0.1 Localization

Definition 1. A multiplicative set is a subset S of a ring R such that the following two conditions hold.

1. S contains the identity, i.e. $1 \in S$.
2. For any two elements $x \in S$ and $y \in S$ it is their prodcut is contained in S, i.e. $xy \in S$.

Example 1.1. In any ring R, the subsets $\{1\}$, and R are multiplicative sets.

Example 1.2. Consider the ring of integers $\mathbb{Z}$. Some of the multiplicative sets in $\mathbb{Z}$ includes

- $\{1, 2, 4, 8, \ldots\} = \{\, 2^n \mid n \in \mathbb{N}_0 \,\}$
- $\{1, 3, 9, 27, \ldots\} = \{\, 3^n \mid n \in \mathbb{N}_0 \,\}$
- $\{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, \ldots\} = \{\, 2^n \cdot 3^m \mid n, m \in \mathbb{N}_0 \,\}$
- $\{1, 4, 16, 64, \ldots\} = \{\, 4^n \mid n \in \mathbb{N} \,\}$
- $\{1, -2, 4, -8, \ldots\} = \{\, (-2)^n \mid n \in \mathbb{N} \,\}$
- $\{1, -1, 2, -2, 4, -4, 8, -8, \ldots\} = \{\, \pm 2^n \mid n \in \mathbb{N} \,\}$

Observation: Given some elements of the ring, we may generate the multiplicative set.

Example 1.3.
- In $\mathbb{Z}/2\mathbb{Z}$, the only multiplicative sets are $\{1\}$ and $\mathbb{Z}/2\mathbb{Z}$
- In $\mathbb{Z}/3\mathbb{Z}$, the multiplicative sets are $\{1\}$, $\{1, 0\}$, $\{1, 2\}$ and $\mathbb{Z}/2\mathbb{Z}$
- In $\mathbb{Z}/4\mathbb{Z}$, the multiplicative sets are $\{1\}$, $\{1, 0\}$, $\{1, 2, 0\}$, $\{1, 3\}$, and $\mathbb{Z}/4\mathbb{Z}$
- In $\mathbb{Z}/5\mathbb{Z}$, the multiplicative sets are $\{1\}$, $\{1, 0\}$, and $\{0, 1, 2, 3, 4\}$
  1. $S(\{1, 2\}) = \{1, 2, 3, 4\}$
  2. $S(\{1, 3\}) = \{1, 2, 3, 4\}$
  3. $S(\{1, 4\}) = \{1, 4\}$
- In $\mathbb{Z}/6\mathbb{Z}$, the multiplicative sets are $\{1\}$, $\{1, 0\}$
  - $S(\{1, 2\}) = \{1, 2, 4\}$
  - $S(\{1, 3\}) = \{1, 3\}$
  - $S(\{1, 4\}) = \{1, 4\}$
  - $S(\{1, 5\}) = \{1, 5\}$
  - $S(\{1, 2, 3\}) = \{0, 1, 2, 3, 4\}$
  - $S(\{1, 2, 5\}) = \{1, 2, 4, 5\}$
  - $S(\{1, 3, 4\}) = \{0, 1, 3, 4\}$
  - $S(\{1, 3, 5\}) = \{1, 3, 5\}$
  - $S(\{1, 4, 5\}) = \{1, 2, 4, 5\}$
- In $\mathbb{Z}/7\mathbb{Z}$, the multiplicative sets are
  - $S(\{1, 2\}) = \{1, 2, 4\}$
  - $S(\{1, 3\}) = \{1, 2, 3, 4, 5, 6\}$
  - $S(\{1, 4\}) = \{1, 2, 4\}$

  - $S(\{1,5\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,6\}) = \{1,6\}$
  - $S(\{1,2,3\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,2,5\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,2,6\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,3,4\}) = \{1,2,3,4,5,6\}$
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  - $S(\{1,4,5\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,4,6\}) = \{1,2,3,4,5,6\}$
  - $S(\{1,5,6\}) = \{1,2,3,4,5,6\}$

- In $\mathbb{Z}/12\mathbb{Z}$
  - $S(\{1,2\}) = \{1,2,4,8\}$
  - $S(\{1,3\}) = \{1,3,9\}$
  - $S(\{1,4\}) = \{1,4\}$

Well, I wrote a python script for this

Remark. The statement: If S is a multiplicative set with 0, then $S \setminus \{0\}$ is a multiplicative set is false.
The converse is (probably) true: If S is multiplicative, then so is $S \cup \{0\}$.

