Series 1

K

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Problem 1

Suppose (X, d_X) is a metric space and $\sim$ is an equivalence relation on X, with the set of equivalence classes denoted by $X/\!\sim$. For the equivalence classes $[x], [y] \in X/\!\sim$ represented by elements $x, y \in X$, define

$$d([x],[y]) := \inf \{\, d_X(x,y) \mid x \in [x],\; y \in [y] \,\}$$

(a)

Show that d is a metric on $X/\!\sim$ if the following assumption is added: for every triple $[x], [y], [z] \in X/\!\sim$, there exist representatives $x \in [x], y \in [y]$ and $z \in [z]$ such that

$$d_X(x,y) = d([x],[y]) \quad \text{and} \quad d_X(y,z) = d([y],[z]).$$

Solution. Clearly, d is a map from $X/\!\sim \times X/\!\sim \longrightarrow$ to $\mathbb{R}$. We verify the axioms of a metric one by one.

1. "$d([x],[y]) \geq 0$": This is simply inherited from d_X.
2. "$d([x],[x]) = 0$": Again, inherited from d_X.
3. "$d([x],[y]) = d([y],[x])$": Also inherited from d_X.
4. "$d([x],[z]) \leq d(x,y) + d(y,z)$":
5. "$d([x],[y]) > 0$ whenever $x \neq y$": By the added assumption, we are guranteed that the value $d([x],[y])$ is taken by some $x \in [x]$ and $y \in [y]$. In other words, $d([x],[y])$ is not just an infimum of $d_X(x,y)$, but also a minimum. Thus, this property is also inherited by d_X.

$\square$