Topology

K

September 21, 2022

Contents

Chapter 1

Introduction

Chapter 2

Topological Spaces

2-1

"$\Rightarrow$": Let $f : X_1 \longrightarrow X_2$ be a homeomorphism and fix a subset (not necessarily open) $U \in \mathcal{T}_1$.

1. Assume U is open in X_1. Because f is continuous, the image of open subsets are again open, thus $f(U)$ lies in $\mathcal{T}_2$.
2. On the other hand, if $f(U)$ is open in X_2, then since f is bijective we have
$$f^{-1}\left(f\left(U\right)\right) = U.$$
Because f is continuous, the preimage of open subsets under f is open. We may therefore conclude U is open in X_1.

We have shown that if f is a homeomorphism, then $f(\mathcal{T}_1) = \mathcal{T}_2$.

"$\Leftarrow$": Let $f : X_1 \longrightarrow X_2$ be a bijective map such that $f(\mathcal{T}_1) = \mathcal{T}_2$. Consider the inverse map f^{-1}. We want to show f^{-1} is continuous. Fix an open subset $U \in \mathcal{T}_1$. It is

$$\left(f^{-1}\right)^{-1}(U) = f(U)$$

because f is bijective. Since $f(\mathcal{T}_1) = \mathcal{T}_2$ and U is open, $f(U)$ is open as well. Hence the preimage of U under f^{-1} is open and f^{-1} is continuous.

Now we show that f is also continuous. Again, fix an open subset $V \in \mathcal{T}_2$. The preimage of V under f is just the image of the inverse function. We have already shown that the inverse is continuous. Thus, $f^{-1}(V)$ is open and f is continuous. Since f and f^{-1} exist and are continuous, f is a homeomorphism as desired.

2-2

a)

We show that $\mathcal{T}$ is a topology by verifying the axioms of a topology.

1. Since $\mathcal{T}$ is the collection of all unions of finite intersections of elements of $\mathcal{B}$, it contains the union of all elements of $\mathcal{B}$ which is just X. The union of empty collection generates the emptyset so $\varnothing \in \mathcal{T}$ as well.
2. Let $\mathcal{U} \subset \mathcal{T}$ be any subset. The elements of $\mathcal{U}$ are unions of finite intersections of elements of $\mathcal{B}$. Thus, $\bigcup_{U \in \mathcal{U}} U$ is again a union of finite intersections of elements of $\mathcal{B}$. In other words, $\mathcal{T}$ is closed under union.
3. $\mathcal{T}$ is stable under finite intersections due to distributive property of sets.

b)