Algebraic Geometry

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Contents

Housekeeping

Notes taken from

- my course

to-do

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Chapter 1

Basics?

Let K be an arbitary field.

Definition 1 (Algebraic Subset)**.** For a subset $M \subset K[X_1, \dots, X_n]$, we define

$$V(M) = \{\, p \in K^n \mid \text{ for all polynomials } f \in M \text{ it is } f(p) = 0 \,\}$$

called the algebraic subset in K^n or an affine algebraic set over K.

Intuition. Imagine the ring of polynomials $K[X_1, \dots, X_n]$ to be a set of locks and the kartesian product of the field K^n to be a set of keys. Then, a subset M of $K[X_1, \dots, X_n]$ is some combination of locks and the algebraic subset $V(M)$ are the set of keys that open all the locks in M.

Example 1.1.

1. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$.
   (a) If we set $M = \left\{\, X^2 - 1 \,\right\} \subset \mathbb{C}[X]$, then $V(\left\{\, X^2 - 1 \,\right\}) = \{\, -1, 1 \,\} \in \mathbb{C}$ is the algebraic subset.
   (b) Now, if we append the set above to $M = \left\{\, X^2 - 1, X - 1 \,\right\} \subset \mathbb{C}[X]$, we have
   $$V(\left\{\, X^2 - 1, X - 1 \,\right\}) = \{\, 1 \,\} \in \mathbb{C}$$
   instead. This example illustrates that appending the set of polynomials makes its algebraic subset smaller.
   (c) In general, the finite subsets of $\mathbb{K}$ are precisely the algebraic subsets of $\mathbb{K}$.
2. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X, Y]$.
   (a) If $M = \{\, X - 1, Y - 1 \,\} \subset \mathbb{C}[X, Y]$, then $V(\{\, X - 1, Y - 1 \,\}) = \{\, (1, 1) \,\} \subset \mathbb{C}^2$.
   (b) If we remove the second polynomial, we get $M = \{\, X - 1 \,\} \subset \mathbb{C}[X, Y]$ and
   $$V(\{\, X - 1 \,\}) = \{\, (1, y) \mid y \in \mathbb{C} \,\}.$$
   It is the first example of an infinite algebraic subset.

Lemma 2. If $M, N \subset K[X_1, \dots, X_n]$ with $N \subset M$, then it is $V(N) \supset V(M)$.

Intuition. N has less locks then M. If N has less locks, then more keys are able to open all locks in N.

Proof. Let $p \in V(M)$. For all $f \in M$, we have $f(p) = 0$. Now, $N \subset M$, thus for all $g \in N$ it is $g \in M$ and therefore $g(p) = 0$. That means $V(M) \subset V(N)$. $\square$

Proposition 3.

1. The empty set $\varnothing$ is an algebraic subset of $\mathbb{K}^n$.
2. The whole set $\mathbb{K}^n$ is an algebraic subset of $\mathbb{K}^n$.
3. An arbitary intersection of algebraic subsets of $\mathbb{K}^n$ is an algebraic subset of $\mathbb{K}^n$.
4. A finite union of algebraic subsets of $\mathbb{K}^n$ is an algebraic subset of $\mathbb{K}^n$.

Example 3.1. The union of non-finite union of algebraic subsets is not algebraic. Take $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$. As we have seen before, the singleton sets $\{x\}$ are algebraic for each $x \in \mathbb{N}$, but their union

$$\bigcup_{x \in \mathbb{N}} \{x\} = \mathbb{N}$$

is not.

The above proposition justifies the following definition.

Definition 4. The unique topology on K^n whose closed subsets are the algebraic subsets are called the Zariski topology on K^n.

##missing I think we didn't prove the uniqueness yet.

Lemma 5. Let $M \subset K[X_1, \dots, X_n]$ be a subset. Define I to be the ideal generated[a] by M. Then we have $V(M) = V(I)$.

[a]By Hilbert's Basis Theorem the polynomial ring is Noetherian and thus this ideal is finitely generated.

Thus, $V(M)$ may be expressed by finite amount of generators $V(M) = V(f_1, \dots, f_n)$.

Example 5.1.

1. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$.
   (a) With the same example as above, set $M = \{ X^2 - 1 \} \subset \mathbb{C}[X]$, then $V(\{ X^2 - 1 \}) = \{ -1, 1 \} \in \mathbb{C}$ is the algebraic subset. Consider the ideal $(X^2 - 1)$. Some elements in this ideal are

   $$3(X^2 - 1), \qquad X(X^2 - 1), \qquad (X + 1)(X^2 - 1)$$

   which all still have the roots $\{ -1, 1 \}$.

Definition 6 (Radical of an Ideal)**.**

Proposition 7. Let I be an ideal in A, then:

1. $\sqrt{I} = \sqrt{\sqrt{I}}$
2. $I = \sqrt{I} \iff A/I$ is a reduced ring, i.e. there are no nilpotent elements in A/I.
3. $A \xrightarrow{\pi} A_{\text{red}} := A/\sqrt{I}$ has the following universal property:

Proof. $\square$

Lemma 8. $V(I) = V(\sqrt{I})$

This implies the surjectivity of the following map:

$$\{\text{ radical ideals in } K[X_1, \ldots, X_n]\,\} \to \{\text{ algebraic sets of } \mathbb{A}^n(K)\,\}$$

(because let V be an algebraic set i.e. $V = V(M)$ then $V(M) = V((M)) = V(\sqrt{(M)}))$

In general not injective. ##missingexample

Theorem 9 (Hilbert Nullstellensatz)**.** If K is algebraically closed, then the above map is a bijection with inverse $V \mapsto \mathrm{Ann}(V)$.

From now on, assume $K = \overline{K}$.

Corollary 1. For any proper ideal $I \subset K[X_1, \ldots, X_n]$ there is a $p \in K^n$ with $f(p) = 0$ for all $p \in I$.

Definition 10 (A-algebra)**.** Let A be a ring. An A-algebra is a ring B together with a rign homomorphism $f : A \longrightarrow B$, making B into an A-module such that scalar multiplication and the product on B are compatible.
If $A = K$ is a field, f is injective, so a K-algebra is a ring containing K as a subring, for example $K[X_1, \ldots, X_n]$.
##missing something about two A-algebras

Definition 11 (Coordinate Ring)**.** Let $V \subset K^n$ be an algebraic subset. Definite the coordinate ring to be $\mathcal{O}(V) := K[X_1, \ldots, X_n]/Ann(V)$ which is a reduced, finitely generated K-algebra (underlying ring is reduced).

Definition 12. Suppose we have two algebraic sets $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. A morphism between those two is a map $f : X \longrightarrow Y$, $p \mapsto (f_1(p), \ldots, f_m(p))$ where $f_i \in \mathcal{O}(X)$ for all $1 \leq i \leq m$.

##missing blue text