

Topology

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Contents

1	Topological Space	7
1.1	Cheat Sheet	7
1.2	Proofs, Remarks, and Examples	7
1.3	Exercises and Notes	14
2	Connected Spaces and Sets	15
2.1	Definition and Theorems	15
2.2	Proofs, Remarks, and Examples	17
2.3	Exercises and Notes	20
2.3.1	Connectedness	20
2.3.2	Path-Connectedness	21
3	Separation Axioms	23
3.1	Definitions and Theorems	23
3.2	Proofs, Remarks, and Examples	24
3.2.1	T_0 Space	24
3.3	Exercises and Notes	25
4	Compact Spaces	27
4.1	Proofs, Remarks, and Examples	28
5	Quotient Space	31
5.1	Definitions and Theorems	31
5.2	Proofs, Remarks, and Examples	31
5.3	Exercises and Notes	34
6	Pasting	35

Conventions

\mathbb{N} contains 0, that is $\mathbb{N} = \{0, 1, 2, \dots\}$.

Chapter 1

Topological Space

1.1 Cheat Sheet

1.2 Proofs, Remarks, and Examples

Definition 1 (Topological Space). A **topological space** is an **ordered pair** (X, \mathcal{O}) , where X is a **set** and \mathcal{O} is a **collection of subsets** that satisfies the following **axioms**.

1. The **empty set** \emptyset and the **entire set** X belongs to \mathcal{O} .
2. Any **arbitrary union** of members of \mathcal{O} belongs to \mathcal{O} .
3. The **intersection** of **finite number** of members of \mathcal{O} belongs to \mathcal{O} .

The **collection** \mathcal{O} is called a **topology** on X and the **elements** of \mathcal{O} are called **open sets**. A **subset** $A \subset X$ is said to be **closed** if its **complement** $X \setminus A$ is **open**. Moreover, a **subset** that is **both open** and **closed** is called **clopen**.

Remark. We often just write X instead of (X, \mathcal{O}) if the given topology is clear.

Remark. 1. One might believe that openness and closedness are mutually exclusive, but this is not true, i.e. there are indeed subsets that are both open and closed. Trivial examples include the emptyset \emptyset and X itself.

2. Furthermore, a subset may neither be open nor closed. If $X = \{1, 2, 3\}$ is a set, then $(X, \{\emptyset, X\})$ is a well-defined topological space and the subsets $\{1\}$ and $\{2, 3\}$ are neither open nor closed.

Example 1.1. Let $X = \{1, 2, 3\}$ be a set with three elements. The power set of X is

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$$

and has eight elements. There are $2^8 = 256$ subsets of $\mathcal{P}(X)$ ¹.

1. There are no singleton topologies because the first axiom of a topology requires at least two subsets. Thus, the only topology on X with two elements is $\{\emptyset, X\}$.

¹The number of subsets of any finite set is 2^n where n is the number of elements because for each element one can choose this element to be inside the subset or not.

2. Any collection of three subsets of X is a well-defined topology as long as it contains the emptyset and the entire set X . Therefore, there are six topologies on X containing exactly three elements.

$$\begin{array}{ll} \{\emptyset, \{1\}, X\}, & \{\emptyset, \{2, 3\}, X\}, \\ \{\emptyset, \{2\}, X\}, & \{\emptyset, \{1, 3\}, X\}, \\ \{\emptyset, \{3\}, X\}, & \{\emptyset, \{1, 2\}, X\}. \end{array}$$

The collection of closed subsets for each of the topologies above corresponds to the other topology in the same row, i.e. $\{\emptyset, \{2, 3\}, X\}$ is the collection of closed subsets for the topology $\{\emptyset, \{1\}, X\}$.

Nonexamples of a topology include $\{\{1\}, \{2\}, X\}$, $\{\emptyset, \{1\}, \{2, 3\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ because these already fail to fulfill the first axiom.

3. Since any topology must include the union of members, the topologies on X with four elements contain a subset that is included in the other one that is not X .

$$\begin{array}{ll} \{\emptyset, \{1\}, \{1, 2\}, X\}, & \{\emptyset, \{3\}, \{2, 3\}, X\}, \\ \{\emptyset, \{1\}, \{1, 3\}, X\}, & \{\emptyset, \{2\}, \{2, 3\}, X\}, \\ \{\emptyset, \{2\}, \{1, 2\}, X\}, & \{\emptyset, \{3\}, \{1, 3\}, X\}, \end{array}$$

Again, the collection of closed subsets for each of the topologies listed are in the same row but on the other side, i.e. $\{\emptyset, \{3\}, \{2, 3\}, X\}$ is the collection of closed subset for $\{\emptyset, \{1\}, \{1, 2\}, X\}$.

However, $\{\emptyset, \{1\}, \{2\}, X\}$ and $\{\emptyset, \{1\}, \{2, 3\}, X\}$ are not topologies because they do not contain the union of its members.

