

Integration and Integration

K

March 30, 2021

Contents

Introduction	5
I σ-algebra and measures	7
1 Family of Sets	9
2 Measure	11
2.1 Content, Premeasure, and Measure	11
2.2 Lebesgue Content	12
2.3 Lebesgue Measure	12
2.4 Measure Space	13
II Lebesgue Integral	15
2.5 Measurable Maps	17
2.6 Lebesgue Integral	17
III Applications	19
IV More Theory	21
3 Lebesgue Space	23
3.1 Lebesgue Space	23
3.2 Convergence Theorems	23
3.3 Convergence	24

Introduction

Part I

σ -algebra and measures

Chapter 1

Family of Sets

Chapter 2

Measure

2.1 Content, Premeasure, and Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \rightarrow [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ -additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu \left(\bigsqcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu(A_k). \quad (2.1)$$

- subadditive if for all $A, B \in \mathcal{R}$ it is $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- σ -subadditive if for all $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is

$$\mu \left(\bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} \mu(A_k). \quad (2.2)$$

- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ -finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in \mathcal{R} with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that

$$\bigcup_{k \in \mathbb{N}} A_k = X. \quad (2.3)$$

- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ -additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in \mathcal{R} is required. For σ -rings and therefore σ -algebras, it is guaranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \rightarrow [0, \infty]$ are σ -finite as X need not be included in a ring of sets.

Definition 2.2 (Content). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \rightarrow [0, \infty]$ is called a content if

1. $\mu(\emptyset) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ -additive content $\mu \rightarrow [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure). Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ -algebra. A σ -additive content $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a measure.

2.2 Lebesgue Content

Definition 2.5 (Lebesgue Content). Let $\mathcal{Q}(\mathbb{R}^n)$ be the ring of sets over \mathbb{R}^n .

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigsqcup_{k=1}^m [a_{1,k}, b_{1,k}) \times \cdots \times [a_{n,k}, b_{n,k}) \mid m \in \mathbb{N}; a_{i,k}, b_{i,k} \in \mathbb{R}; 1 \leq k \leq n \right\} \quad (2.4)$$

Set $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ as

$$\lambda^n(A) := \sum_{k=1}^m \prod_{i=1}^n (b_{i,k} - a_{i,k}) \quad (2.5)$$

λ^n is the Lebesgue content.

Theorem 2.5.1. λ^n is a well-defined finite content.

Theorem 2.5.2. λ^n is a premeasure.

2.3 Lebesgue Measure

CHEET SHEET

1. Content $\mu : \mathcal{R} \rightarrow [0, \infty]$ is empty set 0 and finitely additive.
2. Premeasure $\mu : \mathcal{R} \rightarrow [0, \infty]$ is σ -additive content.
3. First extension $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$
4. Outer measure $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$

$$\mathcal{A} \subset \mathcal{A}^\uparrow \subset \sigma(\mathcal{A}) \subset \hat{\mathcal{A}} \quad (2.6)$$

Definition 2.6. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings. Set

$$\mathcal{R}^\uparrow := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k \in \mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \quad (2.7)$$

Remark. \mathcal{R}^\uparrow is the set of all $A \in \mathcal{P}(X)$ that can be expressed as a countable many unions of sets in \mathcal{R} .

In general, \mathcal{R}^\uparrow is not a set of rings.

Definition 2.7. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets and $\mu : \mathcal{R} \rightarrow [0, \infty]$ a premeasure. For $A_k \uparrow A$ with $A_k \in \mathcal{R}$ for $k \in \mathbb{N}$ define

$$\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty], A \mapsto \tilde{\mu}(A) := \lim_{k \rightarrow \infty} \mu(A_k). \quad (2.8)$$

$\tilde{\mu}$ is called the first extension of the premeasure μ .

Remark. In general, $\tilde{\mu}$ is not a premeasure as \mathcal{R}^\uparrow need not be a ring of sets.

$\tilde{\mu}$ restricted on \mathcal{R} is identical with μ , i.e. $\tilde{\mu}|_{\mathcal{R}} \equiv \mu$.

Lemma 2.7.1. The first extension $\tilde{\mu}$ is well-defined.

Proposition 2.7.1 (Properties of \mathcal{R}^\uparrow).

Proposition 2.7.2 (Properties of the First Extension).

