Chapter 1

Category Theory

Definition 1.1 — Category.
A category C consists of:

1. A class $\mathrm{Ob}(C)$ of objects.
2. For every two objects A and B of C, a set $\mathrm{Hom}_C(A, B)$ of morphism, with the properties listed
   (a) For every object A of C, there exists (at least) one morphism $1_A \in \mathrm{Hom}_C(A, A)$, the identity on A.
   (b) For every triple of objects A, B, and C of C there is a function (of sets)
   $$\mathrm{Hom}_C(A, B) \times \mathrm{Hom}_C(B, C) \longrightarrow \mathrm{Hom}_C(A, C) \tag{1.1}$$
   (c) The composition law is associative.
   (d) The identity morphism is commutative.
   (e) $\mathrm{Hom}_C(A, B) = \mathrm{Hom}_C(C, D)$ if and only if $A = C$ and $B = D$.

Example 1.2. Let S be a set and $\sim$ a relation on S satisfying the reflexive and transitive properties, then $\mathrm{Ob}(C) = S$ and if $a, b \in S$, then let $\mathrm{Hom}(a, b) = \{ (a, b) \}$ if $a \sim b$ and $\mathrm{Hom}(a, b) = \emptyset$.

Chapter 2

Commutative Rings

List of Definitions

1. Rings
2. Ring Homomorphism
3. Ideal
4. prime ideal
5. coprime
6. irreducible
7. zero divisor
8. nilpotent
9. spe
10. jacobson radical
11. ideal operation

Definition 2.1 — Ring.
A ring is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(R, +)$ is an abelian group.
2. $(R, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

From here, all rings are unitary commutative rings.

Definition 2.2 — Ring Homomorphism.

Definition 2.3 — Ideal.

Definition 2.4 — Coprime.
Let R be a ring. We say two elements $x, y \in R$ are coprime if one of the following equivalent condition hold.

1. For each $z \in R$ there exist $a, b \in R$ such that $ax + by = z$ (Bézout's identity).
2. $y + \langle x \rangle$ is a unit in $R \big/ \langle x \rangle$.

3. The principal ideals generated by the elements are comaximal, i.e. $\langle x\rangle + \langle y\rangle = \langle 1\rangle = R$.

Similarly, two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R are coprime if they are comaximal.

Definition 2.5 — Prime Ideal.

Definition 2.6 — Zero Divisor.

Part I

Exercises

Exercise 2.7. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals in A, and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of B. Prove the following statements.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

   Proof. We show $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$. Let $x \in (\mathfrak{a}_1 + \mathfrak{b}_2)^e$, then we have for some index set I

$$x = \sum_{i \in I} \lambda_i x_i, \tag{2.1}$$

   where $\lambda_i \in B$ and $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$ for all $i \in I$. For each $i \in I$ it is $x_i = \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2})$, hence

$$\begin{aligned}
x =& \sum_{i \in I} \lambda_i \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2}) && (2.2)\\
=& \sum_{i \in I} \lambda_i \left(\varphi(\mu_{i,1}a_{i,1}) + \varphi(\mu_{i,2}a_{i,2})\right) \qquad \text{(by linearity)} && (2.3)\\
=& \sum_{i \in I} \lambda_i \left(\mu_{i,1}\varphi(a_{i,1}) + \mu_{i,2}\varphi(a_{i,2})\right) \qquad \text{(by linearity)} && (2.4)\\
=& \sum_{i \in I} \lambda_i \mu_{i,1}\varphi(a_{i,1}) + \lambda_i \mu_{i,2}\varphi(a_{i,2}) \qquad \text{(by distributivity)} && (2.5)\\
=& \sum_{i \in I} \lambda_i \mu_{i,1}\varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2}\varphi(a_{i,2}) \qquad \text{(reordering the sum).} && (2.6)\\
& && (2.7)
\end{aligned}$$

   The last term is exactly the elements expressed by $\mathfrak{a}_1^e + \mathfrak{a}_2^e$, therefore, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.
   I think the above proof should work into both directions. □

2. $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$

   Proof. We have

$$(\mathfrak{b}_1 + \mathfrak{b}_2)^c = \left\{ x \in A \;\middle|\; \exists b_1 \in \mathfrak{b}_1 \exists b_2 \in \mathfrak{b}_2 : \varphi(x) = b_1 + b_2 \right\}. \tag{2.8}$$

