Topology

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Contents

Part I

Rings

Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring). A ring is a set A equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. $(A, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Chapter 2

Ideals

Definition 2 (Ideal)**.** Let A be a ring. A subset $\mathfrak{a} \subset A$ is called an ideal if it satisfies the following two conditions.

1. $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$.
2. For every $r \in A$ and every $x \in \mathfrak{a}$, it is $rx \in \mathfrak{a}$.

Given a subset $S \subset A$, by the ideal (S) that S generates, we mean the smallest ideal containing S. If an ideal is generated by a subset $S \subset A$, then the elements of this subset are called generators.
An ideal that is generated by a single element is called principal.
If an ideal $\mathfrak{a}$ is not the whole ring A, then the ideal is called proper.

Definition 3 (Ideal Operation)**.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring A.

1. The sum of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by
$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \} = (\mathfrak{a}, \mathfrak{b})$$
which is again an ideal. It is the smallest ideal in A that contains $\mathfrak{a}$ and $\mathfrak{b}$.
2. The product of an ideal
3. The intersection of
4. The radical of an ideal $\mathfrak{a}$ is defined by
$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$
which is again an ideal.
5. The transporter

Proof. We verify the statements made in the definition.

1. (a) "$\mathfrak{a} + \mathfrak{b}$ is an ideal.":

□

Example 3.1. The union of two ideals is **not** an ideal in general. Consider (2) and (3) in $\mathbb{Z}$. If $(2) \cup (3)$ was an ideal, then $3 - 2 = 1$ would be contained in $(2) \cup (3)$. But $1 \notin (2)$ and $1 \notin (3)$, thus $1 \notin (2) \cup (3)$.

Proposition 4. Let $\mathfrak{a}$ be an ideal of A.

1. $\mathfrak{a} = A$ if and only if $1 \in \mathfrak{a}$ if and only if $\mathfrak{a}$ contains an unit.
2. $\mathfrak{a}^2 \subset \mathfrak{a}$.
3. $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.
4. $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.

Proposition 5. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a ring A.

1. $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$.
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.
3. If $\mathfrak{a} \subset \mathfrak{b}$, then $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$.
4. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.
5. $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
6. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
7. If $\mathfrak{a} = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ and $n \in \mathbb{N}^+$, then $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Proof. We verify each statement.

1. Let $x \in \mathfrak{a}$, then trivially, $x^1 \in \mathfrak{a}$, so $x \in \sqrt{\mathfrak{a}}$.

2. Since $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$ from above, it suffices to verify the other inclusion. Let $x \in \sqrt{\sqrt{\mathfrak{a}}}$, then $x^n \in \sqrt{\mathfrak{a}}$ and in turn, $(x^n)^m \in \mathfrak{a}$. Thus, $x^{nm} \in \mathfrak{a}$, therefore, $x \in \sqrt{\mathfrak{a}}$.

3. Suppose $\mathfrak{a} \subset \mathfrak{b}$ and let $x \in \sqrt{\mathfrak{a}}$. Then, $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$, thus $x^n \in \mathfrak{b}$. It follows that $x \in \sqrt{\mathfrak{b}}$.

4. "$\Rightarrow$": Let $\sqrt{\mathfrak{a}} = A$, then for all $x \in A$, we have that $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$. In particular, $1^n \in \mathfrak{a}$, but $1^n = 1$ for all $n \in \mathbb{N}^+$. Thus, $\mathfrak{a} = A$.

   "$\Leftarrow$": On the other hand, let $\mathfrak{a} = A$. In general, it is $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, therefore $A \subset \sqrt{\mathfrak{a}}$ which immediately yields the desired equality $A = \sqrt{\mathfrak{a}}$.

