Formal Political Theory

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Pol Sci 505

Overview

Introduction

Motivating example: The Presidential veto

- In the United States and many other places, the President must sign legislation passed by the legislature (or be overridden) before it becomes law.

- Does the presence of a Presidential veto substantially shape the laws that are eventually implemented?

From Chuck Cameron's <u>Veto Bargaining</u>:

> *Consider the extreme rarity of vetoes. Between 1945 and 1992, Congress presented presidents with over 17,000 public general bills.... From this flood of bills, presidents vetoed only 434. In other words, presidents in the post-war period vetoed only 25 public general bills per 1,000 passed....How can a weapon that is hardly ever used shape the content of important legislation under frequently occurring circumstances? (9)*

A simple bargaining model

To specify a model we must:

- Name the relevant players
- Specify what choices those players can make and how those choices lead to different outcomes
- Specify players' preferences over outcomes

Bargaining model

Players Congress and the President.

Choices and order of moves Congress moves first and chooses to pass a left-wing bill (L) or a moderate bill (M). The President moves second, sees what bill is chosen by Congress, and chooses to veto or not.

Outcomes If a bill is passed and not vetoed then that bill becomes law. If a bill is vetoed then it does not become law and the outcome is a status quo policy.

Preferences President prefers the moderate bill to the status quo but prefers the status quo to the left-wing bill. Congress prefers the left-wing bill to the moderate bill and the moderate bill to the status quo.

When does the President veto the bill?

- President can unilaterally kill any bill and implement the status quo.
- $\Rightarrow$ President's choice is between Congress's bill and the status quo.
- President's preferences:

$$\text{Moderate bill} > \text{SQ} > \text{Left-wing bill}$$

- *Rightarrow* President will use the veto when the left-wing bill is passed and not when the Moderate bill is passed.

What bill will Congress pass?

- Congress prefers the left-wing policy to the moderate policy and prefers the moderate policy to the status quo.

- Congress would therefore LIKE to pass the left-wing bill.

- However, Congress knows that the left-wing bill would be vetoed resulting in the status quo but the moderate policy would not be vetoed.

- $\Rightarrow$ Congress passes the moderate bill.

Some quick take-aways

- We predict NO VETOES yet the presence of a veto changes the final policy
- The reason is that Congress is thinking strategically and should try to anticipate the actions of the President
- We could complicate this much further (Congress might make errors, there might be conflicts within Congress, we could allow veto overrides) but this simple model makes this point nicely
- Broader point: We need a theory of the interaction we are studying in order to even begin to think about the meaning of the data we observe.

What are formal models?

This is two questions: What is a model? And, what makes a model formal?

Start with the first question

Models

- Loosely speaking a model is a representation of some target system which the analyst wishes to understand

Some types of models:

- Physical representation (a globe and a map are different models of the earth)
- Verbal analogy (e.g. Wlezien thermostatic model of public opinion)
- Mathematical system (e.g. the models we'll study in this class)

Models and accuracy

- Every model differs from its target system in some deliberate way.
  - Frictionless environments in physics
  - Two-species models in biology
  - Perfectly rational agents in a social scientific model

- A good model should also resemble the target system in some key ways, which will depend on the aspects of the target system the analyst wants to study

What makes a model formal?

Formal model:

- System of symbols and axioms
- Set of rules for manipulating them to derive results

Why?

- Anyone who applies the rules will derive the same results
- This is good! We'd like to accumulate and share knowledge
- We will study mathematical models. This is one type of formal model but not the only type (e.g. computational models are formal models).

Game theory

Most (not all) formal models in political science are game theoretic.

Game theory is the study of mathematical models of strategic interaction among two or more decision-makers.

Questions a game theorist asks about a problem

- Who are the relevant players?
- What decisions can these players make?
- What does each player know at the time that he or she must make a decision? What are the possible outcomes?
- What are players' preferences over these outcomes?

The answers to these questions will be used to create a mathematical system which we call a game.

The rules for making predictions from these games will come in the form of solution concepts which will be the main focus of our technical discussion.

- Questions about syllabus?

- Break.

Individual decision theory

A decision problem

Three components

- Actions that are available to the player
- Outcomes (i.e., consequences as the result of actions and possibly unforeseen/random events). Sometimes this is also referred to as alternatives.
- Preferences (i.e., how the player ranks the outcomes).

Actions

- Actions available to the decisionmaker is a set denoted by A.

- Example: A vote choice model
  - Voter can choose between voting for the Democrat, voting for the Republican, voting for a third-party candidate, or abstaining.
  - $A = \{d, r, t, a\}$

- Example: How much water to drink from a 1 gallon jug
  - Decisionmaker can choose any proportion of the jug from 0 to 1
  - $A = [0, 1]$

Outcomes

- Set of outcomes denoted by X
- Vote choice example:
  - $A = \{d, r, t, a\}$
  - Outcomes: the Democrat wins, the Republican wins, a third party candidate wins: $X = \{D, R, T\}$.

Preferences

- Preferences are defined over outcomes.

- Preferences defined as a binary relation over elements of X.

Binary relation?

- Technically, a set of ordered pairs of elements of X
- Relates any two elements of a set according to some criterion.
- Familiar example: Weak inequality relation "$\geq$" defined on the set of integers, where the expression $x \geq y$ is interpreted as "integer x is at least as big as the integer y."
- ...but back to preferences.

Preference relation

- relates any two outcomes in X as more or less desirable to the agent.
- Our notation: xRy is interpreted as “x is at least as good as y for the agent.
- May say “weak preference relation” xRy does not rule out indifference between x and y.

Preferences: Strict and weak

We can define strict preference and indifference using R:

$$xPy \text{ if and only if } xRy \text{ and not} yRx$$
$$xIy \text{ if and only if } xRy \text{ and } yRx.$$

Preference notation review page

Concept	My notation	Alternative notation	Analogy to numbers
Weak preference	R	$\succeq$	$\geq$
Strict preference	P	$\succ$	$>$
Indifference	I	$\sim$	$=$

Rationality

So far we have placed no restrictions on preferences whatsoever

Unfortunately, without more restrictions, things can get weird

Example: Things get weird

Consider our voter. The relevant outcomes are $X = \{d, r, t, a\}$ (democrat, republican, third party, abstain).

Consider the following strict preference relation:

$$dPr$$
$$rPt$$
$$tPa$$
$$aPd.$$

What should the agent choose?

$$dPr$$
$$rPt$$
$$tPa$$
$$aPd.$$

Consider each possibility:

- d? This cannot be best because aPd
- r? no, dPr
- t? no, rPt
- okay then, a! oh no! tPa!

There is no best choice!

Okay fine, let's make some assumptions

We want to make a sufficient number of assumptions to guarantee that we can find a maximal choice for the decision-maker

Otherwise we cannot say what the decision-maker would do

We will collectively refer to these assumptions as rationality

Maximal outcomes

Definition

An outcome $x \in X$ is a maximal element of X with respect to the binary relation R if and only if xRy for all $y \in X$.

