Topology

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Introduction

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Topological Spaces

2-1

"$\Rightarrow$": Let $f : X_1 \longrightarrow X_2$ be a homeomorphism and fix a subset (not necessarily open) $U \in \mathcal{T}_1$.

1. Assume U is open in X_1. Because f is continuous, the image of open subsets are again open, thus $f(U)$ lies in $\mathcal{T}_2$.

2. On the other hand, if $f(U)$ is open in X_2, then since f is bijective we have

$$f^{-1}\left(f\left(U\right)\right) = U.$$

   Because f is continuous, the preimage of open subsets under f is open. We may therefore conclude U is open in X_1.

We have shown that if f is a homeomorphism, then $f(\mathcal{T}_1) = \mathcal{T}_2$.

"$\Leftarrow$": Let $f : X_1 \longrightarrow X_2$ be a bijective map such that $f(\mathcal{T}_1) = \mathcal{T}_2$. Consider the inverse map f^{-1}. We want to show f^{-1} is continuous. Fix an open subset $U \in \mathcal{T}_1$. It is

$$\left(f^{-1}\right)^{-1}(U) = f(U)$$

because f is bijective. Since $f(\mathcal{T}_1) = \mathcal{T}_2$ and U is open, $f(U)$ is open as well. Hence the preimage of U under f^{-1} is open and f^{-1} is continuous.

Now we show that f is also continuous. Again, fix an open subset $V \in \mathcal{T}_2$. The preimage of V under f is just the image of the inverse function. We have already shown that the inverse is continuous. Thus, $f^{-1}(V)$ is open and f is continuous. Since f and f^{-1} exist and are continuous, f is a homeomorphism as desired.

2-2

a)

We show that $\mathcal{T}$ is a topology by verifying the axioms of a topology.

1. Since $\mathcal{T}$ is the collection of all unions of finite intersections of elements of $\mathcal{B}$, it contains the union of all elements of $\mathcal{B}$ which is just X. The union of empty collection generates the emptyset so $\varnothing \in \mathcal{T}$ as well.

2. Let $\mathcal{U} \subset \mathcal{T}$ be any subset. The elements of $\mathcal{U}$ are unions of finite intersections of elements of $\mathcal{B}$. Thus, $\bigcup_{U \in \mathcal{U}} U$ is again a union of finite intersections of elements of $\mathcal{B}$. In other words, $\mathcal{T}$ is closed under union.

3. $\mathcal{T}$ is stable under finite intersections due to distributive property of sets.

b)

2-3

1.

The collection of subset $\mathcal{T}_1 = \{\, U \subset X \mid X \setminus U \text{ is finite or is all of } X \,\}$ forms a topology. We show this by verifying the axioms of a topology.

1. It is $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ which is finite. Thus, $X \in \mathcal{T}_1$ and $\varnothing \in \mathcal{T}_1$.
2. Let $\mathcal{U} \subset \mathcal{T}$ be a subset. By De Morgan's laws we have

$$X \setminus \left(\bigcup_{U \in \mathcal{U}} U \right) = \bigcap_{U \in \mathcal{U}} (X \setminus U)\,.$$

   Since each $U \in \mathcal{U}$ lies in $\mathcal{T}$, the complement $X \setminus U$ is finite or is all of X. Therefore, the intersection of all $X \setminus U$ is again finite or all of X, and we may conclude that $\mathcal{T}$ is stable under arbitary unions.
3. Use De Morgan's law again.

2.

The collection of subsets $\mathcal{T}_2 = \{\, U \subset X \mid X \setminus U \text{ is infinite or is empty} \,\}$ is not a topology. Take $X = \mathbb{Z}$ for example and consider $A = \{\, 1, 2, 3, \dots \,\}$ and $B = \{\, -1, -2, -3, \dots \,\}$. A and B are open because their complements are the non-positive and the non-negative integers respectively. If $\mathcal{T}_2$ is a topology, it should contain their union $A \cup B = \mathbb{Z} \setminus \{0\}$. However,

$$\mathbb{Z} \setminus (A \cup B) = \mathbb{Z}(\mathbb{Z} \setminus \{0\}) = \{0\}$$

which is not infinite and thus doesn't lie in $\mathcal{T}_2$.

3.

The collection of subsets $\mathcal{T}_3 = \{\, U \subset X \mid X \setminus U \text{ is countable or all of } X \,\}$ is a topology PROBABLY.

2-4

Already did somewhere else.

2-5

1. $\mathrm{id}_1 : X \longrightarrow \mathbb{R}^2$ is continuous probably.
2. $\mathrm{id}_2 : \mathbb{R}^2 \longrightarrow X$ is not continuous probably.

