

Topology

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Part I

Rings

Chapter 1

Rings

1.1 Definition and Theorems

Definition 1 (Ring). A ring is a set A equipped with two binary operations $+$ (addition) and \cdot (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(A, +)$ is an abelian group.
2. (A, \cdot) is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
 - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in A$ (left distributivity).
 - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in A$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Chapter 2

Ideals

Definition 2 (Ideal). Let A be a ring. A subset $\mathfrak{a} \subset A$ is called an ideal if it satisfies the following two conditions.

1. $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$.
2. For every $r \in A$ and every $x \in \mathfrak{a}$, it is $rx \in \mathfrak{a}$.

Given a subset $S \subset A$, by the ideal (S) that S generates, we mean the smallest ideal containing S . If an ideal is generated by a subset $S \subset A$, then the elements of this subset are called generators.

An ideal that is generated by a single element is called principal.

If an ideal \mathfrak{a} is not the whole ring A , then the ideal is called proper.

Definition 3 (Ideal Operation). Let \mathfrak{a} and \mathfrak{b} be ideals of a ring A .

1. The sum of two ideals \mathfrak{a} and \mathfrak{b} is defined by

$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \} = (\mathfrak{a}, \mathfrak{b})$$

which is again an ideal. It is the smallest ideal in A that contains \mathfrak{a} and \mathfrak{b} .

2. The product of an ideal
3. The intersection of
4. The radical of an ideal \mathfrak{a} is defined by

$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$

which is again an ideal.

5. The transporter

Proof. We verify the statements made in the definition.

1. (a) “ $\mathfrak{a} + \mathfrak{b}$ is an ideal.”:

□

Example 3.1. The union of two ideals is **not** an ideal in general. Consider (2) and (3) in \mathbb{Z} . If $(2) \cup (3)$ was an ideal, then $3 - 2 = 1$ would be contained in $(2) \cup (3)$. But $1 \notin (2)$ and $1 \notin (3)$, thus $1 \notin (2) \cup (3)$.

Proposition 4. Let \mathfrak{a} be an ideal of A .

1. $\mathfrak{a} = A$ if and only if $1 \in \mathfrak{a}$ if and only if \mathfrak{a} contains an unit.
2. $\mathfrak{a}^2 \subset \mathfrak{a}$.
3. $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.
4. $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}$.

Proposition 5. Let \mathfrak{a} and \mathfrak{b} be two ideals of a ring A .

1. $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$.
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.
3. If $\mathfrak{a} \subset \mathfrak{b}$, then $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$.
4. $\sqrt{\mathfrak{a}} = A$ if and only if $\mathfrak{a} = A$.
5. $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
6. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
7. If $\mathfrak{a} = \mathfrak{p}^n$ for some prime ideal \mathfrak{p} and $n \in \mathbb{N}^+$, then $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Proof. We verify each statement.

