Topology

K

January 20, 2023

Contents

Part I

Rings

Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring)**.** A ring is a set A equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. $(A, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Chapter 2

Ideals

Definition 2 (Ideal)**.** Let A be a ring. A subset $\mathfrak{a} \subset A$ is called an ideal if it satisfies the following two conditions.

1. $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$.
2. For every $r \in A$ and every $x \in \mathfrak{a}$, it is $rx \in \mathfrak{a}$.

Given a subset $S \subset A$, by the ideal (S) that S generates, we mean the smallest ideal containing S. If an ideal is generated by a subset $S \subset A$, then the elements of this subset are called generators.
An ideal that is generated by a single element is called principal.
If an ideal $\mathfrak{a}$ is not the whole ring A, then the ideal is called proper.

Definition 3 (Ideal Operation)**.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring A.

1. The sum of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by

$$\mathfrak{a} + \mathfrak{b} = \{\, a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \,\} = (\mathfrak{a}, \mathfrak{b})$$

   which is again an ideal. It is the smallest ideal in A that contains $\mathfrak{a}$ and $\mathfrak{b}$.
2. The product of an ideal
3. The intersection of
4. The radical of an ideal $\mathfrak{a}$ is defined by

$$\sqrt{\mathfrak{a}} = \{\, x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \,\}$$

   which is again an ideal.
5. The transporter

Proof. We verify the statements made in the definition.

1. (a) "$\mathfrak{a} + \mathfrak{b}$ is an ideal.":

□

Example 3.1. The union of two ideals is **not** an ideal in general. Consider (2) and (3) in $\mathbb{Z}$. If $(2) \cup (3)$ was an ideal, then $3 - 2 = 1$ would be contained in $(2) \cup (3)$. But $1 \notin (2)$ and $1 \notin (3)$, thus $1 \notin (2) \cup (3)$.

Proposition 4. Let $\mathfrak{a}$ be an ideal of A.

1. $\mathfrak{a} = A$ if and only if $1 \in \mathfrak{a}$ if and only if $\mathfrak{a}$ contains an unit.
2. $\mathfrak{a}^2 \subset \mathfrak{a}$.
3. $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.
4. $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.

Proposition 5. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a ring A.

1. $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$.
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.
3. If $\mathfrak{a} \subset \mathfrak{b}$, then $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$.
4. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.
5. $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
6. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
7. If $\mathfrak{a} = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ and $n \in \mathbb{N}^+$, then $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Proof. We verify each statement.

1. Let $x \in \mathfrak{a}$, then trivially, $x^1 \in \mathfrak{a}$, so $x \in \sqrt{\mathfrak{a}}$.

2. Since $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$ from above, it suffices to verify the other inclusion. Let $x \in \sqrt{\sqrt{\mathfrak{a}}}$, then $x^n \in \sqrt{\mathfrak{a}}$ and in turn, $(x^n)^m \in \mathfrak{a}$. Thus, $x^{nm} \in \mathfrak{a}$, therefore, $x \in \sqrt{\mathfrak{a}}$.

3. Suppose $\mathfrak{a} \subset \mathfrak{b}$ and let $x \in \sqrt{\mathfrak{a}}$. Then, $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$, thus $x^n \in \mathfrak{b}$. It follows that $x \in \sqrt{\mathfrak{b}}$.

4. "$\Rightarrow$": Let $\sqrt{\mathfrak{a}} = A$, then for all $x \in A$, we have that $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$. In particular, $1^n \in \mathfrak{a}$, but $1^n = 1$ for all $n \in \mathbb{N}^+$. Thus, $\mathfrak{a} = A$.

   "$\Leftarrow$": On the other hand, let $\mathfrak{a} = A$. In general, it is $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, therefore $A \subset \sqrt{\mathfrak{a}}$ which immediately yields the desired equality $A = \sqrt{\mathfrak{a}}$.

5. "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cdot \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Since $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, we have $x^n \in \mathfrak{a} \cap \mathfrak{b}$, and it follows that $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$": Let $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Hence it is $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, therefore $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": If $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$. We may write $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, therefore $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.

   "$\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$": Finally, let $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. Then, $x\sqrt{\mathfrak{a}}$ and $x\sqrt{\mathfrak{b}}$, so $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$ for some $n, m \in \mathbb{N}^+$. Say $n \geq m$, then $x^n \in \mathfrak{b}$. This yields $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.