   Now let $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$, then $x = a_1 + a_2$ where $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$. It is

$$\begin{aligned}
\varphi(x) =& \varphi(a_1 + a_2) && (2.9)\\
=& \varphi(a_1) + \varphi(a_2) \qquad \text{(by additivity)} && (2.10)
\end{aligned}$$

   Since $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$ we have that $x \in (\mathfrak{b}_1 + \mathfrak{b}_2)^c$. □

Exercise 2.8. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}$ an ideal of A, and $\mathfrak{b}$ an ideal of B. Prove the following statements:

1. Then $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$.

   Proof. It is

$$\begin{aligned}
\mathfrak{a}^{ec} =& \left\{ x \in A \;\middle|\; \varphi(x) \in \mathfrak{a}^e \right\} && (2.11)\\
=& \left\{ x \in A \;\middle|\; \varphi(x) \in \langle \varphi(\mathfrak{a}) \rangle \right\} && (2.12)\\
=& \left\{ x \in A \;\middle|\; \forall i \in I \, \exists a_i \in \mathfrak{a}_1 : \varphi(x) = \sum_{i \in I} \lambda_i \varphi(a_i) \right\}. && (2.13)
\end{aligned}$$

   Let $a \in \mathfrak{a}$ and choose $I = \{1\}$, λ_1, and $a_i = a$, then $a \in \mathfrak{a}^{ec}$. □

2. $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$.
3. $\mathfrak{a}^{ece} = \mathfrak{a}^e$.
4. $\mathfrak{b}^{cec} = \mathfrak{b}^c$.
5. If $\mathfrak{b}$ is an extension, then $\mathfrak{b}^c$ is the largest ideal of A with extension $\mathfrak{b}$.

6. If two extensions have the same contraction, then they are equal.

   Proof. a ☐

Exercise 2.9. Let A be a ring, $A[\mathcal{X}, \mathcal{Y}]$ the polynomial ring in two sets of variables $\mathcal{X}$ and $\mathcal{Y}$. Show that $\langle \mathcal{X} \rangle$ is prime if and only if A is a domain.

Proof. It should be noted here, that $A[\mathcal{X}]$ does not contain X_1X_2 for example. It does contain X_1+X_2 however. The rest is easy. ☐

Exercise 2.10. Show that, in a PID, nonzero elements x and y are relatively prime (share no prime factor) if and only if they're coprime.

Exercise 2.11. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals, and $\mathfrak{p}$ a prime ideal. Prove that these conditions are equivalent:

1. $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$
2. $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$
3. $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$

Proof. (1) to (2) is easy. Same for (2) to (3). For (3) to (1) show it with contradiction. ☐

Exercise 2.12. Let A be a ring, $\mathfrak{p}$ a prime ideal, and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ maximal ideals with $\mathfrak{m}_1, \dots, \mathfrak{m}_n = 0$. Show $\mathfrak{p} = \mathfrak{m}_i$ for some i.

Proof. By induction. Proof first for m_1m_2, the rest is clear. ☐

Exercise 2.13. Let A be a ring, $\mathfrak{p}$ a prime, and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals.

1. If $\bigcap_{i=1}^n \mathfrak{a}_i \subseteq \mathfrak{p}$, then $\mathfrak{a}_j \subseteq \mathfrak{p}$ for some j.

   Proof. If $\mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq \mathfrak{p}$, then by the exercise above we have the desired result. The rest is induction. ☐

2. If $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{p}$, then $\mathfrak{a}_j \subseteq \mathfrak{p}$ for some j.

   Proof. Clear. ☐

Exercise 2.14. Let A be a ring, $\mathcal{S}$ the set of all ideals that consist entirely of zerodivisors. Show that $\mathcal{S}$ has maximal elements and they're prime. Conclude that $\mathrm{ZD}(A)$ is a union of primes.

Exercise 2.15. Exercise 2.27, proof is silly

Exercise 2.16. Let $A_1 \times A_2$ be a product of two rings. Show that $A_1 \times A_2$ is a domain if and only if either A_1 or A_2 is a domain and the other is 0.

Proof. The back implication is clear.

