Chapter 1

Probability Space

1. Decide if the following statements are true or false.

1. If F^X is a CDF of a random variable X that has only values in $(0, 1)$, then $F^X(0) = 0$.
2. Let $\Omega = \{1, 2, \ldots, 6\}^6$ be the sample space that describes a six consequtive dice roles. The event $\{\omega \in \Omega \mid \omega_1 < \omega_2 < \ldots < \omega_6\}$ is precisely the outcome in which a street $1, 2, \ldots, 6$ is rolled.
3. Every probability measure is σ-finite.

2. For $\lambda > 0$ let $X \sim \mathrm{Exp}(\lambda)$ and let

$$Y := \lceil X \rceil := \min\{\, n \in \mathbb{N} \mid n \geq X \,\} \tag{1.1}$$

Show that for the parameter $p = 1 - e^{-\lambda}$ holds $Y \sim \mathrm{Geo}(p)$.

Solution:
For the distribution of Y we have for all $k \in \mathbb{N}_0$

$$p^Y(k) = \mathbb{P}(Y = k) = \mathbb{P}(\lceil X \rceil = k) = \mathbb{P}(k - 1 < X \leq k) = F^X(k) - F^X(k - 1). \tag{1.2}$$

On the other hand, the CDF of X is

$$F^X(x) = \int_{-\infty}^{x} f^X(y)\,\mathrm{d}y = \int_{-\infty}^{x} \lambda e^{-\lambda y} \mathbb{1}_{[0,\infty)}(y)\,\mathrm{d}y = 1 - e^{-\lambda x}. \tag{1.3}$$

Therefore we have

$$\begin{aligned}
p^Y(k) &= F^X(k) - F^X(k - 1) && (1.4)\\
&= (1 - e^{-\lambda k}) - (1 - e^{-\lambda(k-1)}) && (1.5)\\
&= -e^{-\lambda k} + e^{-\lambda(k-1)} && (1.6)\\
&= -e^{-\lambda} \cdot e^{-\lambda(k-1)} + e^{-\lambda(k-1)} && (1.7)\\
&= e^{-\lambda(k-1)}(1 - e^{-\lambda}) && (1.8)\\
&= \left(1 - \left(1 - e^{-\lambda}\right)\right)^{k-1} (1 - e^{-\lambda}) && (1.9)
\end{aligned}$$

Setting $1 - e^{-\lambda} = p$ yields the desired result $p^Y(k) = (1 - p)^{k-1}\, p$.

3. Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$.

1. Show that $Y \sim N(a\mu + b, a^2\sigma^2)$.

2. How must a and b be chosen so that $Y \sim N(0,1)$ holds?

Solution:

1. Just use Transformationformula.
2. $a = \sigma^{-1}$ and $b = -\mu\sigma^{-1}$.

4. Let $\Phi = \Phi_{0,1}$ the cumulative mass function of $N(0,1)$-distribution and let Φ^{-1} be its inverse. Show that the following equation hold.

$$\Phi^{-1}(p) = -\Phi^{-1}(1-p), \qquad p \in (0,1). \tag{1.10}$$

Solution:
Consider the PMF of the standard normal distribution Φ and let $x \in \mathbb{R}_0^+$. Because of its symmetry on the y-axis, we have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \phi(y)\,\mathrm{d}y && (1.11)\\
&= \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y + \int_{x}^{\infty} \phi(y)\,\mathrm{d}y && (1.12)\\
&= \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y + \int_{-\infty}^{-x} \phi(y)\,\mathrm{d}y && (1.13)\\
&= \Phi(x) + \Phi(-x). && (1.14)
\end{aligned}$$

Hence $\Phi(-x) = 1 - \Phi(x)$. Using this and the bijectivity of Φ we get

$$\begin{aligned}
\Phi^{-1}(p) = -\Phi^{-1}(1-p) &\iff \Phi\left(\Phi^{-1}(p)\right) = \Phi\left(-\Phi^{-1}(1-p)\right) && (1.15)\\
&\iff p = 1 - \Phi(\Phi^{-1}(1-p)) && (1.16)\\
&\iff p = p && (1.17)
\end{aligned}$$

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Chapter 2

Independence

6. Let $X_1, \ldots, X_n$ be real-valued independent random variables with CDFs $F_1, \ldots, F_n$.

1. Show that the CDFs of $M = \max(X_1, \ldots, X_n)$ and $m = \min(X_1, \ldots, X_n)$ are given by

$$F^M(x) = \prod_{i=1}^{n} F_i(x) \qquad \text{and} \qquad F^m(x) = 1 - \prod_{i=1}^{n} (1 - F_i(x))\,. \tag{2.1}$$

Solution:

$$\{X_i \in A_i\} \tag{2.2}$$