Integration and Integration

K

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2.1 Content, Premeasure, and Measure

Definition 2.1. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called

- finitely additive if for all disjoint $A, B \in \mathcal{R}$ it is $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- σ-additive if for all disjoint $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is
$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{2.1}$$
- subadditive if for all $A, B \in \mathcal{R}$ it is $\mu(A \cup B) \leq \mu(A) + \mu(B)$
- σ-subadditive if for all $A_k \in \mathcal{R}$ with $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{R}$ it is
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k). \tag{2.2}$$
- finite if for all $A \in \mathcal{R}$ it is $\mu(A) < \infty$.
- σ-finite if there exists a collection of subsets $\{A_k\}_{k \in \mathbb{N}}$ in $\mathcal{R}$ with $\mu(A_k) < \infty$ for all $k \in \mathbb{N}$ such that
$$\bigcup_{k \in \mathbb{N}} A_k = X. \tag{2.3}$$
- monotonous if for all $A, B \in \mathcal{R}$ with $A \subset B$ it is $\mu(A) \leq \mu(B)$.

Remark. In the definition of σ-additivity, checking whether $\bigsqcup_{k=1}^{\infty} A_k$ is included in $\mathcal{R}$ is required. For σ-rings and therefore σ-algebras, it is guranteed that a countable union of disjoint sets are included.

In general, not all finite set functions $\mu \to [0, \infty]$ are σ-finite as X need not be included in a ring of sets.

Definition 2.2 (Content)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A set function $\mu \to [0, \infty]$ is called a content if

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.

Definition 2.3 (Premeasure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets. A σ-additive content $\mu \to [0, \infty]$ is called a premeasure.

Definition 2.4 (Measure)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. A σ-additive content $\mu : \mathcal{A} \to [0, \infty]$ is called a measure.

2.2 Lebesgue Content

Definition 2.5 (Lebesgue Content)**.** Let $\mathcal{Q}(\mathbb{R}^n)$ be the ring of sets over $\mathbb{R}^n$.

$$\mathcal{Q}(\mathbb{R}^n) = \left\{ \bigsqcup_{k=1}^{m} [a_{1,k},\, b_{1,k}) \times \cdots \times [a_{n,k},\, b_{n,k}) \,\middle|\, m \in \mathbb{N};\, a_{i,k}, b_{i,k} \in \mathbb{R};\, 1 \leq k \leq n \right\} \tag{2.4}$$

Set $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ as

$$\lambda^n(A) := \sum_{k=1}^{m} \prod_{i=1}^{n} (b_{i,k} - a_{i,k}) \tag{2.5}$$

λ^n is the Lebesgue content.

Theorem 2.5.1. λ^n is a well-defined finite content.

Theorem 2.5.2. λ^n is a premeasure.

2.3 Lebesgue Measure

CHEET SHEET

1. Content $\mu : \mathcal{R} \to [0, \infty]$ is empty set 0 and finitely additive.
2. Premeasure $\mu : \mathcal{R} \to [0, \infty]$ is σ-additive content.
3. First extension $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$
4. Outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$

$$\mathcal{A} \subset \mathcal{A}^{\uparrow} \subset \sigma(\mathcal{A}) \subset \hat{\mathcal{A}} \tag{2.6}$$

Definition 2.6. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings. Set

$$\mathcal{R}^{\uparrow} := \{A \in \mathcal{P}(X) \mid \exists (A_k)_{k\in\mathbb{N}} \text{ in } \mathcal{R} \text{ with } A_k \uparrow A\} \subset \mathcal{R}. \tag{2.7}$$

Remark. $\mathcal{R}^{\uparrow}$ is the set of all $A \in \mathcal{P}(X)$ that can be expressed as a countable many unions of sets in $\mathcal{R}$.

In general, $\mathcal{R}^{\uparrow}$ is not a set of rings.

Definition 2.7. Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets and $\mu : \mathcal{R} \to [0, \infty]$ a premeasure. For $A_k \uparrow A$ with $A_k \in \mathcal{R}$ for $k \in \mathbb{N}$ define

$$\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty],\ A \mapsto \tilde{\mu}(A) := \lim_{k\to\infty} \mu(A_k). \tag{2.8}$$

$\tilde{\mu}$ is called the first extension of the premeasure μ.