Reflexivity and completeness

Definition

A binary relation R on X is

1. Reflexive if for all $x \in X$, xRx.
2. Complete if for all $x, y \in X$ such that $x \neq y$, either xRy or yRx (or both).

We need reflexivity and completeness

We want to find assumptions that guarantee that there is always a maximal choice (i.e. there is a maximal choice given any subset of X).

It turns out reflexivity and completeness are necessary.

Our example satisfied reflexivity and completeness, so obviously these two assumptions will not be sufficient.

Lemma

If R violates reflexivity or completeness then there exists some $S \subseteq X$ such that there is no maximal element of S with respect to R.

Proof.

Part 1:
Suppose R violates reflexivity.
Then there exists some $x \in X$ such that xRx does not hold.
This implies that x is not a maximal element of any set
Therefore there is no maximal element for the subset $S = \{x\}$ of X.
Part two: Now suppose that R violates completeness.
Then there exist some $x, y \in X$ such that $\neg xRy$ and $\neg yRx$.
This immediately implies that there is no maximal element of the subset $S = \{x, y\}$ with respect to R. $\square$

We have not yet captured what was wrong with out voting example.

In that example the voter’s preferences were inconsistent in a problematic way: I should not strictly prefer d to r and r to t but then strictly prefer t to d.

The next conditions deal with this problem

Consistency conditions

Definition

A binary relation R on X is:

1. Transitive if for all $x, y, z \in X$, xRy and yRz implies xRz.
2. Quasi-transitive if for all $x, y, z \in X$, xPy and yPz implies xPz.
3. Acyclic if for all $\{x, y, z, \ldots, u, v\} \in X$, $xPy \& yPz \ldots \& uPv$ implies xRv.

The are ordered from strong to weak: Transitivity implies quasi-transitivity and acyclicity but not the reverse, quasi-transitivity implies acyclicity but not the reverse

Example: Quasi-transitive but not transitive

Example

Let $X = \{x, y, z\}$ with xPy, yIz, zIx. Then R is quasitransitive (vacuuously). However, it is not transitive: yRz and zRx but not yRx. $\square$

Example: Acyclic but not quasitransitive

Example

Let $X = \{x, y, z\}$ with xPy and yPz and xIz. Then R is acyclic (since xRz) but not quasitransitive (since zRx). $\square$

Theorem

Let R be reflexive and complete and let X be finite; then the set of maximal outcomes is nonempty for all $S \subseteq X$ if and only if R is acyclic.

To prove this we need to prove both directions of the "if and only if": that is, we need to show that acyclicity is sufficent for a maximal choice and also that it is necessary.

Proof of sufficiency

For any $S \subseteq X$ choose $x \in S$. If, for all $s \in S$, xRs then x is a maximal element and we are done.

Otherwise (since R is reflexive and complete) there must exist $y \in S \backslash \{x\}$ such that yPx.

If for all $s \in S$, yRs then y is a maximal element and again the we are done. Otherwise, there exists $z \in S \backslash \{x, y\}$ such that zPy. By acyclicity, it must be that zRx as well.

If for all $s \in S$, zRs then again we are done. Otherwise there exists $w \in S \backslash \{x, y, z\}$ such that wPz. By acyclicity this means wRy and wRx as well.

Because X (and hence S) is finite, we can continue this logic to conclude that there must exist an alternative weakly preferred to all other alternatives in S. $\square$

Proof of necessity

Let $x_1, x_2, \ldots, x_n$ be elements of X and assume

$$x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n;$$

we wish to show that $x_1 R x_n$ if the set of maximal outcomes is nonempty.

Let $S = \{x_1, \ldots, x_n\} \subseteq X$ and suppose that the set of maximal elements of S with respect to R is nonempty.

Therefore. Because $x_{i-1} P x_i$ for $i = 2, \ldots, n$, we have that x_i is not maximal for any $i = 2, \ldots, n$.

Therefore, if the set of maximal elements is nonempty it must be the case that x_1 is maximal which implies in particular that $x_1 R x_i$ for all $i = 2, \ldots, n$; in particular $x_1 R x_n$ as required. $\square$

Rationality

- This result lays the foundation for a minimal set of requirements on preferences to guarantee that the agent has an optimal choice in any given situation.

- As shorthand, we will say an agent is rational if her preferences are reflexive, transitive, and complete.

- But we will typically make stronger assumptions to make the problem more tractable.

Utility

In practice, we would rather not work with binary relations all the time.

We therefore introduce utility, which lets us work with a mathematical function rather than an ordinal relation

Utility

A utility function is a function $u : X \to \mathbb{R}$ that assigns each outcome in X a real number

A utility function represents a preference relation R if the function assigns higher numbers to outcomes that are ranked higher under R

Definition

A utility function $u : X \to \mathbb{R}$ represents the preference relation R if for any pair $x, y \in X$, $u(x) \geq u(y)$ if and only if xRy.

Remarks on utility functions

1. No utility function <u>uniquely</u> represents a preference relation: since preference relations are only ordinal, any two functions preserving the same order represent the same preferences
   - e.g. if u represents the preference relation R, then so does any increasing transformation of u.

2. For the simplest settings with no uncertainty, the ordinal properties of the utility function will not matter for the solution.

3. We next turn to the question: When can preferences be represented by a utility function?

When can preferences be represented by a utility function?

Theorem

Let X be finite. Then any reflexive, complete, and transitive preference relation over X can be represented by a utility function.

Proof by construction

Let R be a reflexive, complete, and transitive preference relation over X.

Because the preference relation is complete and transitive, we can find a least-preferred outcome $\underline{x} \in X$ such that all other outcomes $y \in X$ are at least as good as $\underline{x}$.

Now define the "worst outcome equivalence set", denote X_1, to include $\underline{x}$ and any other outcome y for which the player is indifferent between y and $\underline{x}$.

Proof by construction (continued)

Then, from the remaining elements $X \backslash X_1$, define the "second worst outcome equivalence set," X_2, and continue in this fashion until the "best outcome equivalence set" X_n, is created.

Because X is finite and R is reflexive, transitive, and complete, such a finite collection of n sets exists.

