Topology

K

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Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring)**.** A ring is a set A equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. $(A, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Definition 2 (Ideal)**.**

Definition 3 (Ideal Operation)**.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring A.

1. The sum of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

   $$\mathfrak{a} + \mathfrak{b} = \{\, a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \,\}$$

   which is again an ideal. It is the smallest ideal in A that contains $\mathfrak{a}$ and $\mathfrak{b}$.
2. The product of an ideal
3. The intersection of
4. The radical of an ideal $\mathfrak{a}$ is defined by

   $$\sqrt{\mathfrak{a}} = \{\, x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \,\}$$

   which is again an ideal.
5. The transporter

Proposition 4. Let $\mathfrak{a}$ be an ideal.

1. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.

Proof. 1. ☐

Definition 5 (Nilpotent Element and Nilradical)**.** An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.
The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\mathrm{Nil}(A)$.

1.2 Exercises and Notes

Example 5.1. Let K be a field and $A = {}^{K[X,Y]}/_{(X-XY^2,Y^3)}$.

1. Compute the nilradical $\mathrm{Nil}(A)$.

   Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

   Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism ${}^{K[X,Y]}/_{(X-XY^2,Y^3)} \simeq {}^{K[Y]}/_{(Y^3)}$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

   Clearly, $Y \in \mathrm{Nil}(A)$ or in other words $(Y) \subset \mathrm{Nil}(A)$. But we also have that ${}^{K[Y]}/_{(Y)} = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \mathrm{Nil}(A)$ conclude $\mathrm{Nil}(A) = (Y)$. ☐