Notes on Algebraic Geometry

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Contents

[1]

Definition 0.0.1. Let k be a fixed algebraically closed field. We define affine n-space over k, denoted $\mathbb{A}^n$ or simply $\mathbb{A}^n$, to be the set of all n-tuples of elements of k. An element $P \in \mathbb{A}^n$ will be called a point, and if $P = (a_1, a_2, \ldots, a_n)$ with $a_i \in k$, then the a_i will be called the coordinates of P.

Definition 0.0.2. For a subset T of the polynomial ring $A = k[X_1, \ldots, X_n]$, we define the zero set of T to be the common zeros of all the elements of T, i.e.

$$Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T \}.$$

Definition 0.0.3. A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subset A = k[X_1, \ldots, X_n]$ such that $Y = Z(T)$.

Definition 0.0.4. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of Y, to be $A/I(Y)$.

Exercise 0.0.1. *Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.*

Solution. By definition 0.0.4, we simply have $A(Y) = k[X, Y]/(Y - X^2)$. The isomorphism follows from the isomorphism theorem and the map $f : k[X, Y] \to k[X]$ where we set $f(Y) = X^2$.

Exercise 0.0.2. *Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.*

Solution. $A(Z) = k[X, Y]/(XY - 1)$

We know $A(Z)$ is an k-algebra (see remark). Consider $f : k[X, Y] \longrightarrow k[T]$. We must have $\ker f = (XY - 1)$, thus $f(XY - 1) = 0$, so $f(X) = 1/f(Y)$

I'll think about the rigorous details later, but basically $A(Z) \cong k[X, X^{-1}]$

Exercise 0.0.3. *Let f be any irreducible quadratic polynomial in $k[X, Y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?*

Solution. Let f be irreducible.

$A(W) = k[X, Y]/(f)$

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.