Now consider n arbitrary numbers such that $u_n > u_{n-1} > \cdots > u_2 > u_1$, and assign payoffs according the rule: for any $k \in \{1, \ldots, n\}$ and $x \in X_k$, $u(x) = u_k$. This payoff function represents the preference relation R and therefore we have proven that such a function exists. $\square$

Example: Two simple decision problems

$X = A = \{a, b, c\}$

Utility function	$u(a) = 1, u(b) = 2, u(c) = 3$	$\tilde{u}(a) = 1, \tilde{u}(b) = \tilde{u}(c) = 3.$
Maximal elements	? $\{c\}$	? $\{b, c\}$

Example: Spatial policy choice

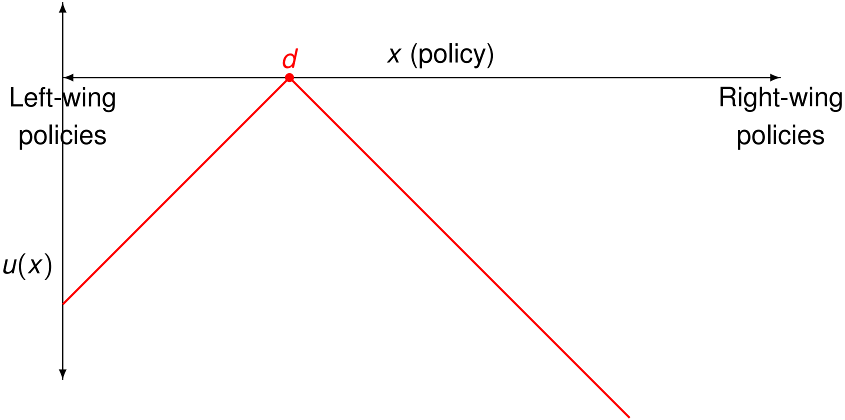
- Political scientists often represent policies and policy preferences as points on the real line.

- Let $x \in \mathbb{R}$ be a policy outcome. We think of smaller values of x as being more left-wing policies and larger values as being more right-wing policies.

- Let $d \in \mathbb{R}$ be the decisionmaker's ideal policy (often called an ideal point). This represents the policy the decisionmaker most prefers. The decisionmaker likes policies closer to d more than policies further away from d.

- We can represent these preferences with the utility function:

$$u(x) = -|x - d|$$



It's immediate that the decisionmaker's optimal policy choice is $x = d$.

Example: Allocating effort

- Consider a bureaucrat asked to allocate effort toward policy implementation
- The policy can be implemented poorly with little effort or implemented better with increasing effort
- The possible effort levels are the set of nonnegative real numbers (i.e. $X = [0, \infty)$.)
- The decisionmaker's preferences are represented by the following utility function:

$$u(x) = x - kx^2.$$

Example: Allocating effort (solution)

How do we solve for the maximal effort?

The first-order condition for a maximum is that the first derivative of the utility function with respect to x be equal to zero:

$$\frac{\partial u(x)}{\partial x} = 1 - 2kx = 0,$$

which is solved by $x = \frac{1}{2k}$.

The second-order condition to ensure that this is indeed a maximum is that the second derivative is negative:

$$\frac{\partial^2 u(x)}{\partial x^2} = -2k < 0,$$

which holds for all values of x since $k > 0$.

Next step: Decisionmaking under uncertainty

- So far we have only considered problems in which the agent faces no uncertainty

- In reality, decisionmakers are not always completely certain of the payoff consequences of their choices. We need to expand our framework to account for this.

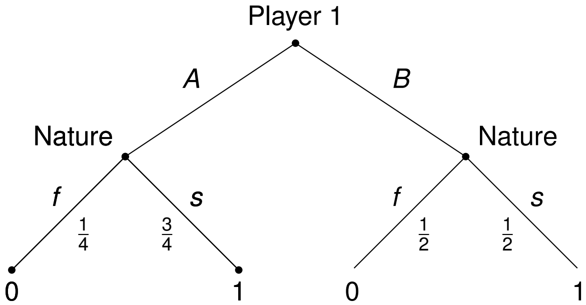
Motivating example: Choosing between two policies

- Two policies, A and B
- Policy A is successful with probability $\frac{3}{4}$ and unsuccessful with probability $\frac{1}{4}$. Policy B is successful with probability $\frac{1}{2}$ and unsuccessful with probability $\frac{1}{2}$.
- Payoff: 1 for a successful policy and 0 for an unsuccessful policy.

Motivating example: Choosing between two policies

- We can think of the decisionmaker as choosing between a lottery that offers a payoff of 0 with probability $\frac{1}{4}$ and of 1 with probability $\frac{3}{4}$ and another lottery that offers a payoff of 1 or 0 each with equal probabilities.

- We may think of this problem using a decision tree with moves by "Nature":


Player 1
A
B
Nature
Nature
f
s
f
s
1/4
3/4
1/2
1/2
0
1
0
1

Introducting lotteries

Definition (Lottery over finite outcomes)

A simple lottery over outcomes $X = \{x_1, x_2, \ldots, x_n\}$ is defined as a probability distribution $p = (p(x_1), p(x_2), \ldots, p(x_n))$, where $p(x_k) \geq 0$ is the probability that x_k occurs and $\sum_{k=1}^{n} p(x_k) = 1$.

Lotteries over infinite sets

Preferences over lotteries

- Conceptually there is no problem with simply defining preferences over lotteries and imposing the same rationality requirements as before

- In practice, it is more useful for us to think about preferences over lotteries as extensions of preferences over fundamental outcomes, combined with beliefs about probabilities

- To this end, we will think of agents as maximized <u>expected</u> utility

Expected utility

Definition

Let $u(x)$ be the player's utility function over outcomes $X = \{x_1, x_2, \ldots, x_n\}$, and let $p = (p_1, p_2, \ldots, p_n)$ be a lottery over X such that $p_k = \Pr[x = x_k]$. Then we define the player's <u>expected utility</u> from the lottery p as

$$\mathbb{E}[u(x)|p] = \sum_{k=1}^{n} p_k u(x_k).$$

Continuous version

Solving the two uncertain policies problem

- Expected utility from choosing A:

$$\mathbb{E}[u(x)|A] = \Pr[s|A]u(s) + \Pr[f|A]u(f) = \frac{3}{4}1 + \frac{1}{4}0 = \frac{3}{4}.$$

- Expected utility from choose B is

$$\mathbb{E}[u(x)|B] = \Pr[s|B]u(s) + \Pr[f|B]u(f) = \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}.$$

- $\Rightarrow$ A is the maximal choice

Allocating effort with uncertainty

- The bureaucrat allocates effort equal to $a \in [0, 1]$.
- The policy is successful ($x = 1$) with probability $\Pr[x = 1|a] = a$ and unsuccessful ($x = 0$) with probability $1 - a$.
- Bureaucrat's payoff:

$$u(x, a) = x - ka^2$$

Solving for optimal effort level

- Expected utility from any effort level a:

$$\begin{aligned}\mathbb{E}[u(x,a)|a] =& a(1-ka^2)+(1-a)(0-ka^2)\\ =& a-ka^2.\end{aligned}$$

- Differentiating and solving for the FOC gives: $a=\frac{1}{2k}$.

- Accounting for the fact that effort is constrained to be in [0, 1]: the agent gives maximum effort of $a=1$ for $k \leq \frac{1}{2}$ and gives $a=\frac{1}{2k}$ if $k>\frac{1}{2}$.

Remarks on expected utility

- The cardinal properties of the utility function matter now!
- The properties of u capture risk preferences. For continuous action sets, concavity of u implies risk aversion, convexity of u implies risk acceptance.
- This is not an innocuous assumption and there are other possible ways to represent preferences over lotteries. Real people sometimes do violate expected utility.

