

# Topology

K

September 21, 2022



# Contents

1	Introduction	5
2	Topological Spaces	7



## Chapter 1

# Introduction



## Chapter 2

# Topological Spaces

### 2-1

“ $\Rightarrow$ ”: Let  $f : X_1 \longrightarrow X_2$  be a homeomorphism and fix a subset (not necessarily open)  $U \in \mathcal{T}_1$ .

1. Assume  $U$  is open in  $X_1$ . Because  $f$  is continuous, the image of open subsets are again open, thus  $f(U)$  lies in  $\mathcal{T}_2$ .
2. On the other hand, if  $f(U)$  is open in  $X_2$ , then since  $f$  is bijective we have

$$f^{-1}(f(U)) = U.$$

Because  $f$  is continuous, the preimage of open subsets under  $f$  is open. We may therefore conclude  $U$  is open in  $X_1$ .

We have shown that if  $f$  is a homeomorphism, then  $f(\mathcal{T}_1) = \mathcal{T}_2$ .

“ $\Leftarrow$ ”: Let  $f : X_1 \longrightarrow X_2$  be a bijective map such that  $f(\mathcal{T}_1) = \mathcal{T}_2$ . Consider the inverse map  $f^{-1}$ . We want to show  $f^{-1}$  is continuous. Fix an open subset  $U \in \mathcal{T}_1$ . It is

$$(f^{-1})^{-1}(U) = f(U)$$

because  $f$  is bijective. Since  $f(\mathcal{T}_1) = \mathcal{T}_2$  and  $U$  is open,  $f(U)$  is open as well. Hence the preimage of  $U$  under  $f^{-1}$  is open and  $f^{-1}$  is continuous.

Now we show that  $f$  is also continuous. Again, fix an open subset  $V \in \mathcal{T}_2$ . The preimage of  $V$  under  $f$  is just the image of the inverse function. We have already shown that the inverse is continuous. Thus,  $f^{-1}(V)$  is open and  $f$  is continuous. Since  $f$  and  $f^{-1}$  exist and are continuous,  $f$  is a homeomorphism as desired.

### 2-2

#### a)

We show that  $\mathcal{T}$  is a topology.

1. Since  $\mathcal{T}$  is the collection of all unions of finite intersections of elements of  $\mathcal{B}$ , it contains the union of all elements of  $\mathcal{B}$  which is just  $X$ . The union of empty collection generates the emptyset so  $\emptyset \in \mathcal{T}$  as well.
2. Let  $I$  be an arbitrary index set and  $U_{i \in I}$