# McMeekin Hill: A new triangle, like Pascal's, bijectively enumerates all paths across any lattice which allows diagonals

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## 1 Abstract

Pascal's Triangle ("PT") had prior discovery in Asia but the eponymous attribution stands. PT enables many basic identities to be explored in the field of enumerative combinatorics and related geometric problems. One such identity, is that PT contains binomial coefficients which enumerate "Manhattan" paths across a lattice. We mean paths which may only go up or right stepwise. That is helpful because recursive additions which build PT are polynomially easier than manually counting possible paths. Each numerical element of PT is paired with exactly one node of the lattice and vice versa and there are no unpaired elements between them. Hence the path enumeration and lattice nodes are bijective. Lattices can represent orders of sets, and so the bijection allows geometric consideration of orders of sets. This is well established by authorities such as De Bucchianico, Knuth and Stanley. To our knowledge there was no easy way to bijectively enumerate paths across a diagonal lattice. In this paper we show one. This author requests it be called "McMeekin Hill", ("MMH") per the triangular shape of hills[1]. PT is rich in applications and we hope MMH is too. The prior authorial context of the current paper is quantifying dis-information in mandatory reporting by hospitals concerning patient restraints. These depended on Mann Whitney U. Mann Whitney U is in practice approximated when large using the normal distribution. Such estimation in turn distorts the count of disinformation, motivating a combinatorially exact treatment[2], which this paper is intended to meet[3].

[1] in honour of his doctoral advisors McMeekin and Hill

[2] perhaps this is like the former situation where Fishers Exact is more exact than the Chi Squared but was for some time computationally more difficult

[3] Separate papers are in late draft on that, forming part of the author's doctoral studies.

# 2 Basic pre-existing Theorem 1: Pascal-Manhattan lattice bijection

This paper's proof is aided by a brief preface. A description of PT's history, relationship to binomials and "almost bewildering" number of identities is given in Concrete Mathematics (Graham et al., 1994, Ch.5). The topic is rich enough, for example, for the chapter to support a "top ten" table of identities, Table 174. Drawing PT, which we shall do, is sufficient to our own descriptive need and to support a brief proof by analogy which extends to MMH.

A discussion of the bijection between number of paths from a node in a lattice and PT can be found in the enumerative combinatorics textbook by Stanley (Stanley, 2011, 1.2.1). He calls it *inter alia* "an important geometric interpretation", and "the most basic result in the vast subject lattice path enumeration". We might phrase this basic result as a theorem which we call Theorem 1 as follows. Take as given by the construction of Pascal's triangle:

- At apex of Pascal's triangle is 1.
- 1 borders each layer of Pascal's triangle.
- Each 1 has $n = 1$ parent of value 1.
- Let $c$ be any number other than 1 in Pascal's triangle.
- In constructing layers $c$ is determined $c$ as the sum of two parents $p_a + p_b$.

**Theorem 1** *In Pascals's Triangle the number of possible paths from c to the apex via parents and grandparents is c.*

This "most basic result" is quite intuitive. Hand searches of the whole book yield no discussion of a bijection which enumerates paths across diagonals, nor do searches in the literature elsewhere, nor enquiring to some experts in the field.

## 2.1 Demonstrating Theorem 1

Though some use $m, n$ for set numbers, including the founding MWU paper, we use $s, t$ following the notation of Knuth. The number of paths across a lattice of dimensions $s, t \mid n = s + t$ without diagonals, also called "Manhattan" paths, relates to the notation "n choose t":

$$\frac{n!}{t!(n-t)!} = \binom{n}{t} = {}^{n}C_t = C_n^t$$

$$\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & \\
 & & 1 & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & \\
 & 1 & & 4 & & \mathbf{6} & & \mathbf{4} & & 1 & \\
1 & & 5 & & 10 & & \mathbf{10} & & 5 & & 1
\end{array}$$

Figure 1: Pascal's triangle with some bold numbers

Note the bold numbers $1, 2, 3, 4, 6, 10$ in a grouping, each the sum of two parents. We may confirm by brute force that there are two ways to get from the 2 to to the apex; 10 steps from the either visible 10 to the apex. Intuitively each number adds all and only the paths of its two parents. These correspond to a lattice Figure 2 when the part of PT with the numbers **above** is diagonally "tilted" 45°.