1. Let $x \in \mathfrak{a}$, then trivially, $x^1 \in \mathfrak{a}$, so $x \in \sqrt{\mathfrak{a}}$.
2. Since $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$ from above, it suffices to verify the other inclusion. Let $x \in \sqrt{\sqrt{\mathfrak{a}}}$, then $x^n \in \sqrt{\mathfrak{a}}$ and in turn, $(x^n)^m \in \mathfrak{a}$. Thus, $x^{nm} \in \mathfrak{a}$, therefore, $x \in \sqrt{\mathfrak{a}}$.
3. Suppose $\mathfrak{a} \subset \mathfrak{b}$ and let $x \in \sqrt{\mathfrak{a}}$. Then, $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$, thus $x^n \in \mathfrak{b}$. It follows that $x \in \sqrt{\mathfrak{b}}$.
4. “ \Rightarrow ”: Let $\sqrt{\mathfrak{a}} = A$, then for all $x \in A$, we have that $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}^+$. In particular, $1^n \in \mathfrak{a}$, but $1^n = 1$ for all $n \in \mathbb{N}^+$. Thus, $\mathfrak{a} = A$.
 “ \Leftarrow ”: On the other hand, let $\mathfrak{a} = A$. In general, it is $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, therefore $A \subset \sqrt{\mathfrak{a}}$ which immediately yields the desired equality $A = \sqrt{\mathfrak{a}}$.
5. “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: If $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cdot \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Since $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$, we have $x^n \in \mathfrak{a} \cap \mathfrak{b}$, and it follows that $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.
 “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: Let $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n \in \mathbb{N}^+$. Hence it is $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, therefore $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$.
 “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: If $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$. We may write $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, therefore $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
 “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: Finally, let $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. Then, $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, so $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$ for some $n, m \in \mathbb{N}^+$. Say $n \geq m$, then $x^n \in \mathfrak{b}$. This yields $x^n \in \mathfrak{a} \cap \mathfrak{b}$, thus $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$.
6. “ $\sqrt{\mathfrak{a} + \mathfrak{b}} \subset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ ”: Let $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$, then $x^n \in \mathfrak{a} + \mathfrak{b}$ for some $n \in \mathbb{N}^+$. By definition of sum of ideals, we have that $x^n = a + b$ for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ and $\mathfrak{b} \subset \sqrt{\mathfrak{b}}$, we have $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, thus $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
 “ $\sqrt{\mathfrak{a} + \mathfrak{b}} \supset \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ ”: Now let $x \in \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$, then $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$ for some $n \in \mathbb{N}^+$. Hence there exists $a \in \sqrt{\mathfrak{a}}$ and $b \in \sqrt{\mathfrak{b}}$ such that $x^n = a + b$. We have that $a^p \in \mathfrak{a}$ and $b^q \in \mathfrak{b}$

for some $p, q \in \mathbb{N}^+$. Consider

$$\begin{aligned} (x^n)^{(p+q-1)} &= (a+b)^{(p+q-1)} \\ &= \sum_{k=0}^{p+q-1} \binom{p+q-1}{k} a^k \cdot b^{p+q-1-k}. \end{aligned}$$

For each $k \in \{0, 1, \dots, p+q-1\}$, we have $a^k \in \mathfrak{a}$ or $b^{p+q-1-k} \in \mathfrak{b}$. Thus, the whole sum lies in $\mathfrak{a} + \mathfrak{b}$ or in other words $x^{n(p+q-1)} \in \mathfrak{a} + \mathfrak{b}$. Conclude $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

7. “ $\sqrt{\mathfrak{a}} \subset \mathfrak{p}$ ”: Let $x \in \sqrt{\mathfrak{a}}$, then $x^m \in \mathfrak{a}$ for some $m \in \mathbb{N}^+$. Because $\mathfrak{a} = \mathfrak{p}^n$, we have $x^m \in \mathfrak{p}^n$. We also have $\mathfrak{p}^n \subset \mathfrak{p}$, thus $x^m \in \mathfrak{p}$ and since \mathfrak{p} is prime, $x \in \mathfrak{p}$.

“ $\sqrt{\mathfrak{a}} \supset \mathfrak{p}$ ”: On the other hand, if $x \in \mathfrak{p}$, then $x^n \in \mathfrak{p}^n = \mathfrak{a}$, therefore $x \in \sqrt{\mathfrak{a}}$.

□

Proposition 6. 1. $\mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$.

Example 6.1. Does $\sqrt{\mathfrak{a}^2} = \mathfrak{a}$ hold?

Chapter 3

Anatomy of Rings

Definition 7 (Nilpotent Element and Nilradical). An element x of a ring A is called nilpotent if there exists some positive integer $n \in \mathbb{N}^+$, called the index or the degree, such that $x^n = 0$.

The set of all nilpotent elements is called the nilradical of the ring and is denoted by $\text{Nil}(A)$.

Definition 8 (Reduced Ring). A ring A is called reduced ring if it has no non-zero nilpotent elements.

Proposition 9. Let A and B be two rings and $A' \subset A$ be a subring of A .

1. If A is reduced, then A' is also reduced.
2. If A and B are reduced, then $A \times B$ is also reduced.

(XXX DOES THIS ALSO HOLD FOR ARBITRARY MANY PRODUCTS?)

3.1 Exercises and Notes

Example 9.1. Let K be a field and $A = K[X, Y]/(X - XY^2, Y^3)$.

