Topology

K

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Contents

Conventions

$\mathbb{N}$ contains 0, that is $\mathbb{N} = \{ 0, 1, 2, \ldots \}$.

Chapter 1

Topological Space

1.1 Cheat Sheet

1.2 Proofs, Remarks, and Examples

1.2.1 Think of a Title

Definition 1 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to $\mathcal{O}$.
2. Any **arbitary** union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The intersection of **finite number** of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The collection $\mathcal{O}$ is called a topology on X and the elements of $\mathcal{O}$ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open. Moreover, a subset that is **both** open and closed is called clopen.

Remark. We often just write X instead of $(X, \mathcal{O})$ if the given topology is clear.

Remark.

1. One might believe that openness and closedness are mutually exclusive, but this is not true, i.e. there are indeed subsets that are both open and closed. Trivial examples include the emptyset $\varnothing$ and X itself.
2. Furthermore, a subset may neither be open nor closed. If $X = \{1, 2, 3\}$ is a set, then $(X, \{\varnothing, X\})$ is a well-defined topological space and the subsets $\{1\}$ and $\{2, 3\}$ are neither open nor closed.

Remark. The definition of a topological space does not require X to be nonempty. If X is the emptyset, then the only topology that can be defined on X is $\mathcal{O} = \{\varnothing\}$.

Example 1.1. Let $X = \{1, 2, 3\}$ be a set with three elements. The power set of X is

$$\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$$

and has eight elements. There are $2^8 = 256$ subsets of $\mathcal{P}(X)$[1].

[1] The number of subsets of any finite set is 2^n where n is the number of elements because for each element one can choose this element to be inside the subset or not.

1. There are no singleton topologies because the first axiom of a topology requires at least two subsets. Thus, the only topology on X with two elements is $\{\varnothing, X\}$.

2. Any collection of three subsets of X is a well-defined topology as long as it contains the emptyset and the entire set X. Therefore, there are six topologies on X containing exactly three elements.

$$\begin{array}{ll} \{\varnothing, \{1\}, X\}, & \{\varnothing, \{2,3\}, X\}, \\ \{\varnothing, \{2\}, X\}, & \{\varnothing, \{1,3\}, X\}, \\ \{\varnothing, \{3\}, X\}, & \{\varnothing, \{1,2\}, X\}. \end{array}$$

   The collection of closed subsets for each of the topologies above corresponds to the other topology in the same row, i.e. $\{\varnothing, \{2,3\}, X\}$ is the collection of closed subsets for the topology $\{\varnothing, \{1\}, X\}$.

   Nonexamples of a topology include $\{\{1\}, \{2\}, X\}$, $\{\varnothing, \{1\}, \{2,3\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ because these already fail to fulfill the first axiom.

3. Since any topology must include the union of members, the topologies on X with four elements contain a subset that is included in the other one that is not X.

$$\begin{array}{ll} \{\varnothing, \{1\}, \{1,2\}, X\}, & \{\varnothing, \{3\}, \{2,3\}, X\}, \\ \{\varnothing, \{1\}, \{1,3\}, X\}, & \{\varnothing, \{2\}, \{2,3\}, X\}, \\ \{\varnothing, \{2\}, \{1,2\}, X\}, & \{\varnothing, \{3\}, \{1,3\}, X\}, \end{array}$$

$$\begin{array}{c} \{\varnothing, \{1\}, \{2,3\}, X\}, \\ \{\varnothing, \{2\}, \{1,3\}, X\}, \\ \{\varnothing, \{3\}, \{1,2\}, X\} \end{array}$$

   Again, the collection of closed subsets for each of the topologies listed are in the same row but on the other side, i.e. $\{\varnothing, \{3\}, \{2,3\}, X\}$ is the collection of closed subset for $\{\varnothing, \{1\}, \{1,2\}, X\}$.

   However, $\{\varnothing, \{1\}, \{2\}, X\}$ and $\{\varnothing, \{1\}, \{2,3\}, X\}$ are not topologies because they do not contain the union of its members.

4. Similary, there are exactly six topologies on X that contain five elements. Those are

$$\begin{array}{ll} \{\varnothing, \{1\}, \{2\}, \{1,2\}, X\}, & \{\varnothing, \{3\}, \{1,3\}, \{2,3\}, X\}, \\ \{\varnothing, \{1\}, \{3\}, \{1,3\}, X\}, & \{\varnothing, \{2\}, \{1,2\}, \{2,3\}, X\}, \\ \{\varnothing, \{2\}, \{3\}, \{2,3\}, X\}, & \{\varnothing, \{1\}, \{1,2\}, \{1,3\}, X\}. \end{array}$$

   Note that the collection of closed subsets corresponds to the one in the same row, i.e. for $\{\varnothing, \{1\}, \{2\}, \{1,2\}, X\}$ the collection of closed subsets are $\{\varnothing, \{3\}, \{1,3\}, \{2,3\}, X\}$ and vice versa.

5. a

$$\begin{array}{lr} \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}, X\} & \{\varnothing, \{2\}, \{3\}, \{1,3\}, \{2,3\}, X\} \\ \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{2,3\}, X\} & \{\varnothing, \{1\}, \{3\}, \{1,3\}, \{2,3\}\} \\ \{\varnothing, \{2\}, \{3\}, \{1,2\}, \{2,3\}, X\} & \{\varnothing, \{1\}, \{3\}, \{1,2\}, \{1,3\}\} \end{array}$$

6. There are no topolgoies on X that contain exactly seven subsets. Take

$$\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, X\}$$

   for example. The missing elements $\{2,3\}$ and $\{1,3\}$ can be generated by $\{2\} \cup \{3\}$ and $\{1\} \cup \{3\}$.

