Part I

Topology

Definition 0.1 — Topology and Topological Space.
Let X be a nonempty set. A set $\mathcal{T}$ of subsets of X is said to be a topology on X if

1. X and the empty set $\emptyset$ belong to $\mathcal{T}$
2. the union of arbitary many number of sets in $\mathcal{T}$ belong to $\mathcal{T}$
3. the intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$

The pair $(X, \mathcal{T})$ is called a topological space.

Definition 0.2 — Discrete Topology.
Let X be any nonemoty set and $\mathcal{T}$ be the collection of all subsets of X. Then $\mathcal{T}$ is called the discrete topology on the set X. The topological space $(X, \mathcal{T})$ is called a discrete space.

Definition 0.3 — Indiscrete Topology.
Let X be any nonempty set and $\mathcal{T} = \{\mathcal{T}, \emptyset\}$. Then $\mathcal{T}$ is called the indiscrete topology and $(X, \mathcal{T})$ is said to be an indiscrete space.

Proposition 1. *If $(X, \mathcal{T})$ is a topological space such that for every $x \in X$ the singleton set $\{x\}$ is in $\mathcal{T}$ then $\mathcal{T}$ is the discrete topology.*

Definition 0.4 — .
Let $(X, \mathcal{T})$ be any topological space. Then the members of $\mathcal{T}$ are said to be open sets.

Proposition 2. *If $(X, \mathcal{T})$ is any topological space, then*

1. *X and $\emptyset$ are open sets.*
2. *The union of arbitary many number of open sets is an open set.*
3. *The intersection of finitely many number of open sets is an open set.*

Definition 0.5 — .
Let $(X, \mathcal{T})$ be a topological space. A subset S of X is said to be a closed set in $(X, \mathcal{T})$ if its complment in X, namely $X - S$ is open in $(X, \mathcal{T})$.

Proposition 3. *If $(X, \mathcal{T})$ is any topological space, then*

1. *$\emptyset$ and X are closed set.*
2. *The intersection of arbitary many number of closed sets is a closed set.*
3. *The union of finitely many number of closed sets is a closed set.*

Definition 0.6 — .
A subset S of a topological space $(X, \mathcal{T})$ is said to be clopen if it is both open and closed in $(X, \mathcal{T})$.

Definition 0.7 — .
Let X be any nonempty set. A topology $\mathcal{T}$ on X is called the finite-closed topology or the cofinite topology if the closed subsets of X are X and all finite subsets of X; that is, the open sets are $\emptyset$ and all subsets of X which have finite complments.

Definition 0.8 — Euclidean Topology.
A subset S of $\mathbb{R}$ is said to be open in the euclidean topology on $\mathbb{R}$ if for each $x \in S$, there exist $a, b \in \mathbb{R}$, with $a < b$, such that $x \in (a, b) \subseteq S$.

Proposition 4. *A subset S of $\mathbb{R}$ is open if and only if it is a union of open intervals.*

Definition 0.9 — Basis for a Topology.
Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of open subsets of X is said to be a basis for the topology $\mathcal{T}$ if every open set is a union of members in $\mathcal{B}$.

Example 0.10. Let $\mathcal{B} = \{\, (a, b) \mid a, b \in \mathbb{R}, a < b \,\}$. Then $\mathcal{B}$ is a basis for the euclidean topology on $\mathbb{R}$.

Example 0.11. Let $(X, \mathcal{T})$ be a discrete space and $\mathcal{B}$ the family of all singleton subsets of X; that is, $\mathcal{B} = \{\, \{x\} \mid x \in X \,\}$.

Example 0.12. Let $X = \{ a, b, c, d, e, f \}$ and

$$\mathcal{T}_1 = \{ X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\} \} \tag{1}$$

Then $\mathcal{B} = \{ a, c, d, b, c, d, e, f \}$ is a basis for $\mathcal{T}_1$ as $\mathcal{B} \subseteq \mathcal{T}_1$ and every member of $\mathcal{T}_1$ can be expressed as a union of members of $\mathcal{B}$. Note that $\mathcal{T}_1$ itself is also a basis for $\mathcal{T}_1$.