5. "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cdot \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Since $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, we have $x^n \in \mathfrak{a} \cap \mathfrak{b}$, and it follows that $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": Let $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Hence it is $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, therefore $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$. We may write $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, therefore $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": Finally, let $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. Then, $x\sqrt{\mathfrak{a}}$ and $x\sqrt{\mathfrak{b}}$, so $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$ for some $n, m \in \mathbb{N}^+$. Say $n \geq m$, then $x^n \in \mathfrak{b}$. This yields $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

6. "$\sqrt{\mathfrak{a} + \mathfrak{b}} \subset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Let $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$, then $x^n \in \mathfrak{a} + \mathfrak{b}$ for some $n \in \mathbb{N}^+$. By definition of sum of ideals, we have that $x^n = a + b$ for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ and $\mathfrak{b} \subset \sqrt{\mathfrak{b}}$, we have $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, thus $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

   "$\sqrt{\mathfrak{a} + \mathfrak{b}} \supset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Now let $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$, then $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$ for some $n \in \mathbb{N}^+$. Hence there exists $a \in \sqrt{\mathfrak{a}}$ and $b \in \sqrt{\mathfrak{b}}$ such that $x^n = a + b$. We have that $a^p \in \mathfrak{a}$ and $b^q \in \mathfrak{b}$

for some $p, q \in \mathbb{N}^+$. Consider

$$\begin{aligned}(x^n)^{(p+q-1)} &= (a+b)^{(p+q-1)} \\ &= \sum_{k=0}^{p+q-1} \binom{p+q-1}{k} a^k \cdot b^{p+q-1-k}.\end{aligned}$$

For each $k \in \{0, 1, \ldots, p+q-1\}$, we have $a^k \in \mathfrak{a}$ or $b^{p+q-1} \in \mathfrak{b}$. Thus, the whole sum lies in $\mathfrak{a} + \mathfrak{b}$ or in other words $x^{n(p+q-1)} \in \mathfrak{a} + \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

7. "$\sqrt{\mathfrak{a}} \subset \mathfrak{p}$": Let $x \in \sqrt{\mathfrak{a}}$, then $x^m \in \mathfrak{a}$ for some $m \in \mathbb{N}^+$. Because $\mathfrak{a} = \mathfrak{p}^n$, we have $x^m \in \mathfrak{p}^n$. We also have $\mathfrak{p}^n \subset \mathfrak{p}$, thus $x^m \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$.

   "$\sqrt{\mathfrak{a}} \supset \mathfrak{p}$": On the other hand, if $x \in \mathfrak{p}$, then $x^n \in \mathfrak{p}^n = \mathfrak{a}$, therefore $x \in \sqrt{\mathfrak{a}}$.

□

Proposition 6. 1. $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$.

Example 6.1. Does $\sqrt{\mathfrak{a}^2} = \mathfrak{a}$ hold?

Chapter 3

Anatomy of Rings

Definition 7 (Nilpotent Element and Nilradical)**.** An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.
The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\operatorname{Nil}(A)$.

Definition 8 (Reduced Ring)**.** A ring A is called reduced ring if it has no non-zero nilpotent elements.

Proposition 9. Let A and B be two rings and $A' \subset A$ be a subring of A.

1. If A is reduced, then A' is also reduced.
2. If A and B are reduced, then $A \times B$ is also reduced.
   (XXX DOES THIS ALSO HOLD FOR ARBITARY MANY PRODUCTS?)

3.1 Exercises and Notes

Example 9.1. Let K be a field and $A = {}^{K[X,Y]}/_{(X - XY^2, Y^3)}$.

1. Compute the nilradical $\operatorname{Nil}(A)$.

   Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

   Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism ${}^{K[X,Y]}/_{(X - XY^2, Y^3)} \simeq {}^{K[Y]}/_{(Y^3)}$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

   Clearly, $Y \in \operatorname{Nil}(A)$ or in other words $(Y) \subset \operatorname{Nil}(A)$. But we also have that ${}^{K[Y]}/_{(Y)} = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \operatorname{Nil}(A)$ conclude $\operatorname{Nil}(A) = (Y)$. $\square$

Chapter 4

Polynomial Rings

Chapter 5

Quotient

Chapter 6

Localization

6.1 Definition and Theorems

Definition 10 (Multiplicative Subset)**.** A subset S of a ring A is called a multiplicative subset if the following conditions hold.

1. $1 \in S$.
2. For all $x, y \in S$ it is $xy \in S$.

Example 10.1. Let A be a ring. Important examples of a multiplicative subset include the following.

1. The set of units $A^\times$ is a multiplicative subset.
2. The set of non-zero-divisors $A \setminus \mathrm{ZD}(A)$ is a multiplicative subset.

Example 10.2. Let A be a ring. Other examples of multiplicative subsets are the following.