6. "$\sqrt{\mathfrak{a} + \mathfrak{b}} \subset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Let $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$, then $x^n \in \mathfrak{a} + \mathfrak{b}$ for some $n \in \mathbb{N}^+$. By definition of sum of ideals, we have that $x^n = a + b$ for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ and $\mathfrak{b} \subset \sqrt{\mathfrak{b}}$, we have $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, thus $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

   "$\sqrt{\mathfrak{a} + \mathfrak{b}} \supset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$": Now let $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$, then $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$ for some $n \in \mathbb{N}^+$. Hence there exists $a \in \sqrt{\mathfrak{a}}$ and $b \in \sqrt{\mathfrak{b}}$ such that $x^n = a + b$. We have that $a^p \in \mathfrak{a}$ and $b^q \in \mathfrak{b}$

for some $p, q \in \mathbb{N}^+$. Consider

$$\begin{aligned}(x^n)^{(p+q-1)} &= (a+b)^{(p+q-1)} \\ &= \sum_{k=0}^{p+q-1} \binom{p+q-1}{k} a^k \cdot b^{p+q-1-k}.\end{aligned}$$

For each $k \in \{0, 1, \dots, p+q-1\}$, we have $a^k \in \mathfrak{a}$ or $b^{p+q-1} \in \mathfrak{b}$. Thus, the whole sum lies in $\mathfrak{a} + \mathfrak{b}$ or in other words $x^{n(p+q-1)} \in \mathfrak{a} + \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

7. "$\sqrt{\mathfrak{a}} \subset \mathfrak{p}$": Let $x \in \sqrt{\mathfrak{a}}$, then $x^m \in \mathfrak{a}$ for some $m \in \mathbb{N}^+$. Because $\mathfrak{a} = \mathfrak{p}^n$, we have $x^m \in \mathfrak{p}^n$. We also have $\mathfrak{p}^n \subset \mathfrak{p}$, thus $x^m \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$.

   "$\sqrt{\mathfrak{a}} \supset \mathfrak{p}$": On the other hand, if $x \in \mathfrak{p}$, then $x^n \in \mathfrak{p}^n = \mathfrak{a}$, therefore $x \in \sqrt{\mathfrak{a}}$.

$\square$

Proposition 6. 1. $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$.

Example 6.1. Does $\sqrt{\mathfrak{a}^2} = \mathfrak{a}$ hold?

Chapter 3

Anatomy of Rings

Definition 7 (Nilpotent Element and Nilradical)**.** An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.
The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\mathrm{Nil}(A)$.

Definition 8 (Reduced Ring)**.** A ring A is called reduced ring if it has no non-zero nilpotent elements.

Proposition 9. Let A and B be two rings and $A' \subset A$ be a subring of A.

1. If A is reduced, then A' is also reduced.
2. If A and B are reduced, then $A \times B$ is also reduced.
   (XXX DOES THIS ALSO HOLD FOR ARBITARY MANY PRODUCTS?)

3.1 Exercises and Notes

Example 9.1. Let K be a field and $A = {}^{K[X,Y]}/_{(X - XY^2, Y^3)}$.

1. Compute the nilradical $\mathrm{Nil}(A)$.

   Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

   Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism ${}^{K[X,Y]}/_{(X - XY^2, Y^3)} \simeq {}^{K[Y]}/_{(Y^3)}$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

   Clearly, $Y \in \mathrm{Nil}(A)$ or in other words $(Y) \subset \mathrm{Nil}(A)$. But we also have that ${}^{K[Y]}/_{(Y)} = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \mathrm{Nil}(A)$ conclude $\mathrm{Nil}(A) = (Y)$. ◻

Chapter 4

Polynomial Rings

Chapter 5

Quotient

Chapter 6

Localization

6.1 Definition and Theorems

Definition 10 (Multiplicative Subset)**.** A subset S of a ring A is called a multiplicative subset if the following conditions hold.

1. $1 \in S$.
2. For all $x, y \in S$ it is $xy \in S$.

Example 10.1. Let A be a ring. Important examples of a multiplicative subset include the following.

1. The set of units $A^\times$ is a multiplicative subset.
2. The set of non-zero-divisors $A \setminus \mathrm{ZD}(A)$ is a multiplicative subset.

Example 10.2. Let A be a ring. Other examples of multiplicative subsets are the following.

1. For any element $x \in A$, the set generated by its power $\left\{1, x, x^2, x^3, \ldots\right\}$ is a multiplicative subset.
2. For any ideal $\mathfrak{a} \subset A$, the set $1 + \mathfrak{a}$ is a multiplicative subset.

Lemma 11. An ideal $\mathfrak{p}$ of a ring A is prime if and only if its complement $A \setminus \mathfrak{p}$ is a multiplicative subset.

Definition 12 (Localization)**.** $S^{-1}A$ is again a ring.

Lemma 13. Let A be a ring and S a multiplicative subset, then the following are equivalent.

1. $S^{-1}A = 0$.
2. S contains a nilpotent element.
3. $0 \in S$.

Proof. "1. $\Rightarrow$ 2.": Let $S^{-1}A = 0$, then for all $x \in A$ and $s \in S$ it is $(x, s) \sim (0, 1)$, thus $x \cdot u = 0$ for some $u \in S$. In particular, this holds for $x = 1$, therefore $1 \cdot u = 0$. Since a unit can never be a zero divisor, we must have $u = 0$ which is nilpotent and lies in S.

