Part I

Commutative Rings

Definition 0.1 — Ring.
A ring is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(R, +)$ is an abelian group.
2. $(R, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Definition 0.2 — Ideal.

Definition 0.3 — Ideal Arithmetic.

Definition 0.4 — Prime Ideal.

Definition 0.5 — Maximal Ideal.

Definition 0.6 — Spectrum.

Definition 0.7 — .
Let A be a ring and $\mathfrak{a}$ an ideal. Let $\mathbf{V}(\mathfrak{a})$ denote the subset of $\mathrm{Spec}(A)$ consisting of those primes that contain $\mathfrak{a}$, i.e.

$$\mathbf{V}(\mathfrak{a}) := \{\, \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p} \,\} \tag{1}$$

We call $\mathbf{V}(\mathfrak{a})$ the variety of $\mathfrak{a}$.

Definition 0.8 — Zariski Topology.
Let $\mathfrak{a} \subseteq A$ be an ideal. Declaring the sets

$$Z(\mathfrak{a}) := \{\, \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p} \,\} \tag{2}$$

to be closed induces a topology on $\mathrm{Spec}(A)$, the Zariski Topology.

Definition 0.9 — Quotient Ring.
Given a ring A and two-sided ideal $\mathfrak{a}$ in A, we may define an congruence relation $\sim$ on A as follows:

$$x \sim y :\Longleftrightarrow x - y \in \mathfrak{a}. \tag{3}$$

The equivalence class of the element x in A is given by

$$[x] = x + \mathfrak{a} := \{\, x + a \mid a \in \mathfrak{a} \,\} \tag{4}$$

and the set of all such equivalence classes is denoted by $A/\mathfrak{a}$; it becomes a ring, the factor ring or the quotient ring of A modulo $\mathfrak{a}$, if one defines

1. $(a + \mathfrak{a}) + (b + \mathfrak{a}) = (a + b) + \mathfrak{a}$
2. $(a + \mathfrak{a})(b + \mathfrak{a}) = (ab) + \mathfrak{a}$

The map $\pi : R \longrightarrow A/\mathfrak{a}$, $x \mapsto \pi(x) := x + \mathfrak{a}$ is a surjective ring homomorphism and is sometimes called the natural quotient map or the canonical homomorphism.

Proposition 1 (Universal Property)**.** *Let A and B be rings, $\mathfrak{a}$ an ideal, and $f : A \longrightarrow B$ a ring homomorphism with $\mathfrak{a} \subseteq \mathrm{Ker}\,(f)$. Then there exists a unique ring homomorphism $\tilde{f} : A/\mathfrak{a} \longrightarrow B$ such that $f = \tilde{f} \circ \pi$.*

Definition 0.10 — Integral Domain.

Theorem 0.11.
- *prime ideal, quotient is integral domain*
- *same as above, but if prime maximal, then quotient is a fields*
- *Maximal ideals are prime ideals.*
- *There is a 1:1 correspondence*

$$\{\, \textit{Ideals in } A/\mathfrak{a} \,\} \longleftrightarrow \{\, \mathfrak{b}/\mathfrak{a} \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq A \,\} \tag{5}$$

0.1 Exercises