

# Topology

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## **Chapter 1**

# **Introduction**



## Chapter 2

# Topological Spaces

### 2-1

“ $\Rightarrow$ ”: Let  $f : X_1 \longrightarrow X_2$  be a homeomorphism and fix a subset (not necessarily open)  $U \in \mathcal{T}_1$ .

1. Assume  $U$  is open in  $X_1$ . Because  $f$  is continuous, the image of open subsets are again open, thus  $f(U)$  lies in  $\mathcal{T}_2$ .
2. On the other hand, if  $f(U)$  is open in  $X_2$ , then since  $f$  is bijective we have

$$f^{-1}(f(U)) = U.$$

Because  $f$  is continuous, the preimage of open subsets under  $f$  is open. We may therefore conclude  $U$  is open in  $X_1$ .

We have shown that if  $f$  is a homeomorphism, then  $f(\mathcal{T}_1) = \mathcal{T}_2$ .

“ $\Leftarrow$ ”: Let  $f : X_1 \longrightarrow X_2$  be a bijective map such that  $f(\mathcal{T}_1) = \mathcal{T}_2$ . Consider the inverse map  $f^{-1}$ . We want to show  $f^{-1}$  is continuous. Fix an open subset  $U \in \mathcal{T}_1$ . It is

$$(f^{-1})^{-1}(U) = f(U)$$

because  $f$  is bijective. Since  $f(\mathcal{T}_1) = \mathcal{T}_2$  and  $U$  is open,  $f(U)$  is open as well. Hence the preimage of  $U$  under  $f^{-1}$  is open and  $f^{-1}$  is continuous.

Now we show that  $f$  is also continuous. Again, fix an open subset  $V \in \mathcal{T}_2$ . The preimage of  $V$  under  $f$  is just the image of the inverse function. We have already shown that the inverse is continuous. Thus,  $f^{-1}(V)$  is open and  $f$  is continuous. Since  $f$  and  $f^{-1}$  exist and are continuous,  $f$  is a homeomorphism as desired.

### 2-2

#### a)

We show that  $\mathcal{T}$  is a topology by verifying the axioms of a topology.

1. Since  $\mathcal{T}$  is the collection of all unions of finite intersections of elements of  $\mathcal{B}$ , it contains the union of all elements of  $\mathcal{B}$  which is just  $X$ . The union of empty collection generates the emptyset so  $\emptyset \in \mathcal{T}$  as well.
2. Let  $\mathcal{U} \subset \mathcal{T}$  be any subset. The elements of  $\mathcal{U}$  are unions of finite intersections of elements of  $\mathcal{B}$ . Thus,  $\bigcup_{U \in \mathcal{U}} U$  is again a union of finite intersections of elements of  $\mathcal{B}$ . In other words,  $\mathcal{T}$  is closed under union.
3.  $\mathcal{T}$  is stable under finite intersections due to distributive property of sets.

b)

**2-3**

1.

The collection of subset  $\mathcal{T}_1 = \{U \subset X \mid X \setminus U \text{ is finite or is all of } X\}$  forms a topology. We show this by verifying the axioms of a topology.

1. It is  $X \setminus \emptyset = X$  and  $X \setminus X = \emptyset$  which is finite. Thus,  $X \in \mathcal{T}_1$  and  $\emptyset \in \mathcal{T}_1$ .
2. Let  $\mathcal{U} \subset \mathcal{T}$  be a subset. By De Morgan's laws we have

$$X \setminus \left( \bigcup_{U \in \mathcal{U}} U \right) = \bigcap_{U \in \mathcal{U}} (X \setminus U).$$

Since each  $U \in \mathcal{U}$  lies in  $\mathcal{T}$ , the complement  $X \setminus U$  is finite or is all of  $X$ . Therefore, the intersection of all  $X \setminus U$  is again finite or all of  $X$ , and we may conclude that  $\mathcal{T}$  is stable under arbitrary unions.

3. Use De Morgan's law again.

2.

The collection of subsets  $\mathcal{T}_2 = \{U \subset X \mid X \setminus U \text{ is infinite or is empty}\}$  is not a topology. Take  $X = \mathbb{Z}$  for example and consider  $A = \{1, 2, 3, \dots\}$  and  $B = \{-1, -2, -3, \dots\}$ .  $A$  and  $B$  are open because their complements are the non-positive and the non-negative integers respectively. If  $\mathcal{T}_2$  is a topology, it should contain their union  $A \cup B = \mathbb{Z} \setminus \{0\}$ . However,

$$\mathbb{Z} \setminus (A \cup B) = \mathbb{Z} \setminus (\mathbb{Z} \setminus \{0\}) = \{0\}$$

which is not infinite and thus doesn't lie in  $\mathcal{T}_2$ .

