

Integration and Integration

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Introduction

Part I

σ -algebra and measures

Chapter 1

Family of Sets

Chapter 2

Measure

2.1 Content, Premeasure, and Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \rightarrow [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ -additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \quad (2.1)$$

- subadditive if for all $A, B \in \mathcal{R}$ it is $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- σ -subadditive if for all $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k). \quad (2.2)$$

- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ -finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in \mathcal{R} with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \quad (2.3)$$

- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ -additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in \mathcal{R} is required. For σ -rings and therefore σ -algebras, it is guaranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \rightarrow [0, \infty]$ are σ -finite as X need not be included in a ring of sets.

Definition 2.2 (Content). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \rightarrow [0, \infty]$ is called a content if

1. $\mu(\emptyset) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ -additive content $\mu \rightarrow [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure). Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ -algebra. A σ -additive content $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a measure.

2.2 Lebesgue Content

Definition 2.5 (Lebesgue Content). Let $\mathcal{Q}(\mathbb{R}^n)$ be the ring of sets over \mathbb{R}^n .

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigcup_{k=1}^m [a_{1,k}, b_{1,k}) \times \cdots \times [a_{n,k}, b_{n,k}) \mid m \in \mathbb{N}; a_{i,k}, b_{i,k} \in \mathbb{R}; 1 \leq k \leq n \right\} \quad (2.4)$$

Set $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ as

$$\lambda^n(A) := \sum_{k=1}^m \prod_{i=1}^n (b_{i,k} - a_{i,k}) \quad (2.5)$$

λ^n is the Lebesgue content.

Theorem 2.5.1. λ^n is a well-defined finite content.

Theorem 2.5.2. λ^n is a premeasure.

2.3 Lebesgue Measure

CHEET SHEET

1. Content $\mu : \mathcal{R} \rightarrow [0, \infty]$ is empty set 0 and finitely additive.
2. Premeasure $\mu : \mathcal{R} \rightarrow [0, \infty]$ is σ -additive content.
3. First extension $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$
4. Outer measure $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$

$$\mathcal{A} \subset \mathcal{A}^\uparrow \subset \sigma(\mathcal{A}) \subset \hat{\mathcal{A}} \quad (2.6)$$

Definition 2.6. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings. Set

$$\mathcal{R}^\uparrow := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k \in \mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \quad (2.7)$$

Remark. \mathcal{R}^\uparrow is the set of all $A \in \mathcal{P}(X)$ that can be expressed as a countable many unions of sets in \mathcal{R} .

In general, \mathcal{R}^\uparrow is not a set of rings.

Definition 2.7. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets and $\mu : \mathcal{R} \rightarrow [0, \infty]$ a premeasure. For $A_k \uparrow A$ with $A_k \in \mathcal{R}$ for $k \in \mathbb{N}$ define

$$\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty], A \mapsto \tilde{\mu}(A) := \lim_{k \rightarrow \infty} \mu(A_k). \quad (2.8)$$

$\tilde{\mu}$ is called the first extension of the premeasure μ .

Remark. In general, $\tilde{\mu}$ is not a premeasure as \mathcal{R}^\uparrow need not be a ring of sets.

$\tilde{\mu}$ restricted on \mathcal{R} is identical with μ , i.e. $\tilde{\mu}|_{\mathcal{R}} \equiv \mu$.

Lemma 2.7.1. The first extension $\tilde{\mu}$ is well-defined.

Proposition 2.7.1 (Properties of \mathcal{R}^\uparrow).

Proposition 2.7.2 (Properties of the First Extension).

Definition 2.8 (Second Extension or the Outer Measure). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets, $\mu : \mathcal{R} \rightarrow [0, \infty]$ a σ -finite premeasure on \mathcal{R} , and $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$ the first extension of μ on \mathcal{R}^\uparrow . Moreover, let $B \subset X$ be a subset of X . Then, the map

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty], B \mapsto \mu^* := \inf \{ \tilde{\mu}(A) \mid A \in \mathcal{R}^\uparrow, A \supset B \} \quad (2.9)$$

is called the outer measure induced by $\tilde{\mu}$ on $\mathcal{P}(X)$.

Proposition 2.8.1 (Properties of the Second Extension).

Proposition 2.8.2 (Properties of the Outer Measure).

Definition 2.9 (Lebesgue Outer Measure). Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty], B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^\uparrow, A \supset B \right\} \quad (2.10)$$

is called the Lebesgue outer measure induced by $\tilde{\lambda}^n$.

