

# Commutative Ring Theory

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# Contents

1	Introduction and Motivation	5
2	Metric Spaces	7
3	Topological Spaces	9
4	Products, Sequential Continuity, and Nets	11
5	Compactness	13



## Chapter 1

# Introduction and Motivation



## Chapter 2

# Metric Spaces

**Definition 1** (Definition 2.1). Metric Space. Metric.

**Definition 2** (Pseudometric Space). Unlike a metric space, points in a pseudometric space need not be distinguishable; that is, one may have  $d(x, y) = 0$  for distinct values  $x \neq y$ .

**Definition 3** (Open Ball). In any metric space  $(X, d)$ , one can define the open ball of radius  $r > 0$  about a given point  $x \in X$  as

$$B_r(x) := \{ y \in X \mid d(x, y) < r \}$$

**Intuition.** The axiom of triangle inequality in the definition of a metric might seem arbitrary. But it is needed to have for example two desirable properties.

1. Open balls are open themselves.
2. The function  $d : X \times X \rightarrow [0, \infty)$  is continuous.

*Proof.* 1. Let  $(X, d)$  be a metric space and  $B_r(x)$  be an open ball. For any  $y \in B_r(x)$  choose  $0 < \epsilon < r - d(x, y)$ .

□

**Intuition.** Without the fifth axiom in the definition of a metric, a singleton  $\{x\}$  need not be closed.

*Proof.* Let  $(X, d)$  be a metric space and  $x \in X$  a point. Consider the complement  $X \setminus \{x\}$ . We want to show  $X \setminus \{x\}$  is open. Indeed, for any  $y \in X \setminus \{x\}$  and for all  $\epsilon < d(x, y)$  the ball  $B_\epsilon(y)$  is contained in  $X \setminus \{x\}$ .

□

**Example 3.1** (Pseudometric).

**Example 3.2.** Show that for the pseudometric space  $X$  in Example 2.2,  $[(0, 0)] \subset X$  is not a closed subset.

*Solution.* Assume  $\{(0,0)\} \subset X$  is a closed subset. Then,  $X \setminus \{(0,0)\}$  must be open, that is, every point in  $X \setminus \{(0,0)\}$  has an open ball centered around it that is contained in  $X \setminus \{(0,0)\}$ . Consider the point  $[(0,1)]$ . We have that  $d([0,0], [0,1]) = 0$ , thus  $X \setminus \{(0,0)\}$  cannot be open.  $\square$

**Definition 4** (Definition 2.4). Convergence of a sequence.

**Definition 5** (Definition 2.5). For two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a map  $f : X \rightarrow Y$  is called continuous if it satisfies any of the following equivalent conditions:

1. epsilon delta
2. For every open subset  $\mathcal{U} \subset Y$ , the preimage

$$f^{-1}(\mathcal{U}) := \{x \in X \mid f(x) \in \mathcal{U}\}$$

is an open subset.

3. For every convergent sequence  $x_n \in X$ ,  $x_n \rightarrow x$  implies  $f(x_n) \rightarrow f(x)$ .

*Proof.* “2.  $\Rightarrow$  3.”: Let  $x_n \rightarrow x$  be a sequence and  $\mathcal{U}$  a neighbourhood of  $f(x)$ .  $\square$

**Remark.** The direction 3. to 2. requires the metric spaces because 2. and 3. are not equivalent in arbitrary topological spaces.

**Definition 6** (Homeomorphism).

**Example 6.1.** Any open ball in  $\mathbb{R}^n$  with the standard Euclidean metric  $d_E$  is homeomorphic to  $(\mathbb{R}^n, d_E)$ . It follows that any open ball in  $\mathbb{R}^n$  are homeomorphic to each other.

*Proof.* We begin by constructing a homeomorphism. Choose any continuous, increasing, bijective function  $f : [0, r) \rightarrow [0, \infty)$  and define  $F : B_r(x) \rightarrow \mathbb{R}^n$  by

$$F(\mathbf{x}) = \mathbf{x} \text{ and } F(\mathbf{x} + \mathbf{y}) = \mathbf{x} + f(|\mathbf{y}|) \frac{\mathbf{y}}{|\mathbf{y}|} \text{ for all } \mathbf{y} \in B_r(0) \setminus \{0\} \subset \mathbb{R}^n$$

