Topology

K

May 12, 2022

Contents

Chapter 1

Topological Space

1.1 Definitions and Theorems

Definition 1 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to $\mathcal{O}$.
2. Any **arbitary** union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The intersection of **finite number** of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The collection $\mathcal{O}$ is called a topology on X and the elements of $\mathcal{O}$ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.
We often just write X instead of $(X, \mathcal{O})$ if the given topology is clear.

Definition 2 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \tag{1.1}$$

Lemma 3. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 4 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. If $Y = X$ we also write $\mathrm{Homeo}(X)$.

Definition 5 (Homeomorphism)**.** Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.

2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 6. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets, then there exists **exactly one** topology $\tau \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \tau$

2. If $\tau' \subset \mathcal{P}(X)$ a topology with $S \subset \tau'$, then $\tau \subset \tau'$.

Definition 7. 1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{\, S \cap U \mid U \in \tau \,\} .$$

2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{\, U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \,\} .$$

3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{\, U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \,\} .$$

Definition 8. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.

2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\mathrm{int}(S)$, is the set of all interior points of S.

3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\mathrm{cl}(S)$, is defined by

$$\mathrm{cl}(S) := X \setminus \mathrm{int}(X \setminus S).$$

1.2 Proofs, Remarks, and Examples

Definition 9 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to $\mathcal{O}$.
2. Any **arbitary** union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The intersection of **finite number** of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The collection $\mathcal{O}$ is called a topology on X and the elements of $\mathcal{O}$ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.
We often just write X instead of $(X, \mathcal{O})$ if the given topology is clear.

Example 9.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the discrete topology. In this case, (X, τ) is called the discrete space. It is the finest topology that can be defined on a set. (The set of all possible topologies on a given set forms a partially ordered set.)
2. $\tau = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology.
3. Let (X, d) be a metric space. Set
$$\tau_d := \{\, U \in X \mid U \text{ is a open subset in the metric space } (X, d) \,\} . \tag{1.2}$$
Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U.
Here, τ is a topology. In other words, a metric induces a topology.
(Proof as homework.)
4. The Zariski-topology.

Example 9.2. List of natural topologies.

1. On $\mathbb{R}^n$ the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of $\mathbb{R}^n$ are arbitary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then
$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{\, x \in \mathbb{R}^n \mid d(p, x) < r \,\} .$$
This definition agrees with the topology endowed on arbitary metric spaces.
2. The matrix space $\mathrm{Mat}_{n \times m}(\mathbb{K})$ for a field $\mathbb{K}$ does not have one canonical topology. Depending on the context and literature different ones are used.
   - Since $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is isomorphic to $\mathbb{R}^{n \cdot m}$, one could use the Euclidean topology as defined above.
   - $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is a metric space via multitude of operator norms. The metric space induces the topology.
   - Another metric on $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is the rank distance for $A, B \in \mathrm{Mat}_{n \times m}$ defined as $d(A, B) := \mathrm{rank}(B - A)$ which again would induce a topology.

Definition 10 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.
$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X . \tag{1.3}$$

Lemma 11. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 12 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. If $Y = X$ we also write $\mathrm{Homeo}(X)$.

Proposition 13. The set of all homeomorphisms of X to itself $\mathrm{Homeo}(X)$ is a group with composition as its operation.

Remark. This lemma does not hold for basis.

Remark. 1. $\tau_{X\times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X\sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall(x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. [1] Let $(X, \mathcal{O})$ be a topological space. A subset that is **both** open and closed is called clopen. Moreover, a subset is clopen if and only if its boundary is empty.

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\mathrm{cl}(A) = A$, but on the other hand, A is open, so we also have $\mathrm{int}(A) = A$. Then, the boundary of A is $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A) = A \setminus A = \varnothing$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. □

[1] The following is a definition and a small proposition.

1.3 Exercises and Notes

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 14. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.

2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.

3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.

4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Lemma 15. Any interval $I \subset \mathbb{R}$ is connected.

Lemma 16. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Definition 17. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 18. Connected components are closed subsets.

Lemma 19 (Lemma 11)**.** Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 20. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 21. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 22. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.
3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Proof. We verify the equivalence of the different definitions. So, let $(X, \mathcal{O})$ be a topological space.

- "*1.* $\Rightarrow$ *2.*": Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X, then $X \setminus A$ is also not the empty set nor X. Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \varnothing$ or $A = X$.
- "*2.* $\Rightarrow$ 1.": Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.
- "*2.* $\iff$ *3.*": This is one of the properties of clopen subsets and was proven in remark XXX.
- "*1.* $\Rightarrow$ *4.*": Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \longrightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.
- "*4.* $\Rightarrow$ *1.*": Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \longrightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

□

Lemma 23. Any interval $I \subset \mathbb{R}$ is connected.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \tag{2.1}$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \mathrm{cl}(B)$ and since the complement of B is an open subset A, so $B = \mathrm{cl}(B)$. It is therefore $x \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. □

Lemma 24. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}B = f^{-1}(A \cap B) = \varnothing$ so $f^{-1}(A) = \varnothing$ or $f^{-1}(B) = \varnothing$, so $A = \varnothing$ or $B = \varnothing$ and we are done. □

Remark. The two lemma above are handy to show that images of functions are connected.

