

# Topology

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# Chapter 1

## Topological Space

### 1.1 Definitions and Theorems

**Definition 1** (Topological Space). A **topological space** is an **ordered pair**  $(X, \tau)$ , where  $X$  is a **set** and  $\tau$  is a **collection of subsets** that satisfies the following **axioms**.

1. The **empty set**  $\emptyset$  and the **entire set**  $X$  belongs to  $\tau$ .
2. Any **arbitrary union** of members of  $\tau$  belongs to  $\tau$ .
3. The **intersection** of **finite number** of members of  $\tau$  belongs to  $\tau$ .

The **collection**  $\tau$  is called a **topology** on  $X$  and the **elements** of  $\tau$  are called **open sets**. A **subset**  $A \subset X$  is said to be **closed** if its **complement**  $X \setminus A$  is **open**.

**Definition 2** (Continuous Maps). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be **topological spaces**. A **map**  $f : X \longrightarrow Y$  is said to be **continuous** if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \quad (1.1)$$

**Lemma 3.** The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if  $X$  and  $Y$  are metric spaces, then  $f : X \longrightarrow Y$  is  $\epsilon$ - $\delta$ -continuous if and only if  $f$  is continuous.

**Definition 4** (Homeomorphism). Let  $X$  and  $Y$  be **topological spaces**.

1. A **map**  $f : X \longrightarrow Y$  is a **homeomorphism** if it has the following properties.
  - (a)  $f$  is **bijective**.
  - (b)  $f$  and the **inverse map**  $f^{-1}$  is **continuous**.
2. Two topological spaces  $X$  and  $Y$  are said to be **homeomorphic** if a homeomorphism exists.
3. We denote the set of all homeomorphisms from  $X$  to  $Y$  by  $\text{Homeo}(X, Y)$ . If  $Y = X$  we also write  $\text{Homeo}(X)$ .

**Definition 5** (Homeomorphism). Let  $(X, \tau)$  a topological space.

1.  $\mathcal{B} \subset \mathcal{O}$  is a **basis** of the topology, if any member of  $\mathcal{O}$  is the **union of subsets** from  $\mathcal{B}$ .
2.  $\mathcal{S} \subset \mathcal{O}$  is a **subbasis** of the topology, if any member of  $\mathcal{O}$  is the **union of finite intersections of subsets** from  $\mathcal{S}$ .

We say that  $\mathcal{B}$  and  $\mathcal{S}$  **generates**  $\mathcal{O}$  and write  $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$ .

**Lemma 6.** Let  $\mathcal{S} \subset \mathcal{P}(X)$  be a **collection of subsets**, then there **exists exactly one** topology  $\tau \subset \mathcal{P}(X)$  of  $X$  such that

1.  $\mathcal{S} \subset \tau$
2. If  $\tau' \subset \mathcal{P}(X)$  a topology with  $\mathcal{S} \subset \tau'$ , then  $\tau \subset \tau'$ .

**Definition 7.** 1. Given  $(X, \tau)$  be a **topological space**,  $S \subset X$  a subset, the **subspace topology** (also the induced topology or the relative topology) on  $S$  is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \}.$$

2. Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be two **topological spaces**. The product topology of  $X$  and  $Y$  is defined by

$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

3. Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be two **topological spaces**. The topological sum of  $X$  and  $Y$  is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

**Definition 8.** Let  $(X, \tau)$  be a topological space.

1. Given a **point**  $p \in X$ , a subset  $U \subset X$  is a neighborhood of  $p$  if there is an open subset  $V \in \tau$  such that  $p \in V$ . If such a neighborhood exists,  $p$  is called a interior point of  $U$ .
2. Let  $S \subset X$  be a subset. The interior of  $S$ , denoted by  $\overset{\circ}{S}$  or  $\text{int}(S)$ , is the **set** of all interior points of  $S$ .
3. Let  $S \subset X$  be a subset. The closure of  $S$ , denoted by  $\overline{S}$  or  $\text{cl}(S)$ , is defined by

$$\text{cl}(S) := X \setminus \text{int}(X \setminus S).$$

## 1.2 Proofs, Remarks, and Examples

**Example 8.1.** Let  $X$  be a set.

1.  $\tau = \mathcal{P}(X)$  is called the **discrete topology**. In this case,  $(X, \tau)$  is called the **discrete space**. It is the **finest topology** that can be defined on a set. (The set of all possible topologies on a given set forms a partially ordered set.)
2.  $\tau = \{\emptyset, \mathcal{P}(X)\}$  is called the **trivial topology**.
3. Let  $(X, d)$  be a **metric space**. Set

$$\tau_d := \{U \in \mathcal{P}(X) \mid U \text{ is an open subset in the metric space } (X, d)\}. \quad (1.2)$$

Recall that  $U$  being an open subset in the metric space  $(X, d)$  means that for all  $x \in U$  there is an  $r > 0$  such that  $B_d(x, r)$  is contained in  $U$ .