7. Lastly, $\mathcal{P}(X)$ itself is a well-defined topology. Each subset of this topology is clopen.

We have shown that on $X = \{1, 2, 3\}$ there are 20 possible topologies. While the total amount of all possible subsets were 256, if we consider that any topology must contain the emptyset $\varnothing$ and X itself, then the number of valid collection of subsets shrinks to $2^6 = 64$. Thus, around 1/3 of sensible collection of subsets were topologies in this example.

There are two key takeaways from this example. Firstly, the collection of closed subsets for each topology were topologies themselves. Another is that the collection of all possible topologies can be partially ordered using the inclusion relation.

Proposition 2. Let $(X, \mathcal{O})$ be a topological space. The collection of all closed subsets of X with regards to $\mathcal{O}$ is a topology.

Proof. Denote $\mathcal{C}$ to be the collection of closed subsets of X.

1. Because $X \setminus \varnothing = X \in \mathcal{O}$ and $X \setminus X = \varnothing \in \mathcal{O}$, the emptyset $\varnothing$ and X are closed subsets.
2. Let I be an arbitary index set and $\{C_i\}_{i \in I}$ be a familiy of closed subsets in $\mathcal{C}$.

$\square$

Definition 3 (Definition of a Topology via Closed Sets)**.** A topology on a set X is a collection $\mathcal{O}$ of subsets of X that satisfies the following axioms.

1. The emptyset $\varnothing$ and the entire set X belong to $\mathcal{O}$.
2. Any arbitary intersection of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The union of finite number of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

In this case, the elements of $\mathcal{O}$ are called closed sets. A subset $A \subset X$ is said to be open if its complement $X \setminus A$ is open.

Definition 4 (Definition of a Topology via Neighborhoods)**.**

Proposition 5. The three definitions of a topology are equivalent.

Definition 6 (Comparison of Topologies)**.** Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on a set X such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then the topology $\mathcal{O}_1$ is said to be coarser (also weaker or smaller) than $\mathcal{O}_2$, and $\mathcal{O}_2$ is said to be finer (also stronger or larger) than $\mathcal{O}_1$. The binary relation $\subseteq$ defines a partial ordering relation on the set of all possible topologies on X.

Example 6.1. Let X be a set.

1. $\mathcal{O} = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology. It is the coarsest topology that can be defined on a set.
2. $\mathcal{O} = \mathcal{P}(X)$ is called the discrete topology. In this case, $(X, \mathcal{O})$ is called the discrete space. It is the finest topology that can be defined on a set.

Remark. The only sets in which the trivial topology and the discrete topology coincide are the emptyset and a singleton set.

Metric Space

Proposition 7. Let (X, d) be a metric space. The collection of subsets

$$\mathcal{O}_d := \{ U \subset X \mid U \text{ is a open subset in the metric space } (X, d) \}$$

defines a topology on X. In other words, a metric induces a topology.

Proof. We will show that $\mathcal{O}_d$ fullfills the axioms of a topology.

1. The emptyset $\varnothing$ is open in the metric space vacuously, hence $\varnothing \in \mathcal{O}_d$. For the entire set X, if $x \in X$, then clearly $B_\epsilon(x) \subset X$ for any $\epsilon \in \mathbb{R}^+$, therefore $X \in \mathcal{O}_d$.

2. Let $S \subset \mathcal{O}_d$ be a collection of subsets. Consider

$$x \in \bigcup_{U \in S} U,$$

then $x \in U_0$ for some set in $\mathcal{O}_d$. U_0 is open in the metric space, therefore, there is an $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon(x) \in U_0$. The ϵ-ball $B_\epsilon(x)$ is also contained in the union of the subsets in S. In other words, any arbitary union of members of $\mathcal{O}_d$ are again in $\mathcal{O}_d$.

3. Let $U, V \in \mathcal{O}_d$ and consider $x \in U \cap V$. We have that $x \in U$ and $x \in V$. Since $U, V \in \mathcal{O}_d$, they are open subsets in the metric space, hence there are $\epsilon_1, \epsilon_2 \in \mathbb{R}^+$ such that $B_{\epsilon_1}(x) \subset U$ and $B_{\epsilon_2}(x) \subset V$. Without loss of generality assume $\epsilon_1 \leq \epsilon_2$. Then, $B_{\epsilon_1}(x) \subset B_{\epsilon_2}(x)$, so $B_{\epsilon_1}(x) \subset V$ also. This implies $B_{\epsilon_1}(x) \subset U \cap V$, so $U \cap V \in \mathcal{O}_d$. By simple induction, we may conclude that the intersection of finite number of members of $\mathcal{O}_d$ is again in $\mathcal{O}_d$.

□

Remark. The proof above coincides with the fact that in a metric space arbitary union of open subsets and finite intersection of open subsets are open.

Example 7.1. The Zariski-topology.

Example 7.2. List of natural topologies.

1. On $\mathbb{R}^n$ the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of $\mathbb{R}^n$ are arbitary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{ x \in \mathbb{R}^n \mid d(p, x) < r \}.$$

This definition agrees with the topology endowed on arbitary metric spaces.

