1 Rigidity Conjecture

Remark. When studying compositions of formal power series, we require that the inner power series $f(X)$ has no constant term, i.e., $f(0) = 0$. This condition ensures that the resulting composition is well-defined in the ring of formal power series $\mathbb{C}[[X]]$, as it prevents infinite contributions to the coefficients.

Consider $f(X) = \sum_{k\geq 1} a_k X^k$ and $g(X) = \sum_{k\geq 0} b_k X^k$. The composition $g(f(X))$ is given by substituting $f(X)$ into $g(X)$:

$$\begin{aligned} g(f(X)) &= b_0 + b_1 f(X) + b_2 f(X)^2 + \cdots \\ &= b_0 + b_1(a_1X + a_2X^2 + \cdots) + b_2(a_1X + a_2X^2 + \cdots)^2 + \cdots \\ &= b_0 + b_1a_1X + (b_1a_2 + b_2a_1^2)X^2 + \cdots , \end{aligned}$$

where we grouped the terms by powers of X in the last step. We observe that the coefficients of X^n in $g(f(X))$ depend only on a finite number of coefficients of $f(X)$ and $g(X)$. This is because, with $f(0) = 0$, each power $f(X)^k$ introduces terms of degree at least k, ensuring that lower-degree terms do not contribute infinitely to higher-order coefficients.

On the other hand, if $f(0) \neq 0$, we write $f(X) = a_0 + \sum_{k\geq 1} a_k X^k$, where $a_0 = f(0)$. In this case,

$$f(X)^k = (a_0 + a_1X + a_2X^2 + \cdots)^k$$

produces a constant term $a_0^k \neq 0$. Consequently, the constant term of $g(f(X))$ depends on infinitely many terms of $g(X)$, and the composition $g(f(X))$ is no longer a formal power series.

Since we are interested in the compositional inverse, it is necessary to extend the condition $f(0) = 0$ to both power series. This ensures that the inverse series $f^{-1}(X)$, when substituted into $f(X)$, results in the identity series X, with no contributions from constant terms that would otherwise make the series ill-defined.

The following proposition and lemma are taken from Enumerative Combinatorics by Richard P. Stanley and Sergey Fomin.

Definition 1. Let $f(X) \in \mathbb{C}[[X]]$ be a power series with no constant term. We call a power series $f^{-1}(X) \in \mathbb{C}[[X]]$ the compositional inverse of f, if it satisfies $f(f^{-1}(X)) = f^{-1}(f(X)) = X$.

Proposition 2. A power series $f(X) = a_1X + a_2X^2 + \cdots \in \mathbb{C}[[X]]$ has a compositional inverse if and only if $a_1 \neq 0$. Moreover, if the compositional inverse exists, then it is unique.

Proof. Assume f has a compositional inverse and denote the compositional inverse by $f^{-1}(X) = b_1X + b_2X^2 + \cdots$. Writing out $f(f^{-1}(X)) = X$ using multinomial thoerem gives

$$\begin{aligned} X &= a_1(b_1X + b_2X^2 + \cdots) + a_2(b_1X + b_2X^2 + \cdots)^2 + \cdots \\ &= (a_1b_1X + a_1b_2X^2 + a_1b_3X^3 + \cdots) + (a_2b_1^2X^2 + 2a_2b_1b_2X^3 + \cdots) \\ &= (a_1b_1)X + (a_1b_2 + a_2b_1^2)X^2 + (a_1b_3 + 2a_2b_1b_2 + a_3b_1^3)X^3 \cdots . \end{aligned}$$

Equating the coefficients on both sides yields a linear system of equations.

$$\begin{aligned} 1 &= a_1b_1 \\ 0 &= a_1b_2 + a_2b_1^2 \\ 0 &= a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 \\ &\vdots \end{aligned}$$

The first equation has a solution if and only if $a_1 \neq 0$. In that case, the solution is unique. Then, the second equation can be solved uniquely for b_2. By this process, we are able to solve the third equation for b_3, the fourth for b_4 and so on. Thus, $f^{-1}(X)$ exists if and only if $a_1 \neq 0$ and in that case, $f^{-1}(X)$ is unique. $\square$

Lemma 3 (Lagrange Inversion Formula)**.** Let $f(X) = \sum_{k\geq 1} a_k X^k \in \mathbb{C}[[X]]$ be a power series with $a_1 \neq 0$ and denote its composition inverse by $f^{-1}(X) = \sum_{k\geq 1} b_k X^k \in \mathbb{C}[[X]]$. The coefficients of the inverse is given by the following formula.

$$b_k = \frac{1}{k}[X^{n-1}]\left(\frac{X}{f(X)}\right)^k$$

Proof. We begin by substituting $f(X)$ into $f^{-1}(X)$. It is

$$X = f^{-1}(f(X)) = \sum_{k\geq 1} b_k f(X)^k.$$

Differentiating and subsequently taking the quotient with $f(X)^n$ for $n \in \mathbb{N}$ on both sides yields

$$\begin{aligned} 1 &= \sum_{k\geq 1} k \cdot b_k \cdot f(X)^{k-1} \cdot f'(X) \\ \Rightarrow \quad \frac{1}{f(X)^n} &= \sum_{k\geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X). \end{aligned}$$

We want to take the coefficient of X^{-1} on both sides. For that, first notice that for $k \neq n$ it is

$$\frac{1}{k-n}\frac{d}{dX}f(X)^{k-n} = f(X)^{k-n-1}f'(X) = \frac{f(X)^k}{f(X)^{n+1}}f'(X).$$

For any Laurent series, its derivative has no X^{-1} term. Thus, for $k \neq n$, it is

$$[X^{-1}]\frac{f(X)^k}{f(X)^{n+1}}f'(X) = [X^{-1}]\frac{1}{k-n}\frac{d}{dX}f(X)^{k-n} = 0.$$

If we now take the coefficient of X^{-1} in #REFMISSING, we get

$$\begin{aligned} [X^{-1}]\frac{1}{f(X)^n} &= [X^{-1}]\sum_{k\geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) && (1) \\ &= \sum_{k\geq 1} k \cdot b_k \cdot [X^{-1}]\frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) && (2) \\ &= n \cdot b_n [X^{-1}]\frac{f(X)^n}{f(X)^{n+1}} \cdot f'(X) && (3) \\ &= n \cdot b_n [X^{-1}]\frac{f'(X)}{f(X)} && (4) \\ &= n \cdot b_n [X^{-1}]\frac{a_1 + 2a_2X + 3a_3X^2 + \cdots}{a_1X + a_2X^2 + a_3X^3 + \cdots} && (5) \\ &= n \cdot b_n [X^{-1}]\frac{1}{X}\frac{a_1 + 2a_2X + 3a_3X^2 + \cdots}{a_1 + a_2X + a_3X^2 + \cdots} && (6) \\ &= n \cdot b_n [X^{-1}]\frac{1}{X}\left(\frac{a_1}{a_1 + a_2X + a_3X^2 + \cdots} + \frac{2a_2X}{a_1 + a_2X + a_3X^2 + \cdots} + \cdots\right) && (7) \end{aligned}$$

Since we are only interested in the coefficient of X^{-1}, in the last expression, we have $a_1 a_1^{-1} X^{-1} = X^{-1}$, hence

$$[X^{-1}] \qquad (8)$$

□