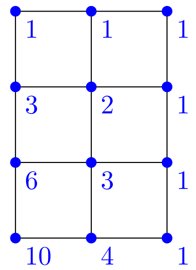



Figure 2: Rotated Pascal's triangle becomes enumeration of Manhattan paths

## 2.2 Statistical relevance

Our own immediate interest in PT stems from the bijection's relevance to Mann Whitney U ("MWU")test. That is a canonical test comparing order or median of two groups (Mann and Whitney, 1947). De Bucchianico exploited the bijection in Manhattan lattices with relevance to MWU (Bucchianico, 1999), noting for example that the metric $U$ giving MWU its name corresponds to the area cut off by the path. See Figure 1 where paths representing a set $s$ and $t$ in various orders with $U = 3$. Manually counting paths amounts to finding all combinations of sets, and needs explosively increasing computation, described since the 1970s as "cumbersome" (Lehmann, 1974, p.8). For our own part as above we have based other methods on Mann Whitney U (Reid and Price, 2022), (Reid, 2022). In essence, the number of paths gives the denominator of a Rational which determines a probability in MWU.

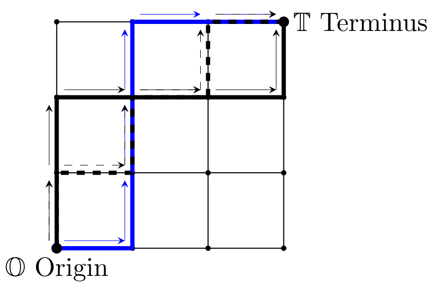



Figure 3: Three paths in $s3t3$ with $U_3$

# 3 Lattices which allow diagonal paths

This geometric topic benefits from illustrations. These are lattices which allow diagonal paths.

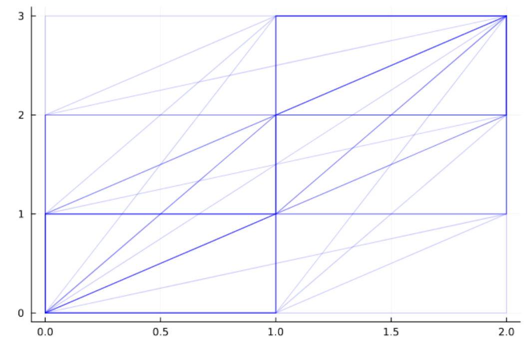



Figure 4: $s2t3$ lattice

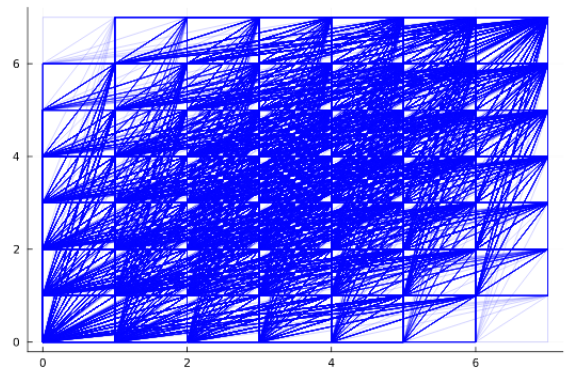



Figure 5: $s7t7$ lattice

# 4 Brute force path counting with diagonals

Brute force enumerations generate a Pascal-like triangle for the number of unique diagonal paths across an $st$-space which allows diagonal traversal. The first author calls it the "McMeekin-Hill" triangle, or simply "McMeekin Hill", ("MMH")[4].

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | **1** | | | | | |
| | | | | **1** | | **1** | | | | |
| | | | **1** | | **3** | | **1** | | | |
| | | 1 | | **6** | | **6** | | **1** | | |
| | 1 | | 10 | | **17** | | **10** | | 1 | |
| 1 | | 15 | | 39 | | **39** | | 15 | | 1 |

Table 1: McMeekin Hill

[4]for his doctoral advisors Prof Peter McMeekin and Associate Prof Michael Hill

## 4.1 Intuitively seeking an additive recurrence

As with PT we rotate MMH to give a lattice enumerating paths.

