

# Notes on Algebraic Geometry

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# Contents

<b>I</b>	<b>Pre</b>	<b>5</b>
<b>II</b>	<b>Algebraic Geometry</b>	<b>9</b>
<b>1</b>	<b>Affine Varieties</b>	<b>11</b>
<b>2</b>	<b>Projective Varieties</b>	<b>15</b>
<b>3</b>	<b>Morphisms</b>	<b>17</b>
	[1]	

TODO

# Part I

## Pre



1. Prove  $R$  int domain, then  $R[X]$  is int domain.

**Proposition 0.0.1.** *If  $R$  is an integral domain, then the polynomial ring  $R[X]$  is again an integral domain.*

*Proof.* 1. Since  $1 \in R \subset R[X]$ , the polynomial ring  $R[X]$  is nonempty.

2. Let  $f, g \in R[X]$  be two nonzero polynomials with

$$f = \sum_{i=0}^m a_i X^i \quad \text{and} \quad g = \sum_{j=0}^n b_j X^j.$$

Consider its product

$$f \cdot g = \sum_{k=0}^{m+n} c_k X^k$$

and suppose  $f \cdot g = 0$ .

3. Since the leading coefficient of the product  $c_{m+n}$  is obtained by multiplying the leading coefficients of  $f$  and  $g$ , we have  $c_{m+n} = a_m \cdot b_n$ .
4. We had  $f \cdot g = 0$ , thus  $c_{m+n} = a_m \cdot b_n = 0$ .
5.  $R$  is an integral domain, therefore  $a_m \cdot b_n = 0$  means  $a_m = 0$  or  $b_n = 0$ .
6. This contradicts that  $f$  and  $g$  were nonzero polynomials.

□





## Part II

# Algebraic Geometry



# Chapter 1

## Affine Varieties

**Definition 1.0.1.** Let  $K$  be an algebraically closed field and let  $n \in \mathbb{N}_0$  be a natural number.

1. The affine  $n$ -space over  $K$  is the set of all  $n$ -tuples of elements of  $K$ .
2. An element  $p$  in  $\mathbb{A}^n$  is called a point.
3. If  $p = (a_1, \dots, a_n) \in \mathbb{A}^n$  is a point, then  $a_i$  is called the coordinate for each  $1 \leq i \leq n$ .

**Intuition 1.0.2.** It's just space with points. But not vectors, because we don't add points.

**Definition 1.0.3.** For each subset  $S$  of polynomials in  $K[X_1, \dots, X_n]$ , we define the zero-locus  $Z(S)$  to be the set of points in the affine  $n$ -space  $\mathbb{A}^n$  on which the functions in  $S$  simultaneously vanish, i.e.

$$Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

**Intuition 1.0.4.** These are just curves.

**Remark 1.0.5.** 1. If  $\mathfrak{a}$  is generated by  $T$ , then  $Z(T) = Z(\mathfrak{a})$ .

2.  $Z(T)$  can be written in finitely many generators.

**Definition 1.0.6.** A subset  $Y$  of  $\mathbb{A}^n$  is an algebraic set if there exists a subset  $T \subset A = k[X_1, \dots, X_n]$  such that  $Y = Z(T)$ .

**Intuition 1.0.7.** So if the points on the space is a curve, then it's an algebraic set.

BOOKMARK

**Definition 1.0.8** (Affine Algebraic Variety). For an algebraically closed field  $K$  and a natural number  $n \in \mathbb{N}_+$ , let  $\mathbb{A}^n$ , be an affine  $n$ -space over  $K$ . The polynomials in  $K[X_1, \dots, X_n]$  can be viewed as  $K$ -valued functions on  $\mathbb{A}^n$ .

1. For each subset  $S$  of polynomials in  $K[X_1, \dots, X_n]$ , define the zero-locus  $Z(S)$  to be the set of points in  $\mathbb{A}^n$  on which the functions in  $S$  simultaneously vanish, i.e.

$$Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

2. A subset  $V$  of  $\mathbb{A}^n$  is called affine algebraic set if  $V = Z(S)$  for some  $S \subset K[X_1, \dots, X_n]$ .
3. A nonempty affine algebraic set is called irreducible if it is not the union of two proper algebraic subsets. An irreducible affine algebraic set is also called an affine variety.

