# 1 Rigidity Conjecture

**Proposition 1** (Division)**.** A formal power series $f(X) = \sum_{k\geq 1} a_k X^k \in \mathbb{C}[[X]]$ is invertible if and only if its constant coefficient $a_0$ is nonzero.

**Lemma 2.** Given two formal power series $f(X) = \sum_{n\geq 0} a_k X^k \in \mathbb{C}[[X]]$ and $g(X) = \sum_{n\geq 0} b_k X^k \in \mathbb{C}[[X]]$ with $b_0 \neq 0$, we may compute their quotient

$$\frac{\sum_{n\geq 0} a_k X^k}{\sum_{n\geq 0} b_k X^k} = \sum_{n\geq 0} c_k X^k \tag{1}$$

by

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{k\geq 1} b_k c_{n-k}\right). \tag{2}$$

*Proof.* #MISSING □

**Remark.** When studying compositions of formal power series, we require that the inner power series $f(X)$ has no constant term, i.e., $f(0) = 0$. This condition ensures that the resulting composition is well-defined in the ring of formal power series $\mathbb{C}[[X]]$, as it prevents infinite contributions to the coefficients.

Consider $f(X) = \sum_{k\geq 1} a_k X^k$ and $g(X) = \sum_{k\geq 0} b_k X^k$. The composition $g(f(X))$ is given by substituting $f(X)$ into $g(X)$:

$$\begin{aligned} g(f(X)) &= b_0 + b_1 f(X) + b_2 f(X)^2 + \cdots \\ &= b_0 + b_1(a_1 X + a_2 X^2 + \cdots) + b_2(a_1 X + a_2 X^2 + \cdots)^2 + \cdots \\ &= b_0 + b_1 a_1 X + (b_1 a_2 + b_2 a_1^2) X^2 + \cdots, \end{aligned}$$

where we grouped the terms by powers of $X$ in the last step. We observe that the coefficients of $X^n$ in $g(f(X))$ depend only on a finite number of coefficients of $f(X)$ and $g(X)$. This is because, with $f(0) = 0$, each power $f(X)^k$ introduces terms of degree at least $k$, ensuring that lower-degree terms do not contribute infinitely to higher-order coefficients.

On the other hand, if $f(0) \neq 0$, we write $f(X) = a_0 + \sum_{k\geq 1} a_k X^k$, where $a_0 = f(0)$. In this case,

$$f(X)^k = (a_0 + a_1 X + a_2 X^2 + \cdots)^k$$

produces a constant term $a_0^k \neq 0$. Consequently, the constant term of $g(f(X))$ depends on infinitely many terms of $g(X)$, and the composition $g(f(X))$ is no longer a formal power series.

Since we are interested in the compositional inverse, it is necessary to extend the condition $f(0) = 0$ to both power series. This ensures that the inverse series $f^{-1}(X)$, when substituted into $f(X)$, results in the identity series $X$, with no contributions from constant terms that would otherwise make the series ill-defined.

The following proposition and lemma are taken from Enumerative Combinatorics by Richard P. Stanley and Sergey Fomin.

**Definition 3.** Let $f(X) \in \mathbb{C}[[X]]$ be a power series with no constant term. We call a power series $f^{-1}(X) \in \mathbb{C}[[X]]$ the compositional inverse of $f$, if it satisfies $f(f^{-1}(X)) = f^{-1}(f(X)) = X$.

**Proposition 4.** A power series $f(X) = a_1 X + a_2 X^2 + \cdots \in \mathbb{C}[[X]]$ has a compositional inverse if and only if $a_1 \neq 0$. Moreover, if the compositional inverse exists, then it is unique.

*Proof.* Assume $f$ has a compositional inverse and denote the compositional inverse by $f^{-1}(X) = b_1 X + b_2 X^2 + \cdots$. Writing out $f(f^{-1}(X)) = X$ using multinomial thoerem gives

$$\begin{aligned} X &= a_1(b_1 X + b_2 X^2 + \cdots) + a_2(b_1 X + b_2 X^2 + \cdots)^2 + \cdots \\ &= (a_1 b_1 X + a_1 b_2 X^2 + a_1 b_3 X^3 + \cdots) + (a_2 b_1^2 X^2 + 2a_2 b_1 b_2 X^3 + \cdots) \\ &= (a_1 b_1) X + (a_1 b_2 + a_2 b_1^2) X^2 + (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) X^3 \cdots. \end{aligned}$$

Equating the coefficients on both sides yields a linear system of equations.

$$\begin{aligned}
1 &= a_1 b_1 \\
0 &= a_1 b_2 + a_2 b_1^2 \\
0 &= a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 \\
&\vdots
\end{aligned}$$

The first equation has a solution if and only if $a_1 \neq 0$. In that case, the solution is unique. Then, the second equation can be solved uniquely for $b_2$. By this process, we are able to solve the third equation for $b_3$, the fourth for $b_4$ and so on. Thus, $f^{-1}(X)$ exists if and only if $a_1 \neq 0$ and in that case, $f^{-1}(X)$ is unique. □

**Lemma 5** (Lagrange Inversion Formula)**.** Let $f(X) = \sum_{k\geq 1} a_k X^k \in \mathbb{C}[[X]]$ be a power series with $a_1 \neq 0$ and denote its composition inverse by $f^{-1}(X) = \sum_{k\geq 1} b_k X^k \in \mathbb{C}[[X]]$. The coefficients of the inverse is given by the following formula.

