Exercise 2 c)

Solution. 1. $\mathcal{B}$ is a subbasis for the discrete topology. Take an arbitary subset $\mathcal{U} \subset \mathbb{R}$. If $\mathcal{U} = \mathbb{R}$, then we simply have

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x, x+1\}$$

as $\{x, x+1\}$ are members of the subbasis $\mathcal{B}$. Similary, if $\mathcal{U} = \mathbb{R} \setminus \{y\}$ for some $y \in \mathbb{R}$, then we have

$$\mathbb{R} \setminus \{y\} = \left(\bigcup_{\substack{x \in \mathbb{R} \\ x+1 \neq y}} \{x, x+1\} \right) \cup \{y-1, y+1\}$$

because again $\{y-1, y+1\}$ lies in $\mathcal{B}$. For any other cases, notice that there are two distinct points $y \neq z$ with $y, z \notin \mathcal{U}$, thus the two sets $\{x, y\}$ and $\{x, z\}$ are members of $\mathcal{B}$. Therefore, we have

$$\begin{aligned} \mathcal{U} &= \bigcup_{x \in \mathcal{U}} \{x\} \\ &= \bigcup_{x \in \mathcal{U}} \{x, y\} \cap \{x, z\}. \end{aligned}$$

In other words, every subset of $\mathbb{R}$ is a union of finite intersections of members in $\mathcal{B}$, thus $\mathcal{B}$ as a subbasis generates the discrete topology.

2. However, $\mathcal{B}$ is not a basis of the discrete topology. Plainly, a singleton set cannot be generated from a union of elements of $\mathcal{B}$.

□

Exercise 3 b)

Suppose $\mathcal{B}$ is a subbasis for a topology $\mathcal{T}$ on a set X. Given another topological space Y, show that a map $f : Y \longrightarrow X$ is continuous if and only if for every $\mathcal{U} \in \mathcal{B}$, $f^{-1}(\mathcal{U})$ is open in Y.

Lemma. The preimage of a map is stable under arbitary unions and finite intersections.

Proof. Let $f : X \longrightarrow Y$ be a map, $\{A_i\}_{i \in I}$ be a family of subsets in Y, and A and B subsets in Y.

1. It is plainly

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{i \in I} A_i \right) &\iff f(x) \in \bigcup_{i \in I} A_i \\ &\iff \text{there is a } i \in I \text{ such that } f(x) \in A_i \\ &\iff \text{there is a } i \in I \text{ such that } x \in f^{-1}(A) \\ &\iff x \in \bigcup_{i \in I} f^{-1}(A). \end{aligned}$$

2. We simply have

$$\begin{aligned} x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\ &\iff f(x) \in A \text{ and } f(x) \in B \\ &\iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

$\square$

Solution. Denote the topology of Y by $\mathcal{S}$.

"$\Rightarrow$": Let $f : Y \longrightarrow X$ be continuous and fix an $\mathcal{U} \in \mathcal{B}$. Since $\mathcal{B}$ is subbasis, all its elements are open subsets, thus $\mathcal{U}$ is open. Then by definition of continuous maps, the preimage $f^{-1}(\mathcal{U})$ is also open in Y. As we have fixed an arbitary $\mathcal{U} \in \mathcal{B}$, we may conclude the desired result.

"$\Leftarrow$": On the other hand, let for every $\mathcal{U} \in \mathcal{B}$ the preimage $f^{-1}(\mathcal{U})$ be open in Y. Consider an arbitary open subset $\mathcal{V} \in \mathcal{T}$. By the definition of a subbasis, $\mathcal{V}$ is a union of finite intersection of members of $\mathcal{B}$, i.e.

