Problem 1

1. Show that the product topology $\mathcal{O}_{X\times Y}$ is the coarsest topology on $X \times Y$ such that both projections are continuous.

   Proof. Assume there is a topology coarser than the product topology that meets the given condition, i.e. say $\mathcal{O} \subset \mathcal{O}_{X\times Y}$. Then, there is a $U \times V$ that is contained in $\mathcal{O}_{X\times Y}$, but not in $\mathcal{O}$. Because both projections are continuous for $\mathcal{O}$, we have

   $$\pi_X^{-1}(U) = U \times Y \in \mathcal{O}$$
   $$\pi_Y^{-1}(V) = X \times V \in \mathcal{O}.$$

   Now, intersection of open subsets should be open again, but $(U \times Y) \cup (X \times V) = U \times V$ will not do. □

2. Show that the topology $\mathcal{O}_{X\sqcup Y}$ of the topological sum is the finest scotch on the disjoint union such that both inclusions are continuous.

   Proof. The idea is the same as above. Again, assume there is a scotch finer than the topological sum and denote it by $\mathcal{O}$. Then, there is a $U \sqcup V$ in $\mathcal{O}$, but not in $\mathcal{O}_{X\sqcup Y}$. We have

   $$i_X^{-1}(U \sqcup 0) = U \in \mathcal{O}_X$$
   $$i_Y^{-1}(1 \sqcup V) = V \in \mathcal{O}_Y$$

   □

Exercise 2

Show that $\mathcal{O} \subset \mathcal{B}(\mathbb{R})$ given by

$$\mathcal{O} := \{\varnothing\} \cup \left\{ \bigcup_{i\in I} [a_i, b_i) \mid -\infty < a_i < b_i < +\infty \right\} \tag{1}$$

defines a topology that is not the discrete topology. Show that the connected components in $(\mathbb{R}, \mathcal{O})$ consists of only one point.

Proof.

1. We show that $\mathcal{O}$ is a topology by verifying the axioms of a topology.
   (a) Clearly, $\varnothing \in \mathcal{O}$. $\mathbb{R}$, on the other hand, is a union of all $[k, k+1)$ with $k \in \mathbb{Z}$, so $\mathbb{R} \in \mathcal{O}$.
   (b) Let I be an arbitary index set and $\{A_i\}_{i\in I}$ be a family of subsets in $\mathcal{O}$. Each A_i consists of unions of right-open intervals, hence $\bigcup_{i\in I} A_i$ is also a union of right-open intervals, and therefore, included in $\mathcal{O}$.
   (c) Now let I be a finite index set and A_i be subsets of $\mathcal{O}$ with $i \in I$. Again, each A_i is a union of right-open intervals. A finite intersection of such subsets will again be an union of right-open intervals (one could show this by going through each possible case). Hence, $\bigcap_{i\in I} A_i \in \mathcal{O}$.
2. $\mathcal{O}$ is not discrete because for example it does not contain (a, b) for $a \neq b$ and $a < b$.
3. We will show that each connected component in $(\mathbb{R}, \mathcal{O})$ is a singleton. Let $A \in \mathcal{O}$ be connected and set $p_1 := \inf A$ and $p_2 := \sup A$. $[p_1, p_2)$ is connected because all intervals in $\mathbb{R}$ are connected. Assume there is a $p_1 \leq c \leq p_2$, but then $[p_1, p_2) = [p_1, c) \cup [c, p_2)$. Because of connectedness of $[p_1, p_2)$ this is only possible if $p_1 = c = p_2$, therefore $A = \{p_1\} = \{p_2\}$.

□

Exercise 3

1. Show that $\left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected, but not path-connected.

   Proof. Denote $S := \left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1]$.

   (a) We show S is connected. Assume otherwise. Then, there are disjoint open subsets $A, B \subset \mathbb{R}^2$ such that $S = A \sqcup B$.

   $\square$

Exercise 4

1. Let $(X, \mathcal{O})$ be a topological space with a basis $\mathcal{B}$. Prove the following.

   (a) For any two basis elements $B_1, B_2 \in \mathcal{B}$ and a point $x \in B_1 \cap B_2$ there exists $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

   Proof. Since the intersection of two open subset is again open, $B_1 \cap B_2$ is open. This means that x is contained in at least one open subset. Now assume that none of the open subsets that contains x and is contained by $B_1 \cap B_2$ is not a basis element of $\mathcal{B}$. But such a subset cannot be constructed through unions of basis elements even though at least one of such subsets must exist. Hence the assumption was false and a $x \in B \subset B_1 \cap B_2$ is contained in $\mathcal{B}$. $\square$

   (b) For any $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

   Proof. Fix an $x \in X$. Since X itself is open, x is contained in at least one open subset. Assume there is no $B \in \mathcal{B}$ that contains x. Then again, none of the open subsets that contain x can be constructed through unions of basis elements, but at least one open subset that contains x exists. As above, conclude that there is a $B \in \mathcal{B}$ with $x \in B$. $\square$

2. Let $\mathcal{B} \subset \mathcal{B}(X)$ satisfy the two conditions above. Show that $\mathcal{B}$ is a basis of a topology on X.

   Proof. Let $A \in \mathcal{O}$. If $A = \varnothing$, then A is a union of the basis elements vacuously. In any other case, for all $x \in A$, we find a $B_x \in \mathcal{B}$ such that $x \in B_x \subset A$ $\square$