Chapter 1

Exact Sequences

Definition 1. exact sequence is a sequence of maps!

Chapter 2

Tensor Products

Definition 2.
1. An element $m \in M$ is called a torsion element of the module if there exists an element $r \in R \setminus \mathrm{ZD}(R)$ such that $rm = 0$.
2. M is called a torsion module if all its elements are torsion elements.
3. M is called torsion-free if zero is the only torsion element.
4. The set of all torsion elements of M is called the torsion module, denoted by $T(M)$ and is a submodule.

Proposition 3. Free modules are torsion-free.

Proof.
1. Let R be a ring.
2. Fix a free R-module F.
3. Assume F is not torsion-free.
4. Then, there is an element $m \in F$ and a non-zero-divisor $r \in R \setminus \mathrm{ZD}(R)$ such that $rm = 0$.
5. Since F is free, it has a basis, say $\{x_i\}_{i \in I}$ for an arbitary index set.
6. The element m has a representation through the basis elements
$$m = \sum_{i \in I} \lambda_i x_i$$
with $\lambda_i \in R$ for all $i \in I$.
7. Thus, we have
$$0 = rm = r \cdot \sum_{i \in I} \lambda_i x_i = \sum_{i \in I} r \lambda_i x_i.$$
8. But that would mean $\{x_i\}_{i \in I}$ are linearly dependent contradicting it being a base.

□

Theorem 4. *Tensor product is a functor.*
Denotation:

- Let R be a ring.
- Let M, M', and M'' be R-modules.
- Let $\varphi : M \to M'$ and $\varphi' : M' \to M''$ be linear maps.
- Let N, N', and N'' be R-modules.
- Let $\psi : N \to N'$ and $\psi' : N' \to N''$ be linear maps.

Result:

1. We have the equality of the two linear maps
$$\mathrm{id}_M \otimes \mathrm{id}_N = \mathrm{id}_{M \otimes_R N}$$
2. We have the equality of the two linear maps
$$(\varphi' \otimes \psi') \circ (\varphi \otimes \psi) = (\varphi' \circ \varphi) \otimes (\psi' \circ \psi)$$
from $M \otimes_R N$ to $M'' \otimes_R N''$.

Theorem 5. *Tensor product is a functor.*
Denotation:

- Let R be a ring.
- Let $\boldsymbol{R}$**-Mod** be the category of modules over R.
- 1. The objects of $\boldsymbol{R}$**-Mod** are R-modules.
  2. The morphisms of $\boldsymbol{R}$**-Mod** are module homomorphisms between R-modules.
- Let $\boldsymbol{R}$**-Mod** $\times$ $\boldsymbol{R}$**-Mod** be the product category of two category of modules over R.

Result: The tensor produt

$$\begin{aligned}\otimes : \boldsymbol{R}\textbf{-Mod} \times \boldsymbol{R}\textbf{-Mod} &\to \boldsymbol{R}\textbf{-Mod},\\ (M, N) &\mapsto M \otimes N.\end{aligned}$$

is a bifunctor.

Proof. □

Theorem 6. Denotation:

- Let $\varphi : M \to M'$ be an isomorphism.
- Let $\psi : N \to N'$ be an isomorphism.

Result: The tensor product $\varphi \otimes \psi$ is an isomorphism.

The following statement is weaker than the one above, because the one above gives the explicit construction of the isomorphism!!

Theorem 7. Denotation:

- Let $M \cong M'$ be an isomorphism.
- Let $N \cong N'$ be an isomorphism.

Result: There is an isomorphism $M \otimes N \cong M' \otimes N'$

Theorem 8. *The tensor product preserves surjectivity.*
Denotation:

Example 8.1. • Define an injective R-linear map

$$\begin{aligned}\alpha : \mathbb{Z}/p\mathbb{Z} &\to \mathbb{Z}/p^2\mathbb{Z}\\ x &\mapsto px.\end{aligned}$$

1. Tensoring with $\mathbb{Z}/p\mathbb{Z}$ gives

$$\begin{aligned}1 \otimes \alpha : \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} &\to \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/p^2\mathbb{Z}\\ a \otimes x &\mapsto a \otimes px\end{aligned}$$

   which is not injective.

