Integration and Integration

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Contents

Introduction

One problem of the Riemann integral is that some functions are not Riemann integratable.

Example 0.0.1 (Dirichlet function)**.** For $[a, b] \subset \mathbb{R}$, define the Dirchlet function as

$$g : [a, b] \to \mathbb{R}, x \mapsto g(x) := \begin{cases} 1 \text{ for } x \in \mathbb{Q}, \\ 0 \text{ for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{1}$$

What are the properties a generalized concept of volumina should have?

1. positive valued
2. null empty set
3. monotonous
4. translationinvariance
5. normalization

Definition 0.1. Let $\mu : \mathcal{P}(\mathbb{R}^n) \to \overline{\mathbb{R}}_0^+$.

- μ is monotonous.
- μ is translationinvariant.
- μ is σ-additive.

Theorem 0.1.1 (Vitali's Theorem)**.**

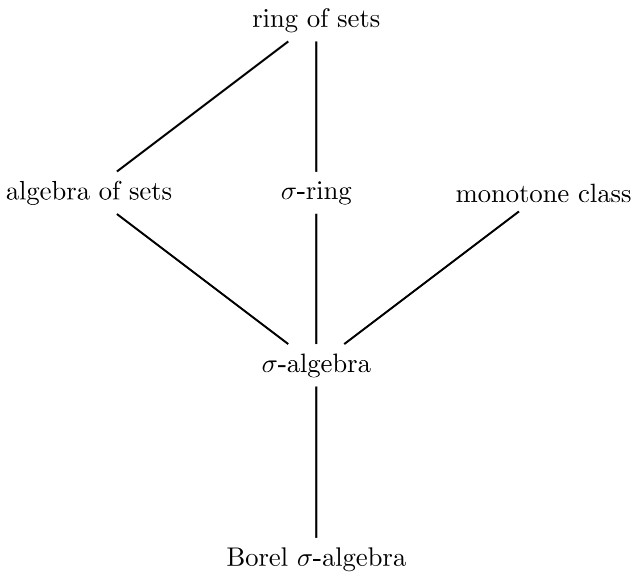
Part I

σ-algebra and measures

Chapter 1

Family of Sets

We have the following tree of inclusion. NOTATION GUIDE:



1. X as the superset
2. $\mathcal{P}(X)$ is the power set of X.
3. $A, B \in \mathcal{P}(X)$ as subsets
4. $\mathcal{R}, \mathcal{A} \subset \mathcal{P}(X)$ system of subsets

1.1 Symmetric Difference

Definition 1.1 (Symmetric difference)**.** Let A, B be sets. The binary set operation symmetric difference is defined as

$$A \triangle B := (A \setminus B) \cup (B \setminus A). \tag{1.1}$$

In other words, $x \in A \triangle B$ implies x is either in A or B, but not in both.

Proposition 1.1.1 (Properties of Symmetric Difference)**.** Let A, B, C, X and Y be sets. Moreover, let A_i and X_i be sets with an arbitary non-empty index set $i \in I$. Then, the following identities hold.

1. $A \triangle B = (A \cup B) \setminus (A \cap B)$.
2. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. (Symmetric difference is associative.)
3. $A \triangle B = B \triangle A$. (Symmetric difference is commutative.)
4. $A \triangle \emptyset = A$ and $A \triangle A = \emptyset$
5. $(A \triangle B) \cup C = (A \cup C) \triangle (B \cup C)$.
6. $A \cap B = \emptyset \Rightarrow A \triangle B = A \cup B$.
7. $B \subset A \Rightarrow A \triangle B = A \setminus B$.
8. $X \cap Y = \emptyset \Rightarrow A \cap B \subset (X \triangle A) \cup (Y \triangle B)$.
9. $(\bigcup_{i \in I} X_i) \triangle (\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} (X_i \triangle A_i)$

Proof. Elementary. □

1.2 Ring of Sets

Definition 1.2 (Ring of sets)**.** There are two equivalent definitions. Let X be a set and $\mathcal{R} \subset \mathcal{P}(X)$ a system of subsets. Then $\mathcal{R}$ is a ring of sets over X, if

1. the following axioms are met.
   (a) $\mathcal{R} \neq \emptyset$ ($\mathcal{R}$ is nonempty.)
   (b) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$ ($\mathcal{R}$ is closed under relative complement.)
   (c) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$ ($\mathcal{R}$ is closed under finite unions.)
2. $(\mathcal{R}, \triangle, \cap)$ is a ring in the algebraic sense, with $\triangle$ as addition and $\cap$ as multiplication.

Proof. We show that the two definitions above are indeed equivalent.

