Exercise Sheet 1

Exercise 1

Use the Norm to find factorizations of $3 + 7i$ and $23 + 14i$ into irreducible elements in $\mathbb{Z}[i]$.

Solution

$\underline{3 + 7i}$

We want to find $a, b \in \mathbb{Z}[i]$ such that $ab = 3 + 7i$. We have $N(3 + 7i) = 9 + 49 = 58$ and the prime factorization is $58 = 2 \cdot 29$. Because of the multiplicative property of the Norm, we have $N(3 + 7i) = N(a)N(b) = 2 \cdot 29$ or in other words, $N(a) = 2$ and $N(b) = 29$.

Set $a = 1 + i$ which is irreducible as $N(a) = 2$. Then, we have $b = (3 + 7i) \cdot (1 + i)^{-1} = 5 + 2i$ which again is irreducible as $N(b) \equiv 1 \mod 4$.

Thus the solution is

$$3 + 7i = (1 + i)(5 + 2i). \tag{1}$$

$\underline{23 + 14i}$

Similarly, we have $N(23 + 14i) = 725 = 5^2 \cdot 29$. Try out some $a \in \mathbb{Z}[i]$ with $N(a) = 29$ until $(23 + 14i)a^{-1}$ is in $\mathbb{Z}[i]$.

$$(23 + 14i)(5 + 2i)^{-1} \notin \mathbb{Z}[i] \tag{2}$$

$$(23 + 14i)(-5 + 2i)^{-1} = -3 - 4i \in \mathbb{Z}[i] \tag{3}$$

It is $(23+14i)(-5+2i)^{-1} = -3-4i$. Now try out some $b \in \mathbb{Z}[i]$ with $N(b) = 5$ until $(-3-4i)b^{-1}$ is in $\mathbb{Z}[i]$.

$$(-3 - 4i)(1 + 2i)^{-1} \notin \mathbb{Z}[i] \tag{4}$$

$$(-3 - 4i)(-1 + 2i)^{-1} = -1 + 2i \in \mathbb{Z}[i] \tag{5}$$

$$\tag{6}$$

As before, $(-5 + 2i)$ and $-1 + 2i)$ are irreducible because the Norm is equivalent to 1 in mod 4. All together, we have

$$23 + 14i = (-1 + 2i)^2(-5 + 2i). \tag{7}$$