Integration and Integration

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Introduction

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2.1 Content, Premeasure, and Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ-additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{2.1}$$

- subadditive if for all $A, B \in \mathcal{R}$ it is $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- σ-subadditive if for all $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k). \tag{2.2}$$

- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ-finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in $\mathcal{R}$ with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \tag{2.3}$$

- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ-additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in $\mathcal{R}$ is required. For σ-rings and therefore σ-algebras, it is guranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \to [0, \infty]$ are σ-finite as X need not be included in a ring of sets.

Definition 2.2 (Content)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called a content if

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ-additive content $\mu \to [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. A σ-additive content $\mu : \mathcal{A} \to [0, \infty]$ is called a measure.

2.2 Lebesgue Content

Definition 2.5 (Lebesgue Content)**.** Let $\mathcal{Q}(\mathbb{R}^n)$ be the ring of sets over $\mathbb{R}^n$.

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigsqcup_{k=1}^{m} [a_{1,k},\, b_{1,k}) \times \cdots \times [a_{n,k},\, b_{n,k}) \,\middle|\, m \in \mathbb{N};\, a_{i,k}, b_{i,k} \in \mathbb{R};\, 1 \leq k \leq n \right\} \tag{2.4}$$

Set $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ as

$$\lambda^n(A) := \sum_{k=1}^{m} \prod_{i=1}^{n} (b_{i,k} - a_{i,k}) \tag{2.5}$$

λ^n is the Lebesgue content.

Theorem 2.5.1. λ^n is a well-defined finite content.

Theorem 2.5.2. λ^n is a premeasure.

2.3 Lebesgue Measure

CHEET SHEET

1. Content $\mu : \mathcal{R} \to [0, \infty]$ is empty set 0 and finitely additive.
2. Premeasure $\mu : \mathcal{R} \to [0, \infty]$ is σ-additive content.
3. First extension $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$
4. Outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$

$$\mathcal{A} \subset \mathcal{A}^{\uparrow} \subset \sigma(\mathcal{A}) \subset \hat{\mathcal{A}} \tag{2.6}$$

Definition 2.6. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings. Set

$$\mathcal{R}^{\uparrow} := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k\in\mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \tag{2.7}$$

Remark. $\mathcal{R}^{\uparrow}$ is the set of all $A \in \mathcal{P}(X)$ that can be expressed as a countable many unions of sets in $\mathcal{R}$.

In general, $\mathcal{R}^{\uparrow}$ is not a set of rings.

Definition 2.7. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets and $\mu : \mathcal{R} \to [0, \infty]$ a premeasure. For $A_k \uparrow A$ with $A_k \in \mathcal{R}$ for $k \in \mathbb{N}$ define

$$\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty],\, A \mapsto \tilde{\mu}(A) := \lim_{k\to\infty} \mu(A_k). \tag{2.8}$$

$\tilde{\mu}$ is called the first extension of the premeasure μ.

Remark. In general, $\tilde{\mu}$ is not a premeasure as $\mathcal{R}^{\uparrow}$ need not be a ring of sets.

$\tilde{\mu}$ restricted on $\mathcal{R}$ is identical with μ, i.e. $\tilde{\mu}\big|_{\mathcal{R}} \equiv \mu$.

Lemma 2.7.1. The first extension $\tilde{\mu}$ is well-defined.

Proposition 2.7.1 (Properties of $\mathcal{R}^{\uparrow}$)**.**

Proposition 2.7.2 (Properties of the First Extension)**.**

Definition 2.8 (Second Extension or the Outer Measure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets, $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$, and $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$ the first extension of μ on $\mathcal{R}^{\uparrow}$. Moreover, let $B \subset X$ be a subset of X. Then, the map

$$\mu^* : \mathcal{P}(X) \to [0, \infty],\, B \mapsto \mu^* := \inf \left\{\tilde{\mu}(A) \mid A \in \mathcal{R}^{\uparrow},\, A \supset B\right\} \tag{2.9}$$

is called the outer measure induced by $\tilde{\mu}$ on $\mathcal{P}(X)$.

Proposition 2.8.1 (Properties of the Second Extension)**.**

Proposition 2.8.2 (Properties of the Outer Measure)**.**

Definition 2.9 (Lebesgue Outer Measure)**.** Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \to [0, \infty],\, B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^\uparrow,\, A \supset B \right\} \tag{2.10}$$

is called the Lebesgue outer measure induced by $\tilde{\lambda^n}$.

Definition 2.10 (Pseudo Metric)**.** Let X be a set. A map $d : X \times X \to \overline{\mathbb{R}},\, (x, y) \mapsto d(x, y)$ is called pseudo metric on X if for all $x, y, z \in X$ it is the following three axioms are met.

1. $x = y \Rightarrow d(x, y) = 0$.
2. $d(x, y) = d(y, x)$. (Symmetry.)
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 2.10.1. The outer measure induces a pseudo metric, i.e.

$$d_{\mu^*} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty],\, (A, B) \mapsto d_{\mu^*}(A, B) := d_{\mu^*}(A \triangle B) \tag{2.11}$$

is a pseudo metric.

Proposition 2.10.2. The outer measure is continuous.

Definition 2.11 (Approximation through elements of Rings)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings, $\mu : \mathcal{R} \to [0, \infty]$ a premeasure on $\mathcal{R}$, and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the outer measure induced by μ. Then, $A \in \mathcal{P}(X)$ is called $\mathcal{R}$-approximatable in respect to μ^* if for all $\epsilon > 0$ there exists an $B \in \mathcal{R}$ such that $d_{\mu^*}(A, B) = \mu^*(A \triangle B) < \epsilon$.

