My notes on "The Strong Factorial Conjecture" by Eric Edo and Arno van den Essen.
See: https://arxiv.org/abs/1304.3956

1 Factorial Conjecture

$\mathbb{C}^{[m]}$ may be viewed as a vecor space over $\mathbb{C}$ with a basis being

$$\left\{ X_1^{l_1} \cdots X_m^{l_m} \;\middle|\; l_k \in \mathbb{N}_0 \text{ for all } 1 \leq k \leq m \right\}.$$

Thus any linear map is fully defined if we set a value for each basis element. Such linear map is the factorial map.

Definition 1. A factorial map is a linear map linear map $\mathcal{L} : \mathbb{C}^{[m]} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{L}(X_1^{l_1} \cdots X_m^{l_m}) = l_1! \cdots l_m! \qquad \text{for all } l_1, \dots, l_m \in \mathbb{N}$$

Example 1.1. Consider $f(X) = 3X - 5XY + 7Y^2 \in \mathbb{C}^{[2]}$. Applying the factorial map yields

$$\begin{aligned} \mathcal{L}(f(X)) &= 3\mathcal{L}(X) - 5\mathcal{L}(XY) + 7\mathcal{L}(Y^2) \\ &= 3 \cdot 1 - 5 \cdot 1 + 7 \cdot 2 \\ &= 12. \end{aligned}$$

Example 1.2. If we limit our selves to a polynomial with one indeterminate, such as $f(X) = \sum_{k=0}^{n} a_k X^k \in \mathbb{C}[X]$ for a fixed $n \in \mathbb{N}_0$, we have

$$\mathcal{L}(f(X)) = \sum_{k=0}^{n} a_k \mathcal{L}(X^k) = \sum_{k=0}^{n} a_k k!$$

Theorem 2 (Conjecture 2.4)**.** Let $f \in \mathbb{C}^{[m]}$ be a polynomial. We have $\mathcal{L}(f^k) = 0$ for all $k \in \mathbb{N}_+$ if and only if $f = 0$.

Remark. The converse is trivial, hence the conjecture is about the forward implication.

2 Rigidity Conjecture

TODO: $\mathbb{C}_0[[X]]$ the set of formal power series with the constant coefficient being 0 forms a $\mathbb{C}$-algebra with composition being the composition.

2.1 Reciprocal of a Power Series

2.2 Formal Differentiation

Definition 3 (Formal Differentiation)**.** Given a formal power series $f(X) = \sum_{k\in\mathbb{N}_0} a_k X^k \in \mathbb{C}[[X]]$ its *formal derivative*, denoted f', is defined by

$$f'(X) := \sum_{k\in\mathbb{N}_0} a_k \cdot k \cdot X^{k-1}.$$

Remark. difference with analytic view of differentiation

Proposition 4 (Linearity of Formal Differentiation)**.** Formal differentiation as an operator is linear, i.e. if we view $\mathbb{C}[[X]]$ as a $\mathbb{C}$-vector space, then $(*)' : \mathbb{C}[[X]] \longrightarrow \mathbb{C}[[X]]$ satisfies additivity and homogeneity.

As expected, the usual rules of differentiation such as the product rule and the chain rule may be transfered one-to-one from the analytical world to the one of algebra and formal power series. For this paper, only the chain rule is of interest. Before we formaly introduce the chain rule however, we require the notion of composition of power series.

2.3 Composition of Formal Power Series

Proposition 5 (Chain Rule)**.** If $f \in \mathbb{C}[[X]]$ and $g \in \mathbb{C}[[X]]$ are two formal power series, then the formal differentiation on their composition may be expressed as

$$(f \circ g)' = (f' \circ g) \cdot g'$$

When we consider compositions of formal power series, we always want the constant term to be 0.

The following example is taken from:

https://math.stackexchange.com/questions/1212053/defining-composition-of-two-formal-series-what-is-going-on

Example 5.1. Let $f = \sum_{k\in\mathbb{N}_0} a_k X^k$ and $g = 1 + X$. Consider $f \circ g$. We have

$$\begin{aligned} f \circ g &= \sum_{k\in\mathbb{N}_0} a_k (1+X)^k \\ &= a_0 + a_1 + a_1 X + a_2 + 2a_2 X + a_2 X^2 + \cdots \end{aligned}$$

If $f \circ g$ is again a formal power series, then we should be able to write $f \circ g = \sum_{k\in\mathbb{N}_+} c_k X^k$ for some $c_k \in \mathbb{C}$. However, we see that c_0 is the sum of all a_k and we cannot evaluate that as algebraists. Thus composition of formal power series only makes sense if the constant coefficient is 0.

Proposition 6. A power series $f(X) = \sum_{k\in\mathbb{N}_+} a_k X^k \in \mathbb{C}[[X]]$ has a compositional inverse $f^{-1}(X)$ if and only if $a_1 \neq 0$, in which case $f^{-1}(X)$ is unique.

Proof. Assume $g(X) = b_1 X + b_2 X^2 + \cdots$ satisfies $f(g(X)) = X$. We then have

$$a_1(b_1 X + b_2 X^2 + \cdots) + a_2(b_1 X + b_2 X^2 + \cdots)^2 + a_3(b_1 X + b_2 X^2 + \cdots)^3 = X$$

Equating coefficients on both sides yields the infinite system of equations

$$\begin{aligned} a_1 b_1 &= 1 \\ a_1 b_2 + a_2 b_1^2 &= 0 \\ a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 &= 0 \\ &\vdots \end{aligned}$$

□

Another proof:

https://www.math.uwaterloo.ca/ dgwagner/co430I.pdf But there is no simple formula for the coefficients of the inverse (see enumerative combinatorics).