Proposition 5. *Let X be a nonempty set and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X if and only if $\mathcal{B}$ has the following properties:*

1. $X = \bigcup_{B \in \mathcal{B}} B$
2. *for any $B_1, B_2 \in \mathcal{B}$ the set $B_1 \cap B_2$ is a union of members of $\mathcal{B}$*

Proposition 6. *Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B}$ of open subsets of X is a basis for $\mathcal{T}$ if and only if for any point x belonging to any open set U there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$*

Proposition 7. *Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on a set X. Then a subset U of X is open if and only if for each $x \in U$ there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proposition 8. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, on a nonempty set X. Then $\mathcal{T}_1 = \mathcal{T}_2$ if and only if*

1. *for each $B \in \mathcal{B}_1$ and each $x \in \mathcal{B}$, there exists a $B' \in \mathcal{B}_2$ such that $x \in B' \subseteq B$*
2. *for each $B \in \mathcal{B}_2$ and each $x \in \mathcal{B}$, there exists a $B' \in \mathcal{B}_1$ such that $x \in B' \subseteq B$*

Part II

Other Stuff

Definition 0.13 — Euler Totient Function.
The Euler totient function counts the positive integers up to a given integer n that are relatively prime to n.

Part III

Commutative Rings

Chapter 1

Rings and Ideals

1.1 Cheat Sheet

Definition 1.1 — Ring.
A ring is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) satisfying the following three sets of axioms, called the ring axioms.

1. $(R, +)$ is an abelian group.
2. $(R, \cdot)$ is a semigroup.
3. Multiplication is distributive with respect to addition, meaning that
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$ (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$ (right distributivity).

A ring is called unitary if it contains the multiplicative identity and commutative if multiplication is commutative.

Definition 1.2 — Unit.

Definition 1.3 — Zerodivisors.

Definition 1.4 — Nilpotent.

Definition 1.5 — Idempotent.

Definition 1.6 — Ideal.

Definition 1.7 — Operations on Ideals.
Let R be a ring $\{\mathfrak{a}_i\}_{i \in I}$ be a collection of ideals in R.

1. $$\sum_{i \in I} \mathfrak{a}_i = \left\{ \sum_{i \in I} a_i \mid a_i \in \mathfrak{a}_i \text{ for all } i \in I, \text{ and } a_i = 0 \text{ for almost all i} \right\} \tag{1.1}$$
2. The transporter of two ideals is defined By
$$(\mathfrak{a} : \mathfrak{b}) := \{\, x \in R \mid x\mathfrak{b} \subset \mathfrak{a} \,\} \tag{1.2}$$

Definition 1.8 — Prime Ideal.

Definition 1.9 — Maximal Ideal.

Definition 1.10 — Quotient Ring.
Given a ring A and two-sided ideal $\mathfrak{a}$ in A, we may define an congruence relation $\sim$ on A as follows:

$$x \sim y :\Longleftrightarrow x - y \in \mathfrak{a}. \tag{1.3}$$

The equivalence class of the element x in A is given by

$$[x] = x + \mathfrak{a} := \{ x + a \mid a \in \mathfrak{a} \} \tag{1.4}$$

and the set of all such equivalence classes is denoted by $A/\mathfrak{a}$; it becomes a ring, the factor ring or the quotient ring of A modulo $\mathfrak{a}$, if one defines

1. $(a + \mathfrak{a}) + (b + \mathfrak{a}) = (a + b) + \mathfrak{a}$
2. $(a + \mathfrak{a})(b + \mathfrak{a}) = (ab) + \mathfrak{a}$

The map $\pi : R \longrightarrow A/\mathfrak{a}$, $x \mapsto \pi(x) := x + \mathfrak{a}$ is a surjective ring homomorphism and is sometimes called the natural quotient map or the canonical homomorphism.