Theorem 2.11.1. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra and $\mu : \mathcal{A} \to \mathbb{R}_0^+$ a finite premeasure. Let the first extension $\tilde{\mu} : \mathcal{A}^\uparrow \to \mathbb{R}_0^+$ also be finite and $\mu^* : \mathcal{P}(X) \to \mathbb{R}_0^+$ the outer measure. Then,

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \tag{2.12}$$

is a σ-algebra on X.

Theorem 2.11.2. Let $\mu, \tilde{\mu}, \mu^*$ and $\mathcal{A}, \mathcal{A}^\uparrow, \hat{\mathcal{A}}$ be given. Then, a finite premeasure $\mu : \mathcal{A} \to \mathbb{R}_0^+$ can be uniquely extended to a finite measure $\hat{\mu} : \hat{\mathcal{A}} \to \mathbb{R}_0^+$ where $\hat{\mu} \equiv \mu^*\big|_{\hat{\mathcal{A}}}$.

Theorem 2.11.3. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings and $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$ and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the outer measure induced by μ. Then, μ can be uniquely extended to a measure $\hat{\mu} : \sigma(\mathcal{R}) \to [0, \infty]$ where $\hat{\mu} \equiv \mu^*\big|_{\sigma(\mathcal{R})}$.

Definition 2.12. Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ a σ-finite Lebesgue premeasure. In this chapter, we constructed a unique extension of λ^n on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathbb{R}^n)$, the Lebesgue-Borel measure $\hat{\lambda} : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$.

2.4 Measure Space

Definition 2.13. Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. The tupel $X, \mathcal{A}$ is called measurable space and the sets in the σ-algebra $A \in \mathcal{A}$ are called measurable sets.

Morover, let $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{P}(X)$. Then, $(X, \mathcal{A}, \mu)$ a measure space.

Definition 2.14 (Null Sets)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the induced outer measure. Then $N \subset X$ with $\mu^*(N) = 0$ is called null set.

For $X = \mathbb{R}^n$ with $\lambda^n(N) = 0$ called Lebesgue null set.

$S = \varnothing$ is called the trivial null set.

Definition 2.15 (Completion of a Measure Space)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. This measure space is called complete if all null sets are included in $\mathcal{A}$, i.e. for all $N \in \mathcal{A}$

$$\mu^* N = 0 \Rightarrow N \in \mathcal{A}. \tag{2.13}$$

Definition 2.16. Let

$$\overline{\mathcal{A}}^\mu := \{A \cup N \mid A \in \mathcal{A},\, N \subset X \text{ with } \mu^*(N) = 0\} \tag{2.14}$$

then $\overline{\mathcal{A}}^\mu$ is called the completion of $(X, \mathcal{A}, \mu)$.

Definition 2.17. The completion of the Lebesgue-Borel measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \hat{\lambda}^n)$ to $(\mathbb{R}^n, \mathcal{B}^{\hat{\lambda}^n}(\mathbb{R}^n), \hat{\lambda}^n)$ or shorter $(\mathbb{R}^n, \overline{\mathcal{B}}^\lambda(\mathbb{R}^n), \lambda^n)$ is called the (completed) Lebesgue measure space.

$B \in \overline{\mathbb{B}}^\lambda(\mathbb{R}^n)$ is called Lebesgue measurable to differentiate from $B \in \mathcal{B}(\mathbb{R}^n)$ Borel measurable.

Part II

Lebesgue Integral

2.5 Measurable Maps

2.6 Lebesgue Integral

Part III

Applications

Part IV

More Theory

Chapter 3

Lebesgue Space

3.1 Lebesgue Space

Definition 3.1 (L^p-Norm)**.** Let $X, \mathcal{A}, \mu$ a measure space, and $f : X \to \overline{\mathbb{R}}$ measurable. Then for $p \in [1, \infty)$ the L^p-norm is defined as

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}. \tag{3.1}$$

Theorem 3.1.1 (Holder Inequality)**.** Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Let $f, g : X \to \overline{\mathbb{R}}$ measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \tag{3.2}$$

Theorem 3.1.2 (Minkowski Inequality)**.** Let $f, g : X \to \overline{\mathbb{R}}$ measurable and $f + g$ well defined on X. Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \tag{3.3}$$

Definition 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty)$. Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \to \mathbb{R} \middle| f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\}. \tag{3.4}$$

Also define

$$L^p(\mu) := \mathcal{L}^p(\mu)/\sim\mu \tag{3.5}$$

Where the equivalent relation means two functions are equivalent iff they agree on every point outside of null sets.

3.2 Convergence Theorems

Theorem 3.2.1 (Lebesgue Monotone Convergence Theorem)**.** *Also called the theorem of Beppo Levi.* Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of measurable functions such that

$$f_n(x) \leq f_{n+1}(x) \tag{3.6}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Define $f : X \to [0, \infty]$ by

$$f(x) := \lim_{n \to \infty} f_n(x). \tag{3.7}$$

Then f is measurable and

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{3.8}$$

Theorem 3.2.2 (Lebesgue Dominated Convergence Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $g : X \to \mathbb{R}_0^+$ be an integrable function, and let $f_n : X \to \mathbb{R}$ be a sequence of integrable functions satisfying

$$|f_n(x)| \leq g(x) \tag{3.9}$$

for all $x \in X$ and $n \in \mathbb{N}$ and converging pointwise to $f : X \to \mathbb{R}$, i.e.

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \in X. \tag{3.10}$$

Then f is integrable and, for every $E \in \mathcal{A}$,

$$\int_E f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu. \tag{3.11}$$

3.3 Convergence