

Chapter 1

Commutative Rings

Definitions

1. prime, coprime, relatively prime, irreducible

Exercise 1.1. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals in A , and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of B . Prove the following statements.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

Proof. We show $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$. Let $x \in (\mathfrak{a}_1 + \mathfrak{a}_2)^e$, then we have for some index set I

$$x = \sum_{i \in I} \lambda_i x_i, \quad (1.1)$$

where $\lambda_i \in B$ and $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$ for all $i \in I$. For each $i \in I$ it is $x_i = \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2})$, hence

$$x = \sum_{i \in I} \lambda_i \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2}) \quad (1.2)$$

$$= \sum_{i \in I} \lambda_i (\varphi(\mu_{i,1}a_{i,1}) + \varphi(\mu_{i,2}a_{i,2})) \quad (\text{by linearity}) \quad (1.3)$$

$$= \sum_{i \in I} \lambda_i (\mu_{i,1}\varphi(a_{i,1}) + \mu_{i,2}\varphi(a_{i,2})) \quad (\text{by linearity}) \quad (1.4)$$

$$= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad (\text{by distributivity}) \quad (1.5)$$

$$= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad (\text{reordering the sum}). \quad (1.6)$$

$$(1.7)$$

The last term is exactly the elements expressed by $\mathfrak{a}_1^e + \mathfrak{a}_2^e$, therefore, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

I think the above proof should work into both directions. \square

2. $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$

Proof. We have

$$(\mathfrak{b}_1 + \mathfrak{b}_2)^c = \left\{ x \in A \mid \exists b_1 \in \mathfrak{b}_1 \exists b_2 \in \mathfrak{b}_2 : \varphi(x) = b_1 + b_2 \right\}. \quad (1.8)$$

Now let $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$, then $x = a_1 + a_2$ where $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$. It is

$$\varphi(x) = \varphi(a_1 + a_2) \quad (1.9)$$

$$= \varphi(a_1) + \varphi(a_2) \quad (\text{by additivity}) \quad (1.10)$$

Since $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$ we have that $x \in (\mathfrak{b}_1 + \mathfrak{b}_2)^c$. \square

Exercise 1.2. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, \mathfrak{a} an ideal of A , and \mathfrak{b} an ideal of B . Prove the following statements:

1. Then $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$.

Proof. It is

$$\mathfrak{a}^{ec} = \left\{ x \in A \mid \varphi(x) \in \mathfrak{a}^e \right\} \quad (1.11)$$

$$= \left\{ x \in A \mid \varphi(x) \in \langle \varphi(\mathfrak{a}) \rangle \right\} \quad (1.12)$$

$$= \left\{ x \in A \mid \forall i \in I \exists a_i \in \mathfrak{a}_1 : \varphi(x) = \sum_{i \in I} \lambda_i \varphi(a_i) \right\}. \quad (1.13)$$

Let $a \in \mathfrak{a}$ and choose $I = \{1\}$, $\lambda_1 = 1$, and $a_i = a$, then $a \in \mathfrak{a}^{ec}$. \square

2. $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$.
3. $\mathfrak{a}^{ece} = \mathfrak{a}^e$.
4. $\mathfrak{b}^{cec} = \mathfrak{b}^c$.
5. If \mathfrak{b} is an extension, then \mathfrak{b}^c is the largest ideal of A with extension \mathfrak{b} .
6. If two extensions have the same contraction, then they are equal.

Proof. a \square

Exercise 1.3. Let A be a ring, $A[\mathcal{X}, \mathcal{Y}]$ the polynomial ring in two sets of variables \mathcal{X} and \mathcal{Y} . Show that $\langle \mathcal{X} \rangle$ is prime if and only if A is a domain.

Proof. It should be noted here, that $A[\mathcal{X}]$ does not contain $X_1 X_2$ for example. It does contain $X_1 + X_2$ however. The rest is easy. \square

Exercise 1.4. Show that, in a PID, nonzero elements x and y are relatively prime (share no prime factor) if and only if they're coprime.

Exercise 1.5. Let \mathfrak{a} and \mathfrak{b} be ideals, and \mathfrak{p} a prime ideal. Prove that these conditions are equivalent:

1. $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$
2. $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$
3. $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$

Proof. (1) to (2) is easy. Same for (2) to (3). For (3) to (1) show it with contradiction. \square