Remark. In general, $\tilde{\mu}$ is not a premeasure as $\mathcal{R}^{\uparrow}$ need not be a ring of sets.

$\tilde{\mu}$ restricted on $\mathcal{R}$ is identical with μ, i.e. $\tilde{\mu}\big|_{\mathcal{R}} \equiv \mu$.

Lemma 2.7.1. The first extension $\tilde{\mu}$ is well-defined.

Proposition 2.7.1 (Properties of $\mathcal{R}^{\uparrow}$)**.**

Proposition 2.7.2 (Properties of the First Extension)**.**

Definition 2.8 (Second Extension or the Outer Measure)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ be a ring of sets, $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$, and $\tilde{\mu} : \mathcal{R}^{\uparrow} \to [0, \infty]$ the first extension of μ on $\mathcal{R}^{\uparrow}$. Moreover, let $B \subset X$ be a subset of X. Then, the map

$$\mu^* : \mathcal{P}(X) \to [0, \infty],\ B \mapsto \mu^* := \inf \left\{ \tilde{\mu}(A) \mid A \in \mathcal{R}^{\uparrow},\, A \supset B \right\} \tag{2.9}$$

is called the outer measure induced by $\tilde{\mu}$ on $\mathcal{P}(X)$.

Proposition 2.8.1 (Properties of the Second Extension)**.**

Proposition 2.8.2 (Properties of the Outer Measure)**.**

Definition 2.9 (Lebesgue Outer Measure)**.** Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ the Lebesgue premeasure. The map

$$\lambda^* : \mathcal{P}(\mathbb{R}^n) \to [0, \infty],\, B \mapsto \lambda^*(B) := \inf \left\{ \tilde{\lambda}^n(B) \mid A \in \mathcal{Q}(\mathbb{R}^n)^\uparrow,\, A \supset B \right\} \tag{2.10}$$

is called the Lebesgue outer measure induced by $\tilde{\lambda^n}$.

Definition 2.10 (Pseudo Metric)**.** Let X be a set. A map $d : X \times X \to \overline{\mathbb{R}}$, $(x, y) \mapsto d(x, y)$ is called pseudo metric on X if for all $x, y, z \in X$ it is the following three axioms are met.

1. $x = y \Rightarrow d(x, y) = 0$.
2. $d(x, y) = d(y, x)$. (Symmetry.)
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Proposition 2.10.1. The outer measure induces a pseudo metric, i.e.

$$d_{\mu^*} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty],\, (A, B) \mapsto d_{\mu^*}(A, B) := d_{\mu^*}(A \triangle B) \tag{2.11}$$

is a pseudo metric.

Proposition 2.10.2. The outer measure is continuous.

Definition 2.11 (Approximation through elements of Rings)**.** Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings, $\mu : \mathcal{R} \to [0, \infty]$ a premeasure on $\mathcal{R}$, and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the outer measure induced by μ. Then, $A \in \mathcal{P}(X)$ is called $\mathcal{R}$-approximatable in respect to μ^* if for all $\epsilon > 0$ there exists an $B \in \mathcal{R}$ such that $d_{\mu^*}(A, B) = \mu^*(A \triangle B) < \epsilon$.

Theorem 2.11.1. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra and $\mu : \mathcal{A} \to \mathbb{R}_0^+$ a finite premeasure. Let the first extension $\tilde{\mu} : \mathcal{A}^\uparrow \to \mathbb{R}_0^+$ also be finite and $\mu^* : \mathcal{P}(X) \to \mathbb{R}_0^+$ the outer measure. Then,

$$\hat{\mathcal{A}} := \{A \in \mathcal{P}(X) \mid A \text{ is } \mathcal{A}\text{-approximatable with } \mu^*\} \tag{2.12}$$

is a σ-algebra on X.

Theorem 2.11.2. Let $\mu, \tilde{\mu}, \mu^*$ and $\mathcal{A}, \mathcal{A}^\uparrow, \hat{\mathcal{A}}$ be given. Then, a finite premeasure $\mu : \mathcal{A} \to \mathbb{R}_0^+$ can be uniquely extended to a finite measure $\hat{\mu} : \hat{\mathcal{A}} \to \mathbb{R}_0^+$ where $\hat{\mu} \equiv \mu^*\big|_{\hat{\mathcal{A}}}$.

