Definition 1. Let $\mathfrak{a}$ be an ideal in R. The variety $\boldsymbol{V}(\mathfrak{a})$ of $\mathfrak{a}$ is the subset of $\operatorname{Spec}(R)$ consisting of all prime ideals that contain $\mathfrak{a}$, i.e.

$$\boldsymbol{V}(\mathfrak{a}) := \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

Definition 2. Given an $f \in R$, we define the distinguished or basic set

$$D(f) := \operatorname{Spec}(R) \setminus V(f)$$

Theorem 3. For $f, g \in R$, we have:

1. $D(f) \cap D(g) = D(f \cdot g)$.
2. $D(f) = \emptyset \iff f$ is nilpotent.
3. $D(f) = \operatorname{Spec}(A) \iff f \in A^\times$

Proof.
1. Let $\mathfrak{p} \in D(f) \cap D(g)$.
2. We have

$$\begin{aligned}
\mathfrak{p} \in D(f) \cap D(g) &\iff \mathfrak{p} \in (\operatorname{Spec}(R) \setminus V(f)) \cap (\operatorname{Spec}(R) \setminus V(g)) \\
&\iff \mathfrak{p} \in \operatorname{Spec}(R) \setminus V(f) \text{ and } \mathfrak{p} \in \operatorname{Spec}(R) \setminus V(g) \\
&\iff \mathfrak{p} \notin V(f) \text{ and } \mathfrak{p} \notin V(g) \\
&\iff \mathfrak{p} \notin \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid (f) \subset \mathfrak{p} \,\} \text{ and } \mathfrak{p} \notin \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid (g) \subset \mathfrak{p} \,\} \\
&\iff (f) \not\subset \mathfrak{p} \text{ and } (g) \not\subset \mathfrak{p} \\
&\iff (f)(g) \not\subset \mathfrak{p}
\end{aligned}$$

2.

$$\begin{aligned}
D(f) = \emptyset &\iff \operatorname{Spec}(R) \setminus V(f) = \emptyset \\
&\iff \operatorname{Spec}(R) = V(f) \\
&\iff \text{All prime ideals } \mathfrak{p} \text{ contain } (f). \\
&\iff (f) \subset \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p} = \operatorname{Nil}(R)
\end{aligned}$$

□

Theorem 4. The set $\{\mathfrak{p}\}$ is closed in $\operatorname{Spec}(R)$ if and only if $\mathfrak{p}$ is a maximal ideal.

Proof.
1. Let $\{\mathfrak{p}\}$ be a closed subset in $\operatorname{Spec}(R)$.
2. By definition, $\{\mathfrak{p}\} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in R.
3. If $\mathfrak{p}$ is not maximal, it is contained in a maximal ideal $\mathfrak{m}$.
4. Thus $V(\mathfrak{a}) \supset \{\mathfrak{p}, \mathfrak{m}\}$

1. trivial

□

Theorem 5. The closure of $\{\mathfrak{p}\}$ is $V(\mathfrak{p})$.

Proof. $\overline{\{\mathfrak{p}\}} \subset V(\mathfrak{p})$

1. trivial since $\mathfrak{p} \subset V(\mathfrak{p})$

$V(\mathfrak{p}) \subset \overline{\{\mathfrak{p}\}}$

1. Let $\mathfrak{p}' \in V(\mathfrak{p}) = \{\, \mathfrak{p}' \in \operatorname{Spec}(R) \mid \mathfrak{p} \subset \mathfrak{p}' \,\}$.
2. So $\mathfrak{p} \subset \mathfrak{p}'$.
3. I think the idea is simply $\overline{\{\mathfrak{p}\}} = V(\mathfrak{a})$

□

Is $\operatorname{Spec}(R)$ a Kolmogorov-space?
Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(R)$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$.

Theorem 6. X is irreducible if and only if every pair of non-empty open sets in X has a non-empty intersection.

Proof.
1. Let X be irreducible, so for any closed subsets X_1 and X_2 with $X = X_1 \cup X_2$ implies $X = X_1$ or $X = X_2$.
2. Let A and B be two non-empty open subsets.
3. Then $X \setminus A$ and $X \setminus B$ is closed.
4. We have $(X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B)$
5. Assume $A \cap B = \emptyset$.
6. contradiction.

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1. Every pair of non-empty open sets in X has a non-empty intersection.
2. Assume X is reducible, i.e. $X = X_1 \cup X_2$
3. $X \setminus X_1$ and $X \setminus X_2$ are open.
4. $(X \setminus X_1) \cap (X \setminus X_2) = X \setminus (X_1 \cup X_2) = \emptyset$
5. contradiction

□

Theorem 7. X is irreducible if and only if every non-empty open subset of X is dense in X.

Proof.
1. Let X be irreducible.
2. Fix an open subset A.
3. $X \setminus A$ is closed.
4. $\operatorname{cl}(A)$ is closed.
5. $X = (X \setminus A) \cup \operatorname{cl}(A)$, so $X = \operatorname{cl}(A)$.

□

Theorem 8. $\operatorname{Spec}(R)$ is irreducible if and only if the nilradical of A is a prime ideal.

Proof.
1. Let $\operatorname{Spec}(R)$ be irreducible.
2. Consider $\bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$
3. $\bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p} = \bigcap_{\mathfrak{m}} V(\mathfrak{m})$
4. $X \setminus \bigcap_{\mathfrak{m}} V(\mathfrak{m}) = \bigcup_{\mathfrak{m}} (X \setminus V(\mathfrak{m}))$
5. $V(\mathfrak{m}) = \emptyset$

□