0.1 Radical of an Ideal

Example 0.1. $\sqrt{(X^2Y, XY^2)} = (XY)$

Proof. 1. Let $x \in \sqrt{(X^2Y, XY^2)}$.

2. Assume $x \notin (XY)$.

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1. We want to show $XY \in \sqrt{(X^2Y, XY^2)}$.
2. $\iff (XY)^2 \in (X^2Y, XY^2)$
3. And this is true because we have $X^2Y^2 = X^2Y \cdot Y$

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1. We want to show $X \in \sqrt{(X^2Y, XY^2)}$.
2. $\iff X^2$

Apparantly there are no easy way to
Maybe another way.

1. What are the nilpotent elements of $K[X,Y]/(X^2Y, XY^2)$?
2.

This is basically asking the same question ... □

0.2 Ideal Quotient

Definition 1. Let I be an ideal in R and J an element in R. The ideal quotient is

$$(I : J) = \{ r \in R \mid rJ \subset I \}$$

Moreover

$$(I : x) = \{ r \in R \mid rx \subset I \}$$

Theorem 2. The ideal quotient are ideals.

Proof. Let $x \in (I : J)$ and $r \in R$. Then $xJ \subset I$ by definition. We also have $rxJ \subset xJ \subset I$.
Let $x, y \in (I : J)$. Then $(x + y)J = xJ + yJ \subset I$. □

Example 2.1. 1. $((2) : 3) = \{ r \in \mathbb{Z} \mid 3r \in (2) \} \Rightarrow ((2) : 3) = (2)$.

2. $((6) : 3) = \{ r \in \mathbb{Z} \mid 3r \in (6) \} \Rightarrow ((6) : 3) = (2)$.
3. $((8) : 2) = \{ r \in \mathbb{Z} \mid 2r \in (8) \} \Rightarrow ((8) : 2) = (4)$.

Example 2.2. Let $I = (X^2Y, XY^2)$ be an ideal in the polynomial ring $R = K[X, Y]$ over a field K.

1. $((X^2Y, XY^2) : XY) = \left\{ r \in R \mid r \cdot XY \subset (X^2Y, XY^2) \right\} = (X, Y) \Rightarrow \sqrt{(X, Y)} = (X, Y)$.
2. $((X^2Y, XY^2) : (X^2, XY)) = (Y) \Rightarrow \sqrt{(Y)} = (Y)$.
3. $((X^2Y, XY^2) : (XY, Y^2)) = (X) \Rightarrow \sqrt{(X)} = (X)$.

It is $\sqrt{(X^2Y, XY^2)} = (XY)$.

Theorem 3.