**Exercise 0.1.** Let $S$ be a multiplicatively closed subset of a ring $A$, and let $M$ be a finitely generated $A$-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

*Proof.* 1. Let $S^{-1}M = 0$, then for all $s \in S$ and for all $m \in M$ we have that

$$\frac{m}{s} = 0, \tag{1}$$

or in other words, $(m, s) \equiv (0, s')$ for some $s' \in S$. By definition, there exists a $t \in S$ such that

$$t(s \cdot 0 - s'm) = 0 \iff t(0 - s'm) = 0 \tag{2}$$
$$\iff ts'm = 0. \tag{3}$$

Choose $ts'$ to be the factor and we get $sM = 0$.

2. If there is an $s \in S$ such that $sM = 0$, then we can write for all $m \in M$ that $s \cdot m = 0$. We have

$$0 = s \cdot m = s(1 \cdot m - 1 \cdot 0) \tag{4}$$

which means again $(m, 1) \equiv (0, 1)$, hence $S^{-1}M = 0$.

□

**Exercise 0.2.** 3.2

Hints

1. To show that $x \in S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, use Proposition 1.9., i.e.

$$x \in \mathrm{Jac}(A) \iff 1 - xy \in A^\times \text{ for all } y \in A. \tag{5}$$

2. If $s \in S$, then the

*Proof.* We want to show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Let $x \in S^{-1}\mathfrak{a}$. From Proposition 1.9. we have that $x \in \mathrm{Jac}(S^{-1}A)$ if and only if $1 - xy \in (S^{-1}A)^\times$ for all $y \in S^{-1}A$. Because $S = 1 + \mathfrak{a}$, we can write

$$x = \frac{a_1}{1 + a_2} \tag{6}$$

for some $a_1, a_2 \in \mathfrak{a}$. With this, it is

$$1 - xy = 1 - \frac{a_1}{1 + a_2}y = \frac{1 + a_2}{1 + a_2} - \frac{a_1 y}{1 + a_2} = \frac{1 + a_2 - a_1 y}{1 + a_2}. \tag{7}$$

Now, $a_2 - a_1 y$ is contained in $\mathfrak{a}$, hence the whole numerator $1 + a_2 - a_1 y$ is contained in $S$, and in turn, this means the whole expression is a unit in $S^{-1}A$.

We want to give an alternative proof to Corollary 2.5. □