Topology

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Contents

Conventions

$\mathbb{N}$ contains 0, that is $\mathbb{N} = \{ 0, 1, 2, \dots \}$.

Chapter 1

Topological Space

1.1 Definitions and Theorems

Definition 1 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to $\mathcal{O}$.
2. Any **arbitary** union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The intersection of **finite number** of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The collection $\mathcal{O}$ is called a topology on X and the elements of $\mathcal{O}$ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.
We often just write X instead of $(X, \mathcal{O})$ if the given topology is clear.

Definition 2 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \tag{1.1}$$

Lemma 3. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 4 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. If $Y = X$ we also write $\mathrm{Homeo}(X)$.

Definition 5 (Base)**.** Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.

2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 6. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets, then there exists **exactly one** topology $\tau \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \tau$

2. If $\tau' \subset \mathcal{P}(X)$ a topology with $S \subset \tau'$, then $\tau \subset \tau'$.

Definition 7. 1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{ S \cap U \mid U \in \tau \} .$$

2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$

3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{ U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \} .$$

Definition 8. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.

2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\operatorname{int}(S)$, is the set of all interior points of S.

3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\operatorname{cl}(S)$, is defined by

$$\operatorname{cl}(S) := X \setminus \operatorname{int}(X \setminus S).$$

1.2 Proofs, Remarks, and Examples

Definition 9 (Topological Space)**.** A topological space is an ordered pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the entire set X belongs to $\mathcal{O}$.
2. Any **arbitary** union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.
3. The intersection of **finite number** of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The collection $\mathcal{O}$ is called a topology on X and the elements of $\mathcal{O}$ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.
We often just write X instead of $(X, \mathcal{O})$ if the given topology is clear.

Definition 10 (Comparison of Topologies)**.** Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on a set X such that $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then the topology $\mathcal{O}_1$ is said to be coarser (also weaker or smaller) than $\mathcal{O}_2$, and $\mathcal{O}_2$ is said to be finer (also stronger or larger) than $\mathcal{O}_1$. The binary relation $\subseteq$ defines a partial ordering relation on the set of all possible topologies on X.

Example 10.1. Let X be a set.

1. $\mathcal{O} = \mathcal{P}(X)$ is called the discrete topology. In this case, $(X, \mathcal{O})$ is called the discrete space. It is the finest topology that can be defined on a set.
2. $\mathcal{O} = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology. It is the coarsest topology that can be defined on a set.

Proposition 11. Let (X, d) be a metric space. The collection of subsets

$$\mathcal{O}_d := \{\, U \subset X \mid U \text{ is a open subset in the metric space } (X, d) \,\}$$

defines a topology on X. In other words, a metric induces a topology.

Proof. We will show that $\mathcal{O}_d$ fullfills the axioms of a topology.

1. The emptyset $\varnothing$ is open in the metric space vacuously, hence $\varnothing \in \mathcal{O}_d$. For the entire set X, if $x \in X$, then clearly $B_\epsilon(x) \subset X$ for any $\epsilon \in \mathbb{R}^+$, therefore $X \in \mathcal{O}_d$.
2. Let $S \subset \mathcal{O}_d$ be a collection of subsets. Consider
$$x \in \bigcup_{U \in S} U,$$
then $x \in U_0$ for some set in $\mathcal{O}_d$. U_0 is open in the metric space, therefore, there is an $\epsilon \in \mathbb{R}^+$ such that $B_\epsilon(x) \in U_0$. The ϵ-ball $B_\epsilon(x)$ is also contained in the union of the subsets in S. In other words, any arbitary union of members of $\mathcal{O}_d$ are again in $\mathcal{O}_d$.
3. Let $U, V \in \mathcal{O}_d$ and consider $x \in U \cap V$. We have that $x \in U$ and $x \in V$. Since $U, V \in \mathcal{O}_d$, they are open subsets in the metric space, hence there are $\epsilon_1, \epsilon_2 \in \mathbb{R}^+$ such that $B_{\epsilon_1}(x) \subset U$ and $B_{\epsilon_2}(x) \subset V$. Without loss of generality assume $\epsilon_1 \leq \epsilon_2$. Then, $B_{\epsilon_1}(x) \subset B_{\epsilon_2}(x)$, so $B_{\epsilon_1}(x) \subset V$ also. This implies $B_{\epsilon_1}(x) \subset U \cap V$, so $U \cap V \in \mathcal{O}_d$. By simple induction, we may conclude that the intersection of finite number of members of $\mathcal{O}_d$ is again in $\mathcal{O}_d$.