Here,  $\tau$  is a topology. In other words, a metric induces a topology.

(Proof as homework.)

4. The Zariski-topology.

**Example 8.2.** List of natural topologies.

1. On  $\mathbb{R}^n$  the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of  $\mathbb{R}^n$  are arbitrary unions of open balls. In other words, if  $A \in \mathcal{O}_{\mathbb{R}^n}$  and  $I$  is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{x \in \mathbb{R}^n \mid d(p, x) < r\}.$$

This definition agrees with the topology endowed on arbitrary metric spaces.

**Remark.** The set of all homeomorphisms of  $X$  to itself  $\text{Homeo}(X)$  is a group with composition as its operation.

**Remark.** This lemma does not hold for basis.

**Remark.** 1.  $\tau_{X \times Y}$  is the most coarse topology for which both of the projections are continuous.

2.  $\tau_{X \sqcup Y}$  is the finest topology for which both the inclusions are continuous.

Note about product topology:  $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$ ; often  $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

**Remark.** <sup>1</sup> Let  $(X, \mathcal{O})$  be a **topological space**. A **subset** that is **both open** and **closed** is called **clopen**. Moreover, a subset is **clopen** if and only if its **boundary** is **empty**.

*Proof.* Let  $A \subset X$  be clopen. Because  $A$  is closed, we have  $\text{cl}(A) = A$ , but on the other hand,  $A$  is open, so we also have  $\text{int}(A) = A$ . Then, the boundary of  $A$  is  $\partial A = \text{cl}(A) \setminus \text{int}(A) = A \setminus A = \emptyset$ . All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement.  $\square$

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<sup>1</sup>The following is a definition and a small proposition.





## Chapter 2

# Connected Spaces and Sets

### 2.1 Definition and Theorems

**Definition 9.** A topological space  $(X, \mathcal{O})$  is said to be **connected**, if one of the following **equivalent** conditions is met.

1.  $X$  is **not** a **union** of two **nonempty**, **disjoint**, and **open subsets**, i.e. there are no open subsets  $A, B \in \mathcal{O}$  with  $A, B \neq \emptyset$  and  $A \cap B = \emptyset$  such that  $A \sqcup B = X$ .
2. The **only subsets** of  $X$  that are **both open** and **closed** (**clopen**) are the empty set  $\emptyset$  and the entire set  $X$ , i.e. if  $A \subset X$  is a subset with  $A \in \mathcal{O}$  and  $X \setminus A \in \mathcal{O}$ , then  $A = \emptyset$  or  $A = X$ .
3. The **only subsets** of  $X$  with empty **boundary** are the emptyset  $\emptyset$  and the entire set  $X$ .
4. All **continuous** maps from  $X$  to the two point space  $\{0, 1\}$  endowed with the **discrete** topology is **constant**.

A **subset** of  $X$  is **connected** if it is a **connected space** when viewed as a **subspace** of  $X$ .

**Lemma 10.** Any **interval**  $I \subset \mathbb{R}$  is **connected**.

**Lemma 11.** Let  $X$  and  $Y$  be **topological spaces** and  $f : X \rightarrow Y$  a **continuous function**. If  $X$  is **connected**, then  $f(X) \subset Y$  is **connected**.

**Definition 12.** A connected component of a topological space is a maximally connected subset  $X_0 \subseteq X$ , i.e.  $X_0$  connected and for all  $X_0 \subsetneq X_1$  then  $X_1$  is not connected.

**Proposition 13.** Connected components are closed subsets.

**Lemma 14** (Lemma 11). Let  $X$  be connected and  $f : X \rightarrow Y$  and locally constant, i.e. for all  $x \in X$  there exists a  $U_x \in \mathcal{O}_X$ ,  $x \in U_x$  such that  $f$  restricted on  $U_x$  is identical to  $f(x)$ , then  $f$  is constant.

**Definition 15.**  $X$  is said to be **path connected**, if for every pair of points  $x$  and  $x_0$  in  $X$  there is a continuous map (called path)  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x_0$  and  $\gamma(1) = x$ .