3.

The collection of subsets  $\mathcal{T}_3 = \{U \subset X \mid X \setminus U \text{ is countable or all of } X\}$  is a topology PROBABLY.

**2-4**

Already did somewhere else.

**2-5**

1.  $\text{id}_1 : X \longrightarrow \mathbb{R}^2$  is continuous probably.
2.  $\text{id}_2 : \mathbb{R}^2 \longrightarrow X$  is not continuous probably.

**2-6**

$f$  is continuous because any preimage of a subset  $U \subset Z$  under  $f$  is open, since any subset in  $X$  is open.

For  $g$ , the only preimages to check are the emptyset  $\emptyset$  and  $Y$ . Simply,  $g^{-1}(\emptyset) = \emptyset$  and  $g^{-1}(Y) = Z$ . Both subsets are open in  $Z$ , therefore  $g$  is continuous.

If  $h$  is constant, say  $h(Y) = \{p\}$ , then  $h^{-1}(U) = Y$  if  $p \in U$  and  $h^{-1}(U) = \emptyset$  if  $p \notin U$ . In both cases the preimages are open, thus  $h$  is continuous. Assume  $h$  is continuous but not constant, i.e. there are points  $x_1, x_2 \in Y$  such that  $h(x_1) \neq h(x_2)$ .  $Z$  is Hausdorff, so there are disjoint neighbourhoods  $U$  of  $h(x_1)$  and  $V$  of  $h(x_2)$ .  $h$  was assumed to be continuous, so  $h^{-1}(U) = Y$  and  $h^{-1}(V) = Y$  which is impossible (REALLY?).



## 2-7

a)

f)

## 2-8

Firstly, any element in  $f(\mathcal{B})$  is open because  $f$  is an open map. Fix an open subset  $V$  in  $Y$  and consider its preimage  $f^{-1}(V)$  under  $f$ . Because  $f$  is continuous, the preimage is open, thus there are base elements  $B_i$  with  $i \in I$  in  $\mathcal{B}$  such that

$$f^{-1}(V) = \bigcup_{i \in I} B_i.$$

The surjectivity of  $f$  grants us  $f(f^{-1}(V)) = V$ , therefore, we have

$$f(f^{-1}(V)) = V = f\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f(B_i).$$

Thus,  $f(\mathcal{B})$  is a basis of  $Y$ .

## 2-9

## 2-10

Fix a point  $y$  in  $Y$ . Since  $f$  is surjective, there is an  $x$  in  $X$  such that  $f(x) = y$ .  $X$  is locally Euclidean, thus there is a neighbourhood  $U$  of  $x$  that is homeomorphic to  $\mathbb{R}^n$ . Moreover,  $f$  is locally homeomorphic, so there is a neighbourhood  $V$  of  $x$  such that the restriction of  $f$  under  $V$  is a homeomorphism. Then, the intersection  $U \cap V = N$  has both of these properties, i.e.  $N$  is a neighbourhood of  $x$  that is homeomorphic to  $\mathbb{R}^n$  and the restriction of  $f$  under  $V$  is a homeomorphism.  $f(N)$  is a neighbourhood of  $y$  that is homeomorphic to  $\mathbb{R}^n$ , therefore  $Y$  is locally Euclidean.

## 2-11

“ $\Rightarrow$ ”: Let  $M^0$  be a 0-manifold and consider a point  $p \in M^0$ . First, we show that  $M^0$  is discrete. Since  $M^0$  is locally Euclidean, there is a neighbourhood  $U$  of  $p$  such that  $U$  is homeomorphic to an open subset of  $\mathbb{R}^0$ . But  $\mathbb{R}^0$  contains only one element, thus the only nonempty open subset is  $\mathbb{R}^0$ . Now, a homeomorphism implies bijectivity, we have that  $U = \{p\}$ . Every singleton set in  $M^0$  is open, so  $M^0$  is a discrete space.

$M^0$  is also countable because being a manifold implies that it has a countable base and any base must contain all the singleton sets.

“ $\Leftarrow$ ”: Let  $M^0$  be a countable discrete space.  $M^0$  is second-countable because the set of singletons form a countable base. It is also  $T_2$  since each point has itself as its neighbourhood which clearly does not contain any other points. Now let  $p \in M^0$  be a point.  $\{p\}$  is a neighbourhood of  $p$  and it is homeomorphic to  $\mathbb{R}^0$  by the mapping  $p \mapsto 0$ , thus  $M^0$  is locally Euclidean.