For the other implication, assume neither is integral domain, this leads to an obvious contradiction.

Now assume neither is 0. Choose $(a, 0) and (0, b)$, contradiction. ☐

Exercise 2.17. Let $A_1 \times A_2$ be a product of rings, $\mathfrak{p} \subset A_1 \times A_2$ an ideal. Show that $\mathfrak{p}$ is prime if and only if either $\mathfrak{p} = \mathfrak{p}_1 \times A_2$ with $\mathfrak{p}_1 \subseteq A_1$ prime or $\mathfrak{p} = A_1 \times \mathfrak{p}_2$ with $\mathfrak{p}_2 \subseteq A_2$ prime.

Proof. If $\mathfrak{p}$ is prime, then for each $(x, y) \in \mathfrak{p}$ we have that $(x, 1) \in \mathfrak{p}$ or $(1, y) \in \mathfrak{p}$. From this the first implication follows.

For the other side is clear. ☐

Exercise 2.18. Let A be a domain, and $x, y \in A$ with $\langle x \rangle = \langle y \rangle$. Show $x = uy$ for some unit u.

Proof. From $\langle x \rangle = \langle y \rangle$ we get that $rx = sy$ for some $r, s \in A$. Because A is a domain, we have $\frac{r}{s}x = y$. This is a unit because $\frac{r}{s} \cdot \frac{s}{r} = 1$. ☐

Exercise 2.19. Let k be a field, R a nonzero ring, $\varphi : k \longrightarrow R$ a ring map. Prove φ is injective.

Proof. The trick here is to know that the kernel is an ideal. Since the kernel contains 0, it must also contain the ideal generated by it. Now, in all fields is the zeroideal maximal, hence the kernel is already maximal and contains only 0. From that we conclude φ is injective. ☐

Exercise 2.20. Let A be a ring, $\mathfrak{p}$ a prime, $\mathcal{X}$ a set of variables. Let $\mathfrak{p}[\mathcal{X}]$ denote the set of polynomials with coefficients in $\mathfrak{p}$. Prove these statements:

1. $\mathfrak{p}R[\mathcal{X}]$ and $\mathfrak{p}[\mathcal{X}]$ and $\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle$ are primes of $R[\mathcal{X}]$, which contract to $\mathfrak{p}$.

   Proof. We have $R[\mathcal{X}] \big/ \mathfrak{p}R[\mathcal{X}] \simeq \left(R/\mathfrak{p}\right)[\mathcal{X}]$. The latter one is a domain because it is the polynomial ring of a domain. Therefore, $\mathfrak{p}R[\mathcal{X}]$ is prime.

   We also have $\mathfrak{p}R[\mathcal{X}] = \mathfrak{p}[\mathcal{X}]$.

   For the contraction let $\varphi : R \longrightarrow R[\mathcal{X}]$. Then $\varphi^{-1}\left(\mathfrak{p}[\mathcal{X}]\right) = \mathfrak{p}$.

   For the last part, consider
   $$\varphi : R[\mathcal{X}] \longrightarrow R/\mathfrak{p} \tag{2.14}$$
   with the natural definition $\varphi\left(a_0 + a_1 X + \ldots + a_n X^n\right) = a_0 + \mathfrak{p}$. Then, the kernel is all the polynomials with $a_0 \in \mathfrak{p}$ so $\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle$. Since φ is obviously surjective, we have the isomorphism
   $$R[\mathcal{X}] \big/ \mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle \simeq R/\mathfrak{p} \tag{2.15}$$
   The latter is a domain, so is the former, hence $\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle$ is prime.

   Again for the contraction we have $\varphi^{-1}(\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle) = \mathfrak{p}$ (because we are basically only caring about a_0). □

2. Assume $\mathfrak{p}$ is maximal. Then $\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle$ is maximal.

   Proof. From above, we have an isomorphism
   $$R[\mathcal{X}] \big/ \mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle \simeq R/\mathfrak{p} \tag{2.16}$$
   therefore, if $\mathfrak{p}$ is maximal so is $\mathfrak{p}R[\mathcal{X}] + \langle \mathcal{X} \rangle$. □

Exercise 2.21. Let R be a ring, X a variable, $H \in P := R[X]$, and $a \in R$. Given $n \geq 1$, show $(X-a)^n$ and H are coprime if and only if $H(a)$ is a unit.

