

# Notes on Algebraic Geometry

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August 21, 2024



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TODO

## Part I

# Pre: Commutative Algebra



1. Prove  $R$  int domain, then  $R[X]$  is int domain.

**Proposition 0.0.1.** *If  $R$  is an integral domain, then the polynomial ring  $R[X]$  is again an integral domain.*

*Proof.* 1. Since  $1 \in R \subset R[X]$ , the polynomial ring  $R[X]$  is nonempty.

2. Let  $f, g \in R[X]$  be two nonzero polynomials with

$$f = \sum_{i=0}^m a_i X^i \quad \text{and} \quad g = \sum_{j=0}^n b_j X^j.$$

Consider its product

$$f \cdot g = \sum_{k=0}^{m+n} c_k X^k$$

and suppose  $f \cdot g = 0$ .

3. Since the leading coefficient of the product  $c_{m+n}$  is obtained by multiplying the leading coefficients of  $f$  and  $g$ , we have  $c_{m+n} = a_m \cdot b_n$ .
4. We had  $f \cdot g = 0$ , thus  $c_{m+n} = a_m \cdot b_n = 0$ .
5.  $R$  is an integral domain, therefore  $a_m \cdot b_n = 0$  means  $a_m = 0$  or  $b_n = 0$ .
6. This contradicts that  $f$  and  $g$  were nonzero polynomials.

□





# Part II

## Topology



**Definition 0.0.2** (Product Topology).  $X = \prod_{i \in I} X_i$

$$\{ p_i^{-1}(U_i) \mid i \in I \text{ and } U_i \subset X_i \text{ is open in } X_i \}$$



## Part III

# Algebraic Geometry



# Chapter 1

## Affine Varieties

### Cheat Sheet

**Definition 1.0.1.** 1. The affine  $n$ -space  $\mathbb{A}^n$  over an algebraically closed field  $K$  is the set of all  $n$ -tuples of elements of  $K$ .

2. For a subset  $S \subset K[X_1, \dots, X_n]$ , we define the zero-locus as

$$Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

3. A subset  $V \subset \mathbb{A}^n$  is an algebraic set if there exists a subset  $S \subset K[X_1, \dots, X_n]$  such that  $V = Z(S)$ .

**Definition 1.0.2.** 1. Zariski topology.

2. Irreducible

## Full Text

### Algebraic Sets

**Definition 1.0.3.** Let  $K$  be an algebraically closed field and let  $n \in \mathbb{N}_0$  be a natural number.

1. The affine  $n$ -space over  $K$  is the set of all  $n$ -tuples of elements of  $K$ .
2. An element  $p$  in  $\mathbb{A}^n$  is called a point.
3. If  $p = (a_1, \dots, a_n) \in \mathbb{A}^n$  is a point, then  $a_i$  is called the coordinate for each  $1 \leq i \leq n$ .

**Intuition 1.0.4.** It's just space with points. But not vectors, because we don't add points.

**Definition 1.0.5.** For each subset  $S$  of polynomials in  $K[X_1, \dots, X_n]$ , we define the zero-locus  $Z(S)$  to be the set of points in the affine  $n$ -space  $\mathbb{A}^n$  on which the functions in  $S$  simultaneously vanish, i.e.

$$Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

**Intuition 1.0.6.** These are just curves.

**Remark 1.0.7.**

1. If  $\mathfrak{a}$  is generated by  $T$ , then  $Z(T) = Z(\mathfrak{a})$ .
2.  $Z(T)$  can be written in finitely many generators.

**Definition 1.0.8** (Algebraic Set). A subset  $V$  of  $\mathbb{A}^n$  is an algebraic set if there exists a subset  $S \subset k[X_1, \dots, X_n]$  such that  $V = Z(S)$ .

**Intuition 1.0.9.** So if the points on the space is a curve, then it's an algebraic set.



## Zariski Topology

**Definition 1.0.10.** Zariski topology on  $\mathbb{A}^n$ . Closed sets are algebraic sets.

**Example 1.0.11.** Consider  $\mathbb{A}^2$ .

1. Any point is a closed subset.
2. Any curve described by a polynomials is a closed subset.

**Definition 1.0.12.** Irreducible subsets

## Affine Algebraic Variety

**Definition 1.0.13.** An affine algebraic variety is an irreducible closed subset of  $\mathbb{A}^n$ . An open subset of an affine variety is a quasi-affine variety.