2. The matrix space $\mathrm{Mat}_{n \times m}(\mathbb{K})$ for a field $\mathbb{K}$ does not have one canonical topology. Depending on the context and literature different ones are used.

   - Since $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is isomorphic to $\mathbb{R}^{n \cdot m}$, one could use the Euclidean topology as defined above.
   - $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is a metric space via multitude of operator norms. The metric space induces the topology.
   - Another metric on $\mathrm{Mat}_{n \times m}(\mathbb{K})$ is the rank distance for $A, B \in \mathrm{Mat}_{n \times m}$ defined as $d(A, B) := \mathrm{rank}(B - A)$ which again would induce a topology.

Definition 8. Convergence in the topological sense

Definition 9 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \tag{1.1}$$

Proposition 10. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Proof. Let X and Y be metric spaces and $f : X \longrightarrow Y$ a function.

1. "$\Rightarrow$": Let f be ϵ-δ-continuous and $V \in \mathcal{O}_Y$ be an open subset. If $f^{-1}(V)$ is empty, then we are finished, so consider $x \in f^{-1}(V)$. We have that $f(x) \in V$. Since V is an open subset, there is an $\epsilon \in \mathbb{R}^+$ such that $B_Y(f(x), \epsilon) \subset V$. Using the ϵ-δ-continuity of f yields

$$f(B_X(x, \delta)) \subset B_Y(f(x), \epsilon) \subset V.$$

   If we apply the definition of a preimage, we get $B_X(x, \delta) \subset f^{-1}(V)$ which implies that $f^{-1}(V)$ is open in the topological sense. Therefore, f is continuous.

2. "$\Leftarrow$": Let f be continuous in the topological sense and consider $x \in X$. The ϵ-ball $B_Y(f(x), \epsilon)$ is open in Y, hence the preimage $f^{-1}(B_Y(f(x), \epsilon))$ is also open and contains x. Now, there exists a $\delta \in \mathbb{R}^+$ such that

$$B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \epsilon)).$$

   Applying the definition of a preimage we get $f(B_X(x, \delta)) \subset B_Y(f(x), \epsilon)$ which means f is ϵ-δ-continuous at x. Since x was chosen arbitary, f is ϵ-δ-continuous.

□

Remark. Again, the proof above coincides with the fact that in a metric space, a function is ϵ-δ-continuous if and only if the preimage of any open subset is open.

Lemma 11. Let X, Y, and Z be topological spaces.

1. Any constant map $f : X \longrightarrow Y$ is continuous.
2. The identity map $\mathrm{id} : X \longrightarrow Y$ is continuous.
3. If $f : X \longrightarrow Y$ is continuous, so is the restriction of f to any open subset of X.
4. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, so is their composition $g \circ f : X \longrightarrow Z$.

Lemma 12. A map $f : X \longrightarrow Y$ between topological spaces is continuous if and only if each point of X is contained in an open subset on which the restriction of f is continuous.

NOTES: Depending on the choice of the topology, the only convergent sequences are the ones that are constant, and in some all sequence converges to any other point!!

Homeomorphism

Definition 13 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. If $Y = X$ we also write $\text{Homeo}(X)$.

Example 13.1. Let $\mathcal{O}_1$ be the discrete topology on $\mathbb{R}$ and $\mathcal{O}_2$ be the trivial topology on $\mathbb{R}$. Then, the identity map on the real numbers

$$\text{id} : (\mathbb{R}, \mathcal{O}_1) \longrightarrow (\mathbb{R}, \mathcal{O}_2)$$

is bijective and continuous, but it is not a homeomorphism.

Proof. Clearly, the identity map id is bijective. For any open set $U \subset \mathcal{O}_2$ we have that $f^{-1}(U) \in \mathcal{O}_1$ because $\mathcal{O}_1$ is discrete, and thus, id is continuous. However, $\{0\}$ is open in $\mathcal{O}_1$, but $\text{id}^{-1}(\{0\}) = \{0\}$ is not open in $\mathcal{O}_2$ since this one is trivial. Therefore, id^{-1} is not continuous. □

Proposition 14. The set of all homeomorphisms of X to itself $\text{Homeo}(X)$ is a group with composition as its operation.

Proof. The identity function is contained in $\text{Homeo}(X)$ and is the identity element. Composition is associative and closed in $\text{Homeo}(X)$. By definition, $\text{Homeo}(X)$ contains the inverse of all its elements. Thus, $\text{Homeo}(X)$ is a group with composition as its operation. □

Definition 15 (Base)**.** Let $(X, \mathcal{O})$ a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Remark. We define the following as convention:

1. The empty union generates the emptyset, i.e.

$$\bigcup_{U \in \varnothing} U = \varnothing$$

2. The empty intersection generates the entire set, i.e.

$$\bigcap_{U \in \varnothing} U = X$$

Example 15.1. 1. $\mathcal{O} = \{ \varnothing, \{a\}, X \}$ then $\mathcal{B} = \{ \{a\}, X \}$

2. $\mathcal{O} = \{ \varnothing, \{a\}, \{a,b\} \}$ then $\mathcal{B} = \{ \{a\}, \{a,b\} \}$ and

Example 15.2. 1. The set Γ of all open intervals in $\mathbb{R}$ form a basis for the Euclidean topology on $\mathbb{R}$. If we require Γ to be of all bounded open intervals, it will still generate the Euclidean topology.

