Topology

K

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Chapter 1

Topological Space

Definition 1.1 (Topological Space)**.** A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to τ.
2. Any **arbitrary** union of members of τ belongs to τ.
3. The intersection of **finite number** of members of τ belongs to τ.

The collection τ is called a topology on X and the elements of τ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.

Example 1.1.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the discrete topology. In this case, (X, τ) is called the discrete space. It is the finest topology that can be defined on a set. (The set of all possible topologies on a given set forms a partially ordered set.)
2. $\tau = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology.
3. Let (X, d) be a metric space. Set

$$\tau_d := \{ U \in X \mid U \text{ is a open subset in the metric space } (X, d) \} . \tag{1.1}$$

   Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U.

   Here, τ is a topology. In other words, a metric induces a topology.

   (Proof as homework.)

4. The Zariski-topology.

Definition 1.2 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X . \tag{1.2}$$

Lemma 1.2.1. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 1.3 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. If $Y = X$ we also write $\mathrm{Homeo}(X)$.

Remark. The set of all homeomorphisms of X to itself $\mathrm{Homeo}(X)$ is a group with composition as its operation.

Definition 1.4 (Homeomorphism)**.** Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 1.4.1 (Lemma 5)**.** Let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets, then there exists **exactly one** topology $\tau \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \tau$
2. If $\tau' \subset \mathcal{P}(X)$ a topology with $S \subset \tau'$, then $\tau \subset \tau'$.

Remark. This lemma does not hold for basis.

Proof. missing □

Definition 1.5. 1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \} .$$

2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by
$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$

3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$

Remark. 1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Definition 1.6. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.

2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\mathrm{int}(S)$, is the set of all interior points of S.

3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\mathrm{cl}(S)$, is defined by

$$\mathrm{cl}(S) := X \setminus \mathrm{int}(X \setminus S).$$

Chapter 2

Connected Spaces and Sets

Definition 2.1 (Def 8)**.** A topological space X is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two disjoint sets.
2. The **only** subsets of X that are **both** open and closed (clopen) are the emptyset $\varnothing$ and the entire set X.

Lemma 2.1.1. Any interval $I \subset \mathbb{R}$ is connected.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \tag{2.1}$$

then $s \in [a, b] \subset I$ because I is an interval. □

Example 2.1.1. The general linear group $\mathrm{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Definition 2.2. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Remark. Let $f : X \longrightarrow Y$ be continuous and X be connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}B = f^{-1}(A \cap B) = \varnothing$ so $f^{-1}(A) = \varnothing$ or $f^{-1}(B) = \varnothing$, so $A = \varnothing$ or $B = \varnothing$ and we are done. □

Proposition 2.2.1. Connected components are closed subsets.

Proof. □

Example 2.2.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Lemma 2.2.1 (Lemma 11)**.** Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Proof. Locally constant implies continuous with regards to the discrete topology on Y. Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \varnothing$. Conclude f is identical to $f(x)$. $\square$

Application: $f : X \longrightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Definition 2.3. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 2.3.1. If X is path connected, then it is also connected.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \longrightarrow X$ be continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We have that γ^{-1} $\square$

Remark. The converse statement is not true in general.

Example 2.3.1. $X = \left\{ (x, \sin(\frac{1}{x})) \mid x > 0 \right\} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected but not path connected.

Proof. Homework $\square$

Remark. missing

Chapter 3

Separation Axioms

Literature: Groessere Liste in Sten, Seibeck

Definition 3.1 (T_1 Space)**.** A topological space (X, τ) is a T_1 space if any two distinct points in X are separable, i.e. if each lies in a neighborhood that does not contain the other point.

Lemma 3.1.1. 1. X is from type 1 if and only if $\{x\}$ is closed.

Remark. The type 1 and type 2 properties are inherited to subspaces, topological sums and products. Metric spaces are from type 2.