Remark. A multiplicative set need not to have a finite generating set.

Question: Do multiplicative sets have countably finite generating sets?
Answer: Most certainly it isn't. There is no way that R has a countable set of elements that generate R just through multiplication.

Theorem 2. An ideal $\mathfrak{p}$ of a ring R is prime if and only if its complement $R \setminus \mathfrak{p}$ is multiplicativly closed.

Remark. Not all multiplicative sets are the complements of a prime ideal. The above theorem gives an alternative definition of prime ideals but does not function as a definition of multiplicative sets.

Proof. Let $\mathfrak{p}$ be an ideal of a ring R.
"$\Rightarrow$": Suppose $\mathfrak{p}$ is prime. Since $\mathfrak{p}$ does not contain 1, its complement $R \setminus \mathfrak{p}$ must contain 1. Fix two elements $x \in R \setminus \mathfrak{p}$ and $y \in R \setminus \mathfrak{p}$ and assume $xy \notin R \setminus \mathfrak{p}$. Then, $xy \in \mathfrak{p}$, but then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ both of which are contradictions.
"$\Leftarrow$": Suppose $R \setminus \mathfrak{p}$ is multiplicativly closed. $\square$

Example 2.1. Consider the ring of integers $\mathbb{Z}$. Some of the multiplicative sets in $\mathbb{Z}$ includes

- $\mathbb{Z} \setminus (2) = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ this corresponds to the high school math observation that odd times odd is odd

Theorem 3. If $\mathfrak{a}$ is an ideal in R, then $1 + \mathfrak{a}$ is a multiplicative set.

What about the converse? Let $1 + \mathfrak{a}$ be multiplicative.

0.2 Localization

Definition 4.

Example 4.1.
- Localization of $\mathbb{Z}$ at $S := \{1, 2, 4, 8, \ldots\}$
  - some elements: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$
  - it is $\frac{1}{2} = \frac{2}{4}$ because $1 \cdot 4 - 2 \cdot 2 = 0$
- Localization of $\mathbb{Z}$ at $S := \{1, -2, 4, -8, \ldots\}$
- some elements: $\frac{-1}{-2}, \frac{2}{4}$
- it is $\frac{-1}{-2} = \frac{2}{4}$ because $-1 \cdot 4 = -2 \cdot 2$

Definition 5.

$$\tau : R \longrightarrow S^{-1}R, x \mapsto \frac{x}{1}$$

Remark. τ is not injective

Why??

Theorem 6. For $\tau : R \longrightarrow S^{-1}R$ it is true:

1. $\ker\tau = \{x \in R \mid sx = 0 \text{ for some } x \in S\}$. This means that the kernel is a subset of the zero divisor
2. For all $s \in S$, we have $\tau(s)$ is a unit in the Localization.
3. $S^{-1}R \neq 0$ iff $0 \notin S$
4. if S contains only units, τ bijective

Proof.
1. Let $x \in \ker(\tau)$, then $\tau(x) = \frac{x}{1} \sim \frac{0}{1} \iff xs = 0$
2. Let $s \in S$, then $\tau(s) = \frac{s}{1}$. We want to find $\frac{p}{q}$ such that $\frac{sp}{q} \sim \frac{1}{1}$ so $(sp - q)t = 0 \iff spt - qt = 0$

   Yeah, so simply:

   Let $s \in S$. The inverse of $\tau(s) = \frac{s}{1}$ is $\frac{1}{s}$ as $\frac{s}{s} \sim \frac{1}{1}$
3. $0 \in S$, then by definition.

$\square$

0.3 Localization of Modules

Definition 7.

Theorem 8. 1. For every R-module M, there is a canonical isomorphism

$$M \otimes_R R_S \xrightarrow{\sim} M_S, (x \otimes \frac{a}{s}) \mapsto \frac{ax}{s}$$

Proof. $\square$