1. Compute the nilradical $\text{Nil}(A)$.

Solution. Denote $(X - XY^2, Y^3) =: \mathfrak{a}$.

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

Thus, $X \in (X - XY^2, Y^3)$. We have therefore the isomorphism $K[X, Y]/(X - XY^2, Y^3) \simeq K[Y]/(Y^3)$. [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

Clearly, $Y \in \text{Nil}(A)$ or in other words $(Y) \subset \text{Nil}(A)$. But we also have that $K[Y]/(Y) = K$ which is a field, therefore (Y) is a maximal ideal. Because $1 \notin \text{Nil}(A)$ conclude $\text{Nil}(A) = (Y)$. \square

Chapter 4

Polynomial Rings

Chapter 5

Quotient

Chapter 6

Localization

6.1 Definition and Theorems

Definition 10 (Multiplicative Subset). A subset S of a ring A is called a multiplicative subset if the following conditions hold.

1. $1 \in S$.
2. For all $x, y \in S$ it is $xy \in S$.

Example 10.1. Let A be a ring. Important examples of a multiplicative subset include the following.

1. The set of units A^\times is a multiplicative subset.
2. The set of non-zero-divisors $A \setminus \text{ZD}(A)$ is a multiplicative subset.

Example 10.2. Let A be a ring. Other examples of multiplicative subsets are the following.

1. For any element $x \in A$, the set generated by its power $\{1, x, x^2, x^3, \dots\}$ is a multiplicative subset.
2. For any ideal $\mathfrak{a} \subset A$, the set $1 + \mathfrak{a}$ is a multiplicative subset.

Lemma 11. An ideal \mathfrak{p} of a ring A is prime if and only if its complement $A \setminus \mathfrak{p}$ is a multiplicative subset.

Definition 12 (Localization). $S^{-1}A$ is again a ring.

Lemma 13. Let A be a ring and S a multiplicative subset, then the following are equivalent.

1. $S^{-1}A = 0$.
2. S contains a nilpotent element.
3. $0 \in S$.

Proof. “1. \Rightarrow 2.”: Let $S^{-1}A = 0$, then for all $x \in A$ and $s \in S$ it is $(x, s) \sim (0, 1)$, thus $x \cdot u = 0$ for some $u \in S$. In particular, this holds for $x = 1$, therefore $1 \cdot u = 0$. Since a unit can never be a zero divisor, we must have $u = 0$ which is nilpotent and lies in S .

“1. \Leftarrow 2.”: On the other hand, let $x \in S$ be nilpotent, i.e. $x^n = 0$ for some $n \in \mathbb{N}^+$. Because S is multiplicatively closed $x^n = 0$ lies in S . Fix an element $(y, s) \in S^{-1}A$, then $y \cdot 1 \cdot 0 = 0 \cdot s \cdot 0$. Hence $(y, s) \sim (0, 1)$ and we have $S^{-1}A = 0$.

“2. \Rightarrow 3.”: Again, let $x \in S$ be nilpotent, thus $x^n = 0$ for some $n \in \mathbb{N}^+$. S is multiplicatively closed and we have $x^n = 0 \in S$.

“2. \Leftarrow 3.”: If $0 \in S$, then S simply contains a nilpotent element because 0 is nilpotent. \square

Remark. In the lemma above, the condition $0 \notin S$ is required because if S contains 0, then $S^{-1}A = 0$ and by definition, an integral domain is a nonzero ring.

Proposition 14. Let A be a ring. A is reduced if and only if all its localizations $A_{\mathfrak{p}}$ at $\mathfrak{p} \in \text{Spec } A$ is reduced.

Proof. “ \Rightarrow ”: We prove the statement by contrapositive. Let $A_{\mathfrak{p}}$ be not reduced for all $\mathfrak{p} \in \text{Spec } A$. Thus, in all $A_{\mathfrak{p}}$, there is an element, say x/s that is nilpotent and not zero, i.e. $(x/s)^n = 0$ for some $n \in \mathbb{N}^+$. By the definition of localization, we get $x^n \cdot u = 0$ for some $u \in A \setminus \mathfrak{p}$. Now, $u \in A \setminus \mathfrak{p}$ cannot be zero, because if it was, $A_{\mathfrak{p}} = 0$ which is reduced. Thus, x is nilpotent and A is not reduced. \square

Lemma 15. Let A be a ring and $S \subset A$ be a multiplicative subset that does not contain 0.