$$b_k = \frac{1}{k}[X^{n-1}]\left(\frac{X}{f(X)}\right)^k$$

*Proof.* We begin by substituting $f(X)$ into $f^{-1}(X)$. It is

$$X = f^{-1}(f(X)) = \sum_{k\geq 1} b_k f(X)^k.$$

Differentiating and subsequently taking the quotient with $f(X)^n$ for $n \in \mathbb{N}$ on both sides yields

$$\begin{aligned}
1 &= \sum_{k\geq 1} k \cdot b_k \cdot f(X)^{k-1} \cdot f'(X) \\
\Rightarrow \qquad \frac{1}{f(X)^n} &= \sum_{k\geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X).
\end{aligned}$$

We want to take the coefficient of $X^{-1}$ on both sides. For that, first notice that for $k \neq n$ it is

$$\frac{1}{k-n}\frac{d}{dX} f(X)^{k-n} = f(X)^{k-n-1} f'(X) = \frac{f(X)^k}{f(X)^{n+1}} f'(X).$$

For any Laurent series, its derivative has no $X^{-1}$ term. Thus, for $k \neq n$, it is

$$[X^{-1}]\frac{f(X)^k}{f(X)^{n+1}} f'(X) = [X^{-1}]\frac{1}{k-n}\frac{d}{dX} f(X)^{k-n} = 0.$$

If we now take the coefficient of $X^{-1}$ in #REFMISSING, we get

$$\begin{aligned}
[X^{-1}]\frac{1}{f(X)^n} &= [X^{-1}]\sum_{k\geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) && (3) \\
&= \sum_{k\geq 1} k \cdot b_k \cdot [X^{-1}]\frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) && (4) \\
&= n \cdot b_n [X^{-1}]\frac{f(X)^n}{f(X)^{n+1}} \cdot f'(X) && (5) \\
&= n \cdot b_n [X^{-1}]\frac{f'(X)}{f(X)} && (6) \\
&= n \cdot b_n [X^{-1}]\frac{a_1 + 2a_2 X + 3a_3 X^2 + \cdots}{a_1 X + a_2 X^2 + a_3 X^3 + \cdots} && (7) \\
&= n \cdot b_n [X^{-1}]\frac{1}{X}\frac{a_1 + 2a_2 X + 3a_3 X^2 + \cdots}{a_1 + a_2 X + a_3 X^2 + \cdots} && (8) \\
&= n \cdot b_n && (9)
\end{aligned}$$

where we used the formula for power series division given in #REFMISSING to compute the constant term of the quotient.

$$\frac{1}{a_1}(a_1 - 0) = 1 \tag{10}$$

Now, by shifting the power of the coefficient to be extracted, we get

$$[X^{-1}]\frac{1}{f(X)^n} = [X^{n-1}]\frac{X^n}{f(X)^n} = [X^{n-1}]\left(\frac{X}{f(X)}\right)^n. \tag{11}$$

Finally, continuing from #REFMIISING, we get

$$n \cdot b_n = [X^{-1}]\frac{1}{f(X)^n} = [X^{n-1}]\left(\frac{X}{f(X)}\right)^n \tag{12}$$

$$\Rightarrow \quad b_n = \frac{1}{n}[X^{n-1}]\left(\frac{X}{f(X)}\right)^n \tag{13}$$

as desired. □

**Lemma 6** (Additive Inversion Formula)**.** For some $n \in \mathbb{N}_+$, let $a(X) = X(1 - (\alpha_1 X + \cdots + \alpha_m X^m)) \in \mathbb{C}[X]$ be a polynomial. The compositional inverse is given by

$$a^{-1}(X) = X\left(1 + \sum_{n \geq 1} \frac{1}{n+1} u_n X^n\right) \tag{14}$$

where

$$u_n = \frac{1}{n!} \sum_{k_1 + 2k_2 + \cdots + mk_m = n} \frac{(n + k_1 + \cdots + k_m)!}{k_1! \cdots k_m!} \alpha_1^{k_1} \cdots \alpha_m^{k_m}. \tag{15}$$

*Proof.* We start with the expression for $u_n$ given by the Lagrange inversion formula.

$$\begin{aligned}
u_n &= [X^n]\left(\frac{X}{a(X)}\right)^{n+1} && (16)\\
&= [X^n]\left(\frac{1}{1 - (\alpha_1 X + \cdots + \alpha_m X^m)}\right)^{n+1} && (17)\\
&= [X^n]\sum_{k \geq 0}\binom{n+k}{n}(\alpha_1 X + \cdots + \alpha_m X^m)^k && (18)\\
&= [X^n]\sum_{k \geq 0}\binom{n+k}{n}\sum_{k_1 + \cdots k_m = k}\frac{k!}{k_1! \cdots k_m!}(\alpha_1 X)^{k_1} \cdots (\alpha_m X^m)^{k_m} && (19)\\
&= \frac{1}{n!}[X^n]\sum_{k \geq 0}(n+k)! \sum_{k_1 + \cdots k_m = k}\frac{1}{k_1! \cdots k_m!}\alpha_1^{k_1} \cdots \alpha_m^{k_m} X^{k_1 + \cdots mk_m} && (20)\\
&= \frac{1}{n!}\sum_{k_1 + 2k_2 + \cdots + mk_m = n}\frac{(n + k_1 + \cdots + k_m)!}{k_1! \cdots k_m!}\alpha_1^{k_1} \cdots \alpha_m^{k_m} && (21)
\end{aligned}$$

We have substituted $a(X)$ in (17), expanded the fraction using the binominal series formula for $(1-z)^{-(n+1)}$ (see Concrete Mathematics 5.56) in (18), used the multinominal theorem in (19), and finally collected terms for $[X^n]$ in (21). □