

# Topology

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# Part I

# Rings



# Chapter 1

## Rings

### 1.1 Definition and Theorems

**Definition 1** (Ring). A ring is a set  $A$  equipped with two binary operations  $+$  (addition) and  $\cdot$  (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1.  $(A, +)$  is an abelian group.
2.  $(A, \cdot)$  is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  for all  $a, b, c \in A$  (left distributivity).
  - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$  for all  $a, b, c \in A$  (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.





# Chapter 2

## Ideals

**Definition 2** (Ideal). Let  $A$  be a ring. A subset  $\mathfrak{a} \subset A$  is called an ideal if it satisfies the following two conditions.

1.  $(\mathfrak{a}, +)$  is a subgroup of  $(A, +)$ .
2. For every  $r \in A$  and every  $x \in \mathfrak{a}$ , it is  $rx \in \mathfrak{a}$ .

Given a subset  $S \subset A$ , by the ideal  $(S)$  that  $S$  generates, we mean the smallest ideal containing  $S$ . If an ideal is generated by a subset  $S \subset A$ , then the elements of this subset are called generators.

An ideal that is generated by a single element is called principal.

If an ideal  $\mathfrak{a}$  is not the whole ring  $A$ , then the ideal is called proper.

**Definition 3** (Ideal Operation). Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals of a ring  $A$ .

1. The sum of two ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  is defined by

$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \} = (\mathfrak{a}, \mathfrak{b})$$

which is again an ideal. It is the smallest ideal in  $A$  that contains  $\mathfrak{a}$  and  $\mathfrak{b}$ .

2. The product of an ideal
3. The intersection of
4. The radical of an ideal  $\mathfrak{a}$  is defined by

$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$

which is again an ideal.

5. The transporter

*Proof.* We verify the statements made in the definition.

1. (a) “ $\mathfrak{a} + \mathfrak{b}$  is an ideal.”:

□

**Example 3.1.** The union of two ideals is **not** an ideal in general. Consider  $(2)$  and  $(3)$  in  $\mathbb{Z}$ . If  $(2) \cup (3)$  was an ideal, then  $3 - 2 = 1$  would be contained in  $(2) \cup (3)$ . But  $1 \notin (2)$  and  $1 \notin (3)$ , thus  $1 \notin (2) \cup (3)$ .

**Proposition 4.** Let  $\mathfrak{a}$  be an ideal of  $A$ .

1.  $\mathfrak{a} = A$  if and only if  $1 \in \mathfrak{a}$  if and only if  $\mathfrak{a}$  contains an unit.
2.  $\mathfrak{a}^2 \subset \mathfrak{a}$ .
3.  $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$ .
4.  $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$  and  $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$ .

**Proposition 5.** Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of a ring  $A$ .

1.  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ .
2.  $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$ .
3. If  $\mathfrak{a} \subset \mathfrak{b}$ , then  $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$ .
4.  $\sqrt{\mathfrak{a}} = A$  if and only if  $\mathfrak{a} = A$ .
5.  $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ .
6.  $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ .
7. If  $\mathfrak{a} = \mathfrak{p}^n$  for some prime ideal  $\mathfrak{p}$  and  $n \in \mathbb{N}^+$ , then  $\sqrt{\mathfrak{a}} = \mathfrak{p}$ .

*Proof.* We verify each statement.

