Exercise 0.1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. 1. Let $S^{-1}M = 0$, then for all $s \in S$ and for all $m \in M$ we have that

$$\frac{m}{s} = 0, \tag{1}$$

or in other words, $(m, s) \equiv (0, s')$ for some $s' \in S$. By definition, there exists a $t \in S$ such that

$$\begin{aligned} t(s \cdot 0 - s'm) = 0 &\iff t(0 - s'm) = 0 \qquad (2)\\ &\iff ts'm = 0. \qquad (3) \end{aligned}$$

Choose ts' to be the factor and we get $sM = 0$.

2. If there is an $s \in S$ such that $sM = 0$, then we can write for all $m \in M$ that $s \cdot m = 0$. We have

$$0 = s \cdot m = s(1 \cdot m - 1 \cdot 0) \tag{4}$$

which means again $(m, 1) \equiv (0, 1)$, hence $S^{-1}M = 0$.

□