Exercise Sheet 8

Exercise 1
Solution to 1.
We want to show that I_α is a nonzero ideal. First we reformulate the statement to be proven. Let $n \in \mathbb{N}$ the smallest integer such that $\alpha^{n+1} \in A$. We have

$$
\begin{aligned}
& I_\alpha = \{a \in A \mid aB \subseteq A[\alpha]\} \neq \{0\} && (1)\\
\iff & \exists\, a \in A \setminus \{0\} : aB \subseteq A[\alpha] && (2)\\
\iff & \exists\, a \in A \setminus \{0\}, \exists\, b \in B, \exists\, \lambda_1, \dots, \lambda_n \in A : ab = \sum_{k=0}^{n} \lambda_k \alpha^k && (3)\\
\iff & \exists\, a \in A \setminus \{0\}, \exists\, b \in B, \exists\, \lambda_1, \dots, \lambda_n \in A : b = \sum_{k=0}^{n} \frac{\lambda_k}{a} \alpha^k. && (4)
\end{aligned}
$$

According to Theorem 4.1.11. (script), B is a Dedekind domain and with Remark 3.1.5.2 (script) it is also an A-module. As B is in particular noetherian, it is according to Corollary 4.1.5.1. finitely generated, i.e. there is a generating set

$$\{b_1, \dots, b_n\} \subseteq B \text{ for some } n \in \mathbb{N}. \tag{5}$$

For each $i \in \mathbb{N}$, we have

$$b_i = a_{i,0} + a_{i,1}\alpha + \cdots + a_{i,m}\alpha^m \tag{6}$$

for $a_{i,j} \in K$ and $j, m \in \mathbb{N}$. As K is a quotient field, we can write $a_{i,j} = \frac{p_{i,j}}{q_{i,j}}$ with $p_{i,j}, q_{i,j} \in A$.