1. Let  $x \in \mathfrak{a}$ , then trivially,  $x^1 \in \mathfrak{a}$ , so  $x \in \sqrt{\mathfrak{a}}$ .
2. Since  $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$  from above, it suffices to verify the other inclusion. Let  $x \in \sqrt{\sqrt{\mathfrak{a}}}$ , then  $x^n \in \sqrt{\mathfrak{a}}$  and in turn,  $(x^n)^m \in \mathfrak{a}$ . Thus,  $x^{nm} \in \mathfrak{a}$ , therefore,  $x \in \sqrt{\mathfrak{a}}$ .
3. Suppose  $\mathfrak{a} \subset \mathfrak{b}$  and let  $x \in \sqrt{\mathfrak{a}}$ . Then,  $x^n \in \mathfrak{a}$  for some  $n \in \mathbb{N}^+$ , thus  $x^n \in \mathfrak{b}$ . It follows that  $x \in \sqrt{\mathfrak{b}}$ .
4. “ $\Rightarrow$ ”: Let  $\sqrt{\mathfrak{a}} = A$ , then for all  $x \in A$ , we have that  $x^n \in \mathfrak{a}$  for some  $n \in \mathbb{N}^+$ . In particular,  $1^n \in \mathfrak{a}$ , but  $1^n = 1$  for all  $n \in \mathbb{N}^+$ . Thus,  $\mathfrak{a} = A$ .  
 “ $\Leftarrow$ ”: On the other hand, let  $\mathfrak{a} = A$ . In general, it is  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ , therefore  $A \subset \sqrt{\mathfrak{a}}$  which immediately yields the desired equality  $A = \sqrt{\mathfrak{a}}$ .
5. “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: If  $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cdot \mathfrak{b}$  for some  $n \in \mathbb{N}^+$ . Since  $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ , we have  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , and it follows that  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ .  
 “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: Let  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cap \mathfrak{b}$  for some  $n \in \mathbb{N}^+$ . Hence it is  $x^n \in \mathfrak{a}$  and  $x^n \in \mathfrak{b}$ , therefore  $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$ . Conclude  $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$ .  
 “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: If  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , thus  $x^n \in \mathfrak{a}$  and  $x^n \in \mathfrak{b}$ . We may write  $x \in \sqrt{\mathfrak{a}}$  and  $x \in \sqrt{\mathfrak{b}}$ , therefore  $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ .  
 “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: Finally, let  $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ . Then,  $x \in \sqrt{\mathfrak{a}}$  and  $x \in \sqrt{\mathfrak{b}}$ , so  $x^n \in \mathfrak{a}$  and  $x^m \in \mathfrak{b}$  for some  $n, m \in \mathbb{N}^+$ . Say  $n \geq m$ , then  $x^n \in \mathfrak{b}$ . This yields  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , thus  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ .
6. “ $\sqrt{\mathfrak{a} + \mathfrak{b}} \subset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ ”: Let  $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} + \mathfrak{b}$  for some  $n \in \mathbb{N}^+$ . By definition of sum of ideals, we have that  $x^n = a + b$  for some  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$ . Since  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$  and  $\mathfrak{b} \subset \sqrt{\mathfrak{b}}$ , we have  $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$ , thus  $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ .  
 “ $\sqrt{\mathfrak{a} + \mathfrak{b}} \supset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ ”: Now let  $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ , then  $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$  for some  $n \in \mathbb{N}^+$ . Hence there exists  $a \in \sqrt{\mathfrak{a}}$  and  $b \in \sqrt{\mathfrak{b}}$  such that  $x^n = a + b$ . We have that  $a^p \in \mathfrak{a}$  and  $b^q \in \mathfrak{b}$

for some  $p, q \in \mathbb{N}^+$ . Consider

$$\begin{aligned} (x^n)^{(p+q-1)} &= (a+b)^{(p+q-1)} \\ &= \sum_{k=0}^{p+q-1} \binom{p+q-1}{k} a^k \cdot b^{p+q-1-k}. \end{aligned}$$

For each  $k \in \{0, 1, \dots, p+q-1\}$ , we have  $a^k \in \mathfrak{a}$  or  $b^{p+q-1-k} \in \mathfrak{b}$ . Thus, the whole sum lies in  $\mathfrak{a} + \mathfrak{b}$  or in other words  $x^{n(p+q-1)} \in \mathfrak{a} + \mathfrak{b}$ . Conclude  $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$ .

7. “ $\sqrt{\mathfrak{a}} \subset \mathfrak{p}$ ”: Let  $x \in \sqrt{\mathfrak{a}}$ , then  $x^m \in \mathfrak{a}$  for some  $m \in \mathbb{N}^+$ . Because  $\mathfrak{a} = \mathfrak{p}^n$ , we have  $x^m \in \mathfrak{p}^n$ . We also have  $\mathfrak{p}^n \subset \mathfrak{p}$ , thus  $x^m \in \mathfrak{p}$  and since  $\mathfrak{p}$  is prime,  $x \in \mathfrak{p}$ .

“ $\sqrt{\mathfrak{a}} \supset \mathfrak{p}$ ”: On the other hand, if  $x \in \mathfrak{p}$ , then  $x^n \in \mathfrak{p}^n = \mathfrak{a}$ , therefore  $x \in \sqrt{\mathfrak{a}}$ .