1. A is an integral domain if and only if $S^{-1}A$ is an integral domain.
2. A is a unique factorization domain if and only if $S^{-1}A$ is a unique factorization domain.

Proof. “ \Rightarrow ”: Let A be an integral domain. Since S does not contain 0, the localization $S^{-1}A$ is a nonzero ring (see EXAMPLE). Let $(x, s) \in S^{-1}A \setminus \{0\}$ be a nonzero element and suppose there is a $(y, t) \in S^{-1}A$ with $(x, s) \cdot (y, t) = 0$. It is $(xy, st) = (0, 1)$ and thus $xy \cdot u = 0$ for some $u \in S$. Because x was nonzero and S does not contain 0 we must have $y = 0$. Hence $S^{-1}A$ is an integral domain.

“ \Leftarrow ”: On the other hand, let $S^{-1}A$ be an integral domain. JUST USE THE CANONIC MAPPING $\varphi_S : A \rightarrow S^{-1}A$. \square

6.2 Exercises and Notes

Example 15.1. Let A_1 and A_2 be rings. Consider $A = A_1 \times A_2$ and set $S := \{(1, 1), (1, 0)\}$. Prove $A_1 \simeq S^{-1}A$.

Solution. I don’t understand the solution? \square

Example 15.2. Find all intermediate rings $\mathbb{Z} \subset A \subset \mathbb{Q}$, and describe each A as a localization of \mathbb{Z} . As a starter, prove $\mathbb{Z}[\frac{2}{3}] = S_3^{-1}\mathbb{Z}$ where $S_3 := \{3^i \mid i \in \mathbb{N}^+\}$.

Chapter 7

Hierarchy of Rings

7.1 Definition and Theorems

7.1.1 Integral Domains

Part II

Modules

Definition 16 (Module).

Example 16.1. 1. If A is a field, then an A -module is a vector space.
2. A \mathbb{Z} -module is just an abelian group.

Definition 17 (Annihilator).

Definition 18 (Radical).

Definition 19 (Simple Modules). Let A be a ring. A nonzero A -module M is called simple if the only submodules are $\{0\}$ and M itself.

Example 19.1. If M is a simple A -module, then any $f \in \text{Hom}_A(M, M) \setminus \{0\}$ is an isomorphism.

Proof. Fix an $f \in \text{Hom}_A(M, M) \setminus \{0\}$. Since $\ker(f)$ is a submodule of M , it must be either $\{0\}$ or whole M . But $\ker(f) = M$ would mean that $f = 0$ which was explicitly excluded, thus $\ker(f) = \{0\}$. By the isomorphism theorem, we also have $\text{im}(f) \cong M/\ker(f) \cong M$. Therefore, f is bijective. \square

Definition 20 (Indecomposable). Let A be a ring. A nonzero A -module M is called indecomposable if it cannot be written as a direct sum of two non-zero submodules.

Proposition 21. Every simple module is indecomposable.

Example 21.1. Not all indecomposable modules are simple. For example, \mathbb{Z} is indecomposable, but is not simple.

7.2 Exercises and Notes

Example 21.2. Let $f : M \rightarrow N$ be a surjective homomorphism of two finitely generated A -modules.

1. If $N \cong A^n$ is a free A -module, show that $M \cong \ker(f) \oplus N$.

Proof. Since N is finitely generated, let (e_1, \dots, e_n) be a set of generators. □

Example 21.3. Let A be a ring, \mathfrak{a} and \mathfrak{b} ideals, M and N A -modules. Set

$$\Gamma_{\mathfrak{a}}(M) := \left\{ m \in M \mid \mathfrak{a} \subset \sqrt{\text{Ann}(m)} \right\}.$$

Prove the following statements.