1. For any element $x \in A$, the set generated by its power $\{1, x, x^2, x^3, \dots\}$ is a multiplicative subset.
2. For any ideal $\mathfrak{a} \subset A$, the set $1 + \mathfrak{a}$ is a multiplicative subset.

Lemma 11. An ideal $\mathfrak{p}$ of a ring A is prime if and only if its complement $A \setminus \mathfrak{p}$ is a multiplicative subset.

Definition 12 (Localization)**.** $S^{-1}A$ is again a ring.

Lemma 13. Let A be a ring and S a multiplicative subset, then the following are equivalent.

1. $S^{-1}A = 0$.
2. S contains a nilpotent element.
3. $0 \in S$.

Proof. "1. $\Rightarrow$ 2.": Let $S^{-1}A = 0$, then for all $x \in A$ and $s \in S$ it is $(x, s) \sim (0, 1)$, thus $x \cdot u = 0$ for some $u \in S$. In particular, this holds for $x = 1$, therefore $1 \cdot u = 0$. Since a unit can never be a zero divisor, we must have $u = 0$ which is nilpotent and lies in S.

"1. $\Leftarrow$ 2.": On the other hand, let $x \in S$ be nilpotent, i.e. $x^n = 0$ for some $n \in \mathbb{N}^+$. Because S is multiplicatively closed $x^n = 0$ lies in S. Fix an element $(y, s) \in S^{-1}A$, then $y \cdot 1 \cdot 0 = 0 \cdot s \cdot 0$. Hence $(y, s) \sim (0, 1)$ and we have $S^{-1}A = 0$.

"2. $\Rightarrow$ 3.": Again, let $x \in S$ be nilpotent, thus $x^n = 0$ for some $n \in \mathbb{N}^+$. S is multiplicatively closed and we have $x^n = 0 \in S$.

"2. $\Leftarrow$ 3.": If $0 \in S$, then S simply contains a nilpotent element because 0 is nilpotent. □

Remark. In the lemma above, the condition $0 \notin S$ is required because if S contains 0, then $S^{-1}A = 0$ and by definition, an integral domain is a nonzero ring.

Proposition 14. Let A be a ring. A is reduced if and only if all its localizations $A_\mathfrak{p}$ at $\mathfrak{p} \in \operatorname{Spec} A$ is reduced.

Proof. "$\Rightarrow$": We prove the statement by contrapositive. Let $A_\mathfrak{p}$ be not reduced for all $\mathfrak{p} \in \operatorname{Spec} A$. Thus, in all $A_\mathfrak{p}$, there is an element, say x/s that is nilpotent and not zero, i.e. $(x/s)^n = 0$ for some $n \in \mathbb{N}^+$. By the definition of localization, we get $x^n \cdot u = 0$ for some $u \in A \setminus \mathfrak{p}$. Now, $u \in A \setminus \mathfrak{p}$ cannot be zero, because if it was, $A_\mathfrak{p} = 0$ which is reduced. Thus, x is nilpotent and A is not reduced. □

Lemma 15. Let A be a ring and $S \subset A$ be a multiplicative subset that does not contain 0.

1. A is an integral domain if and only if $S^{-1}A$ is an integral domain.
2. A is a unique factorization domain if and only if $S^{-1}A$ is a unique factorization domain.

Proof. "$\Rightarrow$": Let A be an integral domain. Since S does not contain 0, the localization $S^{-1}A$ is a nonzero ring (see EXAMPLE). Let $(x, s) \in S^{-1}A \setminus \{0\}$ be a nonzero element and suppose there is a $(y, t) \in S^{-1}A$ with $(x, s) \cdot (y, t) = 0$. It is $(xy, st) = (0, 1)$ and thus $xy \cdot u = 0$ for some $u \in S$. Because x was nonzero and S does not contain 0 we must have $y = 0$. Hence $S^{-1}A$ is an integral domain.

"$\Leftarrow$": On the other hand, let $S^{-1}A$ be an integral domain. JUST USE THE CANONIC MAPPING $\varphi_S : A \longrightarrow S^{-1}A$. □

6.2 Exercises and Notes

Example 15.1. Let A_1 and A_2 be rings. Consider $A = A_1 \times A_2$ and set $S := \{(1, 1), (1, 0)\}$. Prove $A_1 \simeq S^{-1}A$.

