1 Probability Space

Definition 1 (Sample Space)**.** An outcome is a possible result of an experiment or trial. The sample space (also called sample discription space or possibility space) of an experiment or random trial is the set of all possible outcomes or results of that experiment. A subset of a sample space is called an event. Moreover, we say the following.

- The empty event $A = \varnothing$ is called the impossible event.
- The sample space itself as an event $A = \Omega$ is called the certain event.
- The complementary event of any event A is the set of all outcomes not in A.

Definition 2 (Probability Measure)**.** Let Ω be a set and $\mathcal{A}$ a σ-algebra over Ω. A function $\mathbb{P} : \mathcal{A} \to [0, 1]$ is a probability measure if it is a measure and if $\mathbb{P}(\Omega) = 1$.

Definition 3 (Probability Space)**.** A probability space is the structrue $(\Omega, \mathcal{A}, \mathbb{P})$ consisting out of a sample space Ω, a σ-algebra over Ω and a probability measure $\mathbb{P} : \mathcal{A} \to [0, 1]$.

Definition 4 (Random Variable)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(S, \mathcal{S})$ a measureable space. A random variable is a measure function $X : \Omega \to S$. The distribution of a random variable X is given by the probability measure

$$\mathbb{P}^X(B) := \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathcal{S}. \tag{1}$$

Definition 5 (Probability Mass Function)**.**

Definition 6 (Cumulative Distribution Function)**.** For a probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathfrak{B}_\mathbb{R})$ is the respective cumulative distribution function is given by $F(x) := \mathbb{P}((-\infty, x])$ with $x \in \mathbb{R}$.

Lemma 6.1 (Properties of CDF)**.**

2 Discrete Distribution

Definition 7 (Discrete Probability Space)**.**

Lemma 7.1. If $(\Omega, \mathcal{A}), \mathbb{P}$ is a discrete probability space, then $\mathbb{P}$ is uniquely defined by its counting measure $p : \Omega \to [0, 1]$ with $p(\omega) = \mathbb{P}(\{\omega\})$.

Similary, if X is a discretely distributed random varible, then $\mathbb{P}^X$ is uniquely defined by its counting measure $p^X(x) = \mathbb{P}(X = x)$.

Proof. Because Ω is countable, its σ-algebra is simply $\mathcal{P}(\Omega)$. We then have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} \mathbb{P}(\{\omega\}) = \sum_{\omega \in A} p(\omega). \tag{2}$$

Same argument can be applied on $\mathbb{P}^X$. $\square$

Lemma 7.2. Lemma 1.3.3.

Theorem 7.1 (Results from Combinatorics)**.**

Definition 8 (Uniform Distribution)**.**

Definition 9 (Hypergeometric Distribution)**.**

Definition 10 (Bernoulli-Chain).

Definition 11 (Binomial Distribution).

Definition 12 (Multinomial Distribution).

Definition 13 (Geometric Distribution). The geometric distribution with success chance $p \in (0,1]$ on $\Omega = \mathbb{N}$ is given by the discrete probability mass function

$$p_{\mathrm{Geo}(p)}(k) = (1-p)^{k-1}p, \qquad k \in \mathbb{N}. \tag{3}$$

If $p^X = p_{\mathrm{Geo}(p)}$ for a random variable X, we say X is geometrically distributed and denote $X \sim \mathrm{Geo}(p)$.

Definition 14 (Poisson Distribution).

Theorem 14.1 (Poisson Limit Theorem).

3 Continuous Distribution

Definition 15. The exponential distribution with parameter $\lambda > 0$ is given by the continuous probability mass function

$$f_{\mathrm{Exp}(\lambda)}(x) = \lambda e^{-\lambda x}\mathbb{1}_{[0,\infty)}(x), \qquad x \in \mathbb{R}. \tag{4}$$

If $f^X = f_{\mathrm{Exp}(\lambda)}$ for a real-valued random variable X we write $X \sim \mathrm{Exp}(\lambda)$.

Definition 16. The one-dimensional normal distribution with the parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ is given by its probability mass function

$$\phi_{\mu,\sigma^2}(x) := f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}. \tag{5}$$

If $f^X = \phi_{\mu,\sigma^2}$ for a real-valued random variable X, we write $X \sim N(\mu, \sigma^2)$ and say that X is normally distributed.

Lemma 16.1 (Stuff about normal distribution).