Exercise Sheet 1

Exercise 3
Let $R = \mathbb{Z}[i]$.

1. Find a factorization of $2i$ into irreducible elements.
2. Use the factorization $x^2 + 1 = (x - i)(x + i)$ to find all integral solutions to the equation $x^2 + 1 = y^3$.

Solution
<u>1:</u> We have $2i = (1 + i)(1 - i)$. As $N(1 + i) = N(1 - i) = 2$ these are irreducible elements.
<u>2:</u> From Theorem 1.0.1. we have that $y^3 = 2$ or $y^3 \equiv 1 \mod 4$. Since $y^3 = 2$ is not possible, we have $x^2 + 1 = y^3 = 1 + 4n$ for some $n \in \mathbb{N}$. So y is odd and x is even.

Therefore, the only integral solutions is $x = 0$ and $y = 1$.