1. Compute the values of the following sums.

1. $\sum_{k=0}^{\infty} x^k$, for $|x| < 1$,
2. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$,
3. $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Solution:

1. The sum submits to the ratio test and converges to some number. Denote this number with S. We have

$$S = \tag{1}$$

2. For $\lambda > 0$ let $X \sim \mathrm{Exp}(\lambda)$ and let

$$Y := \lceil X \rceil := \min\{\, n \in \mathbb{N} \mid n \geq X \,\} \tag{2}$$

Show that for the parameter $p = 1 - e^{-\lambda}$ holds $Y \sim \mathrm{Geo}(p)$.

Solution:
For the distribution of Y we have for all $k \in \mathbb{N}_0$

$$p^Y(k) = \mathbb{P}(Y = k) = \mathbb{P}(\lceil X \rceil = k) = \mathbb{P}(k - 1 < X \leq k) = F^X(k) - F^X(k-1). \tag{3}$$

On the other hand, the CDF of X is

$$F^X(x) = \int_{-\infty}^{x} f^X(y)\,\mathrm{d}y = \int_{-\infty}^{x} \lambda e^{-\lambda y} \mathbb{1}_{[0,\infty)}(y)\,\mathrm{d}y = 1 - e^{-\lambda x}. \tag{4}$$

Therefore we have

$$\begin{aligned}
p^Y(k) &= F^X(k) - F^X(k-1) && (5)\\
&= (1 - e^{-\lambda k}) - (1 - e^{-\lambda(k-1)}) && (6)\\
&= -e^{-\lambda k} + e^{-\lambda(k-1)} && (7)\\
&= -e^{-\lambda} \cdot e^{-\lambda(k-1)} + e^{-\lambda(k-1)} && (8)\\
&= e^{-\lambda(k-1)}(1 - e^{-\lambda}) && (9)\\
&= \left(1 - \left(1 - e^{-\lambda}\right)\right)^{k-1}(1 - e^{-\lambda}) && (10)
\end{aligned}$$

Setting $1 - e^{-\lambda} = p$ yields the desired result $p^Y(k) = (1 - p)^{k-1}\, p$.