

# 1 Rigidity Conjecture

**Proposition 1** (Division). A formal power series  $f(X) = \sum_{k \geq 1} a_k X^k \in \mathbb{C}[[X]]$  is invertible if and only if its constant coefficient  $a_0$  is nonzero.

**Lemma 2.** Given two formal power series  $f(X) = \sum_{n \geq 0} a_n X^n \in \mathbb{C}[[X]]$  and  $g(X) = \sum_{n \geq 0} b_n X^n \in \mathbb{C}[[X]]$  with  $b_0 \neq 0$ , we may compute their quotient

$$\frac{\sum_{n \geq 0} a_n X^n}{\sum_{n \geq 0} b_n X^n} = \sum_{n \geq 0} c_n X^n \quad (1)$$

by

$$c_n = \frac{1}{b_0} \left( a_n - \sum_{k \geq 1} b_k c_{n-k} \right). \quad (2)$$

*Proof.* #MISSING □

**Remark.** When studying compositions of formal power series, we require that the inner power series  $f(X)$  has no constant term, i.e.,  $f(0) = 0$ . This condition ensures that the resulting composition is well-defined in the ring of formal power series  $\mathbb{C}[[X]]$ , as it prevents infinite contributions to the coefficients.

Consider  $f(X) = \sum_{k \geq 1} a_k X^k$  and  $g(X) = \sum_{k \geq 0} b_k X^k$ . The composition  $g(f(X))$  is given by substituting  $f(X)$  into  $g(X)$ :

$$\begin{aligned} g(f(X)) &= b_0 + b_1 f(X) + b_2 f(X)^2 + \dots \\ &= b_0 + b_1(a_1 X + a_2 X^2 + \dots) + b_2(a_1 X + a_2 X^2 + \dots)^2 + \dots \\ &= b_0 + b_1 a_1 X + (b_1 a_2 + b_2 a_1^2) X^2 + \dots, \end{aligned}$$

where we grouped the terms by powers of  $X$  in the last step. We observe that the coefficients of  $X^n$  in  $g(f(X))$  depend only on a finite number of coefficients of  $f(X)$  and  $g(X)$ . This is because, with  $f(0) = 0$ , each power  $f(X)^k$  introduces terms of degree at least  $k$ , ensuring that lower-degree terms do not contribute infinitely to higher-order coefficients.

On the other hand, if  $f(0) \neq 0$ , we write  $f(X) = a_0 + \sum_{k \geq 1} a_k X^k$ , where  $a_0 = f(0)$ . In this case,

$$f(X)^k = (a_0 + a_1 X + a_2 X^2 + \dots)^k$$

produces a constant term  $a_0^k \neq 0$ . Consequently, the constant term of  $g(f(X))$  depends on infinitely many terms of  $g(X)$ , and the composition  $g(f(X))$  is no longer a formal power series.

Since we are interested in the compositional inverse, it is necessary to extend the condition  $f(0) = 0$  to both power series. This ensures that the inverse series  $f^{-1}(X)$ , when substituted into  $f(X)$ , results in the identity series  $X$ , with no contributions from constant terms that would otherwise make the series ill-defined.

The following proposition and lemma are taken from Enumerative Combinatorics by Richard P. Stanley and Sergey Fomin.

**Definition 3.** Let  $f(X) \in \mathbb{C}[[X]]$  be a power series with no constant term. We call a power series  $f^{-1}(X) \in \mathbb{C}[[X]]$  the compositional inverse of  $f$ , if it satisfies  $f(f^{-1}(X)) = f^{-1}(f(X)) = X$ .

**Proposition 4.** A power series  $f(X) = a_1 X + a_2 X^2 + \dots \in \mathbb{C}[[X]]$  has a compositional inverse if and only if  $a_1 \neq 0$ . Moreover, if the compositional inverse exists, then it is unique.

*Proof.* Assume  $f$  has a compositional inverse and denote the compositional inverse by  $f^{-1}(X) = b_1 X + b_2 X^2 + \dots$ . Writing out  $f(f^{-1}(X)) = X$  using multinomial theorem gives

$$\begin{aligned} X &= a_1(b_1 X + b_2 X^2 + \dots) + a_2(b_1 X + b_2 X^2 + \dots)^2 + \dots \\ &= (a_1 b_1 X + a_1 b_2 X^2 + a_2 b_1^2 X^2 + \dots) + (a_2 b_1^2 X^2 + 2a_2 b_1 b_2 X^3 + \dots) \\ &= (a_1 b_1) X + (a_1 b_2 + a_2 b_1^2) X^2 + (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) X^3 \dots \end{aligned}$$

Equating the coefficients on both sides yields a linear system of equations.

$$\begin{aligned} 1 &= a_1 b_1 \\ 0 &= a_1 b_2 + a_2 b_1^2 \\ 0 &= a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 \\ &\vdots \end{aligned}$$

