Topology

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Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring)**.** A ring is a set A equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. $(A, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
    - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
    - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Chapter 2

Ideals

Definition 2 (Ideal)**.**

Definition 3 (Ideal Operation)**.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring A.

1. The sum of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \}$$

   which is again an ideal. It is the smallest ideal in A that contains $\mathfrak{a}$ and $\mathfrak{b}$.

2. The product of an ideal
3. The intersection of
4. The radical of an ideal $\mathfrak{a}$ is defined by

$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$

   which is again an ideal.

5. The transporter

Proof. □

Example 3.1. The union of two ideals is **not** an ideal in general. Consider (2) and (3) in $\mathbb{Z}$. If $(2) \cup (3)$ was an ideal, then $3 - 2 = 1$ would be contained in $(2) \cup (3)$. But $1 \notin (2)$ and $1 \notin (3)$, thus $1 \notin (2) \cup (3)$.

Proposition 4. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a ring A.

1. $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$.
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.
3. If $\mathfrak{a} \subset \mathfrak{b}$, then $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$.
4. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.
5. $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
6. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

Proof. a.

1. Let $x \in \mathfrak{a}$, then trivially, $x^1 \in \mathfrak{a}$, so $x \in \sqrt{\mathfrak{a}}$.
2. Since $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$ from above, it suffices to verify the other inclusion. Let $x \in \sqrt{\sqrt{\mathfrak{a}}}$, then $x^n \in \sqrt{\mathfrak{a}}$ and in turn, $(x^n)^m \in \mathfrak{a}$. Thus, $x^{nm} \in \mathfrak{a}$, therefore, $x \in \sqrt{\mathfrak{a}}$.
3. Suppose $\mathfrak{a} \subset \mathfrak{b}$ and let $x \in \sqrt{\mathfrak{a}}$. Then, $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$, thus $x^n \in \mathfrak{b}$. It follows that $x \in \sqrt{\mathfrak{b}}$.
4. "$\Rightarrow$": Let $\sqrt{\mathfrak{a}} = A$, then for all $x \in A$, we have that $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$. In particular, $1^n \in \mathfrak{a}$, but $1^n = 1$ for all $n \in \mathbb{N}^+$. Thus, $\mathfrak{a} = A$.

   "$\Leftarrow$": On the other hand, let $\mathfrak{a} = A$. In general, it is $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, therefore $A \subset \sqrt{\mathfrak{a}}$ which immediately yields the desired equality $A = \sqrt{\mathfrak{a}}$.
5. "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cdot \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Since $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, we have $x^n \in \mathfrak{a} \cap \mathfrak{b}$, and it follows that $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": Let $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Hence it is $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, therefore $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$. We may write $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, therefore $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": Finally, let $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. Then, $x\sqrt{\mathfrak{a}}$ and $x\sqrt{\mathfrak{b}}$, so $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$ for some $n, m \in \mathbb{N}^+$. Say $n \geq m$, then $x^n \in \mathfrak{b}$. This yields $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.
6. "$\sqrt{\mathfrak{a} + \mathfrak{b}} \subset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Let $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$, then $x^n \in \mathfrak{a} + \mathfrak{b}$ for some $n \in \mathbb{N}^+$. By definition of sum of ideals, we have that $x^n = a + b$ for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ and $\mathfrak{b} \subset \sqrt{\mathfrak{b}}$, we have $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, thus $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

   "$\sqrt{\mathfrak{a} + \mathfrak{b}} \supset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Now let $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$, then $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$ for some $n \in \mathbb{N}^+$. Hence there exists $a \in \sqrt{\mathfrak{a}}$ and $b \in \sqrt{\mathfrak{b}}$ such that $x^n = a + b$. We have that $a^p \in \mathfrak{a}$ and $b^q \in \mathfrak{b}$ for some $p, q \in \mathbb{N}^+$. Consider

   $$\begin{aligned}(x^n)^{(p+q-1)} &= (a+b)^{(p+q-1)} \\ &= \sum_{k=0}^{p+q-1} \binom{p+q-1}{k} a^k \cdot b^{p+q-1-k}.\end{aligned}$$

   For each $k \in \{0, 1, \ldots, p+q-1\}$, we have $a^k \in \mathfrak{a}$ or $b^{p+q-1} \in \mathfrak{b}$. Thus, the whole sum lies in $\mathfrak{a} + \mathfrak{b}$ or in other words $x^{n(p+q-1)} \in \mathfrak{a} + \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

$\square$

Chapter 3

Anatomy of Rings

Definition 5 (Nilpotent Element and Nilradical)**.** An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.
The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\mathrm{Nil}(A)$.

3.1 Exercises and Notes

Example 5.1. Let K be a field and $A = {}^{K[X,Y]}/_{(X - XY^2, Y^3)}$.

1. Compute the nilradical $\mathrm{Nil}(A)$.

   Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

   $$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

   Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism ${}^{K[X,Y]}/_{(X - XY^2, Y^3)} \simeq {}^{K[Y]}/_{(Y^3)}$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

   Clearly, $Y \in \mathrm{Nil}(A)$ or in other words $(Y) \subset \mathrm{Nil}(A)$. But we also have that ${}^{K[Y]}/_{(Y)} = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \mathrm{Nil}(A)$ conclude $\mathrm{Nil}(A) = (Y)$. □

Chapter 4

Polynomial Rings

Chapter 5

Quotient

Chapter 6

Localization

Chapter 7

Hierarchy of Rings