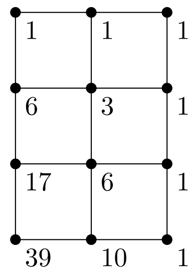



Figure 6: McMeekin Hill rotated enumerates all paths for each point

Pascal's triangle has a recurrence based on addition. MMH looks like it has a similar recursion.

## 4.2 The additive recurrence for McMeekin Hill

See Figure 7 and 8. For MMH, each cell adds those that are "seen" taking into account "occlusion". Point (-1,-1) "sees" $1, 1, 1$ when it "looks" North-East and summing them is assigned value #3. Point (-2,-3) "sees" $sum(1, 1, 1, 3, 6, 10, 17)$ and is allocated 39. As with the classical binomial correspondence, the row and column of ones are consistent with the one path through a multi-set $s$ or $t$, and see only one parent each, recursively. There is only #1 path across the point (-0,-0).

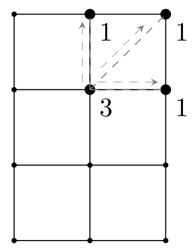



Figure 7: Paths visible from (-1,-1)

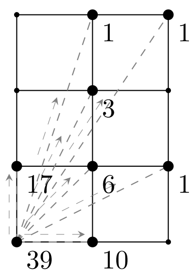



Figure 8: Paths visible from (-2,-3)

### 4.3 Continuing the recurrence

PT elements, being restricted to Manhattan paths, in absence of ties, "see" only two parents. MMH elements are allowed diagonal sight and "see" all non-occluded parents. Analogous to "n choose t" we propose "n see t". Analogous to tall curled brackets we propose tall curled brackets with a circle, denoting an eye and angular options, otherwise preserving binomial notation. Thus we know that for an $s_1t_2$ space there are #6 possible paths including diagonals, without drawing them. Similarly $\# \circ 28 \mid s_1t_6$, and $\# \circ 4197 \mid s_4t_6$. Or, as below, $\# \circ 219,736 \mid s_7t_7$. It is our McMeekin Hill rotated by 45°. The recursion is based on simple addition and can be achieved with unsigned integers.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 36 | 28 | 21 | 15 | 10 | 6 | 3 | 1 |
| 346 | 222 | 135 | 76 | 39 | 17 | 6 | 1 |
| 1997 | 1100 | 566 | 266 | 111 | 39 | 10 | 1 |
| 8673 | 4197 | 1876 | 757 | 266 | 76 | 15 | 1 |
| 31206 | 13469 | 5321 | 1876 | 566 | 135 | 21 | 1 |
| 97971 | 38131 | 13469 | 4197 | 1100 | 222 | 28 | 1 |
| 276913 | 97971 | 31206 | 8673 | 1997 | 346 | 36 | 1 |

### 4.4 Closed expression

$$\circ\binom{s}{t} = \circ^s C_t = \circ C_s^t$$

The idea benefits from a closed expression. The numbers referenced in the occlusion sum that occurs in each recursion, after the base case of $1, 1, 1...$ along the edges are those numbers which are on co-prime coordinates. See this diagram of occlusion rays. For example, $x', y' = 2, 2$ is occluded by $x, y = 1, 1$ and the co-prime pair $1, 1$ occludes the other. This is consistent with the convention that 1 is not prime; but it is co-prime with itself.