Example 8.2. *A tensor product of submodules need not be a submodule.*

1. We have $p\mathbb{Z} \cong \mathbb{Z}$ by

$$\begin{aligned}\varphi : p\mathbb{Z} &\to \mathbb{Z},\\ pn &\mapsto n\end{aligned}$$

2. Tensoring by $\mathbb{Z}/p\mathbb{Z}$ gives

$$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$$

Theorem 9. *When does the tensor product preserve injectivity?*
Denotation:

- Let $\varphi : M \to N$ be injective.
- Let the image of φ be the direct summand of N, i.e.

$$N = \varphi(M) \oplus P$$

  for some submodule P of N.

Result: For $k \in \mathbb{N}_0$, the following maps are injective

$$\varphi \otimes \varphi : M \otimes M \to N \otimes N$$

Chapter 3

Flat Modules

3.1 Flat Modules OLD!!!

Definition 10. An R-module N is called flat if for all injective linear maps $\varphi : M \to M'$ the linear map $1 \otimes \varphi : N \otimes M \to N \otimes M'$ is injective.

Theorem 11. Any free R-module is flat.

Proof.

1. Fix a free R-module F.
2. Fix an injective R-module homomorphism $\varphi : M \to M'$.
3. Consider
$$\begin{aligned} 1 \otimes \varphi : F \otimes M &\to F \otimes M', \\ n \otimes m &\mapsto n \otimes \varphi(m). \end{aligned}$$
We want to show that this map $1 \otimes \varphi$ is injective.
4. The case $F = 0$ is trivial, thus assume $F \neq 0$ with basis $\{e_i\}_{i \in I}$
5.

□

Theorem 12. The fraction field of an integral domain is flat.

Proof.

1. Let R be an integral domain.
2. Let K be the field of fraction of R.
3. Fix two R-modules M and M'.
4. Fix an injective R-module homomorphism $\varphi : M \to M'$.
5. Every tensor in $K \otimes_R M$ is elementary.

□

3.2 Right Exactness of Tensor Products

Theorem 13. Let

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

be an exact sequence. Then, for any R-module N, the sequence

$$M' \otimes_R N \xrightarrow{\varphi \otimes \mathrm{id}_N} M \otimes_R N \xrightarrow{\psi \otimes \mathrm{id}_N} M'' \otimes_R N \longrightarrow 0$$

is exact.

Theorem 14. For a given R-Module N, the functor

$$F_N : \textbf{R-Mod} \to \textbf{R-Mod}, \\ M \mapsto M \otimes_R N$$

is right exact.

Remark. Tensor product is not left exact.

Remark. Saying that tensor product is right exact, but not left exact is probably the same as tensor product preserves surjectivity, but not injectivity.

Example 14.1. Consider the exact sequence

$$2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

tensoring by $\mathbb{Z}/2\mathbb{Z}$ gives

$$2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

which is exact.

Note that the left function is not injective, but the right one is surjective. Thus, tensor is right exact, but not left exact.

Proof. "$\mathrm{im}\{i \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}\} = 0$"

1. We have the module isomorphism $2\mathbb{Z} \cong \mathbb{Z}$.
2. Thus, we have the module isomorphism $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.
3. By the basic properties of the tensor product, it is $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
4. Combining yields the module isomorphism $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
5. Indeed, the only two elements in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ are $0 \otimes 0$ and $2 \otimes 1$. Any other element $2n \otimes 1$ with $n \in \mathbb{N}^+$ reduces to
$$2n \otimes 1 = 2 \cdot (n \otimes 1) = (n \otimes 2) = (n \otimes 0) = 0.$$
6. Similary, the only two elements in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ are $0 \otimes 0$ and $1 \otimes 1$.

7. Now, we have

$$i \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}(2 \otimes 1) = 2 \otimes 1 = 2 \cdot (1 \otimes 1) = (1 \otimes 2) = (1 \otimes 0) = 0.$$

"$\ker\{\pi \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}\} = 0$"

1. This is because

$$\pi \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}(1 \otimes 1) = 1 \otimes 1$$

$\square$

3.3 Flat Modules NEW!!!

Definition 15. Let R be a ring.

