Exercise 1

1. Let (X, d) be a metric space. Prove that the set of subsets

$$\mathcal{O}(d) := \{ U \subset X \mid \forall x \in U \, \exists \epsilon > 0 \text{ with } B_d(x, \epsilon) \subset U \} \tag{1}$$

defines a topology.
Proof. We verify that $\mathcal{O}(d)$ fullfills the axioms of a topology.

   (a) $X \in \mathcal{O}(d)$ since any ball of a point x is contained in X. $\varnothing \in \mathcal{O}(d)$ is true vacuously.

   (b) Let I be an arbitary index set and $\{A_i\}_{i \in I}$ be a family of subsets that belong to $\mathcal{O}(d)$. Consider the union $\bigcup_{i \in I} A_i$. If a point x is in $\bigcup_{i \in I} A_i$, then there is an A_i where this point x is contained. Since A_i is in $\mathcal{O}(d)$, there exists an ϵ such that $B_d(x, \epsilon) \subset A_i \subset \bigcup_{i \in I} A_i$. Therefore, we have that $\bigcup_{i \in I} A_i$ belongs to $\mathcal{O}(d)$.

   (c) Let I be a finite index set and A_i with $i \in I$ be subsets in $\mathcal{O}(d)$. Consider the intersection $\bigcap_{i \in I} A_i$. If a point x is in $\bigcap_{i \in I} A_i$, then x is included in each A_i. Again, A_i is in $\mathcal{O}(d)$, so there is an ϵ_i such that $B_d(x, \epsilon_i) \subset A_i$. Choose the smallest (accordig to the metric d) among all $\epsilon_i \in I$ and denote it as ϵ. We have $B_d(x, \epsilon) \subset B_d(x, \epsilon_i) \subset A_i$ for all $i \in I$. This means $B_d(x, \epsilon) \subset \bigcap_{i \in I} A_i$ as desired.

2. Show that any ball $B_d(x, r) \in \mathcal{O}(d)$ for all $x \in X$ and for all $r > 0$.

   Proof. Fix an $p \in B_d(x, r)$. Set $\epsilon := (r - d(x, p))/2$ (dividing it by two might only be for good measure). Then $B_d(p, \epsilon) \subset B_d(x, r)$, so $B_d(x, r) \in \mathcal{O}(d)$. □

3. Let d_1 and d_2 be equivalent metrics on X. Show that $\mathcal{O}(d_1) = \mathcal{O}(d_2)$.

   Proof. We will show $\mathcal{O}(d_1) \subseteq \mathcal{O}(d_2)$. Symmetry will take care of the other side. Let $A \in \mathcal{O}(d_1)$ and fix a point $x \in A$ (if A is empty, it is immediately included in $\mathcal{O}(d_2)$), then there is an $\epsilon_1 > 0$ such that $B_{d_1}(x, \epsilon_1) \subset A$.

   Since d_1 and d_2 are equivalent, we have some $c \in \mathbb{R}$ such that $cd_1(a, b) \leq d_2(a, b)$ for all $a, b \in X$. Set $\epsilon_2 := c^{-1}\epsilon_1$.

   Now consider $B_{d_2}(x, \epsilon_2)$ and fix a $y \in B_{d_2}(x, \epsilon_2)$. We have $cd_1(x, y) \leq d_2(x, y) < \epsilon_2$ and if we multiply c^{-1} on both ends, we get $d_1(x, y) < c^{-1}\epsilon_2 = \epsilon_1$. Cocnlude that $y \in B_{d_1}(x, \epsilon_1)$, so $B_{d_2}(x, \epsilon_2) \subset B_{d_1}(x, \epsilon_1)$, hence $A \in \mathcal{O}(d_2)$. □

4. Let $f : (X, d_X) \longrightarrow (X, d_Y)$ be a map that is ϵ-δ-continuous in the sense of metric spaces. Show that f is continuous with respect to the topologies $\mathcal{O}_{d_X}$ and $\mathcal{O}_{d_Y}$.

   Proof. Let $V \in \mathcal{O}_{d_Y}$. If V is empty, we are done. In other case, fix an $x \in f^{-1}(V)$. We have that $B_{d_Y}(f(x), \epsilon) \subset V$ for some ϵ since V is open. The ϵ-δ-continuity implies that there is a $\delta > 0$ such that $B_{d_X}(x, \delta) \subset f^{-1}(V)$, so $f^{-1}(V)$ is open and f is also open set continuous. □