**Lemma 16.** If  $X$  is path connected, then it is also connected.

## 2.2 Proofs, Remarks, and Examples

**Definition 17.** A topological space  $(X, \mathcal{O})$  is said to be **connected**, if one of the following **equivalent** conditions is met.

1.  $X$  is **not** a union of two **nonempty**, **disjoint**, and **open** subsets, i.e. there are no open subsets  $A, B \in \mathcal{O}$  with  $A, B \neq \emptyset$  and  $A \cap B = \emptyset$  such that  $A \sqcup B = X$ .
2. The **only** subsets of  $X$  that are **both** **open** and **closed** (**clopen**) are the empty set  $\emptyset$  and the entire set  $X$ , i.e. if  $A \subset X$  is a subset with  $A \in \mathcal{O}$  and  $X \setminus A \in \mathcal{O}$ , then  $A = \emptyset$  or  $A = X$ .
3. The **only** subsets of  $X$  with empty **boundary** are the emptyset  $\emptyset$  and the entire set  $X$ .
4. All **continuous** maps from  $X$  to the two point space  $\{0, 1\}$  endowed with the **discrete** topology is **constant**.

A **subset** of  $X$  is **connected** if it is a **connected** space when viewed as a **subspace** of  $X$ .

*Proof.* We verify the equivalence of the different definitions. So, let  $(X, \mathcal{O})$  be a topological space.

- “1.  $\Rightarrow$  2.”: Assume that  $X$  is not a union of two nonempty, disjoint, and open subsets. Fix a subset  $A \in X$  that is clopen. If  $A$  is neither the empty set nor  $X$ , then  $X \setminus A$  is also not the empty set nor  $X$ . Clearly,  $A$  and  $X \setminus A$  are disjoint and they are also open because  $A$  is clopen. But  $A \sqcup B = X$ , so our assumption was absurd. It must be that  $A = \emptyset$  or  $A = X$ .
- “2.  $\Rightarrow$  1.”: Now let the only clopen set contained in  $X$  be the empty set or  $X$  itself. Assume there are  $A, B \in \mathcal{O}$  with  $A, B \neq \emptyset$  and  $A \cap B = \emptyset$  such that  $A \sqcup B = X$ . Then,  $A$  is open, but also closed because  $X \setminus A = B$  is open. Furthermore,  $A$  is not empty and since  $B$  is also not empty,  $A \neq X$ . Hence our assumption was wrong and there no nonempty, disjoint, and open subsets  $A$  and  $B$  such that  $A \sqcup B = X$ .
- “2.  $\iff$  3.”: This is one of the properties of clopen subsets and was proven in remark XXX.
- “1.  $\Rightarrow$  4.”: Let  $X$  not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function  $f : X \rightarrow \{0, 1\}$  with regards to the discrete topology that is not constant. Then,  $f^{-1}(\{0\})$  and  $f^{-1}(\{1\})$  are nonempty sets that are also disjoint. Since  $f$  is continuous, these are also open subsets. But we also have  $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$ .
- “4.  $\Rightarrow$  1.”: Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets  $A, B \in \mathcal{O}$  such that  $A \sqcup B = X$ . Define  $f : X \rightarrow \{0, 1\}$  as  $f(A) = 0$  and  $f(B) = 1$ . This definition is well-defined because  $A, B \in \mathcal{O}$  are nonempty, disjoint, and  $A \sqcup B = X$ .  $f$  is also continuous as the preimage of  $\{0\}$  and  $\{1\}$  are  $A$  and  $B$  respectively which are open subsets. Hence our assumption was wrong.

□

**Lemma 18.** Any **interval**  $I \subset \mathbb{R}$  is **connected**.

*Proof.* Fix an interval  $I \subset \mathbb{R}$ , and let  $A, B \subset \mathbb{R}$  be two nonempty, open and disjoint subsets such that  $A \sqcup B = I$ . Moreover, let  $a \in A$  and  $b \in B$  and assume without loss of generality that  $a < b$ . If we set

$$s := \inf \{ x \in B \mid a < x \}, \quad (2.1)$$

then  $s \in I$  because  $s$  is between  $a$  and  $b$  and we have  $[a, b] \subset I$ .

Now, on one side, we have  $s \in \text{cl}(B)$  and since the complement of  $B$  is an open subset  $A$ , so  $B = \text{cl}(B)$ . It is therefore  $x \in B$ .

