**Definition 1 — Chart.**
An $n$-dimensional chart $(\mathcal{U}, x)$ on a set $M$ consists of a subset $\mathcal{U} \subseteq \mathcal{M}$ and an injective map $x : \mathcal{U} \longrightarrow \mathbb{R}^n$ whose image $x(\mathcal{U}) \subseteq \mathbb{R}^n$ is an open set.

**Definition 2 — Transition Maps.**
Any two charts $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$ determine a pair of transition maps

$$x(\mathcal{U} \cap \mathcal{V}) \xrightarrow{y \circ x^{-1}} y(\mathcal{U} \cap \mathcal{V}) \tag{1}$$

$$y(\mathcal{U} \cap \mathcal{V}) \xrightarrow{x \circ y^{-1}} x(\mathcal{U} \cap \mathcal{V}) \tag{2}$$

which are inverse to each other, and are thus bijections between subset of $\mathbb{R}^n$.

**Definition 3 — $C^k$-compatible.**
We say that the two charts are $C^k$-compatible for some $k \in \mathbb{N} \cup \{0, \infty\}$ if the sets $x(\mathcal{U} \cap \mathcal{V})$ and $y(\mathcal{U} \cap \mathcal{V})$ are both open and the transition maps $y \circ x^{-1}$ and $x \circ y^{-1}$ are both of class $C^k$. If $k = \infty$, we say the charts are smoothly compatible.

**Definition 4 — Atlas.**
An atlas of class $C^k$ for the set $M$ (or smooth atlas in the case $k = \infty$) is a collection of charts $\mathcal{A} = \{(\mathcal{U}_\alpha, x_\alpha)\}_{\alpha \in I}$ that are all $C^k$-compatible with each other, such that $\bigcup_{\alpha \in I} \mathcal{U} = M$.

**Definition 5 — .**
For a set $M$ with an atlas $\mathcal{A}$ of class $C^k$ and $r \in \mathbb{N} \cup \{0, \infty\}$ with $r \leq k$, a function $f : M \longrightarrow \mathbb{R}$ is said to be of class $C^k$ if and only if the function

$$f \circ x^{-1} : x(\mathcal{U}) \longrightarrow \mathbb{R} \tag{3}$$

is of class $C^r$ for every chart $(\mathcal{U}, x) \in \mathcal{A}$.

**Definition 6 — .**
For $k \in \mathbb{N} \cap \{\infty\}$, a $C^k$-structure or differentiable structure of class $C^k$ on a set $M$ is a maximal atlas $\mathcal{A}$ of class $C^k$ on $M$. In the case $k = \infty$, we also call this a smooth structure on $M$. If $M$ has been endowed with a $C^k$-structure $\mathcal{A}$, then a chart $(\mathcal{U}, x)$ on $M$ will be referred to as a $C^k$-chart if it belongs to the maximal atlas $\mathcal{A}$.

**Definition 7 — Topology.**
A topology on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ satisfying the following axioms:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. For every subcollection $I \subseteq \mathcal{T}$ it is $\bigcup_{\mathcal{U} \in I} \mathcal{U} \in \mathcal{T}$.
3. For every pair $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$ it is $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is then called a topological space, and we call the sets $\mathcal{U} \in \mathcal{T}$ the open subsets in $(X, \mathcal{T})$.

**Definition 8 — Metrizable.**

**Definition 9 — .**
Assume $k \in \mathbb{N} \cup \{\infty\}$. A differentiable manifold of class $C^k$ or $C^k$-manifold is a set $M$ endowed with a $C^k$-structure such that the induced topology on $M$ is metrizable and separable. In the case $k = \infty$, we also call $M$ are smooth manifold.

We say that $M$ is $n$-dimensional and refer to $M$ as an $n$-manifold, written $\dim M = n$, if every chart in its differentiable structure is $n$-dimensional.

**Exercise 1 — .**
Let $S^2 \in \mathbb{R}^3$ be the three dimensional unit sphere and set $p_\pm := (0, 0, \pm 1) \in S^2$. Find two subsets $\mathcal{U}_1, \mathcal{U}_2 \subset S^2$ with $\mathcal{U}_1 \cup \mathcal{U}_2 = S^2 \setminus \{p_+, p_-\}$ such that for $i = 1, 2$,

a) there are 2-dimensional charts of the form $(\mathcal{U}_i, \alpha_i)$ with $\alpha_i = (\theta_i, \phi_i)$, where the coordinate functions $\theta_i, \phi_i : \mathcal{U}_i \longrightarrow \mathbb{R}$ are
   i) continuous
   ii) and satisfy the spherical coordinate relations,
b) and have images

$$\alpha_1(\mathcal{U}_1) = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}^3 \quad \text{and} \quad \alpha_2(\mathcal{U}_2) = (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}^3. \tag{4}$$

*Proof.* hi ☐