(1 $\Rightarrow$ 2) Let $\mathcal{R}$ be nonempty, closed under the relative complement, and closed under finite unions. First, consider $(\mathcal{R}, \triangle)$. Let $A, B \in \mathcal{R}$. It is

1. (Closure under addition) $A \cup B \in \mathcal{R}$ because $\mathcal{R}$ is closed under finite unions. We also have $A \cap B = A \setminus (A \setminus B) \in \mathcal{R}$ as $\mathcal{R}$ is closed under the relative complement. From these it follows that $A \triangle B = (A \cup B) \setminus (A \cap B) \in \mathcal{R}$ by using the closure under the relative complement again.
2. (Associativity)
3. (Commutativity)
4. (Neutral element) $\emptyset$
5. (Inverse element) A

Therefore, $(\mathcal{R}, \triangle)$ is an abelian group. Secondly, consider $(\mathcal{R}, \cap)$. $\cap$ is associative and commutative. The identity element is the union of all sets (does this exist??). □

Remark. Since we have the identity $A \cap B = A \setminus (A \setminus B)$, the condition that $\mathcal{R}$ is closed under the relative complement, i.e.

$$A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R} \tag{1.2}$$

can be replaced with closure under finite intersection, therefore

$$A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}. \tag{1.3}$$

Example 1.2.1. Let X be a set.

1. $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are ring of sets.
2. $\{\emptyset\}$ is a ring of sets.

1.3 Algebra of Sets

Definition 1.3 (Algebra of sets)**.** There are two equivalent definitions. Let X be a set and $\mathcal{R} \subset \mathcal{P}(X)$ a system of subsets. Then $\mathcal{A}$ is a algebra of sets over X,

1. if $\mathcal{A}$ is a ring of sets that contains X, or
2. if the following axioms are met
   (a) $\mathcal{A} \neq \emptyset$ ($\mathcal{A}$ is nonempty.)
   (b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ ($\mathcal{R}$ is closed under the absolute complement.)
   (c) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ ($\mathcal{R}$ is closed under finite unions.)

1.4 σ-Ring

Definition 1.4 (σ-Ring)**.** Let X be set and $\mathcal{R} \subset \mathcal{P}(X)$ a system of subsets. $\mathcal{R}$ is a σ-ring over X, if

1. $\mathcal{R} \neq \emptyset$. ($\mathcal{A}$ is nonempty.)
2. $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$ (closed under the relative complement.)
3. $A_1, A_2, A_3, ... \in \mathcal{R} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ (Closed under countable unions.)

1.5 Monotone Class

Definition 1.5 (Notation for Monotonous Sequence of Sets)**.**

Definition 1.6 (Monotone class)**.** Let $\mathcal{M} \subset \mathcal{P}(\Omega)$ a system of sets and $k \in \mathbb{N}^*$. Then, $\mathcal{M}$ is a monotone class, if

1. Let $X_k \in \mathcal{M}$ with $X_k \uparrow X$, then $X \in \mathcal{M}$.
2. Let $Y_k \in \mathcal{M}$ with $Y_k \downarrow X$, then $Y \in \mathcal{M}$.

Intersection of arbitary many monotonous class is again a monotonous class. Therefore, for all $\mathcal{E} \subset \mathcal{P}(\Omega)$ with $\mathcal{E} \neq \emptyset$ there exists the smallest monotonous class around $\mathcal{E}$

$$\mathcal{M}_{\mathcal{E}} := \bigcap_{\mathcal{M} \text{ is monotonous class}, \mathcal{E} \subset \mathcal{M}} \mathcal{M} \tag{1.4}$$

Remark. All σ-algebras are monotone class.

Theorem 1.6.1. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ an algebra of sets. Then, the following are equivalent

- $\mathcal{A}$ is a σ-algebra.
- For $A_k \uparrow A$, $A \in \mathcal{A}$.

1.6 σ-Algebra

Definition 1.7 (σ-algebra)**.** Let Ω be set and $\mathcal{A} \subset \mathcal{P}(\Omega)$ a system of subsets. $\mathcal{A}$ is a σ-algebra over Ω, if

1. $\mathcal{A} \neq \emptyset$.
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3. $A_1, A_2, A_3, ... \in \mathcal{A} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

Example 1.7.1. Trivial examples for the above structures.

Definition 1.8. Let $\mathcal{E} \subset \mathcal{P}(\Omega)$ be a system of sets. Define

$$\mathcal{F}(\mathcal{E}) := \{\mathcal{A} \subset \mathcal{P}(\Omega) | \mathcal{E} \subset \mathcal{A}, \mathcal{A} \sigma\text{-Algebra}\} \tag{1.5}$$

$$\langle \mathcal{E} \rangle^\sigma := \sigma(\mathcal{E}) := \bigcap_{\mathcal{A} \in \mathcal{F}(\mathcal{E})} \mathcal{A} \tag{1.6}$$

The first is the family of all σ-algebras that contain $\mathcal{E}$. The second is the smallest σ-algebra that contains $\mathcal{E}$.

1.7 Product Algebra??

Definition 1.9. Let Ω_1 and Ω_1 be sets; let $\mathcal{R}_1 \subset \mathcal{P}(\Omega_1)$ and $\mathcal{R}_2 \subset \mathcal{P}(\Omega_2)$ be ring of sets, and $\Omega := \Omega_1 \times \Omega_2$. Define

$$\mathcal{R} := \mathcal{R}_1 \boxtimes \mathcal{R}_2 := \left\{ \bigcup_{i=1}^m A_i \times B_i \middle| A_i \in \mathcal{R}_1, B_i \in \mathcal{R}_2, m \in \mathbb{N} \right\} \tag{1.7}$$

$\mathcal{R}$ is a ring of sets over Ω.