Proof. Let $(X-a)^n$ and H be coprime. This means that
$$\langle (X-a)^n \rangle + \langle H \rangle = R[X] \tag{2.17}$$
With the ring homomorphism φ that substitutes X with a we have
$$R = \left(\langle (X-a)^n \rangle + \langle H \rangle\right)^e = \langle (X-a)^n \rangle^e + \langle H \rangle^e = \langle 0 \rangle + \langle H(a) \rangle \tag{2.18}$$
hence $H(a)$ must be a unit.

Let $H(a)$ be a unit in R. Consider the map $\varphi_a : R[X] \longrightarrow R$ with
$$p(X) \mapsto \varphi_a(p(X)) := p(a). \tag{2.19}$$
φ_a is a ring homomorphism because
$$\varphi_a(p(X) + q(X)) = p(a) + q(a) = \varphi_a(p(X)) + \varphi_a(q(X)) \tag{2.20}$$
and I'm to lazy the show it for the multiplication and $\varphi_a(1) = 1$. Morover, φ_a is surjective. So we have $\varphi_a^{-1}(R^\times) \subseteq (R[X])^\times$. From there, if $H(a)$ is a unit, so must $H(X)$ be. In that case, $H(X)$ and $(X-a)^n$ are obviously coprime. □

Exercise 2.22. Let R be a ring, X a variable, $F \in P := R[X]$, and $a \in R$. Set $F' := \partial F / \partial X$. Show the following statements are equivalent:

1. a is a supersimple root of F. (a is a supersimple root if $F(a) = 0$ and $F'(a) \neq 0$ is a unit.)
2. a is a root of F, and $X - a$ and F' are coprime.
3. $F = (X-a)G$ for some G in P coprime to $X - a$.

Show that if 3. holds, then G is unique.

Proof. "1. to 2.": Immideately, we have that a is a root of F. Since $F'(a)$ is a unit, by the previous exercise, we have that $(X-a)^n$ and F' are coprimes. In particular, if we choose $n=1$, we get the desired result.

"2. to 3.": We have $F' = G(X) + (X-a)G'$ and since this is coprime to $X-a$ we have for $\lambda, \mu \in R[X]$

$$\lambda(X-a) + \mu F'(X) = 1 \tag{2.21}$$
$$\lambda(X-a) + \mu(X-a)G'(X) + \mu G(X) = 1 \tag{2.22}$$
$$(\lambda + \mu G'(X))(X-a) + \mu G(X) = 1 \tag{2.23}$$

If we set $\lambda + \mu G'(X)$ as the factor, we see that $X-a$ and G are again coprime.

"3. to 1.": Clearly, a is a root of F. We also have $\lambda G(X) + \mu(X-a) = 1$, so if we substitute X for a, we get the desired result. □

Exercise 2.23. Let R be a ring, $\mathfrak{p}$ a prime, $\mathcal{X}$ a set of variables, $F, G \in R[\mathcal{X}]$. Let $c(F), c(G), c(FG)$ be the ideals of R generated by the coefficients of F, G, FG.

1. Show that if $\mathfrak{p}$ doesn't contain either $c(F)$ or $c(G)$, then $\mathfrak{p}$ doesn't contain $c(FG)$.

   Proof. Assume $c(FG) \subseteq \mathfrak{p}$, then because of $\mathfrak{p}$ is prime, we must have that the coefficients of F are also in $\mathfrak{p}$. □

2. Show that if $c(F) = R$ and $c(G) = R$, then $c(FG) = R$.

   Proof. □

Chapter 3

Radicals

Exercise 3.1. Let $\varphi : R \longrightarrow R'$ be a ring homomorphism, $\mathfrak{p}$ an ideal of R. Show:

1. there is an ideal $\mathfrak{q}$ of R' with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$ if and only if $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$.

   Proof. Choose $\mathfrak{q} := \langle \varphi(\mathfrak{p}) \rangle = \mathfrak{p}R'$, then $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) = \varphi^{-1}(\mathfrak{p}R')$.