□

**Proposition 6.** 1.  $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$ .

**Example 6.1.** Does  $\sqrt{\mathfrak{a}^2} = \mathfrak{a}$  hold?



## Chapter 3

# Anatomy of Rings

**Definition 7** (Nilpotent Element and Nilradical). An element  $x$  of a ring  $A$  is called nilpotent if there exists some positive integer  $n \in \mathbb{N}^+$ , called the index or the degree, such that  $x^n = 0$ .

The set of all nilpotent elements is called the nilradical of the ring and is denoted by  $\text{Nil}(A)$ .

**Definition 8** (Reduced Ring). A ring  $A$  is called reduced ring if it has no non-zero nilpotent elements.

**Proposition 9.** Let  $A$  and  $B$  be two rings and  $A' \subset A$  be a subring of  $A$ .

1. If  $A$  is reduced, then  $A'$  is also reduced.
2. If  $A$  and  $B$  are reduced, then  $A \times B$  is also reduced.

(XXX DOES THIS ALSO HOLD FOR ARBITRARY MANY PRODUCTS?)

### 3.1 Exercises and Notes

**Example 9.1.** Let  $K$  be a field and  $A = K[X, Y]/(X - XY^2, Y^3)$ .

1. Compute the nilradical  $\text{Nil}(A)$ .

*Solution.* Denote  $(X - XY^2, Y^3) =: \mathfrak{a}$ .

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

Thus,  $X \in (X - XY^2, Y^3)$ . We have therefore the isomorphism  $K[X, Y]/(X - XY^2, Y^3) \simeq K[Y]/(Y^3)$ . [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

Clearly,  $Y \in \text{Nil}(A)$  or in other words  $(Y) \subset \text{Nil}(A)$ . But we also have that  $K[Y]/(Y) = K$  which is a field, therefore  $(Y)$  is a maximal ideal. Because  $1 \notin \text{Nil}(A)$  conclude  $\text{Nil}(A) = (Y)$ .  $\square$



## Chapter 4

# Polynomial Rings





## Chapter 5

# Quotient



# Chapter 6

## Localization

### 6.1 Definition and Theorems

**Definition 10** (Multiplicative Subset). A subset  $S$  of a ring  $A$  is called a multiplicative subset if the following conditions hold.

1.  $1 \in S$ .
2. For all  $x, y \in S$  it is  $xy \in S$ .

**Example 10.1.** Let  $A$  be a ring. Important examples of a multiplicative subset include the following.

1. The set of units  $A^\times$  is a multiplicative subset.
2. The set of non-zero-divisors  $A \setminus \text{ZD}(A)$  is a multiplicative subset.

**Example 10.2.** Let  $A$  be a ring. Other examples of multiplicative subsets are the following.

1. For any element  $x \in A$ , the set generated by its power  $\{1, x, x^2, x^3, \dots\}$  is a multiplicative subset.
2. For any ideal  $\mathfrak{a} \subset A$ , the set  $1 + \mathfrak{a}$  is a multiplicative subset.

**Lemma 11.** An ideal  $\mathfrak{p}$  of a ring  $A$  is prime if and only if its complement  $A \setminus \mathfrak{p}$  is a multiplicative subset.

**Definition 12** (Localization).  $S^{-1}A$  is again a ring.

**Lemma 13.** Let  $A$  be a ring and  $S$  a multiplicative subset, then the following are equivalent.

1.  $S^{-1}A = 0$ .
2.  $S$  contains a nilpotent element.
3.  $0 \in S$ .

*Proof.* “1.  $\Rightarrow$  2.”: Let  $S^{-1}A = 0$ , then for all  $x \in A$  and  $s \in S$  it is  $(x, s) \sim (0, 1)$ , thus  $x \cdot u = 0$  for some  $u \in S$ . In particular, this holds for  $x = 1$ , therefore  $1 \cdot u = 0$ . Since a unit can never be a zero divisor, we must have  $u = 0$  which is nilpotent and lies in  $S$ .

“1.  $\Leftarrow$  2.”: On the other hand, let  $x \in S$  be nilpotent, i.e.  $x^n = 0$  for some  $n \in \mathbb{N}^+$ . Because  $S$  is multiplicatively closed  $x^n = 0$  lies in  $S$ . Fix an element  $(y, s) \in S^{-1}A$ , then  $y \cdot 1 \cdot 0 = 0 \cdot s \cdot 0$ . Hence  $(y, s) \sim (0, 1)$  and we have  $S^{-1}A = 0$ .

“2.  $\Rightarrow$  3.”: Again, let  $x \in S$  be nilpotent, thus  $x^n = 0$  for some  $n \in \mathbb{N}^+$ .  $S$  is multiplicatively closed and we have  $x^n = 0 \in S$ .

