Chapter 1

Interpolation

Chapter 2

Finite Element

Example 0.1. Let $K = [0, 1]$, $\mathcal{P}$ be the set of linear polynomials and $\mathcal{L} = \{L_1, L_2\}$ where $L_1(p) = p(0)$ and $L_2(p) = p(1)$ for all $p \in \mathcal{L}$. Then $(K, \mathcal{P}, \mathcal{L})$ is a finite element.

First proof by verifying linearity. Just check

$$\lambda_1 L_1 + \lambda_2 L_2 = 0$$

□

Second proof by construction of a nodal basis. We construct the nodal basis $\{\phi_1, \phi_2\}$ of $\mathcal{P}$ explicitly. Since ϕ_j must fulfill $L_i(\phi_j) = \delta_{ij}$, we have

$$\begin{aligned}
L_1(\phi_1) = 1 &\iff a_1 \cdot 0 + b_1 = 1 \\
&\iff b_1 = 1 \\
L_2(\phi_1) = 0 &\iff a_1 \cdot 1 + b_1 = 0 \\
&\iff a_1 = -1 \\
L_1(\phi_2) = 0 &\iff a_2 \cdot 0 + b_2 = 0 \\
&\iff b_2 = 0 \\
L_2(\phi_2) = 1 &\iff a_2 \cdot 1 + b_2 = 1 \\
&\iff a_2 = 1
\end{aligned}$$

Set $\phi_1(x) = -x + 1$ and $\phi_2(x) = x$, then $\{\phi_1, \phi_2\}$ is a nodal basis of $\mathcal{P}$ and $(K, \mathcal{P}, \mathcal{L})$ is a finite element. □

We can generalize the previous example.

Example 0.2. Let $K = [a, b]$, $\mathcal{P}_k$ be the set of all polynomials of degree less than or equal to k, and $\mathcal{L} = \{L_0, L_1, \ldots, L_k\}$ where

$$L_i(p) = p\left(a + \frac{(b-a)i}{k}\right) \qquad \text{for all } p \in \mathcal{P}_k \text{ and } i \in \{0, 1, \ldots, k\}. \tag{2.1}$$

Then $K, \mathcal{P}_k, \mathcal{N}_k$ is a finite element.

Proof.

$$L_i(\phi_j) = 0 \iff \sum_{l=0}^{k} c_l \left(a + \frac{(b-a)i}{k}\right)^l = 0 \tag{2.2}$$

□

Example 0.3 (Counter Example to Nonconform P_2-FE)**.** Denote

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \qquad P_3 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \tag{2.3}$$

and write $p(x) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy$. Then, for L_1 we have

$$\begin{aligned} L_1(p) &= p(Q_1) && (2.4) \\ &= p(\mu P_2 + (1-\mu)P_3) && (2.5) \\ &= \mu p(P_2) + p(P_3) - \mu p(P_3) && (2.6) \\ &= \mu \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right) \\ &+ \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right) && (2.7) \\ &- \mu \left(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy\right) \end{aligned}$$

$$\mu \tag{2.8}$$

Example 0.4. Let K be any rectangle, $\mathcal{P} = \mathcal{Q}_k$ and $\mathcal{N}$ denote point evaluations at $\{(t_i, t_j) \mid 0 \leq i, j \leq k\}$ where $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element.