

Algebraic Geometry

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Housekeeping

Notes taken from

- my course

to-do

- 1.

Chapter 1

Basics?

Let K be an arbitrary field.

Definition 1 (Algebraic Subset). For a subset $M \subset K[X_1, \dots, X_n]$, we define

$$V(M) = \{ p \in K^n \mid \text{for all polynomials } f \in M \text{ it is } f(p) = 0 \}$$

called the algebraic subset in K^n or an affine algebraic set over K .

Intuition. Imagine the ring of polynomials $K[X_1, \dots, X_n]$ to be a set of locks and the kartsian product of the field K^n to be a set of keys. Then, a subset M of $K[X_1, \dots, X_n]$ is some combination of locks and the algebraic subset $V(M)$ are the set of keys that open all the locks in M .

Example 1.1. 1. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$.

(a) If we set $M = \{ X^2 - 1 \} \subset \mathbb{C}[X]$, then $V(\{ X^2 - 1 \}) = \{ -1, 1 \} \in \mathbb{C}$ is the algebraic subset.

(b) Now, if we append the set above to $M = \{ X^2 - 1, X - 1 \} \subset \mathbb{C}[X]$, we have

$$V(\{ X^2 - 1, X - 1 \}) = \{ 1 \} \in \mathbb{C}$$

instead. This example illustrates that appending the set of polynomials makes its algebraic subset smaller.

(c) In general, the finite subsets of \mathbb{K} are precisely the algebraic subsets of \mathbb{K} .

2. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X, Y]$.

(a) If $M = \{ X - 1, Y - 1 \} \subset \mathbb{C}[X, Y]$, then $V(\{ X - 1, Y - 1 \}) = \{ (1, 1) \} \subset \mathbb{C}^2$.

(b) If we remove the second polynomial, we get $M = \{ X - 1 \} \subset \mathbb{C}[X, Y]$ and

$$V(\{ X - 1 \}) = \{ (1, y) \mid y \in \mathbb{C} \}.$$

It is the first example of an infinite algebraic subset.

Lemma 2. If $M, N \subset K[X_1, \dots, X_n]$ with $N \subset M$, then it is $V(N) \supset V(M)$.

Intuition. N has less locks than M . If N has less locks, then more keys are able to open all locks in N .

Proof. Let $p \in V(M)$. For all $f \in M$, we have $f(p) = 0$. Now, $N \subset M$, thus for all $g \in N$ it is $g \in M$ and therefore $g(p) = 0$. That means $V(M) \subset V(N)$. \square

- Proposition 3.**
1. The empty set \emptyset is an algebraic subset of \mathbb{K}^n .
 2. The whole set \mathbb{K}^n is an algebraic subset of \mathbb{K}^n .
 3. An arbitrary intersection of algebraic subsets of \mathbb{K}^n is an algebraic subset of \mathbb{K}^n .
 4. A finite union of algebraic subsets of \mathbb{K}^n is an algebraic subset of \mathbb{K}^n .

Example 3.1. The union of non-finite union of algebraic subsets is not algebraic. Take $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$. As we have seen before, the singleton sets $\{x\}$ are algebraic for each $x \in \mathbb{N}$, but their union

$$\bigcup_{x \in \mathbb{N}} \{x\} = \mathbb{N}$$

is not.

The above proposition justifies the following definition.

Definition 4. The unique topology on K^n whose closed subsets are the algebraic subsets are called the Zariski topology on K^n .

##missing I think we didn't prove the uniqueness yet.

Lemma 5. Let $M \subset K[X_1, \dots, X_n]$ be a subset. Define I to be the ideal generated^a by M . Then we have $V(M) = V(I)$.

^aBy Hilbert's Basis Theorem the polynomial ring is Noetherian and thus this ideal is finitely generated.

Thus, $V(M)$ may be expressed by finite amount of generators $V(M) = V(f_1, \dots, f_n)$.

Example 5.1. 1. Let $\mathbb{K} = \mathbb{C}$ and $A = \mathbb{C}[X]$.

- (a) With the same example as above, set $M = \{X^2 - 1\} \subset \mathbb{C}[X]$, then $V(\{X^2 - 1\}) = \{-1, 1\} \in \mathbb{C}$ is the algebraic subset. Consider the ideal $(X^2 - 1)$. Some elements in this ideal are

$$3(X^2 - 1), \quad X(X^2 - 1), \quad (X + 1)(X^2 - 1)$$

which all still have the roots $\{-1, 1\}$.

Definition 6 (Radical of an Ideal).

Proposition 7. Let I be an ideal in A , then:

1. $\sqrt{I} = \sqrt{\sqrt{I}}$
2. $I = \sqrt{I} \iff A/I$ is a reduced ring, i.e. there are no nilpotent elements in A/I .
3. $A \xrightarrow{\pi} A_{\text{red}} := A/\sqrt{I}$ has the following universal property:

Proof.

\square

Lemma 8. $V(I) = V(\sqrt{I})$

Corollary 1. The map

$$\begin{aligned} \{\text{radical ideals in } K[X_1, \dots, X_n]\} &\rightarrow \{\text{algebraic sets of } \mathbb{A}^n(K)\} \\ \sqrt{I} &\mapsto V(\sqrt{I}) \end{aligned}$$

is surjective.

Example 8.1. The map above is not injective in general. ##missing

Theorem 9 (Hilbert Nullstellensatz). If K is algebraically closed, then the above map is a bijection with inverse $V \mapsto \text{Ann}(V)$.

From now on, assume $K = \overline{K}$.

Corollary 2. For any proper ideal $I \subset K[X_1, \dots, X_n]$ there is a $p \in K^n$ with $f(p) = 0$ for all $p \in I$.

Definition 10 (A -algebra). Let A be a ring. An A -algebra is a ring B together with a ring homomorphism $f : A \rightarrow B$, making B into an A -module such that scalar multiplication and the product on B are compatible.

If $A = K$ is a field, f is injective, so a K -algebra is a ring containing K as a subring, for example $K[X_1, \dots, X_n]$.

##missing something about two A -algebras

Definition 11 (Coordinate Ring). Let $V \subset K^n$ be an algebraic subset. Define the coordinate ring to be $\mathcal{O}(V) := K[X_1, \dots, X_n] / \text{Ann}(V)$ which is a reduced, finitely generated K -algebra (underlying ring is reduced).

Definition 12. Suppose we have two algebraic sets $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. A morphism between those two is a map $f : X \rightarrow Y$, $p \mapsto (f_1(p), \dots, f_m(p))$ where $f_i \in \mathcal{O}(X)$ for all $1 \leq i \leq m$.

##missing blue text