## 2-12

a

It is  $L(b) \cap R(a) = \{c \in X \mid c < b \text{ and } c > a\} = (a, b)$ , thus  $(a, b)$  is open.

Moreover, we have  $L(a) \cup R(b) = \{c \in X \mid c < a \text{ or } c > b\} = X \setminus [a, b]$  which is open, so  $[a, b]$  is closed.

**b**

Let  $a, b \in X$  be two distinct points and assume without loss of generality  $a < b$ . Then,  $L(b)$  is open and contains the point  $a$ , while  $R(a)$  is also open and contains the point  $b$ , but  $L(b)$  and  $R(a)$  are disjoint. Thus,  $X$  is Hausdorff.

**c**

Fix two points  $a, b \in X$ . By definition, it is

$$\overline{(a, b)} = \bigcap \{ C \subset X \mid (a, b) \subset C \text{ and } C \text{ is closed in } X \}.$$

We have shown that  $[a, b]$  is closed in  $a$ ) and clearly contains  $(a, b)$ , thus

$$[a, b] \in \{ C \subset X \mid (a, b) \subset C \text{ and } C \text{ is closed in } X \}$$

or in other words

$$\bigcap \{ C \subset X \mid (a, b) \subset C \text{ and } C \text{ is closed in } X \} \subset [a, b]$$

as desired.

When does  $\overline{(a, b)} = [a, b]$  hold? Maybe it is pertinent to ask when does it not hold? The equality does not hold if and only if  $(a, b)$  is already closed. That means  $X \setminus (a, b) = (-\infty, a] \cup [b, \infty)$  is open. I'm not sure, maybe  $X$  needs to be countable, finite?

## 2-13

Let  $X$  be a second countable topological space and fix a collection of disjoint open subsets  $\mathcal{S}$ , i.e.

$$\mathcal{S} = \{ U \subset X \mid U \text{ is open and for all } U, V \in \mathcal{S} \text{ it is } U \cap V = \emptyset \}.$$

We want to show  $\mathcal{S}$  is countable. If  $\mathcal{B}$  is a base for  $X$ , then for any two members of the collection  $U, V \in \mathcal{S}$ , we have

$$U = \bigcup_{i \in \mathbb{N}} B_i \quad V = \bigcup_{j \in \mathbb{N}} B_j.$$

Since  $U$  and  $V$  are disjoint,  $B_i$  and  $B_j$  are also disjoint for all  $i, j \in \mathbb{N}$ . Thus, any  $U \in \mathcal{S}$  is a union of base elements that is different from any other  $V \in \mathcal{S}$ .  $\mathcal{B}$  is countable, therefore  $\mathcal{S}$  must also be.

## 2-14

Let  $X$  be a locally Euclidean space. We show that  $X$  is first-countable. Let  $p \in X$  be a point, then since  $X$  is locally Euclidean, there is a neighbourhood  $N$  of  $p$  such that  $N$  is homeomorphic to  $\mathbb{R}^n$ . Thus, we have a sequence of neighbourhoods as  $\mathbb{R}^n$  is first-countable, yada yada yada.

Let  $M$  be a metric space. I've shown that this is first-countable already.

## 2-15

**a)**

Let  $X$  be a second-countable space. We want to show that  $X$  contains a dense subset that is countable.

## Chapter 3

# New Spaces from Old

3-1



## Chapter 4

# Simplicial Complexes

### Exercise 5.1

**Definition 1** (Simplex). Given points  $v_0, \dots, v_k$  in general position in  $\mathbb{R}^n$ , simplex spanned by them is the set of all points in  $\mathbb{R}^n$  of the form:

$$\sum_{i=0}^k t_i v_i \quad \text{where } 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^k t_i = 1.$$

**Definition 2** (Convex Hull). Let  $X$  be a subset of  $\mathbb{R}^n$ , then the convex hull of  $X$  is the intersection of all convex sets containing  $X$ .