1. If $\mathfrak{a} \supset \mathfrak{b}$, then $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{b}}(M)$.

Proof. The proof is a matter of verification. Let $m \in \Gamma_{\mathfrak{a}}(M)$. It is

$$\begin{aligned} m \in \Gamma_{\mathfrak{a}}(M) &\Rightarrow \mathfrak{a} \subset \sqrt{\text{Ann}(m)} \\ &\Rightarrow \text{For all } a \in \mathfrak{a} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \in \text{Ann}(m). \\ &\Rightarrow \text{For all } a \in \mathfrak{a} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \cdot m = 0. \end{aligned}$$

Since $\mathfrak{a} \supset \mathfrak{b}$, the last statement is true for all $a \in \mathfrak{b}$. We have

$$\begin{aligned} &\Rightarrow \text{For all } a \in \mathfrak{b} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \cdot m = 0. \\ &\Rightarrow \text{For all } a \in \mathfrak{b} \text{ there is a } n \in \mathbb{N}^+ \text{ such that } a^n \in \text{Ann}(m). \\ &\Rightarrow \mathfrak{b} \subset \sqrt{\text{Ann}(m)} \\ &\Rightarrow m \in \Gamma_{\mathfrak{b}}(M) \end{aligned}$$

Thus, $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{b}}(M)$. □

2. If $M \subset N$, then $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(N) \cap M$.

Proof. Again, the proof is a matter of verification.

“ \subset ”: $M \subset N$ implies $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{a}}(N)$. Moreover, it is $\Gamma_{\mathfrak{a}}(M) \subset M$. Thus, $\Gamma_{\mathfrak{a}}(M) \subset \Gamma_{\mathfrak{a}}(N) \cap M$.

“ \supset ”: Let $m \in \Gamma_{\mathfrak{a}}(N) \cap M$. It is

$$\begin{aligned} m \in \Gamma_{\mathfrak{a}}(N) \cap M &\Rightarrow \mathfrak{a} \subset \sqrt{\text{Ann}(m)} \text{ and } m \in M. \\ &\Rightarrow m \in \Gamma_{\mathfrak{a}}(M). \end{aligned}$$

Hence, $\Gamma_{\mathfrak{a}}(N) \cap M \subset \Gamma_{\mathfrak{a}}(M)$. □

3. In general, it is $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(M)) = \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M) \cap \Gamma_{\mathfrak{b}}(M)$.

4. In general, it is $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\sqrt{\mathfrak{a}}}(M)$.

5. If \mathfrak{a} is finitely generated, then

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 1} \{ m \in M \mid \mathfrak{a}^n m = 0 \}.$$

Example 21.4. Let A be a ring, M a module, $x \in \text{Rad}(M)$, and $m \in M$. If $(1+x)m = 0$, then $m = 0$.

Proof. By definition of radical of a module, it is

$$\text{Rad} (A/\text{Ann}(M)) = \text{Rad}(M)/\text{Ann}(M).$$

Thus, if $x \in \text{Rad}(M)$, then its residue $x' := x + \text{Ann}(M)$ lies in $\text{Rad} (A/\text{Ann}(M))$ which means x' is nilpotent. SOME THEOREM yields $(1 + x')$ is an unit in $A/\text{Ann}(M)$. \square

Chapter 8

Tensor Product

8.1 Definition and Theorems

Definition 22. Let M and N be A -modules. Their tensor product is a pair $(M \otimes_A N, \theta)$ where

1. $M \otimes_A N$ is an A -module.
2. $\theta : M \times N \rightarrow M \otimes_A N$ is an A -bilinear mapping.

satisfying the universal property, for every pair (P, ω) of an A -module and an A -bilinear mapping $\omega : M \times N \rightarrow P$, there exists a unique A -module homomorphism $f : M \otimes_A N \rightarrow P$ with $\omega = f \circ \theta$.

Definition 23. Let M and N be A -modules. Their tensor product is the pair $(M \otimes_A N, \theta)$, where

1. $M \otimes_A N$ is the quotient of the free A -module $A^{M \times N}$ on the direct product $M \times N$, by the submodule generated by the set of elements of the form:

$$\begin{aligned} &(\lambda m_1 + m_2, n) - \lambda(m_1, n) - (m_2, n) \\ &(m, \lambda n_1 + n_2) - \lambda(m, n_1) - (m, n_2) \end{aligned}$$

for $m, m_1, m_2 \in M$; $n, n_1, n_2 \in N$; and $\lambda \in A$, where we denote (m, n) for its image under the canonical mapping $M \times N \rightarrow A^{(M \times N)}$.