$\square$

Remark. The proof above coincides with the fact that in a metric space arbitary union of open subsets and finite intersection of open subsets are open.

Example 11.1. The Zariski-topology.

Example 11.2. List of natural topologies.

1. On $\mathbb{R}^n$ the canonical topology, called the Euclidean topology, is generated by the basis that is formed by open balls, i.e. open subsets of $\mathbb{R}^n$ are arbitary unions of open balls. In other words, if $A \in \mathcal{O}_{\mathbb{R}^n}$ and I is an index set, then

$$A = \bigcup_{i \in I} B_r(p) = \bigcup_{i \in I} \{ x \in \mathbb{R}^n \mid d(p,x) < r \} .$$

   This definition agrees with the topology endowed on arbitary metric spaces.

2. The matrix space $\mathrm{Mat}_{n\times m}(\mathbb{K})$ for a field $\mathbb{K}$ does not have one canonical topology. Depending on the context and literature different ones are used.

   - Since $\mathrm{Mat}_{n\times m}(\mathbb{K})$ is isomorphic to $\mathbb{R}^{n\cdot m}$, one could use the Euclidean topology as defined above.
   - $\mathrm{Mat}_{n\times m}(\mathbb{K})$ is a metric space via multitude of operator norms. The metric space induces the topology.
   - Another metric on $\mathrm{Mat}_{n\times m}(\mathbb{K})$ is the rank distance for $A, B \in \mathrm{Mat}_{n\times m}$ defined as $d(A,B) := \mathrm{rank}(B - A)$ which again would induce a topology.

Definition 12 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.

$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X . \tag{1.2}$$

Proposition 13. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Proof. Let X and Y be metric spaces and $f : X \longrightarrow Y$ a function.

1. "$\Rightarrow$": Let f be ϵ-δ-continuous and $V \in \mathcal{O}_Y$ be an open subset. If $f^{-1}(V)$ is empty, then we are finished, so consider $x \in f^{-1}(V)$. We have that $f(x) \in V$. Since V is an open subset, there is an $\epsilon \in \mathbb{R}^+$ such that $B_Y(f(x), \epsilon) \subset V$. Using the ϵ-δ-continuity of f yields

$$f(B_X(x,\delta)) \subset B_Y(f(x), \epsilon) \subset V .$$

   If we apply the definition of a preimage, we get $B_X(x,\delta) \subset f^{-1}(V)$ which implies that $f^{-1}(V)$ is open in the topological sense. Therefore, f is continuous.

2. "$\Leftarrow$": Let f be continuous in the topological sense and consider $x \in X$. The ϵ-ball $B_Y(f(x), \epsilon)$ is open in Y, hence the preimage $f^{-1}(B_Y(f(x), \epsilon))$ is also open and contains x. Now, there exists a $\delta \in \mathbb{R}^+$ such that

$$B_X(x,\delta) \subset f^{-1}(B_Y(f(x), \epsilon)).$$

   Applying the definition of a preimage we get $f(B_X(x,\delta)) \subset B_Y(f(x), \epsilon)$ which means f is ϵ-δ-continuous at x. Since x was chosen arbitary, f is ϵ-δ-continuous.

$\square$

Remark. Again, the proof above coincides with the fact that in a metric space, a function is ϵ-δ-continuous if and only if the preimage of any open subset is open.

Definition 14 (Homeomorphism)**.** Let X and Y be topological spaces.

1. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.
   (a) f is bijective.
   (b) f and the inverse map f^{-1} is continuous.
2. Two topological spaces X and Y are said to be homeomorphic if a homeomorphism exists.
3. We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. If $Y = X$ we also write $\mathrm{Homeo}(X)$.

Proposition 15. The set of all homeomorphisms of X to itself $\mathrm{Homeo}(X)$ is a group with composition as its operation.

Proof. The identity function is contained in $\mathrm{Homeo}(X)$ and is the identity element. Composition is associative and closed in $\mathrm{Homeo}(X)$. By definition, $\mathrm{Homeo}(X)$ contains the inverse of all its elements. Thus, $\mathrm{Homeo}(X)$ is a group with composition as its operation. $\square$

Definition 16 (Base)**.** Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Example 16.1.
1. The set Γ of all open intervals in $\mathbb{R}$ form a basis for the Euclidean topology on $\mathbb{R}$. If we require Γ to be of all bounded open intervals, it will still generate the Euclidean topology.

