Chapter 1

Commutative Rings

Definitions

1. prime, coprime, relatively prime, irreducible

Exercise 1.1. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals in A, and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of B. Prove the following statements.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

   Proof. We show $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$. Let $x \in (\mathfrak{a}_1 + \mathfrak{b}_2)^e$, then we have for some index set I

   $$x = \sum_{i \in I} \lambda_i x_i, \tag{1.1}$$

   where $\lambda_i \in B$ and $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$ for all $i \in I$. For each $i \in I$ it is $x_i = \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2})$, hence

   $$\begin{aligned} x =& \sum_{i \in I} \lambda_i \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2}) && (1.2) \\ =& \sum_{i \in I} \lambda_i \left(\varphi(\mu_{i,1} a_{i,1}) + \varphi(\mu_{i,2} a_{i,2})\right) && \text{(by linearity)} \quad (1.3) \\ =& \sum_{i \in I} \lambda_i \left(\mu_{i,1} \varphi(a_{i,1}) + \mu_{i,2} \varphi(a_{i,2})\right) && \text{(by linearity)} \quad (1.4) \\ =& \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \lambda_i \mu_{i,2} \varphi(a_{i,2}) && \text{(by distributivity)} \quad (1.5) \\ =& \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) && \text{(reordering the sum).} \quad (1.6) \\ & && (1.7) \end{aligned}$$

   The last term is exactly the elements expressed by $\mathfrak{a}_1^e + \mathfrak{a}_2^e$, therefore, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.
   I think the above proof should work into both directions. □

2. $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$

   Proof. We have

   $$(\mathfrak{b}_1 + \mathfrak{b}_2)^c = \left\{ x \in A \;\middle|\; \exists b_1 \in \mathfrak{b}_1 \exists b_2 \in \mathfrak{b}_2 : \varphi(x) = b_1 + b_2 \right\}. \tag{1.8}$$

   Now let $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$, then $x = a_1 + a_2$ where $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$. It is

   $$\begin{aligned} \varphi(x) =& \varphi(a_1 + a_2) && (1.9) \\ =& \varphi(a_1) + \varphi(a_2) && \text{(by additivity)} \quad (1.10) \end{aligned}$$

   Since $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$ we have that $x \in (\mathfrak{b}_1 + \mathfrak{b}_2)^c$. □

Exercise 1.2. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}$ an ideal of A, and $\mathfrak{b}$ an ideal of B. Prove the following statements:

1. Then $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$.

Proof. It is

$$\mathfrak{a}^{ec} = \left\{ x \in A \;\middle|\; \varphi(x) \in \mathfrak{a}^e \right\} \tag{1.11}$$
$$= \left\{ x \in A \;\middle|\; \varphi(x) \in \langle \varphi(\mathfrak{a}) \rangle \right\} \tag{1.12}$$
$$= \left\{ x \in A \;\middle|\; \forall i \in I \, \exists a_i \in \mathfrak{a}_1 : \varphi(x) = \sum_{i \in I} \lambda_i \varphi(a_i) \right\}. \tag{1.13}$$

Let $a \in \mathfrak{a}$ and choose $I = \{ 1 \}$, λ_1, and $a_i = a$, then $a \in \mathfrak{a}^{ec}$. □

2. $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$.
3. $\mathfrak{a}^{ece} = \mathfrak{a}^e$.
4. $\mathfrak{b}^{cec} = \mathfrak{b}^c$.
5. If $\mathfrak{b}$ is an extension, then $\mathfrak{b}^c$ is the largest ideal of A with extension $\mathfrak{b}$.
6. If two extensions have the same contraction, then they are equal.

   Proof. a □

Exercise 1.3. Let A be a ring, $A[\mathcal{X}, \mathcal{Y}]$ the polynomial ring in two sets of variables $\mathcal{X}$ and $\mathcal{Y}$. Show that $\langle \mathcal{X} \rangle$ is prime if and only if A is a domain.

Proof. It should be noted here, that $A[\mathcal{X}]$ does not contain X_1X_2 for example. It does contain $X_1 + X_2$ however. The rest is easy. □

Exercise 1.4. Show that, in a PID, nonzero elements x and y are relatively prime (share no prime factor) if and only if they're coprime.

Exercise 1.5. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals, and $\mathfrak{p}$ a prime ideal. Prove that these conditions are equivalent:

1. $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$
2. $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$
3. $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$

Proof. (1) to (2) is easy. Same for (2) to (3). For (3) to (1) show it with contradiction. □

Exercise 1.6. Let A be a ring, $\mathfrak{p}$ a prime ideal, and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ maximal ideals with $\mathfrak{m}_1, \ldots, \mathfrak{m}_n = 0$. Show $\mathfrak{p} = \mathfrak{m}_i$ for some i.

Proof. By induction. Proof first for $m_1 m_2$, the rest is clear. □

Exercise 1.7. Let A be a ring, $\mathfrak{p}$ a prime, and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ ideals.

1. If $\bigcap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$, then $\mathfrak{a}_j \subseteq \mathfrak{p}$ for some j.

   Proof. If $\mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq \mathfrak{p}$, then by the exercise above we have the desired result. The rest is induction. □

2. If $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{p}$, then $\mathfrak{a}_j \subseteq \mathfrak{p}$ for some j.

   Proof. Clear. □