

My notes on "The Strong Factorial Conjecture" by Eric Edo and Arno van den Essen.  
 See: <https://arxiv.org/abs/1304.3956>

## 1 Factorial Conjecture

$\mathbb{C}^{[m]}$  may be viewed as a vector space over  $\mathbb{C}$  with a basis being

$$\left\{ X_1^{l_1} \cdots X_m^{l_m} \mid l_k \in \mathbb{N}_0 \text{ for all } 1 \leq k \leq m \right\}.$$

Thus any linear map is fully defined if we set a value for each basis element. Such linear map is the factorial map.

**Definition 1.** A factorial map is a linear map  $\mathcal{L} : \mathbb{C}^{[m]} \rightarrow \mathbb{C}$  defined by

$$\mathcal{L}(X_1^{l_1} \cdots X_m^{l_m}) = l_1! \cdots l_m! \quad \text{for all } l_1, \dots, l_m \in \mathbb{N}$$

**Example 1.1.** Consider  $f(X) = 3X - 5XY + 7Y^2 \in \mathbb{C}^{[2]}$ . Applying the factorial map yields

$$\begin{aligned} \mathcal{L}(f(X)) &= 3\mathcal{L}(X) - 5\mathcal{L}(XY) + 7\mathcal{L}(Y^2) \\ &= 3 \cdot 1 - 5 \cdot 1 + 7 \cdot 2 \\ &= 12. \end{aligned}$$

**Example 1.2.** If we limit our selves to a polynomial with one indeterminate, such as  $f(X) = \sum_{k=0}^n a_k X^k \in \mathbb{C}[X]$  for a fixed  $n \in \mathbb{N}_0$ , we have

$$\mathcal{L}(f(X)) = \sum_{k=0}^n a_k \mathcal{L}(X^k) = \sum_{k=0}^n a_k k!$$

**Theorem 2** (Conjecture 2.4). Let  $f \in \mathbb{C}^{[m]}$  be a polynomial. We have  $\mathcal{L}(f^k) = 0$  for all  $k \in \mathbb{N}_+$  if and only if  $f = 0$ .

**Remark.** The converse is trivial, hence the conjecture is about the forward implication.

## 2 Rigidity Conjecture

When we consider compositions of formal power series, we always want the constant term to be 0.

The following example is taken from:

<https://math.stackexchange.com/questions/1212053/defining-composition-of-two-formal-series-what-is-going-on>

**Example 2.1.** Let  $f = \sum_{k \in \mathbb{N}_0} a_k X^k$  and  $g = 1 + X$ . Consider  $f \circ g$ . We have

$$\begin{aligned} f \circ g &= \sum_{k \in \mathbb{N}_0} a_k (1 + X)^k \\ &= a_0 + a_1 + a_1 X + a_2 + 2a_2 X + a_2 X^2 + \dots \end{aligned}$$

If  $f \circ g$  is again a formal power series, then we should be able to write  $f \circ g = \sum_{k \in \mathbb{N}_+} c_k X^k$  for some  $c_k \in \mathbb{C}$ . However, we see that  $c_0$  is the sum of all  $a_k$  and we cannot evaluate that as algebraists. Thus composition of formal power series only makes sense if the constant coefficient is 0.

**Proposition 3.** A power series  $f(X) = \sum_{k \in \mathbb{N}_+} a_k X^k \in \mathbb{C}[[X]]$  has a compositional inverse  $f^{-1}(X)$  if and only if  $a_1 \neq 0$ , in which case  $f^{-1}(X)$  is unique.

*Proof.* Let  $g(X) = b_1 X + b_2 X^2 + \dots \in \mathbb{C}[[X]]$  be some power series that satisfies  $f(g(X)) = X$ . We have

$$X = a_1(b_0 + b_1 X + b_2 X^2 + \dots) + a_2(b_0 + b_1 X + b_2 X^2 + \dots)^2 + \dots \quad (1)$$

$$= a_1 b_0 + a_2 b_0^2 + \dots + \quad (2)$$

$$a_1 b_1 X \quad (3)$$

□

**Theorem 4** (Conjecture 2.13). Let  $a(X) \in \mathbb{C}[X]$  be a polynomial of degree less or equal to  $m + 1 \in \mathbb{N}_+$  such that  $a(X) \equiv X \pmod{X^2}$ . If  $m$  consecutive coefficient of the compositional inverse  $a^{-1}(X)$  vanish, i.e.  $b_{n+1} = b_{n+2} = \dots = b_{n+m} = 0$  for some  $n \in \mathbb{N}_+$  then  $a(X) = X$ .