Theorem 2.11.3. Let $\mathcal{R} \subset \mathcal{P}(X)$ a set of rings and $\mu : \mathcal{R} \to [0, \infty]$ a σ-finite premeasure on $\mathcal{R}$ and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the outer measure induced by μ. Then, μ can be uniquely extended to a measure $\hat{\mu} : \sigma(\mathcal{R}) \to [0, \infty]$ where $\hat{\mu} \equiv \mu^*\big|_{\sigma(\mathcal{R})}$.

Definition 2.12. Let $\lambda^n : \mathcal{Q}(\mathbb{R}^n) \to \mathbb{R}_0^+$ a σ-finite Lebesgue premeasure. In this chapter, we constructed a unique extension of λ^n on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathbb{R}^n)$, the Lebesgue-Borel measure $\hat{\lambda} : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$.

2.4 Measure Space

Definition 2.13. Let $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra. The tupel $X, \mathcal{A}$ is called measurable space and the sets in the σ-algebra $A \in \mathcal{A}$ are called measurable sets.

Morover, let $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{P}(X)$. Then, $(X, \mathcal{A}, \mu)$ a measure space.

Definition 2.14 (Null Sets)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ the induced outer measure. Then $N \subset X$ with $\mu^*(N) = 0$ is called null set.

For $X = \mathbb{R}^n$ with $\lambda^n(N) = 0$ called Lebesgue null set.

$S = \varnothing$ is called the trivial null set.

Definition 2.15 (Completion of a Measure Space)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. This measure space is called complete if all null sets are included in $\mathcal{A}$, i.e. for all $N \in \mathcal{A}$

$$\mu^* N = 0 \Rightarrow N \in \mathcal{A}. \tag{2.13}$$

Definition 2.16. Let

$$\overline{\mathcal{A}}^{\mu} := \{A \cup N \mid A \in \mathcal{A},\, N \subset X \text{ with } \mu^*(N) = 0\} \tag{2.14}$$

then $\overline{\mathcal{A}}^{\mu}$ is called the completion of $(X, \mathcal{A}, \mu)$.

Definition 2.17. The completion of the Lebesgue-Borel measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \hat{\lambda}^n)$ to $(\mathbb{R}^n, \mathcal{B}^{\hat{\lambda}^n}(\mathbb{R}^n), \hat{\lambda}^n)$ or shorter $(\mathbb{R}^n, \overline{\mathcal{B}}^{\lambda}(\mathbb{R}^n), \lambda^n)$ is called the (completed) Lebesgue measure space.

$B \in \overline{\mathbb{B}}^{\lambda}(\mathbb{R}^n)$ is called Lebesgue measurable to differentiate from $B \in \mathcal{B}(\mathbb{R}^n)$ Borel measurable.

Part II

Lebesgue Integral

2.5 Measurable Maps

There is measurable, Borel measurable and Lebesgue measurable.

Definition 2.18 (Measurable Function)**.** Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces. A map $f : X \to Y$ is called measurable if the pre-image of every measurable subset of Y under f is measurable subset of X, i.e.

$$B \in \mathcal{A}_Y \Rightarrow f^{-1}(B) \in \mathcal{A}_X. \tag{2.15}$$

Definition 2.19. Let $(X, \mathcal{A}_X)$ be a measurable space. A function $f : X \to \overline{\mathbb{R}}$ is called measurable if it is measurable with respect to the Borel σ-algebra on $\overline{\mathbb{R}}$ associated to the standard topology.

Definition 2.20 (Borel Measurable Maps)**.** Let $X, \mathcal{U}_X$ and $Y, \mathcal{U}_Y$ be topological spaces. A map $f : X \to Y$ is called Borel measurable if the pre-image of every Borel measurable subset of Y under f is a Borel measurable subset of X.

