Definition 1 — Group of Units.
Let A be a ring. An element $a \in A$ is called an unit if there is an element $b \in A$ such that $a \cdot b = 1$.

We denote the set of all units as following.

$$A^\times := \{ a \in A \mid \exists b \in A : a \cdot b = 1 \} \tag{1}$$

$A^\times$ forms a group.

1. Let $a, b \in A^\times$. Then, there are a' and b' in $A^\times$ such that $a \cdot a' = 1$ and $b \cdot b' = 1$ respectively. We have $a \cdot b \cdot a' \cdot b' = 1$ hence $a \cdot b \in A^\times$. In other words, $A^\times$ is closed under multiplication.
2. Associativity is inherited from the ring A.
3. The identity element is 1. It is included in $A^\times$ as $1 \cdot 1 = 1$. And the identity property $a \cdot 1 = a$ for all $a \in A^\times$ is inherited from A.
4. Let $a \in A^\times$. Then, there is a $b \in A^\times$ such that $a \cdot b = 1$. This b is precisely the inverse element of a.

If A is commutativ, then A' is commutative.

My guess is that A' being a commutative group does not imply that A is commutative.

Also, if A isn't commutative, there probably is a left unit group and a right unit group. Or are they the same?

Examples:

1. $\mathbb{Z}^\times = \{-1, 1\}$
2. For any field $\mathbb{K}$, it is $\mathbb{K}^\times = \mathbb{K}$.
3. Let $A = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Then, the group of units $A^\times$ is the set of all invertible matricies also called the general linear group $\mathrm{GL}_2(\mathbb{R})$. This should be true of the general case $A = \mathrm{Mat}_{n\times n}(\mathbb{K})$.
4. Let $\mathbb{Q}[X]$ be a polynomial ring.

Definition 2 — Set of Zero Divisors.

$$\mathrm{ZD}(A) := \{ a \in A \mid \exists b \in A \setminus \{0\} : a \cdot b = 0 \}. \tag{2}$$

Examples:

1. $\mathrm{ZD}(\mathbb{Z}) = \{0\}$.
2. For any field $\mathbb{K}$, it is $\mathrm{ZD}(\mathbb{K}) = \{0\}$.
3.

Proof of above: Let $\mathbb{K}$ be a field and assume there is a nonzero $x \in \mathbb{K}$ such that $x \cdot b = 0$ for a $b \in \mathbb{K}$. The issue here is that $\mathbb{K}$ contains the inverse of b and so we have $x = 0 \cdot b^{-1} = 0$.

Definition 3 — Integral Domain.
A ring A with $\mathrm{ZD}(A) = \{0\}$ is called an integral domain.

Definition 4 — Set of Nilpotent Elements.

$$\mathrm{Nil}(A) := \{ a \in A \mid \exists n \in \mathbb{N} : a^n = 0 \} \tag{3}$$

Definition 5 — Reduced Ring.
A ring A with $\mathrm{Nil}(A) = \{0\}$ is called a reduced ring.

Here some lemmas.

$A \setminus \mathrm{ZD}(A)$ is a semigroup containing $A^\times$.

Proof:

1. Let $x, y \in A \setminus \mathrm{ZD}(A)$. Then $x \cdot a \neq 0$ and $y \cdot b \neq 0$ for all $a, b \in A$. Assume there exists a $c \in A$ such that $x \cdot y \cdot c = 0$. This implies $x \cdot c = 0$ or $y \cdot c = 0$, but this is impossible. Conclude $x \cdot y \in A \setminus \mathrm{ZD}(A)$.

2. Let $x \in A^{\times}$. By definition we have for some $a \in A$ that $x \cdot a = 1$. Assume $x \in \mathrm{ZD}(A)$. Then we have $x \cdot b = 0$ for some $b \in A \setminus \{0\}$. With the previous equation we get

$$\begin{aligned} x \cdot a = 1 &\iff x \cdot a \cdot b = 1 \cdot b && (4)\\ &\iff x \cdot b \cdot a = b && (5)\\ &\iff 0 = b && (6)\end{aligned}$$

   But this is a contradiction. Hence $x \notin \mathrm{ZD}(A)$.