**Definition 1.0.9.** An affine algebraic variety is an irreducible closed subset of  $\mathbb{A}^n$ . An open subset of an affine variety is a quasi-affine variety.

**Corollary 1.0.10.** *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

**Definition 1.0.11.** If  $Y \subset \mathbb{A}^n$  is an affine algebraic set, we define the affine coordinate ring  $A(Y)$  of  $Y$ , to be  $A/I(Y)$ .

**Definition 1.0.12.** If  $X$  is a topological space, we define the dimension of  $X$  (denoted  $\dim X$ ) to be the supremum of all integers  $n$  such that there exists a chain  $Z_0 \subset Z_1 \subset \dots \subset Z_n$  of distinct irreducible closed subsets of  $X$ . We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

**Exercise 1.0.1.** Let  $Y$  be the plane curve  $y = x^2$  (i.e.,  $Y$  is the zero set of the polynomial  $f = y - x^2$ ). Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.* By definition 1.0.11, we simply have  $A(Y) = k[X, Y]/(Y - X^2)$ . The isomorphism follows from the isomorphism theorem and the map  $f : k[X, Y] \rightarrow k[X]$  where we set  $f(Y) = X^2$ .

**Exercise 1.0.2.** Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.*  $A(Z) = k[X, Y]/(XY - 1)$

We know  $A(Z)$  is an  $k$ -algebra (see remark). Consider  $f : k[X, Y] \rightarrow k[T]$ . We must have  $\ker f = (XY - 1)$ , thus  $f(XY - 1) = 0$ , so  $f(X) = 1/f(Y)$

I'll think about the rigorous details later, but basically  $A(Z) \cong k[X, X^{-1}]$

**Exercise 1.0.3.** Let  $f$  be any irreducible quadratic polynomial in  $k[X, Y]$ , and let  $W$  be the conic defined by  $f$ . Show that  $A(W)$  is isomorphic to  $A(Y)$  or  $A(Z)$ . Which one is it when?

*Solution.* Let  $f$  be irreducible.

$$A(W) = k[X, Y]/(f)$$

isn't this kinda clear ...? I'll come back to write it down rigorously, but in general ...

**Exercise 1.0.4.** Let  $V \subset \mathbb{A}^3$  be the set  $Y = \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}$ .

1. Show that  $V$  is an affine variety of dimension 1.

*Solution.* 1. We show that  $V$  is an algebraic set.

- (a) Consider the ideal  $(Y - X^2, Z - X^3) \subset K[X, Y, Z]$  and its zero set  $Z(Y - X^2, Z - X^3)$ .
- (b) Writing out the definition of the zero set gives

$$\begin{aligned} Z(Y - X^2, Z - X^3) &= \{ (x, y, z) \in \mathbb{A}^3 \mid y - x^2 = 0, z - x^3 = 0 \} \\ &= \{ (x, y, z) \in \mathbb{A}^3 \mid y = x^2, z = x^3 \} \\ &= \{ (x, x^2, x^3) \in \mathbb{A}^3 \mid x \in K \}. \end{aligned}$$

Thus,  $V$  is the zero set of the ideal  $(Y - X^2, Z - X^3)$ .

- (c) Hence, by definition,  $V = Z(Y - X^2, Z - X^3)$  is an algebraic set.

2. Here, we prove that  $V$  is irreducible.

- (a) Consider the quotient  $K[X, Y, Z]/(Y - X^2, Z - X^3)$ .
- (b) By substitution, we get the isomorphism

$$K[X, Y, Z]/(Y - X^2, Z - X^3) \cong K[X, X^2, X^3] = K[X].$$

- (c) Since  $K$  is a field it is in particular an integral domain and so is  $K[X]$ .
- (d) Thus,  $(Y - X^2, Z - X^3)$  is prime in  $K[X, Y, Z]$ .
- (e) With corollary 1.0.10 we may conclude the variety  $V = Z(Y - X^2, Z - X^3)$  is irreducible.

