

# Chapter 1

## Probability Space

1. Decide if the following statements are true or false.

1. If  $F^X$  is a CDF of a random variable  $X$  that has only values in  $(0, 1)$ , then  $F^X(0) = 0$ .
2. Let  $\Omega = \{1, 2, \dots, 6\}^6$  be the sample space that describes a six consecutive dice roles. The event  $\{\omega \in \Omega \mid \omega_1 < \omega_2 < \dots < \omega_6\}$  is precisely the outcome in which a street  $1, 2, \dots, 6$  is rolled.
3. Every probability measure is  $\sigma$ -finite.

2. For  $\lambda > 0$  let  $X \sim \text{Exp}(\lambda)$  and let

$$Y := \lceil X \rceil := \min\{n \in \mathbb{N} \mid n \geq X\} \quad (1.1)$$

Show that for the parameter  $p = 1 - e^{-\lambda}$  holds  $Y \sim \text{Geo}(p)$ .

**Solution:**

For the distribution of  $Y$  we have for all  $k \in \mathbb{N}_0$

$$p^Y(k) = \mathbb{P}(Y = k) = \mathbb{P}(\lceil X \rceil = k) = \mathbb{P}(k-1 < X \leq k) = F^X(k) - F^X(k-1). \quad (1.2)$$

On the other hand, the CDF of  $X$  is

$$F^X(x) = \int_{-\infty}^x f^X(y) dy = \int_{-\infty}^x \lambda e^{-\lambda y} \mathbb{1}_{[0, \infty)}(y) dy = 1 - e^{-\lambda x}. \quad (1.3)$$

Therefore we have

$$p^Y(k) = F^X(k) - F^X(k-1) \quad (1.4)$$

$$= (1 - e^{-\lambda k}) - (1 - e^{-\lambda(k-1)}) \quad (1.5)$$

$$= -e^{-\lambda k} + e^{-\lambda(k-1)} \quad (1.6)$$

$$= -e^{-\lambda} \cdot e^{-\lambda(k-1)} + e^{-\lambda(k-1)} \quad (1.7)$$

$$= e^{-\lambda(k-1)}(1 - e^{-\lambda}) \quad (1.8)$$

$$= (1 - (1 - e^{-\lambda}))^{k-1} (1 - e^{-\lambda}) \quad (1.9)$$

Setting  $1 - e^{-\lambda} = p$  yields the desired result  $p^Y(k) = (1 - p)^{k-1} p$ .

3. Let  $X \sim N(\mu, \sigma^2)$  and  $Y = aX + b$  for some  $a, b \in \mathbb{R}$  with  $a \neq 0$ .

1. Show that  $Y \sim N(a\mu + b, a^2\sigma^2)$ .

2. How must  $a$  and  $b$  be chosen so that  $Y \sim N(0, 1)$  holds?

**Solution:**

1. Just use Transformationformula.
2.  $a = \sigma^{-1}$  and  $b = -\mu\sigma^{-1}$ .

4. Let  $\Phi = \Phi_{0,1}$  the cumulative mass function of  $N(0, 1)$ -distribution and let  $\Phi^{-1}$  be its inverse. Show that the following equation hold.

$$\Phi^{-1}(p) = -\Phi^{-1}(1 - p), \quad p \in (0, 1). \quad (1.10)$$

**Solution:**

Consider the PMF of the standard normal distribution  $\Phi$  and let  $x \in \mathbb{R}_0^+$ . Because of its symmetry on the  $y$ -axis, we have

$$1 = \int_{-\infty}^{\infty} \phi(y) \, dy \quad (1.11)$$

$$= \int_{-\infty}^x \phi(y) \, dy + \int_x^{\infty} \phi(y) \, dy \quad (1.12)$$

$$= \int_{-\infty}^x \phi(y) \, dy + \int_{-\infty}^{-x} \phi(y) \, dy \quad (1.13)$$

$$= \Phi(x) + \Phi(-x). \quad (1.14)$$

Hence  $\Phi(-x) = 1 - \Phi(x)$ . Using this and the bijectivity of  $\Phi$  we get

$$\Phi^{-1}(p) = -\Phi^{-1}(1 - p) \iff \Phi(\Phi^{-1}(p)) = \Phi(-\Phi^{-1}(1 - p)) \quad (1.15)$$

$$\iff p = 1 - \Phi(\Phi^{-1}(1 - p)) \quad (1.16)$$

$$\iff p = p \quad (1.17)$$

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## Chapter 2

# Independence

6. Let  $X_1, \dots, X_n$  be real-valued independent random variables with CDFs  $F_1, \dots, F_n$ .

1. Show that the CDFs of  $M = \max(X_1, \dots, X_n)$  and  $m = \min(X_1, \dots, X_n)$  are given by

$$F^M(x) = \prod_{i=1}^n F_i(x) \quad \text{and} \quad F^m(x) = 1 - \prod_{i=1}^n (1 - F_i(x)). \quad (2.1)$$

**Solution:**

$$\{X_i \in A_i\} \quad (2.2)$$



## Chapter 3

# Convergence of Random Variables

**7.** Let  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  be two sequence of random variables that converges in distribution to  $X$  and  $Y$  respectively. Moreover, let  $X$  and  $Y$  as well as  $X_n$  and  $Y_n$  for each  $n \in \mathbb{N}$  independent. Show that  $X_n + Y_n$  converges in distribution to  $X + Y$ .