But we also have  $s \in A$  because the infimum cannot be contained in an open set, but  $s \in I = A \sqcup B$ .  $\square$

**Lemma 19.** Let  $X$  and  $Y$  be **topological spaces** and  $f : X \rightarrow Y$  a **continuous function**. If  $X$  is **connected**, then  $f(X) \subset Y$  is **connected**.

*Proof.* Let  $f(X) = A \sqcup B$  with  $A$  and  $B$  being two open disjoint sets.  $f^{-1}(A)$  and  $f^{-1}(B)$  are open since  $f$  is continuous. We also have  $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$  so  $f^{-1}(A) = \emptyset$  or  $f^{-1}(B) = \emptyset$ , so  $A = \emptyset$  or  $B = \emptyset$  and we are done.  $\square$

**Remark.** The two lemma above are handy to show that images of functions are connected.

**Example 19.1.** The general linear group  $\text{GL}_n(K)$  for a field  $K$  and  $n \in \mathbb{N}$  is not connected for  $K = \mathbb{R}$  and  $K = \mathbb{C}$ .

**Definition 20.** A connected component of a topological space is a maximally connected subset  $X_0 \subseteq X$ , i.e.  $X_0$  connected and for all  $X_0 \subsetneq X_1$  then  $X_1$  is not connected.

**Example 20.1.** For  $\mathbb{Q} \subset \mathbb{R}$  the connected components are points and those are not open.

**Proposition 21.** Connected components are closed subsets.

*Proof.* Locally constant implies continuous with regards to the discrete topology on  $Y$ . Let  $x \in X$ ,  $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$  is a disjoint union and since  $X$  is connected  $f^{-1}(Y \setminus \{f(x)\}) = \emptyset$ . Conclude  $f$  is identical to  $f(x)$ .  $\square$

**Application:**  $f : X \rightarrow \{0, 1\}$ ,  $X$  is connected,  $f$  locally constant, there is a  $x \in X$  such that  $f(x) = 1$ , then  $f$  is identical to 1.

*Proof.* Let  $A$  and  $B$  two disjoint open sets such that  $A \sqcup B = X$ , and let  $a \in A$  and  $b \in B$ . Let  $\gamma : [0, 1] \rightarrow X$  be continuous path with  $\gamma(0) = a$  and  $\gamma(1) = b$ . We have that  $\gamma^{-1}$   $\square$

## 2.3 Exercises and Notes

**Remark.** If  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  are two connected topological spaces, then their product  $X \times Y$  with the product topology  $\mathcal{O}_{X \times Y}$  is also connected.

*Proof.* We will use the definition that all continuous maps from  $X \times Y$  to  $\{0, 1\}$  endowed with the discrete topology must be constant. Fix a continuous  $f : X \times Y \rightarrow \{0, 1\}$ .

First, consider the image  $f(\{x\} \times Y)$  with  $x \in X$ . Assume  $f$  is not constant on  $\{x\} \times Y$ , then  $f(\{x\} \times Y) = \{0, 1\}$ . So we have the preimages  $f^{-1}(\{0\}) = \{x\} \times U$  and  $f^{-1}(\{1\}) = \{x\} \times V$  with  $U, V \subset Y$ ,  $U, V \neq \emptyset$ , and  $U \cap V = \emptyset$ . Because  $f$  is continuous,  $U$  and  $V$  must also be open. This would however mean that  $U \sqcup V = Y$  and  $Y$  would not be connected, therefore,  $f$  is constant on  $\{x\} \times Y$ . Similarly, we get that  $f$  is constant on  $X \times \{y\}$  for all  $y \in Y$ .

Let  $(x, y) \in X \times Y$  and  $(x', y') \in X \times Y$  be two arbitrary points. We have  $f(x, y) = f(x, y')$  because  $f$  is constant on  $\{x\} \times Y$  and similarly  $f(x, y') = f(x', y')$  because  $f$  is constant on  $X \times \{y'\}$ . Putting everything together, it is  $f(x, y) = f(x', y')$ , therefore all continuous  $f : X \times Y \rightarrow \{0, 1\}$  are constant.  $\square$

**Example 21.1.** Clearly, the union of two connected sets need not be connected. Take for example  $[0, 1] \subset \mathbb{R}$  and  $[2, 3] \subset \mathbb{R}$ . Their union  $[0, 1] \cup [2, 3]$  is not connected.

