

Topology

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Chapter 1

Topological Space

Definition 1.1 (Topological Space). A **topological space** is an **ordered pair** (X, τ) , where X is a **set** and τ is a **collection of subsets** that satisfies the following **axioms**.

1. The **empty set** \emptyset and the **entire set** X belongs to τ .
2. Any **arbitrary union** of members of τ belongs to τ .
3. The **intersection** of **finite number** of members of τ belongs to τ .

The **collection** τ is called a **topology** on X and the **elements** of τ are called **open sets**. A **subset** $A \subset X$ is said to be **closed** if its **complement** $X \setminus A$ is **open**.

Remark. We write X for (X, τ) .

Example 1.1.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the **discrete topology**. In this case, (X, τ) is called the **discrete space**. It is the finest topology. (One can define an ordering of topologies.)
2. $\tau = \{\emptyset, \mathcal{P}(X)\}$ is called the **trivial topology**.
3. Let (X, d) be a **metric space**. Set

$$\tau_d := \{U \in \mathcal{P}(X) \mid U \text{ is an open subset in the metric space } (X, d)\}. \quad (1.1)$$

Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U .

Here, τ is a topology. In other words, a metric induces a topology.

(Proof as homework.)

4. The Zariski-topology.

Definition 1.2 (Continuous Maps). Let (X, τ_X) and (Y, τ_Y) be **topological spaces**. A **map** $f : X \rightarrow Y$ is said to be **continuous** if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \quad (1.2)$$

Lemma 1.2.1. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \rightarrow Y$ is ϵ - δ -continuous if and only if f is continuous.

Definition 1.3 (Homeomorphism). Let X and Y be topological spaces.

1. A map $f : X \rightarrow Y$ is a homeomorphism if it has the following properties.
 - (a) f is bijective.
 - (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. If $Y = X$ we also write $\text{Homeo}(X)$.

Remark. The set of all homeomorphisms of X to itself $\text{Homeo}(X)$ is a group with composition as its operation.

Definition 1.4. Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of \mathcal{O} is the union of subsets from \mathcal{B} .
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of \mathcal{O} is the union of finite intersections of subsets from \mathcal{S} .

We say that \mathcal{B} and \mathcal{S} generates \mathcal{O} and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 1.4.1. Let $\mathcal{S} \subset \mathcal{P}(X)$, then there exists exactly one topology $\mathcal{O} \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \mathcal{O}$

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Chapter 2

Connected Spaces and Sets

Definition 2.1 (Def 9). A topological space X is said to be **connected**, if one of the following **equivalent** conditions is met.

1. X is **not** a **union** of two **disjoint** sets.
2. The **only** **subsets** of X that are **both** **open** and **closed** (**clopen**) are the emptyset \emptyset and the entire set X .

Proof. missing. □

Lemma 2.1.1. Any **interval** $I \subset \mathbb{R}$ is **connected**.

Proof. Let $I = A \cup B$ with A and B being nonempty disjoint sets in \mathbb{R} that are open, and let $a \in A$ and $b \in B$. Without loss of generality, assume $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \} \quad (2.1)$$

then $s \in [a, b] \subset I$ because I is an interval. □

Example 2.1.1. The general linear group $\text{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Definition 2.2. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Remark. Let $f : X \rightarrow Y$ be continuous and X be connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$ so $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$, so $A = \emptyset$ or $B = \emptyset$ and we are done. □

Proposition 2.2.1. Connected components are closed subsets.

Proof. □

Example 2.2.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Lemma 2.2.1 (Lemma 11). Let X be connected and $f : X \rightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$, then f is constant.

Proof. Locally constant implies continuous with regards to the discrete topology on Y . Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \emptyset$. Conclude f is identical to $f(x)$. □

Application: $f : X \rightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Definition 2.3. X is said to be **path connected**, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 2.3.1. If X is path connected, then it is also connected.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \rightarrow X$ be continuous path with $\gamma(0) = a$ and $\gamma(1) = b$. We have that γ^{-1} □

Remark. The converse statement is not true in general.

Example 2.3.1. $X = \{ (x, \sin(\frac{1}{x})) \mid x > 0 \} \cup \{0\} \times [-1, 1] \subset \mathbb{R}^2$ is connected but not path connected.

Proof. Homework □

Remark. missing

Chapter 3

Trennungsaxiome

Literature: Groessere Liste in Sten, Seibeck

Definition 3.1. (X, τ) fullfills

1. For all $x \in X$ and $y \in X$ with $x \neq y$ there is a subset $U \in \tau$ open such that $y \in U$ but $x \notin U$.
2. Hausdorff

Lemma 3.1.1. 1. X is from type 1 if and only if $\{x\}$ is closed.

Remark. The type 1 and type 2 properties are inherited to subspaces, topological sums and products. Metric spaces are from type 2.