The first equation has a solution if and only if  $a_1 \neq 0$ . In that case, the solution is unique. Then, the second equation can be solved uniquely for  $b_2$ . By this process, we are able to solve the third equation for  $b_3$ , the fourth for  $b_4$  and so on. Thus,  $f^{-1}(X)$  exists if and only if  $a_1 \neq 0$  and in that case,  $f^{-1}(X)$  is unique.  $\square$

**Lemma 5** (Lagrange Inversion Formula). Let  $f(X) = \sum_{k \geq 1} a_k X^k \in \mathbb{C}[[X]]$  be a power series with  $a_1 \neq 0$  and denote its composition inverse by  $f^{-1}(X) = \sum_{k \geq 1} b_k X^k \in \mathbb{C}[[X]]$ . The coefficients of the inverse is given by the following formula.

$$b_k = \frac{1}{k} [X^{n-1}] \left( \frac{X}{f(X)} \right)^k$$

*Proof.* We begin by substituting  $f(X)$  into  $f^{-1}(X)$ . It is

$$X = f^{-1}(f(X)) = \sum_{k \geq 1} b_k f(X)^k.$$

Differentiating and subsequently taking the quotient with  $f(X)^n$  for  $n \in \mathbb{N}$  on both sides yields

$$\begin{aligned} 1 &= \sum_{k \geq 1} k \cdot b_k \cdot f(X)^{k-1} \cdot f'(X) \\ \Rightarrow \quad \frac{1}{f(X)^n} &= \sum_{k \geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X). \end{aligned}$$

We want to take the coefficient of  $X^{-1}$  on both sides. For that, first notice that for  $k \neq n$  it is

$$\frac{1}{k-n} \frac{d}{dX} f(X)^{k-n} = f(X)^{k-n-1} f'(X) = \frac{f(X)^k}{f(X)^{n+1}} f'(X).$$

For any Laurent series, its derivative has no  $X^{-1}$  term. Thus, for  $k \neq n$ , it is

$$[X^{-1}] \frac{f(X)^k}{f(X)^{n+1}} f'(X) = [X^{-1}] \frac{1}{k-n} \frac{d}{dX} f(X)^{k-n} = 0.$$

If we now take the coefficient of  $X^{-1}$  in #REFMISSING, we get

$$[X^{-1}] \frac{1}{f(X)^n} = [X^{-1}] \sum_{k \geq 1} k \cdot b_k \cdot \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) \tag{3}$$

$$= \sum_{k \geq 1} k \cdot b_k \cdot [X^{-1}] \frac{f(X)^k}{f(X)^{n+1}} \cdot f'(X) \tag{4}$$

$$= n \cdot b_n [X^{-1}] \frac{f(X)^n}{f(X)^{n+1}} \cdot f'(X) \tag{5}$$

$$= n \cdot b_n [X^{-1}] \frac{f'(X)}{f(X)} \tag{6}$$

$$= n \cdot b_n [X^{-1}] \frac{a_1 + 2a_2 X + 3a_3 X^2 + \dots}{a_1 X + a_2 X^2 + a_3 X^3 + \dots} \tag{7}$$

$$= n \cdot b_n [X^{-1}] \frac{1}{X} \frac{a_1 + 2a_2 X + 3a_3 X^2 + \dots}{a_1 + a_2 X + a_3 X^2 + \dots} \tag{8}$$

$$= n \cdot b_n \tag{9}$$

where we used the formula for power series division given in #REFMISSING to compute the constant term of the quotient.

$$\frac{1}{a_1} (a_1 - 0) = 1 \quad (10)$$

Now, by shifting the power of the coefficient to be extracted, we get

$$[X^{-1}] \frac{1}{f(X)^n} = [X^{n-1}] \frac{X^n}{f(X)^n} = [X^{n-1}] \left( \frac{X}{f(X)} \right)^n. \quad (11)$$

Finally, continuing from #REFMIISING, we get

$$n \cdot b_n = [X^{-1}] \frac{1}{f(X)^n} = [X^{n-1}] \left( \frac{X}{f(X)} \right)^n \quad (12)$$

$$\Rightarrow b_n = \frac{1}{n} [X^{n-1}] \left( \frac{X}{f(X)} \right)^n \quad (13)$$

as desired.  $\square$

**Lemma 6.** For a formal power series  $a(X) =$

*Proof.*

$$u_n = [X^n] \left( \frac{X}{a(X)} \right)^{n+1} \quad (14)$$

$$= [X^n] \left( \frac{1}{1 - (\alpha_1 X + \dots + \alpha_m X^m)} \right)^{n+1} \quad (15)$$

$$= [X^n] \sum_{k \geq 0} \binom{n+k}{n} (\alpha_1 X + \dots + \alpha_m X^m)^k \quad (16)$$

1. substituted  $f(X)$

$$\left( \frac{1}{1-z} \right)^{n+1} = \left( \sum_{k \geq 0} z^k \right)^{n+1} \quad (17)$$

$\square$