3. We have to prove associativity and the identity element, but both are clear.

More lemma: cancelation lemma, clear.
Here is one interesting:
$\mathrm{Nil}(A)$ is an ideal in A.
Proof. Let $x \in \mathrm{Nil}(A)$ and $a \in A$. Then $x \cdot a \in \mathrm{Nil}(A)$ (duh, obviously).
We have to show that $\mathrm{Nil}(A)$ is an addtive subgroup of A.

1. Let $x, y \in \mathrm{Nil}(A)$. Then $a^n = 0$ and $b^m = 0$ for some $n \in \mathbb{N}$. With the binominal theorem we get $(a+b)^{n+m} = 0$

I need the latex thingy for quotient ring.
Another lemma. The set $A_{\text{red}} := A/\mathrm{Nil}(A)$ is a reduced ring.
Proof. Assume there is an $\overline{x} \in \mathrm{Nil}(A_{\text{red}})$ but $\overline{x} \neq 0$. So $\overline{x}^n = 0$ for a suitable $n \in \mathbb{N}$. We have $0 = \overline{x}^n = (x + \mathrm{Nil}(A))^n =$

Definition 6 — Sum of Ideals.
Let A be a ring and $\{\mathfrak{a}_i\}_{i\in I}$ be a collection of ideals. We define the smallest ideal in A which contains each $\mathfrak{a}_i$ by $\sum_{i\in I} \mathfrak{a}_i$, i.e.

$$\sum_{i\in I} \mathfrak{a}_i := \left\{ \sum_{i\in I} a_i \;\middle|\; a_i \in \mathfrak{a}_i \text{ for all } i \in I, \text{ and } a_i = 0 \text{ for almost all i} \right\} \tag{7}$$

This makes sense to me.

Definition 7 — Intersection of Ideals.
We define the largest ideal in A containing each $\mathfrak{a}_i$ by

$$\bigcap_{i\in I} \mathfrak{a}_i \tag{8}$$

Definition 8 — Product of Ideals.

Definition 9 — Radical of Ideals.
The radical of an ideal $\mathfrak{a}$ is given by

$$\sqrt{\mathfrak{a}} := \{\, b \in A \mid \exists n \in \mathbb{N} : b^n = a \,\} \tag{9}$$

Again some lemmas.
$\sqrt{\mathfrak{a}}$ is an ideal.

1. We prove that $\sqrt{\mathfrak{a}}$ is an additive subgroup of A.

   (a) Let $x, y \in \sqrt{\mathfrak{a}}$. Then for some $n, m \in \mathbb{N}$ we have that $x^n = y^m = a$. Consider

   $$(x+y)^{n+m} \tag{10}$$

   This is a sum and product out of the elements in $\sqrt{\mathfrak{a}}$.

   (b) Associativity and identity is inherited.

   (c) Inverse element is clear.

   This is also clear.

Alternate way:

If $\mathfrak{a} = A$, then $\sqrt{\mathfrak{a}} = A$ and this is an ideal. Consider the case $\mathfrak{a} \neq A$. Let $\pi : A \longrightarrow A/\mathfrak{a}$ be the natural projection. Since $A/\mathfrak{a}$ is an ideal, we can apply the lemma above and we know that $\mathrm{Nil}(A/\mathfrak{a})$ is an ideal.

The point here is that

$$\pi^{-1}(\mathrm{Nil}(A/\mathfrak{a})) = \sqrt{\mathfrak{a}} \tag{11}$$

The Chinese Remainder theorem

Let A be a ring, $n \geq 2$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in A.

1. If the $\mathfrak{a}_i$ are pairwise coprime, then $\prod_{i=1}^n \mathfrak{a}_i = \bigcap_{i=1}^n \mathfrak{a}_i$