My notes on "The Strong Factorial Conjecture" by Eric Edo and Arno van den Essen.
See: https://arxiv.org/abs/1304.3956

1 Factorial Conjecture

$\mathbb{C}^{[m]}$ may be viewed as a vecor space over $\mathbb{C}$ with a basis being

$$\left\{ X_1^{l_1} \cdots X_m^{l_m} \,\middle|\, l_k \in \mathbb{N}_0 \text{ for all } 1 \leq k \leq m \right\}.$$

Thus any linear map is fully defined if we set a value for each basis element. Such linear map is the factorial map.

Definition 1. A factorial map is a linear map linear map $\mathcal{L} : \mathbb{C}^{[m]} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{L}(X_1^{l_1} \cdots X_m^{l_m}) = l_1! \cdots l_m! \qquad \text{for all } l_1, \ldots, l_m \in \mathbb{N}$$

Example 1.1. Consider $f(X) = 3X - 5XY + 7Y^2 \in \mathbb{C}^{[2]}$. Applying the factorial map yields

$$\begin{aligned}\mathcal{L}(f(X)) &= 3\mathcal{L}(X) - 5\mathcal{L}(XY) + 7\mathcal{L}(Y^2) \\ &= 3 \cdot 1 - 5 \cdot 1 + 7 \cdot 2 \\ &= 12.\end{aligned}$$

Example 1.2. If we limit our selves to a polynomial with one indeterminate, such as $f(X) = \sum_{k=0}^{n} a_k X^k \in \mathbb{C}[X]$ for a fixed $n \in \mathbb{N}_0$, we have

$$\mathcal{L}(f(X)) = \sum_{k=0}^{n} a_k \mathcal{L}(X^k) = \sum_{k=0}^{n} a_k k!$$

Theorem 2 (Conjecture 2.4)**.** Let $f \in \mathbb{C}^{[m]}$ be a polynomial. We have $\mathcal{L}(f^k) = 0$ for all $k \in \mathbb{N}_+$ if and only if $f = 0$.

Remark. The converse is trivial, hence the conjecture is about the forward implication.

2 Rigidity Conjecture

TODO: $\mathbb{C}_0[[X]]$ the set of formal power series with the constant coefficient being 0 forms a $\mathbb{C}$-algebra with composition being the composition.

When we consider compositions of formal power series, we always want the constant term to be 0.

The following example is taken from:

https://math.stackexchange.com/questions/1212053/defining-composition-of-two-formal-series-what-is-going-on

Example 2.1. Let $f = \sum_{k\in\mathbb{N}_0} a_k X^k$ and $g = 1 + X$. Consider $f \circ g$. We have

$$\begin{aligned} f \circ g &= \sum_{k\in\mathbb{N}_0} a_k(1+X)^k \\ &= a_0 + a_1 + a_1X + a_2 + 2a_2X + a_2X^2 + \cdots \end{aligned}$$

If $f \circ g$ is again a formal power series, then we should be able to write $f \circ g = \sum_{k\in\mathbb{N}_+} c_k X^k$ for some $c_k \in \mathbb{C}$. However, we see that c_0 is the sum of all a_k and we cannot evaluate that as algebraists. Thus composition of formal power series only makes sense if the constant coefficient is 0.

Proposition 3. A power series $f(X) = \sum_{k\in\mathbb{N}_+} a_k X^k \in \mathbb{C}[[X]]$ has a compositional inverse $f^{-1}(X)$ if and only if $a_1 \neq 0$, in which case $f^{-1}(X)$ is unique.

Proof. Assume $g(X) = b_1X + b_2X^2 + \cdots$ satisfies $f(g(X)) = X$. We then have

$$a_1(b_1X + b_2X^2 + \cdots) + a_2(b_1X + b_2X^2 + \cdots)^2 + a_3(b_1X + b_2X^2 + \cdots)^3 = X$$

Equating coefficients on both sides yields the infinite system of equations

$$\begin{aligned} a_1b_1 &= 1 \\ a_1b_2 + a_2b_1^2 &= 0 \\ a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 &= 0 \\ &\vdots \end{aligned}$$

□

Another proof:

https://www.math.uwaterloo.ca/ dgwagner/co430I.pdf But there is no simple formula for the coefficients of the inverse (see enumerative combinatorics).

Theorem 4 (Conjecture 2.13)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If m consecutive coefficient of the compositional inverse $a^{-1}(X)$ vanish, i.e. $b_{n+1} = b_{n+2} = \cdots = b_{n+m} = 0$ for some $n \in \mathbb{N}_+$ then $a(X) = X$.

Remark. If we denote the polynomial $a(X)$ by $\sum_{k\in\mathbb{N}_0} a_k X^k$ for some $a_k \in \mathbb{C}$ for all $k \in \mathbb{N}_0$, then the condition $a(X) \equiv X \mod X^2$ amounts to $a_0 = 0$ and $a_1 = 1$.

