Topology

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Definition 0.1 (Topological Space)**.** A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets that satisfies the following axioms.

1. The empty set $\varnothing$ and the whole set X belongs to τ.
2. Any arbitary union of members of τ belongs to τ.
3. The intersection of finite number of members of τ belongs to τ.

The collection τ is called a topology on X and the elements of τ are called open sets. A subset $A \subset X$ is said to be closed if its complement $X \setminus A$ is open.

Example 0.1.1. Let X be a set.

1. $\tau = \mathcal{P}(X)$ is called the discrete topology. In this case, (X, τ) is called the discrete space. It is the finest topology. (One can define an ordering of topologies.)
2. $\tau = \{\varnothing, \mathcal{P}(X)\}$ is called the trivial topology.
3. Let (X, d) be a metric space. Set
$$\tau_d := \{\, U \in X \mid U \text{ is a open subset in the metric space } (X, d) \,\}\,. \tag{1}$$
Recall that U being an open subset in the metric space (X, d) means that for all $x \in U$ there is an $r > 0$ such that $B_d(x, r)$ is contained in U.
Here, τ is a topology. In other words, a metric induces a topology.
(Proof as homework.)
4. The Zariski-topology.

Definition 0.2 (Continuous Maps)**.** Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \longrightarrow Y$ is said to be continuous if the preimage of an open subset is again open, i.e.
$$\text{for all } U \in \tau_Y \text{ it is } f^{-1}(U) \in \tau_X. \tag{2}$$

Lemma 0.2.1. The different definitions of continuity in a topological space and a metric space are equivalent, i.e. if X and Y are metric spaces, then $f : X \longrightarrow Y$ is ϵ-δ-continuous if and only if f is continuous.

Definition 0.3 (Homeomorphism)**.** Let X and Y be topological spaces. A map $f : X \longrightarrow Y$ is a homeomorphism if it has the following properties.

1. f is bijective.
2. f is continuous.
3. The inverse map f^{-1} is continuous.

If such function exists, X and Y are said to be homeomorphic.

We denote the set of all homeomorphisms from X to Y by $\mathrm{Homeo}(X, Y)$. The set of all homeomorphisms of X to itself $\mathrm{Homeo}(X)$ is a group with composition as its operation.

Definition 0.4. Let (X, τ) a topological space.

1. $\mathcal{B} \subset \mathcal{O}$ is a basis of the topology, if any member of $\mathcal{O}$ is the union of subsets from $\mathcal{B}$.
2. $\mathcal{S} \subset \mathcal{O}$ is a subbasis of the topology, if any member of $\mathcal{O}$ is the union of finite intersections of subsets from $\mathcal{S}$.

We say that $\mathcal{B}$ and $\mathcal{S}$ generates $\mathcal{O}$ and write $\overline{\mathcal{S}} = \overline{\mathcal{B}} = \mathcal{O}$.

Lemma 0.4.1. Let $\mathcal{S} \subset \mathcal{P}(X)$, then there exists exactly one topology $\mathcal{O} \subset \mathcal{P}(X)$ of X such that

1. $\mathcal{S} \subset \mathcal{O}$