0.1 Integral Dependence

Definition 0.1. Let $A \subset B$ be an extension of rings.

1. An element $x \in B$ is integral over A if there is a monic polynomial $f(T) \in A[T]$ such that $f(x) = 0 \in B$. That is, if for certain $n \in \mathbb{N}_0$ there are elements $a_1, \ldots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0. \quad (1)$$

   Such a polynomial $f(T) = T^n + a_1 T^{n-1} + \cdots + a_n$ is called an equation of integral dependence of x over A.

2. The integral closure of A in B is the subset

$$\bar{A} := \{x \in B \mid x \text{ is integral over A}\} \subset B. \quad (2)$$

3. The extension $A \subset B$ is called integral if every element of B is integral over A.

Lemma 0.1.1. All UFDs are integrally closed.

Definition 0.2 (A-Module)**.**

Remark. 1. The submodules of the ring A itself are precisely the ideals of A.

2. In an extension of rings $A \subset B$, B is an A-module.

Proposition 0.2.1. Let $A \subset B$ be an extension of rings and $x \in B$. Then the following are equivalent.

- x is integral over A.
- The subring $A[x] := \{f(x) \mid f(T) \in A[T]\} \subset S$ is a finitely generated A-module.
- There is a finitely generated A-submodule $M \subset B$ such that $1 \in M$ and $x \cdot M \subset M$.

0.2 Rings of Integers of Number Fields

Definition 0.3. 1. A number field is a finite extension K of $\mathbb{Q}$.

2. The ring of integers $\mathcal{O}_K$ of a number field K is the integral closure of $\mathbb{Z}$ in K.

Example 0.3.1. Let $K = \mathbb{Q}(i)$ be a number field. Then, $\mathcal{O}_K = \mathbb{Z}[i]$.

Proposition 0.3.1. The field L and the ring B are related to each other as follows.

1. $B = \{\alpha \in L \mid \text{the minimal polynomial }\}$