Exercise 1.1

Let $A \subset B$ be an integral extension of rings and assume that B is an integral domain. Suppose $\mathfrak{q} \subset B$ is a prime ideal and let $\mathfrak{p} := \mathfrak{q} \cap A \subset A$.

1. Prove that A is a field if and only if B is a field.

   Proof. Assume A is a field and let's do this a little different. Let $\mathfrak{m}$ be a maximal ideal in B and fix a nonzero element $b \in \mathfrak{m}$. Because b is integral over A, we have an expression

   $$0 = a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n \iff -a_0 = a_1 b + a_2 b^2 + \cdots a_n b^n.$$

   On the right side, for each $1 \leq i \leq n$, we have that $a_i b^i$ is in $\mathfrak{m}$, so the whole sum is in $\mathfrak{m}$. This implies the absurdity that $-a_0$, an unit, is contained in $\mathfrak{m}$. So there is no such thing as nonzero b in $\mathfrak{m}$ and B is a field.

   For the other direction of the implication, we will do it traditionally. Let B be a field and fix an $x \in A$. x is a unit in B, so there is a $y \in B$ with $xy = 1$. Again, for y we have the expression

   $$0 = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n$$

   and if we multiply x^{n-1} on both sides, we yield

   $$\begin{aligned} & 0 = a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + a_n y \\ \iff & -a_0 x^{n-1} - a_1 x^{n-2} - a_2 x^{n-3} = a_n y \\ \iff & a_n^{-1}(-a_0 x^{n-1} - a_1 x^{n-2} - a_2 x^{n-3}) = y \end{aligned}$$

   In other words, y is in A or in different words, A is a field. □

2. Show that $\mathfrak{p}$ is a prime ideal of A and that $A/\mathfrak{p}$ can be viewed as a subring of $B/\mathfrak{q}$.

   Proof. (a) We simply have $A/(\mathfrak{q} \cap A) \cong (A + \mathfrak{q})/\mathfrak{q}$ and since $\mathfrak{q}$ is prime in $A + \mathfrak{q}$, $A/\mathfrak{p}$ is an integral domain.

   (b) That $A/\mathfrak{p}$ is a subring of $B/\mathfrak{q}$ follows from above.

   □

3. Show that $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$.

   Proof. Should be clear. $\square$

4. Deduce that $\mathfrak{q}$ is maximal in B if and only if $\mathfrak{p}$ is maximal A.

   Proof. This is clear. $\square$