Commutative Ring Theory

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Chapter 1

Introduction and Motivation

Chapter 2

Metric Spaces

Definition 1 (Definition 2.1.: Metric Space)**.** Metric Space. Metric.

Definition 2 (Definition 2.1.: Pseudometric Space)**.** Unlike a metric space, points in a pseudometric space need not be distinguishable; that is, one may have $d(x, y) = 0$ for distinct values $x \neq y$.

Definition 3 (Definition 2.1.: Open Ball)**.** In any metric space (X, d), one can define the open ball of radius $r > 0$ about a given point $x \in X$ as

$$B_r(x) := \{\, y \in X \mid d(x, y) < r \,\}$$

Intuition. The axiom of triangle inequality in the definition of a metric might seem arbitary. But it is needed to have for example two desireable properties.

1. Open balls are open themselves.
2. The function $d : X \times X \longrightarrow [0, \infty)$ is continuous.

Proof. 1. Let (X, d) be a metric space and $B_r(x)$ be an open ball. For any $y \in B_r(x)$ choose $0 < \epsilon < r - d(x, y)$.

□

Intuition. Without the fifth axiom in the definition of a metric, a singleton $\{x\}$ need not be closed.

Proof. Let (X, d) be a metric space and $x \in X$ a point. Consider the complement $X \setminus \{x\}$. We want to show $X \setminus \{x\}$ is open. Indeed, for any $y \in X \setminus \{x\}$ and for all $\epsilon < d(x, y)$ the ball $B_\epsilon(y)$ is contained in $X \setminus \{x\}$. □

Example 3.1 (Pseudometric)**.**

Example 3.2. Show that for the pseudometric space X in Example 2.2, $[(0, 0)] \subset X$ is not a closed subset.

Solution. Assume $\{[(0,0)]\} \subset X$ is a closed subset. Then, $X \setminus \{[(0,0)]\}$ must be open, that is, every point in $X \setminus \{[(0,0)]\}$ has an open ball centered around it that is contained in $X \setminus \{[(0,0)]\}$. Consider the point $[(0,1)]$. We have that $d([0,0],[0,1]) = 0$, thus $X \setminus \{[(0,0)]\}$ cannot be open. □

Definition 4 (Definition 2.4)**.** Convergence of a sequence.

Definition 5 (Definition 2.5)**.** For two metric spaces (X, d_X) and (Y, d_Y), a map $f : X \longrightarrow Y$ is called continuous if it satisfies any of the following equivalent conditions:

1. epsilon delta
2. For every open subset $\mathcal{U} \subset Y$, the preimage

$$f^{-1}(\mathcal{U}) := \{\, x \in X \mid f(x) \in \mathcal{U} \,\}$$

   is an open subset.
3. For every convergent sequence $x_n \in X$, $x_n \to x$ implies $f(x_n) \to f(x)$.

Proof. "2. $\Rightarrow$ 3.": Let $x_n \to x$ be a sequence and $\mathcal{U}$ a neighbourhood of $f(x)$. □

Remark. The direction 3. to 2. requires the metric spaces because 2. and 3. are not equivalent in arbitary topological spaces.

Definition 6 (Homeomorphism)**.**

Example 6.1. Any open ball in $\mathbb{R}^n$ with the standard Euclidean metric d_E is homeomorphic to $(\mathbb{R}^n, d_E)$. It follows that any open ball in $\mathbb{R}^n$ are homeomorphic to each other.

Proof. We begin by constructing a homeomorphism. Choose any continuous, increasing, bijective function $f : [0, r) \longrightarrow [0, \infty)$ and define $F : B_r(x) \longrightarrow \mathbb{R}^n$ by

$$F(\mathbf{x}) = \mathbf{x} \text{ and } F(\mathbf{x}+\mathbf{y}) = \mathbf{x} + f(|\mathbf{y}|)\frac{\mathbf{y}}{|\mathbf{y}|} \text{ for all } \mathbf{y} \in B_r(0) \setminus \{0\} \subset \mathbb{R}^n$$

F and F^{-1} is continuous and bijective. □

Definition 7 (Compactness)**.**

Theorem 8. If $f : X \longrightarrow Y$ is continuous and $K \subset X$ is compact, then so is $f(K) \subset Y$.

