Topology

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Contents

Chapter 1

Topological Space

1.1 Definitions and Theorems

Definition 1 (Topological Space)**.** A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to τ.
2. Any **arbitary** union of members of τ belongs to τ.
3. The intersection of **finite number** of members of τ belongs to τ.

The collection τ is called a topology on X and the elements of τ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.

Definition 2 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \tag{1.1}$$

Lemma 3. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 4 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. If $Y = X$ we also write $\text{Homeo}(X)$.

Definition 5 (Homeomorphism)**.** Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 6. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets, then there exists **exactly one** topology $\tau \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \tau$
2. If $\tau' \subset \mathcal{P}(X)$ a topology with $S \subset \tau'$, then $\tau \subset \tau'$.

Definition 7.

1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by
$$\tau_S = \{ S \cap U \mid U \in \tau \} .$$
2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by
$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$
3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by
$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$

Definition 8. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.
2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\mathrm{int}(S)$, is the set of all interior points of S.
3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\mathrm{cl}(S)$, is defined by
$$\mathrm{cl}(S) := X \setminus \mathrm{int}(X \setminus S).$$

1.2 Proofs, Remarks, and Examples

Example 8.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the discrete topology. In this case, (X, τ) is called the discrete space. It is the finest topology that can be defined on a set. (The set of all possible topologies on a given set forms a partially ordered set.)

2. $\tau = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology.

3. Let (X, d) be a metric space. Set

$$\tau_d := \{\, U \in X \mid U \text{ is a open subset in the metric space } (X, d) \,\} . \tag{1.2}$$

   Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U.

   Here, τ is a topology. In other words, a metric induces a topology.

   (Proof as homework.)

4. The Zariski-topology.

Example 8.2. List of natural topologies.

1. On $\mathbb{R}^n$ the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of $\mathbb{R}^n$ are arbitary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{\, x \in \mathbb{R}^n \mid d(p, x) < r \,\} .$$

   This definition agrees with the topology endowed on arbitary metric spaces.

Remark. The set of all homeomorphisms of X to itself $\mathrm{Homeo}(X)$ is a group with composition as its operation.

Remark. This lemma does not hold for basis.

Remark. 1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{\, U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y \,\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. [1] Let $(X, \mathcal{O})$ be a topological space. A subset that is **both** open and closed is called clopen. Moreover, a subset is clopen if and only if its boundary is empty.

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\mathrm{cl}(A) = A$, but on the other hand, A is open, so we also have $\mathrm{int}(A) = A$. Then, the boundary of A is $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A) = A \setminus A = \varnothing$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. □

[1] The following is a definition and a small proposition.

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 9. A topological space X is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two disjoint sets.
2. The **only** subsets of X that are **both** open and closed (clopen) are the emptyset $\varnothing$ and the entire set X.
3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

Lemma 10. Any interval $I \subset \mathbb{R}$ is connected.

Definition 11. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 12. Connected components are closed subsets.

Lemma 13 (Lemma 11)**.** Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 14. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 15. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 16. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.
3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Proof. We verify the equivalence of the different definitions. So, let $(X, \mathcal{O})$ be a topological space.

- "*1.* $\Rightarrow$ *2.*": Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X, then $X \setminus A$ is also not the empty set nor X. Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \varnothing$ or $A = X$.
- "*2.* $\Rightarrow$ 1.": Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.
- "*2.* $\iff$ *3.*": This is one of the properties of clopen subsets and was proven in remark XXX.
- "*1.* $\Rightarrow$ *4.*": Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \longrightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.
- "*4.* $\Rightarrow$ *1.*": Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \longrightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

□

Lemma 17. Any interval $I \subset \mathbb{R}$ is connected.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \tag{2.1}$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \mathrm{cl}(B)$ and since the complement of B is an open subset A, so $B = \mathrm{cl}(B)$. It is therefore $x \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. $\square$

Example 17.1. The general linear group $\mathrm{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Remark. Let $f : X \longrightarrow Y$ be continuous and X be connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}B = f^{-1}(A \cap B) = \varnothing$ so $f^{-1}(A) = \varnothing$ or $f^{-1}(B) = \varnothing$, so $A = \varnothing$ or $B = \varnothing$ and we are done. $\square$

Proof. $\square$

Example 17.2. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proof. Locally constant implies continuous with regards to the discrete topology on Y. Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \varnothing$. Conclude f is identical to $f(x)$. $\square$

Application: $f : X \longrightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \longrightarrow X$ be continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We have that γ^{-1} $\square$

Remark. The converse statement is not true in general.

Example 17.3. $X = \left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected but not path connected.

Proof. Homework $\square$

Remark. missing

2.3 Exercises and Notes

Remark. If X and Y are two connected topological spaces, then their product $X \times Y$ is also connected.

Proof. $\square$

Example 17.4. Clearly, the union of two connected sets need not be connected. Take for example $[0,1] \subset \mathbb{R}$ and $[2,3] \subset \mathbb{R}$. Their union $[0,1] \cup [2,3]$ is not connected.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \left\{ (x,y) \mid x^2 + y^2 = 1 \right\}$ and another unit circle around $(1,0)$ $A := \left\{ (x,y) \mid (x-1)^2 + y^2 = 1 \right\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \right\}$$

which is not connected.

Chapter 3

Separation Axioms

Literature: Groessere Liste in Sten, Seibeck

Definition 18 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 19. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 20 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 21. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{\, (x, x) \mid x \in X \,\}$ is closed as a subset of the product space $X \times X$.

Proposition 22. T_2 spaces are also T_1 spaces.

Chapter 4

Compact Spaces

Definition 23.
1. A topological space X is called compact if each of its open cover has a **finite** subcover.
2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Theorem 24. Satz 17

Theorem 25. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 26. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 27. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \longrightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Lemma 28. $[0,1] \subset \mathbb{R}$ is compact.