1 Question 1

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two absolutely convergent series. Does $\sum_{k=1}^{\infty} a_k b_k$ converge absolutely as well?

Solution. Yes. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two absolutely convergent series. We aim to show that $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely as well.

Since $\sum_{k=1}^{\infty} b_k$ converges absolutely, the sequence $\{b_k\}$ is bounded, i.e., there exists $M > 0$ such that $|b_k| \leq M$ for all $k \in \mathbb{N}$. Similarly, the absolute convergence of $\sum_{k=1}^{\infty} a_k$ implies that $\sum_{k=1}^{\infty} |a_k| < \infty$.

Consider the series

$$\sum_{k=1}^{\infty} |a_k b_k| = \sum_{k=1}^{\infty} |a_k| \cdot |b_k|.$$

Using the boundedness of $\{b_k\}$, we have

$$|a_k b_k| \leq |a_k| \cdot M.$$

Thus,

$$\sum_{k=1}^{\infty} |a_k b_k| \leq M \sum_{k=1}^{\infty} |a_k|.$$

Since $\sum_{k=1}^{\infty} |a_k| < \infty$, it follows that $\sum_{k=1}^{\infty} |a_k b_k|$ converges. Hence, $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely. □

2 Question 2

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two convergent series. Moreover, let a_k and b_k be two null sequences. Does $\sum_{k=1}^{\infty} a_k b_k$ converge?

Solution. No. Consider

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}.$$

a_k and b_k are clearly null sequences, and by alternating series test, both series converge. However,

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \cdot \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which is the harmonic series and hence does not converge. □

3 Question 3

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two convergent series. Moreover, let a_k and b_k be monotonic sequences. Does $\sum_{k=1}^{\infty} a_k b_k$ converge absolutely?

Solution. Yes. If a_k is monotonic and $\sum_{k=1}^{\infty} a_k$ converge, then $\sum_{k=1}^{\infty} a_k$ converges absolutely. Similarly, $|b_k|$ is bounded by b_1. Now, we have

$$\sum_{k=1}^{\infty} |a_k b_k| = \sum_{k=1}^{\infty} |a_k||b_k| \leq b_1 \sum_{k=1}^{\infty} |a_k|$$

Thus, $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely. □