2. $\theta : M \times N \rightarrow M \otimes_A N$ is the composition of the canonical mapping $M \times N \rightarrow A^{(M \times N)}$ with the quotient module homomorphism $A^{(M \times N)} \rightarrow M \otimes_A N$.

Example 23.1. It is $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$.

Proof. Let's show this in multiple concrete ways.

Method 1: I want to do this concretely. First, we have

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \{ (0, 0); (0, 1); (0, 2); (1, 0); (1, 1); (1, 2) \}.$$

Thus, the elements of $\mathbb{Z}^{(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})}$ are in the form

$$(x_{(0,0)}, x_{(0,1)}, x_{(0,2)}, x_{(1,0)}, x_{(1,1)}, x_{(1,2)})$$

where $x_{(i,j)} \in \mathbb{Z}$ with $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$.

Now, we want to find the submodule generated by the rules in the definition.

1. Set $m_1 = m_2 = n = \lambda = 0$, then

$$(0 \cdot 0 + 0, 0) + 0 \cdot (0, 0) - (0, 0) = (0, 0) = 1 \cdot (0, 0) \rightarrow (1, 0, 0, 0, 0, 0).$$

2. Set $m = n_2 = 0$, $n_1 = 1$, and $\lambda = 2$, then

$$\begin{aligned} (0, 2 \cdot 1 + 0) - 2 \cdot (0, 1) - (0, 0) &= (0, 2) - (2 \cdot 0, 1) \\ &= (0, 2) - (0, 1) \\ &= (0, 1) \\ &= 1 \cdot (0, 1) \\ &\rightarrow (0, 1, 0, 0, 0, 0) \end{aligned}$$

3. I think the rest is clear for now.

We may conclude that the submodule generated by the rules defined is the whole module, thus $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = 0$.

Method 2: <https://www.math.brown.edu/reschwar/M153/tensor.pdf>

□

Proposition 24. Let A be a ring, and M, N and P be A -modules.

1. (identity) $A \otimes_A M = M$.
2. (commutative law) $M \otimes_A N = N \otimes_A M$.

Proof. As in the proposition, let A be a ring, and M, N and P be A -modules.

1. Define $\beta : A \times M \rightarrow M$ by $\beta(x, m) := xm$. Clearly, β is bilinear.

□

8.2 Exercises and Notes

Example 24.1. Let $A \rightarrow B \rightarrow C$ be ring homomorphisms and M and N be A -modules. Show the following.

1. $(M \otimes_A B) \otimes_B C \cong M \otimes_A C$

Proof. It is

$$\begin{aligned} (M \otimes_A B) \otimes_B C &\cong M \otimes_A (B \otimes_B C) \\ &\cong M \otimes_A C \end{aligned}$$

□

2. $(M \otimes_A N) \otimes_A B \cong (M \otimes_A B) \otimes_B (N \otimes_A B)$

Proof. trivial

□

Example 24.2. Let A be a ring.

1. If M, N are A -modules, then $\text{Hom}_A(M, N)$ may be viewed as an A -module via

$$a \cdot \varphi := (m \mapsto a \cdot \varphi(m))$$

for $a \in A$ and $\varphi \in \text{Hom}_A(M, N)$.

Proof. this is trivial

□

2. If M, N, L are A -modules, then there exists a natural isomorphism of A -modules

$$\text{Hom}_A(L \otimes_A M, N) \cong \text{Hom}_A(L, \text{Hom}_A(M, N))$$

Example 24.3. Let A be a ring, \mathfrak{a} an ideal of A , and M an A -module.

1. Show that $M/\mathfrak{a}M \cong M \otimes_A A/\mathfrak{a}$.

Proof. Define $\varphi : M \otimes_A A/\mathfrak{a} \rightarrow M/\mathfrak{a}M$ by

$$m \otimes_A \bar{x} \mapsto x \cdot m + \mathfrak{a}M.$$

□