Definition 2.8 (Second Extension or the Outer Measure). Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets, $\mu : \mathcal{R} \rightarrow [0, \infty]$ a σ -finite premeasure on \mathcal{R} , and $\tilde{\mu} : \mathcal{R}^\uparrow \rightarrow [0, \infty]$ the first extension of μ on \mathcal{R}^\uparrow . Moreover, let $B \subset X$ be a subset of X . Then, the map

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty], B \mapsto \mu^* := \inf \{ \tilde{\mu}(A) \mid A \in \mathcal{R}^\uparrow, A \supset B \} \quad (2.9)$$

is called the outer measure induced by $\tilde{\mu}$ on $\mathcal{P}(X)$.

Proposition 2.8.1 (Properties of the Second Extension).

Proposition 2.8.2 (Properties of the Outer Measure).

Definition 2.9 (Lebesgue Outer Measure). Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty], B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^\uparrow, A \supset B \right\} \quad (2.10)$$

is called the Lebesgue outer measure induced by $\tilde{\lambda}^n$.

Definition 2.10 (Pseudo Metric). Let X be a set. A map $d : X \times X \rightarrow \overline{\mathbb{R}}$, $(x, y) \mapsto d(x, y)$ is called pseudo metric on X if for all $x, y, z \in X$ it is the following three axioms are met.

1. $x = y \Rightarrow d(x, y) = 0$.
2. $d(x, y) = d(y, x)$. (Symmetry.)
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 2.10.1. The outer measure induces a pseudo metric, i.e.

$$d_{\mu^*} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty], (A, B) \mapsto d_{\mu^*}(A, B) := d_{\mu^*}(A \triangle B) \quad (2.11)$$

is a pseudo metric.

Proposition 2.10.2. The outer measure is continuous.

Definition 2.11 (Approximation through elements of Rings). Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings, $\mu : \mathcal{R} \rightarrow [0, \infty]$ a premeasure on \mathcal{R} , and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the outer measure induced by μ . Then, $A \in \mathcal{P}(X)$ is called \mathcal{R} -approximatable in respect to μ^* if for all $\epsilon > 0$ there exists an $B \in \mathcal{R}$ such that $d_{\mu^*}(A, B) = \mu^*(A \triangle B) < \epsilon$.

Theorem 2.11.1. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ -algebra and $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$ a finite premeasure. Let the first extension $\tilde{\mu} : \mathcal{A}^\uparrow \rightarrow \mathbb{R}_0^+$ also be finite and $\mu^* : \mathcal{P}(X) \rightarrow \mathbb{R}_0^+$ the outer measure. Then,

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \quad (2.12)$$

is a σ -algebra on X .

Theorem 2.11.2. Let $\mu, \tilde{\mu}, \mu^*$ and $\mathcal{A}, \mathcal{A}^\uparrow, \hat{\mathcal{A}}$ be given. Then, a finite premeasure $\mu : \mathcal{A} \rightarrow \mathbb{R}_0^+$ can be uniquely extended to a finite measure $\hat{\mu} : \hat{\mathcal{A}} \rightarrow \mathbb{R}_0^+$ where $\hat{\mu} \equiv \mu^*|_{\hat{\mathcal{A}}}$.

Theorem 2.11.3. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings and $\mu : \mathcal{R} \rightarrow [0, \infty]$ a σ -finite premeasure on \mathcal{R} and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the outer measure induced by μ . Then, μ can be uniquely extended to a measure $\hat{\mu} : \sigma(\mathcal{R}) \rightarrow [0, \infty]$ where $\hat{\mu} \equiv \mu^*|_{\sigma(\mathcal{R})}$.

Definition 2.12. Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \rightarrow \mathbb{R}_0^+$ a σ -finite Lebesgue premeasure. In this chapter, we constructed a unique extension of λ^n on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathbb{R}^n)$, the Lebesgue-Borel measure $\hat{\lambda} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$.

2.4 Measure Space

Definition 2.13. Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ -algebra. The tuple (X, \mathcal{A}) is called measurable space and the sets in the σ -algebra $A \in \mathcal{A}$ are called measurable sets.

Moreover, let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a measure on $\mathcal{P}(X)$. Then, (X, \mathcal{A}, μ) a measure space.

Definition 2.14 (Null Sets). Let (X, \mathcal{A}, μ) be a measure space and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ the induced outer measure. Then $N \subset X$ with $\mu^*(N) = 0$ is called null set.

For $X = \mathbb{R}^n$ with $\lambda^n(N) = 0$ called Lebesgue null set.

$S = \emptyset$ is called the trivial null set.

Definition 2.15 (Completion of a Measure Space). Let (X, \mathcal{A}, μ) be a measure space. This measure space is called complete if all null sets are included in \mathcal{A} , i.e. for all $N \in \mathcal{A}$

$$\mu^* N = 0 \Rightarrow N \in \mathcal{A}. \quad (2.13)$$

Definition 2.16. Let

$$\overline{\mathcal{A}}^\mu := \{A \cup N \mid A \in \mathcal{A}, N \subset X \text{ with } \mu^*(N) = 0\} \quad (2.14)$$

then $\overline{\mathcal{A}}^\mu$ is called the completion of (X, \mathcal{A}, μ) .