Dynamic decision problems

- We have analyzed only static problems
- Sometimes, we are interested in how a decisionmaker makes multiple interconnected choices at different points in time
- We will look in these cases for an entire plan that maximizes expected utility at every step.

Policy then effort choice

First, decisionmaker chooses between two policies:

- Safe policy: Implemented with no effort
- Risky policy: Better than the safe policy when it's successful, worse when it is not. Effort helps increase the likelihood of success.

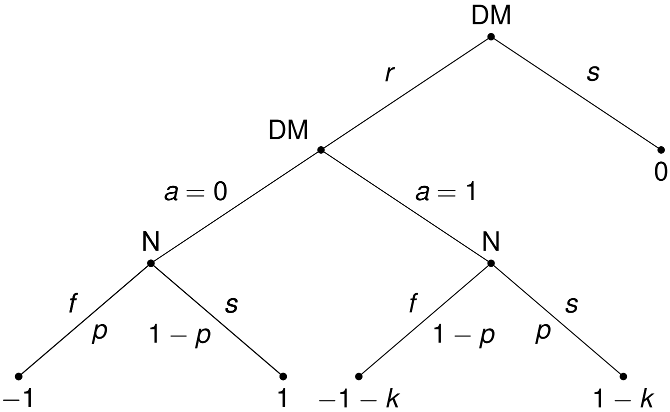
Second, decisionmaker chooses effort $a \in \{0, 1\}$.

The probability that the risky policy succeeds is $p > \frac{1}{2}$ if $a = 1$ and $1 - p$ if $a = 0$.

The policymaker's payoff is 0 from the safe policy and

$$u(r, a) = \begin{cases} 1 - ka & \text{if the policy succeeds} \\ -1 - ka & \text{if the policy fails} \end{cases}$$

from the risky policy, where $k > 0$ is a cost of effort.


DM
r
s
DM
0
a = 0
a = 1
N
N
f
s
f
s
p
1 − p
1 − p
p
−1
1
−1 − k
1 − k

Solving by backward induction

Start with the effort decision:

$$\mathbb{E}[u|r, a = 0] = p(-1) + (1 - p)1 = 1 - 2p.$$

$$\mathbb{E}[u|r, a = 1] = p(1 - k) + (1 - p)(-1 - k) = 2p - 1 - k.$$

$\Rightarrow$ policymaker makes an effort if

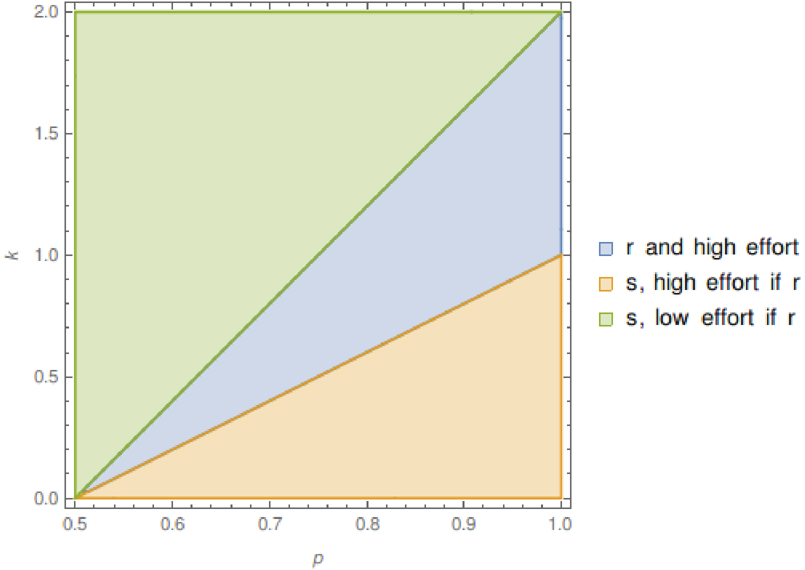
$$2p - 1 - k \geq 1 - 2p$$

Back one step: The policy decision

- Case 1: $k \leq 4p - 2$.
  - DM will make an effort if she chooses r, so her expected utility for choosing r is $2p - 1 - k$.
  - r is optimal in this case if $2p - 1 - k > 0$
  - $\Rightarrow$ if $2p - 1 < k \leq 4p - 2$ then the decisionmaker's optimal choice is s. If $k \leq 2p - 1$ then the optimal choice is r.

- Case 2: $k > 4p - 2$.
  - The agent would not make an effort after choosing r.
  - Then her expected utility for choosing r is $1 - 2p < 0$
  - Thus, the decisionmaker should always choose s in this case.

Solution pictured



Decision theory wrap up

- Rationality requirements: Acyclicity or transitivity, completeness, reflexivity
- Basic problems: Choose the maximal choice from a set
- Wrinkles: Uncertainty, dynamics

The rest of the class: Multiple decisionmakers (games)

Static games of Complete Information

A policy problem: Defending against natural disasters

Consider the mayor of a town: 

She must allocate resources to prepare for a potential natural disaster at two sites.

The sites:

	City Hall	The Lookout
Chances of disaster	75%	33%
Damages from disaster	50	200

Resource allocation lessens the damage in some way proportional to the amount of resources

- This is a decision-theoretic problem and is fairly easy to solve

- We know the probabilities and the outcomes, we just compute the expected damage and allocate funds to optimize the Mayor's objective function

For instance, if her objective is simply to minimize expected damages she should allocate all of the resources for the Lookout.

Related problem: Defending against a terrorist attack

Why is this problem so different?

Attacks are strategic!

The Mayor needs to think about what the bad kitties are going to do. Just as importantly...

The Mayor should assume that the cats are thinking about what she is going to do!

The solution must be different!

- Suppose Mayor Goodway devotes all of her resources to protecting the Lookout, meaning no damage will occur
- Then the Kitten Catastrophe Crew will for sure not attack the lookout and will instead target City Hall
- But then the original allocation of resources is no longer optimal!
- This is a game theoretic problem. Politics and policy is full of them. The rest of this class is about how to solve them.

Strategic form games

Definition

A game in strategic (or normal) form has three elements:

1. The set of players $N = \{1, 2, \ldots, n\}$, with a particular player denoted by $i \in N$.
2. A set of pure strategies S_i for each players, with the set of all pure strategies denoted $S = \{S_1, S_2, \ldots, S_n\}$.
3. A set of payoff functions $\{u_1, u_2, \ldots, u_n\}$ that give play i's von Neumann-Morgenstern utility $u_i(s)$ for every profile $(s_1, s_2, \ldots, s_n)$ of pure strategies.

Common knowledge

In a game of complete information, all aspects of the game are common knowledge

An event E is common knowledge if everyone knows E, everyone knows E, everyone knows that everyone knows E, and so on ad infinitum.

Important for how we understand games: I choose my actions anticipating that you understand the game, that you believe that I understand the game, that you believe that I believe that you understand the game, and so on.

