Chapter 1

Commutative Rings

Exercise 1.1. Let $\varphi : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals in A, and $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of B. Prove the following statements.

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.

   Proof. We show $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$. Let $x \in (\mathfrak{a}_1 + \mathfrak{b}_2)^e$, then we have for some index set I

$$x = \sum_{i \in I} \lambda_i x_i, \tag{1.1}$$

   where $\lambda_i \in B$ and $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$ for all $i \in I$. For each $i \in I$ it is $x_i = \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2})$, hence

$$\begin{aligned}
x =& \sum_{i \in I} \lambda_i \varphi(\mu_{i,1} a_{i,1} + \mu_{i,2} a_{i,2}) && (1.2)\\
=& \sum_{i \in I} \lambda_i \left(\varphi(\mu_{i,1} a_{i,1}) + \varphi(\mu_{i,2} a_{i,2})\right) \quad \text{(by linearity)} && (1.3)\\
=& \sum_{i \in I} \lambda_i \left(\mu_{i,1} \varphi(a_{i,1}) + \mu_{i,2} \varphi(a_{i,2})\right) \quad \text{(by linearity)} && (1.4)\\
=& \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad \text{(by distributivity)} && (1.5)\\
=& \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad \text{(reordering the sum).} && (1.6)\\
& && (1.7)
\end{aligned}$$

   The last term is exactly the elements expressed by $\mathfrak{a}_1^e + \mathfrak{a}_2^e$, therefore, $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$.
   I think the above proof should work into both directions. □

2. $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$

   Proof. We have

$$(\mathfrak{b}_1 + \mathfrak{b}_2)^c = \{ x \in A \mid \exists b_1 \in \mathfrak{b}_1 \exists b_2 \in \mathfrak{b}_2 : \varphi(x) = b_1 + b_2 \}. \tag{1.8}$$

   Now let $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$, then $x = a_1 + a_2$ where $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$. It is

$$\begin{aligned}
\varphi(x) =& \varphi(a_1 + a_2) && (1.9)\\
=& \varphi(a_1) + \varphi(a_2) \quad \text{(by additivity)} && (1.10)
\end{aligned}$$

   Since $\varphi(a_1) \in \mathfrak{b}_1$ and $\varphi(a_2) \in \mathfrak{b}_2$ we have that $x \in (\mathfrak{b}_1 + \mathfrak{b}_2)^c$. □