Definition 2.17. The completion of the Lebesgue-Borel measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \hat{\lambda}^n)$ to $(\mathbb{R}^n, \mathcal{B}^{\hat{\lambda}^n}(\mathbb{R}^n), \hat{\lambda}^n)$ or shorter $(\mathbb{R}^n, \overline{\mathcal{B}}^\lambda(\mathbb{R}^n), \lambda^n)$ is called the (completed) Lebesgue measure space.

$B \in \overline{\mathcal{B}}^\lambda(\mathbb{R}^n)$ is called Lebesgue measurable to differentiate from $B \in \mathcal{B}(\mathbb{R}^n)$ Borel measurable.

Part II

Lebesgue Integral

2.5 Measurable Maps

2.6 Lebesgue Integral

Part III

Applications

Part IV

More Theory

Chapter 3

Lebesgue Space

3.1 Lebesgue Space

Definition 3.1 (L^p -Norm). Let X, \mathcal{A}, μ a measure space, and $f : X \rightarrow \overline{\mathbb{R}}$ measurable. Then for $p \in [1, \infty)$ the L^p -norm is defined as

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}. \quad (3.1)$$

Theorem 3.1.1 (Holder Inequality). Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Let $f, g : X \rightarrow \overline{\mathbb{R}}$ measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \quad (3.2)$$

Theorem 3.1.2 (Minkowski Inequality). Let $f, g : X \rightarrow \overline{\mathbb{R}}$ measurable and $f + g$ well defined on X . Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (3.3)$$

Definition 3.2. Let (X, \mathcal{A}, μ) be a measure space and $p \in [1, \infty)$. Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \rightarrow \overline{\mathbb{R}} \mid f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\}. \quad (3.4)$$

Also define

$$L^p(\mu) := \mathcal{L}^p(\mu) / \sim \mu \quad (3.5)$$

Where the equivalent relation means two functions are equivalent iff they agree on every point outside of null sets.

3.2 Convergence Theorems

Theorem 3.2.1 (Lebesgue Monotone Convergence Theorem). *Also called the theorem of Beppo Levi.* Let (X, \mathcal{A}, μ) be a measure space and let $f_n : X \rightarrow [0, \infty]$ be a sequence of measurable functions such that

$$f_n(x) \leq f_{n+1}(x) \quad (3.6)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Define $f : X \rightarrow [0, \infty]$ by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x). \quad (3.7)$$

Then f is measurable and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu. \quad (3.8)$$

Theorem 3.2.2 (Lebesgue Dominated Convergence Theorem). Let (X, \mathcal{A}, μ) be a measure space, let $g : X \rightarrow \mathbb{R}_0^+$ be an integrable function, and let $f_n : X \rightarrow \mathbb{R}$ be a sequence of integrable functions satisfying

$$|f_n(x)| \leq g(x) \quad (3.9)$$

for all $x \in X$ and $n \in \mathbb{N}$ and converging pointwise to $f : X \rightarrow \mathbb{R}$, i.e.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for all } x \in X. \quad (3.10)$$

Then f is integrable and, for every $E \in \mathcal{A}$,

$$\int_E f \, d\mu = \lim_{n \rightarrow \infty} \int_E f_n \, d\mu. \quad (3.11)$$

3.3 Convergence

Definition 3.3. Let (X, \mathcal{A}, μ) be a measure space.

1. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ be a sequence of function, and let $f : X \rightarrow \overline{\mathbb{R}}$. f_m converges to f almost everywhere, written $f_m \xrightarrow{a.e.} f$, if there exists a null set $N \subset X$ such that for all $x \in X \setminus N$ it is

$$\lim_{m \rightarrow \infty} f_m(x) = f(x). \quad (3.12)$$

2. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and let $f : X \rightarrow \overline{\mathbb{R}}$ also with $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. f_m is L^1 -convergent to f , written $f_m \xrightarrow{L^1} f$, if

$$\lim_{m \rightarrow \infty} \|f - f_m\|_{L^1} = 0. \quad (3.13)$$

3. For all $m \in \mathbb{N}$ let $f_m : X \rightarrow \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. $(f_m)_{m \in \mathbb{N}}$ is called L^1 -Cauchy sequence if for all $\epsilon > 0$ there exists a $m_0(\epsilon)$ such that for all $m, k \geq m_0(\epsilon)$ it is $\|f_m - f_k\|_{L^1} < \epsilon$.