4. Similarly, there are exactly six topologies on X that contain five elements. Those are

$$\begin{array}{ll} \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}, & \{\emptyset, \{3\}, \{1, 3\}, \{2, 3\}, X\}, \\ \{\emptyset, \{1\}, \{3\}, \{1, 3\}, X\}, & \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, X\}, \\ \{\emptyset, \{2\}, \{3\}, \{2, 3\}, X\}, & \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, X\}. \end{array}$$

Note that the collection of closed subsets corresponds to the one in the same row, i.e. for $\{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ the collection of closed subsets are $\{\emptyset, \{3\}, \{1, 3\}, \{2, 3\}, X\}$ and vice versa.

5. There are no topologies on X that contain exactly six or seven subsets. Take

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, X\}$$

for example. The missing elements $\{2, 3\}$ and $\{1, 3\}$ can be generated by $\{2\} \cup \{3\}$ and $\{1\} \cup \{3\}$.

6. Lastly, $\mathcal{P}(X)$ itself is a well-defined topology. Each subset of this topology is clopen.

We have shown that on $X = \{1, 2, 3\}$ there are 20 possible topologies. While the total amount of all possible subsets were 256, if we consider that any topology must contain the emptyset \emptyset and X itself, then the number of valid collection of subsets shrinks to $2^6 = 64$. Thus, around 1/3 of sensible collection of subsets were topologies in this example.

There are two key takeaways from this example. Firstly, the collection of closed subsets for each topology were topologies themselves. Another is that the collection of all possible topologies can be partially ordered using the inclusion relation.

Proposition 2. Let (X, \mathcal{O}) be a topological space. The collection of all closed subsets of X with regards to \mathcal{O} is a topology.

Proof.

□

Definition 3 (Definition of a Topology via Closed Sets). A topology on a set X is a collection \mathcal{O} of subsets of X that satisfies the following axioms.

1. The emptyset \emptyset and the entire set X belong to \mathcal{O} .
2. Any arbitrary intersection of members of \mathcal{O} belongs to \mathcal{O} .
3. The union of finite number of members of \mathcal{O} belongs to \mathcal{O} .

In this case, the elements of \mathcal{O} are called closed sets. A subset $A \subset X$ is said to be open if its complement $X \setminus A$ is open.

Definition 4 (Definition of a Topology via Neighborhoods).

Proposition 5. The three definitions of a topology are equivalent.

Definition 6 (Comparison of Topologies). Let \mathcal{O}_1 and \mathcal{O}_2 be two topologies on a set X such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then the topology \mathcal{O}_1 is said to be coarser (also weaker or smaller) than \mathcal{O}_2 , and \mathcal{O}_2 is said to be finer (also stronger or larger) than \mathcal{O}_1 . The binary relation \subseteq defines a partial ordering relation on the set of all possible topologies on X .

Example 6.1. Let X be a set.

1. $\mathcal{O} = \{\emptyset, \mathcal{P}(X)\}$ is called the **trivial topology**. It is the coarsest topology that can be defined on a set.
2. $\mathcal{O} = \mathcal{P}(X)$ is called the **discrete topology**. In this case, (X, \mathcal{O}) is called the **discrete space**. It is the **finest topology** that can be defined on a set.

Proposition 7. Let (X, d) be a metric space. The collection of subsets

$$\mathcal{O}_d := \{U \subset X \mid U \text{ is a open subset in the metric space } (X, d)\}$$

defines a topology on X . In other words, a metric induces a topology.

Proof. We will show that \mathcal{O}_d fullfills the axioms of a topology.

1. The emptyset \emptyset is open in the metric space vacuously, hence $\emptyset \in \mathcal{O}_d$. For the entire set X , if $x \in X$, then clearly $B_\epsilon(x) \subset X$ for any $\epsilon \in \mathbb{R}^+$, therefore $X \in \mathcal{O}_d$.
2. Let $S \subset \mathcal{O}_d$ be a collection of subsets. Consider

$$x \in \bigcup_{U \in S} U,$$

then $x \in U_0$ for some set in \mathcal{O}_d . U_0 is open in the metric space, therefore, there is an $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon(x) \subset U_0$. The ϵ -ball $B_\epsilon(x)$ is also contained in the union of the subsets in S . In other words, any arbitrary union of members of \mathcal{O}_d are again in \mathcal{O}_d .

3. Let $U, V \in \mathcal{O}_d$ and consider $x \in U \cap V$. We have that $x \in U$ and $x \in V$. Since $U, V \in \mathcal{O}_d$, they are open subsets in the metric space, hence there are $\epsilon_1, \epsilon_2 \in \mathbb{R}^+$ such that $B_{\epsilon_1}(x) \subset U$

and $B_{\epsilon_2}(x) \subset V$. Without loss of generality assume $\epsilon_1 \leq \epsilon_2$. Then, $B_{\epsilon_1}(x) \subset B_{\epsilon_2}(x)$, so $B_{\epsilon_1}(x) \subset V$ also. This implies $B_{\epsilon_1}(x) \subset U \cap V$, so $U \cap V \in \mathcal{O}_d$. By simple induction, we may conclude that the intersection of finite number of members of \mathcal{O}_d is again in \mathcal{O}_d . \square

Remark. The proof above coincides with the fact that in a metric space arbitrary union of open subsets and finite intersection of open subsets are open.