Definition 2.10 (Pseudo Metric). Let X be a set. A map $d : X \times X \rightarrow \overline{\mathbb{R}}$, $(x, y) \mapsto d(x, y)$ is called pseudo metric on X if for all $x, y, z \in X$ it is the following three axioms are met.

1. $x = y \Rightarrow d(x, y) = 0$.
2. $d(x, y) = d(y, x)$. (Symmetry.)
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 2.10.1. The outer measure induces a pseudo metric, i.e.

$$d_{\mu^*} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty], (A, B) \mapsto d_{\mu^*}(A, B) := d_{\mu^*}(A \triangle B) \quad (2.11)$$

is a pseudo metric.

Proposition 2.10.2. The outer measure is continuous.

Definition 2.11 (Approximation through elements of Rings). Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings, $\mu : \mathcal{R} \rightarrow [0, \infty]$ a premeasure on \mathcal{R} , and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the outer measure induced by μ . Then, $A \in \mathcal{P}(X)$ is called \mathcal{R} -approximatable in respect to μ^* if for all $\epsilon > 0$ there exists an $B \in \mathcal{R}$ such that $d_{\mu^*}(A, B) = \mu^*(A \triangle B) < \epsilon$.

Theorem 2.11.1. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ -algebra and $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$ a finite premeasure. Let the first extension $\tilde{\mu} : \mathcal{A}^\uparrow \rightarrow \mathbb{R}_0^+$ also be finite and $\mu^* : \mathcal{P}(X) \rightarrow \mathbb{R}_0^+$ the outer measure. Then,

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \quad (2.12)$$

is a σ -algebra on X .

Theorem 2.11.2. Let $\mu, \tilde{\mu}, \mu^*$ and $\mathcal{A}, \mathcal{A}^\uparrow, \hat{\mathcal{A}}$ be given. Then, a finite premeasure $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$ can be uniquely extended to a finite measure $\hat{\mu} : \hat{\mathcal{A}} \rightarrow \mathbb{R}_0^+$ where $\hat{\mu} \equiv \mu^*|_{\hat{\mathcal{A}}}$.

Theorem 2.11.3. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings and $\mu : \mathcal{R} \rightarrow [0, \infty]$ a σ -finite premeasure on \mathcal{R} and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the outer measure induced by μ . Then, μ can be uniquely extended to a measure $\hat{\mu} : \sigma(\mathcal{R}) \rightarrow [0, \infty]$ where $\hat{\mu} \equiv \mu^*|_{\sigma(\mathcal{R})}$.

Definition 2.12. Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ a σ -finite Lebesgue premeasure. In this chapter, we constructed a unique extension of λ^n on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathbb{R}^n)$, the Lebesgue-Borel measure $\hat{\lambda} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$.

2.4 Measure Space

Definition 2.13. Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ -algebra. The tuple (X, \mathcal{A}) is called measurable space and the sets in the σ -algebra $A \in \mathcal{A}$ are called measurable sets.

Moreover, let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a measure on $\mathcal{P}(X)$. Then, (X, \mathcal{A}, μ) a measure space.

Definition 2.14 (Null Sets). Let (X, \mathcal{A}, μ) be a measure space and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the induced outer measure. Then $N \subset X$ with $\mu^*(N) = 0$ is called null set.

For $X = \mathbb{R}^n$ with $\lambda^n(N) = 0$ called Lebesgue null set.

$S = \emptyset$ is called the trivial null set.

Definition 2.15 (Completion of a Measure Space). Let (X, \mathcal{A}, μ) be a measure space. This measure space is called complete if all null sets are included in \mathcal{A} , i.e. for all $N \in \mathcal{A}$

$$\mu^* N = 0 \Rightarrow N \in \mathcal{A}. \quad (2.13)$$

Definition 2.16. Let

$$\overline{\mathcal{A}}^\mu := \{A \cup N \mid A \in \mathcal{A}, N \subset X \text{ with } \mu^*(N) = 0\} \quad (2.14)$$

then $\overline{\mathcal{A}}^\mu$ is called the completion of (X, \mathcal{A}, μ) .

Definition 2.17. The completion of the Lebesgue-Borel measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \hat{\lambda}^n)$ to $(\mathbb{R}^n, \mathcal{B}^{\hat{\lambda}^n}(\mathbb{R}^n), \hat{\lambda}^n)$ or shorter $(\mathbb{R}^n, \overline{\mathcal{B}}^\lambda(\mathbb{R}^n), \lambda^n)$ is called the (completed) Lebesgue measure space.