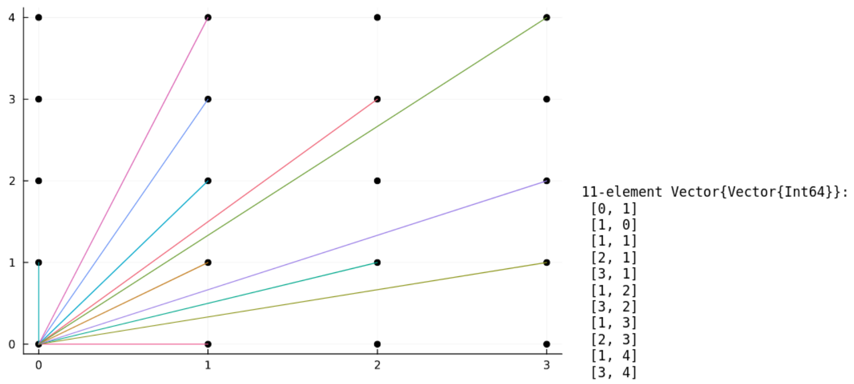


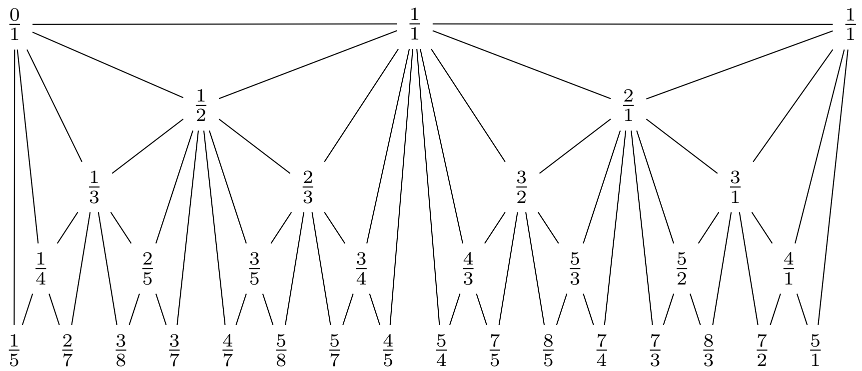

Figure 9: occlusion rays and coordinates

In these diagrams the coordinates are the increment to the $s, t$ element of equal dimensions to our lattice, i.e. the relative coordinates of the seen and occluded elements. We use the verb "s see t" analogously to "n choose k". For notation of co-primality we follow Concrete Mathematics, saying “x is prime to y” where $x, y$ are co-prime, and denote co-primality $x \perp y$ (Graham et al., 1994). The number of parents from a point is the sum of the paths of the points that are not occluded.

$$\sum_{x,y=0,0}^{s,t} x, y \mid x \perp y = \circ\binom{s}{t} = \circ^{s}C_t = \circ C_s^t$$

### 4.5 Correspondence with Stern-Brocot

As shown above the pairs of co-prime coordinates give the short occluded path and counting these gives the recursion. One way to do this is to make the Cartesian product of the series $1:s, 1:t$ and exclude pairs which have are not co-prime. They will have Greatest Common Denominator other than 1. As noted by Graham et al, there is common content with the Stern=Brocot tree. It may be computationally simpler to build a Stern-Brocot tree with largest node $\frac{t}{1}$. Conventionally $t$ is larger than $s$ if they differ. Here we have the Stern-Brocot tree with largest node $\frac{5}{1}$, i.e. 5, elegantly typeset by an anonymous user on StackExchange [5]




As with McMeekin Hill, this can be done with simple addition of integers. The current author greatly valuing both languages, has a an open repository with a Stern-Brocot tree generator implemented in Julia and Python at github keithreid-sfw.

# 5 Proposition 1: McMeekin Hill bjiectively enumerates paths which allow diagonals

The current author being a clinician with an interest in extending statistical measures is not a pure mathematician. He has tried to adhere to Appendix C of

[5] https://tex.stackexchange.com/questions/595573/stern-brocot-tree-where-each-node-has-two-fathers

Cumming's Long Form textbook[6] (Cummings, 2021) on proofs[7]. Take as given by the construction of McMeekin Hill:

- At apex of McMeekin Hill is 1.
- 1 borders each layer of McMeekin Hill.
- Each 1 has $n = 1$ parent of value 1.
- Let $c$ be any number other than 1 in McMeekin Hill.
- In constructing layers $c$ is determined $c$ as the sum of all numbers in rows $p_a + p_b ... p_\omega$ which can be seen by $c$ looking up towards the apex.

