0.1 Ideal norms and the Dedekind-Kummer thoerem

Theorem 0.1 (Dedekind-Kummer Theorem) Assume $AKLB$ with $L = K(\alpha)$ and $\alpha \in B$, let $f \in A[X]$ be the minimal polynomial of α and assume $B = A[\alpha]$. Suppose $g_1, \ldots, g_r \in A[X]$ are monic polynomials for which

$$\overline{f} = \overline{g_1}^{e_1} \cdots \overline{g_r}^{e_r} \tag{1}$$

is a complete factorization of $\overline{f} \in (A/\mathfrak{p})[X]$, where $\bar{\cdot}$ denotes reduction modulo $\mathfrak{p}$, and let $\mathfrak{q}_i := (\mathfrak{p}, g_i(\alpha))$ be the B-ideal genereated by $\mathfrak{p}$ and $g_i(\alpha)$. Then

$$pB = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_r^{e^r}, \tag{2}$$

is the prime factorization of $\mathfrak{p}B$ in B and the residue degree of $\mathfrak{q}_i$ is $f_i := \deg g_i$.

Definition 1 (Conductor)
Let S/R be an extension of rings. The conductor $\mathfrak{c}$ of R in S is the largest S-ideal that is also an R-ideal, i.e.,

$$\mathfrak{c} := \mathfrak{c}_{S/R} := \{\, \alpha \in S \mid \alpha S \subseteq R \,\} = \{\, \alpha \in R \mid aS \subseteq R \,\}. \tag{3}$$

Proposition 1.1 The assumption $B = A[\alpha]$ in the Dedekind-Kummer theorem can be replaced with the assumption that $\mathfrak{p}A[\alpha]$ is prime to the conductor of $A[\alpha]$ in B, i.e,

$$asdf \tag{4}$$

0.2 Orders in Dedekind domains, primes in Galois extensions

Lemma 1.1 Let R be a noetherian domain. The conductor of R in its integral closure S is nonzero if and only if S is finitely generated as an R-module.

Definition 2 (Order)
An order $\mathcal{O}$ is a noetherian domain of dimension one whose conductor is nonzero, equivalently, whose integral closure is finitely generated as an $\mathcal{O}$-module.

Lemma 2.1 Let $\mathcal{O}$ be an order with integral closure B and conductor $\mathfrak{c}$. A prime $\mathfrak{q}$ of B contains $\mathfrak{c}$ if and only if the prime $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}$ of $\mathfrak{O}$ contains $\mathfrak{c}$. In particular, only finitely many primes $\mathfrak{p}$ of $\mathcal{O}$ contain $\mathfrak{c}$.