Exercise 2

Show that $\mathcal{O} \subset \mathcal{B}(\mathbb{R})$ given by

$$\mathcal{O} := \{\varnothing\} \cup \left\{ \bigcup_{i \in I} [a_i, b_i) \mid -\infty < a_i < b_i < +\infty \right\} \tag{1}$$

defines a topology that is not the discrete topology. Show that the connected components in $(\mathbb{R}, \mathcal{O})$ consists of only one point.

Proof. 1. We show that $\mathcal{O}$ is a topology by verifying the axioms of a topology.

   (a) Clearly, $\varnothing \in \mathcal{O}$. $\mathbb{R}$, on the other hand, is a union of all $[k, k+1)$ with $k \in \mathbb{Z}$, so $\mathbb{R} \in \mathcal{O}$.

   (b) Let I be an arbitary index set and $\{A_i\}_{i \in I}$ be a family of subsets in $\mathcal{O}$. Each A_i consists of unions of right-open intervals, hence $\bigcup_{i \in I} A_i$ is also a union of right-open intervals, and therefore, included in $\mathcal{O}$.

   (c) Now let I be a finite index set and A_i be subsets of $\mathcal{O}$ with $i \in I$. Again, each A_i is a union of right-open intervals. A finite intersection of such subsets will again be an union of right-open intervals (one could show this by going through each possible case). Hence, $\bigcap_{i \in I} A_i \in \mathcal{O}$.

2. $\mathcal{O}$ is not discrete because for example it does not contain (a, b) for $a \neq b$ and $a < b$.

3. We will show that each connected component in $(\mathbb{R}, \mathcal{O})$ is a singleton. Let $A \in \mathcal{O}$ be connected and set $p_1 := \inf A$ and $p_2 := \sup A$. $[p_1, p_2)$ is connected because all intervals in $\mathbb{R}$ are connected. Assume there is a $p_1 \leq c \leq p_2$, but then $[p_1, p_2) = [p_1, c) \cup [c, p_2)$. Because of connectedness of $[p_1, p_2)$ this is only possible if $p_1 = c = p_2$, therefore $A = \{p_1\} = \{p_2\}$.

□

Exercise 3

1. Show that $\left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected, but not path-connected.

   Proof. Denote $S := \left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1]$.

   (a) We show S is connected. Assume otherwise. Then, there are disjoint open subsets $A, B \subset \mathbb{R}^2$ such that $S = A \sqcup B$.

   □