Definition 1 — Chart.
An n-dimensional chart $(\mathcal{U}, x)$ on a set M consists of a subset $\mathcal{U} \subseteq \mathcal{M}$ and an injective map $x : \mathcal{U} \longrightarrow \mathbb{R}^n$ whose image $x(\mathcal{U}) \subseteq \mathbb{R}^n$ is an open set.

Definition 2 — Transition Maps.
Any two charts $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$ determine a pair of transition maps

$$x(\mathcal{U} \cap \mathcal{V}) \xrightarrow{y \circ x^{-1}} y(\mathcal{U} \cap \mathcal{V}) \tag{1}$$

$$y(\mathcal{U} \cap \mathcal{V}) \xrightarrow{x \circ y^{-1}} x(\mathcal{U} \cap \mathcal{V}) \tag{2}$$

which are inverse to each other, and are thus bijections between subset of $\mathbb{R}^n$.

Definition 3 — C^k-compatible.
We say that the two charts are C^k-compatible for some $k \in \mathbb{N} \cup \{0, \infty\}$ if the sets $x(\mathcal{U} \cap \mathcal{V})$ and $y(\mathcal{U} \cap \mathcal{V})$ are both open and the transition maps $y \circ x^{-1}$ and $x \circ y^{-1}$ are both of class C^k. If $k = \infty$, we say the charts are smoothly compatible.

Definition 4 — Atlas.
An atlas of class C^k for the set M (or smooth atlas in the case $k = \infty$) is a collection of charts $\mathcal{A} = \{(\mathcal{U}_\alpha, x_\alpha)\}_{\alpha \in I}$ that are all C^k-compatible with each other, such that $\bigcup_{\alpha \in I} \mathcal{U} = M$.

Definition 5 — .
For a set M with an atlas $\mathcal{A}$ of class C^k and $r \in \mathbb{N} \cup \{0, \infty\}$ with $r \leq k$, a function $f : M \longrightarrow \mathbb{R}$ is said to be of class C^k if and only if the function

$$f \circ x^{-1} : x(\mathcal{U}) \longrightarrow \mathbb{R} \tag{3}$$

is of class C^r for every chart $(\mathcal{U}, x) \in \mathcal{A}$.

Definition 6 — .
For $k \in \mathbb{N} \cap \{\infty\}$, a C^k-structure or differentiable structure of class C^k on a set M is a maximal atlas $\mathcal{A}$ of class C^k on M. In the case $k = \infty$, we also call this a smooth structure on M. If M has been endowed with a C^k-structure $\mathcal{A}$, then a chart $(\mathcal{U}, x)$ on M will be referred to as a C^k-chart if it belongs to the maximal atlas $\mathcal{A}$.

Definition 7 — Topology.
A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. For every subcollection $I \subseteq \mathcal{T}$ it is $\bigcup_{\mathcal{U} \in I} \mathcal{U} \in \mathcal{T}$.
3. For every pair $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$ it is $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is then called a topological space, and we call the sets $\mathcal{U} \in \mathcal{T}$ the open subsets in $(X, \mathcal{T})$.

Definition 8 — Metrizable.

Definition 9 — .
Assume $k \in \mathbb{N} \cup \{\infty\}$. A differentiable manifold of class C^k or C^k-manifold is a set M endowed with a C^k-structure such that the induced topology on M is metrizable and separable. In the case $k = \infty$, we also call M are smooth manifold.

We say that M is n-dimensional and refer to M as an n-manifold, written $\dim M = n$, if every chart in its differentiable structure is n-dimensional.