Exercise Sheet 1

Exercise 1
A polynomial $f(X) \in \mathbb{Z}[X]$ is primitive if the greatest common divisor of its coefficients is 1. Show the following:

1. If $f(X), g(X) \in \mathbb{Z}[X]$ are primitive, then the product $f(X)g(X)$ is also primitive.

Solution
1.
Denote the coefficients of f and g with a_i and b_j for $1 \leq i \leq \deg f$ and $1 \leq j \leq \deg g$ such that

$$f(X) = \sum_{i=0}^{\deg f} a_i X^i \qquad\qquad g(X) = \sum_{j=0}^{\deg g} b_j X^j \tag{1}$$

Assume there is a prime $x \in \mathbb{Z}$ that divides all coefficients of fg and let a_n and b_m be the first coefficients in f and g respectively that are not divisble by x.
Consider X^{n+m} in the polynomial fg. The coefficient for this term is the sum of products of a_i and b_j for which $i + j = n + m$, i.e.

$$a_n b_m + a_{n-1} b_{m+1} + a_{n+1} b_{m-1} + a_{n-2} b_{m+2} + \dots. \tag{2}$$

This coefficient is however not divisible by x as x divides all but the first term. Hence we have a contradiction.
2.
On the other hand, let f be primitive and irreducible in $\mathbb{Q}[X]$, but assume it is reducible in $\mathbb{Z}[X]$.

If f is a constant, then it it $f(X) = \pm 1$ as f is primitive. This is a contradiction, however, because ± 1 is a unit in $\mathbb{Q}[X]$.

Consider the case where $\deg \geq 1$. From the assumption, we have a factorization $f(X) = g(X)h(x)$ with $g, h \in \mathbb{Z}[X]$ but $g, h \neq \pm 1$.

Assume g is a constant, then g divides all coefficient of f in $\mathbb{Z}$. This cannot be since f is primitive. Therefore, we have $\deg g \geq 1$ which means that g is not a unit in $\mathbb{Q}[X]$.

Apply the same argument for h and we have $f(X) = g(X)h(X)$ is a non-trivial factorization in $\mathbb{Q}[X]$. This is a contradiction with the first assumption.