2-6

f is continuous because any preimage of a subset $U \subset Z$ under f is open, since any subset in X is open.

For g, the only preimages to check are the emptyset $\varnothing$ and Y. Simply, $g^{-1}(\varnothing) = \varnothing$ and $g^{-1}(Y) = Z$. Both subsets are open in Z, therefore g is continuous.

If h is constant, say $h(Y) = \{p\}$, then $h^{-1}(U) = Y$ if $p \in U$ and $h^{-1}(U) = \varnothing$ if $p \in U$. In both cases the preimages are open, thus h is continuous. Assume h is continuous but not constant, i.e. there are points $x_1, x_2 \in Y$ such that $h(x_1) \neq h(x_2)$. Z is Hausdorff, so there are disjoint neighbourhoods U of $h(x_1)$ and V of $h(x_2)$. h was assumed to be continuous, so $h^{-1}(U) = Y$ and $h^{-1}(V) = Y$ which is impossible (REALLY?).

2-7

a)

f)

2-8

Firstly, any element in $f(\mathcal{B})$ is open because f is an open map. Fix an open subset V in Y and consider its preimage $f^{-1}(V)$ under f. Because f is continuous, the preimage is open, thus there are base elements B_i with $i \in I$ in $\mathcal{B}$ such that

$$f^{-1}(V) = \bigcup_{i\in I} B_i.$$

The surjectivity of f grants us $f(f^{-1}(V)) = V$, therefore, we have

$$f(f^{-1}(V)) = V = f\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f(B_i).$$

Thus, $f(\mathcal{B})$ is a basis of Y.

2-9

2-10

Fix a point y in Y. Since f is surjective, there is an x in X such that $f(x) = y$. X is locally Euclidean, thus there is a neighbourhood U of x that is homeomorphic to $\mathbb{R}^n$. Moreover, f is locally homeomorphic, so there is a neighbourhood V of x such that the restriction of f under V is a homeomorphism. Then, the intersection $U \cap V = N$ has both of these properties, i.e. N is a neighbourhood of x that is homeomorphic to $\mathbb{R}^n$ and the restriction of f under V is a homeomorphism. $f(N)$ is a neighbourhood of y that is homeomorphic to $\mathbb{R}^n$, therefore Y is locally Euclidean.

2-11

"$\Rightarrow$": Let M^0 be a 0-manifold and consider a point $p \in M^0$. First, we show that M^0 is discrete. Since M^0 is locally Euclidean, there is a neighbourhood U of p such that U is homeomorphic to an open subset of $\mathbb{R}^0$. But $\mathbb{R}^0$ contains only one element, thus the only nonempty open subset is $\mathbb{R}^0$. Now, a homeomorphism implies bijectivity, we have that $U = \{p\}$. Every singleton set in M^0 is open, so M^0 is a discrete space.

M^0 is also countable because being a manifold implies that it has a countable base and any base must contain all the singleton sets.

"$\Leftarrow$": Let M^0 be a countable discrete space. M^0 is second-countable because the set of singletons form a countable base. It is also T_2 since each point has itself as its neighbourhood which clearly does not contain any other points. Now let $p \in M^0$ be a point. $\{p\}$ is a neighbourhood of p and it is homeomorphic to $\mathbb{R}^0$ by the mapping $p \mapsto 0$, thus M^0 is locally Euclidean.

2-12

a

It is $L(b) \cap R(a) = \{\, c \in X \mid c < b \text{ and } c > a \,\} = (a, b)$, thus (a, b) is open.

Moreover, we have $L(a) \cup R(b) = \{\, c \in X \mid c < a \text{ or } c > b \,\} = X \setminus [a, b]$ which is open, so $[a, b]$ is closed.

b

Let $a, b \in X$ be two distinct points and assume without loss of generality $a < b$. Then, $L(b)$ is open and contains the point a, while $R(a)$ is also open and contains the point b, but $L(b)$ and $R(a)$ are disjoint. Thus, X is Hausdorff.

c

Fix two points $a, b \in X$. By definition, it is

$$\overline{(a,b)} = \bigcap \{\, C \subset X \mid (a,b) \subset C \text{ and } C \text{ is closed in } X \,\}.$$

We have shown that $[a, b]$ is closed in a) and clearly contains (a, b), thus

$$[a,b] \in \{\, C \subset X \mid (a,b) \subset C \text{ and } C \text{ is closed in } X \,\}$$

or in other words

$$\bigcap \{\, C \subset X \mid (a,b) \subset C \text{ and } C \text{ is closed in } X \,\} \subset [a,b]$$

as desired.