   Solution from the book: Given $\mathfrak{q}$ note $\varphi(\mathfrak{p}) \subseteq \mathfrak{q}$, as always $\varphi(\varphi^{-1}(\mathfrak{q})) = \mathfrak{q}$. So $\mathfrak{p}R' \subseteq \mathfrak{q}$. Hence $\varphi^{-1}(\mathfrak{p}R \subseteq \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$

   On the other hand, if $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$, then define $\mathfrak{q} := \mathfrak{p}R'$ and we have $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. $\square$

Exercise 3.2. Use Zorn's lemma to prove that any prime ideal $\mathfrak{p}$ contains a prime ideal $\mathfrak{q}$ that is minimal containing any given subset $\mathfrak{s} \subseteq \mathfrak{p}$.

Proof. $\square$

Exercise 3.3. Let R be a ring, $\mathfrak{a} \subseteq \mathrm{Jac}(R)$ an ideal, $w \in R$, and $w' \in {}^{R}/_{\mathfrak{a}}$ its residue. Prove that $w \in R^\times$ if and only if $w' \in ({}^{R}/_{\mathfrak{a}})^\times$. What if $\mathfrak{a} \neq\notin \mathrm{Jac}(R)$?

Proof. Let $w \in R^\times$ $\square$

Chapter 4

Exact Sequences

Theorem 4.1 (Snake Lemma)**.** *Given a following commutative diagram with exact rows*

$$
\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 &
\end{tikzcd}
$$

Then the following sequence is exact.

$$\mathrm{Ker}\,(\gamma_1) \longrightarrow \mathrm{Ker}\,(\gamma_2) \longrightarrow \mathrm{Ker}\,(\gamma_3) \longrightarrow \mathrm{Coker}\,(\gamma_1) \longrightarrow \mathrm{Coker}\,(\gamma_2) \longrightarrow \mathrm{Coker}\,(\gamma_3) \tag{4.1}$$

Proof. We first name the arrows.

$$
\begin{tikzcd}
 & \mathrm{Ker}\,(\gamma_1) \arrow[r, "\varphi_1"] \arrow[d] & \mathrm{Ker}\,(\gamma_2) \arrow[r, "\varphi_2"] \arrow[d] & \mathrm{Ker}\,(\gamma_3) \arrow[d] \arrow[dddll, "\delta"] & \\
 & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_3 \arrow[r] \arrow[d, "\gamma_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\beta_1"] \arrow[d] & N_2 \arrow[r, "\beta_2"] \arrow[d] & N_3 \arrow[d] & \\
 & \mathrm{Coker}\,(\gamma_1) \arrow[r, "\psi_1"] & \mathrm{Coker}\,(\gamma_2) \arrow[r, "\psi_2"] & \mathrm{Coker}\,(\gamma_3) &
\end{tikzcd}
$$

1. Define $\varphi_1 : \mathrm{Ker}\,(\gamma_1) \longrightarrow \mathrm{Ker}\,(\gamma_2)$ as $\varphi_1 := \alpha_1\big|_{\mathrm{Ker}(\gamma_1)}$ and $\varphi_2 : \mathrm{Ker}\,(\gamma_2) \longrightarrow \mathrm{Ker}\,(\gamma_3)$ as $\varphi_2 := \alpha_2\big|_{\mathrm{Ker}(\gamma_2)}$. Let $x \in \mathrm{Ker}\,(\gamma_1)$, then it is

$$\gamma_2(\varphi_1(x)) = \gamma_2\left(\alpha_1\big|_{\mathrm{Ker}(\gamma_1)}(x)\right) \qquad \text{(definition of } \varphi_1) \tag{4.2}$$
$$= \beta_1(\gamma_1(x)) \qquad \text{(diagram is commutative)} \tag{4.3}$$
$$= \beta_1(0) \qquad (a \in \mathrm{Ker}\,(\gamma_1)) \tag{4.4}$$
$$= 0 \qquad (\beta_1 \text{ is } A\text{-linear}) \tag{4.5}$$

therefore, $x \in \mathrm{Ker}\,(\gamma_2)$ and we see that φ_1 is well-defined.

