Exercise 0.1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Proof. 1. Let $S^{-1}M = 0$, then for all $s \in S$ and for all $m \in M$ we have that

$$\frac{m}{s} = 0, \tag{1}$$

or in other words, $(m, s) \equiv (0, s')$ for some $s' \in S$. By definition, there exists a $t \in S$ such that

$$\begin{aligned} t(s \cdot 0 - s'm) = 0 &\iff t(0 - s'm) = 0 \quad (2)\\ &\iff ts'm = 0. \quad (3)\end{aligned}$$

Choose ts' to be the factor and we get $sM = 0$.

2. If there is an $s \in S$ such that $sM = 0$, then we can write for all $m \in M$ that $s \cdot m = 0$. We have

$$0 = s \cdot m = s(1 \cdot m - 1 \cdot 0) \tag{4}$$

which means again $(m, 1) \equiv (0, 1)$, hence $S^{-1}M = 0$.

□

Exercise 0.2. 3.2
Hints

1. To show that $x \in S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$, use Proposition 1.9., i.e.

$$x \in \mathrm{Jac}(A) \iff 1 - xy \in A^\times \text{ for all } y \in A. \tag{5}$$

2. If $s \in S$, then $f(s)$ is a unit in $S^{-1}A$.

Proof. We want to show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. Let $x \in S^{-1}\mathfrak{a}$. From Proposition 1.9. we have that $x \in \mathrm{Jac}(S^{-1}A)$ if and only if $1 - xy \in (S^{-1}A)^\times$ for all $y \in S^{-1}A$. Because $S = 1 + \mathfrak{a}$, we can write

$$x = \frac{a_1}{1 + a_2} \tag{6}$$

for some $a_1, a_2 \in \mathfrak{a}$. With this, it is

$$1 - xy = 1 - \frac{a_1}{1 + a_2} y = \frac{1 + a_2}{1 + a_2} - \frac{a_1 y}{1 + a_2} = \frac{1 + a_2 - a_1 y}{1 + a_2}. \tag{7}$$

Now, $a_2 - a_1 y$ is contained in $\mathfrak{a}$, hence the whole numerator $1 + a_2 - a_1 y$ is contained in S, and in turn, this means the whole expression is a unit in $S^{-1}A$.

We want to give an alternative proof to Corollary 2.5. If $\mathfrak{a}M = M$, then $(S^{-1}\mathfrak{a})(S^{-1}M) = (S^{-1}M)$ because modules of fractions are determined uniquely up to isomorphisms. From above, we have that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. By Nakayama's lemma, we get $S^{-1}M = 0$. Now applying Exercise 3.1 yields that there is a $s \in S$ such that $sM = 0$. Since $S = 1 + \mathfrak{a}$, it is clearly $s \equiv 1 \mod \mathfrak{a}$. □

Exercise 0.3 (3.5.i))**.**

Proof. Let S be saturated. We show $A - S$ is a union of prime ideals.

1. Let $x \in A - S$ and Σ be a collection of ideals that contain a, but disjoint with S.

2. We show that Σ is nonempty has a maximal elements.

   (a) Consider the ideal (a). Its elements are in the form xa with $x \in A$. If $xa \in S$, then because S is saturated, we would have $a \in S$ which is impossible. So $(a) \cap S = \emptyset$ and we have $(a) \in \Sigma$.

   (b) For each ideal chain ordered by inclusion, we can construct an upper bound via the union of the ideals in the chain. Zorn's lemma now gurantees the existence of maximal elements.

3. Let $\mathfrak{p} \in \Sigma$ be a maximal element. We show $\mathfrak{p}$ is prime by contrapositive.

   (a) Let $x, y \notin \mathfrak{p}$.

   (b) $(x) + \mathfrak{p}$ and $(y) + \mathfrak{p}$ contain $\mathfrak{p}$ strictly.

   (c) Since $\mathfrak{p}$ was maximal in Σ, these two ideals $(x) + \mathfrak{p}$ and $(y) + \mathfrak{p}$ are not in Σ.

   (d) But a is contained in $(x) + \mathfrak{p}$ and $(y) + \mathfrak{p}$, therefore these two ideals are not disjoint from S.

   (e) Let $s \in ((x) + \mathfrak{p}) \cap S$ and $t \in ((y) + \mathfrak{p}) \cap S$.

Let $A - S$ be a union of prime ideals. We show S is saturated.

1. Let $xy \in S$. We show that $x \in S$ and $y \in S$ by contradiction.
   (a) Let $xy \in S$ and assume $x \notin S$.
   (b) The assumptions implies $x \in A - S$.
   (c) Since $A - S$ is a union of prime ideals, there is a prime ideal $\mathfrak{p}$ that contains x.
   (d) If an ideal contains x, it must also contain xy.
   (e) So $xy \in A - S$, or in other words, $xy \notin S$ which is a contradiction.
2. Let $x \in S$ and $y \in S$, clearly $xy \in S$ because S is multiplicatively closed.

$\square$