Solution. I don't understand the solution? □

Example 15.2. Find all intermediate rings $\mathbb{Z} \subset A \subset \mathbb{Q}$, and describe each A as a localization of $\mathbb{Z}$. As a starter, prove $\mathbb{Z}\left[\frac{2}{3}\right] = S_3^{-1}\mathbb{Z}$ where $S_3 := \left\{ 3^i \mid i \in \mathbb{N}^+ \right\}$.

Chapter 7

Hierarchy of Rings

7.1 Definition and Theorems

7.1.1 Integral Domains

Part II

Modules

Definition 16 (Module).

Example 16.1. 1. If A is a field, then an A-module is a vector space.

2. A $\mathbb{Z}$-module is just an abelian group.

Proposition 17. Let M and N be an A-module, and $\varphi : M \to N$ be an A-module homomorphism.

1. $\operatorname{im}(\varphi)$ is a submodule of M.
2. $\ker(\varphi)$ is a submodule of N.
3. For any submodule N' of N, its preimage $\varphi^{-1}(N')$ is a submodule of M.

Definition 18 (Annihilator).

Definition 19 (Radical).

Definition 20 (Simple Modules)**.** Let A be a ring. A nonzero A-module M is called simple if the only submodules are $\{0\}$ and M itself.

Example 20.1. If M is a simple A-module, then any $f \in \operatorname{Hom}_A(M, M) \setminus \{0\}$ is an isomorphism.

Proof. Fix an $f \in \operatorname{Hom}_A(M, M) \setminus \{0\}$. Since $\ker(f)$ is a submodule of M, it must be either $\{0\}$ or whole M. But $\ker(f) = M$ would mean that $f = 0$ which was explicitly excluded, thus $\ker(f) = \{0\}$. By the isomorphism theorem, we also have $\operatorname{im}(f) \cong {}^{M}/_{\ker(f)} \cong M$. Therefore, f is bijective. $\square$

Definition 21 (Indecomposable)**.** Let A be a ring. A nonzero A-module M is called indecomposable if it cannot be written as a direct sum of two non-zero submodules.

Proposition 22. Every simple module is indecomposable.

Example 22.1. Not all indecomposable modules are simple. For example, $\mathbb{Z}$ is indecomposable, but is not simple.

7.2 Exercises and Notes

Example 22.2. Let $f : M \to N$ be a surjective homomorphism of two finitely generated A-modules.

1. If $N \cong A^n$ is a free A-module, show that $M \cong \ker(f) \oplus N$.

   Proof. Since N is finitely generated, let $(e_1, \dots, e_n)$ be a set of generators. □

Example 22.3. Let A be a ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals, M and N A-modules. Set

$$\Gamma_{\mathfrak{a}}(M) := \left\{ m \in M \mid \mathfrak{a} \subset \sqrt{\operatorname{Ann}(m)} \right\}.$$

Prove the following statements.

1. If $\mathfrak{a} \supset \mathfrak{b}$, then $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{b}}(M)$.

   Proof. The proof is a matter of verification. Let $m \in \Gamma_{\mathfrak{a}}(M)$. It is

$$\begin{aligned} m \in \Gamma_{\mathfrak{a}}(M) &\Rightarrow \mathfrak{a} \subset \sqrt{\operatorname{Ann}(m)} \\ &\Rightarrow \text{For all } a \in \mathfrak{a} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \in \operatorname{Ann}(m). \\ &\Rightarrow \text{For all } a \in \mathfrak{a} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \cdot m = 0. \end{aligned}$$

   Since $\mathfrak{a} \supset \mathfrak{b}$, the last statement is true for all $a \in \mathfrak{b}$. We have

$$\begin{aligned} &\Rightarrow \text{For all } a \in \mathfrak{b} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \cdot m = 0. \\ &\Rightarrow \text{For all } a \in \mathfrak{b} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \in \operatorname{Ann}(m). \\ &\Rightarrow \mathfrak{b} \subset \sqrt{\operatorname{Ann}(m)} \\ &\Rightarrow m \in \Gamma_{\mathfrak{b}}(M) \end{aligned}$$