Lemma 17. For any collection of subsets $S \subset \mathcal{P}(X)$, there exists exactly one topology $\mathcal{O} \subset \mathcal{P}(X)$ that contains S and is the coarsest topology to do so, i.e.

1. $\mathcal{S} \subset \mathcal{O}$, and
2. if $\mathcal{O}' \subset \mathcal{P}(X)$ is an another topology with $S \subset \mathcal{O}'$, then $\mathcal{O} \subset \mathcal{O}'$.

Proof. Let $S \subset \mathcal{P}(X)$ be a collection of subsets and $\mathcal{O}(S)$ be the set of topologies that contain S, i.e.

$$\mathcal{O}(S) = \{\, \tau \subset \mathcal{P}(X) \mid \tau \text{ is a topology and } S \subset \tau \,\}\,.$$

We know that $\mathcal{O}(S)$ is not empty because $\mathcal{P}(X) \in \mathcal{O}(S)$. Now define

$$\mathcal{O} := \bigcap_{\tau \in \mathcal{O}(S)} \tau.$$

Our claim is that this $\mathcal{O}$ is a topology.

1. The emptyset $\varnothing$ and the entire set X is contained in each $\tau \in \mathcal{O}(S)$ since these are topolgoies. Thus, the empty set and the entire set lie also in the intersection, i.e. $\varnothing, X \in \mathcal{O}$.
2. Let $\{U_i\}_{i \in I}$ be a familiy of subsets in $\mathcal{O}$ for an arbitary index set I. This means for each $i \in I$ it is $U_i \subset \mathcal{O}$, therefore, again for each $i \in I$ we have $U_i \in \tau$. Since τ was a topology, the arbitary union of $\{U_i\}_{i \in I}$ will lie in τ.
3. Similar for the finite intersection.

In particular, $\mathcal{O}$ lies in $\mathcal{O}(S)$.

MISSING THAT IT IS UNIQUE! □

Definition 18. 1. Given (X, τ) be a topological space, $S \subset X$ a subset, the subspace topology (also the induced topology or the relative topology) on S is defined by

$$\tau_S = \{\, S \cap U \mid U \in \tau \,\}.$$

2. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The product topology of X and Y is defined by

$$\tau_{X \times Y} := \{\, U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \,\}.$$

3. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The topological sum of X and Y is defined by

$$\tau_{X \sqcup Y} := \{\, U \sqcup V \mid U \in \tau_X \text{ and } V \in \tau_Y \,\}.$$

Definition 19. Let (X, τ) be a topological space.

1. Given a point $p \in X$, a subset $U \subset X$ is a neighborhood of p if there is an open subset $V \in U$ such that $p \in V$. If such a neighborhood exists, p is called a interior point of U.
2. Let $S \subset X$ be a subset. The interior of S, denoted by $\mathring{S}$ or $\operatorname{int}(S)$, is the set of all interior points of S.
3. Let $S \subset X$ be a subset. The closure of S, denoted by $\overline{S}$ or $\operatorname{cl}(S)$, is defined by

$$\operatorname{cl}(S) := X \setminus \operatorname{int}(X \setminus S).$$

Remark. This lemma does not hold for basis.

Remark. 1. $\tau_{X \times Y}$ is the most coarse topology for which both of the projections are continuous.

2. $\tau_{X \sqcup Y}$ is the finest topology for which both the inclusions are continuous.

Note about product topology: $\{\, U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y \,\}$; often $W \subset X \times Y \iff \forall (x, y) \in W \exists U_X \in \mathcal{O}_X, V_Y \in \mathcal{O}_Y, x \in U_X, y \in V_Y$

Remark. Let $(X, \mathcal{O})$ be a topological space. A subset that is **both** open and closed is called clopen. Moreover, a subset is clopen if and only if its boundary is empty.

Proof. Let $A \subset X$ be clopen. Because A is closed, we have $\mathrm{cl}(A) = A$, but on the other hand, A is open, so we also have $\mathrm{int}(A) = A$. Then, the boundary of A is $\partial A = \mathrm{cl}(A) \setminus \mathrm{int}(A) = A \setminus A = \varnothing$. All steps we have taken are not just implications, but equivalencies, therefore we have proven the statement. $\square$

1.3 Exercises and Notes

Definition 20 (Metric Space)**.**

Definition 21 (Open and Closed Subsets)**.**

Theorem 22 (Union and Intersection of Open Subsets)**.**

Definition 23. There are many equivalent ways to define continuity.

- *ϵ-δ-continuity*:
- *sequential continuity*:

Chapter 2

Connected Spaces and Sets

2.1 Definition and Theorems

Definition 24. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.
2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.
3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.
4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Lemma 25. Any interval $I \subset \mathbb{R}$ is connected.