**Proposition 1.0.14** (1.2.).

**Theorem 1.0.15** (Hilbert's Nullstellensatz).

**Corollary 1.0.16.** *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

**Example 1.0.17.**  $\mathbb{A}^n$  is irreducible, since it corresponds to the zero ideal in  $K[X_1, \dots, X_n]$ , which is prime.

**Example 1.0.18.** Let  $f$  be an irreducible polynomial in  $K[X, Y]$ . Then  $f$  generates a prime ideal in  $K[X, Y]$ , since  $K[X, Y]$  is a UFD, so the zero set  $V = Z(f)$  is irreducible.

**Example 1.0.19.** 1. Consider  $Z((X + 1)(Y + 1)) = Z(XY + X + Y + 1)$  in  $\mathbb{A}^2$ . Clearly,  $X^2 - 1$  is not prime, hence it is not irreducible. We must have  $Z(XY + X + Y + 1) = Z(X + 1) \cup Z(Y + 1)$ . Indeed, the above example is just two lines crossing.

2. Consider  $Z((X^2 - Y)(Y + 1)) = Z(X^2Y + X^2 - Y^2 - Y)$ . This is a parabola and a line.

**Definition 1.0.20.** If  $f$  is an irreducible polynomial in  $K[X_1, \dots, X_n]$  then the affine variety  $H = Z(f)$

## Affine Coordinate Ring

**Definition 1.0.21.** If  $Y \subset \mathbb{A}^n$  is an affine algebraic set, we define the affine coordinate ring  $A(Y)$  of  $Y$ , to be  $A/I(Y)$ .

## Dimension

**Definition 1.0.22** (Noetherian Topological Space). A topological space  $X$  is called Noetherian if it satisfies the descending chain condition for closed subsets, i.e. for any sequence  $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \dots$  becomes stationary.

**Proposition 1.0.23** (1.5.). *In a Noetherian topological space  $X$ , every nonempty closed subset  $\mathcal{V}$  can be expressed as finite union of irreducible, closed subsets.*

**Corollary 1.0.24.** *Every algebraic set in  $\mathbb{A}^n$  can be expressed uniquely as a union of varieties, no one containing another.*

**Definition 1.0.25** (Dimension of a Topological Space).

**Definition 1.0.26** (Height of a Prime Ideal).

**Definition 1.0.27** (Dimension of a Ring).

**Theorem 1.0.28.** *Let  $K$  be a field, and let  $B$  be an integral domain which is a finitely generated  $K$ -algebra. Then:*

1. *the dimension of  $B$  is equal to the transcendence degree of the quotient field  $K(B)$  of  $B$  over  $K$*
2. *For any prime ideal  $\mathfrak{p}$  in  $B$ , we have*

$$\text{height } \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

BOOKMARK

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**Definition 1.0.29** (Affine Algebraic Variety). For an algebraically closed field  $K$  and a natural number  $n \in \mathbb{N}_+$ , let  $\mathbb{A}^n$  be an affine  $n$ -space over  $K$ . The polynomials in  $K[X_1, \dots, X_n]$  can be viewed as  $K$ -valued functions on  $\mathbb{A}^n$ .

1. For each subset  $S$  of polynomials in  $K[X_1, \dots, X_n]$ , define the zero-locus  $Z(S)$  to be the set of points in  $\mathbb{A}^n$  on which the functions in  $S$  simultaneously vanish, i.e.

$$Z(S) = \{ x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \}.$$

2. A subset  $V$  of  $\mathbb{A}^n$  is called affine algebraic set if  $V = Z(S)$  for some  $S \subset K[X_1, \dots, X_n]$ .
3. A nonempty affine algebraic set is called irreducible if it is not the union of two proper algebraic subsets. An irreducible affine algebraic set is also called an affine variety.

**Definition 1.0.30.** If  $X$  is a topological space, we define the dimension of  $X$  (denoted  $\dim X$ ) to be the supremum of all integers  $n$  such that there exists a chain  $Z_0 \subset Z_1 \subset \dots \subset Z_n$  of distinct irreducible closed subsets of  $X$ . We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

## 1.1 Exercises

**Exercise 1.1.1** (1.1. (a)). Let  $Y$  be the plane curve  $y = x^2$  (i.e.,  $Y$  is the zero set of the polynomial  $f = y - x^2$ ). Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.* By definition 1.0.21 of a coordinate ring, we simply have  $A(Y) = k[X, Y]/(Y - X^2)$ . The isomorphism follows from the isomorphism theorem and the map  $f : k[X, Y] \rightarrow k[X]$  where we set  $f(Y) = X^2$ .