Proposition 9 (Universal Property)**.** *Let A and B be rings, $\mathfrak{a}$ an ideal, and $f : A \longrightarrow B$ a ring homomorphism with $\mathfrak{a} \subseteq \mathrm{Ker}\,(f)$. Then there exists a unique ring homomorphism $\tilde{f} : A/\mathfrak{a} \longrightarrow B$ such that $f = \tilde{f} \circ \pi$.*

Definition 1.11 — Integral Domain.

Theorem 1.12.
- *prime ideal, quotient is integral domain*
- *same as above, but if prime maximal, then quotient is a fields*
- *Maximal ideals are prime ideals.*
- *There is a 1:1 correspondence*

$$\{ \textit{Ideals in } A/\mathfrak{a} \} \longleftrightarrow \{ \mathfrak{b}/\mathfrak{a} \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq A \} \tag{1.5}$$

Definition 1.13 — Unique Factorization Domain.

Definition 1.14 — Principal Ideal Domain.

Proposition 10. *Commutative Rings $\supset$ Unique Factorization Domain $\supset$ Principal Ideal Domain $\supset$ Fields*

Theorem 1.15.
- *prime ideal, quotient is integral domain*
- *same as above, but if prime maximal, then quotient is a fields*
- *Maximal ideals are prime ideals.*
- *There is a 1:1 correspondence*

$$\{ \textit{Ideals in } A/\mathfrak{a} \} \longleftrightarrow \{ \mathfrak{b}/\mathfrak{a} \mid \mathfrak{a} \subseteq \mathfrak{b} \subseteq A \} \tag{1.6}$$

Proof. For the last point, it is easier to understand with extension and contraction. □

1.2 Examples

Example 1.16.
1. $\mathbb{Z}$
2. All fields.
3. Let S be any set, then $(\mathcal{P}(S), \triangle, \cap)$ is a ring.
4. continuous $f : I \longrightarrow \mathbb{R}$ with a real interval I forms a ring.
5. cartesian product of rings

Example 1.17. Let S be any set, then $(2^S, \triangle, \cap)$ is a ring.

1. $0 = \emptyset$ and $-A = A$

2. The neutral element of the multiplication is S.
3. $(2^S)^\times = \{S\}$
4. $\mathrm{ZD}(2^S) = 2^S - S$ since $A \cap A^c = \emptyset$ (also minus the empty set) (this seems to be true for all boolean rings)
5. $\mathrm{Nil}(2^S) = \emptyset$ (seems to be true for all boolean rings)
6. $\langle A \rangle = 2^A$ contains all subset of A

Example 1.18 (Integral Domains)**.** 1. $\mathbb{Z}[i]$ the Gaussian integers, $(\mathbb{Z}[i])^\times = \{\pm 1, \pm i\}$

2. $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^\times = \{\, \overline{a} \mid \gcd(a, n) = 1 \,\}$, $|(\mathbb{Z}/n\mathbb{Z})^\times| = \phi(n)$ (Euler totient function)

1.3 Proofs

1.4 Exercises

Chapter 2

Radicals

2.1 Cheat Sheet

2.2 Proofs

2.3 Exercises

Exercise 2.1. Let R be a ring, $\mathfrak{a} \subset \operatorname{Jac}(R)$ an ideal, $u \in R$, and $u + \mathfrak{a}$ its residue in R. Prove that $u \in R^\times$ if and only if $u + \mathfrak{a} \in (R/\mathfrak{a})^\times$. What if $\mathfrak{a} \not\subset \operatorname{Jac}(R)$?

Chapter 3

Zariski Topology

Definition 3.1 — Spectrum.
Let R be a ring. We denote the set of all prime ideals of R by $\mathrm{Spec}(R)$ and the set of all maximal ideals of R by $\mathrm{Spm}(R)$.