Example 7.1. The Zariski-topology.

Example 7.2. List of natural topologies.

1. On \mathbb{R}^n the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of \mathbb{R}^n are arbitrary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{x \in \mathbb{R}^n \mid d(p, x) < r\}.$$

This definition agrees with the topology endowed on arbitrary metric spaces.

2. The matrix space $\text{Mat}_{n \times m}(\mathbb{K})$ for a field \mathbb{K} does not have one canonical topology. Depending on the context and literature different ones are used.
 - Since $\text{Mat}_{n \times m}(\mathbb{K})$ is isomorphic to $\mathbb{R}^{n \cdot m}$, one could use the Euclidean topology as defined above.
 - $\text{Mat}_{n \times m}(\mathbb{K})$ is a metric space via multitude of operator norms. The metric space induces the topology.
 - Another metric on $\text{Mat}_{n \times m}(\mathbb{K})$ is the rank distance for $A, B \in \text{Mat}_{n \times m}$ defined as $d(A, B) := \text{rank}(B - A)$ which again would induce a topology.

Definition 8 (Continuous Maps). Let (X, τ_X) and (Y, τ_Y) be **topological spaces**. A **map** $f : X \longrightarrow Y$ is said to be **continuous** if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \quad (1.1)$$

Proposition 9. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ - δ -continuous if and only if f is continuous.

Proof. Let X and Y be metric spaces and $f : X \longrightarrow Y$ a function.

1. “ \Rightarrow ”: Let f be ϵ - δ -continuous and $V \in \mathcal{O}_Y$ be an open subset. If $f^{-1}(V)$ is empty, then we are finished, so consider $x \in f^{-1}(V)$. We have that $f(x) \in V$. Since V is an open subset, there is an $\epsilon \in \mathbb{R}^+$ such that $B_Y(f(x), \epsilon) \subset V$. Using the ϵ - δ -continuity of f yields

$$f(B_X(x, \delta)) \subset B_Y(f(x), \epsilon) \subset V.$$

If we apply the definition of a preimage, we get $B_X(x, \delta) \subset f^{-1}(V)$ which implies that $f^{-1}(V)$ is open in the topological sense. Therefore, f is continuous.

2. “ \Leftarrow ”: Let f be continuous in the topological sense and consider $x \in X$. The ϵ -ball $B_Y(f(x), \epsilon)$ is open in Y , hence the preimage $f^{-1}(B_Y(f(x), \epsilon))$ is also open and contains x . Now, there exists a $\delta \in \mathbb{R}^+$ such that

$$B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \epsilon)).$$

Applying the definition of a preimage we get $f(B_X(x, \delta)) \subset B_Y(f(x), \epsilon)$ which means f is ϵ - δ -continuous at x . Since x was chosen arbitrary, f is ϵ - δ -continuous. \square

Remark. Again, the proof above coincides with the fact that in a metric space, a function is ϵ - δ -continuous if and only if the preimage of any open subset is open.

Definition 10 (Homeomorphism). Let X and Y be topological spaces.

1. A map $f : X \rightarrow Y$ is a **homeomorphism** if it has the following properties.
 - (a) f is **bijective**.
 - (b) f and the **inverse map** f^{-1} is **continuous**.
2. Two topological spaces X and Y are said to be **homeomorphic** if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\text{Homeo}(X, Y)$. If $Y = X$ we also write $\text{Homeo}(X)$.

Proposition 11. The set of all homeomorphisms of X to itself $\text{Homeo}(X)$ is a group with composition as its operation.

Proof. The identity function is contained in $\text{Homeo}(X)$ and is the identity element. Composition is associative and closed in $\text{Homeo}(X)$. By definition, $\text{Homeo}(X)$ contains the inverse of all its elements. Thus, $\text{Homeo}(X)$ is a group with composition as its operation. \square

Definition 12 (Base). Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a **basis** of the topology, if any member of \mathcal{O} is the **union of subsets** from \mathcal{B} .
2. $\mathcal{S} \subset \mathcal{O}$ is a **subbasis** of the topology, if any member of \mathcal{O} is the **union of finite intersections of subsets** from \mathcal{S} .

We say that \mathcal{B} and \mathcal{S} **generates** \mathcal{O} and write $\bar{\mathcal{S}} = \bar{\mathcal{B}} = \mathcal{O}$.

Example 12.1. 1. The set Γ of all open intervals in \mathbb{R} form a basis for the Euclidean topology on \mathbb{R} . If we require Γ to be of all bounded open intervals, it will still generate the Euclidean topology.