**Exercise 1.1.2** (1.1. (b)). Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.*  $A(Z) = k[X, Y]/(XY - 1)$

We know  $A(Z)$  is an  $k$ -algebra (see remark). Consider  $f : k[X, Y] \rightarrow k[T]$ . We must have  $\ker f = (XY - 1)$ , thus  $f(XY - 1) = 0$ , so  $f(X) = 1/f(Y)$

I'll think about the rigorous details later, but basically  $A(Z) \cong k[X, X^{-1}]$

**Exercise 1.1.3** (1.1. (c)). Let  $f$  be any irreducible quadratic polynomial in  $k[X, Y]$ , and let  $W$  be the conic defined by  $f$ . Show that  $A(W)$  is isomorphic to  $A(Y)$  or  $A(Z)$ . Which one is it when?

*Solution.* Let  $f$  be irreducible.

$$A(W) = k[X, Y]/(f)$$

isn't this kinda clear ...? I'll come back to write it down rigorously, but in general ...

**Exercise 1.1.4.** Let  $V \subset \mathbb{A}^3$  be the set  $V = \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}$ .

1. Show that  $V$  is an affine variety of dimension 1.
2. Find generators for the ideal  $I(V)$ .

*Solution.* An [affine variety](#) is an irreducible, closed subset of  $\mathbb{A}^n$  with regard to the Zariski topology.

1. We show that  $V$  is a closed subset with regard to the Zariski topology.
  - (a) Since any algebraic set is immediately a closed subset, it is enough to show that  $V$  is the zero-locus of some subset of polynomials in  $K[X_1, \dots, X_n]$ .
  - (b) Consider the ideal  $(Y - X^2, Z - X^3) \subset K[X, Y]$  and it's zero set  $Z(Y - X^2, Z - X^3)$ .
  - (c) Writing out the definition of the zero set gives

$$\begin{aligned} Z(Y - X^2, Z - X^3) &= \{ (x, y, z) \in \mathbb{A}^3 \mid y - x^2 = 0, z - x^3 = 0 \} \\ &= \{ (x, y, z) \in \mathbb{A}^3 \mid y = x^2, z = x^3 \} \\ &= \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}. \end{aligned}$$

Thus,  $V$  is the zero set of the ideal  $(Y - X^2, Z - X^3)$ .

(d) Hence, by definition,  $V = Z(Y - X^2, Z - X^3)$  is an algebraic set.

2. Here, we prove that  $V$  is irreducible.

(a) Consider the quotient  $K[X, Y, Z]/(Y - X^2, Z - X^3)$ .

(b) By substitution, we get the isomorphism

$$K[X, Y, Z]/(Y - X^2, Z - X^3) \cong K[X, X^2, X^3] = K[X].$$

(c) Since  $K$  is a field it is in particular an integral domain and so is  $K[X]$ .

(d) Thus,  $(Y - X^2, Z - X^3)$  is prime in  $K[X, Y, Z]$ .

(e) With corollary 1.0.16 we may conclude the variety  $V = Z(Y - X^2, Z - X^3)$  is irreducible.

3. We show that  $V$  is of dimension 1.

(a) By proposition 1.7, the dimension of  $V$  corresponds with the dimension of its affine coordinate ring  $A(V)$ .

(b) It is

$$A(V) = K[X, Y, Z]/(Y - X^2, Z - X^3) \cong K[X].$$

(c)  $K[X]$  is a PID. All prime ideals are generated by irreducible elements. Thus dimension is 1.

**Exercise 1.1.5** (1.3.). *Let  $V$  be the algebraic set in  $\mathbb{A}^3$  defined by two polynomials  $X^2 - YZ$  and  $XZ - X$ . Show that  $V$  is a union of three irreducible components. Describe them and find their prime ideals.*

*Solution.*  $V = Z(X^2 - YZ, XZ - X)$

If  $z = 0$ , then  $x = 0$  and  $y$  can be any thing, so one irreducible component is the  $y$ -axis. This is described by  $V_1 = Z(x, z)$ .  $V_1$  is irreducible because its ideal  $(x, z)$  is prime because  $K[X, Y, Z]/(x, z) \cong K[Y]$  is an integral domain.

If  $x = 0$ , then  $yz = 0$ . If  $z = 0$ , then see above.  $y = 0$  gives the  $z$ -axis  $V_2 = Z(x, y)$

If  $Z = 1$ , then  $X^2 - Y = 0$ , so  $X^2 = Y$ . We have  $V_3 = Z(X^2 - Y, Z - 1)$ . This is also irreducible because  $K[X, Y, Z]/(X^2 - Y, Z - 1) \cong K[X, Y]/(X^2 - Y) \cong K[X, X^2] = K[X]$ .