Definition 3.2 — Variety.
Let R be a ring and $\mathfrak{a}$ an ideal in R. Let $\mathbf{V}(\mathfrak{a})$ denote the subset of $\mathrm{Spec}(R)$ consisting of those primes that contain $\mathfrak{a}$, i.e.

$$\mathbf{V}(\mathfrak{a}) := \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p} \,\}\,. \tag{3.1}$$

We call $\mathbf{V}(\mathfrak{a})$ the variety of $\mathfrak{a}$.

Proposition 11. *Let R be a ring, and $\mathfrak{a}$ and $\mathfrak{b}$ two ideals in R.*

1. *If $\mathfrak{a} \subset \mathfrak{b}$, then $\mathbf{V}(b) \subset \mathbf{V}(a)$.*
2. *If $\mathbf{V}(b) \subset \mathbf{V}(a)$, then $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$.*
3. *$\mathbf{V}(\mathfrak{a}) = \mathbf{V}(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.*
4. *$\mathbf{V}(\mathfrak{a}) \cup \mathbf{V}(\mathfrak{b}) = \mathbf{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathbf{V}(\mathfrak{a}\mathfrak{b})$.*
5. *For any index set I, it is $\bigcap_{i\in I} \mathbf{V}(\mathfrak{a}_i) = \mathbf{V}(\sum_{i\in I} \mathfrak{a}_i)$.*
6. *$\mathbf{V}(\langle 0\rangle) = \mathrm{Spec}(R)$.*

Definition 3.3 — Zariski Topology.
Declaring $\mathbf{V}(\mathfrak{a})$ to be closed sets induces a topology on $\mathrm{Spec}(R)$, the Zariski topology.

Given an element $f \in R$, we call the open set

$$\mathbf{D}(f) := \mathrm{Spec}(R) - \mathbf{V}(\langle f\rangle) \tag{3.2}$$

a principal open set. These sets form a basis for the topology of $\mathrm{Spec}(R)$; indeed, given any prime $\mathfrak{a} \not\subset \mathfrak{p}$, there is an $f \in \mathfrak{a} - \mathfrak{p}$, and so $\mathfrak{p} \in \mathbf{D}(f) \subset \mathrm{Spec}(R) - \mathbf{V}(\mathfrak{a})$. Further, $f, g \notin \mathfrak{p}$ if and only if $fg \notin \mathfrak{p}$ for any $f, g \in R$ and prime $\mathfrak{p}$, in other words

$$\mathbf{D}(f) \cap \mathbf{D}(g) = \mathbf{D}(fg) \tag{3.3}$$

3.1 Proofs

Proposition 12. *Let R be a ring, and $\mathfrak{a}$ and $\mathfrak{b}$ two ideals in R.*

1. *If $\mathfrak{a} \subset \mathfrak{b}$, then $\mathbf{V}(b) \subset \mathbf{V}(a)$.*
2. *If $\mathbf{V}(b) \subset \mathbf{V}(a)$, then $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$.*
3. *$\mathbf{V}(\mathfrak{a}) = \mathbf{V}(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.*
4. *$\mathbf{V}(\mathfrak{a}) \cup \mathbf{V}(\mathfrak{b}) = \mathbf{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathbf{V}(\mathfrak{a}\mathfrak{b})$.*

Proof.

$$\begin{aligned}\mathbf{V}(\mathfrak{a}) \cup \mathbf{V}(\mathfrak{b}) &= \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p} \,\} \cup \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{b} \subset \mathfrak{p} \,\} && (3.4)\\ &= \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p} \text{ or } \mathfrak{b} \subset \mathfrak{p} \,\} && (3.5)\\ &= \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \,\} && (3.6)\\ &= \mathbf{V}(\mathfrak{a} \cap \mathfrak{b}) && (3.7)\end{aligned}$$

□

5. *For any index set I, it is $\bigcap_{i \in I} \mathbf{V}(\mathfrak{a}_i) = \mathbf{V}(\sum_{i \in I} \mathfrak{a}_i)$.*
6. $\mathbf{V}(R) = \emptyset$.