Lemma 13. For any collection of subsets $S \subset \mathcal{P}(X)$, there exists exactly one topology $\mathcal{O} \subset \mathcal{P}(X)$ that contains S and is the coarsest topology to do so, i.e.

1. $S \subset \mathcal{O}$, and
2. if $\mathcal{O}' \subset \mathcal{P}(X)$ is an another topology with $S \subset \mathcal{O}'$, then $\mathcal{O} \subset \mathcal{O}'$.

Proof. Let $S \subset \mathcal{P}(X)$ be a collection of subsets and $\mathcal{O}(S)$ be the set of topologies that contain S , i.e.

$$\mathcal{O}(S) = \{ \tau \subset \mathcal{P}(X) \mid \tau \text{ is a topology and } S \subset \tau \}.$$

We know that $\mathcal{O}(S)$ is not empty because $\mathcal{P}(X) \in \mathcal{O}(S)$. Now define

$$\mathcal{O} := \bigcap_{\tau \in \mathcal{O}(S)} \tau.$$

Our claim is that this \mathcal{O} is a topology.

1. The emptyset \emptyset and the entire set X is contained in each $\tau \in \mathcal{O}(S)$ since these are topologies. Thus, the empty set and the entire set lie also in the intersection, i.e. $\emptyset, X \in \mathcal{O}$.
2. Let $\{U_i\}_{i \in I}$ be a family of subsets in \mathcal{O} for an arbitrary index set I . This means for each $i \in I$ it is $U_i \subset \mathcal{O}$, therefore, again for each $i \in I$ we have $U_i \in \tau$. Since τ was a topology, the arbitrary union of $\{U_i\}_{i \in I}$ will lie in τ .
3. Similar for the finite intersection.

In particular, \mathcal{O} lies in $\mathcal{O}(S)$.

MISSING THAT IT IS UNIQUE! □

Definition 14. 1. Given (X, τ) be a **topological space**, $S \subset X$ a subset, the **subspace topology** (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \}.$$

2. Let (X, τ_X) and (Y, τ_Y) be two **topological spaces**. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

3. Let (X, τ_X) and (Y, τ_Y) be two **topological spaces**. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \}.$$

Definition 15. Let (X, τ) be a topological space.

1. Given a **point** $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in \tau$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U .
2. Let $S \subset X$ be a subset. The interior of S , denoted by $\overset{\circ}{S}$ or $\text{int}(S)$, is the **set** of all interior points of S .
3. Let $S \subset X$ be a subset. The closure of S , denoted by \overline{S} or $\text{cl}(S)$, is defined by

$$\text{cl}(S) := X \setminus \text{int}(X \setminus S).$$

Remark. This lemma does not hold for basis.

Remark. 1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$; often $W \subset X \times Y \iff \forall(x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. Let (X, \mathcal{O}) be a topological space. A subset that is both open and closed is called clopen. Moreover, a subset is clopen if and only if its boundary is empty.

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\text{cl}(A) = A$, but on the other hand, A is open, so we also have $\text{int}(A) = A$. Then, the boundary of A is $\partial A = \text{cl}(A) \setminus \text{int}(A) = A \setminus A = \emptyset$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. \square

1.3 Exercises and Notes

Definition 16 (Metric Space).

Definition 17 (Open and Closed Subsets).

Theorem 18 (Union and Intersection of Open Subsets).

Definition 19. There are many equivalent ways to define continuity.

- ϵ - δ -continuity:
- *sequential continuity*:

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 20. A topological space (X, \mathcal{O}) is said to be **connected**, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** open and closed (**clopen**) are the empty set \emptyset and the entire set X , i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \emptyset$ or $A = X$.
3. The **only** subsets of X with empty boundary are the emptyset \emptyset and the entire set X .
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is **connected** if it is a connected space when viewed as a subspace of X .

Lemma 21. Any interval $I \subset \mathbb{R}$ is **connected**.

Lemma 22. Let X and Y be topological spaces and $f : X \rightarrow Y$ a continuous function. If X is **connected**, then $f(X) \subset Y$ is **connected**.

Definition 23. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 24. Connected components are closed subsets.

Lemma 25. Let X be connected and $f : X \rightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$, then f is constant.

Definition 26. X is said to be **path connected**, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 27. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 28. A topological space (X, \mathcal{O}) is said to be **connected**, if one of the following **equivalent** conditions is met.

1. X is **not** a union of two **nonempty**, **disjoint**, and **open** subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** **open** and **closed** (**clopen**) are the empty set \emptyset and the entire set X , i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \emptyset$ or $A = X$.
3. The **only** subsets of X with empty **boundary** are the emptyset \emptyset and the entire set X .
4. All **continuous** maps from X to the two point space $\{0, 1\}$ endowed with the **discrete** topology is **constant**.

A **subset** of X is **connected** if it is a **connected** space when viewed as a **subspace** of X .

Proof. We verify the equivalence of the different definitions. So, let (X, \mathcal{O}) be a topological space.