Important example

A battle of wits

Example: Paper, Rock, Scissors

- Players: Player 1 and Player 2 ($N = \{1, 2\}$)
- $S_1 = S_2 = \{\text{Rock}, \text{Paper}, \text{Scissors}\}$
- Rock beats Scissors, Paper beats Rock, and Scissors beats Paper. Each player gets 1 for a win, -1 for a loss, and 0 for a draw.

Matrix Form

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

Strategies

- A mixed strategy σ_i for player $i \in N$ is a probability distribution over i's pure strategies S_i.
- i.e. $\sigma_i(s_i)$ is the probability that i's strategy assigns to s_i.
- Let Σ_i be the set of all of player i's mixed strategies

- Each pure strategy can be represented as a degenerate case of a mixed strategy: a strategy in which i always plays some action s is simply a mixed strategy assigning $\sigma_i(s) = 1$ and $\sigma_i(s') = 0$ for all $s' \neq s$.

Back to Rock, Paper, Scissors

Examples of mixed strategy profiles:

$$\sigma_1(\text{Rock}) = \frac{1}{3}, \sigma_1(\text{Paper}) = \frac{1}{3}, \sigma_1(\text{Scissors}) = \frac{1}{3}$$
$$\sigma_2(\text{Rock}) = \frac{1}{2}, \sigma_2(\text{Paper}) = \frac{1}{4}, \sigma_2(\text{Scissors}) = \frac{1}{4}.$$

$$\sigma_1(\text{Rock}) = 1, \sigma_1(\text{Paper}) = 0, \sigma_1(\text{Scissors}) = 0$$
$$\sigma_2(\text{Rock}) = 0, \sigma_2(\text{Paper}) = 0, \sigma_2(\text{Scissors}) = 1,$$

(ok just to write $s_1 = \text{Rock}, s_2 = \text{Scissors}$.)

Payoffs and expected payoffs given strategies

- $u_i(s)$ is player i's payoff when every player's action follows the strategy profile s.
  - e.g. $s = (\text{Rock}, \text{Scissors})$ is a particular strategy profile in the RPS game, for which we have $u_1(s) = 1$ and $u_2(s) = -1$
- Convention: For a profile s, let $s = (s_i, s_{-i})$ for some player i, where $s_{-i} \in S_{-i}$ is the strategy used by every player other than i
- Hence, $u_i(s) = u_i(s_i, s_{-i})$
- Similarly, (σ_i, σ_{-i}) can denote a particular mixed strategy profile

Expected payoffs from arbitrary mixed strategies

Player i's expected utility from choosing the pure strategy $s_i \in S_i$ when her opponents choose the mixed strategy $\sigma_{-i} \in \Sigma_{-i}$ is

$$\begin{aligned} U_i(s_i, \sigma_{-i}) &= \sum_{s \in S_{-i}} \Pr[s|\sigma_{-i}] u_i(s_i, s_{-i}) \\ &= \sum_{s \in S_{-i}} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s_i, s_{-i}). \end{aligned}$$

Expected payoffs from arbitrary mixed strategies (cont)

Player i's expected utility from playing the pure strategy σ_i when her opponents play σ_{-i} is therefore

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) U(_i s_i, \sigma_{-i}), \tag{1}$$

by the law of iterated expectations.

Back to rock, paper scissors

Consider this strategy:

$$\sigma_1(\text{Rock}) = \frac{1}{3}, \sigma_1(\text{Paper}) = \frac{1}{3}, \sigma_1(\text{Scissors}) = \frac{1}{3}$$
$$\sigma_2(\text{Rock}) = \frac{1}{2}, \sigma_2(\text{Paper}) = \frac{1}{4}, \sigma_2(\text{Scissors}) = \frac{1}{4}.$$

$$\begin{aligned} U_1(\text{Rock}, \sigma_2) =& \sigma_2(\text{Rock})0 + \sigma_2(\text{Paper})(-1) + \sigma_2(\text{Scissors})1 \\ =& \frac{1}{2}0 + \frac{1}{4}(-1) + \frac{1}{4}(1) \\ =& 0, \end{aligned}$$

$$\begin{aligned} U_1(\text{Paper}, \sigma_2) =& \sigma_2(\text{Rock})1 + \sigma_2(\text{Paper})0 + \sigma_2(\text{Scissors})(-1) \\ =& \frac{1}{2}1 + \frac{1}{4}(0) + \frac{1}{4}(-1) \\ & \frac{1}{} \end{aligned}$$

Which strategies are better?

We now need to develop a notion of how agents should choose strategies

The choice is not obvious because we have to take into account beliefs about what other players will do, knowing that these other players are also trying to choose the best strategies

A set of conditions that allow us to choose the best strategy of every player will be called a solution concept. We will work through several.

Dominance

- We will start with a very strong way in which one strategy might be better than another
- We do this by assuming nothing about what i believes the other players will do: can we still say that some strategies are better than others?
- Sometimes, yes: we will say a strategy dominates another one if that strategy is better no matter what the other players do

Dominance in pure strategies

Definition

Let $s_i \in S_i$ and $s_i' \in S_i$ be possible strategies for player i. s_i' is <u>strictly dominated</u> by s_i if

$$u_i(s_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Note: We may also refer to weak dominance, which occurs when the inequality above holds weakly for all $s_{-i} \in S_{-i}$ and strictly for some particular s_{-i}.

Remarks on dominance

- Dominance avoids the difficulties associated with beliefs about other players' strategies: a rational player should never play a dominated strategy, regardless of what they believe the other players will do
- Disadvantage: Often we will have a pair of strategies for which no strategy strictly dominates the other (consider rock, paper, scissors), in which case dominance does not make a prediction

Strictly dominant strategies

Definition

$s_i \in S_i$ is a strictly dominant strategy if every other strategy of i is strictly dominated by it:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$.

In words: Regardless of what the other players do, s_i is the best choice

Dominant strategy equilibrium

This leads us to our first solution concept for games:

Definition

The strategy profile $s^D \in S$ is a strict dominant strategy equilibrium if $s_i^D \in S_i$ is a strict dominant strategy for all $i \in N$.

Example: Prisoner's Dilemma

The story:

- Two people are taken in for questioning
- Each can stay quiet ("cooperate" with the other person) or provide evidence against the other person ("defect")
- There is not enough evidence to convince on the principal charge, but enough to convict each on a lesser charge
- Both people offered a deal: defect on the other and avoid jail
- If both defect, the testimony is no longer needed so both are convicted, perhaps with some slight leniency in sentencing for helping prosecutors
- If one defects and the other cooperates, the cooperator gets the maximum sentence and the other gets off

A matrix form for the prisoner's dilemma

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	$-1, -1$	$-3, 0$
	Defect	$0, -3$	$-2, -2$

Dominance in the PD

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	$-1, -1$	$-3, 0$
	Defect	$0, -3$	$-2, -2$

Does Player 1 have a dominant strategy?

If Player 2 cooperates, Player 1 is better off defecting.

If player 2 defects, Player 1 is better off defecting.

That is, Player 1 is better off defecting <u>no matter what Player 2 does</u>. A dominant strategy!

Notice that the same is true for Player 2.

