**Theorem 1.** Let $M$ be a finitely generated $R$-module and $I \subset \operatorname{Jac}(R)$ an ideal such that $IM = M$. Then $M = 0$.

*Proof.* We prove the statement through contradiction.

1. Assume $M \neq 0$.
2. Since $M$ is finitely generated, it has a generating system $m_1, \dots, m_n \in M$. Assert that this generating system is minimal.
3. With $IM = M$, there is an equation
$$m_1 = a_1 m_1 + \cdots a_n m_n$$
for some $a_1, \dots, a_n \in I$.
4. We may rewrite
$$\begin{aligned} m_1 = a_1 m_1 + \cdots a_n m_n &\iff m_1 - a_1 m_1 = a_2 m_2 + \cdots + a_n m_n \\ &\iff (1 - a_1) m_1 = a_2 m_2 + \cdots + a_n m_n \end{aligned}$$
5. Since $a_1 \in I \subset \operatorname{Jac}(A)$, $1 - a_1$ is a unit in $R$.
6. We arrive at a contradiction with the minimality of the genrating system.

$\square$

**Theorem 2.** Let $M$ be a finitely generated $R$-module, $I \subset \operatorname{Jac}(R)$ an ideal, and $N$ a submodule of $M$ such that $M = N + IM$. Then $M = N$.

*Proof.*
1. $M = N + IM$ implies $M/N = (N + IM)/N = 0 + I(M/N) = I(M/N)$.
2. $M/N$ is finitely generated.
3. Applying Nakayama yields
4. $M/N = 0$
5. Thus $M = N$.

$\square$

**Theorem 3.** Let $R$ be a local ring with maximal ideal $\mathfrak{m}$ and $M$ a finitely generated $R$-module.

1. Then $M/\mathfrak{m}M$ is a vector space over the field $R/\mathfrak{m}$.
2. If $x_1, \dots, x_n \in M$ are elements such that $x_1, \dots, x_n \in M/\mathfrak{m}M$ generates this vector space, then $M = \sum_{i=1}^n Rx_i$

**Theorem 4.** Let $I$ be an nilpotent ideal in $R$ and $M$ be an $R$-module such that $IM = M$. Then $M = 0$.

*Proof.*
1. $IM = M$ implies $I \cdot IM = IM = M$.
2. Induction yields $I^n M = M = 0$.

$\square$