Lemma 16. For any collection of subsets $S \subset \mathcal{P}(X)$, there exists exactly one topology $\mathcal{O} \subset \mathcal{P}(X)$ that contains S and is the coarsest topology to do so, i.e.

1. $\mathcal{S} \subset \mathcal{O}$, and
2. if $\mathcal{O}' \subset \mathcal{P}(X)$ is an another topology with $S \subset \mathcal{O}'$, then $\mathcal{O} \subset \mathcal{O}'$.

Proof. Let $S \subset \mathcal{P}(X)$ be a collection of subsets and $\mathcal{O}(S)$ be the set of topologies that contain S, i.e.

$$\mathcal{O}(S) = \{ \tau \subset \mathcal{P}(X) \mid \tau \text{ is a topology and } S \subset \tau \}.$$

We know that $\mathcal{O}(S)$ is not empty because $\mathcal{P}(X) \in \mathcal{O}(S)$. Now define

$$\mathcal{O} := \bigcap_{\tau \in \mathcal{O}(S)} \tau.$$

Our claim is that this $\mathcal{O}$ is a topology.

1. The emptyset $\varnothing$ and the entire set X is contained in each $\tau \in \mathcal{O}(S)$ since these are topolgoies. Thus, the empty set and the entire set lie also in the intersection, i.e. $\varnothing, X \in \mathcal{O}$.

2. Let $\{U_i\}_{i \in I}$ be a familiy of subsets in $\mathcal{O}$ for an arbitary index set I. This means for each $i \in I$ it is $U_i \subset \mathcal{O}$, therefore, again for each $i \in I$ we have $U_i \in \tau$. Since τ was a topology, the arbitary union of $\{U_i\}_{i \in I}$ will lie in τ.

3. Similar for the finite intersection.

In particular, $\mathcal{O}$ lies in $\mathcal{O}(S)$.

MISSING THAT IT IS UNIQUE! □

Definition 17. 1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \}.$$

2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

Definition 18. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.
2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\text{int}(S)$, is the set of all interior points of S.
3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\text{cl}(S)$, is defined by

$$\text{cl}(S) := X \setminus \text{int}(X \setminus S).$$

Remark. This lemma does not hold for basis.

Remark.

1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.
2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. Let $(X, \mathcal{O})$ be a topological space. A subset that is **both** open and closed is called clopen. Moreover, a subset is clopen if and only if its boundary is empty.

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\text{cl}(A) = A$, but on the other hand, A is open, so we also have $\text{int}(A) = A$. Then, the boundary of A is $\partial A = \text{cl}(A) \setminus \text{int}(A) = A \setminus A = \varnothing$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. $\square$

1.3 Exercises and Notes

Definition 19 (Metric Space)**.**

Definition 20 (Open and Closed Subsets)**.**

Theorem 21 (Union and Intersection of Open Subsets)**.**

Definition 22. There are many equivalent ways to define continuity.

- ϵ-δ-*continuity*:
- *sequential continuity*:

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 23. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.
3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Lemma 24. Any interval $I \subset \mathbb{R}$ is connected.

Lemma 25. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Definition 26. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 27. Connected components are closed subsets.

Lemma 28. Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 29. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 30. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 31. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.

2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.

3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.

4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Proof. We verify the equivalence of the different definitions. So, let $(X, \mathcal{O})$ be a topological space.

- "*1.* $\Rightarrow$ *2.*": Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X, then $X \setminus A$ is also not the empty set nor X. Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \varnothing$ or $A = X$.

- "*2.* $\Rightarrow$ 1.": Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.

- "*2.* $\iff$ *3.*": This is one of the properties of clopen subsets and was proven in remark XXX.

- "*1.* $\Rightarrow$ *4.*": Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \longrightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.

- "*4.* $\Rightarrow$ *1.*": Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \longrightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

□

Lemma 32. Any interval $I \subset \mathbb{R}$ is connected.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \tag{2.1}$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \mathrm{cl}(B)$ and since the complement of B is an open subset A, so $B = \mathrm{cl}(B)$. It is therefore $x \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. □

Lemma 33. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}B = f^{-1}(A \cap B) = \varnothing$ so $f^{-1}(A) = \varnothing$ or $f^{-1}(B) = \varnothing$, so $A = \varnothing$ or $B = \varnothing$ and we are done. □

Remark. The two lemma above are handy to show that images of functions are connected.

Example 33.1. The general linear group $\mathrm{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Proof. Define the following partition of $\mathrm{GL}_n(\mathbb{K})$

$$
\begin{aligned}
A &:= \{\, M \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \mid \det(M) > 0 \,\} \\
B &:= \{\, M \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \mid \det(M) < 0 \,\},
\end{aligned}
$$

then, A and B are disjoint, nonempty, and $\mathrm{GL}_n(\mathbb{K}) = A \sqcup B$. We show that A and B are open sets.

The determinant function $\det : \mathrm{Mat}_{n\times n}(\mathbb{K}) \longrightarrow \mathbb{C}$ is continuous because it is a multivariate polynomial. $\mathbb{R}^+$ is an interval, therefore open, and so $\det^{-1}(\mathbb{R}^+) = A$ is also open. Similary B is an open subset. Hence $\mathrm{GL}_n(\mathbb{K})$ is not connected. □

Remark. In the proof above, the topology of $\mathrm{Mat}_{n\times n}(\mathbb{K})$ matters because the continuity of the determinant function depends on the underlying topology.