**Theorem 2** *In McMeekin Hill the number of possible paths passing through any ancestors from c to the apex via parents and grandparents is c.*

1. Let us say that any Manhattan path which can only go up or right is "monotonic", knowing it it is sometimes called "North-East".
2. Any monotonic Manhattan path starts at an origin point in the lattice $\mathbb{O}$.
3. Any monotonic Manhattan path ends at a terminus point in the lattice $\mathbb{T}$.
4. Any individual path is an option which traverses the space monotonically from its $\mathbb{O}$ to $\mathbb{T}$.
5. The base case of a point representing the possible orders of two empty sets has 1 way to traverse itself; we otherwise call it trivial.
6. Any non-trivial Manhattan Path makes $n = s + t$ ordered steps up or[8] right from $\mathbb{O}$.
7. The set of paths along the edge of PT are in a simple sense repeated straight lines in one direction and can only have one path; this geometric point bears combinatorial analogy because there is only one order of a non-empty set mixed with an empty-set.
8. Whether trivial or not $n = s + t$ because $n = 0$ only arises when $s = 0$ and $t = 0$.
9. Non-trivial paths have a first step.
10. For brevity, from here, "path" means "non-trivial path".
11. Generalising, progression from step $i$ to step $i + 1$ reduces or retains $i$'s enumeration by excluding the complement of the chosen options.
12. This remains true for lines which have an empty complement, i.e. "no option" but to progress linearly to $\mathbb{T}$ and hence suffer no reduction in options when they go from node enumerated 1 to node enumerated 1.

[6] The Red One

[7] thanks also to Prof De Bucchianico for gracious feedback on an unrelated and failed proof.

[8] exclusive, xor

13. For any path which is not a line or a point there is a choice between up and right.

14. The two options of movement "up" vs "right" have identity with two options of "reducing to the enumeration of options held by the point above" and "reducing to the enumeration of options held by the point right".

15. Working back from step $i+1$, to $i$ we make a reverse induction that the enumeration at $i$ must have the enumeration of paths which is the sum of the enumerations of points "up" and "right".

16. This bears recursion and gives us PT.

17. Hence we accept PT enumerates Manhattan Paths over a lattice which is Theorem 1.

18. Denote a step "up" 1 in the numerator but 0 in the denominator, relax about division by zero[9], and treat "up" if it is $\frac{1}{0}$ the parent entity in Stern-Brocot Tree.

19. Denote a step "right" 0 in the numerator but 1 in the denominator and treat as if it is $\frac{0}{1}$ the parent number in Stern-Brocot Tree.

20. Denote a step "up and right" $\frac{1}{1}$ and consider it equivalent to all $\frac{n}{n}$ which it occludes by visual proof and because the angle of $\frac{n}{n}$ must have identity with the angle of $\frac{1}{1}$.

21. Call any other step $\frac{a}{b}$ and consider it equivalent to all $\frac{na}{mb}$ which it occludes. Denote repeats of these simplest fractions by repeats of the simplest fractions.

22. By analogy with Pascal's triangle and our steps 1. to 17., any non-occluded step reduces the options from its own enumeration to those of its various monotonically visible nodes if steps to them.

23. It follows by analogy that the overall recursion enumerates all possible paths from $\mathbb{O}$ to $\mathbb{T}$ if diagonals are allowed, over our lattice of dimensions $s, t$.

24. By analogy this is is the number of possible unique orderings of two sets $s$ and $t$ if ties are allowed.

25. McMeekin Hill bjiectively enumerates paths which allow diagonals $\square$

The author's doctoral thesis[10] elaborate this in various ways helpful to extending MWU.

[9] We never execute the division we just do mediant addition

[10] undefended at time of writing in April 2023; the existing title above is medical degree

## 6 Discussion

This technique has been developed in parallel with other combinatorics advances aiding MWU which are in the grey literature awaiting viva. PT and binomial techniques have wide application as reflected in the summaries by Graham, Stanley, et al. The MMH was chosen for first submission due to potential wider interest than those interested in fine tuning MWU algorithms. In any event, it seems counting all diagonal paths is no longer combinatorially explosive. This author hopes that MMH will also have a rich future, which must be developed by others with more pure-mathematical expertise. The sums of rows of Pascal's triangle are the powers of 2. Those of MMH are $1, 2, 5, 14, 39, 110, 307, 860$. This yields no OEIS series. Fascinatingly, is mimics OEIS series A026135, which has relations to the geometrically interesting Motzkin numbers, for 6 terms then diverges: A026135 is $1, 2, 5, 14, 39, 110, *312*, *890*$.

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