Lemma 26. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Definition 27. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Proposition 28. Connected components are closed subsets.

Lemma 29. Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 30. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 31. If X is path connected, then it is also connected.

2.2 Proofs, Remarks, and Examples

Definition 32. A topological space $(X, \mathcal{O})$ is said to be connected, if one of the following equivalent conditions is met.

1. X is **not** a union of two nonempty, disjoint, and open subsets, i.e. there are no open subsets $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$.

2. The **only** subsets of X that are **both** open and closed (clopen) are the empty set $\varnothing$ and the entire set X, i.e. if $A \subset X$ is a subset with $A \in \mathcal{O}$ and $X \setminus A \in \mathcal{O}$, then $A = \varnothing$ or $A = X$.

3. The **only** subsets of X with empty boundary are the emptyset $\varnothing$ and the entire set X.

4. All continuous maps from X to the two point space $\{0, 1\}$ endowed with the discrete topology is constant.

A subset of X is connected if it is a connected space when viewed as a subspace of X.

Proof. We verify the equivalence of the different definitions. So, let $(X, \mathcal{O})$ be a topological space.

- "*1.* $\Rightarrow$ *2.*": Assume that X is not a union of two nonempty, disjoint, and open subsets. Fix a subset $A \in X$ that is clopen. If A is neither the empty set nor X, then $X \setminus A$ is also not the empty set nor X. Clearly, A and $X \setminus A$ are disjoint and they are also open because A is clopen. But $A \sqcup B = X$, so our assumption was absurd. It must be that $A = \varnothing$ or $A = X$.

- "*2.* $\Rightarrow$ 1.": Now let the only clopen set contained in X be the empty set or X itself. Assume there are $A, B \in \mathcal{O}$ with $A, B \neq \varnothing$ and $A \cap B = \varnothing$ such that $A \sqcup B = X$. Then, A is open, but also closed because $X \setminus A = B$ is open. Furthermore, A is not empty and since B is also not empty, $A \neq X$. Hence our assumption was wrong and there no nonempty, disjoint, and open subsets A and B such that $A \sqcup B = X$.

- "*2.* $\iff$ *3.*": This is one of the properties of clopen subsets and was proven in remark XXX.

- "*1.* $\Rightarrow$ *4.*": Let X not be a union of two nonempty, disjoint, and open subsets. Assume there exists a continuous function $f : X \longrightarrow \{0, 1\}$ with regards to the discrete topology that is not constant. Then, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are nonempty sets that are also disjoint. Since f is continuous, these are also open subsets. But we also have $f^{-1}(\{0\}) \sqcup f^{-1}(\{1\}) = X$.

- "*4.* $\Rightarrow$ *1.*": Let all continuous functions with regards to the discrete topology be constant. Assume there are two nonempty, disjoint, and open subsets $A, B \in \mathcal{O}$ such that $A \sqcup B = X$. Define $f : X \longrightarrow \{0, 1\}$ as $f(A) = 0$ and $f(B) = 1$. This definition is well-defined because $A, B \in \mathcal{O}$ are nonempty, disjoint, and $A \sqcup B = X$. f is also continuous as the preimage of $\{0\}$ and $\{1\}$ are A and B respectively which are open subsets. Hence our assumption was wrong.

$\square$

Lemma 33. Any interval $I \subset \mathbb{R}$ is connected.

Proof. Fix an interval $I \subset \mathbb{R}$, and let $A, B \subset \mathbb{R}$ be two nonempty, open and disjoint subsets such that $A \sqcup B = I$. Moreover, let $a \in A$ and $b \in B$ and assume without loss of generality that $a < b$. If we set

$$s := \inf \{ x \in B \mid a < x \}, \tag{2.1}$$

then $s \in I$ because s is between a and b and we have $[a, b] \subset I$.

Now, on one side, we have $s \in \mathrm{cl}(B)$ and since the complement of B is an open subset A, so $B = \mathrm{cl}(B)$. It is therefore $x \in B$.

But we also have $s \in A$ because the infimum cannot be contained in an open set, but $s \in I = A \sqcup B$. $\square$

Lemma 34. Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous function. If X is connected, then $f(X) \subset Y$ is connected.

Proof. Let $f(X) = A \sqcup B$ with A and B being two open disjoint sets. $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous. We also have $f^{-1}(A) \cap f^{-1}B = f^{-1}(A \cap B) = \varnothing$ so $f^{-1}(A) = \varnothing$ or $f^{-1}(B) = \varnothing$, so $A = \varnothing$ or $B = \varnothing$ and we are done. $\square$

Remark. The two lemma above are handy to show that images of functions are connected.