- “1. \Rightarrow 2.”: Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X , then $X \setminus A$ is also not the empty set nor X . Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \emptyset$ or $A = X$.
- “2. \Rightarrow 1.”: Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.
- “2. \iff 3.”: This is one of the properties of clopen subsets and was proven in remark XXX.
- “1. \Rightarrow 4.”: Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \rightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.
- “4. \Rightarrow 1.”: Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \rightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

□

Lemma 29. Any **interval** $I \subset \mathbb{R}$ is **connected**.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \quad (2.1)$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \text{cl}(B)$ and since the complement of B is an open subset A , so $B = \text{cl}(B)$. It is therefore $x \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. \square

Lemma 30. Let X and Y be topological spaces and $f : X \rightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$ so $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$, so $A = \emptyset$ or $B = \emptyset$ and we are done. \square

Remark. The two lemma above are handy to show that images of functions are connected.

Example 30.1. The general linear group $\text{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Proof. Define the following partition of $\text{GL}_n(\mathbb{K})$

$$\begin{aligned} A &:= \{ M \in \text{Mat}_{n \times n}(\mathbb{K}) \mid \det(M) > 0 \} \\ B &:= \{ M \in \text{Mat}_{n \times n}(\mathbb{K}) \mid \det(M) < 0 \}, \end{aligned}$$

then, A and B are disjoint, nonempty, and $\text{GL}_n(\mathbb{K}) = A \sqcup B$. We show that A and B are open sets.

The determinant function $\det : \text{Mat}_{n \times n}(\mathbb{K}) \rightarrow \mathbb{C}$ is continuous because it is a multivariate polynomial. \mathbb{R}^+ is an interval, therefore open, and so $\det^{-1}(\mathbb{R}^+) = A$ is also open. Similarly B is an open subset. Hence $\text{GL}_n(\mathbb{K})$ is not connected. \square

Remark. In the proof above, the topology of $\text{Mat}_{n \times n}(\mathbb{K})$ matters because the continuity of the determinant function depends on the underlying topology.

Definition 31. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Example 31.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proof. Assume there is a connected set $A \subset \mathbb{Q}$ that contains more than one point. Let $x \in A$ be a point in A . We show that $\{x\}$ is a clopen set.

Denote another point in A that is closest to x as x_0 , i.e. for all $y \in A$ it is $d(x, y) \geq d(x, x_0)$. Now set $\epsilon := d(x, x_0)$. Then, $B_\epsilon(x) \cap \mathbb{Q} = \{x\}$ is an open subset.

I think showing closedness is quite similar. \square

Proposition 32. Connected components are closed subsets.

Proof. Let X be a set and $C \subset X$ be a connected component. Consider $\text{Cl}(C)$. Clearly, $C \subset \text{Cl}(C)$. Moreover, $\text{Cl}(C)$ is connected hence $\text{Cl}(C) \subset C$. We have $\text{cl}(C) = C$ so C is connected. \square

Lemma 33. Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 34. X is said to be **path connected**, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 35. If X is path connected, then it is also connected.

Proof. Locally constant implies continuous with regards to the discrete topology on Y . Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \emptyset$. Conclude f is identical to $f(x)$. \square

Application: $f : X \longrightarrow \{0, 1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0, 1] \longrightarrow X$ be continuous path with $\gamma(0) = a$ and $\gamma(1) = b$. We have that γ^{-1} \square

2.3 Exercises and Notes

2.3.1 Connectedness

Lemma 36. If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are two connected topological spaces, then their product $X \times Y$ with the product topology $\mathcal{O}_{X \times Y}$ is also connected.

Proof. We will use the definition that all continuous maps from $X \times Y$ to $\{0, 1\}$ endowed with the discrete topology must be constant. Fix a continuous $f : X \times Y \rightarrow \{0, 1\}$.

First, consider the image $f(\{x\} \times Y)$ with $x \in X$. Assume f is not constant on $\{x\} \times Y$, then $f(\{x\} \times Y) = \{0, 1\}$. So we have the preimages $f^{-1}(\{0\}) = \{x\} \times U$ and $f^{-1}(\{1\}) = \{x\} \times V$ with $U, V \subset Y$, $U, V \neq \emptyset$, and $U \cap V = \emptyset$. Because f is continuous, U and V must also be open. This would however mean that $U \sqcup V = Y$ and Y would not be connected, therefore, f is constant on $\{x\} \times Y$. Similarly, we get that f is constant on $X \times \{y\}$ for all $y \in Y$.

Let $(x, y) \in X \times Y$ and $(x', y') \in X \times Y$ be two arbitrary points. We have $f(x, y) = f(x, y')$ because f is constant on $\{x\} \times Y$ and similarly $f(x, y') = f(x', y')$ because f is constant on $X \times \{y'\}$. Putting everything together, it is $f(x, y) = f(x', y')$, therefore all continuous $f : X \times Y \rightarrow \{0, 1\}$ are constant. \square

Example 36.1. Clearly, the union of two connected sets need not be connected. Take for example $[0, 1] \subset \mathbb{R}$ and $[2, 3] \subset \mathbb{R}$. Their union $[0, 1] \cup [2, 3]$ is not connected.