   Proof. $\mathbf{V}(R) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid R \subset \mathfrak{p} \,\} = \emptyset$ because by definition a prime ideal must not be the whole ring. □
7. $\mathbf{V}(\langle 0 \rangle) = \operatorname{Spec}(R)$.

   Proof. $\mathbf{V}(\langle 0 \rangle) = \{\, \mathfrak{p} \in \operatorname{Spec}(R) \mid \langle 0 \rangle \subset \mathfrak{p} \,\} = \operatorname{Spec}(R)$ because all ideals contain the zeroideal. □

Proposition 13. *The Zariski topology is indeed a topology.*

Proof. □

3.2 Exercises

Exercise 3.4. Let R be a ring and $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$. Show:

1. The closure $\overline{\{\mathfrak{p}\}}$ of $\mathfrak{p}$ is equal to $\mathbf{V}(\mathfrak{p})$; that is, $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.

   Proof. Let $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$. If $f \in R - \mathfrak{p}$, then $\mathfrak{q} \in \mathbf{D}(f)$. □

Exercise 3.5. Describe $\operatorname{Spec}(\mathbb{R})$, $\operatorname{Spec}(\mathbb{Z})$, $\operatorname{Spec}(\mathbb{C}[X])$, and $\operatorname{Spec}(\mathbb{R}[X])$.

Proof.
1. $\operatorname{Spec}(\mathbb{R}) = \{\, \langle 0 \rangle \,\}$ because the only ideals in a field are the zeroideal and the field itself.
2. $\operatorname{Spec}(\mathbb{Z}) = \{\, p\mathbb{Z} \mid p \text{ is a prime number} \,\}$.
3. $\operatorname{Spec}(\mathbb{C}[X]) = \{\, \langle X - z \rangle \mid z \in \mathbb{C} \,\}$ because $\mathbb{C}[X]$ is a PID and because of the fundamental theorem of algebra.
4. $\operatorname{Spec}(\mathbb{R}[X])$ has the ideals above and all polynomials of degree two with complex roots.

For any PID R, the points x_p of $\operatorname{Spec}(R)$ represents the ideals $\langle p \rangle$ with p prime or 0. The closed sets are the $\mathbf{V}(\langle a \rangle)$ with $a \in R$; moreover, $\mathbf{V}(\langle a \rangle) = \emptyset$ if a is a unit, $\mathbf{V}(\langle 0 \rangle) = R$, and $\mathbf{V}(\langle a \rangle) = x_{p_1} \cup \ldots \cup x_{p_s}$ if $a = p_1^{n_1} \cdots p_s^{n_s}$ with p_i a prime and $n_i \leq 1$. □

Exercise 3.6. Let R be a ring, and let $X_1, X_2 \subset \operatorname{Spec}(R)$ closed subsets. Show that the following four statements are equivalent:

1. Then $X_1 \sqcup X_2 = \operatorname{Spec}(R)$; that is, $X_1 \cup X_2 = \operatorname{Spec}(R)$ and $X_1 \cap X_2 = \emptyset$.
2. There are complementary idempotents $e_1, e_2 \in R$ with $\mathrm{V}(\langle e_i \rangle) = X_i$.

Proof. "1. to 2." Since X_1 and X_2 are closed subsets, there are ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that

$$\operatorname{Spec}(R) = \mathbf{V}(\mathfrak{a}_1) \cup \mathbf{V}(\mathfrak{a}_2) = \mathbf{V}(\mathfrak{a}_1\mathfrak{a}_2) \tag{3.8}$$
$$\emptyset = \mathbf{V}(\mathfrak{a}_1) \cap \mathbf{V}(\mathfrak{a}_2) = \mathbf{V}(\mathfrak{a}_1 + \mathfrak{a}_2) \tag{3.9}$$

If two variety are equal, the radical of the generating ideals are equal, hence $\sqrt{\langle 0 \rangle} = \sqrt{\mathfrak{a}_1\mathfrak{a}_2}$ and $\sqrt{R} = \sqrt{\mathfrak{a}_1 + \mathfrak{a}_2}$. □

Part IV

Modules