Set difference of connected sets are also not necessarily connected, e.g.  $[0, 2] \subset \mathbb{R}$  and  $\{1\} \subset \mathbb{R}$  are connected, but  $[0, 2] \setminus \{1\} = [0, 1] \cup (1, 2]$  is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin  $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$  and another unit circle around  $(1, 0)$   $A := \{(x, y) \mid (x - 1)^2 + y^2 = 1\}$ . They are both connected, but their intersection is a two point set

$$\left\{ \left( \frac{1}{2}, \frac{1}{2}\sqrt{3} \right), \left( \frac{1}{2}, -\frac{1}{2}\sqrt{3} \right) \right\}$$

which is not connected.

**Example 21.2.** Connectedness does not imply path-connectedness. Let  $\mathbb{R}^2$  be endowed with the Euclidean topology and consider

$$X = \left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \cup (\{0\} \times [-1, 1]) \subset \mathbb{R}^2.$$

and see figure XXX.  $X$  is connected, but it is not path-connected.

*Proof.* Denote

$$A := \left\{ \left( x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \quad B := \{0\} \times [-1, 1],$$

then  $X = A \sqcup B$ .

1. First, define  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^2$  as

$$f(x) := \left( x, \sin\left(\frac{1}{x}\right) \right).$$

$f$  is continuous,  $\mathbb{R}^+$  is an interval, therefore connected, so  $f(\mathbb{R}^+) = A$  is connected. On the other hand,  $\{0\}$  and  $[-1, 1]$  are connected and so is their product  $B$ .

Assume there is a clopen subset  $S \subset X$  that is not empty. Without loss of generality, we have that  $(0, 0) \in U$  (otherwise, consider the complement of  $U$  which also must be clopen). Since  $A$  is clopen in  $A$ , the intersection  $A \cap U$  must also be clopen in  $A$ , but  $A$  is connected, so  $A$  is contained in  $U$ .

Moreover, the closure of  $A$  is also contained in  $U$ . So there is an  $\epsilon > 0$  such that the ball  $B(p, \epsilon)$  that contains  $(0, 0)$  is in  $U$ . I got lazy to go into the details, but this ball contains a point of  $B$ . Follow the same reason as above.

- 2.

$\square$



## Chapter 3

# Separation Axioms

Literature: Groessere Liste in Sten, Seibeck

**Definition 22** ( $T_1$  Space). Let  $X$  be a topological space.

1. We say that two points  $x$  and  $y$  can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space  $X$  is a  $T_1$  space if any two distinct points in  $X$  are separated.

**Proposition 23.** Let  $X$  be a topological space. Then, the following are equivalent.

1.  $X$  is a  $T_1$  space.
2. Points are closed in  $X$ , i.e. given any  $x \in X$ , the singleton set  $\{x\}$  is a closed set.

**Definition 24** ( $T_2$  Space). Let  $X$  be a topological space.

1. Points  $x$  and  $y$  in  $X$  can be separated by neighborhood if there exists a neighborhood  $U$  of  $x$  and a neighborhood  $V$  of  $y$  such that  $U$  and  $V$  are disjoint, i.e.  $U \cap V = \emptyset$ .
2. A topological space  $X$  is a  $T_2$  space if any two distinct points in  $X$  are separated by neighborhood.

**Proposition 25.** Let  $X$  be a topological space. Then, the following are equivalent.

1.  $X$  is a  $T_2$  space.
2. Any singleton set  $\{x\}$  is the intersection of all closed neighborhoods of  $x$ .
3. The diagonal  $\Delta = \{(x, x) \mid x \in X\}$  is closed as a subset of the product space  $X \times X$ .

**Proposition 26.**  $T_2$  spaces are also  $T_1$  spaces.





## Chapter 4

# Compact Spaces

**Definition 27.** 1. A topological space  $X$  is called compact if each of its open cover has a finite subcover.

2. A topological space  $X$  is called sequentially compact if every sequence in  $X$  has a convergent subsequence whose limit is in  $X$ .

**Theorem 28.** Satz 17

**Theorem 29.** Let  $A \subset \mathbb{R}^n$  be a subset.  $A$  is compact if and only if it is closed and bounded.

**Theorem 30.** Let  $X$  be a  $T_2$  space. If a subset  $K \subset X$  is compact, then it is closed.

**Theorem 31.** Let  $X$  and  $Y$  be topological spaces,  $X$  compact, and  $Y$  be a  $T_2$  space. If  $f : X \rightarrow Y$  is bijective and continuous, then the inverse function  $f^{-1}$  is continuous.

## 4.1 Proofs, Remarks, and Examples

**Lemma 32.**  $[0, 1] \subset \mathbb{R}$  is compact.