Theorem 5 (Conjecture 2.14)**.** Let $a(X) \in \mathbb{C}[X]$ be a polynomial of degree less or equal to $m+1 \in \mathbb{N}_+$ such that $a(X) \equiv X \mod X^2$. If the coefficients of $X^{n+1}, \dots, X^{n+m}$ of the compositional inverse vanish, then $a(X) = X$.

Remark. $R(m)$ if and only if $R(m)_n$ for all $n \in \mathbb{N}_+$.

Lemma 6 (Lemma 2.16)**.** Let $f \in \mathbb{C}[[X]]$ and $g \in \mathbb{C}[[X]]$ be two formal series such that $f(X) \equiv g(X) \mod X^2$, i.e. the constant and the coefficient of the first degree agree. If $f(X) \equiv g(X) \mod X^n$ for some integer $n \geq 2$ then $f^{-1}(X) \equiv g^{-1}(X) \mod X^n$.

Proof. □

Proposition 7. 1. The polynomial $a(X)$ is invertible for the composition.

2. For all $i \in \{1, \dots, \deg(a-1)\}$, the coefficient a_i is nilpotent element in A. I just don't see this ...

The following lemma and proof are due to #XXX.

Lemma 8 (Lagrange Inversion Formula)**.** Let K be a field of charateristic

$$f^{-1}(X) = \sum_{n \in \mathbb{N}_+} b_n X^n$$
$$\text{where } b_n = \frac{1}{n} \cdot [X^{n-1}] \left(\frac{X}{f(X)} \right)^n$$

Proof. We will prove that the given formula for b_n, i.e. the n-th coefficient of the compositional inverse, is merited. Thus begin by fixing an arbitary integer $n \in \mathbb{N}_+$.

By proposition #XXX, f is guranteed to have a unique compositional inverse which we will denote by $f^{-1}(X) = \sum_{k \in \mathbb{N}_+} b_k X^k$ with $b_k \in \mathbb{C}$ for all $k \in \mathbb{N}_+$. Applying the original f to both sides yields $f(f^{-1}(X)) = X$ on the left side and on the right we have

$$f\left(\sum_{k \in \mathbb{N}_+} b_k X^k \right) = \sum_{k \in \mathbb{N}_+} b_k f(X)^k$$

due to the linearity of f as a map, thus $X = \sum_{k \in \mathbb{N}_+} b_k f(X)^k$. Now, formal differentiation with the chain rule #sure? gives

$$1 = \sum_{k \in \mathbb{N}_+} k \cdot b_k \cdot f(X)^{k-1} \cdot f'(X).$$

Let $n \in \mathbb{N}$ #with0? be an integer. #moremotivation Dividing the above equation with the n-th power of the reciprocal produces

$$f(X)^{-n} = \sum_{k \in \mathbb{N}_+} k \cdot b_k \cdot f(X)^{k-n-1} \cdot f'(X).$$

After □

Example 8.1 (See 5.4.4)**.** $f(X) = Xe^{-X} = X \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} X^k$

$$[X^n] f^{-1}(X) = \frac{1}{n} [X^{n-1}] e^{nX}$$

Lemma 9 (Lemma 2.20 (Additive Inversion Formula))**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers. The formal inverse of $a(X) = X(1 - (\alpha_1 X + \cdots + \alpha_m X^m))$ is given by the following formula

$$a^{-1}(X) = X \left(1 + \frac{1}{n+1} \sum_{n \geq 1} u_n X^n \right)$$

where

$$u_n = \frac{1}{n!} \sum_{j_1 + 2j_2 + \cdots + m j_m = n} \frac{(n + j_1 + \cdots + j_m)!}{j_1! \cdots j_m!} \alpha_1^{j_1} \cdots \alpha_m^{j_m}$$

Proposition 10 (Proposition 2.23)**.** Let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be complex numbers and let $(u_n)_{n \in \mathbb{N}_+}$ be a sequence defined by AIF in Lemma 2.20. For all $n \in \mathbb{N}_+$, the Rigidity Conjecture $R(m)_n$ is equivalent to the following implication: If $u_n = \cdots = u_{n+m-1} = 0$ then $\alpha_1 = \cdots = \alpha_m = 0$.

Proof. □

Theorem 11. 1. The inclusion $E^{[m]} \subset F_n^{[m]}$ implies $R(m)_n$

Definition 12.

$$E^{[m]} = \{ X_1 \cdots X_m (\mu_1 X_1 + \cdots + \mu_m X_m) \mid \mu_1, \dots, \mu_m \in \mathbb{C} \} \subset$$
$$F_n^{[m]} = \left\{ f \in \mathbb{C}^{[m]} \setminus \{0\} \;\middle|\; \mathcal{L}(f^k) \neq 0 \text{ for some } n \leq k \leq \mathcal{N}(f) - 1 \right\} \cup \{0\}$$