Definition 34. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Example 34.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proof. Assume there is a connected set $A \subset \mathbb{Q}$ that contains more than one point. Let $x \in A$ be a point in A. We show that $\{x\}$ is a clopen set.

Denote another point in A that is closest to x as x_0, i.e. for all $y \in A$ it is $d(x, y) \geq d(x, x_0)$. Now set $\epsilon := d(x, x_0)$. Then, $B_\epsilon(x) \cap \mathbb{Q} = \{x\}$ is an open subset.

I think showing closedness is quite similar. □

Proposition 35. Connected components are closed subsets.

Proof. Let X be a set and $C \subset X$ be a connected component. Consider $\mathrm{Cl}(C)$. Clearly, $C \subset \mathrm{Cl}(C)$. Moreover, $\mathrm{Cl}(C)$ is connected hence $\mathrm{Cl}(C) \subset C$. We have $\mathrm{cl}(C) = C$ so C is connected. □

Lemma 36. Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 37. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 38. If X is path connected, then it is also connected.

Proof. Locally constant implies continuous with regards to the discrete topology on Y. Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \varnothing$. Conclude f is identical to $f(x)$. □

Application: $f : X \longrightarrow \{0,1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0,1] \longrightarrow X$ be continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We have that γ^{-1} □

2.3 Exercises and Notes

2.3.1 Connectedness

Lemma 39. If $(X, \mathcal{O}_{\mathcal{X}})$ and $(Y, \mathcal{O}_Y)$ are two connected topological spaces, then their product $X \times Y$ with the product topology $\mathcal{O}_{X \times Y}$is also connected.

Proof. We will use the definition that all continuous maps from $X \times Y$ to $\{0, 1\}$ endowed with the discrete topology must be constant. Fix a continuous $f : X \longrightarrow \{0, 1\}$.

First, consider the image $f(\{x\} \times Y)$ with $x \in X$. Assume f is not constant on $\{x\} \times Y$, then $f(\{x\} \times Y) = \{0, 1\}$. So we have the preimages $f^{-1}(\{0\}) = \{x\} \times U$ and $f^{-1}(\{1\}) = \{x\} \times V$ with $U, V \subset Y$, $U, V \neq \varnothing$, and $U \cap V = \varnothing$. Because f is continuous, U and V must also be open. This would however mean that $U \sqcup V = Y$ and Y would not be connected, therefore, f is constant on $\{x\} \times Y$. Similary, we get that f is constant on $X \times \{y\}$ for all $y \in Y$.

Let $(x, y) \in X \times Y$ and $(x', y') \in X \times Y$ be two arbitary points. We have $f(x, y) = f(x, y')$ because f is constant on $\{x\} \times Y$ and similary $f(x, y') = f(x', y')$ because f is constant on $X \times \{y\}$. Putting everything together, it is $f(x, y) = f(x', y')$, therefore all continuous $f : X \times Y \longrightarrow \{0, 1\}$ are constant. □

Example 39.1. Clearly, the union of two connected sets need not be connected. Take for example $[0, 1] \subset \mathbb{R}$ and $[2, 3] \subset \mathbb{R}$. Their union $[0, 1] \cup [2, 3]$ is not connected.

Set difference of connected sets are also not necessarily connected, e.g. $[0, 2] \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ are connected, but $[0, 2] \setminus \{1\} = [0, 1) \cup (1, 2]$ is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \left\{ (x, y) \mid x^2 + y^2 = 1 \right\}$ and another unit circle around $(1, 0)$ $A := \left\{ (x, y) \mid (x - 1)^2 + y^2 = 1 \right\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \right\}$$

which is not connected.

Proposition 40. 1. Every trivial topological space is connected.

2. Every discrete topological space with at least two elements is disconnected.
3. Trivially, every singleton set and the empty set are connected spaces vacuously.

Proof. 1. Let X be an arbitary set and $\mathcal{O} = \{ \varnothing, X \}$ be the trivial topology. If $S \subset X$ is a clopen subset, then it is trivially either $\varnothing$ or X, therefore, X is connected.

2. Let X be a set containing more than one element and $\mathcal{O} = \mathcal{P}(X)$ be the discrete topology of X. Let $A \subset X$ be a nonempty proper subset, then $B := X \setminus A$ is also not empty. Both are open subsets, but $A \sqcup B = X$, so X is not connected.

□

Proposition 41. Every singleton set in $\mathbb{R}^n$ endowed with the Euclidean topology is clopen. ??? IDK IF THIS IS TRUE

2.3.2 Path-Connectedness

Example 41.1. Connectedness does not imply path-connectedness. Let $\mathbb{R}^2$ be endowed with the Euclidean topology and consider

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \cup (\{0\} \times [-1,1]) \subset \mathbb{R}^2.$$

and see figure XXX. X is connected, but it is not path-connected.

Proof. Denote

$$A := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \qquad B := \{0\} \times [-1,1],$$

then $X = A \cup B$.

1. First, define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^2$ as

   $$f(x) := \left(x, \sin\left(\frac{1}{x}\right) \right).$$

   f is continuous, $\mathbb{R}^+$ is an interval, therefore connected, so $f(\mathbb{R}^+) = A$ is connected. Similary, define $g : [-1,1] \longrightarrow 0 \times [-1,1]$ as

   $$g(x) := (0, x).$$

   Again, g is continuous and $[-1,1]$ is connected, so $g([-1,1]) = 0 \times [-1,1] = B$ is also connected.

   Now assume $X = U \sqcup V$ for some nonempty, disjoint, open subsets of $\mathbb{R}^2$. Since A is connected, either U or V must contain A, say U. On the other hand, $(0,0) \in V$, but since V is open, it contains all the interior points, i.e.

   Assume there is a clopen subset $S \subset X$ that is not empty. Without loss of generality, we have that $(0,0) \in U$ (otherwise, consider the complement of U which also must be clopen). Since A is clopen in A, the intersection $A \cap U$ must also be clopen in A, but A is connected, so A is contained in U.

   Moreover, the closure of A is also contained in U. So there is an $\epsilon > 0$ such that the ball $B(p, \epsilon)$ that contains $(0,0)$ is in U. I got lazy to go into the details, but this ball contains a point of B. Follow the same reason as above.

2. Assume X is path-connected.

   Choose two points $x_0 = (0,1) \in A$ and $x_1 = (1,1) \in B$ and a path $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Let $\epsilon \in (0,1)$, then $B_\epsilon(x_0) \cap X$ is an open subset that contains x_0, therefore, $\gamma^{-1}(B_\epsilon(x_0) \cap X)$ is also open.

□

As motivation, the proof of intermediate value theorem becomes very elegant.

Chapter 3

Separation Axioms

3.1 Definitions and Theorems

Definition 42 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 43. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 44 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 45. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{ (x, x) \mid x \in X \}$ is closed as a subset of the product space $X \times X$.

Proposition 46. T_2 spaces are also T_1 spaces.

3.2 Proofs, Remarks, and Examples

3.2.1 T_0 Space

Definition 47. A topological space $(X, \mathcal{O})$ is a T_0 space (or Kolmogorov space) if for every pair of distinct points of X, at least one of them has a neighborhood not containing the other (this property is called topologically distinguishable).

Definition 48. A topological space $(X, \mathcal{O})$ is a T_1 space (also called accessible space or a space with Fréchet topology) if one of the following equivalent conditions are met.

1. Any two distinct points in X are separated, i.e. if $x, y \in X$ are points with $x \neq y$, then there are neighborhoods U_x and U_y of x and y respectively such that $y \notin U_x$ and $x \notin U_y$.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.
3. Every subset of X is the intersection of all the open sets containing it.
4. Every finite set is closed.
5. Every cofinite set of X is open.

Definition 49 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 50. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 51 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 52. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{\, (x, x) \mid x \in X \,\}$ is closed as a subset of the product space $X \times X$.

Proposition 53. T_2 spaces are also T_1 spaces.

3.3 Exercises and Notes

Chapter 4

Compact Spaces

Definition 54.
1. A topological space X is called compact if each of its open cover has a **finite** subcover.
2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Theorem 55. Satz 17

Theorem 56. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 57. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 58. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \longrightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Definition 59. 1. A topological space X is called compact if each of its open cover has a **finite** subcover. That is, X is compact if for every arbitary collection $\mathcal{C}$ of open subsets of X such that

$$X = \bigcup_{U \in \mathcal{C}} U$$

there is a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that

$$X = \bigcup_{U \in \mathcal{F}} U.$$

2. A subset is said to be compact if it is compact as a subspace.

Definition 60. X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Remark. The notion of compact and sequentially compact are not equivalent.

Example 60.1. 1. Example of a space that is compact but not sequentially compact.

2. Example of a space that is sequentially compact but not compact.

Proposition 61. Let X and Y be two topological spaces.

1. Continuous functions preserve compactness, i.e. if $f : X \longrightarrow Y$ is continuous and X is compact, then $f(X) \subset Y$ is compact.

2. In a compact space, every closed subset is compact, i.e. if X is compact and $A \subset X$ is a closed subset, then A is compact.

3. The product of compact spaces is again compact. If X and Y are both compact, so is $X \times Y$.

Proof. 1. Let $f : X \longrightarrow Y$ be continuous and X compact. Denote the open cover of the continuous image of X by $\mathcal{C}$, i.e.

$$f(X) \subset \bigcup_{U \in \mathcal{C}} U.$$

Because f is continuous, each of the preimages $f^{-1}(U)$ with $U \in \mathcal{C}$ is open. Now, X is compact, there are finitely many $f^{-1}(U)$ such that

$$X \subset \bigcup_{U \in \mathcal{F}} f^{-1}(U)$$

Conclude that $f(X)$ is compact.

2. Let $\mathcal{U}$ be an open cover of A. Every open set in $\mathcal{U}$ is in the form $U \cap A$ for some open set $U \subset X$. Define

$$\mathcal{V} := \{ U \in \mathcal{O} \mid U \cap A \in \mathcal{U} \}$$

then $\mathcal{V}$ is an open cover of A as well. Since A is closed, $X \setminus A$ is open, so $\mathcal{V} \cup (X \setminus A)$ is an open cover of X. By compactness of X, there is a finite subcover that XXXXX.

3. I think this is clear.

□

Lemma 62. $[0,1] \subset \mathbb{R}$ is compact.

Proof. skipped □

Theorem 63 (Heine-Borel)**.** A compact subset of a Euclidean space $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition 64. Let X be a T_2-space. If $K \subset X$, then K is closed.

Proof. for the two things above, skipped. □

Lemma 65. Let $f : X \longrightarrow Y$ be continuous and bijective. If X is compact and Y is a T_2-space, then f^{-1} is continuous.

Chapter 5

Quotient Space

5.1 Definitions and Theorems

5.2 Proofs, Remarks, and Examples

Definition 66. Let $(X, \mathcal{O})$ be a topological space, and let $\sim$ be an equivalence relation on X. The quotient set, $X/\sim$ is the set of equivalence classes of elements of X. The equivalence class of $x \in X$ is denoted $[x]$. The projection map (also quotient or canonical map) associated with $\sim$ refers to the following surjective map:

$$\pi : X \longrightarrow X/\sim, \qquad x \mapsto [x]$$

For any subset $S \subset X/\sim$ (so in particular, $s \subset X$ for every $s \in S$) the following holds.

$$q^{-1}(S) = \{\, x \in X \mid [x] \in S \,\} = \bigcup_{s \in S} s.$$

The quotient space under $\sim$ is the quotient set $X/\sim$ equipped with the quotient topology, which is the topology whose open sets are subsets $U \subset X/\sim$ such that

$$\{\, x \in X \mid [x] \in U \,\} = \bigcup_{u \in U} u$$

is an open subset of $(X, \mathcal{O}_X)$; that is, $U \subset X/\sim$

Proof. We will show that $\mathcal{O}_\sim$ is a topology.

1. Clearly, $\pi^{-1}(\varnothing) = \varnothing \in \mathcal{O}$, thus $\varnothing \in \mathcal{O}_\sim$. Moreover, $\pi(X) = X/\sim$, hence $\pi^{-1}(X/\sim) = X \in \mathcal{O}$, and it is $X/\sim \in \mathcal{O}_\sim$.

2. Let I be an arbitary index set and $\{U_i\}_{i \in I}$ a family of open sets in $X/\sim$. It is

$$\pi^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \pi^{-1}(U_i). \tag{5.1}$$

   Since U_i is open and π is continuous, $\pi^{-1}(U_i)$ is open for all $i \in I$. Thus $\bigcup_{i \in I} U_i \in \mathcal{O}_\sim$.

3. Similar as above as preimages preserve unions and intersections.

Let $\mathcal{O}_\sim \subsetneq \mathcal{O}'$. Then there is a open set $U \in \mathcal{O}'$ but $U \notin \mathcal{O}_\sim$. So $\pi^{-1}(U)$ does not lie in $\mathcal{O}$. But this is a contradiction. $\square$

Example 66.1.

1. $\mathbb{R}/\mathbb{Z}$

   This space is homeomorph to S^{-1} and is compact.

2. $(\mathbb{R}/\mathbb{Q}, \mathcal{O}_{\mathbb{R}/\mathbb{Q}})$ is the trivial topology.

Proposition 67. $\mathcal{O}_{X/\sim}$ is the finest topology in which the projection map $\pi : X \longrightarrow X/\sim$ is continuous.

Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a surjective map. The map is a quotient map (also said strong continuity) if one of the equivalent condition hold.

1. A subset $U \subset Y$ is open in Y if and only if the preimage $p^{-1}(U)$ is open in X.
2. A subset $U \subset Y$ is closed in Y if and only if the preimage $p^{-1}(U)$ is closed in X.

Remark. Quotient maps are continuous. There are quotient maps that are neither open nor closed maps.

Theorem 68. Let Y be a topological space. Then the following are equivalent.

1. $f : X/\sim \longrightarrow Y$ continuous
2. $f \circ \pi : X \longrightarrow Y$ is continuous.

Moreover, if X is connected, then $X/\sim$ is connected. Same is true for path-connectedness and compactness.

Definition 69. A topological group G is a topological space that is also a group such that the group operation

$$\circ : G \times G \longrightarrow G, (x, y) \mapsto x \circ y$$

and the inversion map

$$^{-1} : G \longrightarrow G, x \mapsto x^{-1}$$

are continuous. Here $G \times G$ is viewed as a topological space with the product topology. Such a topoloy is said to be compatible with the group operations and is called a group topology.

Remark. About homoemorphism of G-space.

Definition 70. Consider a group acting on a set X. The orbit of an element x in X is the set of elements in X to which x can be moved by the elements of G. The orbit of x is denoted by $G \cdot x$.

$$G \cdot x := \{ g \cdot x \mid g \in G \}.$$

Definition 71. $X/G := X/\sim$ such that $x \sim y$ if and only if there is a $g \in G$ such that $x = gy$.

Definition 72 (Hilbert)**.**

- Matrix A is semi-stable if A is diagonizable.
- Matrix A is stable, if it is semi-stable and all Eigenvalues are distinct.

Definition 73 (Stabilisator)**.** $x \in X$
$G \supset G_x := \{ h \mid h \cdot x = x \}$

Lemma 74.

1. $G_x \subset G$ is a subgroup.
2. $G/G_x \longrightarrow G_x$ is well-defined and $[x] \mapsto gx$ is a continuous bijection (repsective of the quotient topology on G/G_x).

Corollary 1. If G is compact and X is T_2, then $f : G/G_x \longrightarrow G$ is a homeomorphism.

Definition 75.

5.3 Exercises and Notes

Example 75.1. $D^n/\partial D^n$ is homeomorph to S^n where D^n is the closed n-ball and S^n is the n-sphere.

Proof. What works is this

$$\text{if } ||x|| = 1 \Rightarrow f(x) = \text{ north pole}$$
$$\text{if } ||x|| = 0 \Rightarrow f(x) = \text{ south pole}$$

So how to find $y = (y_0, y_1, \ldots, y_n)$? We have $y = (2||x|| - 1, \ldots)$ and y_k is a multiple of x_k such that $||y|| = 1$. $\square$

Chapter 6

Pasting

Definition 76. Let X_0 and X be topological spaces and $\varphi : X_0 \longrightarrow X$ a continuous function. Set $X_\varphi := X/\sim$ where $\sim$ is generated by

$$\{ x \sim \varphi(x) \mid x \in X_0 \} .$$

Example 76.1. idk

Definition 77. Abbildungstorus

Definition 78. Let X be a topological space.

1. X is said to be a first-countable space or to satisfy the first axiom of countability if each point has a countable neighbourhood basis (local base). That is, for each point $x \in X$ there exists a sequence $N_1, N_2, \ldots$ of neighbourhoods of x such that for any neighbourhood N of x there exists an integer i with N_i contained in N. Since every neighbourhood of any point contains an open neighbourhood of that point, the neighbourhood basis can be chosen without loss of generality to consist of open neighbourhoods.

2. X is said to be a second-countable space, also called completely separable space, or to satisfy the second axiom of countability if it has a countable base.

Remark. The majority of 'everyday' spaces in mathematics are first-countable.

Example 78.1. Every metric space is first-countable.

Proof. Let (M, d) be a metric space and fix a point $x \in M$. The open balls $\{B_n(x)\}_{n\in\mathbb{N}}$ around x where

$$B_n(x) := \left\{ y \in M \mid d(x, y) < \frac{1}{n} \right\}$$

defines a sequence of neighbourhoods of x. Now, any neighbourhood N of x is contained in at least one of these balls. Thus, every metric space is first-countable. $\square$

Example 78.2. Not all metric spaces are second-countable. For example, $\mathbb{R}$ equipped with the discrete metric is not second-countable.

Remark. A countable subbase induces a countable base.

Lemma 79. The second-countable axiom implies the first.

Example 79.1. The converse of proposition XXX is not true. For example, the Sorgenfrey line $(\mathbb{R}, \mathcal{O})$ where the topology $\mathcal{O}$ is generated by all the right-open, left-closed intervals, i.e.

$$\mathcal{O} = \{\varnothing\} \cup \left\{ \bigcup_{i \in I} [a_i, b_i) \subset \mathbb{R} \mid -\infty < a_i < b_i < \infty \right\}$$

satisfies the first-countable axiom but not the second.

Proof. Let $(\mathbb{R}, \mathcal{O})$ be the Sorgenfrey line as defined abvoe.

1. We show that $(\mathbb{R}, \mathcal{O})$ satisfies the first-countable axiom. Fix a point $x \in \mathbb{R}$ and set

$$I_n(x) := \left[x, x + \frac{1}{n} \right).$$

   Then, $\{I_n\}_{n \in \mathbb{N}}$ is a sequence of neighbourhoods of x. For any neighbourhood N of x there is a $n \in \mathbb{N}$ large enough such that $I_n \subset N$, and thus, the Sorgenfrey line $(\mathbb{R}, \mathcal{O})$ is first-countable.

2. We show that $(\mathbb{R}, \mathcal{O})$ is not second-countable. Consider a point $x \in \mathbb{R}$. Because $[x, \infty)$ is open, there must be a base element $B \subset [x, \infty]$. This is true for all points on $\mathbb{R}$ and all the base elements are distinct. $\mathbb{R}$ is not countable, thus $(\mathbb{R}, \mathcal{O})$ is not second-countable.

□

Lemma 80. Is X a first-countable space, then

1. all sequentially-continuous function is continuous.
2. all compact spaces are also sequentially compact.

Definition 81. A topological mannifold $\mathcal{M}^n$ of dimension n is a topological space that is T_2, is a second-countable space, and for each point has a neighbourhood that is homeomorph to a open subset of

$$\mathbb{H}^n := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0 \}$$

Definition 82 (Pasting)**.** Let X and Y be two disjoint topological spaces, $A \subset X$ a closed subset, and $f : A \longrightarrow Y$ continuous. Define a equivalence relation on $X \cup Y$ by

$$v \sim w : \begin{cases} v = w \text{ if } v, w \in X \cup Y \\ f(v) = f(w) \text{ if } v, w \in A \\ v = f(w) \text{ if } v \in Y, w \in A \\ w = f(v) \text{ if } v \in A, w \in Y \end{cases}$$

We write

$$Y \cup_f X := X \cup Y / \sim$$

Example 82.1. Consider $X = [0, 1]$ on the x-axis and $Y = [0, 1]$.

Example 82.2. Portals can be thought of a Pasting

Example 82.3. Let $D^n := \{ x \in \mathbb{R}^n \mid ||x|| \leq 1 \} \in \mathbb{R}^n$ be the n-dimensional ball and define

$$id : \partial D^n \longrightarrow \partial D^n, x \mapsto x$$

It is

$$D^n \cup_\varphi D^n = D^n \cup D^n / \sim$$

Lemma 83. Any open subset of an n-manifold is an n-manifold.