"1. $\Leftarrow$ 2.": On the other hand, let $x \in S$ be nilpotent, i.e. $x^n = 0$ for some $n \in \mathbb{N}^+$. Because S is multiplicatively closed $x^n = 0$ lies in S. Fix an element $(y, s) \in S^{-1}A$, then $y \cdot 1 \cdot 0 = 0 \cdot s \cdot 0$. Hence $(y, s) \sim (0, 1)$ and we have $S^{-1}A = 0$.

"2. $\Rightarrow$ 3.": Again, let $x \in S$ be nilpotent, thus $x^n = 0$ for some $n \in \mathbb{N}^+$. S is multiplicatively closed and we have $x^n = 0 \in S$.

"2. $\Leftarrow$ 3.": If $0 \in S$, then S simply contains a nilpotent element because 0 is nilpotent. □

Remark. In the lemma above, the condition $0 \notin S$ is required because if S contains 0, then $S^{-1}A = 0$ and by definition, an integral domain is a nonzero ring.

Proposition 14. Let A be a ring. A is reduced if and only if all its localizations $A_\mathfrak{p}$ at $\mathfrak{p} \in \operatorname{Spec} A$ is reduced.

Proof. "$\Rightarrow$": We prove the statement by contrapositive. Let $A_\mathfrak{p}$ be not reduced for all $\mathfrak{p} \in \operatorname{Spec} A$. Thus, in all $A_\mathfrak{p}$, there is an element, say x/s that is nilpotent and not zero, i.e. $(x/s)^n = 0$ for some $n \in \mathbb{N}^+$. By the definition of localization, we get $x^n \cdot u = 0$ for some $u \in A \setminus \mathfrak{p}$. Now, $u \in A \setminus \mathfrak{p}$ cannot be zero, because if it was, $A_\mathfrak{p} = 0$ which is reduced. Thus, x is nilpotent and A is not reduced. □

Lemma 15. Let A be a ring and $S \subset A$ be a multiplicative subset that does not contain 0.

1. A is an integral domain if and only if $S^{-1}A$ is an integral domain.
2. A is a unique factorization domain if and only if $S^{-1}A$ is a unique factorization domain.

Proof. "$\Rightarrow$": Let A be an integral domain. Since S does not contain 0, the localization $S^{-1}A$ is a nonzero ring (see EXAMPLE). Let $(x, s) \in S^{-1}A \setminus \{0\}$ be a nonzero element and suppose there is a $(y, t) \in S^{-1}A$ with $(x, s) \cdot (y, t) = 0$. It is $(xy, st) = (0, 1)$ and thus $xy \cdot u = 0$ for some $u \in S$. Because x was nonzero and S does not contain 0 we must have $y = 0$. Hence $S^{-1}A$ is an integral domain.

"$\Leftarrow$": On the other hand, let $S^{-1}A$ be an integral domain. JUST USE THE CANONIC MAPPING $\varphi_S : A \longrightarrow S^{-1}A$. □

6.2 Exercises and Notes

Example 15.1. Let A_1 and A_2 be rings. Consider $A = A_1 \times A_2$ and set $S := \{ (1, 1), (1, 0) \}$. Prove $A_1 \simeq S^{-1}A$.

Solution. I don't understand the solution? □

Example 15.2. Find all intermediate rings $\mathbb{Z} \subset A \subset \mathbb{Q}$, and describe each A as a localization of $\mathbb{Z}$. As a starter, prove $\mathbb{Z}\left[\frac{2}{3}\right] = S_3^{-1}\mathbb{Z}$ where $S_3 := \left\{ 3^i \mid i \in \mathbb{N}^+ \right\}$.

Chapter 7

Hierarchy of Rings

7.1 Definition and Theorems

7.1.1 Integral Domains

Part II

Modules

Definition 16 (Module)**.**

Example 16.1. 1. If A is a field, then an A-module is a vector space.

2. A $\mathbb{Z}$-module is just an abelian group.

Definition 17 (Annihilator)**.**

Definition 18 (Radical)**.**

Definition 19 (Simple Modules)**.** Let A be a ring. A nonzero A-module M is called simple if the only submodules are $\{0\}$ and M itself.

Example 19.1. If M is a simple A-module, then any $f \in \mathrm{Hom}_A(M, M) \setminus \{0\}$ is an isomorphism.

Proof. Fix an $f \in \mathrm{Hom}_A(M, M) \setminus \{0\}$. Since $\ker(f)$ is a submodule of M, it must be either $\{0\}$ or whole M. But $\ker(f) = M$ would mean that $f = 0$ which was explicitly excluded, thus $\ker(f) = \{0\}$. By the isomorphism theorem, we also have $\mathrm{im}(f) \cong {}^{M}/_{\ker(f)} \cong M$. Therefore, f is bijective. $\square$

Definition 20 (Indecomposable)**.** Let A be a ring. A nonzero A-module M is called indecomposable if it cannot be written as a direct sum of two non-zero submodules.

Proposition 21. Every simple module is indecomposable.

Example 21.1. Not all indecomposable modules are simple. For example, $\mathbb{Z}$ is indecomposable, but is not simple.