When does $\overline{(a,b)} = [a,b]$ hold? Maybe it is pertinent to ask when does it not hold? The equality does not hold if and only if (a, b) is already closed. That means $X \setminus (a,b) = (-\infty, a] \cup [b, \infty)$ is open. I'm not sure, maybe X needs to be countable, finite?

2-13

Let X be a second countable topological space and fix a collection of disjoint open subsets $\mathcal{S}$, i.e.

$$\mathcal{S} = \{\, U \subset X \mid U \text{ is open and for all } U, V \in \mathcal{S} \text{ it is } U \cap V = \varnothing \,\}.$$

We want to show $\mathcal{S}$ is countable. If $\mathcal{B}$ is a base for X, then for any two members of the collection $U, V \in \mathcal{S}$, we have

$$U = \bigcup_{i \in \mathbb{N}} B_i \qquad V = \bigcup_{j \in \mathbb{N}} B_j.$$

Since U and V are disjoint, B_i and B_j are also disjoint for all $i, j \in \mathbb{N}$. Thus, any $U \in \mathcal{S}$ is a union of base elements that is different from any other $V \in \mathcal{S}$. $\mathcal{B}$ is countable, therefore $\mathcal{S}$ must also be.

2-14

Let X be a locally Euclidean space. We show that X is first-countable. Let $p \in X$ be a point, then since X is locally Euclidean, there is a neighbourhood N of p such that N is homeomorphic to $\mathbb{R}^n$. Thus, we have a sequence of neighbourhoods as $\mathbb{R}^n$ is first-countable, yada yada yada.
Let M be a metric space. I've shown that this is first-countable already.

2-15

a)

Let X be a second-countable space. We want to show that X contains a dense subset that is countable.

Chapter 3

New Spaces from Old

3-1

Chapter 4

Simplicial Complexes

Exercise 5.1

Definition 1 (Simplex)**.** Given points $v_0, \dots, v_k$ in general position in $\mathbb{R}^n$, simplex spanned by them is the set of all points in $\mathbb{R}^n$ of the form:

$$\sum_{i=0}^{k} t_i v_i \qquad \text{where } 0 \leq t_1 \leq 1 \text{ and } \sum_{i=0}^{k} t_i = 1.$$

Definition 2 (Convex Hull)**.** Let X be a subset of $\mathbb{R}^n$, then the convex hull of X is the intersection of all convex sets containing X.

Definition 3 (Convex Set)**.** A subset X of $\mathbb{R}^n$ is convex if for all $x, y \in X$ and for all $t \in [0,1]$ it is

$$(1-t)x + ty \in X.$$

Proof. Let σ be a simplex and denote its vertices by $v_0, \dots, v_k$. Fix a point $p \in \sigma$, then by definition

$$p = \sum_{i=0}^{k} t_i v_i$$

for some $t \in [0,1]$ and $\sum_{i=0}^{k} t_i = 1$. We show that p lies in any convex set containing $v_0, \dots, v_k$ by induction. In the initial case, $p = t_i v_i = v_i$ for some i which clearly lies in all convex set containing $v_0, \dots, v_k$. Consider the case where $n+1$ number of $t_0, \dots, t_k$ is nonzero. Without loss of generality, we may reorder the indecies for notational ease and yield

$$p = \sum_{i=0}^{n+1} t_i v_i = t_{n+1} v_{n+1} + \sum_{i=0}^{n} t_i v_i.$$

By induction hypothesis, $\sum_{i=0}^{n} t_i v_i$ is contained in any convex set containing $v_0, \dots, v_k$. □

Exercise 5.2

a)

Fix two simplices σ and τ, and denote the set of their vertices by $\mathrm{vert}(\sigma)$ and $\mathrm{vert}(\tau)$ respectively. Let $f_0 : \mathrm{vert}(\sigma) \longrightarrow \mathrm{vert}(\tau)$ be any map and consider a point $p \in \sigma$. p may be represented by a linear combination of the vertices, thus

$$p = \sum_{i=1}^{k} v_i$$

which allows us to define

$$f(p) := \sum_{i=1}^{k} f_0(v_i).$$

Since a simplex is the convex hull of its vertices, $f(p)$ lies in τ.

Unsure, but should be the right direction.

b)

sounds reasonable

c)

more suprising

Exercise 5.3

Example 5.2 a)

Let K be the collection of a n-simplex σ and its faces. Trivially, the faces of σ lies in K, and the faces of its faces are just faces of σ, and thus are also members of K. The any intersection of σ and its faces are again faces or empty. (IF I UNDERSTOOD THIS CORRECTLY) since K is already finite the third condition also applies.

Exercise 5.4

Exercise 5.5