Theorem 1.9.1. In above definition, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are algebra of sets, then $\mathcal{R}$ is a algebra of set.

Theorem 1.9.2.

$$\mathfrak{Q}(\mathbb{R}^n) \tag{1.8}$$

is a ring of sets.

Remark. From $\mathfrak{Q}(\mathbb{R}^n)$ we can construct one very important σ-algebra, the Borel-Algebra of $\mathbb{R}^n$.

Definition 1.10 (Products of σ-algebras)**.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be σ-algebras on Ω_1, Ω_2. Then, let

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \tag{1.9}$$

Example 1.10.1.

$$\mathcal{B}(\mathbb{R}^{n+m}) = \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \tag{1.10}$$

Definition 1.11. Let $(X_k)_{k \in \mathbb{N}^*}$ be a sequence of sets with $X_1 \subset X_2 \subset X_3 \subset \dots$ and $X := \lim_{k \to \infty} := \bigcup_{k \in \mathbb{N}*} X_k$. Similar for monotonously decreasing.

1.8 Rectangles

Example 1.11.1. Let

$$\mathfrak{Q}(\mathbb{R}) := \left\{ \bigcup_{i=1}^m [a_i, b_i) \middle| m \in \mathbb{N}; a_i, b_i \in \mathbb{R} \right\} \tag{1.11}$$

be the set of all unions of finitely many right half open intervals on $\mathbb{R}$. Then, $\mathfrak{Q}(\mathbb{R})$ is a set of rings. Similary for the left half open sets, but not for open or closed intervals! $\mathfrak{Q}(\mathbb{R})$ is neither σ-ring, σ-algebra nor an algebra of sets. One can generalize this to higher dimensions.

1.9 Borel σ-algebra

Definition 1.12. Let Ω be a set. A collection $\mathcal{U} \subset \mathcal{P}(\Omega)$ of subsets of X is called a topology on X if it satisfies the following axioms.

1. $\emptyset, X \in \mathcal{U}$.

2. If $n \in \mathbb{N}$ and $U_1, \dots U_n \in \mathcal{U}$ then $\bigcap_{i=1}^n U_i \in \mathcal{U}$.
3. If I is any index set and $U_i \in \mathcal{U}$ for $i \in I$ then $\bigcup_{i\in I} U_i \in \mathcal{U}$.

A topological space is a pair $(\Omega, \mathcal{U})$ consisting of a set Ω and a topology $\mathcal{U} \in \mathcal{P}(\Omega)$.

Example 1.12.1 (Standard Topology on $\overline{\mathbb{R}}$)**.** The set of open subsets $\mathcal{T}$ of $\overline{\mathbb{R}}$ is the standard topology on $\overline{\mathbb{R}}$. Concretely, $\mathcal{T}$ contains countable union of open intervals in $\mathbb{R}$ and sets of the form $(a, \infty]$ or $[-\infty, b)$ for $a, b \in \mathbb{R}$.

Definition 1.13 (Borel algebra)**.** Let $(\Omega, \mathcal{T})$ be a topological space, then $\mathcal{B}(\Omega) := \sigma(\mathcal{T})$ is the Borel σ-algebra of Ω. The elments of $\mathcal{B}$ are called Borel (measurable) sets. There are many ways to generate this algebra.

Theorem 1.13.1. Let $(\Omega, \mathcal{T})$ be a topological space. Then the following holds.

1. Every closed subset $F \subset \Omega$ is a Borel set.
2. Every countable union $\bigcup_{i=1}^\infty F_i$ of closed subsets $F_i \subset \Omega$ is a Borel set.
3. Every countable intersection $\bigcap_{i=1}^\infty F_i$ of open subsets $F_i \subset \Omega$ is a Borel set.

Theorem 1.13.2. It is

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathfrak{Q}(\mathbb{R}^n)) \tag{1.12}$$

Moreover, define

$$\mathfrak{Q}_{\mathbb{Q}}(\mathbb{R}^n) := \left\{ \bigcup_{i=1}^m [a_{1,i}, b_{1,i}) \times \dots [a_{n,i} \times b_{n,i}) \middle| m \in \mathbb{N}; a_{\nu,i}, b_{\nu,i} \in \mathbb{Q}; \nu = 1, \dots, n \right\} \tag{1.13}$$

the ring of sets of finite unions of quadern with rational edge points. Then, we even have

$$\mathcal{R}(\mathbb{R}^n) = \sigma(\mathfrak{Q}_{\mathbb{Q}}(\mathbb{R}^n)) \tag{1.14}$$

Lemma 1.13.1. Open subsets $U \subset \mathbb{R}^n$ are disjoint union of countably many right half open dices with edge points in $\mathbb{Q}^n$

1.10 Exercises

Exercise 1.1
Let X be a nonempty set and for all $1 \leq i \leq m$ with $m \in \mathbb{N}$ let $A_i \subset X$ be a finite amount of subsets. Set

$$S := A_1 \triangle A_2 \triangle \dots \triangle A_m. \tag{1.15}$$

Because of the associative property of the symmetric difference, S is uniquely defined regardless of the order of the operations.

Show that $x \in X$ belongs to S if and only if x belongs to an odd number of sets A_k, i.e. when the number of indices $k \in \{1, 2, \dots, m\}$ with $x \in A_k$ is odd.

Solution 1.1

Exercise 1.2
Let X be a nonempty set and $R := \{f : X \to \mathbb{F}_2\}$ where $\mathbb{F}_2 = \{0, 1\}$ is a field of two elements equipped with the common addition and the common multiplication. Moreover, define the operations

$$(f \oplus g)(x) := f(x) + g(x) \tag{1.16}$$
$$(f \otimes g)(x) := f(x) \cdot g(x). \tag{1.17}$$

Show the following statements.

1. $(R, \oplus, \otimes)$ is a commutative ring with the identity element.
2. The map $\mathcal{P}(X) \to R, A \mapsto \chi_A$ that maps a subset $A \subset X$ to its characteristic function is bijective.
3. For all $A, B \in \mathcal{P}(X)$ we have
$$\chi_{A \triangle B} = \chi_A \oplus \chi_B \qquad \chi_{A \cap B} = \chi_A \otimes \chi_B. \tag{1.18}$$
4. Conclude from the statements above that $\mathcal{P}(X)$ is isomorphic to R as a ring with $\triangle$ as its addition and with $\cap$ as its multiplication.
5. A subset $\mathcal{R} \subset \mathcal{P}(X)$ is a ring of sets if and only if $\mathcal{R}$ is a subring of $\mathcal{P}(X)$ with respects to the ring structure defined above.

Solution 1.2

Exercise 1.3

Exercise 1.4

Exercise 1.5
Show explicitly that the following subsets generate the same σ-algebra on $\mathbb{R}$.
$$\mathcal{E}_1 := \{(a, b) \mid a, b \in \mathbb{R},\, a \leq b\} \qquad \mathcal{E}_2 := \{[a, b) \mid a, b \in \mathbb{Q},\, a \leq b\} \tag{1.19}$$
$$\mathcal{E}_3 := \{[a, b) \mid a, b \in \mathbb{R},\, a \leq b\} \qquad \mathcal{E}_4 := \{(-\infty, b) \mid a, b \in \mathbb{Q},\, a \leq b\} \tag{1.20}$$

Solution 1.5
We want to proof
$$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_3) = \sigma(\mathcal{E}_4). \tag{1.21}$$
We will do this by showing four inclusions. In each step, our goal is to show that an arbitary interval from the generator of the superset is included in the σ-algebra of the subset.

1. First we show $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$. Fix $a, b \in \mathbb{Q}$ with $a \leq b$ and consider the interval $[a, b)$. If $a = b$, then the interval is empty and $[a, b) = \varnothing \in \sigma(\mathcal{E}_1)$ immediately. Now let $x, y \in \mathbb{R}$ with $x < y < a$. The set $(x, a)^c \cap (y, b)$ is included in $\sigma(\mathcal{E}_1)$ as σ-algebras are closed under absolute complements and intersections. We also have
$$(x, a)^c \cap (y, b) = ((-\infty, x] \cup [a, \infty]) \cap (y, b) = [a, b). \tag{1.22}$$
Therefore, it follows that $[a, b) \in \sigma(\mathcal{E}_1)$ and hence $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$.
2. Next, we show $\sigma(\mathcal{E}_2) \subset \sigma(\mathcal{E}_3)$. As before, fix $a, b \in \mathbb{R}$ with $a \leq b$ and consider the interval $[a, b)$. If this interval is empty, it is included in $\sigma(\mathcal{E}_2)$, so assume $a < b$. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ sequences in $\mathbb{Q}$ with $a < a_k$ and $b_k < b$ and with a and b as their limits respectively. Since a σ-algebra is closed under countable unions, $\bigcup_{k=1}^{\infty} [a_k, b_k)$ is included in $\sigma(\mathcal{E}_2)$, but we also have
$$\bigcup_{k=1}^{\infty} [a_k, b_k) = \lim_{k \to \infty} [a_k, b_k) = [a, b) \tag{1.23}$$
We conclude that $[a, b) \in \sigma(\mathcal{E}_2)$ and therefore, $\sigma(\mathcal{E}_2) \subset \sigma(\mathcal{E}_3)$.
3. Now we will show $\sigma(\mathcal{E}_3) \subset \sigma(\mathcal{E}_4)$. Again, fix $b \in \mathbb{Q}$ and consider $(-\infty, b)$. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $x_k < b$ for each $k \in \mathbb{N}$ and diverging to negative infinity. As σ-algebras are closed under countable unions, we have $\bigcup_{k=1}^{\infty} (x_k, b) \in \sigma(\mathcal{E}_3)$. On the other hand, it is
$$\bigcup_{k=1}^{\infty} (x_k, b) = \lim_{k \to \infty} (x_k, b) = (-\infty, b). \tag{1.24}$$
This means that $(-\infty, b) \in \sigma(\mathcal{E}_3)$ and from this we have $\sigma(\mathcal{E}_3) \subset \sigma(\mathcal{E}_4)$.

4. Lastly, we want to show $\mathcal{E}_4 \subset \mathcal{E}_1$. Fix $a, b \in \mathbb{R}$ with $a \leq b$ and consider (a, b). Again, if $a = b$, then the interval is empty and included in $\sigma(\mathcal{E}_4)$. Otherwise,

Chapter 2

Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(\Omega)$ a ring of sets, and let $\mu : \mathcal{R} \to [0, \infty]$ be an application. μ is called a content, if

1. $\mu(\emptyset) = 0$.

2. $\mu(A \dot{\cup} B) = \mu(A) + \mu(B)$

An σ-additive content is called premeasure.
A premeasure $\mu : \mathcal{A} \to [0, \infty]$ on σ-algebra $\mathcal{A}$ is called a measure.
μ is finite if for all $A \in \mathcal{R} : \mu(A) < \infty$.
μ is σ-finite if there exists are sequence $(A_m)_{m \in \mathbb{N}^*}$ in $\mathcal{R}$ with $\mu(A_m) < \infty$ and $\bigcup_{m \in \mathbb{N}^*} A_m = \Omega$.

Lemma 2.1.1. If $\mu(A \cap B) < \infty$, then

$$\mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cup B) \tag{2.1}$$

Theorem 2.1.1 (Properties of premeasure)**.**

Example 2.1.1 (Dirac-measure)**.** Let $\Omega \neq \emptyset$. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ a σ-algebra. Define for all $x \in \Omega$ a $\delta_x : \mathcal{A} \to \mathbb{R}_0^+$ with

$$\delta_x(A) := \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ else.} \end{cases} \tag{2.2}$$

δ_x is a finite measure, called the Dirac-measure.

Definition 2.2. Let

$$\mathfrak{Q}(\mathbb{R}^n) := \left\{ \dot{\bigcup}_{i=1}^{m} [a_{1,i}, b_{1,i}) \times \ldots [a_{n,i} \times b_{n,i}) \middle| m \in \mathbb{N}; a_{\nu,i}, b_{\nu,i} \in \mathbb{R}; \nu = 1, \ldots, n \right\} \tag{2.3}$$

define

$$\lambda^n : \mathfrak{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+, A \mapsto \lambda^n(A) := \sum_{i=1}^{m} \prod_{\nu=1}^{n} (b_{\nu,i} - a_{\nu,i}) \tag{2.4}$$

is a premeasure.

Definition 2.3.

$$\mathcal{R}^{\uparrow} := \{A \in \mathcal{P}(\Omega) | \exists (A_k)_{k \in \mathbb{N}^*} \subset \mathcal{R} \text{ with } A_k \uparrow A\} \tag{2.5}$$

$\mathcal{R}^{\uparrow}$ is the set of all $A \in \mathcal{P}(\Omega)$ that can be expressed as countably many sets from $\mathcal{R}$. $\mathcal{R}^{\uparrow}$ is not a ring of sets.

Definition 2.4. Let $\mu : \mathcal{R} \to [0, \infty]$ be a premeasure on $\mathcal{R}$, and $A_k \uparrow A$. Then,

$$\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty], A \mapsto := \tilde{\mu}(A) = \lim_{k \to \infty} \mu(A_k) \tag{2.6}$$

is an extension of μ on $\mathcal{R}^{\uparrow}$. This is not in general a premeasure.

Theorem 2.4.1 (Properties of the first extension)**.**

Definition 2.5. Let $\mathcal{R} \subset \mathcal{P}(\Omega)$ a set of rings, $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$, and $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$ the first extension on $\mathcal{R}^{\uparrow}$. Moreover, let $X \subset \Omega$ a subset of Ω. Then,

$$\mu^* : \mathcal{P}(\Omega) \to [0, \infty], X \mapsto \mu^*(X) := \inf \left\{ \tilde{\mu}(A) \middle| A \in \mathcal{R}^{\uparrow}, X \subset A \right\} \tag{2.7}$$

is the outer measure.

Theorem 2.5.1 (Properties of the second extension)**.**

Bla Bla bla

Definition 2.6 (Lebesgue measure)**.**

Part II

Lebesgue Integral

Chapter 3

Measurable Functions

There is measurable, Borel measurable and Lebesgue measurable.

Definition 3.1 (Measurable Function)**.** Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces. A map $f : X \to Y$ is called measurable if the pre-image of every measurable subset of Y under f is measurable subset of X, i.e.

$$B \in \mathcal{A}_Y \Rightarrow f^{-1}(B) \in \mathcal{A}_X. \tag{3.1}$$

Definition 3.2. Let $(X\mathcal{A}_X)$ be a measurable space. A function $f : X \to \overline{\mathbb{R}}$ is called measurable if it is measurable with respect to the Borel σ-algebra on $\overline{\mathbb{R}}$

Definition 3.3 (Borel Measurable Maps)**.**

Theorem 3.3.1. Let $(\Omega, \mathcal{A})$ be a measurable space, and $\mathcal{B} = \sigma(\mathcal{E})$ for a generator $\mathcal{E} \subset \mathcal{P}(\Omega)$. If for all $E \in \mathcal{E}$ it is $f^{-1}(E) \in \mathcal{A}$, then f is measurable.

Example 3.3.1. Let $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ defined as

$$f(x) := \begin{cases} 1 x \in Q \\ -1 x \notin Q \end{cases} \tag{3.2}$$

for a $Q \notin \mathcal{B}(\mathbb{R})$. Then, $f^{-1}(1) = Q \notin \mathcal{B}$ and therefore, f is not measurable even though $|f| = 1$ is measurable.

