

# Notes on Algebraic Geometry

Kei Thoma

June 30, 2024



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**Definition 0.0.1.** Let  $K$  be an algebraically closed field and let  $n \in \mathbb{N}_0$  be a natural number.

1. The affine  $n$ -space over  $K$  is the set of all  $n$ -tuples of elements of  $K$ .
2. An element  $p$  in  $\mathbb{A}^n$  is called a point.
3. If  $p = (a_1, \dots, a_n) \in \mathbb{A}^n$  is a point, then  $a_i$  is called the coordinate for each  $1 \leq i \leq n$ .

**Intuition 0.0.2.** It's just space with points. But not vectors, because we don't add points.

**Definition 0.0.3.** 1. The set of zeros for  $f \in A = K[X_1, X_2, \dots, X_n]$  is

$$\{p \in \mathbb{A}^n \mid f(p) = 0\}.$$

2. For a subset  $T$  of the polynomial ring  $A = k[X_1, \dots, X_n]$ , we define the zero set of  $T$  to be the common zeros of all the elements of  $T$ , i.e.

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

**Intuition 0.0.4.** These are just curves.

**Remark 0.0.5.** 1. If  $\mathfrak{a}$  is generated by  $T$ , then  $Z(T) = Z(\mathfrak{a})$ .

2.  $Z(T)$  can be written in finitely many generators.

**Definition 0.0.6.** A subset  $Y$  of  $\mathbb{A}^n$  is an algebraic set if there exists a subset  $T \subset A = k[X_1, \dots, X_n]$  such that  $Y = Z(T)$ .

**Intuition 0.0.7.** So if the points on the space is a curve, then it's an algebraic set.

**Definition 0.0.8** (Affine Algebraic Variety). For an algebraically closed field  $K$  and a natural number  $n \in \mathbb{N}_+$ , let  $\mathbb{A}^n$  be an affine  $n$ -space over  $K$ . The polynomials in  $K[X_1, \dots, X_n]$  can be viewed as  $K$ -valued functions on  $\mathbb{A}^n$ .

1. For each subset  $S$  of polynomials in  $K[X_1, \dots, X_n]$ , define the zero-locus  $Z(S)$  to be the set of points in  $\mathbb{A}^n$  on which the functions in  $S$  simultaneously vanish, i.e.

$$Z(S) = \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

2. A subset  $V$  of  $\mathbb{A}^n$  is called affine algebraic set if  $V = Z(S)$  for some  $S \subset K[X_1, \dots, X_n]$ .
3. A nonempty affine algebraic set is called irreducible if it is not the union of two proper algebraic subsets. An irreducible affine algebraic set is also called an affine variety.

**Definition 0.0.9.** An affine algebraic variety is an irreducible closed subset of  $\mathbb{A}^n$ . An open subset of an affine variety is a quasi-affine variety.

**Corollary 0.0.10.** *An algebraic set is irreducible if and only if its ideal is a prime ideal.*

**Definition 0.0.11.** If  $Y \subset \mathbb{A}^n$  is an affine algebraic set, we define the affine coordinate ring  $A(Y)$  of  $Y$ , to be  $A/I(Y)$ .

**Definition 0.0.12.** If  $X$  is a topological space, we define the dimension of  $X$  (denoted  $\dim X$ ) to be the supremum of all integers  $n$  such that there exists a chain  $Z_0 \subset Z_1 \subset \dots \subset Z_n$  of distinct irreducible closed subsets of  $X$ . We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

**Exercise 0.0.1.** *Show that  $k$ -algebra  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ , for some  $n$ , if and only if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.*



# Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.