Exercise Sheet 8

Exercise 1
Solution to 1.
According to Theorem 4.1.11. (script), B is a Dedekind domain and with Remark 3.1.5.2 (script) it is also an A-module. As B is in particular noetherian, it is according to Corollary 4.1.5.1. finitely generated, i.e. there is a generating set

$$\{b_1, \ldots, b_n\} \subseteq B \text{ for some } n \in \mathbb{N}. \tag{1}$$

For each $i \in \mathbb{N}$, we have

$$b_i = a_{i,0} + a_{i,1}\alpha + \cdots + a_{i,m}\alpha^m \tag{2}$$

with $a_{i,j} \in K$ and $j, m \in \mathbb{N}$. As K is a quotient field, we can write $a_{i,j} = p_{i,j}q_{i,j}^{-1}$ with $p_{i,j}, q_{i,j} \in A$. From the equation above, we get

$$b_i \prod_{j=0}^{m} q_{i,j} = p_{i,0} + p_{i,1}\alpha + \cdots + p_{i,m}\alpha^m. \tag{3}$$

Set $\prod_{j=0}^{m} q_{i,j} = C_i \in A$. As $p_{i,j} \in A$, we have that $b_i C_i \in A[\alpha]$ for all $1 \leq i \leq m$. Now set $\prod_{i=1}^{n} C_i = C \in A$ and we have $b_i C \in A[\alpha]$.

We found a factor C so that the generating set of B lies in $A[\alpha]$. So for every $b \in B$ we have that $Cb \in A[\alpha]$.

We have $C \in I_\alpha \neq \varnothing$.

Solution to 2.
We first show $B/\mathfrak{p}B \cong \mathbb{F}_\mathfrak{p}[T]/(\overline{f(T)})$.

Define a homomorphism $\varphi : A[\alpha] \to B/\mathfrak{p}B$, $x \mapsto \varphi(x) \equiv y \mod \mathfrak{p}$. Its kernel is simply $\mathfrak{p}A[\alpha]$. From the condition, we have $\mathfrak{p} + I_\alpha = A$ and hence also $\mathfrak{p}B + I_\alpha = B$ (as $A \subseteq B$). With $I_\alpha \in A[\alpha]$ we have $\mathfrak{p}B + A[\alpha] = B$. Therefore, φ is surjective and we have with the isomorphism theorem for rings $B/\mathfrak{p}B \cong A[\alpha]/\mathfrak{p}A[\alpha]$.

Again define a natural homomorphism $\psi : A[T] \to \mathbb{F}_\mathfrak{p}/(\overline{f(T)})$. This mapping is already surjective. For the kernel, we have the ideal generated by $(\mathfrak{p}, \overline{f(T)})$. Moreover, we have that $\overline{A[\alpha]} = A[T]/(f(T))$. With the isomorphism theorem for rings we have $\mathbb{F}_\mathfrak{p}[T]/(\overline{f(T)}) \cong A[T]/(\mathfrak{p}, \overline{f(T)}) \cong A[\alpha]/\mathfrak{p}A[\alpha] \cong B/\mathfrak{p}B$.

To conclude that the condition $\mathfrak{p} + I_\alpha = A$ can be substituted for $B = A[\alpha]$, we follow the proof from the lecture. In the proof, $\mathfrak{p} + I_\alpha = A$ is used to show

$$B/\mathfrak{p}B \cong A[T]/(f(T), \mathfrak{p}) \cong \mathbb{F}_\mathfrak{p}[T]/(\overline{f(T)}). \tag{4}$$

which we have already shown. So the prime factorization of $\mathfrak{p}B$ corresponds to the factorization as in the case $A[\alpha] = B$.