Exercise Sheet 2

Exercise 1
Solution

1. $\mathbb{Z} \times \mathbb{Z}$ is not a Dedekind domain as it is not even an integral domain. Take $(1,0) \in \mathbb{Z} \times \mathbb{Z}$ and $(0,1) \in \mathbb{Z} \times \mathbb{Z}$ for example. $(1,0) \cdot (0,1) = (0,0)$ even though we chose nonzero elements.

2. $\mathbb{Z}[X]/(X^2+3)$ is not a Dedekind domain as it is not integrally closed.

   First, define a ring homomorphism $\varphi : \mathbb{Z}[X] \to \mathbb{Z}$ that substitues X with $\sqrt{-3}$. We have $\varphi(\mathbb{Z}[X]) = \mathbb{Z}[\sqrt{-3}]$ and $\ker(\varphi) = X^2 + 3$. With the isomorphism theorem, we have $\mathbb{Z}[X]/(X^2+3) \cong \mathbb{Z}[\sqrt{-3}]$.

   Consider

   $$\alpha := \frac{1}{2} + \frac{1}{2}\sqrt{-3} \in \mathrm{Quot}(\mathbb{Z}[\sqrt{-3}]) = \mathbb{Q}(\sqrt{-3}). \tag{1}$$

   From example 3.2.5. (script), we know that

   $$\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}[\alpha]. \tag{2}$$

   Therefore, α is integral over $\mathbb{Z}$ and hence over $\mathbb{Z}[\sqrt{-3}]$ as well, but it does not lie in $\mathbb{Z}[\sqrt{-3}]$. We conclude $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed.