

Topology

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Definition 0.1 (Topological Space). A **topological space** is an ordered pair (X, τ) , where X is a **set** and τ is a **collection of subsets** that satisfies the following axioms.

1. The **empty set** \emptyset and the **whole set** X belongs to τ .
2. Any **arbitrary union** of members of τ belongs to τ .
3. The **intersection of finite number** of members of τ belongs to τ .

The **collection** τ is called a **topology** on X and the **elements** of τ are called **open sets**. A **subset** $A \subset X$ is said to be **closed** if its **complement** $X \setminus A$ is **open**.

Example 0.1.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the **discrete topology**. In this case, (X, τ) is called the **discrete space**. It is the finest topology. (One can define an ordering of topologies.)
2. $\tau = \{\emptyset, \mathcal{P}(X)\}$ is called the **trivial topology**.
3. Let (X, d) be a **metric space**. Set

$$\tau_d := \{U \in \mathcal{P}(X) \mid U \text{ is an open subset in the metric space } (X, d)\}. \quad (1)$$

Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U .

Here, τ is a topology. In other words, a metric induces a topology.

(Proof as homework.)

4. The Zariski-topology.

Definition 0.2 (Continuous Maps). Let (X, τ_X) and (Y, τ_Y) be **topological spaces**. A **map** $f : X \rightarrow Y$ is said to be **continuous** if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \quad (2)$$

Lemma 0.2.1. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \rightarrow Y$ is ϵ - δ -continuous if and only if f is continuous.

Definition 0.3 (Homeomorphism). Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is a homeomorphism if it has the following properties.

1. f is **bijective**.
2. f is **continuous**.
3. The inverse map f^{-1} is **continuous**.

If such function exists, X and Y are said to be **homeomorphic**.

We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. The set of all homeomorphisms of X to itself $\text{Homeo}(X)$ is a group with composition as its operation.