Theorem 7 (Conjecture 2.13)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If m consecutive coefficient of the compositional inverse $a^{-1}(X)$ vanish, i.e. $b_{n+1} = b_{n+2} = \cdots = b_{n+m} = 0$ for some $n \in \mathbb{N}_+$ then $a(X) = X$.

Remark. If we denote the polynomial $a(X)$ by $\sum_{k\in\mathbb{N}_0} a_k X^k$ for some $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}_0$, then the condition $a(X) \equiv X \mod X^2$ amounts to $a_0 = 0$ and $a_1 = 1$.

Theorem 8 (Conjecture 2.14)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the coefficients of $X^{n+1}, \ldots, X^{n+m}$ of the compositional inverse vanish, then $a(X) = X$.

Remark. $R(m)$ if and only if $R(m)_n$ for all $n \in \mathbb{N}_+$.

Lemma 9 (Lemma 2.16)**.** Let $f \in \mathbb{C}[[X]]$ and $g \in \mathbb{C}[[X]]$ be two formal series such that $f(X) \equiv g(X) \mod X^2$, i.e. the constant and the coefficient of the first degree agree. If $f(X) \equiv g(X) \mod X^n$ for some integer $n \geq 2$ then $f^{-1}(X) \equiv g^{-1}(X) \mod X^n$.

Proof. □

Proposition 10. 1. The polynomial $a(X)$ is invertible for the composition.

2. For all $i \in \{1, \ldots, \deg(a-1)\}$, the coefficient a_i is nilpotent element in A. I just don't see this ...

The following lemma and proof are due to #XXX.

Lemma 11 (Lagrange Inversion Formula)**.** Let K be a field of charateristic

$$f^{-1}(X) = \sum_{n\in\mathbb{N}_+} b_n X^n$$
$$\text{where } b_n = \frac{1}{n} \cdot [X^{n-1}] \left(\frac{X}{f(X)}\right)^n$$

Proof. We will prove that the given formula for b_n, i.e. the n-th coefficient of the compositional inverse, is merited. Thus begin by fixing an arbitary integer $n \in \mathbb{N}_+$.

By proposition #XXX, f is guranteed to have a unique compositional inverse which we will denote by $f^{-1}(X) = \sum_{k\in\mathbb{N}_+} b_k X^k$ with $b_k \in \mathbb{C}$ for all $k \in \mathbb{N}_+$. Applying the original f to both sides yields $f(f^{-1}(X)) = X$ on the left side and on the right we have

$$f\left(\sum_{k\in\mathbb{N}_+} b_k X^k\right) = \sum_{k\in\mathbb{N}_+} b_k f(X)^k$$

due to the linearity of f as a map, thus $X = \sum_{k\in\mathbb{N}_+} b_k f(X)^k$. Now, formal differentiation with the chain rule #sure? gives

$$1 = \sum_{k\in\mathbb{N}_+} k \cdot b_k \cdot f(X)^{k-1} \cdot f'(X).$$

Let $n \in \mathbb{N}$ #with0? be an integer. #moremotivation Dividing the above equation with the n-th power of the reciprocal produces

$$f(X)^{-n} = \sum_{k\in\mathbb{N}_+} k \cdot b_k \cdot f(X)^{k-n-1} \cdot f'(X).$$

After □

Example 11.1 (See 5.4.4). $f(X) = Xe^{-X} = X \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} X^k$

$$[X^n]f^{-1}(X) = \frac{1}{n}[X^{n-1}]e^{nX}$$

Lemma 12 (Lemma 2.20 (Additive Inversion Formula)). Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers. The formal inverse of $a(X) = X(1 - (\alpha_1 X + \cdots + \alpha_m X^m))$ is given by the following formula

$$a^{-1}(X) = X\left(1 + \frac{1}{n+1}\sum_{n\geq 1} u_n X^n\right)$$

where

$$u_n = \frac{1}{n!} \sum_{j_1+2j_2+\cdots+mj_m=n} \frac{(n+j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\alpha_1^{j_1}\cdots\alpha_m^{j_m}$$

Proposition 13 (Proposition 2.23). Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers and let $(u_n)_{n\in\mathbb{N}_+}$ be a sequence defined by AIF in Lemma 2.20. For all $n \in \mathbb{N}_+$, the Rigidity Conjecture $R(m)_n$ is equivalent to the following implication: If $u_n = \cdots = u_{n+m-1} = 0$ then $\alpha_1 = \cdots = \alpha_m = 0$.

Proof. □

Theorem 14. 1. The inclusion $E^{[m]} \subset F_n^{[m]}$ implies $R(m)_n$

Definition 15.

$$\begin{aligned} E^{[m]} &= \{\, X_1 \cdots X_m(\mu_1 X_1 + \cdots + \mu_m X_m) \mid \mu_1, \dots, \mu_m \in \mathbb{C} \,\} \subset \\ F_n^{[m]} &= \left\{ f \in \mathbb{C}^{[m]} \setminus \{0\} \;\middle|\; \mathcal{L}(f^k) \neq 0 \text{ for some } n \leq k \leq \mathcal{N}(f) - 1 \right\} \cup \{0\} \end{aligned}$$