If Player 1 cooperates, Player 2 is better of defecting.

If Player 1 cooperates, player 2 is better off defecting

Therefore, Player 2 also has a dominant strategy to defect.

Thus, $(\text{Defect}, \text{Defect})$ is a strict dominant strategy equilibrium: for both players, Defect is a strictly dominant strategy.

Mixed strategy dominance

- We defined dominance for pure strategies but we should extend the idea to mixed strategies

- Fact 1: If s_i' is strictly dominated by s_i then any mixed strategy that assigns positive probability to s_i' is also strictly dominated.

- Fact 2: A strategy s_i' may be strictly dominated by some mixed strategy even if it is not strictly dominated by any pure strategy.

Mixed strategy dominance example

Player 2

Player 1		L	C	R
	U	5, 1	1, 4	1, 0
	M	3, 2	0, 0	3, 5
	D	4, 3	4, 4	0, 3

No pure strategy is dominated by any other pure strategy (check)
For player 2, I claim that L is dominated by $(0, \frac{1}{2}, \frac{1}{2})$ (randomizing evenly between C and R).
We can check this for each pure strategy of player 1:

	L	$(0, 1/2, 1/2)$	
U	1	$\frac{1}{2}4 + \frac{1}{2}0 =$	2
M	2	$\frac{1}{2}0 + \frac{1}{2}5 =$	2.5
D	3	$\frac{1}{2}4 + \frac{1}{2}3 =$	3.5

Dominated strategy

Definition

Let $\sigma_i \in \Delta S_i$ and $s_i' \in S_i$ be possible strategies for player i. We say s' is strictly dominated by σ_i if

$$u_i(\sigma_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.

s_i' is a strictly dominated strategy if there exists a strategy $\sigma_i \in \Delta S_i$ such that σ_i strictly dominates s_i'.

Iterated Elimination of Strictly Dominated Strategies

- Strict dominated strategy equilibrium has the same strength and weakness: it relies only on player rationality

- This is a strength because we do not have to make strong assumptions about what players believe

- This is a weakness because it very often means we cannot make a prediction

- Furthermore: if players are rational, we might think that players should believe that other players are rational. This lets us take one step further: players will not play dominated strategies AND they will not believe that other players will play dominated strategies.

Iterated Elimination of Strictly Dominated Strategies

- Adding common knowledge of rationality suggests an iterative procedure that we can use to eliminate strategies (called IESDS):
  1. Eliminate strictly dominated strategies from the original game
  2. Consider the new game formed after eliminating those strategies: are there any strictly dominated strategies? If so delete them.
  3. Continue this process until there are no strictly dominate strategies remaining.

- Any strategy profile that survives IESDS is called an iterated-elimination equilibrium.

IESDS example

		Player 2		
		Left	Middle	Right
Player 1	Up	1, 0	1, 2	0, 1
	Down	0, 3	0, 1	2, 0

		Player 2	
		Left	Middle
Player 1	Up	1, 0	1, 2
	Down	0, 3	0, 1

		Player 2	
		Left	Middle
Player 1	Up	1, 0	1, 2

		Player 2
		Middle
Player 1	Up	1, 2

Best response

- Both of the solution concepts we have used so far are based on eliminating actions that players should never play
- Alternatively we might ask: What strategies might players choose to play and under what conditions?
- To move in that direction we ask: What strategy should a player choose if she believes the other players are playing some strategy σ_{-i}? This is the concept of best response.

Best response: Definition

Definition

A strategy σ_i is a best response to an opponent strategy profile σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$$

for all $s_i \in S_i$.

Nash equilibrium

Definition

A mixed strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Sigma_i$.

Nash equilibrium: Remarks

- One way we think about Nash equilibrium. We require:
  1. People best respond to their beliefs about what others will do
  2. Their beliefs are correct

- Mathematically: A Nash equilibrium is a <u>fixed point</u> of the best response correspondence
  1. Fixed point of a function: x such that $f(x) = x$.
  2. Significant for studying stability in different systems

- Relatedly, we can reach Nash equilibria as the stable state of simple dynamic learning processes

Nash's theorem

Theorem

Every game in which each player has finitely many actions has a mixed strategy Nash equilibrium.

Remarks:

- Recall a pure strategy equilibrium is a mixed strategy equilibrium in which all the probabilities are zero or one, so this includes games with only PSNE
- Finitely many actions is important: for games with a continuum of actions we need more assumptions to guarantee existence of Nash equilibria.
- This is an important result: Shows that Nash equilibrium is a complete solution, at least for finite games.

Nash equilibrium in pure strategies

A pure strategy profile is a Nash equilibrium if for all i we have $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

These do not always exist but when they do we can find them by analyzing the best response correspondences of all the players.

Example: Bach or Stravinsky

		Player 2	
		Bach	Stravinsky
Player 1	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

Two pure strategy Nash equilibria: (Bach, Bach), (Stravinsky, Stravinsky).

Example: 3x3 Game

		Player 2		
		L	C	R
Player 1	U	7, 7	4, 2	1, 8
	M	2, 4	5, 5	2, 3
	D	8, 1	3, 2	0, 0

One pure strategy Nash equilibrium: (M, C).

Tragedy of the commons (2 players)

- Two individuals share a common pool resource (say, a fishery). $K > 0$ represents the total amount of resource available.
- Each decides how much of the resource to consume.
- Each player gets a private benefit from consumption but also cares about conservation:

$$u_i(s_i, s_{-i}) = \underbrace{\log(s_i)}_{\text{consumption}} + \underbrace{\log(K - s_i - s_{-i})}_{\text{conservation}}$$

Tragedy of the commons: Best responses

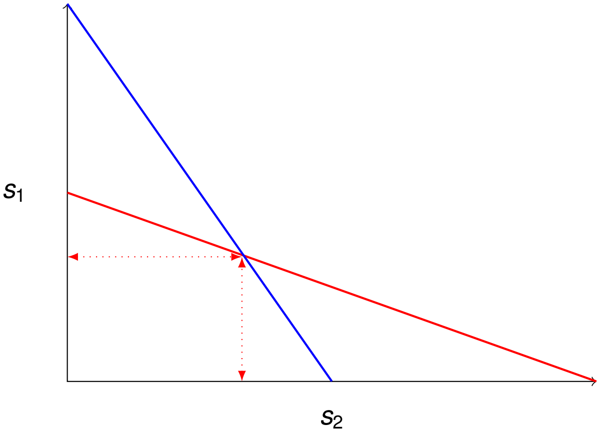
The best response of player i so a consumption level s_{-i} is found using some calculus. First-order condition:

$$\frac{\partial u(s_i, s_{-i})}{\partial s_i} = \frac{1}{s_i} - \frac{1}{K - s_i - s_{-i}} = 0$$

Solving for s:

$$s_i^*(s_{-i}) = \frac{K - s_{-i}}{2}$$

Visualizing best responses



Player 1's best response: $s_1 = \frac{K - s_2}{2}$
Player 2's best response: $s_2 = \frac{K - s_1}{2}$
Equilibrium is at intersection of these best response functions: this is the point at which both players are playing best responses

Solving mathematically

Given our best responses, a Nash Equilibrium occurs when both of the following are true:

$$s_1 = \frac{K - s_2}{2}$$
$$s_2 = \frac{K - s_1}{2}.$$

We can solve this system for s_1 and s_2. Plugging in the formula for s_2 into the first equation:

$$s_1 = \frac{K - \frac{K - s_1}{2}}{2} = \frac{K + s_1}{4}$$
$$\frac{3}{4} s_1 = \frac{K}{4}$$
$$\Rightarrow s_1 = \frac{K}{3}.$$

Solving mathematically (continued)

Plugging in our solution to P2's best response function:

$$
\begin{aligned}
s_2 &= \frac{K - \frac{K}{3}}{2} \\
&= \frac{\frac{2}{3}K}{2} \\
&= \frac{K}{3}.
\end{aligned}
$$

Therefore, the Nash equilibrium to this game is one in which both players consume $\frac{K}{3}$.