“2.  $\Leftarrow$  3.”: If  $0 \in S$ , then  $S$  simply contains a nilpotent element because 0 is nilpotent.  $\square$

**Remark.** In the lemma above, the condition  $0 \notin S$  is required because if  $S$  contains 0, then  $S^{-1}A = 0$  and by definition, an integral domain is a nonzero ring.

**Proposition 14.** Let  $A$  be a ring.  $A$  is reduced if and only if all its localizations  $A_{\mathfrak{p}}$  at  $\mathfrak{p} \in \text{Spec } A$  is reduced.

*Proof.* “ $\Rightarrow$ ”: We prove the statement by contrapositive. Let  $A_{\mathfrak{p}}$  be not reduced for all  $\mathfrak{p} \in \text{Spec } A$ . Thus, in all  $A_{\mathfrak{p}}$ , there is an element, say  $x/s$  that is nilpotent and not zero, i.e.  $(x/s)^n = 0$  for some  $n \in \mathbb{N}^+$ . By the definition of localization, we get  $x^n \cdot u = 0$  for some  $u \in A \setminus \mathfrak{p}$ . Now,  $u \in A \setminus \mathfrak{p}$  cannot be zero, because if it was,  $A_{\mathfrak{p}} = 0$  which is reduced. Thus,  $x$  is nilpotent and  $A$  is not reduced.  $\square$

**Lemma 15.** Let  $A$  be a ring and  $S \subset A$  be a multiplicative subset that does not contain 0.

1.  $A$  is an integral domain if and only if  $S^{-1}A$  is an integral domain.
2.  $A$  is a unique factorization domain if and only if  $S^{-1}A$  is a unique factorization domain.

*Proof.* “ $\Rightarrow$ ”: Let  $A$  be an integral domain. Since  $S$  does not contain 0, the localization  $S^{-1}A$  is a nonzero ring (see EXAMPLE). Let  $(x, s) \in S^{-1}A \setminus \{0\}$  be a nonzero element and suppose there is a  $(y, t) \in S^{-1}A$  with  $(x, s) \cdot (y, t) = 0$ . It is  $(xy, st) = (0, 1)$  and thus  $xy \cdot u = 0$  for some  $u \in S$ . Because  $x$  was nonzero and  $S$  does not contain 0 we must have  $y = 0$ . Hence  $S^{-1}A$  is an integral domain.

“ $\Leftarrow$ ”: On the other hand, let  $S^{-1}A$  be an integral domain. JUST USE THE CANONIC MAPPING  $\varphi_S : A \rightarrow S^{-1}A$ .  $\square$

## 6.2 Exercises and Notes

**Example 15.1.** Let  $A_1$  and  $A_2$  be rings. Consider  $A = A_1 \times A_2$  and set  $S := \{(1, 1), (1, 0)\}$ . Prove  $A_1 \simeq S^{-1}A$ .

*Solution.* I don’t understand the solution?  $\square$

**Example 15.2.** Find all intermediate rings  $\mathbb{Z} \subset A \subset \mathbb{Q}$ , and describe each  $A$  as a localization of  $\mathbb{Z}$ . As a starter, prove  $\mathbb{Z}[\frac{2}{3}] = S_3^{-1}\mathbb{Z}$  where  $S_3 := \{3^i \mid i \in \mathbb{N}^+\}$ .

## Chapter 7

# Hierarchy of Rings

### 7.1 Definition and Theorems

#### 7.1.1 Integral Domains



# Part II

## Modules





**Definition 16** (Simple Modules). Let  $A$  be a ring. A nonzero  $A$ -module  $M$  is called simple if the only submodules are  $\{0\}$  and  $M$  itself.

**Example 16.1.** If  $M$  is a simple  $A$ -module, then any  $f \in \text{Hom}_A(M, M) \setminus \{0\}$  is an isomorphism.

*Proof.* Fix an  $f \in \text{Hom}_A(M, M) \setminus \{0\}$ . Since  $\ker(f)$  is a submodule of  $M$ , it must be either  $\{0\}$  or whole  $M$ . But  $\ker(f) = M$  would mean that  $f = 0$  which was explicitly excluded, thus  $\ker(f) = \{0\}$ . By the isomorphism theorem, we also have  $\text{im}(f) \cong M/\ker(f) \cong M$ . Therefore,  $f$  is bijective.  $\square$