

# Chapter 1

## Commutative Rings

### Definitions

1. prime, coprime, relatively prime, irreducible

**Exercise 1.1.** Let  $\varphi : A \longrightarrow B$  be a ring homomorphism,  $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$  ideals in  $A$ , and  $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$  ideals of  $B$ . Prove the following statements.

1.  $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$ .

*Proof.* We show  $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$ . Let  $x \in (\mathfrak{a}_1 + \mathfrak{a}_2)^e$ , then we have for some index set  $I$

$$x = \sum_{i \in I} \lambda_i x_i, \quad (1.1)$$

where  $\lambda_i \in B$  and  $x_i \in \varphi(\mathfrak{a}_1 + \mathfrak{a}_2)$  for all  $i \in I$ . For each  $i \in I$  it is  $x_i = \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2})$ , hence

$$x = \sum_{i \in I} \lambda_i \varphi(\mu_{i,1}a_{i,1} + \mu_{i,2}a_{i,2}) \quad (1.2)$$

$$= \sum_{i \in I} \lambda_i (\varphi(\mu_{i,1}a_{i,1}) + \varphi(\mu_{i,2}a_{i,2})) \quad (\text{by linearity}) \quad (1.3)$$

$$= \sum_{i \in I} \lambda_i (\mu_{i,1}\varphi(a_{i,1}) + \mu_{i,2}\varphi(a_{i,2})) \quad (\text{by linearity}) \quad (1.4)$$

$$= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad (\text{by distributivity}) \quad (1.5)$$

$$= \sum_{i \in I} \lambda_i \mu_{i,1} \varphi(a_{i,1}) + \sum_{i \in I} \lambda_i \mu_{i,2} \varphi(a_{i,2}) \quad (\text{reordering the sum}). \quad (1.6)$$

$$(1.7)$$

The last term is exactly the elements expressed by  $\mathfrak{a}_1^e + \mathfrak{a}_2^e$ , therefore,  $(\mathfrak{a}_1 + \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1)^e + (\mathfrak{a}_2)^e$ .

I think the above proof should work into both directions.  $\square$

2.  $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$

*Proof.* We have

$$(\mathfrak{b}_1 + \mathfrak{b}_2)^c = \left\{ x \in A \mid \exists b_1 \in \mathfrak{b}_1 \exists b_2 \in \mathfrak{b}_2 : \varphi(x) = b_1 + b_2 \right\}. \quad (1.8)$$

Now let  $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$ , then  $x = a_1 + a_2$  where  $\varphi(a_1) \in \mathfrak{b}_1$  and  $\varphi(a_2) \in \mathfrak{b}_2$ . It is

$$\varphi(x) = \varphi(a_1 + a_2) \quad (1.9)$$

$$= \varphi(a_1) + \varphi(a_2) \quad (\text{by additivity}) \quad (1.10)$$

Since  $\varphi(a_1) \in \mathfrak{b}_1$  and  $\varphi(a_2) \in \mathfrak{b}_2$  we have that  $x \in (\mathfrak{b}_1 + \mathfrak{b}_2)^c$ .  $\square$

**Exercise 1.2.** Let  $\varphi : A \longrightarrow B$  be a ring homomorphism,  $\mathfrak{a}$  an ideal of  $A$ , and  $\mathfrak{b}$  an ideal of  $B$ . Prove the following statements:

1. Then  $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ .

*Proof.* It is

$$\mathfrak{a}^{ec} = \left\{ x \in A \mid \varphi(x) \in \mathfrak{a}^e \right\} \quad (1.11)$$

$$= \left\{ x \in A \mid \varphi(x) \in \langle \varphi(\mathfrak{a}) \rangle \right\} \quad (1.12)$$

$$= \left\{ x \in A \mid \forall i \in I \exists a_i \in \mathfrak{a}_1 : \varphi(x) = \sum_{i \in I} \lambda_i \varphi(a_i) \right\}. \quad (1.13)$$

Let  $a \in \mathfrak{a}$  and choose  $I = \{1\}$ ,  $\lambda_1 = 1$ , and  $a_i = a$ , then  $a \in \mathfrak{a}^{ec}$ .  $\square$

2.  $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$ .
3.  $\mathfrak{a}^{ece} = \mathfrak{a}^e$ .
4.  $\mathfrak{b}^{cec} = \mathfrak{b}^c$ .
5. If  $\mathfrak{b}$  is an extension, then  $\mathfrak{b}^c$  is the largest ideal of  $A$  with extension  $\mathfrak{b}$ .
6. If two extensions have the same contraction, then they are equal.

*Proof.* a  $\square$

**Exercise 1.3.** Let  $A$  be a ring,  $A[\mathcal{X}, \mathcal{Y}]$  the polynomial ring in two sets of variables  $\mathcal{X}$  and  $\mathcal{Y}$ . Show that  $\langle \mathcal{X} \rangle$  is prime if and only if  $A$  is a domain.

*Proof.* It should be noted here, that  $A[\mathcal{X}]$  does not contain  $X_1 X_2$  for example. It does contain  $X_1 + X_2$  however. The rest is easy.  $\square$

**Exercise 1.4.** Show that, in a PID, nonzero elements  $x$  and  $y$  are relatively prime (share no prime factor) if and only if they're coprime.

**Exercise 1.5.** Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals, and  $\mathfrak{p}$  a prime ideal. Prove that these conditions are equivalent:

1.  $\mathfrak{a} \subseteq \mathfrak{p}$  or  $\mathfrak{b} \subseteq \mathfrak{p}$
2.  $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$
3.  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$

*Proof.* (1) to (2) is easy. Same for (2) to (3). For (3) to (1) show it with contradiction.  $\square$

**Exercise 1.6.** Let  $A$  be a ring,  $\mathfrak{p}$  a prime ideal, and  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$  maximal ideals with  $\mathfrak{m}_1, \dots, \mathfrak{m}_n = 0$ . Show  $\mathfrak{p} = \mathfrak{m}_i$  for some  $i$ .

*Proof.* By induction. Proof first for  $m_1 m_2$ , the rest is clear.  $\square$