Exercise 3 b)

Suppose $\mathcal{B}$ is a subbasis for a topology $\mathcal{T}$ on a set X. Given another topological space Y, show that a map $f : Y \longrightarrow X$ is continuous if and only if for every $\mathcal{U} \in \mathcal{B}$, $f^{-1}(\mathcal{U})$ is open in Y.

Solution. Denote the topology of Y by $\mathcal{S}$.

"$\Rightarrow$": Let $f : Y \longrightarrow X$ be continuous and fix an $\mathcal{U} \in \mathcal{B}$. Since $\mathcal{B}$ is subbasis, all its elements are open subsets, thus $\mathcal{U}$ is open. Then by definition of continuous maps, the preimage $f^{-1}(\mathcal{U})$ is also open in Y. As we have fixed an arbitary $\mathcal{U} \in \mathcal{B}$, we may conclude the desired result.

"$\Leftarrow$": On the other hand, let for every $\mathcal{U} \in \mathcal{B}$ the preimage $f^{-1}(\mathcal{U})$ be open in Y. Consider an arbitary open subset $\mathcal{V} \in \mathcal{T}$. By the definition of a subbasis, $\mathcal{V}$ is a finite intersection of members of $\mathcal{B}$, i.e.

$$\mathcal{V} = \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n$$

with $n \in \mathbb{N}$. The preimage of $\mathcal{V}$ therefore is

$$\begin{aligned} f^{-1}(\mathcal{V}) &= f^{-1}(\mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n) \\ &= f^{-1}(\mathcal{U}_1) \cap \ldots \cap f^{-1}(\mathcal{U}_n) \end{aligned}$$

where we applied the lemma on the last step. Now, $f^{-1}(\mathcal{U}_i)$ are open subsets for all $1 \leq i \leq n$. By the definition of topological spaces, finite intersections of open subsets are also open, hence $f^{-1}(\mathcal{V})$ is open. Thus, f is continuous. $\square$