Example 34.1. The general linear group $\mathrm{GL}_n(K)$ for a field K and $n \in \mathbb{N}$ is not connected for $K = \mathbb{R}$ and $K = \mathbb{C}$.

Proof. Define the following partition of $\mathrm{GL}_n(\mathbb{K})$

$$\begin{aligned} A &:= \{ M \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \mid \det(M) > 0 \} \\ B &:= \{ M \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \mid \det(M) < 0 \}, \end{aligned}$$

then, A and B are disjoint, nonempty, and $\mathrm{GL}_n(\mathbb{K}) = A \sqcup B$. We show that A and B are open sets.

The determinant function $\det : \mathrm{Mat}_{n\times n}(\mathbb{K}) \longrightarrow \mathbb{C}$ is continuous because it is a multivariate polynomial. $\mathbb{R}^+$ is an interval, therefore open, and so $\det^{-1}(\mathbb{R}^+) = A$ is also open. Similary B is an open subset. Hence $\mathrm{GL}_n(\mathbb{K})$ is not connected. $\square$

Remark. In the proof above, the topology of $\mathrm{Mat}_{n\times n}(\mathbb{K})$ matters because the continuity of the determinant function depends on the underlying topology.

Definition 35. A connected component of a topological space is a maximally connected subset $X_0 \subseteq X$, i.e. X_0 connected and for all $X_0 \subsetneq X_1$ then X_1 is not connected.

Example 35.1. For $\mathbb{Q} \subset \mathbb{R}$ the connected components are points and those are not open.

Proof. Assume there is a connected set $A \subset \mathbb{Q}$ that contains more than one point. Let $x \in A$ be a point in A. We show that $\{x\}$ is a clopen set.

Denote another point in A that is closest to x as x_0, i.e. for all $y \in A$ it is $d(x, y) \geq d(x, x_0)$. Now set $\epsilon := d(x, x_0)$. Then, $B_\epsilon(x) \cap \mathbb{Q} = \{x\}$ is an open subset.

I think showing closedness is quite similar. $\square$

Proposition 36. Connected components are closed subsets.

Proof. Let X be a set and $C \subset X$ be a connected component. Consider $\mathrm{Cl}(C)$. Clearly, $C \subset \mathrm{Cl}(C)$. Moreover, $\mathrm{Cl}(C)$ is connected hence $\mathrm{Cl}(C) \subset C$. We have $\mathrm{cl}(C) = C$ so C is connected. $\square$

Lemma 37. Let X be connected and $f : X \longrightarrow Y$ and locally constant, i.e. for all $x \in X$ there exists a $U_x \in \mathcal{O}_X$, $x \in U_x$ such that f restricted on U_x is identical to $f(x)$., then f is constant.

Definition 38. X is said to be path connected, if for every pair of points x and x_0 in X there is a continuous map (called path) $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Lemma 39. If X is path connected, then it is also connected.

Proof. Locally constant implies continuous with regards to the discrete topology on Y. Let $x \in X$, $X = f^{-1}(f(x)) \cup f^{-1}(Y \setminus \{f(x)\})$ is a disjoint union and since X is connected $f^{-1}(Y \setminus \{f(x)\}) = \varnothing$. Conclude f is identical to $f(x)$. □

Application: $f : X \longrightarrow \{0,1\}$, X is connected, f locally constant, there is a $x \in X$ such that $f(x) = 1$, then f is identical to 1.

Proof. Let A and B two disjoint open sets such that $A \sqcup B = X$, and let $a \in A$ and $b \in B$. Let $\gamma : [0,1] \longrightarrow X$ be continuous path with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We have that γ^{-1} □

2.3 Exercises and Notes

2.3.1 Connectedness

Lemma 40. If $(X, \mathcal{O}_\mathcal{X})$ and $(Y, \mathcal{O}_Y)$ are two connected topological spaces, then their product $X \times Y$ with the product topology $\mathcal{O}_{X \times Y}$is also connected.

Proof. We will use the definition that all continuous maps from $X \times Y$ to $\{0, 1\}$ endowed with the discrete topology must be constant. Fix a continuous $f : X \longrightarrow \{0, 1\}$.

