Chapter 1

Rings

Example 1

1. $(\mathbb{Z}, +, \cdot)$
2. All fields, such as $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, and $(\mathbb{C}, +, \cdot)$, are rings.
3. Let R be a commutative ring, then $R[X]$, the set of polynomials with coefficients in R, is again a ring, e.g. $\mathbb{Z}[X]$, $\mathbb{Q}[X]$, and $\mathbb{R}[X]$.
4. For any ring R and for any $n \in \mathbb{N}$, the set of all square n-by-n matricies with entries from R, forms a ring with matrix addition and matrix multiplication as operations. If $n = 1$, this matrix ring is isomorphic to R itself. For $n > 1$ (and R not a zero ring), this matrix is noncommutative. More concretely, $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ is a noncommutative ring.

1.1 No idea yet

Definition 1 (Fractional Ideal)
Let A be an integral domain.

1. A fractional ideal of A is an A-submodule $I \subset \mathrm{Quot}(A)$ such that $dI \subset A$ for some denominator $d \in A \setminus \{0\}$.
2. A principal fractional ideal is a fractional ideal of the form $(r) = rA = \{ar \mid a \in A\}$

Example 2

- All ordinary ideals $I \subset A$ are also fractional ideals with denominator $d = 1$, and are often referred to as integral ideals.
- The subset
$$\frac{3}{25}\mathbb{Z} = \left\{ \frac{3n}{25} \in \mathbb{Q} \,\middle|\, n \in \mathbb{Z} \right\} \subset \mathbb{Q} \tag{1.1}$$
is a principal fractional ideal of $\mathbb{Z}$

Example 3
The subset
$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ a_0 + a_1\frac{1}{2} + a_2\frac{1}{2^2} + \cdots + a_n\frac{1}{2^n} \,\middle|\, a_0, \ldots, a_n \in \mathbb{Z} \subset \mathbb{Q} \right\} \tag{1.2}$$
is not a fractional ideal, because the denominators are not bounded.

Lemma 1.1 If $I \subset \mathrm{Quot}(A)$ is an A-submodule and $d \in \mathrm{Quot}(A)$, then $dI \subset \mathrm{Quot}(A)$ is also an A-module. Thus $I \subset K$ is a fractional ideal if and only if $I = \frac{1}{d}J$ for some $d \in A \setminus \{0\}$ and some integral ideal $J \subset A$ (just take d a denominator of I and $J = dI$).

Lemma 1.2 Let A be an integral domain and denote its field of fraction with $\text{Quot}(A) = K$.

1. If $I \subset K$ is a finitely generated A-submodule, then I is a fractional ideal.
2. Conversely, if A is noetherian and $I \subset K$ is a fractional ideal, then I is a finitely generated A-module.
3. If $I, J \subset K$ are fractional ideals, then $I \cap J, I + J, IJ, \subset K$ are also fractional ideals.
4. If $I, J \subset K$ are fractional ideals and $J \neq 0$, then the generalized ideal quotient

$$(I : J) := \{\, x \in K \mid xJ \subset I \,\} \tag{1.3}$$

is also a fractional ideal. Moreover, it satisfies $(I : J)J \subset I$.

The nonzero fractional ideals form an abelian semigroup with neutral element A with respect to the multiplication. We will now show that, if A is a Dedekind domain, every nonzero fractional ideal has an inverse hence they forme an abelian group $Id(A)$.

Definition 2
Let A be an integral domain. A fractional ideal $I \subset K$ is invertible if $IJ = A$ for some fractional ideal J called the inverse of I.

The following result shows characterizes invertible fractional ideals and their inverses (which are unique).

Lemma 2.1 A fractional ideal I is invertible if and only if $I(A : I) = A$, in which case $I^{-1} := (A : I)$ is the unique inverse.

The main result of this section is to prove that, in a Dedekind domain, every nonzero ideal is invertible. To this aim we need first a technical result.

Lemma 2.2 Let A be a Dedekind domain and $I \subset A$ a nonzero integral ideal. Then there are not necessarily distinct nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset A$ such that $\mathfrak{p}_1 \cdot \cdots \cdot \mathfrak{p}_n \subset I$.

Let

$$\Sigma = \{\, I \neq \{0\} \mid I \subset A \text{ ideal. } I \text{ does not contain any product of nonzero prime ideals.} \,\}. \tag{1.4}$$

If $\Sigma \neq \varnothing$, let $I \in \Sigma$ be a maximal element which must exist since A is noetherian. In particular, I is not prime, i.e. there exists $a, b \in A \setminus I$ with $a \cdot b \in I$.

Because of the maximility of I, the ideals $I + (a), I + (b) \supsetneq I$ don't lie in I, i.e. there exists nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}_1, \ldots, \mathfrak{q}_m$ such that

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subseteq I + (a) \tag{1.5}$$
$$\mathfrak{q}_1, \ldots, \mathfrak{q}_n \subseteq I + (b). \tag{1.6}$$

We have

$$\mathfrak{p}_1 \cdot \cdots \cdot \mathfrak{p}_n \cdot \mathfrak{q}_1 \cdot \cdots \cdot \mathfrak{q}_m \subseteq (I + (a))(I + (b)) \subseteq I \tag{1.7}$$

which is a contradiction. Hence $\Sigma = \varnothing$.

Theorem 2.1 Let A be a Dedekind domain, I a nonzero ideal, and $\mathfrak{p}$ a prime ideal such that $I \subseteq \mathfrak{p}$. Set

$$\mathfrak{p}^{-1} := (A : \mathfrak{p}) = \{x \in \text{Quot}(A) \mid x\mathfrak{p} \subseteq A\}. \tag{1.8}$$

Then, $I \subsetneq \mathfrak{p}^{-1} I \subseteq A$. In particular, $A \subsetneq \mathfrak{p}^{-1}$ and $\mathfrak{p}^{-1}\mathfrak{p} = A$, i.e. $\mathfrak{p}$ is invertible.

Corollary 1 Let A be a Dedekind domain and

$$Id(A) = \{I \subseteq K \mid I \text{ is a nonzero fractional ideal.}\}. \tag{1.9}$$

1. Every nonzero fractional ideal $I \in Id(A)$ is invertible. In particular, $Id(A)$ is an abelian group with respect to the product of ideals, and the trivial ideal $(1) = A$ as neutral element.

2. Moreover, the map

$$\varphi : K^* \to Id(A), \quad \frac{a}{b} \mapsto \left(\frac{a}{b}\right) = \left\{\frac{ac}{b} \middle| c \in A\right\} \subseteq K, \tag{1.10}$$

   is a group homomorphism, whose image is the subgroup P_A of nonzero principal fractional ideals.

Definition 3
The (ideal) class group of a Dedekind domain A is the quotient $Cl(A) = Id(A)/P_A$ which is naturally an abelian group.