Note that the equilibrium is suboptimal: Their payoffs would both be higher if they could commit to only consuming $\frac{K}{4}$ (you can check this later). The "tragedy" is that they cannot coordinate on a better outcome because of externalities from consumption.

Example: Rock, Paper, Scissors

		Player 2	
	Rock	Paper	Scissors
Player 1 Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Uh oh. No PSNE! We need to look for mixed strategy Nash equilibria. Coming right up.

Mixed strategy in games

Imagine $\sigma_1(T) = p$ and $\sigma_2(L) = q$. Knowing this, we can compute the likelihood of each outcome:

		P2	
		L (q)	R ($1-q$)
P1	T (p)	pq	$p(1-q)$
	B ($1-p$)	$(1-p)q$	$(1-p)(1-q)$

Figure: Probabilities of outcomes

- Outcome (T, L) occurs with probability $p \times q$
- Outcome (B, L) occurs with probability $(1-p) \times q$
- Outcome (T, R) occurs with probability $p \times (1-q)$
- Outcome (B, R) occurs with probability $(1-p) \times (1-q)$

An example: American football

		Defense	
		Pass	Run
Offense	Pass	$-1, 1$	$1, -1$
	Run	$1, -1$	$-1, 1$

- Is there a PSNE? No.

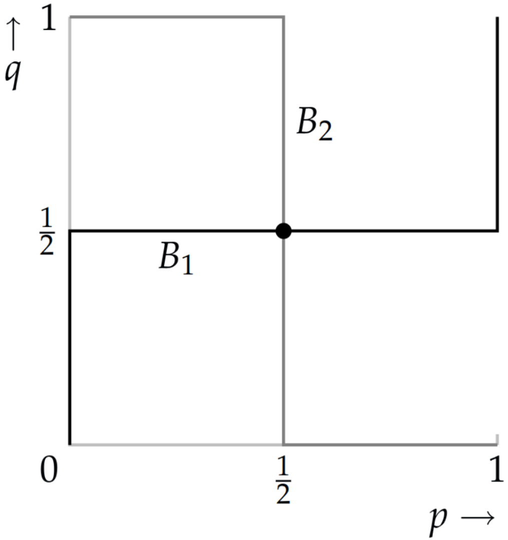
- So what do we do? Look for MSNE.

- In this game, the MSNE is given by $\left(\left(\frac{1}{2}\text{Pass}, \frac{1}{2}\text{Run}\right) ; \left(\frac{1}{2}\text{Pass}, \frac{1}{2}\text{Run}\right) \right)$

See also: Techmo Bowl

See also: Techmo Bowl

Visualizing best responses and equilibrium



Solving the football game

		Defense: Pass (q)	Defense: Run ($1-q$)
Offense	Pass (p)	$-1, 1$	$1, -1$
	Run ($1-p$)	$1, -1$	$-1, 1$

- Offense chooses p to make Defense indifferent (so it can't be taken advantage of) and Defense does the same
- So, to get optimal p we need to think about Defense's payoffs:

$$
\begin{aligned}
EU_D(\text{Pass}) &= p(1) + (1-p)(-1) = p - (1-p) = 2p - 1 \\
EU_D(\text{Run}) &= p(-1) + (1-p)(1) = (1-p) - p = 1 - 2p \\
EU_D(\text{Pass}) &= EU_D(\text{Run}) \text{ [INDIFFERENCE]} \\
2p - 1 &= 1 - 2p \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}
\end{aligned}
$$

- We can do the same solving for q

Mixed strategy Nash equilibrium: the weird part

- Notice that we made a strange move to solve for the mixed Nash Equilibrium: to compute one player's strategy, we used the OTHER player's payoff

- The reasoning:
  - For Player 1's best response to be a mixed strategy, she must be indifferent between the actions over which she is mixing. Otherwise she should choose the action she strictly prefers rather than randomizing
  - Player 1's indifference depends on Player 2's strategy, so we choose a randomization probability for Player 2 that would make Player 1 indifferent between her actions
  - But for Player 2 to randomize, she also must be indifferent, so then we must choose a strategy for Player 1 that makes this so.

Remark 1: Intepreting mixed strategies

- Interpretation 1: The players literally randomize their actions.
  - This probably works well for something like rock paper scissors but not so well for other games
- Interpretation 2: There are a distribution of players and each player is uncertain what type of player they are facing.
  - Here we can interpret a mixed strategy as a proportion playing a particular pure strategy in the population
  - This interpretation is sometimes preferred in biology and might make sense for games describing day-to-day interations
- Purification (Harsanyi 1973): Shows that mixed strategies are the limit of pure strategy equilibria to games with small amounts of noise in the payoffs.

Another example: Regulation

		Warren G	
		Regulate	Do Nothing
Nate Dogg	Regulate	$2, 2$	$0, 1$
	Do nothing	$1, 0$	$1, 1$

- Do you recognize this game?
- What are the PURE strategy Nash equilibria?
- HINT: Almost all games have an odd number of Nash equilibria, so if you find two PSNE keep looking.
- Finding the mixed Nash equilibrium. Let $p = \Pr[\text{Nate Dogg regulates}]$ and $q = \Pr[\text{Warren G regulates}]$:
  1. Find Nate Dogg's strategy by making Warren G indifferent: $p2 + (1-p)0 = p1 + (1-p)1 \Rightarrow p2 = 1 \Rightarrow p = \frac{1}{2}$
  2. Find Warren G's strategy by making Nate Dogg indifferent: $q2 + (1-q)0 = 1 \Rightarrow q = \frac{1}{2}$
- $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is a MSNE.