Chapter 4

Convergence Theorems

Theorem 4.0.1 (Beppo Levi)**.** Let $(\Omega, \mathcal{A}, \mu)$ a measure space, and for $k \in \mathbb{N}^*$, let $f_k : \Omega \to \mathbb{R}$ be a sequence of integratable functions such that

$$\forall x \in \Omega, \forall n \in \mathbb{N} : f_n(x) \leq f_{n+1}(x). \tag{4.1}$$

Moreover, if there exists $M \in \mathbb{R}$ with $\forall k : \int f_k d\mu \leq M$, then

$$f := \lim_{k \to \infty} f_k : \Omega \to \overline{\mathbb{R}} \tag{4.2}$$

integratable with

$$\int f d\mu = \lim_{k \to \infty} \int f_k d\mu \tag{4.3}$$

Theorem 4.0.2. If the Riemann integral exists, it matches the Lebesgue integral.

Theorem 4.0.3. Let $(\Omega, \mathcal{A}, \mu)$ a measure space, let $g : X \to [0, \infty)$ be an integrable function, and let $f_n : X \to \mathbb{R}$ be a sequence of integrable functions satisying

$$|f_n(x)| \leq g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N} \tag{4.4}$$

and converging pointwise to $f : X \to \mathbb{R}$. Then f is integrable and, for every $E \in \mathcal{A}$

$$\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu \tag{4.5}$$

Part III

Applications

Chapter 5

Cavalieri's Principle

Definition 5.1 (Cross-section)**.** Let $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ with $n, k, l \in \mathbb{N}^*$, and $A \subset \mathbb{R}^n$. Then for a $y \in \mathbb{R}^l$

$$A_y := \left\{ x \in \mathbb{R}^k \middle| (x, y) \in A \right\} \tag{5.1}$$

is the l-dimensional cross-sections of A.

Remark. Immediately from the definition above, we have

$$A = \dot{\bigcup}_{y \in \mathbb{R}^l} (A_y, y). \tag{5.2}$$

In other words, $\{(A_y, y)\}_{y \in \mathbb{R}^l}$ is a patition of A.

Theorem 5.1.1 (Cavalieri's principle)**.** Let $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ with $n, k, l \in \mathbb{N}^*$, let $A \subset \mathbb{R}^k \times \mathbb{R}^l$ a Borel subset of $\mathbb{R}^n$, and let $\{(A_y, y)\}_{y \in \mathbb{R}^l}$ be a patition of A via cross-sections. Then we have the following

1. For all $y \in \mathbb{R}^l$, A_y is Borel subset of $\mathbb{R}^k$.
2. Let $F_A : \mathbb{R}^l \to [0, \infty], y \mapsto F_A(y) := Vol_K(A_y) = \lambda^k(A_y)$ be the k-dimensional volume of A_y. Then F_A is Borel measurable on $\mathbb{R}^l$.
3. $\mathrm{Vol}_n(A) := \int_{\mathbb{R}^l} \mathrm{Vol}_k(A_y)$

Proof. 1. Fix $y \in \mathbb{R}^l$

Theorem 5.1.2. For $K \subset \mathbb{R}^{\ltimes}$ compact, we have

$$\mathrm{Vol}_n(K) = \int_{\mathbb{R}} \mathrm{Vol}_{n-1}(K_t) \tag{5.3}$$

Chapter 6

Finding Volume by Rotation

Definition 6.1. $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is rotationally symmetric in $\mathbb{R}^n$ if there exists a $f : [0, \infty) \to \overline{\mathbb{R}}$ such that for all $x \in \mathbb{R}^n$ it is $F(x) = f(\|x\|)$.

Theorem 6.1.1. The volume of the unit sphare is

$$\tau_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \tag{6.1}$$

Theorem 6.1.2. Let $B \subset [0, \infty)$ a Borel subset and $A := \{x \in \mathbb{R}^n | \|x\| \in B\}$. Then the Lebesgue measure of A is

$$\lambda^n(A) = n\tau_n \int_B r^{n-1} dr \tag{6.2}$$

where τ_n is the volume of the unit sphere.

Theorem 6.1.3. Let $f : [0, \infty) \to \overline{\mathbb{R}}$ is Borel measurable. Then the following are equivalent.

1. $F : \mathbb{R}^n \to \overline{\mathbb{R}}, x \mapsto F(x) := f(\|x\|)$ is Lebesgue integrable over $\mathbb{R}^n$.
2. $r^{n-1} f : [0, \infty) \to \overline{\mathbb{R}}, r \mapsto r^{n-1} f(r)$ is Lebesgue integrable over $[0, \infty)$.