**Exercise 1.0.5.** Let  $Y$  be the algebraic set in  $\mathbb{A}^3$  defined by two polynomials  $x^2 - yz$  and  $xz - x$ . Show that  $Y$  is a union of three irreducible components. Describe them and find their prime ideals.

*Solution.*  $Y = Z(x^2 - yz, xz - x)$

If  $z = 0$ , then  $x = 0$  and  $y$  can be any thing, so one irreducible component is the  $y$ -axis. This is described by  $(x, z)$ .

If  $x = 0$ , then  $yz = 0$ . If  $z = 0$ , then see above.  $y = 0$  gives the  $z$ -axis as above.

If  $z \neq 0$ , then  $x^2 = yz$  and  $x(z - 1) = 0$ . And we have a parabola ...

**Exercise 1.0.6.** *If we identify  $\mathbb{A}^2$  with  $\mathbb{A}^1 \times \mathbb{A}^1$  in the natural way, show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology of the Zariski topologies on the two copies of  $\mathbb{A}^1$ .*

*Solution.* In  $\mathbb{A}^1$ , the only closed sets are finite sets and  $\mathbb{A}^1$  itself. Take  $\{p_1, p_2\} \subset \mathbb{A}^1$ . Consider

$$\{(0, 0); (0, 1); (1, 0); (1, 1)\}$$

No, this is probably okay

Rather consider this example:  $Z(X^2 - Y)$ , then the preimages of the projections give the x-axis and the non-negative y-axis, but the latter is not an algebraic set.

**Exercise 1.0.7.** *Show that  $k$ -algebra  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ , for some  $n$ , if and only if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.*

## Chapter 2

# Projective Varieties





## Chapter 3

# Morphisms

**Definition 3.0.1.** Let  $X$  be a quasi-affine variety in  $\mathbb{A}_K^n$  and  $f : X \rightarrow K$  a function.

1.  $f$  is regular at a point  $p \in X$  if there is an open neighborhood  $\mathcal{U} \subset X$  of  $p$ , and polynomials  $g, h \in K[X_1, \dots, X_n]$ , such that  $h(x) \neq 0$  for all  $x \in \mathcal{U}$ , and  $f = g/h$  on  $\mathcal{U}$ .
2.  $f$  is regular on  $X$  if it is regular at every point on  $X$ .

**Lemma 3.0.2.** *A regular function is continuous, when  $K$  is identified with  $\mathbb{A}_K^1$  in its Zariski topology.*

**Definition 3.0.3** (Germ). Given a point  $p$  of a topological space  $X$ , and two maps  $f, g : X \rightarrow Y$  where  $Y$  is any set, then  $f$  and  $g$  define the same germ at  $p$  if there is a neighbourhood  $\mathcal{U}$  of  $p$  such that restricted to  $\mathcal{U}$ ,  $f$  and  $g$  are equal, i.e.

$$f(x) = g(x) \text{ for all } x \in \mathcal{U}.$$

**Definition 3.0.4.** Let  $X$  be a variety.

1. We denote the ring of all regular functions on  $X$  by  $\mathcal{O}(X)$ .
2. If  $p$  is a point on  $X$ , we define the local ring of  $p$  on  $X$ ,  $\mathcal{O}_p$  to be the ring of germs of regular functions on  $X$  near  $p$ . In other words, an element of  $\mathcal{O}_p$  is a pair  $(\mathcal{U}, f)$  where  $\mathcal{U}$  is an open subset of  $X$  containing  $p$ , and  $f$  is a regular function on  $\mathcal{U}$ , and where we identify two such pairs  $(\mathcal{U}, f)$  and  $(\mathcal{V}, g)$  if  $f = g$  on  $\mathcal{U} \cap \mathcal{V}$ .

**Theorem 3.0.5.** *Let  $X \subset \mathbb{A}^n$  be an affine variety.*

1. *The ring of all regular functions on  $X$  is isomorphic to the coordinate ring of  $X$ , i.e.*

$$\mathcal{O}(X) \cong A(X).$$

2. *There is a one-to-one correspondence between the points of  $X$  and the maximal ideals of  $A(X)$ .*
3. *The localization of the ring of all regular functions at  $p \in X$*

# Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.