Let (X, d) be a metric space. Prove that the set of subsets

$$\mathcal{O}(d) := \{\, U \subset X \mid \forall x\, \exists \epsilon > 0 \text{ with } B_d(x, \epsilon) \subset U \,\} \tag{1}$$

defines a topology.

Proof. We verify that $\mathcal{O}(d)$ fullfills the axioms of a topology.

1. $X \in \mathcal{O}(d)$ since any ball of a point x is contained in X. $\varnothing \in \mathcal{O}(d)$ is true vacuously.
2. Let I be an arbitary index set and $\{A_i\}_{i\in I}$ be a family of subsets belong to $\mathcal{O}(d)$. Consider the union $\bigcup_{i\in I} A_i$. If a point x is in $\bigcup_{i\in I} A_i$, then there is an A_i where this point x is contained. Since A_i is in $\mathcal{O}(d)$, there exists an ϵ such that $B_d(x, \epsilon) \subset A_i \subset \bigcup_{i\in I} A_i$. Therefore, we have that $\bigcup_{i\in I} A_i$ belongs to $\mathcal{O}(d)$.
3. Let I be a finite index set and A_i with $i \in I$ be subsets in $\mathcal{O}(d)$. Consider the intersection $\bigcap_{i\in I} A_i$. If a point x is in $\bigcap_{i\in I} A_i$, then x is included in each A_i. Again, A_i is in $\mathcal{O}(d)$, so there is an ϵ_i such that $B_d(x, \epsilon_i) \subset A_i$. Choose the smallest (accordig to the metric d) among all $\epsilon_i \in I$ and denote it as ϵ. We have $B_d(x, \epsilon) \subset B_d(x, \epsilon_i) \subset A_i$ for all $i \in I$. This means $B_d(x, \epsilon) \subset \bigcap_{i\in I} A_i$ as desired.

Show that any ball $B_d(x, r) \in \mathcal{O}(d)$ for all $x \in X$ and for all $r > 0$.

Proof. Fix an $p \in B_d(x, r)$. Set $\epsilon := (r - d(x, p))/2$ (dividing it by two might only be for good measure). Then $B_d(p, \epsilon) \subset B_d(x, r)$, so $B_d(x, r) \in \mathcal{O}(d)$. □

Let d_1 and d_2 be equivalent metrics on X. Show that $\mathcal{O}(d_1) = \mathcal{O}(d_2)$.

Proof. We will show $\mathcal{O}(d_1) \subseteq \mathcal{O}(d_2)$. Symmetry will take care of the other side. Let $A \in \mathcal{O}(d_1)$ and fix a point $x \in A$. □