

Notes on Algebraic Geometry

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For this section, let K be a fixed algebraically closed field, and $A = K[X_1, X_2, \dots, X_n]$

Definition 0.0.1. We define affine n -space over K , denoted \mathbb{A}^n or simply \mathbb{A}^n , to be the set of all n -tuples of elements of K . An element $p \in \mathbb{A}^n$ will be called a point, and if $p = (a_1, a_2, \dots, a_n)$ with $a_i \in k$, then the a_i will be called the coordinates of p .

Definition 0.0.2. 1. The set of zeros for $f \in A = K[X_1, X_2, \dots, X_n]$ is

$$\{p \in \mathbb{A}^n \mid f(p) = 0\}.$$

2. For a subset T of the polynomial ring $A = k[X_1, \dots, X_n]$, we define the zero set of T to be the common zeros of all the elements of T , i.e.

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

Remark 0.0.3. 1. If \mathfrak{a} is generated by T , then $Z(T) = Z(\mathfrak{a})$.

2. $Z(T)$ can be written in finitely many generators.

Definition 0.0.4. A subset Y of \mathbb{A}^n is an algebraic set if there exists a subset $T \subset A = k[X_1, \dots, X_n]$ such that $Y = Z(T)$.

Definition 0.0.5. An affine algebraic variety is an irreducible closed subset of \mathbb{A}^n . An open subset of an affine variety is a quasi-affine variety.

Corollary 0.0.6. *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

Definition 0.0.7. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of Y , to be $A/I(Y)$.

Definition 0.0.8. If X is a topological space, we define the dimension of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \dots \subset Z_n$ of distinct irreducible closed subsets of X . We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

Exercise 0.0.1. Let $Y \subset \mathbb{A}^3$ be the set $Y = \{t, t^2, t^3 \mid t \in k\}$.

1. Show that Y is an affine variety of dimension 1.

Solution. By definition 0.0.5, affine algebraic varieties are irreducible closed subsets of \mathbb{A}^n .

1. The topology on \mathbb{A}^n is the Zariski topology which is defined through closed sets. Closed sets are algebraic sets. Algebraic sets are zeros of some polynomials. Choose $(Y - X^2, Z - X^3)$.

2. By corollary [0.0.6](#), we want to show $(Y - X^2, Z - X^3)$ is prime.

Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.