   Thus, $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{b}}(M)$. □

2. If $M \subset N$, then $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(N) \cap M$.

   Proof. Again, the proof is a matter of verification.
   "$\subset$": $M \subset N$ implies $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{a}}(N)$. Moreover, it is $\Gamma_{\mathfrak{a}}(M) \subset M$. Thus, $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{a}}(N) \cap M$.
   "$\supset$": Let $m \in \Gamma_{\mathfrak{a}}(N) \cap M$. It is

$$\begin{aligned} m \in \Gamma_{\mathfrak{a}}(N) \cap M &\Rightarrow \mathfrak{a} \subset \sqrt{\operatorname{Ann}(m)} \text{ and } m \in M. \\ &\Rightarrow m \in \Gamma_{\mathfrak{a}}(M). \end{aligned}$$

   Hence, $\Gamma_{\mathfrak{a}}(N) \cap M \subset \Gamma_{\mathfrak{a}}(M)$. □

3. In general, it is $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M) \cap \Gamma_{\mathfrak{b}}(M)$.

4. In general, it is $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\sqrt{\mathfrak{a}}}(M)$.

5. If $\mathfrak{a}$ is finitely generated, then

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 1} \{ m \in M \mid \mathfrak{a}^n m = 0 \}.$$

Example 22.4. Let A be a ring, M a module, $x \in \operatorname{Rad}(M)$, and $m \in M$. If $(1 + x)m = 0$, then $m = 0$.

Proof. By definition of radical of a module, it is

$$\operatorname{Rad}\left({}^{A}\!/_{\operatorname{Ann}(M)}\right) = {}^{\operatorname{Rad}(M)}\!/_{\operatorname{Ann}(M)}.$$

Thus, if $x \in \operatorname{Rad}(M)$, then its residue $x' := x + \operatorname{Ann}(M)$ lies in $\operatorname{Rad}\left({}^{A}\!/_{\operatorname{Ann}(M)}\right)$ which means x' is nilpotent. SOME THEOREM yields $(1 + x')$ is an unit in ${}^{A}\!/_{\operatorname{Ann}(M)}$. $\square$

Chapter 8

Tensor Product

8.1 Definition and Theorems

Definition 23. Let M and N be A-modules. Their tensor product is a pair $(M \otimes_A N, \theta)$ where

1. $M \otimes_A N$ is an A-module.
2. $\theta : M \times N \to M \otimes_A N$ is an A-bilinear mapping.

satisfying the universal property, for every pair (P, ω) of an A-module and an A-bilinear mapping $\omega : M \times N \to P$, there exists a unique A-module homomorphism $f : M \otimes_A N \to P$ with $\omega = f \circ \theta$.

Definition 24. Let M and N be A-modules. Their tensor product is the pair $(M \otimes_A N, \theta)$, where

1. $M \otimes_A N$ is the quotient of the free A-module $A^{M \times N}$ on the direct product $M \times N$, by the submodule generated by the set of elements of the form:

$$(\lambda m_1 + m_2, n) - \lambda(m_1, n) - (m_2, n)$$
$$(m, \lambda n_1 + n_2) - \lambda(m, n_1) - (m, n_2)$$

   for $m, m_1, m_2 \in M$; $n, n_1, n_2 \in N$; and $\lambda \in A$, where we denote (m, n) for its image under the canonical mapping $M \times N \to A^{(M \times N)}$.

2. $\theta : M \times N \to M \otimes_A N$ is the composition of the canonical mapping $M \times N \to A^{(M \times N)}$ with the quotient module homomorphism $A^{(M \times N)} \to M \otimes_A N$.

Example 24.1. It is $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$.

Proof. Let's show this in multiple concrete ways.
Method 1: I want to do this conretely. First, we have

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \{ (0,0); (0,1); , (0,2); (1,0); (1,1); (1,2) \} .$$

Thus, the elements of $\mathbb{Z}^{(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})}$ are in the form

$$(x_{(0,0)}, x_{(0,1)}, x_{(0,2)}, x_{(1,0)}, x_{(1,1)}, x_{(1,2)})$$

where $x_{(i,j)} \in \mathbb{Z}$ with $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$.

Now, we want to find the submodule generated by the rules in the definition.

