Series 2

K

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Exercise 3

Suppose $\mathcal{B}$ is a subbase for a topology $\mathcal{T}$ on a set X.

(a)

Show that a sequence $x_n \in X$ converges to $x \in X$ if and only if for every $\mathcal{U} \in \mathcal{B}$ containing x it is $x_n \in \mathcal{U}$ for all n sufficiently large.

Solution. Let $\mathcal{B}$ be a subbase for a topology $\mathcal{T}$ on a set X.

"$\Rightarrow$": Let $x_n \in X$ be a sequence that converges to a point $x \in X$. By the definition of convergence, we have that for every neighbourhood $\mathcal{N} \subset X$ of x it is $x_n \in \mathcal{N}$ for all $n \in \mathbb{N}$ sufficiently large.

Fix a subset $\mathcal{U} \in \mathcal{B}$ that contains x. Since any subbase consists of only open sets, $\mathcal{U}$ is open as well. Open subsets that contain x are themselves a neighbourhood of x, thus $x_n \in \mathcal{U}$ for all $n \in \mathbb{N}$ sufficiently large as desired.

"$\Leftarrow$": Let $x_n \in X$ be a sequence and $x \in X$ be a point such that for every $\mathcal{U} \in \mathcal{B}$ containing x, it is $x_n \in \mathcal{U}$ for all n sufficiently large. Fix a neighbourhood $\mathcal{N} \subset X$ of x. By the definition of a neighbourhood, there is an open set $\mathcal{V} \in \mathcal{T}$ such that $x \in \mathcal{V} \subset \mathcal{N}$. Now by the definition of a subbase, we may write

$$\mathcal{V} = \bigcup_{\alpha \in I} \left(\mathcal{U}_\alpha^1 \cap \cdots \cap \mathcal{U}_\alpha^{N_\alpha}\right)$$

for some collection of subsets $\mathcal{U}_\alpha \subset X$ indexed by a set I, for each $\alpha \in I$ it is $N_\alpha \in \mathbb{N}$, and it is $\mathcal{U}_\alpha^1, \ldots, \mathcal{U}_\alpha^{N_\alpha} \in \mathcal{B}$. Since $x \in \mathcal{V}$, there is an $\beta \in I$ such that $x \in \mathcal{U}_\beta^1 \cap \cdots \cap \mathcal{U}_\beta^{N_\beta}$ and hence $x \in \mathcal{U}_\beta^i$ for all $1 \leq i \leq N_\beta$. By the given condition, this means that $x_n \in \mathcal{U}_\beta^i$ for all $1 \leq i \leq N_\beta$ and for all n sufficiently large. In turn, this gives $x_n \in \mathcal{U}_\beta^1 \cap \cdots \cap \mathcal{U}_\beta^{N_\beta}$ for all n sufficiently large. Thus, $x_n \in \mathcal{V}$ for all n sufficiently larg. Conclude $x_n \in \mathcal{N}$ for all n sufficiently large which means x_n converges to x. $\square$