Set difference of connected sets are also not necessarily connected, e.g. $[0, 2] \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ are connected, but $[0, 2] \setminus \{1\} = [0, 1) \cup (1, 2]$ is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and another unit circle around $(1, 0)$ $A := \{(x, y) \mid (x - 1)^2 + y^2 = 1\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3} \right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3} \right) \right\}$$

which is not connected.

- Proposition 37.**
1. Every trivial topological space is connected.
 2. Every discrete topological space with at least two elements is disconnected.
 3. Trivially, every singleton set and the empty set are connected spaces vacuously.

Proof. 1. Let X be an arbitrary set and $\mathcal{O} = \{\emptyset, X\}$ be the trivial topology. If $S \subset X$ is a clopen subset, then it is trivially either \emptyset or X , therefore, X is connected.

2. Let X be a set containing more than one element and $\mathcal{O} = \mathcal{P}(X)$ be the discrete topology of X . Let $A \subset X$ be a nonempty proper subset, then $B := X \setminus A$ is also not empty. Both are open subsets, but $A \sqcup B = X$, so X is not connected. \square

Proposition 38. Every singleton set in \mathbb{R}^n endowed with the Euclidean topology is clopen.
 ??? IDK IF THIS IS TRUE

2.3.2 Path-Connectedness

Example 38.1. Connectedness does not imply path-connectedness. Let \mathbb{R}^2 be endowed with the Euclidean topology and consider

$$X = \left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) \mid x > 0 \right\} \cup (\{0\} \times [-1, 1]) \subset \mathbb{R}^2.$$

and see figure XXX. X is connected, but it is not path-connected.

Proof. Denote

$$A := \left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) \mid x > 0 \right\} \quad B := \{0\} \times [-1, 1],$$

then $X = A \cup B$.

1. First, define $f : \mathbb{R}^+ \rightarrow \mathbb{R}^2$ as

$$f(x) := \left(x, \sin \left(\frac{1}{x} \right) \right).$$

f is continuous, \mathbb{R}^+ is an interval, therefore connected, so $f(\mathbb{R}^+) = A$ is connected. Similarly, define $g : [-1, 1] \rightarrow 0 \times [-1, 1]$ as

$$g(x) := (0, x).$$

Again, g is continuous and $[-1, 1]$ is connected, so $g([-1, 1]) = 0 \times [-1, 1] = B$ is also connected.

Now assume $X = U \sqcup V$ for some nonempty, disjoint, open subsets of \mathbb{R}^2 . Since A is connected, either U or V must contain A , say U . On the other hand, $(0, 0) \in V$, but since V is open, it contains all the interior points, i.e.

Assume there is a clopen subset $S \subset X$ that is not empty. Without loss of generality, we have that $(0, 0) \in U$ (otherwise, consider the complement of U which also must be clopen). Since A is clopen in A , the intersection $A \cap U$ must also be clopen in A , but A is connected, so A is contained in U .

Moreover, the closure of A is also contained in U . So there is an $\epsilon > 0$ such that the ball $B(p, \epsilon)$ that contains $(0, 0)$ is in U . I got lazy to go into the details, but this ball contains a point of B . Follow the same reason as above.

2. Assume X is path-connected.

Choose two points $x_0 = (0, 1) \in A$ and $x_1 = (1, 1) \in B$ and a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Let $\epsilon \in (0, 1)$, then $B_\epsilon(x_0) \cap X$ is an open subset that contains x_0 , therefore, $\gamma^{-1}(B_\epsilon(x_0) \cap X)$ is also open.

□

As motivation, the proof of intermediate value theorem becomes very elegant.

Chapter 3

Separation Axioms

3.1 Definitions and Theorems

Definition 39 (T_1 Space). Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 40. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X , i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 41 (T_2 Space). Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \emptyset$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 42. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x .
3. The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed as a subset of the product space $X \times X$.

Proposition 43. T_2 spaces are also T_1 spaces.

3.2 Proofs, Remarks, and Examples

3.2.1 T_0 Space

Definition 44. A topological space (X, \mathcal{O}) is a T_0 space (or Kolmogorov space) if for every pair of distinct points of X , at least one of them has a neighborhood not containing the other (this property is called **topologically distinguishable**).

Definition 45. A topological space (X, \mathcal{O}) is a T_1 space (also called **accessible space** or a space with **Fréchet topology**) if one of the following **equivalent** conditions are met.

1. Any two distinct points in X are separated, i.e. if $x, y \in X$ are points with $x \neq y$, then there are neighborhoods U_x and U_y of x and y respectively such that $y \notin U_x$ and $x \notin U_y$.
2. Points are closed in X , i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.
3. Every subset of X is the intersection of all the open sets containing it.
4. Every finite set is closed.
5. Every cofinite set of X is open.

Definition 46 (T_1 Space). Let X be a topological space.