Definition 2.21 (Pushforward)**.** Let $f : X \to Y$ be any map. Then the set

$$f_*\mathcal{A}_X := \{B \subset Y \mid f^{-1}(B) \in \mathcal{A}_X\} \tag{2.16}$$

is a σ-algebra on Y, called the pushforward of $\mathcal{A}_X$ under f.

Theorem 2.21.1. Let $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$, and $(Z, \mathcal{A}_Z)$ be measurable spaces.

1. The identity map $\mathrm{id}_X : X \to X$ is measurable.
2. If $f : X \to Y$ and $g : Y \to Z$ are measurable maps then so is the composition $g \circ f : X \to Z$.
3. A map $f : X \to Y$ is measurable if and only if $\mathcal{A}_Y \subset f_*\mathcal{A}_X$.
4. A map $f : X \to Y$ is measurable if and only if the pre-image of every oben subset $V \subset Y$ under f is measurable, i.e.
$$V \in \mathcal{U}_Y \Rightarrow f^{-1}(V) \in \mathcal{A}_X. \tag{2.17}$$
5. Assume $\mathcal{U}_X \subset \mathcal{P}(X)$ is a topology on X such that $\mathcal{A}_X$ is a Borel σ-algebra of $(X, \mathcal{U}_X)$. Then every continuous map $f : X \to Y$ is (Borel) measurable.
6. Let $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ be a function. Then f is measurable if and only if $f_i : X \to \mathbb{R}$ is measurable for each i.

Theorem 2.21.2. Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \overline{\mathbb{R}}$ be any function. Then the following are equivalent.

- f is measurable.
- $f^{-1}((a, \infty])$ is a measurable subset of X for every $a \in \mathbb{R}$.
- $f^{-1}([a, \infty])$ is a measurable subset of X for every $a \in \mathbb{R}$.
- $f^{-1}([-\infty, b))$ is a measurable subset of X for every $b \in \mathbb{R}$.
- $f^{-1}([-\infty, b])$ is a measurable subset of X for every $b \in \mathbb{R}$.

Lemma 2.21.1. Let $(X, \mathcal{A})$ be a measurable space and let $u, v : X \to \mathbb{R}$ be measurable functions. If $\phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous then the function $h : X \to \mathbb{R}$, defined by $h(x) := \phi(u(x), v(x))$ for $x \in X$, is measurable.

Theorem 2.21.3. Let $X, \mathcal{A}$ be a measurable space.

1. If $f, g : X \to \mathbb{R}$ are measurable functions then so are the functions
$$f + g, \qquad fg, \qquad \max\{f, g\}, \qquad |f|. \tag{2.18}$$

2. Let $f_k : X \to \overline{\mathbb{R}}$, $k \in \mathbb{B}$ be a sequence of measurable functions. Then the following functions from X to $\overline{\mathbb{R}}$ are measurable

$$\inf_k f_k, \qquad \sup_k f_k, \qquad \limsup_{k\to\infty} f_k, \qquad \liminf_{k\to\infty} f_k. \tag{2.19}$$

Theorem 2.21.4. Let $(\Omega, \mathcal{A})$ be a measurable space, and $\mathcal{B} = \sigma(\mathcal{E})$ for a generator $\mathcal{E} \subset \mathcal{P}(\Omega)$. If for all $E \in \mathcal{E}$ it is $f^{-1}(E) \in \mathcal{A}$, then f is measurable.

Example 2.21.1. Let $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ defined as

$$f(x) := \begin{cases} 1 x \in Q \\ -1 x \notin Q \end{cases} \tag{2.20}$$

for a $Q \notin \mathcal{B}(\mathbb{R})$. Then, $f^{-1}(1) = Q \notin \mathcal{B}$ and therefore, f is not measurable even though $|f| = 1$ is measurable.

