Definition 1. Let B a ring, A a subring of B.

1. An element x of B is said to be integral over A if x is a root of a monic polynomial with coefficients in A, that is if x satisfies an equation of the form
$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$
where a_i are the elements of A.
2. The set of elements of B that are integral over A is called the integral closure of A in B. The integral closure is, itself, a subring of B and contais A.
3. If each element of A is integral over itself, then A is said to be integrally closed in B.
4. If the integral closure of A in B is whole B, we say the ring B is integral over A.
5. An integral domain is said to be integrally closed (without qualification) if it is integrally closed in its field of fractions.
6. An element of B is said to be integral over $\mathfrak{a}$ if it satisfies an equation of integral dependence over A in which all the coefficients lie in $\mathfrak{a}$.
7. The integral closure of $\mathfrak{a}$ in B is the set of all elements of B which are integral over $\mathfrak{a}$.

Theorem 2. The following are equivalent:

1. $x \in B$ is integral over A
2. $A[x]$ is a finitely generated A-module
3. $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module
4. There exsists a faithful $A[x]$-module M which is finitely generated as an A-module

Proof. 1.

□

Theorem 3. Let x_i be elements of B, each integral over A. Then the ring $A[x_1, \ldots, x_n]$ is finitely generated A-module.

Theorem 4. The set C of elements of B which are integral over A is a subring of B containing A.

Definition 5. 1. The ring C is called the integral closure of A in B.

2. If $C = A$, then A is said to be integrally closed in B.
3. If $C = B$ the ring B is said to be integral over A.

Theorem 6. If $A \subset B \subset C$ are rings and if B is integral over A, and C is integral over B, then C is integral over A.

Theorem 7. Let $A \subset B$ be rings and let C be the integral closure of A in B. Then C is integrally closed in B.

Theorem 8. Let $A \subset B$ be rings, B integral over A.

1. If $\mathfrak{b}$ is an ideal of B and $\mathfrak{a} = \mathfrak{b}^c = A \cap \mathfrak{b}$ then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.
2. If S is a multiplicatively closed subset of A, then $S^{-1}B$ is integral over $S^{-1}A$.

Theorem 9. Let $A \subset B$ be integral domains, B integral over A. Then B is a field if and only if A is a field.

Theorem 10. Let $A \subset B$ be rings, B integral over A, let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q}^c = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.

Theorem 11. Let $A \subset B$ be rings, B integral over A, let $\mathfrak{q}, \mathfrak{q}'$ be prime ideals of B such that $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q}^c = \mathfrak{q}'^c$. Then $\mathfrak{q} = \mathfrak{q}'$.

Theorem 12. Let $A \subset B$ be rings, B integral over A, and let $\mathfrak{p}$ be a prime ideal of A. Then there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$.

Theorem 13. Let $A \subset B$ be rings, B integral over B, let $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{q} \subset \cdots \subset \mathfrak{q}_m$ $(m < n)$ a chain of prime ideals of B such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ $(1 \leq i \leq m)$. Then the chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m$ can be extended to a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $(1 \leq i \leq n)$.