Integration and Integration

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March 28, 2021

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Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ-additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{2.1}$$

- subadditive if XXXXX
- σ-subadditive if XXXXX
- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ-finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in $\mathcal{R}$ with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \tag{2.2}$$

- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ-additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in $\mathcal{R}$ is required. For σ-rings and therefore σ-algebras, it is guranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \to [0, \infty]$ are σ-finite as X need not be included in a ring of sets.

Definition 2.2 (Content)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called a content if

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ-additive content $\mu \to [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. A σ-additive content $\mu : \mathcal{A} \to [0, \infty]$ is called a measure.