Topology

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September 21, 2022

Contents

Chapter 1

Introduction

Chapter 2

Topological Spaces

2-1

"$\Rightarrow$": Let $f : X_1 \longrightarrow X_2$ be a homeomorphism and fix a subset (not necessarily open) $U \in \mathcal{T}_1$.

1. Assume U is open in X_1. Because f is continuous, the image of open subsets are again open, thus $f(U)$ lies in $\mathcal{T}_2$.

2. On the other hand, if $f(U)$ is open in X_2, then since f is bijective we have

$$f^{-1}\left(f\left(U\right)\right) = U.$$

   Because f is continuous, the preimage of open subsets under f is open. We may therefore conclude U is open in X_1.

We have shown that if f is a homeomorphism, then $f(\mathcal{T}_1) = \mathcal{T}_2$.

"$\Leftarrow$": Let $f : X_1 \longrightarrow X_2$ be a bijective map such that $f(\mathcal{T}_1) = \mathcal{T}_2$. Consider the inverse map f^{-1}. We want to show f^{-1} is continuous. Fix an open subset $U \in \mathcal{T}_1$. It is

$$\left(f^{-1}\right)^{-1}(U) = f(U)$$

because f is bijective. Since $f(\mathcal{T}_1) = \mathcal{T}_2$ and U is open, $f(U)$ is open as well. Hence the preimage of U under f^{-1} is open and f^{-1} is continuous.

Now we show that f is also continuous. Again, fix an open subset $V \in \mathcal{T}_2$. The preimage of V under f is just the image of the inverse function. We have already shown that the inverse is continuous. Thus, $f^{-1}(V)$ is open and f is continuous. Since f and f^{-1} exist and are continuous, f is a homeomorphism as desired.

2-2