

Problem 1

a)

Give an example of retract which is not a deformation retract.

Proof. Consider the sets

$$\begin{aligned} X &= \{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 1 \text{ and } y \neq 0 \} \\ A &= \{ (x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 1 \text{ and } y > 0 \}. \end{aligned}$$

Then, $A \subset X$ and we may define a retraction $r : X \rightarrow A$ by

$$r(x, y) = (x, |y|).$$

But A is not a deformation retract of X because X has two connected components while A has one, and connectedness is a homotopy invariant. \square

b)

Let

$$X := \left\{ (x, y) \in \mathbb{R}^2 \mid x = \frac{1}{n}, n \in \mathbb{Z} \text{ or } x = 0, y \in [0, 1] \right\} \cup ([0, 1] \times \{0\}).$$

Show that $\{(0, 1)\}$ a deformation retract but not a strong deformation retract.