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My notes on "The Strong Factorial Conjecture" by Eric Edo and Arno van den Essen.
See: https://arxiv.org/abs/1304.3956

1 Rigidity Conjecture

Proposition 1. A power series $f(X) = \sum_{k\in\mathbb{N}_0} a_k X^k \in \mathbb{C}[[X]]$ has a compositional inverse $f^{-1}(X)$ if and only if $a_1 \neq 0$, in which case $f^{-1}(X)$ is unique.

Proof. Let $g(X) = b_0 + b_1 X + b_2 X^2 + \cdots \in \mathbb{C}[[X]]$ be some power series that satisfies $f(g(X)) = X$. We have

$$X = a_0 + a_1(b_0 + b_1 X + b_2 X^2 + \cdots) + a_2(b_0 + b_1 X + b_2 X^2 + \cdots)^2 + \cdots \quad (1)$$
$$= a_0 + a_1 b_0 + a_2 b_0 + \cdots + \quad (2)$$
$$a_1 b_1 X \quad (3)$$

□

Theorem 2 (Conjecture 2.13)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the first m consecutive coefficient of the compositional inverse $a^{-1}(X)$ vanish, then $a(X) = X$.

Theorem 3 (Conjecture 2.14)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the coefficients of $X^{n+1}, \ldots, X^{n+m}$ of the compositional inverse vanish, then $a(X) = X$.

Remark. $R(m)$ if and only if $R(m)_n$ for all $n \in \mathbb{N}_+$.

Proof. Let $R(m)$ be true for a $m \in \mathbb{N}_0$.
Then $R(m)_1$ is true, i.e. if $\deg(a) \leq m+1$ and if the □

Remark. If we denote the polynomial $a(X)$ by $\sum_{k\in\mathbb{N}_0} a_k X^k$ for some $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}_0$, then the condition $a(X) \equiv X \mod X^2$ amounts to $a_0 = 0$ and $a_1 = 1$.

Moreover, we have this:

A power series has a compositional inverse if and only if $a_1 \neq 0$. In that case, the inverse is unique.

See
https://www.amazon.com/dp/B00HMUGS4S
https://math.stackexchange.com/questions/2520744/finding-compositional-inverses-for-formal-power-series

My questions:

1. What if $a_0 \neq 0$? Pick $a_0 = 3$.

   Let $f \in \mathbb{C}[X]$ be a polynomial with $a_0 \neq 0$. Then we may write $f(X) = g(X) + a_0$ where g has a compositional inverse. Thus it it

$$\begin{aligned} g^{-1}(g(X) + a_0) &= g^{-1}(g(X)) + g^{-1}(a_0) \\ &= X + g^{-1}(a_0) \end{aligned}$$

$$\begin{aligned} h(X) &= g^{-1}(X) + g^{-1}(a_0) \\ h(f(X)) &= h(g(X) + a_0) \\ &= g^{-1}(g(X) + a_0) + g^{-1}(a_0) \\ &= X \end{aligned}$$

   Let $f \in \mathbb{C}[X]$ be a polynomial with $a_1 \neq 1$ and $a_1 \neq 0$. Then we may write $f(X) =$

https://www.math.uwaterloo.ca/ dgwagner/co430I.pdf
proof

Proposition 4. 1. The polynomial $a(X)$ is invertible for the composition.

2. For all $i \in \{1, \dots, \deg(a-1)\}$, the coefficient a_i is nilpotent element in A. I just don't see this ...

Lemma 5 (Lagrange Inversion Formula)**.** Let K be a field of charateristic

Example 5.1 (See 5.4.4)**.** $f(X) = Xe^{-X} = X\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} X^k$

$$[X^n]f^{-1}(X) = \frac{1}{n}[X^{n-1}]e^{nX}$$

Lemma 6 (Lemma 2.20 (Additive Inversion Formula))**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers. The formal inverse of $a(X) = X(1 - (\alpha_1 X + \cdots + \alpha_m X^m))$ is given by the following formula

$$a^{-1}(X) = X\left(1 + \frac{1}{n+1}\sum_{n\geq 1} u_n X^n\right)$$

where

$$u_n = \frac{1}{n!}\sum_{j_1+2j_2+\cdots+mj_m=n} \frac{(n+j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\alpha_1^{j_1}\cdots\alpha_m^{j_m}$$

Proposition 7 (Proposition 2.23)**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers and let $(u_n)_{n\in\mathbb{N}_+}$ be a sequence defined by AIF in Lemma 2.20. For all $n \in \mathbb{N}_+$, the Rigidity Conjecture $R(m)_n$ is equivalent to the following implication: If $u_n = \cdots = u_{n+m-1} = 0$ then $\alpha_1 = \cdots = \alpha_m = 0$.

Proof. □

Theorem 8. 1. The inclusion $E^{[m]} \subset F_n^{[m]}$ implies $R(m)_n$

Definition 9.

$$E^{[m]} = \{ X_1 \cdots X_m(\mu_1 X_1 + \cdots + \mu_m X_m) \mid \mu_1, \dots, \mu_m \in \mathbb{C} \} \subset$$
$$F_n^{[m]} = \left\{ f \in \mathbb{C}^{[m]} \setminus \{0\} \;\middle|\; \mathcal{L}(f^k) \neq 0 \text{ for some } n \leq k \leq \mathcal{N}(f) - 1 \right\} \cup \{0\}$$