If  $Z \neq 1$ , then  $X(Z - 1) = 0$  gives  $X = 0$ .

We will find the irreducible components by investigating cases.

1. Consider the subset  $V_1 = \{(x, y, z) \in V \mid z = 1\}$ .

(a) If  $Z = 1$ , then the polynomials reduce to

$$X^2 - YZ \Rightarrow X^2 - Y \quad \text{and} \quad XZ - X \Rightarrow X - X \Rightarrow 0.$$

(b) Thus,  $V_1$  is the zero-locus of the ideal  $(X^2 - Y, Z - 1)$ .

(c) This ideal  $(X^2 - Y, Z - 1)$  is prime because

$$K[X, Y, Z]/(X^2 - Y, Z - 1) \cong K[X, Y]/(X^2 - Y) \cong K[X, X^2] = K[X]$$

is an integral domain.

(d) Hence  $V_1$  is irreducible.

2. Consider the subset  $V_1 = \{(x, y, z) \in V \mid z \neq 1\}$ .

(a) If  $Z \neq 1$ , then we have for the second polynomial

$$XZ - X = 0 \Rightarrow X(Z - 1) = 0 \Rightarrow X = 0$$

(b) The first polynomial gives

$$YZ = 0$$

**Exercise 1.1.6.** *If we identify  $\mathbb{A}^2$  with  $\mathbb{A}^1 \times \mathbb{A}^1$  in the natural way, show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology of the Zariski topologies on the two copies of  $\mathbb{A}^1$ .*

*Solution.* Consider  $V = Z(X^2 - Y)$ .

1. We get the two projections

$$p_1(V) = \mathbb{A}^1 \quad \text{and} \quad p_2(V) = [0, \infty).$$

2. In  $\mathbb{A}^1$ , the only closed sets are finite sets and  $\mathbb{A}^1$  itself.

3. Thus  $p_2(V) = [0, \infty)$  is not closed.

**Exercise 1.1.7 (1.5.).** *Show that  $k$ -algebra  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ , for some  $n$ , if and only if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.*

*Solution.*      •  $B$  isomorphic to some coordinate ring

•  $B$  finitely generated with no nilpotent elements

Let  $B$  be a  $K$ -algebra that is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$  for some  $n$ . We show that  $B$  is finitely generated with no nilpotent elements.

1. I mean, any coordinate ring is finitely generated by  $1, X_1, X_2, \dots, X_n$ .

2. Isn't it just if an element is nilpotent, it is in the nilradical, thus in any prime ideal and this prime ideal lies in the ideal?



3. So let  $A/I(V)$  be a coordinate ring and assume  $x \in A/I(V)$  be nilpotent.
4. So  $x^n \in I(V)$ .

Let  $B$  be a finitely generated  $K$ -algebra with no nilpotent elements. We show that  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ .

1. Firstly, let  $1, b_1, \dots, b_n$  generate  $B$ .
2. Consider the map

$$\begin{aligned} \varphi : K[X_1, \dots, X_n] &\rightarrow B \\ b_i &\mapsto X_i \end{aligned}$$

3. We now need the kernel.

$$\ker(\varphi) = \{ f \in K[X_1, \dots, X_n] \mid \varphi(f) = 0 \}$$

**Exercise 1.1.8** (1.6.). *Any nonempty open subset of an irreducible topological space is dense and irreducible. If  $Y$  is a subset of a topological space  $X$ , which is irreducible in its induced topology, then the closure  $\text{cl}(Y)$  is also irreducible.*

*Solution.* Let  $X$  be a irreducible topological space and let  $\mathcal{U}$  be an open subset of  $X$ . We show  $\mathcal{U}$  is dense.

1. Assume  $\mathcal{U}$  does not lie dense in  $X$ , i.e.  $\text{cl}(\mathcal{U}) \neq X$ . In other words, there is a point  $p \in X$  with  $p \notin \text{cl}(\mathcal{U})$ .
2. If all open neighborhoods  $\mathcal{U}_p$  of  $p$  intersects  $\mathcal{U}$ , i.e.  $\mathcal{U} \cap \mathcal{V}_p \neq \emptyset$ , then by definition,  $p$  must lie in the closure  $\text{cl}(\mathcal{U})$ . Hence let  $\mathcal{V}_p$  be an open neighborhood of  $p$  with  $\mathcal{U} \cap \mathcal{V}_p = \emptyset$ .
3. This contradicts the irreducibility of  $X$  hence  $\mathcal{U}$  lies dense in  $X$ .

