Problem 01.2

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and $(M, \mathcal{F})$ a measurable space. Moreover, let $X : \Omega \to M$ a $(\mathcal{A}, \mathcal{F})$-measurable random variable. Show that

$$\mathbb{P}^X(B) := \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathcal{F} \tag{1}$$

defines a probability measure on $(M, \mathcal{F})$.

Solution

1. We have

$$\begin{aligned}
\mathbb{P}^X(M) &\overset{\text{def.}}{=} \mathbb{P}(X \in M) && (2)\\
&\overset{\text{def.}}{=} \mathbb{P}(\{\omega \in M \mid X(\omega) \in M\}) && (3)\\
&= \mathbb{P}(\{\omega \in M\}) && (4)\\
&= \mathbb{P}(M) && (5)\\
&\overset{\text{def.}}{=} 1. && (6)
\end{aligned}$$

   In (4), we used that the codomain of X is M and in the last step, we used the normed property of the probability measure $\mathbb{P}$.

2. Let $A_i \in \mathcal{F}$ with $i \in \mathbb{N}$ disjoint subsets. We have

$$\begin{aligned}
\mathbb{P}^X\left(\bigcup_{i=1}^{\infty} A_i\right) &\overset{\text{def.}}{=} \mathbb{P}\left(X \in \bigcup_{i=1}^{\infty} A_i\right) && (7)\\
&\overset{\text{def.}}{=} \mathbb{P}\left(\left\{\omega \in M \,\middle|\, X(\omega) \in \bigcup_{i=1}^{\infty} A_i\right\}\right). && (8)
\end{aligned}$$

   As A_i are disjoint, $X(\omega)$ is included in one and only one A_i. Therefore with the σ-additivity of $\mathbb{P}$, we have

$$\begin{aligned}
&= \mathbb{P}\left(\bigcup_{i=1}^{\infty} \{\omega \in M \,|\, X(\omega) \in A_i\}\right) && (9)\\
&\overset{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}\{\omega \in M \mid X(\omega) \in A_i\} && (10)\\
&\overset{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}(X \in A_i) && (11)\\
&\overset{\text{def.}}{=} \sum_{i=1}^{\infty} \mathbb{P}^X(A_i). && (12)
\end{aligned}$$

   In short, $\mathbb{P}^X$ is σ-additive.

From above, it follows that $\mathbb{P}^X$ is a probability measure.

Problem 01.3 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $n \in \mathbb{N}$ and $A_k \in \mathcal{A}$ for all $k \in \{1, \dots, n\}$. Prove the following formula.

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n}\left((-1)^{k-1} \sum_{I \subset \{1,\dots,n\}, |I|=k} \mathbb{P}\left(\bigcap_{i \in I} A_i\right)\right) \tag{13}$$

Solution We will prove the statement by induction. Let $n = 1$, then we simply have

$$\mathbb{P}\left(\bigcup_{k=1}^{1} A_k\right) = \mathbb{P}(A_1) = \sum_{k=1}^{1}\left((-1)^{k-1} \sum_{I \subset \{1\}, |I|=k} \mathbb{P}\left(\bigcap_{i \in I} A_i\right)\right). \tag{14}$$

Now consider the statement for the case $n+1$.

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{k=1}^{n+1} A_k\right) &= \mathbb{P}\left(A_{n+1} \cup \left(\bigcup_{k=1}^{n} A_k\right)\right) && (15)\\
&= \mathbb{P}(A_{n+1}) + \mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) - \mathbb{P}\left(A_{n+1} \cap \left(\bigcup_{k=1}^{n} A_k\right)\right) && (16)\\
& && (17)
\end{aligned}$$

The following must hold

$$\begin{aligned}
\mathbb{P}(A_{n+1}) - \mathbb{P}\left(A_{n+1} \cap \left(\bigcup_{k=1}^{n} A_k\right)\right) =& \mathbb{P}\left(\bigcup_{k=1}^{n+1} A_k\right) - \mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) && (18)\\
=& (-1)^n \sum_{I \subset \{1,\dots,n+1\}, |I|=n+1} \mathbb{P}\left(\bigcap_{i \in I} A_i\right) && (19)\\
=& (-1)^n \mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right) && (20)
\end{aligned}$$

$$\mathbb{P}(A_{n+1}) - \mathbb{P}\left(A_{n+1} \cap \left(\bigcup_{k=1}^{n} A_k\right)\right) = \mathbb{P}(A_{n+1}) \tag{21}$$