Exercise 1.1

Let $A \subset B$ be an integral extension of rings and assume that B is an integral domain. Suppose $\mathfrak{q} \subset B$ is a prime ideal and let $\mathfrak{p} := \mathfrak{q} \cap A \subset A$.

1. Prove that A is a field if and only if B is a field.

   Proof. Let's do this a little different. Let $\mathfrak{m}$ be a maximal ideal in B and fix a nonzero element $b \in \mathfrak{m}$. Because b is integral over A, we have an expression

   $$0 = a_0 + a_1 b + a_2 b^2 + \cdots a_n b^n \iff -a_0 = a_1 b + a_2 b^2 + \cdots a_n b^n.$$

   On the right side, for each $1 \leq i \leq n$, we have that $a_i b^i$ is in $\mathfrak{m}$, so the whole sum is in $\mathfrak{m}$. This implies the absurdity that $-a_0$ which is a unit is contained in $\mathfrak{m}$. $\square$