Exercise Sheet 7

Exercise 4
Let $\alpha = \sqrt[3]{2} \in \mathbb{R}$ and $K = \mathbb{Q}(\alpha)$.

1. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
2. Find the factorizations in prime ideals of the ideals
$$2\mathcal{O}_K, \qquad 3\mathcal{O}_K, \qquad 5\mathcal{O}_K, \qquad \text{and } 7\mathcal{O}_K. \tag{1}$$
and describe the relative degree of each factor.
3. Is there a prime number $p \in \mathbb{Z}$ such that $(p) \subset \mathbb{Z}$ completely splits in $\mathbb{O}_K$?

Solution 1.
The one side of the inclusion $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$ is clear. We will prove the other direction.
Let $x \in \mathcal{O}_K$.