1. Set $m_1 = m_2 = n = \lambda = 0$, then

$$(0 \cdot 0 + 0, 0) + 0 \cdot (0, 0) - (0, 0) = (0, 0) = 1 \cdot (0, 0) \to (1, 0, 0, 0, 0, 0).$$

2. Set $m = n_2 = 0$, $n_1 = 1$, and $\lambda = 2$, then

$$\begin{aligned}(0, 2 \cdot 1 + 0) - 2 \cdot (0, 1) - (0, 0) &= (0, 2) - (2 \cdot 0, 1) \\ &= (0, 2) - (0, 1) \\ &= (0, 1) \\ &= 1 \cdot (0, 1) \\ &\to (0, 1, 0, 0, 0, 0)\end{aligned}$$

3. I think the rest is clear for now.

We may conclude that the submodule generated by the rules defined is the whole module, thus $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$.
Method 2: https://www.math.brown.edu/reschwar/M153/tensor.pdf □

Proposition 25. Let A be a ring, and M, N and P be A-modules.

1. (identity) $A \otimes_A M = M$.
2. (commutative law) $M \otimes_A N = N \otimes_A M$.

Proof. As in the proposition, let A be a ring, and M, N and P be A-modules.

1. Define $\beta : A \times M \to M$ by $\beta(x, m) := xm$. Clearly, β is bilinear.

□

8.2 Exercises and Notes

Example 25.1. Let $A \to B \to C$ be ring homomorphisms and M and N be A-modules. Show the following.

1. $(M \otimes_A B) \otimes_B C \cong M \otimes_A C$

   Proof. It is

   $$\begin{aligned}(M \otimes_A B) \otimes_B C &\cong M \otimes_A (B \otimes_B C)\\ &\cong M \otimes_A C\end{aligned}$$

   □

2. $(M \otimes_A N) \otimes_A B \cong (M \otimes_A B) \otimes_B (N \otimes_A B)$

   Proof. trivial □

Example 25.2. Let A be a ring.

1. If M, N are A-modules, then $\mathrm{Hom}_A(M, N)$ may be viewed as an A-module via

   $$a \cdot \varphi := (m \mapsto a \cdot \varphi(m))$$

   for $a \in A$ and $\varphi \in \mathrm{Hom}_A(M, N)$.

   Proof. this is trivial □

2. If M, N, L are A-modules, then there exists a natural isomorphism of A-modules

   $$\mathrm{Hom}_A(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N))$$

Example 25.3. Let A be a ring, $\mathfrak{a}$ an ideal of A, and M an A-module.

1. Show that $M/\mathfrak{a}M \cong M \otimes_A A/\mathfrak{a}$.

   Proof. Define $\varphi : M \otimes_A A/\mathfrak{a} \to M/\mathfrak{a}M$ by

   $$m \otimes_A \overline{x} \mapsto x \cdot m + \mathfrak{a}M.$$

   φ is an homomorphism because

   (a) $\varphi((m_1 \otimes_A \overline{x_1}) + (m_2 \otimes_A \overline{x_2})) =$

   □

Chapter 9

Exact Sequences

9.1 Definition and Theorems

Definition 26. Exact at, exact sequence, short exact sequence

Example 26.1. Let M and N be A-modules. Then, the sequence

$$0 \to M \to M \oplus N \to N \to 0$$

is short exact.

Chapter 10

Noetherian Modules

Definition 27. An A-module M is called Noetherian if one of the following equivalent conditions hold.

1. Its submodules satisfies the asending chain condition, i.e. MISSING.
2. All submodules of M are finitely generated.

Proof. "$\Rightarrow$": Let M be an A-module that satisfies the ascending chain condition and assume a submodule N is not finitely generated. In this case, we may construct a chain of submodules

$$N_1 \subset N_2 \subset \cdots N_i \subset \cdots$$

where $N_i = (n_1, n_2, \ldots, n_{i-1})$ with $n_i \in N$ and $n_i \notin N_i$ for all $i \in \mathbb{N}^+$. This chain never stabilizes, thus N must be finitely generated.
"$\Leftarrow$": $\square$

Lemma 28. Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules. Then N is Noetherian if and only if M and P are Noetherian.

Proof. Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules.
"$\Rightarrow$": Let N be Noetherian.

1. We show that M is Noetherian by verifying all its submodules are finitely generated. Let M' be a submodule of M. In that case, $\alpha(M')$
2. We show that P is Noetherian by verifying all its submodules are finitely generated. Let P' be a submodule of P. Since β is surjective, we have $P' = \beta\left(\beta^{-1}(P')\right)$. $\beta^{-1}(P')$ is a submodule of N and it is finitely generated because N is Noetherian.

$\square$