1 Rigidity Conjecture

Remark. When studying compositions of formal power series, we require that the inner power series $f(X)$ to have no constant term, i.e., $f(0) = 0$.

$$\begin{aligned}
f(g(X)) &= a_0 + a_1(b_0 + b_1 X + b_2 X^2 + \cdots) + a_2(b_0 + b_1 X + b_2 X^2 + \cdots)^2 + \cdots \\
&= a_0 + (a_1 b_0 + a_1 b_1 X + a_1 b_2 X^2 + \cdots) + (a_2 b_0^2 + 2a_2 b_0 b_1 X + (2a_2 b_0 b_2 + a_2 b_1^2) X^2 + \cdots) \\
&= a_0 + a_1 b_0 + a_2 b_0^2 + \cdots
\end{aligned}$$

Definition 1. Let $f(X) \in \mathbb{C}[[X]]$ be a power series. We call a power series $f^{-1}(X) \in \mathbb{C}[[X]]$ the compositional inverse of f, if it satisfies $f(f^{-1}(X)) = f^{-1}(f(X)) = X$.

Proposition 2. A power series $f(X) = a_0 + a_1 X + \cdots \in \mathbb{C}[[X]]$ has a compositional inverse if and only if $a_0 = 0$ and $a_1 \neq 0$. Moreover, if the compositional inverse exists, then it is unique.

Proof. Assume f has a compositional inverse and denote the compositional inverse by $f^{-1}(X) = b_0 + b_1 X + b_2 X^2 + \cdots$. Writing out $f(f^{-1}(X)) = X$ using multinomial thoerem gives

$$\begin{aligned}
X &= a_0 + a_1(b_0 + b_1 X + b_2 X^2 + \cdots) + a_2(b_0 + b_1 X + b_2 X^2 + \cdots)^2 + \cdots \\
&= a_0 + (a_1 b_0 + a_1 b_1 X + a_1 b_2 X^2 + \cdots) + (a_2 b_0^2 + 2a_2 b_0 b_1 X + \cdots).
\end{aligned}$$

Equating the coefficients on both sides yields a linear system of equations.

$$\begin{aligned}
0 &= a_0 + a_1 b_0 + a_2 b_0^2 + \cdots \\
1 &= a_1 b_1 +
\end{aligned}$$

□