1. We say that two points x and y can be **separated** if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are **separated**.

Proposition 47. Let X be a topological space. Then, the following are **equivalent**.

1. X is a T_1 space.
2. Points are **closed** in X , i.e. given any $x \in X$, the **singleton** set $\{x\}$ is a **closed** set.

Definition 48 (T_2 Space). Let X be a topological space.

1. Points x and y in X can be **separated by neighborhood** if there exists a neighborhood U of x and a neighborhood V of y such that U and V are **disjoint**, i.e. $U \cap V = \emptyset$.
2. A topological space X is a T_2 space if any two distinct points in X are **separated by neighborhood**.

Proposition 49. Let X be a topological space. Then, the following are **equivalent**.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x .
3. The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed as a subset of the product space $X \times X$.

Proposition 50. T_2 spaces are also T_1 spaces.

3.3 Exercises and Notes

Chapter 4

Compact Spaces

Definition 51. 1. A topological space X is called compact if each of its open cover has a finite subcover.

2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X .

Theorem 52. Satz 17

Theorem 53. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 54. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 55. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \rightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Definition 56. 1. A topological space X is called **compact** if each of its **open cover** has a **finite subcover**. That is, X is compact if for every arbitrary collection \mathcal{C} of open subsets of X such that

$$X = \bigcup_{U \in \mathcal{C}} U$$

there is a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that

$$X = \bigcup_{U \in \mathcal{F}} U.$$

2. A subset is said to be compact if it is compact as a subspace.

Definition 57. X is called **sequentially compact** if every **sequence** in X has a **convergent subsequence** whose limit is in X .

Remark. The notion of compact and sequentially compact are not equivalent.

Example 57.1. 1. Example of a space that is compact but not sequentially compact.

2. Example of a space that is sequentially compact but not compact.

Proposition 58. Let X and Y be two topological spaces.

1. Continuous functions preserve compactness, i.e. if $f : X \rightarrow Y$ is continuous and X is compact, then $f(X) \subset Y$ is compact.
2. In a compact space, every closed subset is compact, i.e. if X is compact and $A \subset X$ is a closed subset, then A is compact.
3. The product of compact spaces is again compact. If X and Y are both compact, so is $X \times Y$.

Proof. 1. Let $f : X \rightarrow Y$ be continuous and X compact. Denote the open cover of the continuous image of X by \mathcal{C} , i.e.

$$f(X) \subset \bigcup_{U \in \mathcal{C}} U.$$

Because f is continuous, each of the preimages $f^{-1}(U)$ with $U \in \mathcal{C}$ is open. Now, X is compact, there are finitely many $f^{-1}(U)$ such that

$$X \subset \bigcup_{U \in \mathcal{F}} f^{-1}(U)$$

Conclude that $f(X)$ is compact.

2. Let \mathcal{U} be an open cover of A . Every open set in \mathcal{U} is in the form $U \cap A$ for some open set $U \subset X$. Define

$$\mathcal{V} := \{U \in \mathcal{O} \mid U \cap A \in \mathcal{U}\}$$

then \mathcal{V} is an open cover of A as well. Since A is closed, $X \setminus A$ is open, so $\mathcal{V} \cup (X \setminus A)$ is an open cover of X . By compactness of X , there is a finite subcover that XXXXX.

3. I think this is clear.

□

Lemma 59. $[0, 1] \subset \mathbb{R}$ is compact.

Proof. skipped

□

Theorem 60 (Heine-Borel). A compact subset of a Euclidean space $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition 61. Let X be a T_2 -space. If $K \subset X$, then K is closed.

Proof. for the two things above, skipped.

□

Lemma 62. Let $f : X \rightarrow Y$ be continuous and bijective. If X is compact and Y is a T_2 -space, then f^{-1} is continuous.

Chapter 5

Quotient Space

5.1 Definitions and Theorems

5.2 Proofs, Remarks, and Examples

Definition 63. Let (X, \mathcal{O}) be a topological space, and let \sim be an equivalence relation on X . The quotient set, X/\sim is the set of equivalence classes of elements of X . The equivalence class of $x \in X$ is denoted $[x]$. The projection map (also quotient or canonical map) associated with \sim refers to the following surjective map:

$$\pi : X \longrightarrow X/\sim, \quad x \mapsto [x]$$

For any subset $S \subset X/\sim$ (so in particular, $s \subset X$ for every $s \in S$) the following holds.

$$\pi^{-1}(S) = \{x \in X \mid [x] \in S\} = \bigcup_{s \in S} s.$$

The quotient space under \sim is the quotient set X/\sim equipped with the quotient topology, which is the topology whose open sets are subsets $U \subset X/\sim$ such that

$$\{x \in X \mid [x] \in U\} = \bigcup_{u \in U} u$$

is an open subset of (X, \mathcal{O}_X) ; that is, $U \subset X/\sim$

Proof. We will show that \mathcal{O}_\sim is a topology.

