Lagrange Inversion Formula:

- $f(X) = a_1 X + a_2 X^2 + \cdots \in X\mathbb{C}[[X]]$ with $a_1 \neq 0$
- $k, n \in \mathbb{Z}$

$$\Rightarrow n[X^n] f^{-1}(X)^k = k[X^{n-k}]\left(\frac{X}{f(X)}\right)^n = k[X^{-k}] f(X)^{-n}$$

If we choose $k=1$, then

$$n[X^n] f^{-1}(X) = [X^{n-1}]\left(\frac{X}{f(X)}\right)^n = [X^{-1}] f(X)^{-n}$$

$$\Rightarrow [X^n] f^{-1}(X) = \frac{1}{n}[X^{n-1}]\left(\frac{X}{f(X)}\right)^n$$ this matches the one from Edo & van den Essen

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Proof. Set $f^{-1}(X) = \sum_{i=1}^{\infty} p_i X^i \Rightarrow X = \sum_{i=0}^{\infty} p_i f(X)^i$ | Statement proven for $k=1$

Differentiate: $\Rightarrow 1 = \sum_{i=1}^{\infty} i \cdot p_i \cdot f(X)^{i-1} \cdot f'(X)$

$\cdot f(X)^{-n}$
$$\Rightarrow \quad f(X)^{-n} = \sum_{i=1}^{\infty} i \cdot p_i \cdot f(X)^{i-n-1} f'(X)$$

We have: $f(X)^{-n} = (a_1 X + a_2 X^2 + \ldots)^{-n}$

$$= X^{-n}(a_1 + a_2 X + a_3 X^2 + \ldots)^{-n}$$

OTOH: $f(X)^{i-n-1} f'(X) = \frac{1}{i-n} \cdot \frac{d}{dX} f(X)^{i-n}$

Because: $\frac{d}{dX} f(X)^{i-n} = (i-n) \cdot f(X)^{i-n-1} \cdot f'(X)$ due to the chain rule

RHS was $= \sum_{i=1}^{\infty} i \cdot p_i \, f(x)^{i-n-1} \, f'(x)$ ← n was arbitrary chosen

$$\Rightarrow [x^{-1}] \sum_{i=1}^{\infty} i \cdot p_i \cdot \underbrace{f(x)^{i-n-1} \cdot f'(x)}_{\frac{1}{i-n} \frac{d}{dx} f(x)^{i-n}} = [x^{-1}]\, n \cdot p_n \, f(x)^{-1} \cdot f'(x)$$

$$[x^{-1}] \sum_{i=1}^{\infty} \frac{i}{i-n} \, p_i \, \frac{d}{dx} f(x)^{i-n}$$

Wait

$$[x^{-1}] \sum_{i=1}^{\infty} i \cdot p_i \cdot f(x)^{i-n-1} \cdot f'(x)$$

look at $i = n$

$$[x^{-1}]\, n \cdot p_n \cdot f(x)^{-1} \cdot f'(x)$$

$$[x^{-1}]\, n \cdot p_n \frac{a_1 + 2a_2 x + \dots}{a_1 x + a_2 x^2 + \dots} \quad \begin{matrix} = f'(x) \\ = f(x) \end{matrix}$$

$$= n \cdot p_n \cdot \left(\frac{1}{x} + \dots \right)$$

$$= n \cdot p_n$$

$$\Rightarrow [x^{-1}] \frac{x^{-1}}{f(x)^n} = n \cdot p_n = n\,[x^n]\, f^{-1}(x)$$