Exercise 2 c)

Solution. 1. $\mathcal{B}$ is a subbasis for the discrete topology. Take an arbitary subset $\mathcal{U} \subset \mathbb{R}$. Notice that for any $x \in \mathbb{R}$ the two sets $\{x, 0\}$ and $\{x, 1\}$ are members of the subbasis $\mathcal{B}$. Thus, it is

$$\begin{aligned} \mathcal{U} &= \bigcup_{x\in\mathcal{U}} \{x\} \\ &= \bigcup_{x\in\mathcal{U}} \{x, 0\} \cap \{x, 1\}. \end{aligned}$$

In other words, every subset of $\mathbb{R}$ is a union of finite intersections of members in $\mathcal{B}$, thus $\mathcal{B}$ as a subbasis generates the discrete topology.

2. However, $\mathcal{B}$ is not a basis of the discrete topology. Plainly, a singleton set cannot be generated from a union of elements of $\mathcal{B}$.

□

Exercise 3 b)

Suppose $\mathcal{B}$ is a subbasis for a topology $\mathcal{T}$ on a set X. Given another topological space Y, show that a map $f : Y \longrightarrow X$ is continuous if and only if for every $\mathcal{U} \in \mathcal{B}$, $f^{-1}(\mathcal{U})$ is open in Y.

Solution. Denote the topology of Y by $\mathcal{S}$.

"⇒": Let $f : Y \longrightarrow X$ be continuous and fix an $\mathcal{U} \in \mathcal{B}$. Since $\mathcal{B}$ is subbasis, all its elements are open subsets, thus $\mathcal{U}$ is open. Then by definition of continuous maps, the preimage $f^{-1}(\mathcal{U})$ is also open in Y. As we have fixed an arbitary $\mathcal{U} \in \mathcal{B}$, we may conclude the desired result.

"⇐": On the other hand, let for every $\mathcal{U} \in \mathcal{B}$ the preimage $f^{-1}(\mathcal{U})$ be open in Y. Consider an arbitary open subset $\mathcal{V} \in \mathcal{T}$. By the definition of a subbasis, $\mathcal{V}$ is a finite intersection of members of $\mathcal{B}$, i.e.

$$\mathcal{V} = \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n$$

with $n \in \mathbb{N}$. The preimage of $\mathcal{V}$ therefore is

$$\begin{aligned} f^{-1}(\mathcal{V}) &= f^{-1}(\mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n) \\ &= f^{-1}(\mathcal{U}_1) \cap \ldots \cap f^{-1}(\mathcal{U}_n) \end{aligned}$$

where we applied the aforementioned lemma on the last step. Now, $f^{-1}(\mathcal{U}_i)$ are open subsets for all $1 \leq i \leq n$. By the definition of topological spaces, finite intersections of open subsets are also open, hence $f^{-1}(\mathcal{V})$ is open. Thus, f is continuous. □

Exercise 3 c)

Now suppose $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in I}$ is a collection of topological spaces, $(X, \mathcal{T})$ is $\prod_{\alpha\in I} X_\alpha$ with the product topology, and the subbase $\mathcal{B} \subset \mathcal{T}$ is taken to consist of all sets of the form

$$\{\, x_{\alpha \alpha\in I} \mid x_\beta \in \mathcal{U}_\beta \,\} \subset \prod_\alpha X_\alpha$$

for arbitary $\beta \in I$ and $\mathcal{U}_\beta \in \mathcal{T}_\beta$.

Show that a sequence $\{x_\alpha^n\}_{\alpha\in I} \in X$ converges to $\{x_\alpha\}_{\alpha\in X} \in X$ as $n \longrightarrow \infty$ if and only if $x_\alpha^n \longrightarrow x_\alpha$ for every $\alpha \in I$.

Solution. "⇒": Let the sequence $\{x_\alpha^n\}_{\alpha\in I} \in X$ converge to $\{x_\alpha\}_{\alpha\in I} \in X$. By the definition of convergence, we have that every neighbourhood $\mathcal{U} \subset X$ of $\{x_\alpha\}_{\alpha\in I}$ it is $\{x_\alpha^n\}_{\alpha\in I} \in \mathcal{U}$ for $n \in \mathbb{N}$ sufficiently large.

"⇐": On the other hand, let $x_\alpha^n \in X_\alpha$ converge to $x_\alpha \in X_\alpha$ for every $\alpha \in I$. By exercise 3 a), we have that for every member of a subbasis $\mathcal{U}_\alpha \in \mathcal{B}_\alpha$ containing x_α, it is $x_\alpha^n \in \mathcal{U}_\alpha$ for $n \in \mathbb{N}$ sufficiently large. □