Moreover, if one of the above is true, then we have the formula

$$\int_{\mathbb{R}^n} f(\|x\|) d^n x = n\tau_n \int_{[0,\infty)} r^{n-1} f(r) dr \tag{6.3}$$

Example 6.1.1. For a $R \in \mathbb{R}^+$ and $1 \leq i \leq n$ let

$$I_i := \int_{\|x\| \leq R} x_i^2 d^n x. \tag{6.4}$$

We Immediately have $I_i = I_j =: I$ for all i, j.

$$I = \frac{1}{n} \sum_{i=1}^{n} I_i \tag{6.5}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{\|x\| \leq R} x_i^2 d^n x \tag{6.6}$$

$$= \frac{1}{n} \int_{\|x\| \leq R} \sum_{i=1}^{n} x_i^2 d^n x \tag{6.7}$$

$$= \frac{1}{n} \int_{\|x\| \leq R} \|x\|^2 d^n x \tag{6.8}$$

$$\tag{6.9}$$

Now with the formula above, we have

$$I = \frac{1}{n} \cdot n \cdot \tau_n \int_0^R r^{n-1} r^2 dr \tag{6.10}$$
$$= \tau_n \int_0^R r^{n+1} dr \tag{6.11}$$
$$= \tau_n \frac{R^{n+2}}{n+2} \tag{6.12}$$

Example 6.1.2.

$$\int_0^\infty \exp(-x^2) = \frac{\sqrt{\pi}}{2} \tag{6.13}$$

Proof. Define

$$I = \int_{-\infty}^{\infty} \exp(-x^2) dx \tag{6.14}$$

Consider

$$I^2 = \left(\int_{-\infty}^{\infty} \exp(-x^2) dx \right) \left(\int_{-\infty}^{\infty} \exp(-y^2) dy \right) \tag{6.15}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-x^2) \exp(-y^2) dx dy \tag{6.16}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-(x^2 + y^2)) dx dy \tag{6.17}$$
$$= \int_{\mathbb{R}^2} e^{-\|x\|^2} d^2\lambda \tag{6.18}$$
$$= \int_0^\infty r e^{-r^2} dr \tag{6.19}$$

Example 6.1.3. Let $B_1 := \left\{ x \in \mathbb{R}^2 \middle| \|x\| < 1 \right\}$ be the open unit disk. Find the integral

$$\int_{B_1} \frac{1}{\sqrt{1 - \|x\|^2}} d\lambda^2(x) \tag{6.20}$$

Proof. Define $f : [0, \infty) \to \overline{\mathbb{R}}$ as

$$f(x) = \frac{1}{\sqrt{1 - x^2}} \chi_{[0,1)}(x). \tag{6.21}$$

As $[0, 1)$ is a Borel set of $\mathbb{R}$, $\chi_{[0,1)}$ is Borel measurable. On the other hand, $\frac{1}{\sqrt{1-x^2}}$ is continuous for all $x \in [0, 1)$, so the composition of these two functions f is again Borel measurable.
Now consider, $rf(r)$. We have

$$\int |rf(r)| dr = \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr \tag{6.22}$$
$$= -\sqrt{1 - r^2} \tag{6.23}$$
$$= 0 + 1 \tag{6.24}$$
$$= 1 \tag{6.25}$$

Example 6.1.4. Compute the following integral

$$f(\xi, \eta) := \int_{B_1} \frac{\exp(i(x\xi + y\eta))}{\sqrt{1 - x^2 - y^2}} dx dy \tag{6.26}$$

Chapter 7

Transformation Formula

Theorem 7.0.1. Suppose $\phi : U \to V$ is a C^1-diffeomorphism between open subsets of $\mathbb{R}^n$. If $f : V \to \mathbb{R}$ is Lebesgue integrable OR continuous with a compact support, then

$$\int_U (f \circ \phi)|\det(d\phi)|dm = \int_V f dm. \tag{7.1}$$

Example 7.0.1. (2D) From polar coordinates to cartesian coordinates.

$$\phi : \mathbb{R}_0^+ \times [0, 2\pi) \to \mathbb{R}^2, (r, \varphi) \mapsto \phi(r, \varphi) := (r\cos\varphi, r\sin\varphi) \tag{7.2}$$

$$D\phi(r, \varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \tag{7.3}$$

$$\det D\phi(r, \varphi) = r \tag{7.4}$$

(3D) From spherical coordinates to cartesian coordinates.

$$\phi : \mathbb{R}_0^+ \times [0, \pi] \times [0, \pi) \to \mathbb{R}^3 \tag{7.5}$$

$$(r, \theta, \varphi) \mapsto \phi(r, \theta, \varphi) := (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \tag{7.6}$$

$$D\phi(r, \theta, \varphi) := \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \tag{7.7}$$

$$\det D\phi(r, \theta, \varphi) = r^2 \sin\theta \tag{7.8}$$

(3D) From cylindrical coordinates to cartesian coordinates.

$$\phi : \mathbb{R} \times \mathbb{R} \times [0, 2\pi) \to \mathbb{R}^3 \tag{7.9}$$

$$x = r\cos\theta \tag{7.10}$$

$$y = r\sin\theta \tag{7.11}$$

$$z = z \tag{7.12}$$

$$D\phi(r, \theta, z) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.13}$$

$$\det D\phi(r, \theta, z) = r \tag{7.14}$$

Part IV

More Theory

Chapter 8

Lebesgue Space

Definition 8.1 (L^p-Norm)**.** Let $X, \mathcal{A}, \mu$ a measure space, and $f : X \to \overline{\mathbb{R}}$ measurable. Then for $p \in [1, \infty)$ the L^p-norm is defined as

