Integration and Integration

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Solving Integrals

Theorem 3.0.1 (Important Identities).

$$\int x^\alpha \,\mathrm{d}x = \frac{1}{\alpha+1}x^{\alpha+1} + c \qquad \text{For all } \alpha \in \mathbb{N} \tag{3.1}$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \qquad \text{If } x \neq 0. \tag{3.2}$$

$$\int e^x \,\mathrm{d}x = e^x + c \tag{3.3}$$

$$\int \cos x \,\mathrm{d}x = \sin x + c \tag{3.4}$$

$$\int \sin x \,\mathrm{d}x = -\cos x + c \tag{3.5}$$

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(x) + c \tag{3.6}$$

$$\int \tag{3.7}$$

Exercise 3.1

$$\int (1-t)^9 \,\mathrm{d}x \tag{3.8}$$

Solution 3.1
Substitute $1 - t = u \Rightarrow -1\,\mathrm{d}t = \mathrm{d}u$, then we have

$$\int (1-t)^9 \,\mathrm{d}x = -\int u^9 \,\mathrm{d}u \qquad \text{(Substitution.)} \tag{3.9}$$

$$= -\frac{u^{10}}{10} + c \qquad \text{(Important identities: 3.1)} \tag{3.10}$$

$$= -\frac{(1-t)^{10}}{10} + c. \qquad (u = 1 - t.) \tag{3.11}$$

Exercise 3.2

$$\int (x^2+1)^2 \,\mathrm{d}x \tag{3.12}$$

Solution 3.2
Substitute $x^2 + 1 = u \Rightarrow 2x\,\mathrm{d}x = \mathrm{d}u$.

$$\int (x^2+1)^2\,\mathrm{d}x = \int x^4 + 2x^2 + 1\,\mathrm{d}x \tag{3.13}$$

$$= \int x^4\,\mathrm{d}x + 2\int x^2\,\mathrm{d}x + \int 1\,\mathrm{d}x \tag{3.14}$$

$$= \frac{1}{5}x^5 + \frac{2}{3}x^3 + x \tag{3.15}$$

Exercise 3.3

$$\int \frac{1}{x^2 - x + 1}\,\mathrm{d}x \tag{3.16}$$

Solution 3.3
Substitute $x - \frac{1}{2} = u \Rightarrow 1\,\mathrm{d}x = \mathrm{d}u$.

$$\int \frac{1}{x^2 - x + 1}\,\mathrm{d}x = \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}\,\mathrm{d}x \tag{3.17}$$

$$= \int \frac{1}{u^2 + \frac{3}{4}}\,\mathrm{d}x \tag{3.18}$$

$$= \frac{4}{3}\int \frac{1}{\frac{4}{3}u^2 + 1}\,\mathrm{d}u \tag{3.19}$$

The last step was done in order to force the integrand to be in a form similar to

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{x^2+1} \tag{3.20}$$

Lastly, substitute in the equation above $\frac{2}{\sqrt{3}}u = v \Rightarrow \frac{2}{\sqrt{3}}\,\mathrm{d}u = \mathrm{d}v$. It is $u = \frac{\sqrt{3}}{2}v$ and $\mathrm{d}u = \frac{\sqrt{3}}{2}\,\mathrm{d}v$, therefore, we have

$$u^2 = \frac{3}{4}v^2 \Rightarrow u = \frac{\sqrt{3}}{2}v \Rightarrow \mathrm{d}u = \frac{\sqrt{3}}{2}\,\mathrm{d}v \tag{3.21}$$

$$\frac{4}{3}\int \frac{1}{\frac{4}{3}u^2+1}\,\mathrm{d}u = \frac{4}{3}\int \frac{1}{\frac{4}{3}(\frac{\sqrt{3}}{2}v)^2 + 1}\frac{\sqrt{3}}{2}\,\mathrm{d}v \tag{3.22}$$

$$= \frac{4}{3}\frac{\sqrt{3}}{2}\int \frac{1}{v^2+1}\,\mathrm{d}v \tag{3.23}$$

$$= \frac{2\sqrt{3}}{3}\arctan v \tag{3.24}$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{2}u\right) \tag{3.25}$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{2}(x-\frac{1}{2})\right) \tag{3.26}$$