Integration and Integration

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Contents

Introduction

Part I

σ-algebra and measures

Chapter 1

Family of Sets

Chapter 2

Measure

2.1 Content, Premeasure, and Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ-additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{2.1}$$

- subadditive if for all $A, B \in \mathcal{R}$ it is $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- σ-subadditive if for all $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k). \tag{2.2}$$

- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ-finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in $\mathcal{R}$ with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \tag{2.3}$$

- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ-additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in $\mathcal{R}$ is required. For σ-rings and therefore σ-algebras, it is guranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \to [0, \infty]$ are σ-finite as X need not be included in a ring of sets.

Definition 2.2 (Content)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called a content if

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ-additive content $\mu \to [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. A σ-additive content $\mu : \mathcal{A} \to [0, \infty]$ is called a measure.

2.2 Lebesgue Content

Definition 2.5 (Lebesgue Content)**.** Let $\mathcal{Q}(\mathbb{R}^n)$ be the ring of sets over $\mathbb{R}^n$.

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigsqcup_{k=1}^{m} [a_{1,k},\, b_{1,k}) \times \cdots \times [a_{n,k},\, b_{n,k}) \;\middle|\; m \in \mathbb{N};\, a_{i,k}, b_{i,k} \in \mathbb{R};\, 1 \le k \le n \right\} \tag{2.4}$$

Set $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ as

$$\lambda^n(A) := \sum_{k=1}^{m} \prod_{i=1}^{n} (b_{i,k} - a_{i,k}) \tag{2.5}$$

λ^n is the Lebesgue content.

Theorem 2.5.1. λ^n is a well-defined finite content.

Theorem 2.5.2. λ^n is a premeasure.

2.3 Lebesgue Measure

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Definition 2.6. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings. Set

$$\mathcal{R}^{\uparrow} := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k\in\mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \tag{2.6}$$

Remark. $\mathcal{R}^{\uparrow}$ is the set of all $A \in \mathcal{P}(X)$ that can be expressed as a countable many unions of sets in $\mathcal{R}$.

In general, $\mathcal{R}^{\uparrow}$ is not a set of rings.

Definition 2.7. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets and $\mu : \mathcal{R} \to [0, \infty]$ a premeasure. For $A_k \uparrow A$ with $A_k \in \mathcal{R}$ for $k \in \mathbb{N}$ define

$$\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty],\, A \mapsto \tilde{\mu}(A) := \lim_{k\to\infty} \mu(A_k). \tag{2.7}$$

$\tilde{\mu}$ is called the first extension of the premeasure μ.

Remark. In general, $\tilde{\mu}$ is not a premeasure as $\mathcal{R}^{\uparrow}$ need not be a ring of sets.

$\tilde{\mu}$ restricted on $\mathcal{R}$ is identical with μ, i.e. $\tilde{\mu}\big|_{\mathcal{R}} \equiv \mu$.

Lemma 2.7.1. The first extension $\tilde{\mu}$ is well-defined.

Proposition 2.7.1 (Properties of $\mathcal{R}^{\uparrow}$)**.**

Proposition 2.7.2 (Properties of the First Extension)**.**

Definition 2.8 (Second Extension or the Outer Measure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets, $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$, and $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$ the first extension of μ on $\mathcal{R}^{\uparrow}$. Moreover, let $B \subset X$ be a subset of X. Then, the map

$$\mu^* : \mathcal{P}(X) \to [0, \infty],\, B \mapsto \mu^* := \inf \left\{\tilde{\mu}(A) \mid A \in \mathcal{R}^{\uparrow},\, A \supset B\right\} \tag{2.8}$$

is called the outer measure induced by $\tilde{\mu}$ on $\mathcal{P}(X)$.

Proposition 2.8.1 (Properties of the Second Extension)**.**

Proposition 2.8.2 (Properties of the Outer Measure)**.**

Definition 2.9 (Lebesgue Outer Measure)**.** Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \to [0, \infty],\ B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^{\uparrow},\ A \supset B \right\} \tag{2.9}$$

is called the Lebesgue outer measure induced by $\tilde{\lambda^n}$.

Definition 2.10 (Pseudo Metric)**.**

Proposition 2.10.1. The outer measure induces a pseudo metric.

Proposition 2.10.2. The outer measure is continuous.

Definition 2.11 (Approximation through elements of Rings)**.**

Theorem 2.11.1.

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \tag{2.10}$$

is a σ-algebra on X.