2.6 Lebesgue Integral

Part III

Applications

Part IV

More Theory

Chapter 3

Lebesgue Space

3.1 Lebesgue Space

Definition 3.1 (L^p-Norm)**.** Let $X, \mathcal{A}, \mu$ a measure space, and $f : X \to \overline{\mathbb{R}}$ measurable. Then for $p \in [1, \infty)$ the L^p-norm is defined as

$$\|f\|_p := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}. \tag{3.1}$$

Theorem 3.1.1 (Holder Inequality)**.** Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$. Let $f, g : X \to \overline{\mathbb{R}}$ measurable. Then we have

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q \tag{3.2}$$

Theorem 3.1.2 (Minkowski Inequality)**.** Let $f, g : X \to \overline{\mathbb{R}}$ measurable and $f + g$ well defined on X. Then

$$\forall p \in [1, \infty) : \|f + g\|_p \leq \|f\|_p + \|g\|_p \tag{3.3}$$

Definition 3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty)$. Define

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \to \mathbb{R} \middle| f \text{ is } \mathcal{A}\text{-measurable and } \|f\|_p < \infty \right\}. \tag{3.4}$$

Also define

$$L^p(\mu) := \mathcal{L}^p(\mu)/{\sim}\mu \tag{3.5}$$

Where the equivalent relation means two functions are equivalent iff they agree on every point outside of null sets.

3.2 Convergence Theorems

Theorem 3.2.1 (Lebesgue Monotone Convergence Theorem)**.** *Also called the theorem of Beppo Levi.* Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of measurable functions such that

$$f_n(x) \leq f_{n+1}(x) \tag{3.6}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Define $f : X \to [0, \infty]$ by

$$f(x) := \lim_{n \to \infty} f_n(x). \tag{3.7}$$

Then f is measurable and

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu. \tag{3.8}$$

Theorem 3.2.2 (Lebesgue Dominated Convergence Theorem)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, let $g : X \to \mathbb{R}_0^+$ be an integrable function, and let $f_n : X \to \mathbb{R}$ be a sequence of integrable functions satisfying

$$|f_n(x)| \leq g(x) \tag{3.9}$$

for all $x \in X$ and $n \in \mathbb{N}$ and converging pointwise to $f : X \to \mathbb{R}$, i.e.

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \in X. \tag{3.10}$$

Then f is integrable and, for every $E \in \mathcal{A}$,

$$\int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu. \tag{3.11}$$

3.3 Convergence

Definition 3.3. Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. For all $m \in \mathbb{N}$ let $f_m : X \to \overline{\mathbb{R}}$ be a sequence of function, and let $f : X \to \overline{\mathbb{R}}$. f_m converges to f almost everywhere, written $f_m \to^{a.e.} f$, if there exists a null set $N \subset X$ such that for all $x \in X \setminus N$ it is
$$\lim_{m\to\infty} f_m(x) = f(x). \tag{3.12}$$

2. For all $m \in \mathbb{N}$ let $f_m : X \to \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ and let $f : X \to \overline{\mathbb{R}}$ also with $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. f_m is L^1-convergent to f, written $f_m \to^{L^1} f$, if
$$\lim_{m\to\infty} \|f - f_m\|_{L^1} = 0. \tag{3.13}$$

3. For all $m \in \mathbb{N}$ let $f_m : X \to \overline{\mathbb{R}}$ with $f_m \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. $(f_m)_{m\in\mathbb{N}}$ is called L^1-Cauchy sequence if for all $\epsilon > 0$ there exists a $m_0(\epsilon)$ such that for all $m, k \geq m_0(\epsilon)$ it is $\|f_m - f_k\|_{L^1} < \epsilon$.

Proposition 3.3.1 (Properties of Convergence)**.** 1. Let $f_m \to^{a.e.} f$ and $f_m \to^{a.e.} g$, then $f = g$ almost everywhere.

2. Let $f_m \to^{L^1} f$ and $f_m \to^{L^1} g$, then $f = g$ almost everywhere.

3. Let $f_m \to^{L^1} f$, then $((f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence.

Chapter 4

Fourier

4.1 Fourier Series

Definition 4.1. Let Y be a set and $f : \mathbb{R} \to Y$ be a function. f is called periodic with periodicity $L \in \mathbb{R}^+$ if for all $x \in \mathbb{R}$ it is $f(x + L) = f(x)$.

Remark. In the following, if the periodicity of the function is not given, let it be 2π.

Definition 4.2. For all $k \in \mathbb{N}$ let $a_k, b_k \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \tag{4.1}$$

is called the trigonometric polynomial of the order n.

Remark.

- f sets the constants a_k and b_k uniquely.
- f is indeed a polynomial with the degree $2n$.

