# Notes on Algebraic Geometry

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# Contents

For this section, let $K$ be a fixed algebraically closed field, and $A = K[X_1, X_2, \ldots, X_n]$

**Definition 0.0.1.** We define affine $n$-space over $K$, denoted $\mathbb{A}^n$ or simply $\mathbb{A}^n$, to be the set of all $n$-tuples of elements of $K$. An element $p \in \mathbb{A}^n$ will be called a point, and if $p = (a_1, a_2, \ldots, a_n)$ with $a_i \in k$, then the $a_i$ will be called the coordinates of $p$.

**Definition 0.0.2.**

1. The set of zeros for $f \in A = K[X_1, X_2, \ldots, X_n]$ is
$$\{ p \in \mathbb{A}^n \mid f(p) = 0 \}.$$
2. For a subset $T$ of the polynomial ring $A = k[X_1, \ldots, X_n]$, we define the zero set of $T$ to be the common zeros of all the elements of $T$, i.e.
$$Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T \}.$$

**Remark 0.0.3.**

1. If $\mathfrak{a}$ is generated by $T$, then $Z(T) = Z(\mathfrak{a})$.
2. $Z(T)$ can be written in finitely many generators.

**Definition 0.0.4.** A subset $Y$ of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subset A = k[X_1, \ldots, X_n]$ such that $Y = Z(T)$.

**Definition 0.0.5.** An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a quasi-affine variety.

**Definition 0.0.6.** If $Y \subset \mathbb{A}^n$ is an affine algebraic set, we define the affine coordinate ring $A(Y)$ of $Y$, to be $A/I(Y)$.

**Definition 0.0.7.** If $X$ is a topological space, we define the dimension of $X$ (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

**Exercise 0.0.1.** *Let $Y \subset \mathbb{A}^3$ be the set $Y = \{ t, t^2, t^3 \mid t \in k \}$.*

1. *Show that $Y$ is an affine variety of dimension* $1$.

*Solution.*

1. By definition 0.0.5, affine algebraic varieties are irreducible closed subsets of $\mathbb{A}^n$. Thus we have to show that $Y = \{ t, t^2, t^3 \mid t \in k \}$ is irreducible and closed in $\mathbb{A}^n$. In $\mathbb{A}^n$, we have the Zariski topology, hence closed sets are algebraic sets. Which means we have to show $Y$ is an algebraic set. Choose $f(X, Y, Z) = XY - Z^3$. Then $Y$ is closed.

   Is $XY - Z^3$ prime?

# Bibliography

[Har77] Robin Hartshorne. *Algebraic Geometry*. New York: Springer, 1977.