1. Clearly, $\pi^{-1}(\emptyset) = \emptyset \in \mathcal{O}$, thus $\emptyset \in \mathcal{O}_\sim$. Moreover, $\pi(X) = X/\sim$, hence $\pi^{-1}(X/\sim) = X \in \mathcal{O}$, and it is $X/\sim \in \mathcal{O}_\sim$.

2. Let I be an arbitrary index set and $\{U_i\}_{i \in I}$ a family of open sets in X/\sim . It is

$$\pi^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \pi^{-1}(U_i). \quad (5.1)$$

Since U_i is open and π is continuous, $\pi^{-1}(U_i)$ is open for all $i \in I$. Thus $\bigcup_{i \in I} U_i \in \mathcal{O}_\sim$.

3. Similar as above as preimages preserve unions and intersections.

□

Example 63.1. 1. \mathbb{R}/\mathbb{Z}

This space is homeomorph to S^{-1} and is compact.

2. $(\mathbb{R}/\mathbb{Q}, \mathcal{O}_{\mathbb{R}/\mathbb{Q}})$ is the trivial topology.

Proposition 64. $\mathcal{O}_{X/\sim}$ is the **finest topology** in which the **projection map** $\pi : X \longrightarrow X/\sim$ is **continuous**.

Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a surjective map. The map is a quotient map (also said strong continuity) if one of the equivalent condition hold.

1. A subset $U \subset Y$ is open in Y if and only if the preimage $p^{-1}(U)$ is open in X .
2. A subset $U \subset Y$ is closed in Y if and only if the preimage $p^{-1}(U)$ is closed in X .

Remark. Quotient maps are continuous. There are quotient maps that are neither open nor closed maps.

Theorem 65. Let Y be a topological space. Then the following are equivalent.

1. $f : X/\sim \longrightarrow Y$ continuous
2. $f \circ \pi : X \longrightarrow Y$ is continuous.

Moreover, if X is connected, then X/\sim is connected. Same is true for path-connectedness and compactness.

Definition 66. A topological group G is a topological space that is also a group such that the group operation

$$\circ : G \times G \longrightarrow G, (x, y) \mapsto x \circ y$$

and the inversion map

$$^{-1} : G \longrightarrow G, x \mapsto x^{-1}$$

are continuous. Here $G \times G$ is viewed as a topological space with the product topology. Such a topology is said to be compatible with the group operations and is called a group topology.

Remark. About homeomorphism of G-space.

Definition 67. Consider a group acting on a set X . The orbit of an element x in X is the set of elements in X to which x can be moved by the elements of G . The orbit of x is denoted by $G \cdot x$.

$$G \cdot x := \{g \cdot x \mid g \in G\}.$$

Definition 68. $X/G := X/\sim$ such that $x \sim y$ if and only if there is a $g \in G$ such that $x = gy$.

Definition 69 (Hilbert). • Matrix A is semi-stable if A is diagonalizable.
 • Matrix A is stable, if it is semi-stable and all Eigenvalues are distinct.

Definition 70 (Stabilisator). $x \in X$
 $G \supset G_x := \{ h \mid h \cdot x = x \}$

Lemma 71. 1. $G_x \subset G$ is a subgroup.
 2. $G/G_x \longrightarrow G_x$ is well-defined and $[x] \mapsto gx$ is a continuous bijection (repspective of the quotient topology on G/G_x).

Corollary 1. If G is compact and X is T_2 , then $f : G/G_x \longrightarrow G$ is a homeomorphism.

Definition 72.

5.3 Exercises and Notes

Chapter 6

Pasting

Definition 73. Let X_0 and X be topological spaces and $\varphi : X_0 \longrightarrow X$ a continuous function. Set $X_\varphi := X / \sim$ where \sim is generated by

$$\{x \sim \varphi(x) \mid x \in X_0\}.$$

Example 73.1. idk

Definition 74. Abbildungstorus

Definition 75. Let X be a topological space.

1. X is said to be a first-countable space or to satisfy the first axiom of countability if each point has a countable neighbourhood basis (local base). That is, for each point $x \in X$ there exists a sequence N_1, N_2, \dots of neighbourhoods of x such that for any neighbourhood N of x there exists an integer i with N_i contained in N . Since every neighbourhood of any point contains an open neighbourhood of that point, the neighbourhood basis can be chosen without loss of generality to consist of open neighbourhoods.
2. X is said to be a second-countable space, also called completely separable space, or to satisfy the second axiom of countability if it has a countable base.

Remark. A countable subbase induces a countable base.

Lemma 76. The second-countable axiom implies the first.

Lemma 77. If X is a first-countable space, then

1. all sequentially-continuous function is continuous.
2. all compact spaces are also sequentially compact.

Definition 78. A topological manifold \mathcal{M}^n of dimension n is a topological space that is T_2 , is a second-countable space, and for each point has a neighbourhood that is homeomorphic to an open subset of

$$\mathbb{H}^n := \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0 \}$$