Example 24.1. The general linear group $\mathrm{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Proof. □

Definition 25. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Example 25.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proposition 26. Connected components are closed subsets.

Proof. Locally constant implies continuous with regards to the discrete topology on Y. Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \varnothing$. Conclude f is identical to $f(x)$. □

Application: $f : X \longrightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \longrightarrow X$ be continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We have that γ^{-1} □

2.3 Exercises and Notes

Remark. If $(X, \mathcal{O}_{\mathcal{X}})$ and $(Y, \mathcal{O}_Y)$ are two connected topological spaces, then their product $X \times Y$ with the product topology $\mathcal{O}_{X\times Y}$is also connected.

Proof. We will use the definition that all continuous maps from $X \times Y$ to $\{0,1\}$ endowed with the discrete topology must be constant. Fix a continuous $f : X \longrightarrow \{0,1\}$.

First, consider the image $f(\{x\} \times Y)$ with $x \in X$. Assume f is not constant on $\{x\} \times Y$, then $f(\{x\} \times Y) = \{0,1\}$. So we have the preimages $f^{-1}(\{0\}) = \{x\} \times U$ and $f^{-1}(\{1\}) = \{x\} \times V$ with $U, V \subset Y$, $U, V \neq \varnothing$, and $U \cap V = \varnothing$. Because f is continuous, U and V must also be open. This would however mean that $U \sqcup V = Y$ and Y would not be connected, therefore, f is constant on $\{x\} \times Y$. Similary, we get that f is constant on $X \times \{y\}$ for all $y \in Y$.

Let $(x, y) \in X \times Y$ and $(x', y') \in X \times Y$ be two arbitary points. We have $f(x, y) = f(x, y')$ because f is constant on $\{x\} \times Y$ and similary $f(x, y') = f(x', y')$ because f is constant on $X \times \{y\}$. Putting everything together, it is $f(x, y) = f(x', y')$, therefore all continuous $f : X \times Y \longrightarrow \{0,1\}$ are constant. □

Example 26.1. Clearly, the union of two connected sets need not be connected. Take for example $[0,1] \subset \mathbb{R}$ and $[2,3] \subset \mathbb{R}$. Their union $[0,1] \cup [2,3]$ is not connected.

Set difference of connected sets are also not necessarily connected, e.g. $[0,2] \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ are connected, but $[0,2] \setminus \{1\} = [0,1) \cup (1,2]$ is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \left\{ (x,y) \mid x^2 + y^2 = 1 \right\}$ and another unit circle around $(1,0)$ $A := \left\{ (x,y) \mid (x-1)^2 + y^2 = 1 \right\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \right\}$$

which is not connected.

Example 26.2. Connectedness does not imply path-connectedness. Let $\mathbb{R}^2$ be endowed with the Euclidean topology and consider

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid x > 0 \right\} \cup (\, \{0\} \times [-1,1]\,) \subset \mathbb{R}^2.$$

and see figure XXX. X is connected, but it is not path-connected.

Proof. Denote

$$A := \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid x > 0 \right\} \qquad B := \{0\} \cup [-1,1],$$

then $X = A \sqcup B$.

1. First, define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^2$ as

   $$f(x) := \left(x,\, \sin\left(\frac{1}{x}\right)\right).$$

   f is continuous, $\mathbb{R}^+$ is an interval, therefore connected, so $f(\mathbb{R}^+) = A$ is connected. On the other hand, $\{0\}$ and $[-1,1]$ are connected and so is their product B.

   Assume there is a clopen subset $S \subset X$ that is not empty. Without loss of generality, we have that $(0,0) \in U$ (otherwise, consider the complement of U which also must be clopen). Since A is clopen in A, the intersection $A \cap U$ must also be clopen in A, but A is connected, so A is contained in U.

   Moreover, the closure of A is also contained in U. So there is an $\epsilon > 0$ such that the ball $B(p, \epsilon)$ that contains $(0,0)$ is in U. I got lazy to go into the details, but this ball contains a point of B. Follow the same reason as above.

2.

□

Chapter 3

Separation Axioms

Literature: Groessere Liste in Sten, Seibeck

Definition 27 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 28. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 29 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 30. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{ (x,x) \mid x \in X \}$ is closed as a subset of the product space $X \times X$.

Proposition 31. T_2 spaces are also T_1 spaces.

Chapter 4

Compact Spaces

Definition 32. 1. A topological space X is called compact if each of its open cover has a **finite** subcover.

2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Theorem 33. Satz 17

Theorem 34. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 35. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 36. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \longrightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Lemma 37. $[0, 1] \subset \mathbb{R}$ is compact.