**Definition 3** (Convex Set). A subset  $X$  of  $\mathbb{R}^n$  is convex if for all  $x, y \in X$  and for all  $t \in [0, 1]$  it is

$$(1-t)x + ty \in X.$$

*Proof.* Let  $\sigma$  be a simplex and denote its vertices by  $v_0, \dots, v_k$ . Fix a point  $p \in \sigma$ , then by definition

$$p = \sum_{i=0}^k t_i v_i$$

for some  $t \in [0, 1]$  and  $\sum_{i=0}^k t_i = 1$ . We show that  $p$  lies in any convex set containing  $v_0, \dots, v_k$  by induction. In the initial case,  $p = t_i v_i = v_i$  for some  $i$  which clearly lies in all convex set containing  $v_0, \dots, v_k$ . Consider the case where  $n+1$  number of  $t_0, \dots, t_k$  is nonzero. Without loss of generality, we may reorder the indices for notational ease and yield

$$p = \sum_{i=0}^{n+1} t_i v_i = t_{n+1} v_{n+1} + \sum_{i=0}^n t_i v_i.$$

By induction hypothesis,  $\sum_{i=0}^n t_i v_i$  is contained in any convex set containing  $v_0, \dots, v_k$ .  $\square$

### Exercise 5.2

a)

Fix two simplices  $\sigma$  and  $\tau$ , and denote the set of their vertices by  $\text{vert}(\sigma)$  and  $\text{vert}(\tau)$  respectively. Let  $f_0 : \text{vert}(\sigma) \rightarrow \text{vert}(\tau)$  be any map and consider a point  $p \in \sigma$ .  $p$  may be represented by a linear combination of the vertices, thus

$$p = \sum_{i=1}^k v_i$$

which allows us to define

$$f(p) := \sum_{i=1}^k f_0(v_i).$$

Since a simplex is the convex hull of its vertices,  $f(p)$  lies in  $\tau$ .

Unsure, but should be the right direction.

b)

sounds reasonable

c)

more suprising

### Exercise 5.3

#### Example 5.2 a)

Let  $K$  be the collection of a  $n$ -simplex  $\sigma$  and its faces. Trivially, the faces of  $\sigma$  lies in  $K$ , and the faces of its faces are just faces of  $\sigma$ , and thus are also members of  $K$ . The any intersection of  $\sigma$  and its faces are again faces or empty. (IF I UNDERSTOOD THIS CORRECTLY) since  $K$  is already finite the third condition also applies.

### Exercise 5.4

### Exercise 5.5

**Definition 4** (Simplex). Given points  $v_0, \dots, v_k$  in general position in  $\mathbb{R}^n$ , simplex spanned by them is the set of all points in  $\mathbb{R}^n$  of the form:

$$\sum_{i=0}^k t_i v_i \quad \text{where } 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^k t_i = 1.$$

**Definition 5** (Euclidean Simplicial Complex). A Euclidean simplicial complex is a collection  $K$  of simplices in some Euclidean space  $\mathbb{R}^n$  satisfying the following conditions:

1. If  $\sigma \in K$ , then every face of  $\sigma$  is in  $K$ .
2. The intersection of any two simplices in  $K$  is either empty or a face of each.
3. Every point in a simplex of  $K$  has a neighborhood that intersects at most finitely many simplices of  $K$ .

**Definition 6** (Abstract Complex). A collection  $\mathcal{K}$  of nonempty finite sets is called abstract complex if for all  $\sigma \in \mathcal{K}$  nonempty subsets of  $\sigma$  also lie in  $\mathcal{K}$ .

**Definition 7** (Vertex Scheme). Let  $K$  be an Euclidean simplicial complex and set

$$\mathcal{K} := \left\{ \{v_0, \dots, v_k\} \subset K^{(0)} \mid v_0, \dots, v_k \text{ are vertices of some simplex of } K \right\}$$

where  $K^{(0)}$  is the 0-skeleton of  $K$ , i.e. it is the set of all simplices of dimension 0 in  $K$ . In this case,  $\mathcal{K}$  is an abstract simplicial complex and it is called the vertex scheme of  $K$ .

Every finite abstract complex is the vertex scheme of a Euclidean simplicial complex.

*Proof.* Let  $\mathcal{K}$  be a finite abstract complex. □

## Exercise 5.6





## Chapter 5

# Homotopy and the Fundamental Group

### Exercise 7.1.

a)

Let  $f \sim g$ , then there is a homotopy  $H : I \times [0, 1] \longrightarrow X$  such that

$$H(x, 0) = f(x) \quad H(x, 1) = g(x).$$

On the other side of the equivalence, we have

$$\begin{aligned} f \cdot g^{-1}(s) &= \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g^{-1}(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(1 - (2s - 1)) & \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2 - 2s) & \frac{1}{2} \leq s \leq 1 \end{cases} \end{aligned}$$

Set therefore

$$G : I \times [0, 1] \longrightarrow X, \quad G(x, t) = \begin{cases} f \cdot g^{-1}(2x) & 0 \leq t \leq \frac{1}{2} \\ c_p & \frac{1}{2} \leq t \leq 1 \end{cases}$$