Exercise Sheet 2

Exercise 1
Which of the following rings are Dedekind domains?

1. $\mathbb{Z} \times \mathbb{Z}$.
2. $\mathbb{Z}[X]/(X^2 + 3)$.
3. $\mathbb{F}_{11}[X]$.
4. $\mathbb{R}[X, Y]$.
5. $\mathbb{C}[X, Y]/(X^5 + Y - 13)$.

Solution

1. $\mathbb{Z} \times \mathbb{Z}$ is not a Dedekind domain as it is not even an integral domain. Take $(1, 0) \in \mathbb{Z} \times \mathbb{Z}$ and $(0, 1) \in \mathbb{Z} \times \mathbb{Z}$ for example. $(1, 0) \cdot (0, 1) = (0, 0)$ even though we chose nonzero elements.

2. $\mathbb{Z}[X]/(X^2 + 3)$ is not a Dedekind domain as it is not integrally closed.

   First, define a ring homomorphism $\varphi : \mathbb{Z}[X] \to \mathbb{Z}$ that substitues X with $\sqrt{-3}$. We have $\varphi(\mathbb{Z}[X]) = \mathbb{Z}[\sqrt{-3}]$ and $\ker(\varphi) = (X^2 + 3)$. With the first isomorphism theorem for rings, we have $\mathbb{Z}[X]/(X^2 + 3) \cong \mathbb{Z}[\sqrt{-3}]$.

   Consider

   $$\alpha := \frac{1}{2} + \frac{1}{2}\sqrt{-3} \in \mathrm{Quot}(\mathbb{Z}[\sqrt{-3}]) \cong \mathbb{Q}(\sqrt{-3}). \tag{1}$$

   From example 3.2.5. (script), we know that

   $$\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[\alpha]. \tag{2}$$

   Therefore, α is integral over $\mathbb{Z}$ and hence over $\mathbb{Z}[\sqrt{-3}]$ as well, but it does not lie in $\mathbb{Z}[\sqrt{-3}]$. We conclude $\mathbb{Z}[\sqrt{-3}]$ and with it $\mathbb{Z}[X]/(X^2 + 3)$ are not integrally closed.

3. $\mathbb{F}_{11}[X]$ is a Dedekind domain.

   From remark 1.0.3. / 2. (script), we have that the ring of polynomials in one variable over a field is a Euclidean ring, so $\mathbb{F}_{11}[X]$ is a Euclidean ring. Every Euclidean ring is a principal ideal domain (remark 1.0.3. / 3. from the script) and every principal ideal domain is a Dedekind domain (example 4.1.10. from the script). Hence, $\mathbb{F}_{11}[X]$ is a Dedekind domain.

4. $\mathbb{R}[X, Y]$ is not a Dedekind domain because not all nonzero prime ideals are maximal.

   Consider the ideal $(X^2 - Y)$ and the quotient generated $\mathbb{R}[X, Y]/(X^2 - Y)$. If we can show that the quotient is an integral domain, but not a field, then the ideal $(X^2 - Y)$ is prime, but not maximal.

   Define a ring homomorphism $\varphi : \mathbb{R}[X, Y] \to \mathbb{R}[X]$ that substitues the variable Y with X^2. We have that $\varphi(\mathbb{R}[X, Y]) = \mathbb{R}[X]$ and $\ker(\varphi) = (X^2 - Y)$. From the first isomorphism theorem for rings, we get $\mathbb{R}[X, Y]/(X^2 - Y) \cong \mathbb{R}[X]$.

   As $\mathbb{R}[X]$ is a integral domain (polynomial rings over integral domains are again an integral domain), but not a field, the ideal $(X^2 - Y)$ is a non-maximal prime ideal. Hence $\mathbb{R}[X, Y]$ is not a Dedekind domain.

5. $\mathbb{C}[X, Y]/(X^5 + Y - 13)$ is a Dedekind domain.

   As we did in 2., define a ring homomorphism $\varphi : \mathbb{C}[X, Y] \to \mathbb{C}[X]$ as

   $$\varphi(p(X, Y)) := p(X, X^5 - 13). \tag{3}$$

φ is surjective and the kernel is $(X^5 + Y - 13)$. With the first isomorphism theorem for rings, we have the isomorphism $\mathbb{C}[X, Y]/(X^5 + Y - 13) \cong \mathbb{C}[X]$.

Using similar argument as in 3., $\mathbb{C}[X]$ is a principal ideal domain and hence $\mathbb{C}[X, Y]/(X^5 + Y - 13)$ is a Dedekind domain.