + 3 + 3x - 2 = 1 \quad (9.21)$$

So,  $DF$  has a rank of 2, therefore surjective. With this,  $C$  is a submanifold of  $\mathbb{R}^3$ .