First, consider the image $f(\{x\} \times Y)$ with $x \in X$. Assume f is not constant on $\{x\} \times Y$, then $f(\{x\} \times Y) = \{0, 1\}$. So we have the preimages $f^{-1}(\{0\}) = \{x\} \times U$ and $f^{-1}(\{1\}) = \{x\} \times V$ with $U, V \subset Y$, $U, V \neq \varnothing$, and $U \cap V = \varnothing$. Because f is continuous, U and V must also be open. This would however mean that $U \sqcup V = Y$ and Y would not be connected, therefore, f is constant on $\{x\} \times Y$. Similary, we get that f is constant on $X \times \{y\}$ for all $y \in Y$.

Let $(x, y) \in X \times Y$ and $(x', y') \in X \times Y$ be two arbitary points. We have $f(x, y) = f(x, y')$ because f is constant on $\{x\} \times Y$ and similary $f(x, y') = f(x', y')$ because f is constant on $X \times \{y\}$. Putting everything together, it is $f(x, y) = f(x', y')$, therefore all continuous $f : X \times Y \longrightarrow \{0, 1\}$ are constant. $\square$

Example 40.1. Clearly, the union of two connected sets need not be connected. Take for example $[0, 1] \subset \mathbb{R}$ and $[2, 3] \subset \mathbb{R}$. Their union $[0, 1] \cup [2, 3]$ is not connected.

Set difference of connected sets are also not necessarily connected, e.g. $[0, 2] \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ are connected, but $[0, 2] \setminus \{1\} = [0, 1) \cup (1, 2]$ is not.

More interestingly, the intersection of two connected sets also need not be connected. Consider the unit circle around the origin $S^1 = \left\{ (x, y) \mid x^2 + y^2 = 1 \right\}$ and another unit circle around $(1, 0)$ $A := \left\{ (x, y) \mid (x-1)^2 + y^2 = 1 \right\}$. They are both connected, but their intersection is a two point set

$$\left\{ \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right) \right\}$$

which is not connected.

Proposition 41.

1. Every trivial topological space is connected.
2. Every discrete topological space with at least two elements is disconnected.
3. Trivially, every singleton set and the empty set are connected spaces vacuously.

Proof.

1. Let X be an arbitary set and $\mathcal{O} = \{ \varnothing, X \}$ be the trivial topology. If $S \subset X$ is a clopen subset, then it is trivially either $\varnothing$ or X, therefore, X is connected.
2. Let X be a set containing more than one element and $\mathcal{O} = \mathcal{P}(X)$ be the discrete topology of X. Let $A \subset X$ be a nonempty proper subset, then $B := X \setminus A$ is also not empty. Both are open subsets, but $A \sqcup B = X$, so X is not connected.

$\square$

Proposition 42. Every singleton set in $\mathbb{R}^n$ endowed with the Euclidean topology is clopen. ??? IDK IF THIS IS TRUE

2.3.2 Path-Connectedness

Example 42.1. Connectedness does not imply path-connectedness. Let $\mathbb{R}^2$ be endowed with the Euclidean topology and consider

$$X = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \cup (\{0\} \times [-1, 1]) \subset \mathbb{R}^2.$$

and see figure XXX. X is connected, but it is not path-connected.

Proof. Denote

$$A := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x > 0 \right\} \qquad B := \{0\} \times [-1, 1],$$

then $X = A \cup B$.

1. First, define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^2$ as

   $$f(x) := \left(x, \sin\left(\frac{1}{x}\right) \right).$$

   f is continuous, $\mathbb{R}^+$ is an interval, therefore connected, so $f(\mathbb{R}^+) = A$ is connected. Similary, define $g : [-1, 1] \longrightarrow 0 \times [-1, 1]$ as

   $$g(x) := (0, x).$$

   Again, g is continuous and $[-1, 1]$ is connected, so $g([-1, 1]) = 0 \times [-1, 1] = B$ is also connected.

   Now assume $X = U \sqcup V$ for some nonempty, disjoint, open subsets of $\mathbb{R}^2$. Since A is connected, either U or V must contain A, say U. On the other hand, $(0, 0) \in V$, but since V is open, it contains all the interior points, i.e.

   Assume there is a clopen subset $S \subset X$ that is not empty. Without loss of generality, we have that $(0, 0) \in U$ (otherwise, consider the complement of U which also must be clopen). Since A is clopen in A, the intersection $A \cap U$ must also be clopen in A, but A is connected, so A is contained in U.

   Moreover, the closure of A is also contained in U. So there is an $\epsilon > 0$ such that the ball $B(p, \epsilon)$ that contains $(0, 0)$ is in U. I got lazy to go into the details, but this ball contains a point of B. Follow the same reason as above.