We show  $\mathcal{U}$  is irreducible.

1. Assume  $\mathcal{U}$  is not irreducible, i.e. we find two open sets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  with  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$  and  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{U}$ .
2. I mean this is obvious.

**Exercise 1.1.9** (1.7.). *Show that the following conditions are equivalent for a topological space  $X$ .*

1.  $X$  is Noetherian.
2. Every nonempty family of closed subset has a minimal element.

3.  $X$  satisfies the ascending chain condition for open subset.

4. Every nonempty family of open subsets has a maximal element.

*Solution.* This is simple.

**Exercise 1.1.10** (1.7. (b)). *A Noetherian topological space is quasi-compact, i.e. every open cover has a finite subcover.*

*Solution.* This is also just simple topology.

**Exercise 1.1.11** (1.8.). *Let  $V$  be an affine variety of dimension  $d$  in  $\mathbb{A}^n$ . Let  $H$  be a hypersurface in  $\mathbb{A}^n$ , and assume  $V \not\subset H$ . Then every irreducible component of  $V \cap H$  has dimension  $d - 1$ .*

*Solution.* I need to understand hypersurfaces first.

1. Consider any chain of  $Z_0 \subset Z_1 \subset \cdots \subset Z_k$  distinct irreducible closed subsets in  $V$ . By definition, we have  $k \leq d$  for all chains.
2. Consider now the chain

$$Z_0 \cap H \subset Z_1 \cap H \subset \cdots \subset Z_k \cap H$$

**Exercise 1.1.12.** *Let  $\mathfrak{a} \subset A = K[X_1, \dots, X_n]$  be an ideal which can be generated by  $r$  elements. Then every irreducible component of  $Z(\mathfrak{a})$  has dimension  $\geq n - r$ .*

## Chapter 2

# Projective Varieties



## Chapter 3

# Morphisms

**Definition 3.0.1.** Let  $X$  be a quasi-affine variety in  $\mathbb{A}_K^n$  and  $f : X \rightarrow K$  a function.

1.  $f$  is regular at a point  $p \in X$  if there is an open neighborhood  $\mathcal{U} \subset X$  of  $p$ , and polynomials  $g, h \in K[X_1, \dots, X_n]$ , such that  $h(x) \neq 0$  for all  $x \in \mathcal{U}$ , and  $f = g/h$  on  $\mathcal{U}$ .
2.  $f$  is regular on  $X$  if it is regular at every point on  $X$ .

**Lemma 3.0.2.** *A regular function is continuous, when  $K$  is identified with  $\mathbb{A}_K^1$  in its Zariski topology.*

**Definition 3.0.3** (Germ). Given a point  $p$  of a topological space  $X$ , and two maps  $f, g : X \rightarrow Y$  where  $Y$  is any set, then  $f$  and  $g$  define the same germ at  $p$  if there is a neighbourhood  $\mathcal{U}$  of  $p$  such that restricted to  $\mathcal{U}$ ,  $f$  and  $g$  are equal, i.e.

$$f(x) = g(x) \text{ for all } x \in \mathcal{U}.$$

**Definition 3.0.4.** Let  $X$  be a variety.

1. We denote the ring of all regular functions on  $X$  by  $\mathcal{O}(X)$ .
2. If  $p$  is a point on  $X$ , we define the local ring of  $p$  on  $X$ ,  $\mathcal{O}_p$  to be the ring of germs of regular functions on  $X$  near  $p$ . In other words, an element of  $\mathcal{O}_p$  is a pair  $(\mathcal{U}, f)$  where  $\mathcal{U}$  is an open subset of  $X$  containing  $p$ , and  $f$  is a regular function on  $\mathcal{U}$ , and where we identify two such pairs  $(\mathcal{U}, f)$  and  $(\mathcal{V}, g)$  if  $f = g$  on  $\mathcal{U} \cap \mathcal{V}$ .

**Theorem 3.0.5.** *Let  $X \subset \mathbb{A}^n$  be an affine variety.*

1. *The ring of all regular functions on  $X$  is isomorphic to the coordinate ring of  $X$ , i.e.*

$$\mathcal{O}(X) \cong A(X).$$

2. *There is a one-to-one correspondence between the points of  $X$  and the maximal ideals of  $A(X)$ .*
3. *The localization of the ring of all regular functions at  $p \in X$*

# Bibliography

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