

# Topology

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# Chapter 1

## Rings

### 1.1 Definition and Theorems

**Definition 1** (Ring). A ring is a set  $A$  equipped with two binary operations  $+$  (addition) and  $\cdot$  (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1.  $(A, +)$  is an abelian group.
2.  $(A, \cdot)$  is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  for all  $a, b, c \in A$  (left distributivity).
  - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$  for all  $a, b, c \in A$  (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.



## Chapter 2

# Ideals

**Definition 2** (Ideal).

**Definition 3** (Ideal Operation). Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be ideals of a ring  $A$ .

1. The sum of two ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  is defined by

$$\mathfrak{a} + \mathfrak{b} = \{ a + b \mid a \in \mathfrak{a} \text{ and } b \in \mathfrak{b} \}$$

which is again an ideal. It is the smallest ideal in  $A$  that contains  $\mathfrak{a}$  and  $\mathfrak{b}$ .

2. The product of an ideal
3. The intersection of
4. The radical of an ideal  $\mathfrak{a}$  is defined by

$$\sqrt{\mathfrak{a}} = \{ x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}^+ \}$$

which is again an ideal.

5. The transporter

**Proposition 4.** Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two ideals of a ring  $A$ .

1.  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ .
2.  $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$ .
3. If  $\mathfrak{a} \subset \mathfrak{b}$ , then  $\sqrt{\mathfrak{a}} \subset \sqrt{\mathfrak{b}}$ .
4.  $\sqrt{\mathfrak{a}} = A$  if and only if  $\mathfrak{a} = A$ .
5.  $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ .
6.  $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ .

*Proof.* a.

1. Let  $x \in \mathfrak{a}$ , then trivially,  $x^1 \in \mathfrak{a}$ , so  $x \in \sqrt{\mathfrak{a}}$ .
2. Since  $\sqrt{\sqrt{\mathfrak{a}}} \supset \sqrt{\mathfrak{a}}$  from above, it suffices to verify the other inclusion. Let  $x \in \sqrt{\sqrt{\mathfrak{a}}}$ , then  $x^n \in \sqrt{\mathfrak{a}}$

3. Suppose  $\mathfrak{a} \subset \mathfrak{b}$  and let  $x \in \sqrt{\mathfrak{a}}$ . Then,  $x^n \in \mathfrak{a}$  for some  $n \in \mathbb{N}^+$ , thus  $x^n \in \mathfrak{b}$ . It follows that  $x \in \sqrt{\mathfrak{b}}$ .
4. “ $\Rightarrow$ ”: Let  $\sqrt{\mathfrak{a}} = A$ , then for all  $x \in A$ , we have that  $x^n \in \mathfrak{a}$  for some  $n \in \mathbb{N}^+$ . In particular,  $1^n \in \mathfrak{a}$ , but  $1^n = 1$  for all  $n \in \mathbb{N}^+$ . Thus,  $\mathfrak{a} = A$ .
- “ $\Leftarrow$ ”: On the other hand, let  $\mathfrak{a} = A$ . In general, it is  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ , therefore  $A \subset \sqrt{\mathfrak{a}}$  which immediately yields the desired equality  $A = \sqrt{\mathfrak{a}}$ .
5. “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \subset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: If  $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cdot \mathfrak{b}$  for some  $n \in \mathbb{N}^+$ . Since  $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ , we have  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , and it follows that  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ .
- “ $\sqrt{\mathfrak{a} \cdot \mathfrak{b}} \supset \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ ”: Let  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cap \mathfrak{b}$  for some  $n \in \mathbb{N}^+$ . Hence it is  $x^n \in \mathfrak{a}$  and  $x^n \in \mathfrak{b}$ , therefore  $x^n \cdot x^n = x^{2n} \in \mathfrak{a} \cdot \mathfrak{b}$ . Conclude  $x \in \sqrt{\mathfrak{a} \cdot \mathfrak{b}}$ .
- “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: If  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ , then  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , thus  $x^n \in \mathfrak{a}$  and  $x^n \in \mathfrak{b}$ . We may write  $x \in \sqrt{\mathfrak{a}}$  and  $x \in \sqrt{\mathfrak{b}}$ , therefore  $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ .
- “ $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \supset \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ ”: Finally, let  $x \in \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$ . Then,  $x \in \sqrt{\mathfrak{a}}$  and  $x \in \sqrt{\mathfrak{b}}$ , so  $x^n \in \mathfrak{a}$  and  $x^m \in \mathfrak{b}$  for some  $n, m \in \mathbb{N}^+$ . Say  $n \geq m$ , then  $x^n \in \mathfrak{b}$ . This yields  $x^n \in \mathfrak{a} \cap \mathfrak{b}$ , thus  $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ .

□



## Chapter 3

# Anatomy of Rings

**Definition 5** (Nilpotent Element and Nilradical). An element  $x$  of a ring  $A$  is called nilpotent if there exists some positive integer  $n \in \mathbb{N}^+$ , called the index or the degree, such that  $x^n = 0$ .

The set of all nilpotent elements is called the nilradical of the ring and is denoted by  $\text{Nil}(A)$ .

### 3.1 Exercises and Notes

**Example 5.1.** Let  $K$  be a field and  $A = K[X, Y]/(X - XY^2, Y^3)$ .

1. Compute the nilradical  $\text{Nil}(A)$ .

*Solution.* Denote  $(X - XY^2, Y^3) =: \mathfrak{a}$ .

$$\begin{aligned} X + \mathfrak{a} &= XY^2 + \mathfrak{a} && \text{because } X - XY^2 \Rightarrow X \sim XY^2. \\ &= XY^2Y^2 + \mathfrak{a} && \text{because } XY^2 - XY^2Y^2 = Y^2(X - XY^2) = 0 \Rightarrow XY^2 \sim XY^2Y^2 \\ &= XY \cdot Y^3 + \mathfrak{a} \\ &= XY \cdot 0 + \mathfrak{a} \\ &= 0 + \mathfrak{a}. \end{aligned}$$

Thus,  $X \in (X - XY^2, Y^3)$ . We have therefore the isomorphism  $K[X, Y]/(X - XY^2, Y^3) \simeq K[Y]/(Y^3)$ . [I WANT A ELEGANT REASON FOR THIS. PROBABLY ISOMORPHISM THEOREM.]

Clearly,  $Y \in \text{Nil}(A)$  or in other words  $(Y) \subset \text{Nil}(A)$ . But we also have that  $K[Y]/(Y) = K$  which is a field, therefore  $(Y)$  is a maximal ideal. Because  $1 \notin \text{Nil}(A)$  conclude  $\text{Nil}(A) = (Y)$ .  $\square$



## Chapter 4

# Hierarchy of Rings