$B \in \overline{\mathcal{B}}^\lambda(\mathbb{R}^n)$ is called Lebesgue measurable to differentiate from $B \in \mathcal{B}(\mathbb{R}^n)$ Borel measurable.

Part II

Lebesgue Integral

2.5 Measurable Maps

2.6 Lebesgue Integral

Part III

Applications

Part IV

More Theory

Chapter 3

Lebesgue Space

3.1 Lebesgue Space

Definition 3.1 (L^p -Norm). Let X, \mathcal{A}, μ a measure space, and $f : X \rightarrow \overline{\mathbb{R}}$ measurable. Then for $p \in [1, \infty)$ the L^p -norm is defined as

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}. \quad (3.1)$$

Theorem 3.1.1 (Holder Inequality). Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Let $f, g : X \rightarrow \overline{\mathbb{R}}$ measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \quad (3.2)$$

Theorem 3.1.2 (Minkowski Inequality). Let $f, g : X \rightarrow \overline{\mathbb{R}}$ measurable and $f + g$ well defined on X . Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (3.3)$$

Definition 3.2. Let (X, \mathcal{A}, μ) be a measure space and $p \in [1, \infty)$. Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \rightarrow \overline{\mathbb{R}} \mid f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\}. \quad (3.4)$$

Also define

$$L^p(\mu) := \mathcal{L}^p(\mu) / \sim \mu \quad (3.5)$$

Where the equivalent relation means two functions are equivalent iff they agree on every point outside of null sets.

3.2 Convergence Theorems

Theorem 3.2.1 (Lebesgue Monotone Convergence Theorem). *Also called the theorem of Beppo Levi.* Let (X, \mathcal{A}, μ) be a measure space and let $f_n : X \rightarrow [0, \infty]$ be a sequence of measurable functions such that

$$f_n(x) \leq f_{n+1}(x) \quad (3.6)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Define $f : X \rightarrow [0, \infty]$ by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x). \quad (3.7)$$

Then f is measurable and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu. \quad (3.8)$$

Theorem 3.2.2 (Lebesgue Dominated Convergence Theorem). Let (X, \mathcal{A}, μ) be a measure space, let $g : X \rightarrow \mathbb{R}_0^+$ be an integrable function, and let $f_n : X \rightarrow \mathbb{R}$ be a sequence of integrable functions satisfying

$$|f_n(x)| \leq g(x) \quad (3.9)$$

for all $x \in X$ and $n \in \mathbb{N}$ and converging pointwise to $f : X \rightarrow \mathbb{R}$, i.e.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for all } x \in X. \quad (3.10)$$

Then f is integrable and, for every $E \in \mathcal{A}$,

$$\int_E f \, d\mu = \lim_{n \rightarrow \infty} \int_E f_n \, d\mu. \quad (3.11)$$

3.3 Convergence

Definition 3.3. Let (X, \mathcal{A}, μ) be a measure space.

1. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ be a sequence of function, and let $f : X \rightarrow \overline{\mathbb{R}}$. f_m converges to f almost everywhere, written $f_m \rightarrow^{a.e.} f$, if there exists a null set $N \subset X$ such that for all $x \in X \setminus N$ it is

$$\lim_{m \rightarrow \infty} f_m(x) = f(x). \quad (3.12)$$

2. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and let $f : X \rightarrow \overline{\mathbb{R}}$ also with $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. f_m is L^1 -convergent to f , written $f_m \rightarrow^{L^1} f$, if

$$\lim_{m \rightarrow \infty} \|f - f_m\|_{L^1} = 0. \quad (3.13)$$

3. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. $(f_m)_{m \in \mathbb{N}}$ is called L^1 -Cauchy sequence if for all $\epsilon > 0$ there exists a $m_0(\epsilon)$ such that for all $m, k \geq m_0(\epsilon)$ it is $\|f_m - f_k\|_{L^1} < \epsilon$.

Proposition 3.3.1 (Properties of Convergence). 1. Let $f_m \rightarrow^{a.e.} f$ and $f_m \rightarrow^{a.e.} g$, then $f = g$ almost everywhere.

2. Let $f_m \rightarrow^{L^1} f$ and $f_m \rightarrow^{L^1} g$, then $f = g$ almost everywhere.

3. Let $f_m \rightarrow^{L^1} f$, then $((f_m)_{m \in \mathbb{N}})$ is a Cauchy sequence.