

Chapter 1

Rings

Example 1

1. $(\mathbb{Z}, +, \cdot)$
2. All fields, such as $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, and $(\mathbb{C}, +, \cdot)$, are rings.
3. Let R be a commutative ring, then $R[X]$, the set of polynomials with coefficients in R , is again a ring, e.g. $\mathbb{Z}[X]$, $\mathbb{Q}[X]$, and $\mathbb{R}[X]$.
4. For any ring R and for any $n \in \mathbb{N}$, the set of all square n -by- n matrices with entries from R , forms a ring with matrix addition and matrix multiplication as operations. If $n = 1$, this matrix ring is isomorphic to R itself. For $n > 1$ (and R not a zero ring), this matrix is noncommutative. More concretely, $\text{Mat}_{3 \times 3}(\mathbb{R})$ is a noncommutative ring.

1.1 No idea yet

Definition 1 (Fractional Ideal)

Let A be an integral domain.

1. A fractional ideal of A is an A -submodule $I \subset \text{Quot}(A)$ such that $dI \subset A$ for some denominator $d \in A \setminus \{0\}$.
2. A principal fractional ideal is a fractional ideal of the form $(r) = rA = \{ar \mid a \in A\}$

Example 2

- All ordinary ideals $I \subset A$ are also fractional ideals with denominator $d = 1$, and are often referred to as integral ideals.

- The subset

$$\frac{3}{25}\mathbb{Z} = \left\{ \frac{3n}{25} \in \mathbb{Q} \mid n \in \mathbb{Z} \right\} \subset \mathbb{Q} \quad (1.1)$$

is a principal fractional ideal of \mathbb{Z}

Example 3

The subset

$$\mathbb{Z} \left[\frac{1}{2} \right] = \left\{ a_0 + a_1 \frac{1}{2} + a_2 \frac{1}{2^2} + \cdots + a_n \frac{1}{2^n} \mid a_0, \dots, a_n \in \mathbb{Z} \subset \mathbb{Q} \right\} \quad (1.2)$$

is not a fractional ideal, because the denominators are not bounded.

Lemma 1.1 If $I \subset \text{Quot}(A)$ is an A -submodule and $d \in \text{Quot}(A)$, then $dI \subset \text{Quot}(A)$ is also an A -module. Thus $I \subset K$ is a fractional ideal if and only if $I = \frac{1}{d}J$ for some $d \in A \setminus \{0\}$ and some integral ideal $J \subset A$ (just take d a denominator of I and $J = dI$).

Lemma 1.2 Let A be an integral domain and denote its field of fraction with $\text{Quot}(A) = K$.

1. If $I \subset K$ is a finitely generated A -submodule, then I is a fractional ideal.
2. Conversely, if A is noetherian and $I \subset K$ is a fractional ideal, then I is a finitely generated A -module.
3. If $I, J \subset K$ are fractional ideals, then $I \cap J, I + J, IJ, \subset K$ are also fractional ideals.
4. If $I, J \subset K$ are fractional ideals and $J \neq 0$, then the generalized ideal quotient

$$(I : J) := \{x \in K \mid xJ \subset I\} \quad (1.3)$$

is also a fractional ideal. Moreover, it satisfies $(I : J)J \subset I$.

The nonzero fractional ideals form an abelian semigroup with neutral element A with respect to the multiplication. We will now show that, if A is a Dedekind domain, every nonzero fractional ideal has an inverse hence they form an abelian group $\text{Id}(A)$.

Definition 2

Let A be an integral domain. A fractional ideal $I \subset K$ is invertible if $IJ = A$ for some fractional ideal J called the inverse of I .

The following result shows characterizes invertible fractional ideals and their inverses (which are unique).

Lemma 2.1 A fractional ideal I is invertible if and only if $I(A : I) = A$, in which case $I^{-1} := (A : I)$ is the unique inverse.

The main result of this section is to prove that, in a Dedekind domain, every nonzero ideal is invertible. To this aim we need first a technical result.

Lemma 2.2 Let A be a Dedekind domain and $I \subset A$ a nonzero integral ideal. Then there are not necessarily distinct nonzero prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset A$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subset I$.