Notes on Algebraic Geometry

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Contents

Definition 0.0.1. Let K be an algebraically closed field and let $n \in \mathbb{N}_0$ be a natural number.

1. The affine n-space over K is the set of all n-tuples of elements of K.
2. An element p in $\mathbb{A}^n$ is called a point.
3. If $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is a point, then a_i is called the coordinate for each $1 \leq i \leq n$.

Intuition 0.0.2. It's just space with points. But not vectors, because we don't add points.

Definition 0.0.3. For each subset S of polynomials in $K[X_1, \ldots, X_n]$, we define the zero-locus $Z(S)$ to be the set of points in the affine n-space $\mathbb{A}^n$ on which the functions in S simultaneously vanish, i.e.

$$Z(S) = \{\, x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \,\}.$$

Intuition 0.0.4. These are just curves.

Remark 0.0.5.
1. If $\mathfrak{a}$ is generated by T, then $Z(T) = Z(\mathfrak{a})$.
2. $Z(T)$ can be written in finitely many generators.

Definition 0.0.6. A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subset A = k[X_1, \ldots, X_n]$ such that $Y = Z(T)$.

Intuition 0.0.7. So if the points on the space is a curve, then it's an algebraic set.

BOOKMARK

Definition 0.0.8 (Affine Algebraic Variety)**.** For an algebraically closed field K and a natural number $n \in \mathbb{N}_+$, let $\mathbb{A}^n$, be an affine n-space over K. The polynomials in $K[X_1, \ldots, X_n]$ can be viewed as K-valued functions on $\mathbb{A}^n$.

1. For each subset S of polynomials in $K[X_1, \ldots, X_n]$, define the zero-locus $Z(S)$ to be the set of points in $\mathbb{A}^n$ on which the functions in S simultaneously vanish, i.e.
$$Z(S) = \{ x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \} .$$
2. A subset V of $\mathbb{A}^n$ is called affine agebraic set if $V = Z(S)$ for some $S \subset K[X_1, \ldots, X_n]$.
3. A nonempty affine algebraic set is called irreducible if it is not the union of two proper algebraic subsets. An irreducible affine algebraic set is also called an affine variety.

Definition 0.0.9. An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a quasi-affine variety.

Corollary 0.0.10. *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

Definition 0.0.11. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of Y, to be $A/I(Y)$.

Definition 0.0.12. If X is a topological space, we define the dimension of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

Exercise 0.0.1. *Show that k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.*

Chapter 1

Morphisms

Definition 1.0.1. Let X be a quasi-affine variety in $\mathbb{A}^n_K$ and $f : X \to K$ a function.

1. f is regular at a point $p \in X$ if there is an open neighborhood $\mathcal{U} \subset X$ of p, and polynomials $g, h \in K[X_1, \ldots, X_n]$, such that $h(x) \neq 0$ for all $x \in \mathcal{U}$, and $f = g/h$ on $\mathcal{U}$.

2. f is regular on X if it is regular at every point on X.

Lemma 1.0.2. *A regular function is continuous, when K is identified with $\mathbb{A}^1_K$ in its Zariski topology.*

Definition 1.0.3 (Germ)**.** Given a point p of a topological space X, and two maps $f, g : X \to Y$ where Y is any set, then f and g define the same germ at p if there is a neighbourhood $\mathcal{U}$ of p such that restricted to $\mathcal{U}$, f and g are equal, i.e.

$$f(x) = g(x) \text{ for all } u \in \mathcal{U}.$$

Definition 1.0.4. Let X be a variety.

1. We denote the ring of all regular functions on X by $\mathcal{O}(X)$.

2. If p is a point on X, we define the local ring of p on X, $\mathcal{O}_p$ to be the ring of germs of regular functions on X near p. In other words, an element of $\mathcal{O}_p$ is a pair $(\mathcal{U}, f)$ where $\mathcal{U}$ is an open subset of X containing p, and f is a regular function on $\mathcal{U}$, and where we identify two such pairs $(\mathcal{U}, f)$ and $(\mathcal{V}, g)$ if $f = g$ on $\mathcal{U} \cap \mathcal{V}$.

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.