2. Define $\psi_1 : \mathrm{Coker}\,(\gamma_1) \longrightarrow \mathrm{Coker}\,(\gamma_2)$ and $\psi_2 : \mathrm{Coker}\,(\gamma_2) \longrightarrow \mathrm{Coker}\,(\gamma_3)$ as

$$\psi_1(n_1 + \mathrm{Im}\,(\gamma_1)) = \beta_1(n_1) + \mathrm{Im}\,(\beta) \tag{4.6}$$
$$\psi_2(n_2 + \mathrm{Im}\,(\gamma_2)) = \beta_2(n_2) + \mathrm{Im}\,(\gamma_3) \tag{4.7}$$

3. We now define $\delta : \mathrm{Ker}\,(\gamma_3) \longrightarrow \mathrm{Coker}\,(\gamma_1)$.

   Let $x_3 \in \mathrm{Ker}\,(\gamma_3)$, then since α_2 is surjective (because of the exact sequence), there is a $x_2 \in M_2$ such that $\alpha_2(x_2) = x_3$.

$$\begin{aligned}\beta_2(\gamma_2(x_2)) &= \gamma_3(\alpha_2(x_2)) && (4.8)\\ &= \gamma_3(x_3) && (4.9)\\ &= 0 && (4.10)\end{aligned}$$

   So $\gamma_2(x_2) \in \mathrm{Ker}\,(\beta_2) = \mathrm{Im}\,(\beta_1)$ because the second row is exact. Therefore, we find $y_1 \in N_1$ such that $\beta_1(y_1) = \gamma_2(x_2)$. We know that β_1 is injective, therefore $y_1 \in N_1$ with $\beta_1(y_1) = \gamma_2(x_2)$ is unique.

   Now define

$$\delta(x_3) = y_1 + \mathrm{Im}\,(\gamma_1) \tag{4.11}$$

   We have constructed y_1 in the following manner: $\alpha_2(x_2) = x_3$ for some $x_2 \in M_2$ and $\gamma_2(x_2) = \beta_1(y_1)$ for unique $y_1 \in N_1$

4. We show δ as defined above is well-defined.

□

Exercise 4.2. Let M' and M'' be modules, $N \subseteq M'$ a submodule. Prove

$$M/N \simeq M'/N \oplus M'' \tag{4.12}$$

Proof. Consider the two sequences

$$\begin{aligned}0 \longrightarrow N \longrightarrow M' \longrightarrow M'/N \longrightarrow 0 && (4.13)\\ 0 \longrightarrow 0 \longrightarrow M'' \longrightarrow M'' \longrightarrow M'' \longrightarrow 0. && (4.14)\end{aligned}$$

with

$$\begin{aligned}f_0(x) = 0 f_1(x) = x \qquad && f_2(x) = x + N f_3(x) = 0 && (4.15)\\ g_0(x) = 0 g_1(x) = 0 \qquad && g_2(x) = x g_3 = 0, && (4.16)\end{aligned}$$

then the two sequences are exact because

$$\begin{aligned}\mathrm{im}(f_0) = 0 = \ker(f_1)\mathrm{im}(f_1) = N = \ker(f_2) \qquad && \mathrm{im}(f_2) = M'/N = \ker(f_3) && (4.17)\\ \mathrm{im}(g_0) = 0 = \ker(g_1)\mathrm{im}(g_1) = 0\ker(g_2) \qquad && \mathrm{im}(g_2) = M'' = \ker(g_3). && (4.18)\end{aligned}$$

Since both sequences are exact, so is their direct sum. Note that $N \oplus 0 \simeq N$ and the maps between the chain are the composition of f and g.

$$0 \longrightarrow N \longrightarrow M' \oplus M'' \longrightarrow M'/N \oplus M'' \longrightarrow 0 \tag{4.19}$$

Both f_2 and g_2 are surjective, so with the first isomorphism theorem, we have

$$M'/N \oplus M'' \simeq M/N \tag{4.20}$$

as desired. □

Exercise 4.3. Let M' and M'' be modules, and set $M = M' \oplus M''$. Let N be a submodule of M containing M', and set $N'' := N \cup M''$. Prove $N = M' \oplus N''$.

Proof. Consider the sequence

$$N \cup M'' \longrightarrow N \longrightarrow M' \tag{4.21}$$

This splits because the first part has a retraction and the second part is surjective, therefore we have the isomorphism as desired. □

Exercise 4.4. Five Lemma

1. If γ_3 and γ_1 are surjective and if γ_0 is injective, then γ_2 is surjective.