$$\mathcal{V} = \bigcup_{\alpha \in I} (\mathcal{U}_1^\alpha \cap \dots \cap \mathcal{U}_{n_\alpha}^\alpha)$$

with I being an arbitary index set, and $n_\alpha \in \mathbb{N}$ for each $\alpha \in I$. The preimage of $\mathcal{V}$ therefore is

$$\begin{aligned} f^{-1}(\mathcal{V}) &= f^{-1}\left(\bigcup_{\alpha \in I} (\mathcal{U}_1^\alpha \cap \dots \cap \mathcal{U}_{n_\alpha}^\alpha) \right) \\ &= \bigcup_{\alpha \in I} (f^{-1}(\mathcal{U}_1^\alpha) \cap \ldots \cap f^{-1}(\mathcal{U}_{n_\alpha}^\alpha)) \end{aligned}$$

where we applied the aforementioned lemma on the last step. Now, $f^{-1}(\mathcal{U}_i)$ are open subsets for all $i \in \mathbb{N}$. By the definition of topological spaces, unions of finite intersections of open subsets are also open, hence $f^{-1}(\mathcal{V})$ is open. Thus, f is continuous. $\square$

Exercise 3 c)

Now suppose $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ is a collection of topological spaces, $(X, \mathcal{T})$ is $\prod_{\alpha \in I} X_\alpha$ with the product topology, and the subbase $\mathcal{B} \subset \mathcal{T}$ is taken to consist of all sets of the form

$$\{ x_{\alpha \alpha \in I} \mid x_\beta \in \mathcal{U}_\beta \} \subset \prod_\alpha X_\alpha$$

for arbitary $\beta \in I$ and $\mathcal{U}_\beta \in \mathcal{T}_\beta$.

Show that a sequence $\{x_\alpha^n\}_{\alpha \in I} \in X$ converges to $\{x_\alpha\}_{\alpha \in X} \in X$ as $n \longrightarrow \infty$ if and only if $x_\alpha^n \longrightarrow x_\alpha$ for every $\alpha \in I$.

Solution. "$\Rightarrow$": Let the sequence $\{x_\alpha^n\}_{\alpha \in I} \in X$ converge to $\{x_\alpha\}_{\alpha \in I} \in X$. By exercise 3 a), we have that for all subbase $\mathcal{U} \in \mathcal{B}$ with $\{x_\alpha\}_{\alpha \in I} \in \mathcal{U}$ it is $\{x_\alpha^n\}_{\alpha \in I} \in \mathcal{U}$ for sufficiently large n. The members of the subase was in the form

$$\{\{x_\alpha\}_{\alpha \in I} \mid x_\beta \in \mathcal{U}_\beta\} \subset \prod_{\alpha \in I} X_\alpha.$$

Thus, for each $\alpha \in I$, we have $x_\alpha^n \to x_\alpha$.

"$\Leftarrow$": On the other hand, let x_α^n converge to x_α for every $\alpha \in I$. By definition of convergence, we have that for every neighbourhood $\mathcal{V}_\alpha$ of x_α, it is $x_\alpha^n \in \mathcal{V}_\alpha$ for all sufficiently large n. Denote the open subsets of these neighbourhoods by $\mathcal{U}_\alpha$. Then, $\prod_{\alpha \in I} \mathcal{U}_\alpha$ is a neighbourhood of $\{x_\alpha\}_{\alpha \in I}$ in the product topology and also contains all $\{x_\alpha^n\}_{\alpha \in I}$ if n is sufficiently large enough. Thus, $\{x_\alpha^n\}_{\alpha \in I}$ converges to $\{x_\alpha\}_{\alpha \in I}$. $\square$

Exercise 7

Solution. 1. The error is in the following part.

> "which means that x_n cannot enter arbitary neighbourhoods of $x \in X$ for arbitary large values of n, i.e. there exists $N_x \in \mathbb{N}$ and an open neighbourhood of $\mathcal{U}_x \subset X$ of x such that $x_n \notin \mathcal{U}_x$ for every $n \geq N_x$"

The definition of convergence of a sequence was that for every neighbourhood $\mathcal{U} \subset X$ of x it is $x_n \in \mathcal{U}$ for all $n \in \mathbb{N}$ sufficiently large. Thus, if we negate the definition, we have that for every neighbourhood $\mathcal{U} \subset X$ of x it is $x_n \notin \mathcal{U}$ for some $n \in \mathbb{N}$ sufficiently large.

This mistakes makes the last conclusion false. The proof says "each of these finitely many subsets contains at most finitely many terms of x_n", but in actuality the subsets may contain infinitely many terms of x_n.

2. It sufficies to require X to be a first-countable space. Then,

$\square$