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My notes on ”The Strong Factorial Conjecture” by Eric Edo and Arno van den Essen.

See: https://arxiv.org/abs/1304.3956

# 1 Rigidity Conjecture

When we consider compositions of formal power series, we always want the constant term to be 0.

The following example is taken from:

https://math.stackexchange.com/questions/1212053/defining-composition-of-two-formal-series-what-is-going-on

**Example 0.1.** Let $f = \sum_{k\in\mathbb{N}_0} a_k X^k$ and $g = 1 + X$. Consider $f \circ g$. We have

$$\begin{aligned} f \circ g &= \sum_{k\in\mathbb{N}_0} a_k (1+X)^k \\ &= a_0 + a_1 + a_1 X + a_2 + 2a_2 X + a_2 X^2 + \cdots \end{aligned}$$

If $f \circ g$ is again a formal power series, then we should be able to write $f \circ g = \sum_{k\in\mathbb{N}_+} c_k X^k$ for some $c_k \in \mathbb{C}$. However, we see that $c_0$ is the sum of all $a_k$ and we cannot evaluate that as algebraists. Thus composition of formal power series only makes sense if the constant coefficient is 0.

**Proposition 1.** A power series $f(X) = \sum_{k\in\mathbb{N}_+} a_k X^k \in \mathbb{C}[[X]]$ has a compositional inverse $f^{-1}(X)$ if and only if $a_1 \neq 0$, in which case $f^{-1}(X)$ is unique.

*Proof.* Let $g(X) = b_1 X + b_2 X^2 + \cdots \in \mathbb{C}[[X]]$ be some power series that satisfies $f(g(X)) = X$. We have

$$\begin{aligned} X =& a_1(b_0 + b_1 X + b_2 X^2 + \cdots) + a_2(b_0 + b_1 X + b_2 X^2 + \cdots)^2 + \cdots && (1) \\ =& a_1 b_0 + a_2 b_0 + \cdots + && (2) \\ & a_1 b_1 X && (3) \end{aligned}$$

□

**Theorem 2** (Conjecture 2.13)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the first $m$ consecutive coefficient of the compositional inverse $a^{-1}(X)$ vanish, then $a(X) = X$.

**Theorem 3** (Conjecture 2.14)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the coefficients of $X^{n+1}, \ldots, X^{n+m}$ of the compositional inverse vanish, then $a(X) = X$.

**Remark.** $R(m)$ if and only if $R(m)_n$ for all $n \in \mathbb{N}_+$.

*Proof.* Let $R(m)$ be true for a $m \in \mathbb{N}_0$.

Then $R(m)_1$ is true, i.e. if $\deg(a) \leq m+1$ and if the □

**Remark.** If we denote the polynomial $a(X)$ by $\sum_{k\in\mathbb{N}_0} a_k X^k$ for some $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}_0$, then the condition $a(X) \equiv X \mod X^2$ amounts to $a_0 = 0$ and $a_1 = 1$.

Moreover, we have this:

A power series has a compositional inverse if and only if $a_1 \neq 0$. In that case, the inverse is unique.

See

https://www.amazon.com/dp/B00HMUGS4S

https://math.stackexchange.com/questions/2520744/finding-compositional-inverses-for-formal-power-series

My questions:

1. What if $a_0 \neq 0$? Pick $a_0 = 3$.

   Let $f \in \mathbb{C}[X]$ be a polynomial with $a_0 \neq 0$. Then we may write $f(X) = g(X) + a_0$ where $g$ has a compositional inverse. Thus it it

$$
\begin{aligned}
g^{-1}(g(X) + a_0) &= g^{-1}(g(X)) + g^{-1}(a_0) \\
&= X + g^{-1}(a_0)
\end{aligned}
$$

$$
\begin{aligned}
h(X) &= g^{-1}(X) + g^{-1}(a_0) \\
h(f(X)) &= h(g(X) + a_0) \\
&= g^{-1}(g(X) + a_0) + g^{-1}(a_0) \\
&= X
\end{aligned}
$$

   Let $f \in \mathbb{C}[X]$ be a polynomial with $a_1 \neq 1$ and $a_1 \neq 0$. Then we may write $f(X) =$

https://www.math.uwaterloo.ca/ dgwagner/co430I.pdf
proof

**Proposition 4.** 1. The polynomial $a(X)$ is invertible for the composition.

2. For all $i \in \{1, \dots, \deg(a-1)\}$, the coefficient $a_i$ is nilpotent element in $A$. I just don't see this ...

**Lemma 5** (Lagrange Inversion Formula)**.** Let $K$ be a field of charateristic

**Example 5.1** (See 5.4.4)**.** $f(X) = Xe^{-X} = X \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} X^k$

$$
[X^n]f^{-1}(X) = \frac{1}{n}[X^{n-1}]e^{nX}
$$

**Lemma 6** (Lemma 2.20 (Additive Inversion Formula))**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers. The formal inverse of $a(X) = X(1 - (\alpha_1 X + \cdots + \alpha_m X^m))$ is given by the following formula

$$
a^{-1}(X) = X\left(1 + \frac{1}{n+1}\sum_{n \geq 1} u_n X^n\right)
$$

where

$$
u_n = \frac{1}{n!} \sum_{j_1 + 2j_2 + \cdots + mj_m = n} \frac{(n + j_1 + \cdots + j_m)!}{j_1! \cdots j_m!} \alpha_1^{j_1} \cdots \alpha_m^{j_m}
$$

**Proposition 7** (Proposition 2.23)**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers and let $(u_n)_{n \in \mathbb{N}_+}$ be a sequence defined by AIF in Lemma 2.20. For all $n \in \mathbb{N}_+$, the Rigidity Conjecture $R(m)_n$ is equivalent to the following implication: If $u_n = \cdots = u_{n+m-1} = 0$ then $\alpha_1 = \cdots = \alpha_m = 0$.

*Proof.* □

**Theorem 8.** 1. The inclusion $E^{[m]} \subset F_n^{[m]}$ implies $R(m)_n$

**Definition 9.**

$$
\begin{aligned}
E^{[m]} &= \{ X_1 \cdots X_m(\mu_1 X_1 + \cdots + \mu_m X_m) \mid \mu_1, \dots, \mu_m \in \mathbb{C} \} \subset \\
F_n^{[m]} &= \left\{ f \in \mathbb{C}^{[m]} \setminus \{0\} \;\middle|\; \mathcal{L}(f^k) \neq 0 \text{ for some } n \leq k \leq \mathcal{N}(f) - 1 \right\} \cup \{0\}
\end{aligned}
$$