1. An R-module N is called flat if for **every** injective R-module homomorphism $M' \longrightarrow M$ the map $M' \otimes_R N \longrightarrow M \otimes_R N$ obtained by tensoring over R with N is injective.
2. A ring homomorphism $\varphi : R \longrightarrow R'$ is called flat if R' viewed as an R-module via φ is flat.

Theorem 16. For an R-module N the following conditions are equivalent.

1. N is flat.
2. If $0 \to M' \to M \to M'' \to 0$ is short exact, then $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is short exact.
3. If $M' \to M \to M''$ is exact, then $M' \otimes N \to M \otimes N \to M'' \otimes N$ is exact.

Corollary 1. Let N be a flat R-module and $\varphi : M' \to M$ be a linear map. Then, we have the isomorphisms

1. $(\ker \varphi) \otimes_R N \cong \ker(\varphi \otimes \mathrm{id}_N)$,
2. $(\mathrm{coker} \varphi) \otimes_R N \cong \mathrm{coker}(\varphi \otimes \mathrm{id}_N)$,
3. $(\mathrm{im} \varphi) \otimes_R N \cong \mathrm{im}(\varphi \otimes \mathrm{id}_N)$.

Theorem 17. Let $(N_i)_{i \in I}$ be a family of R-modules. The direct sum $\bigoplus_{i \in I} N_i$ is flat if and only if N_i is flat for all $i \in I$.

Corollary 2. Every free module is flat. Every polynomial ring is flat.

Theorem 18. Let N be a a flat R-module. If $r \in R$ is not a zero divisor in R, then an equation $rn = 0$ for some $n \in N$ implies $n = 0$.

Proof.

1. Fix an element $r \in R$ that is not a zero divisor and an equation $rn = 0$ for some $n \in N$.
2. Define a linear map $\varphi_r : N \longrightarrow N, x \mapsto rx$.
3. The equation $rn = 0$ may be rewritten as $\varphi_r(n) = \varphi_r(0) = 0$.
4. If φ_r is injective, we may conclude $n = 0$. Therefore, we will claim φ_r is injective.
5. Consider the injective linear map $\psi_r : R \longrightarrow R, x \mapsto rx$.
6. Tensoring with N gives $\psi \otimes \mathrm{id}_N : R \otimes_R N \longrightarrow R \otimes_R N, x \otimes n \mapsto rx \otimes n$.
7. Applying the isomorphism $R \otimes_R N \cong N$ yields $N \longrightarrow N, xn \mapsto rxn$.
8. Rewriting the above map gives $N \longrightarrow N, n \mapsto rn$.
9. Since N was flat and φ_r was injective, the resulting map is also injective.

□

Theorem 19. Let R be a principal ideal domain and N be an R-module. N is flat if and only if any equation $ax = 0$ for $a \in R$ and $x \in N$ implies $a = 0$ or $x = 0$.

Theorem 20. An R-module N is flat if and only if for every inclusion $\mathfrak{a} \longrightarrow R$ the induced map $\mathfrak{a} \otimes_R N \longrightarrow R \otimes_R N$ injective.

Some rumbling of Bosch

1. every ideal in PID is in the form $\mathfrak{a} = (a)$
2. as R-modules, $\mathfrak{a}$ and R are isomorphic because both are free and generated by exactly one element
3.

Definition 21.

1. An R-module N is called faithfully flat if the following conditions are satisfied.
   (a) N is flat.
   (b) If M is an R-module such that $M \otimes_R N = 0$, then $M = 0$.
2. A ring homomorphism $\varphi : R \longrightarrow R'$ is called faithfully flat if R' viewed as an R-module via φ is faithfully flat.

Theorem 22. For an R-module N, the following conditions are equivalent.

1. N is faithfully flat.
2. (a) N is flat
   (b) For any $\varphi : M' \longrightarrow M$ such that $M' \otimes_R N \longrightarrow M \otimes_R N$ is the zero morphism, then $\varphi = 0$.
3. $M' \to M \to M''$ is exact $\iff$ $M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N$ is exact
4. (a) N is flat
   (b) for every maximal ideal $\mathfrak{m} \subset R$ we have $\mathfrak{m}N \neq N$