**Theorem 5** (Conjecture 2.14). Let  $a(X) \in \mathbb{C}[X]$  be a polynomial of degree less or equal to  $m + 1 \in \mathbb{N}_+$  such that  $a(X) \equiv X \pmod{X^2}$ . If the coefficients of  $X^{n+1}, \dots, X^{n+m}$  of the compositional inverse vanish, then  $a(X) = X$ .

**Remark.**  $R(m)$  if and only if  $R(m)_n$  for all  $n \in \mathbb{N}_+$ .

*Proof.* Let  $R(m)$  be true for a  $m \in \mathbb{N}_0$ .

Then  $R(m)_1$  is true, i.e. if  $\deg(a) \leq m + 1$  and if the

□

**Remark.** If we denote the polynomial  $a(X)$  by  $\sum_{k \in \mathbb{N}_0} a_k X^k$  for some  $a_k \in \mathbb{C}$  for all  $k \in \mathbb{N}_0$ , then the condition  $a(X) \equiv X \pmod{X^2}$  amounts to  $a_0 = 0$  and  $a_1 = 1$ .

Moreover, we have this:

A power series has a compositional inverse if and only if  $a_1 \neq 0$ . In that case, the inverse is unique.

See

<https://www.amazon.com/dp/B00HMUGS4S>

<https://math.stackexchange.com/questions/2520744/finding-compositional-inverses-for-formal-power-series>

My questions:

1. What if  $a_0 \neq 0$ ? Pick  $a_0 = 3$ .

Let  $f \in \mathbb{C}[X]$  be a polynomial with  $a_0 \neq 0$ . Then we may write  $f(X) = g(X) + a_0$  where  $g$  has a compositional inverse. Thus it is

$$\begin{aligned} g^{-1}(g(X) + a_0) &= g^{-1}(g(X)) + g^{-1}(a_0) \\ &= X + g^{-1}(a_0) \end{aligned}$$

$$\begin{aligned}
h(X) &= g^{-1}(X) + g^{-1}(a_0) \\
h(f(X)) &= h(g(X) + a_0) \\
&= g^{-1}(g(X) + a_0) + g^{-1}(a_0) \\
&= X
\end{aligned}$$

Let  $f \in \mathbb{C}[X]$  be a polynomial with  $a_1 \neq 1$  and  $a_1 \neq 0$ . Then we may write  $f(X) =$

<https://www.math.uwaterloo.ca/dgwagner/co430I.pdf>  
proof

**Proposition 6.** 1. The polynomial  $a(X)$  is invertible for the composition.

2. For all  $i \in \{1, \dots, \deg(a-1)\}$ , the coefficient  $a_i$  is nilpotent element in  $A$ . I just don't see this ...

**Lemma 7** (Lagrange Inversion Formula). Let  $K$  be a field of characteristic

**Example 7.1** (See 5.4.4).  $f(X) = Xe^{-X} = X \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} X^k$

$$[X^n]f^{-1}(X) = \frac{1}{n}[X^{n-1}]e^{nX}$$

**Lemma 8** (Lemma 2.20 (Additive Inversion Formula)). Let  $\alpha_1, \dots, \alpha_m \in \mathbb{C}$  be complex numbers. The formal inverse of  $a(X) = X(1 - (\alpha_1 X + \dots + \alpha_m X^m))$  is given by the following formula

$$a^{-1}(X) = X \left( 1 + \frac{1}{n+1} \sum_{n \geq 1} u_n X^n \right)$$

where

$$u_n = \frac{1}{n!} \sum_{j_1+2j_2+\dots+mj_m=n} \frac{(n+j_1+\dots+j_m)!}{j_1! \dots j_m!} \alpha_1^{j_1} \dots \alpha_m^{j_m}$$

**Proposition 9** (Proposition 2.23). Let  $\alpha_1, \dots, \alpha_m \in \mathbb{C}$  be complex numbers and let  $(u_n)_{n \in \mathbb{N}_+}$  be a sequence defined by AIF in Lemma 2.20. For all  $n \in \mathbb{N}_+$ , the Rigidity Conjecture  $R(m)_n$  is equivalent to the following implication: If  $u_n = \dots = u_{n+m-1} = 0$  then  $\alpha_1 = \dots = \alpha_m = 0$ .

*Proof.*

□

**Theorem 10.** 1. The inclusion  $E^{[m]} \subset F_n^{[m]}$  implies  $R(m)_n$

**Definition 11.**

$$\begin{aligned}
E^{[m]} &= \{ X_1 \cdots X_m (\mu_1 X_1 + \dots + \mu_m X_m) \mid \mu_1, \dots, \mu_m \in \mathbb{C} \} \subset \\
F_n^{[m]} &= \left\{ f \in \mathbb{C}^{[m]} \setminus \{0\} \mid \mathcal{L}(f^k) \neq 0 \text{ for some } n \leq k \leq \mathcal{N}(f) - 1 \right\} \cup \{0\}
\end{aligned}$$