2. Assume X is path-connected.

   Choose two points $x_0 = (0, 1) \in A$ and $x_1 = (1, 1) \in B$ and a path $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Let $\epsilon \in (0, 1)$, then $B_\epsilon(x_0) \cap X$ is an open subset that contains x_0, therefore, $\gamma^{-1}(B_\epsilon(x_0) \cap X)$ is also open.

$\square$

As motivation, the proof of intermediate value theorem becomes very elegant.

Chapter 3

Separation Axioms

3.1 Definitions and Theorems

Definition 43 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 44. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 45 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 46. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{\, (x, x) \mid x \in X \,\}$ is closed as a subset of the product space $X \times X$.

Proposition 47. T_2 spaces are also T_1 spaces.

3.2 Proofs, Remarks, and Examples

3.2.1 T_0 Space

Definition 48. A topological space $(X, \mathcal{O})$ is a T_0 space (or Kolmogorov space) if for every pair of distinct points of X, at least one of them has a neighborhood not containing the other (this property is called topologically distinguishable).

Definition 49. A topological space $(X, \mathcal{O})$ is a T_1 space (also called accessible space or a space with Fréchet topology) if one of the following equivalent conditions are met.

1. Any two distinct points in X are separated, i.e. if $x, y \in X$ are points with $x \neq y$, then there are neighborhoods U_x and U_y of x and y respectively such that $y \notin U_x$ and $x \notin U_y$.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.
3. Every subset of X is the intersection of all the open sets containing it.
4. Every finite set is closed.
5. Every cofinite set of X is open.

Definition 50 (T_1 Space)**.** Let X be a topological space.

1. We say that two points x and y can be separated if each lies in a neighborhood that does **not** contain the other point.
2. A topological space X is a T_1 space if any two distinct points in X are separated.

Proposition 51. Let X be a topological space. Then, the following are equivalent.

1. X is a T_1 space.
2. Points are closed in X, i.e. given any $x \in X$, the singleton set $\{x\}$ is a closed set.

Definition 52 (T_2 Space)**.** Let X be a topological space.

1. Points x and y in X can be separated by neighborhood if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint, i.e. $U \cap V = \varnothing$.
2. A topological space X is a T_2 space if any two distinct points in X are separated by neighborhood.

Proposition 53. Let X be a topological space. Then, the following are equivalent.

1. X is a T_2 space.
2. Any singleton set $\{x\}$ is the intersection of all closed neighborhoods of x.
3. The diagonal $\Delta = \{\, (x, x) \mid x \in X \,\}$ is closed as a subset of the product space $X \times X$.

Proposition 54. T_2 spaces are also T_1 spaces.

3.3 Exercises and Notes

Chapter 4

Compact Spaces

Definition 55.
1. A topological space X is called compact if each of its open cover has a **finite** subcover.
2. A topological space X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Theorem 56. Satz 17

Theorem 57. Let $A \subset \mathbb{R}^n$ be a subset. A is compact if and only if it is closed and bounded.

Theorem 58. Let X be a T_2 space. If a subset $K \subset X$ is compact, then it is closed.

Theorem 59. Let X and Y be topological spaces, X compact, and Y be a T_2 space. If $f : X \longrightarrow Y$ is bijective and continuous, then the inverse function f^{-1} is continuous.

4.1 Proofs, Remarks, and Examples

Definition 60. 1. A topological space X is called compact if each of its open cover has a **finite** subcover. That is, X is compact if for every arbitary collection $\mathcal{C}$ of open subsets of X such that

$$X = \bigcup_{U \in \mathcal{C}} U$$

there is a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that

$$X = \bigcup_{U \in \mathcal{F}} U.$$

2. A subset is said to be compact if it is compact as a subspace.

Definition 61. X is called sequentially compact if every sequence in X has a convergent subsequence whose limit is in X.

Remark. The notion of compact and sequentially compact are not equivalent.

Example 61.1. 1. Example of a space that is compact but not sequentially compact.

2. Example of a space that is sequentially compact but not compact.

Proposition 62. Let X and Y be two topological spaces.

1. Continuous functions preserve compactness, i.e. if $f : X \longrightarrow Y$ is continuous and X is compact, then $f(X) \subset Y$ is compact.
2. In a compact space, every closed subset is compact, i.e. if X is compact and $A \subset X$ is a closed subset, then A is compact.
3. The product of compact spaces is again compact. If X and Y are both compact, so is $X \times Y$.