Definition 4.3. Let $u, v : [a, b] \to \mathbb{R}$ integratable. Then $\varphi : [a, b] \to \mathbb{C}$, $x \mapsto \varphi(x) := u(x) + iv(x)$ integratable with

$$\int_a^b \varphi(x) := \int_a^b u(x)\,\mathrm{d}x + i \int_a^b v(x)\,\mathrm{d}x. \tag{4.2}$$

Theorem 4.3.1. something

Definition 4.4 (Fourier Series)**.** Let $f : \mathbb{R} \to \mathbb{C}$ periodic and integratable on $[0, 2\pi]$. Then the constants

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{ikx}\,\mathrm{d}x \tag{4.3}$$

are called the Fourier-coefficients of f. The series

$$\mathcal{F}[f](x) := \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{4.4}$$

is called the Fourier-series of f.

4.2 Fourier Integrals

Definition 4.5. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ be L-integrable. Then, $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ with

$$\xi \mapsto \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-i<x,\xi>} \,\mathrm{d}x \tag{4.5}$$

$$:= \int f(x) e^{ix\xi} \tilde{\mathrm{d}x} \tag{4.6}$$

$$:= \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \frac{\mathrm{d}^n x}{(2\pi)^{\frac{n}{2}}} \tag{4.7}$$

Example 4.5.1. 1.

$$f(x) := e^{-\frac{1}{2}\|x\|^2} \Rightarrow \tag{4.8}$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|^2} e^{-ix\xi} \frac{\mathrm{d}^n x}{(2\pi)^{\frac{n}{2}}} \tag{4.9}$$

$$= \int_{\mathbb{R}^n} e^{-\frac{1}{2}x_1^2} \dots e^{-\frac{1}{2}x_n^2} \cdot e^{-ix_1\xi_1} \dots e^{-ix_n\xi_n} \frac{\mathrm{d}^n x}{(2\pi)^{\frac{n}{2}}} \tag{4.10}$$

$$= \prod_{k=1}^{n} \left(\int_{\mathbb{R}} e^{-\frac{1}{2}x_k^2} e^{-ix_k\xi_k} \frac{\mathrm{d}^n x_k}{(2\pi)^{\frac{n}{2}}} \right) \tag{4.11}$$

$$= \prod_{k=1}^{n} e^{\frac{1}{2}\xi_k^2} \tag{4.12}$$

$$= e^{-\frac{1}{2}\|\xi\|^2} \tag{4.13}$$

In the last step, we used

$$\frac{1}{2\pi} \int_{\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{itx} \,\mathrm{d}x = e^{\frac{1}{2}t^2} \tag{4.14}$$

So it is $f = \hat{f}$.

2. Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^*$ and $g(x) := f(\alpha x)$. Let $y = \alpha x$ and use the transformation formula.

$$\Rightarrow \hat{g}(\xi) = \int f(\alpha x) e^{-ix\xi} \frac{\mathrm{d}^n x}{(2\pi)^{\frac{n}{2}}} \tag{4.15}$$

$$= \frac{1}{|\alpha|} \int f(y) e^{-\frac{1}{\alpha} y \cdot \xi} \frac{\mathrm{d}^n y}{(2\pi)^{\frac{n}{2}}} \tag{4.16}$$

$$= \frac{1}{|\alpha|^n} \hat{f}\left(\frac{\xi}{\alpha}\right) \tag{4.17}$$

3. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) := e^{-|x|}$

$$\Rightarrow \hat{f} = \int_{\mathbb{R}} e^{-|x|} e^{-ix\xi} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_0^{\infty} e^{-x} \left(e^{-ix\xi} + e^{ix\xi}\right) \tag{4.18}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x(i+i\xi)}}{-1-i\xi} + \frac{e^{-x(1-i\xi)}}{-1+i\xi} \right]_{x=0}^{x=\infty} \tag{4.19}$$

$$= \frac{1}{\sqrt{2pi}} \left(\frac{1}{1+i\xi} + \frac{1}{1-i\xi} \right) \tag{4.20}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2} \tag{4.21}$$

Proposition 4.5.1. Let $f, g \in \mathcal{L}^1(\mathbb{R}^n)$

1. Moreover, let