Problem 1

Proof. $ii) \Rightarrow i)$ is self-evident. We will focus on $i) \Rightarrow ii)$. Let $\overline{\xi} \in \mathbb{R}^{d+1}$ an arbitrage opportunity. Construct another arbitrage opportunity $\overline{\xi_*} = (\xi_*^0, \xi_*^1, \ldots, \xi_*^d)$ by setting

$$\begin{aligned}\xi_*^0 =&\xi^0 - \overline{\xi} \cdot \overline{\pi} \\ \xi_*^i =&\xi^i \text{ for all } i \geq 1.\end{aligned}$$

For the newly constructed arbitrage opportunity, we have

$$\begin{aligned}\overline{\xi_*} \cdot \overline{\pi} =& \sum_{i=0}^{d} \xi_*^i \cdot \pi^i \\ =& \left(\xi_0 - \overline{\xi} \cdot \overline{\pi}\right) \pi^0 + \sum_{i=1}^{d} \xi^i \cdot \pi^i \\ =&\xi_0 \cdot \pi_0 + \sum_{i=1}^{d} \xi^i \cdot \pi^i - \overline{\xi} \cdot \overline{\pi} \cdot \pi^0 \\ =&\overline{\xi} \cdot \overline{\pi} - \overline{\xi} \cdot \overline{\pi} \qquad \text{because } \pi^0 = 1 \\ =&0.\end{aligned}$$

In particular, ξ_* fullfills the given condition.

Moreover, it is

$$\begin{aligned}\overline{\xi_*} \cdot \overline{S}(\omega) =& \sum_{i=0}^{d} \xi_*^i S^i(\omega) \\ =&(\xi_0 - \overline{\xi} \cdot \overline{\pi}) S^0(\omega) + \sum_{i=1}^{d} \xi^i \cdot S^i(\omega) \\ =&\overline{\xi} \cdot \overline{S}(\omega) - \overline{\xi}\overline{\pi} S^0(\omega)\end{aligned}$$

Now, $\overline{\xi} \cdot \overline{S}(\omega) \geq 0$ $\mathbb{P}$-almost surely by definition, $\overline{\xi} \cdot \overline{\pi} \leq 0$ and $S^0(\omega) > 0$. Thus, $\overline{\xi_*} \cdot \overline{S}(\omega) \geq 0$ $\mathbb{P}$-almost surely.

Futhermore, since $\overline{\xi_*} \cdot \overline{S}(\omega) > \overline{\xi} \cdot \overline{S}(\omega)$, we also have $\mathbb{P}(\overline{\xi_*} \cdot \overline{S}(\omega) > 0) \geq \mathbb{P}(\overline{\xi} \cdot \overline{S}(\omega) > 0) > 0$ as desired.

Clearly, $i)$ or $ii)$ implies $iii)$. The other direction is false. Consider a market with $d = 1$, $r = 0$, and $\pi = (1)$. $\Omega = \mathbb{N}$ and $S(0) = 2$, and $S(\omega) = 1$ for all $\omega \neq 0$. $\xi = (1, -1)$, so $\overline{\xi} \cdot \pi = 0$ and yada yada yada, but the probability that you make money is 0. $\square$

Problem 2

a)

Simply consider $\overline{\xi} = (-2, 0, 1)$. We have $\overline{\xi} \cdot \overline{\pi} = 0$ and

$$\overline{\xi} \cdot \overline{S}(\omega_1) = -2 \cdot 1.1 + 0 + 3 = 0.8$$
$$\overline{\xi} \cdot \overline{S}(\omega_1) = -2 \cdot 1.1 + 0 + 4 = 1.8$$

b)

Consider $\overline{\xi} = (-4, 1, 1)$. We have $\overline{\xi} \cdot \overline{\pi} = -4 + 2 + 2 = 0$ and

$$\overline{\xi} \cdot \overline{S}(\omega_1) = -2 \cdot 1.1 + 3 + 1 = 1.8$$
$$\overline{\xi} \cdot \overline{S}(\omega_2) = -2 \cdot 1.1 + 1 + 3 = 1.8$$