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}. \tag{8.1}$$

Theorem 8.1.1 (Holder Inequality)**.** Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Let $f, g : X \to \overline{\mathbb{R}}$ measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \tag{8.2}$$

Theorem 8.1.2 (Minkowski Inequality)**.** Let $f, g : X \to \overline{\mathbb{R}}$ measurable and $f + g$ well defined on X. Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \tag{8.3}$$

Definition 8.2. Let $X, \mathcal{A}, \mu$ be a measure space and $p \in [1, \infty)$. Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = \left\{ f : X \to \mathbb{R} \middle| f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\} \tag{8.4}$$

Part V

Manifolds

Definition 8.3. $M \subset \mathbb{R}^n$ is a k-dimensional submanifold, if

- For all $a \in M$ there exists an open neighbourhood U around a in $\mathbb{R}^n$ and there exists a $n-k$ continuously partial differentiable $f_j : U \to \mathbb{R}$ for $j = 1, \ldots, n-k$ such that

$$M \cap U = \{x \in U | f_1(x) = \cdots = f_{n-k}(x) = 0\} \tag{8.5}$$

and for all $x \in U$

$$\operatorname{rank} \frac{\partial(f_1, \ldots, f_{n-k})}{\partial(x_1, \ldots, x_n)}(x) = n - k \tag{8.6}$$

Example 8.3.1. Let's construct the simplest submanifold. Let $n = 2$ and $k = 1$.

$$M = \left\{x \in \mathbb{R}^2 \middle| f(x, y) = c\right\} \tag{8.7}$$

Theorem 8.3.1. If $M \subset \mathbb{R}^n$ is a k-dimensional submanifold then the following are equivalent.

1. For all points $a \in M$ there exists a open neighbourhood $U \in \mathcal{U}_a(\mathbb{R})$, and there exists a function $f_i : U \to \mathbb{R}$ with $1 \leq i \leq n-k$that is $n-k$ continuously (partially) differentiable such that

$$M \cap U = \{x \in U | f_1(x) = \cdots = f_{n-k}(x) = 0\} \tag{8.8}$$

and for all $x \in U$ $Df(x) = n - k$.

Example 8.3.2. The figure eight is described by $f : \mathbb{R} \to \mathbb{R}^2$, $f(t) := (\cos t, \sin 2t)$. Define

$$M := \{x \in \mathbb{R} | \cos x = 0, \sin 2x = 0\} \tag{8.9}$$

then

$$D\phi(x) = \begin{pmatrix} -\sin t \\ 2\cos 2t \end{pmatrix} \tag{8.10}$$

Definition 8.4. A submanifold is k-dimensional of the class $\mathcal{C}^\alpha$ if the $n-k$ functions that describe the submanifold is α times continuously differentiable.

Theorem 8.4.1. Let $M \subset \mathbb{R}^n$ a k-dimensional submanifold of the class $\mathcal{C}^\alpha$. Let $i = 1, 2$ $T_i \subset \mathbb{R}^k$ open and $\varphi_i : T_i \to V_i \subset M$ KARTEN, i.e. in parameter form of the class $\mathcal{C}^\alpha$ with $V := V_1 \cap V_2 \neq \emptyset$.

Exercise 8.1
Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$f(x, y, z) := x^2 + xy - y - z \qquad g(x, y, z) := 2x^2 + 3xy - 2y - 3z \tag{8.11}$$

Show that

$$C := \left\{(x, y, z) \in \mathbb{R}^3 \middle| f(x, y, z) = g(x, y, z) = 0\right\} \tag{8.12}$$

is a submanifold of $\mathbb{R}^3$ and that

$$\phi : \mathbb{R} \to \mathbb{R}^3, \phi(t) := (t, t^2, t^3) \tag{8.13}$$

is a global parametization of C.

Solution 8.1
Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ as $F(x, y, z) = (f(x, y, z), g(x, y, z))$, then C can be rewritten as

$$C = \left\{(x, y, z) \in \mathbb{R}^3 \middle| F(x, y, z) = 0\right\}. \tag{8.14}$$

We have

$$\partial_x f(x, y, z) = 2x + y \qquad \partial_x g(x, y, z) = 4x + 3y \tag{8.15}$$
$$\partial_y f(x, y, z) = x - 1 \qquad \partial_y g(x, y, z) = 3x - 2 \tag{8.16}$$
$$\partial_z f(x, y, z) = -1 \qquad \partial_z g(x, y, z) = -3 \tag{8.17}$$

therefore

$$\mathrm{D}F(x, y, z) = \begin{pmatrix} 2x + y & x - 1 & -1 \\ 4x + 3y & 3x - 2 & -3 \end{pmatrix} \tag{8.18}$$

To check if $\mathrm{D}F$ surjective, it is enough to show that

$$\begin{pmatrix} x - 1 \\ 3x - 2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \tag{8.19}$$

are linearely independent. For that, we compute the determinant of the matrix created by the two vectors.

$$\det \begin{pmatrix} x - 1 & -1 \\ 3x - 2 & -3 \end{pmatrix} = -3x + 3 + 3x - 2 = 1 \tag{8.20}$$

So, $\mathrm{D}F$ has a rank of 2, therefore surjective. With this, C is a submanifold of $\mathbb{R}^3$.