Proof. 1. Let $f : X \longrightarrow Y$ be continuous and X compact. Denote the open cover of the continuous image of X by $\mathcal{C}$, i.e.

$$f(X) \subset \bigcup_{U \in \mathcal{C}} U.$$

Because f is continuous, each of the preimages $f^{-1}(U)$ with $U \in \mathcal{C}$ is open. Now, X is compact, there are finitely many $f^{-1}(U)$ such that

$$X \subset \bigcup_{U \in \mathcal{F}} f^{-1}(U)$$

Conclude that $f(X)$ is compact.

2. Let $\mathcal{U}$ be an open cover of A. Every open set in $\mathcal{U}$ is in the form $U \cap A$ for some open set $U \subset X$. Define

$$\mathcal{V} := \{ U \in \mathcal{O} \mid U \cap A \in \mathcal{U} \}$$

then $\mathcal{V}$ is an open cover of A as well. Since A is closed, $X \setminus A$ is open, so $\mathcal{V} \cup (X \setminus A)$ is an open cover of X. By compactness of X, there is a finite subcover that XXXXX.

3. I think this is clear.

□

Lemma 63. $[0,1] \subset \mathbb{R}$ is compact.

Proof. skipped □

Theorem 64 (Heine-Borel)**.** A compact subset of a Euclidean space $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition 65. Let X be a T_2-space. If $K \subset X$, then K is closed.

Proof. for the two things above, skipped. □

Lemma 66. Let $f : X \longrightarrow Y$ be continuous and bijective. If X is compact and Y is a T_2-space, then f^{-1} is continuous.

Chapter 5

Quotient Space

5.1 Definitions and Theorems

Definition 67. Let $(X, \mathcal{O})$ be a topological space, and let $\sim$ be an equivalence relation on X. The quotient set, $X/\sim$ is the set of equivalence classes of elements of X. The equivalence class of $x \in X$ is denoted $[x]$. The projection map (also quotient or canonical map) associated with $\sim$ refers to the following surjective map:

$$\pi : X \longrightarrow X/\sim, \qquad x \mapsto [x]$$

For any subset $S \subset X/\sim$ (so in particular, $s \subset X$ for every $s \in S$) The quotient space under $\sim$ is the quotient set $X/\sim$ equipped with the quotient topology, which is the topology whose open sets are subsets $U \subset Y = X/\sim$ such that $\{ x \in X \mid [x] \in U \} = \bigcup_{u \in U} u$ is an open subset of $(X, \mathcal{O}_X)$; that is, $U \subset X/\sim$

Proposition 68. $\mathcal{O}_{X/\sim}$ is the finest topology in which the projection map $\pi : X \longrightarrow X/\sim$ is continuous.

Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a surjective map. The map is a quotient map (also said strong continuity) if one of the equivalent condition hold.

1. A subset $U \subset Y$ is open in Y if and only if the preimage $p^{-1}(U)$ is open in X.
2. A subset $U \subset Y$ is closed in Y if and only if the preimage $p^{-1}(U)$ is closed in X.

5.2 Proofs, Remarks, and Examples

Definition 69. Let $(X, \mathcal{O})$ be a topological space, and let $\sim$ be an equivalence relation on X. The quotient set, $X/\sim$ is the set of equivalence classes of elements of X. The equivalence class of $x \in X$ is denoted $[x]$. The projection map (also quotient or canonical map) associated with $\sim$ refers to the following surjective map:

$$\pi : X \longrightarrow X/\sim, \qquad x \mapsto [x]$$

For any subset $S \subset X/\sim$ (so in particular, $s \subset X$ for every $s \in S$) The quotient space under $\sim$ is the quotient set $X/\sim$ equipped with the quotient topology, which is the topology whose open sets are subsets $U \subset Y = X/\sim$ such that $\{ x \in X \mid [x] \in U \} = \bigcup_{u \in U} u$ is an open subset of $(X, \mathcal{O}_X)$; that is, $U \subset X/\sim$

Proposition 70. $\mathcal{O}_{X/\sim}$ is the finest topology in which the projection map $\pi : X \longrightarrow X/\sim$ is continuous.

Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a surjective map. The map is a quotient map (also said strong continuity) if one of the equivalent condition hold.

1. A subset $U \subset Y$ is open in Y if and only if the preimage $p^{-1}(U)$ is open in X.
2. A subset $U \subset Y$ is closed in Y if and only if the preimage $p^{-1}(U)$ is closed in X.

Remark. Quotient maps are continuous. There are quotient maps that are neither open nor closed maps.

5.3 Exercises and Notes