$F$  and  $F^{-1}$  is continuous and bijective.  $\square$

**Definition 7** (Compactness).

**Theorem 8.** If  $f : X \rightarrow Y$  is continuous and  $K \subset X$  is compact, then so is  $f(K) \subset Y$ .

**Intuition.** The above theorem means that compactness is a topologically invariant property.

*Proof.* Let  $f : X \rightarrow Y$  be continuous,  $K \subset X$  compact, and  $\{\mathcal{U}_\alpha\}_{\alpha \in I}$  be a open cover of  $f(K)$ . Since  $f$  is continuous,  $f^{-1}(\mathcal{U}_\alpha)$  is open for all  $\alpha \in I$ . Thus,  $\{f^{-1}(\mathcal{U}_\alpha)\}_{\alpha \in I}$  is an open cover of  $K$ .  $K$  was compact, so this open cover has a finite subcover  $\{f^{-1}(\mathcal{U}_\alpha)\}_{\alpha \in J}$  with  $J \subset I$  finite. Then  $\{\mathcal{U}_\alpha\}_{\alpha \in J}$  is a finite subcover of  $f(K)$ .  $\square$



## Chapter 3

# Topological Spaces



## Chapter 4

# Products, Sequential Continuity, and Nets

**Lemma 9** (Lemma 4.15). In any space  $X$ , a subset  $A \subset X$  is open if and only if every point  $x \in A$  has a neighbourhood  $\mathcal{V} \subset X$  that is contained in  $A$ .

*Proof.* “ $\Rightarrow$ ”: If  $A$  is open, then  $A$  itself can be taken as the desired neighbourhood of every  $x \in A$ . “ $\Leftarrow$ ”: Let every point  $x \in A$  have a neighbourhood  $\mathcal{V} \subset X$  that is contained in  $A$ . Denote the open sets of these neighbourhoods by  $\mathcal{U}_x$ . Then,  $A$  is the union of all these open sets  $\mathcal{U}_x$  and thus open.  $\square$

**Lemma 10** (Lemma 4.16). In any first-countable topological space  $X$ , a subspace  $A \subset X$  is not open if and only if there exists a point  $x \in A$  and a sequence  $x_n \in X \setminus A$  such that  $x_n \rightarrow x$ .

*Proof.* “ $\Leftarrow$ ”: (Proof by contraposition.) If  $A \subset X$  is open, then for every  $x \in A$  and sequence  $x_n \in X$  converging to  $x$ , we cannot have  $x_n \in X \setminus A$  for all  $n$  since  $A$  is a neighbourhood of  $x$ . This is true so far for all topological spaces, with or without first-countability axiom, but the latter will be needed to prove the converse.

“ $\Rightarrow$ ”: So suppose now that  $A \subset X$  is not open, which by Lemma 4.15, means there exists a point  $x \in A$  such that no neighbourhood  $\mathcal{V} \subset X$  of  $x$  is contained in  $A$ . Fix a countable neighbourhood base  $\mathcal{U}_1, \mathcal{U}_2, \dots$  for  $x$ . XXX

Observe that since none of the neighbourhoods  $\mathcal{U}_n$  can be contained in  $A$ , there exists a sequence of points

$$x_n \in \mathcal{U}_n \text{ such that } x_n \notin A.$$

This sequence converges to  $x$  since every neighbourhood  $\mathcal{V} \subset X$  of  $x$  contains one of  $\mathcal{U}_N$ , implying that for all  $n \geq N$ ,

$$x_n \in \mathcal{U}_n \subset \mathcal{U}_N \subset \mathcal{V}.$$

$\square$

**Definition 11.** A **directed set**  $(I, \prec)$  consists of a set  $I$  with a partial order  $\prec$  such that for every pair  $\alpha, \beta \in I$ , there exists an element  $\gamma \in I$  with  $\gamma \prec \alpha$  and  $\gamma \prec \beta$ .

**Definition 12.** Given a space  $X$ , a net  $\{x_\alpha\}_{\alpha \in I}$  in  $X$  is a function  $I \rightarrow X : \alpha \mapsto x_\alpha$  where  $(I, \prec)$  is a directed set.



## Chapter 5

# Compactness

**Definition 13.** A subset  $A \subset X$  is compact if every open cover of  $A$  has a finite subcover, i.e. given an arbitrary open cover  $\{\mathcal{U}_\alpha\}_{\alpha \in I}$  of  $A$ , one can always find a finite subset  $\{\alpha_1, \dots, \alpha_N\} \subset I$  such that  $A \subset \mathcal{U}_{\alpha_1} \cup \dots \cup \mathcal{U}_{\alpha_N}$ . We say that  $X$  itself is a compact space if  $X$  is compact subset of itself.