Topology

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Introduction

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Topological Spaces

2-1

"⇒": Let $f : X_1 \longrightarrow X_2$ be a homeomorphism and fix a subset (not necessarily open) $U \in \mathcal{T}_1$.

1. Assume U is open in X_1. Because f is continuous, the image of open subsets are again open, thus $f(U)$ lies in $\mathcal{T}_2$.

2. On the other hand, if $f(U)$ is open in X_2, then since f is bijective we have

$$f^{-1}(f(U)) = U.$$

   Because f is continuous, the preimage of open subsets under f is open. We may therefore conclude U is open in X_1.

We have shown that if f is a homeomorphism, then $f(\mathcal{T}_1) = \mathcal{T}_2$.

"⇐": Let $f : X_1 \longrightarrow X_2$ be a bijective map such that $f(\mathcal{T}_1) = \mathcal{T}_2$. Consider the inverse map f^{-1}. We want to show f^{-1} is continuous. Fix an open subset $U \in \mathcal{T}_1$. It is

$$\left(f^{-1}\right)^{-1}(U) = f(U)$$

because f is bijective. Since $f(\mathcal{T}_1) = \mathcal{T}_2$ and U is open, $f(U)$ is open as well. Hence the preimage of U under f^{-1} is open and f^{-1} is continuous.

Now we show that f is also continuous. Again, fix an open subset $V \in \mathcal{T}_2$. The preimage of V under f is just the image of the inverse function. We have already shown that the inverse is continuous. Thus, $f^{-1}(V)$ is open and f is continuous. Since f and f^{-1} exist and are continuous, f is a homeomorphism as desired.

2-2

a)

We show that $\mathcal{T}$ is a topology by verifying the axioms of a topology.

1. Since $\mathcal{T}$ is the collection of all unions of finite intersections of elements of $\mathcal{B}$, it contains the union of all elements of $\mathcal{B}$ which is just X. The union of empty collection generates the emptyset so $\varnothing \in \mathcal{T}$ as well.

2. Let $\mathcal{U} \subset \mathcal{T}$ be any subset. The elements of $\mathcal{U}$ are unions of finite intersections of elements of $\mathcal{B}$. Thus, $\bigcup_{U \in \mathcal{U}} U$ is again a union of finite intersections of elements of $\mathcal{B}$. In other words, $\mathcal{T}$ is closed under union.

3. $\mathcal{T}$ is stable under finite intersections due to distributive property of sets.

b)

2-3

1.

The collection of subset $\mathcal{T}_1 = \{ U \subset X \mid X \setminus U \text{ is finite or is all of } X \}$ forms a topology. We show this by verifying the axioms of a topology.

1. It is $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ which is finite. Thus, $X \in \mathcal{T}_1$ and $\varnothing \in \mathcal{T}_1$.
2. Let $\mathcal{U} \subset \mathcal{T}$ be a subset. By De Morgan's laws we have
$$X \setminus \left(\bigcup_{U \in \mathcal{U}} U \right) = \bigcap_{U \in \mathcal{U}} (X \setminus U) \,.$$
Since each $U \in \mathcal{U}$ lies in $\mathcal{T}$, the complement $X \setminus U$ is finite or is all of X. Therefore, the intersection of all $X \setminus U$ is again finite or all of X, and we may conclude that $\mathcal{T}$ is stable under arbitary unions.
3. Use De Morgan's law again.

2.

The collection of subsets $\mathcal{T}_2 = \{ U \subset X \mid X \setminus U \text{ is infinite or is empty} \}$ is not a topology. Take $X = \mathbb{Z}$ for example and consider $A = \{ 1, 2, 3, \dots \}$ and $B = \{ -1, -2, -3, \dots \}$. A and B are open because their complements are the non-positive and the non-negative integers respectively. If $\mathcal{T}_2$ is a topology, it should contain their union $A \cup B = \mathbb{Z} \setminus \{0\}$. However,
$$\mathbb{Z} \setminus (A \cup B) = \mathbb{Z}(\mathbb{Z} \setminus \{0\}) = \{0\}$$
which is not infinite and thus doesn't lie in $\mathcal{T}_2$.

3.

The collection of subsets $\mathcal{T}_3 = \{ U \subset X \mid X \setminus U \text{ is countable or all of } X \}$ is a topology PROBABLY.

2-4

Already did somewhere else.

2-5

1. $\mathrm{id}_1 : X \longrightarrow \mathbb{R}^2$ is continuous probably.
2. $\mathrm{id}_2 : \mathbb{R}^2 \longrightarrow X$ is not continuous probably.

2-6

f is continuous because any preimage of a subset $U \subset Z$ under f is open, since any subset in X is open.

For g, the only preimages to check are the emptyset $\varnothing$ and Y. Simply, $g^{-1}(\varnothing) = \varnothing$ and $g^{-1}(Y) = Z$. Both subsets are open in Z, therefore g is continuous.

If h is constant, say $h(Y) = \{p\}$, then $h^{-1}(U) = Y$ if $p \in U$ and $h^{-1}(U) = \varnothing$ if $p \in U$. In both cases the preimages are open, thus h is continuous. Assume h is continuous but not constant, i.e. there are points $x_1, x_2 \in Y$ such that $h(x_1) \neq h(x_2)$. Z is Hausdorff, so there are disjoint neighbourhoods U of $h(x_1)$ and V of $h(x_2)$. h was assumed to be continuous, so $h^{-1}(U) = Y$ and $h^{-1}(V) = Y$ which is impossible (REALLY?).

2-7

a)

f)

2-8

Firstly, any element in $f(\mathcal{B})$ is open because f is an open map. Fix an open subset V in Y and consider its preimage $f^{-1}(V)$ under f. Because f is continuous, the preimage is open, thus there are base elements B_i with $i \in I$ in $\mathcal{B}$ such that

$$f^{-1}(V) = \bigcup_{i \in I} B_i.$$

The surjectivity of f grants us $f(f^{-1}(V)) = V$, therefore, we have

$$f(f^{-1}(V)) = V = f\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f(B_i).$$

Thus, $f(\mathcal{B})$ is a basis of Y.

2-9

2-10

Fix a point y in Y. Since f is surjective, there is an x in X such that $f(x) = y$. X is locally Euclidean, thus there is a neighbourhood U of x that is homeomorphic to $\mathbb{R}^n$. Moreover, f is locally homeomorphic, so there is a neighbourhood V of x such that the restriction of f under V is a homeomorphism. Then, the intersection $U \cap V = N$ has both of these properties, i.e. N is a neighbourhood of x that is homeomorphic to $\mathbb{R}^n$ and the restriction of f under V is a homeomorphism. $f(N)$ is a neighbourhood of y that is homeomorphic to $\mathbb{R}^n$, therefore Y is locally Euclidean.