Exercise 1.1

Let $A \subseteq B$ be an integral extension of rings and assume that B is an integral domain. Suppose $\mathfrak{q} \subset B$ is a prime ideal and let $\mathfrak{p} := \mathfrak{q} \cap A \subseteq A$.

1. Prove that A is a field if and only if B is a field.

   Proof. (a) "$\Rightarrow$": Let A be a field, and $b \in B$ an element. Since B is an integral extension of rings, b is integral over A, thus we have for some $a_0, \dots, a_{n-1}$ that

   $$\begin{aligned} & b^n + a_{n-1}b^{n-1} + \dots + a_1 b + a_0 = 0 \\ \iff & b^n + a_{n-1}b^{n-1} + \dots + a_1 b = -a_0 \\ \iff & b(b^{n-1} + a_{n-1}b^{n-2} + \dots + a_1) = -a_0 \\ \iff & b\left(\frac{-(b^{n-1} + a_{n-1}b^{n-2} + \dots + a_1)}{a_0}\right) = 1. \end{aligned}$$

   The last step was possible because if $a_0 = 0$, then the above would not have been the minimal polynomial. The equation above shows b is invertible, hence B is a field.

   (b) "$\Leftarrow$": Let B be a field, and $y \in A$ an element. Since B is a field, y has an inverse x in B that is integral over A, i.e.

   $$x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0 \quad (1)$$

   $$\iff x + a_{n-1} + \dots + a_1 y^{n-2} + a_0 y^{n-1} = 0 \quad (2)$$

   $$\iff x = -(a_{n-1} + \dots + a_1 y^{n-2} + a_0 y^{n-1}) \quad (3)$$

   so $x \in A$ and A is a field.

   □

2. Show that $\mathfrak{p}$ is a prime ideal of A and that $A/\mathfrak{p}$ can be viewed as a subring of $B/\mathfrak{q}$.

   Proof. □