Intuition. The above theorem means that compactness is a topologically invariant property.

Proof. Let $f : X \longrightarrow Y$ be continuous, $K \subset X$ compact, and $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be a open cover of $f(K)$. Since f is continuous, $f^{-1}(\mathcal{U}_\alpha)$ is open for all $\alpha \in I$. Thus, $\{f^{-1}(\mathcal{U})_\alpha\}_{\alpha \in I}$ is an open cover of K. K was compact, so this open cover has a finite subcover $\{f^{-1}(\mathcal{U})_\alpha\}_{\alpha \in J}$ with $J \subset I$ finite. Then $\{\mathcal{U}_\alpha\}_{\alpha \in J}$ is a finite subcover of $f(K)$. □

Example 8.1. Show that any set with the discrete metric d_D every subset is open.

Definition 9. Two metrics d and d' on the same set X are called topologically equivalent if the identity map from (X, d) to (X, d') is a homeomorphism.

Chapter 3

Topological Spaces

Chapter 4

Products, Sequential Continuity, and Nets

Lemma 10 (Lemma 4.15)**.** In any space X, a subset $A \subset X$ is open if and only if every point $x \in A$ has a neighbourhood $\mathcal{V} \subset X$ that is contained in A.

Proof. "$\Rightarrow$": If A is open, then A itself can be taken as the desired neighbourhood of every $x \in A$. "$\Leftarrow$": Let every point $x \in A$ have a neighbourhood $\mathcal{V} \subset X$ that is contained in A. Denote the open sets of these neighbourhoods by $\mathcal{U}_x$. Then, A is the union of all these open sets $\mathcal{U}_x$ and thus open. □

Lemma 11 (Lemma 4.16)**.** In any first-countable topological space X, a subspace $A \subset X$ is not open if and only if there exists a point $x \in A$ and a sequence $x_n \in X \setminus A$ such that $x_n \to x$.

Proof. "$\Leftarrow$": (Proof by contraposition.) If $A \subset X$ is open, then for every $x \in A$ and sequence $x_n \in X$ converging to x, we cannot have $x_n \in X \setminus A$ for all n since A is a neighbourhood of x. This is true so far for all topological spaces, with or without first-countability axiom, but the latter will be needed to prove the converse.

"$\Rightarrow$": So suppose now that $A \subset X$ is not open, which by Lemma 4.15, means there exists a point $x \in A$ such that no neighbourhood $\mathcal{V} \subset X$ of x is contained in A. Fix a countable neighbourhood base $\mathcal{U}_1, \mathcal{U}_2, \ldots$ for x. XXX

Observe that since none of the neighbourhoods $\mathcal{U}_n$ can be contained in A, there exists a sequence of points

$$x_n \in \mathcal{U}_n \text{ such that } x_n \notin A.$$

This sequence converges to x since every neighbourhood $\mathcal{V} \subset X$ of x contains one of $\mathcal{U}_N$, implying that for all $n \geq N$,

$$x_n \in \mathcal{U}_n \subset \mathcal{U}_n \subset \mathcal{V}.$$

□

Definition 12. A directed set $(I, \prec)$ consists of a set I with a partial order $\prec$ such that for every pair $\alpha, \beta \in I$, there exists an element $\gamma \in I$ with $\gamma \prec \alpha$ and $\gamma \prec \beta$.

Definition 13. Given a space X, a net $\{x_\alpha\}_{\alpha \in I}$ in X is a function $I \longrightarrow X : \alpha \mapsto x_\alpha$ where $(I, \prec)$ is a directed set.

Chapter 5

Compactness

Definition 14. A subset $A \subset X$ is compact if every open cover of A has a finite subcover, i.e. given an arbitary open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of A, one can always find a finite subset $\{\alpha_1, \ldots, \alpha_N\} \subset I$ such that $A \subset \mathcal{U}_{\alpha_1} \cup \cdots \cup \mathcal{U}_{\alpha_N}$. We say that X itself is a compact space if X is compact subset of itself.