Bach or Stravinsky

		Player 2	
		Bach	Stravinsky
Player 1	Bach	$1, 2$	$0, 0$
	Stravinsky	$0, 0$	$2, 1$

- What are the pure strategy Nash equilibria?
- Computing the mixed strategy equilibrium, letting $p = \Pr[\text{P1 plays Bach}]$ and $q = \Pr[\text{P2 plays Bach}]$:
  1. Find P1's strategy by making P2 indifferent:

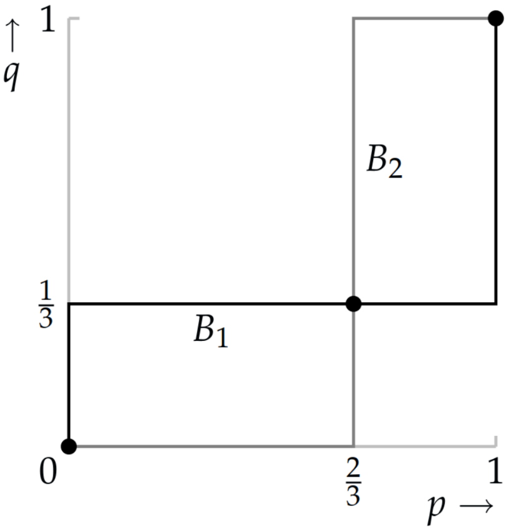
$$p2 + (1-p)0 = p0 + (1-p)1 \Rightarrow 2p = (1-p) \Rightarrow p = \frac{1}{3}$$

  2. Find P2's strategy by making P1 indifferent:

$$q1 + (1-q)0 = q0 + (1-q)2 \Rightarrow q = 2(1-q) \Rightarrow q = \frac{2}{3}$$

- So $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$ is a MSNE.

Visualizing best responses and equilibrium



Whistleblowers

Suppose some number of White House and DOJ staff observe what they see as wrongdoing by the President.

Federal whistleblower protections provide a way for these staff to notify Congress while maintaining anonymity. But blowing the whistle is still costly. And also...

Amy Fiscus @amyfiscus · Sep 26

NEW: The whistleblower is a CIA officer who was detailed to the White House at one point. His complaint suggests he's an analyst. @adamgoldmanNYT @nytmike @julianbarnes

White House Knew of Whistle-Blower's Allegations Soon After Trum…

The whistle-blower, a C.I.A. officer detailed to the White House at one point, first expressed his concerns anonymously to the agency's top lawyer.

nytimes.com

174 171 327

Amy Fiscus
@amyfiscus

Follow

And yes, it's a he.

Whistleblowers

So even among people who agree that Congress should be made aware of wrongdoing, it is better if someone else does it.

The game:
Players: n potential whistleblowers
Actions: Each player can blow the whistle or not.
Payoffs: Each player gets:

- v - c for blowing the whistle
- 0 if nobody blows the whistle
- v if somebody else blows the whistle

Assume $v > c > 0$, so each person is in principle willing to blow the whistle.

Whistleblowers: PSNE

Claim 1: There is no PSNE in which nobody blows the whistle.

Proof: Consider a strategy profile in which all players choose “don’t blow the whistle.” All players get a payoff of 0 in this profile. However, any player could get a payoff of $v - c > 0$ for switching to “blow the whistle”. Therefore, this profile cannot be a Nash equilibrium.

Big idea: Any player would blow the whistle if she knew she was <u>pivotal</u> for whether or not the crime was reported at all. If nobody is blowing the whistle then <u>every</u> player is pivotal, so any player would be willing to blow the whistle.

Whistleblowers: PSNE (cont)

Claim 2: There is no PSNE in which everybody blows the whistle.

Proof: Consider a strategy profile in which all players choose “blow the whistle.” All players get $v - c$ in this profile. However, if some player unilaterally changed to “don’t blow the whistle” then the crime would still be reported, so that person would get $v > v - c$. Therefore, this profile cannot be a Nash equilibrium.

Intuition: Since everyone is blowing the whistle and only one whistleblower is needed, NO player is pivotal. Therefore, any player would switch to “don’t blow the whistle” and free-ride off of the others.

Whistleblowers: PSNE (cont)

Claim 3: There is no PSNE in which more than one player blows the whistle.

Proof: This is the exact same reasoning as before. The whistleblowers get $v - c$ and the players who do not blow the whistle get v. Since more than one person is blowing the whistle, one of the whistleblowers could switch to "don't blow the whistle" and get $v > v - c$.

The intuition once again relies on the idea of pivotality: if I can avoid the cost of whistleblowing without changing the outcome (i.e. I am not pivotal) then I will do so.

Whistleblowers: PSNE (cont)

Claim 4: There are n PSNE in which exactly one person blows the whistle.

Proof: Consider any profile in which one players chooses "blow the whistle" and the rest choose "don't blow the whistle." In such a profile, the whistleblower gets a payoff of $v - c$ and the others get a payoff of v. If any non-whistleblower switches to blowing the whistle, they get $v - c < v$, so this deviation is not profitable. If the whistleblower switches to "don't blow the whistle" he (the NYT tells me it's a he) would get 0 rather than $v - c$, so this deviation also is not profitable. Therefore any such is a Nash equilibrium. Since the whistleblower could be any of the players, there are n such PSNE.

Intuition from pivotality: We can only have one person blow the whistle because that guarantees that the person reporting is a pivotal player so they are willing to report.

Whistleblowers: MSNE

The MSNE of this game also builds on the idea of pivotality.

Each player blows the whistle with some probability, with the key equilibrium condition being that the probability of being pivotal (i.e. that nobody else blows the whistle) is just large enough to make every player indifferent between reporting and not reporting.

We will look for symmetric equilibria, meaning that every player reports with the same probability p.

Whistleblowers: MSNE

The key equilibrium condition:

$$
\begin{aligned}
EU(\text{report}) =& EU(\text{ don't report}) \\
v - c =& 0 \Pr[\text{no other whistleblowers}] + v \Pr[\text{at least one other whistleblower}] \\
v - c =& v(1 - \Pr[\text{no other whistleblowers}]) \\
\frac{c}{v} =& \Pr[\text{no other whistleblowers}].
\end{aligned}
$$

To find the equilibrium, we just find $\Pr[\text{no other whistleblowers}]$ as a function of p and then solve for p.

Whistleblowers: MSNE

Consider computing this probability from the perspective of player 1:

$$\begin{aligned}\Pr[\text{no other whistleblowers}] =& \Pr[\text{player 2 abstains}] \times \cdots \times \Pr[\text{player n abstains}]\\ =&(1-p)\times(1-p)\times\cdots\times(1-p)\\ =&(1-p)^{n-1}.\end{aligned}$$

The equilibrium condition is now

$$\frac{c}{v} = (1-p)^{n-1}.$$

Whistleblowers: MSNE

Solving for the mixed strategies:

$$\begin{aligned}
\frac{c}{v} &= (1-p)^{n-1} && \text{(eqbm condition)}\\
\left(\frac{c}{v}\right)^{1/(n-1)} &= 1-p && \text{(raise both sides to } \tfrac{1}{n-1}\text{)}\\
1-\left(\frac{c}{v}\right)^{1/(n-1)} &= p. && \text{(rearrange)}
\end{aligned}$$

$\Rightarrow$ There is a MSNE in which all players blow the whistle with probability $1-\left(\frac{c}{v}\right)^{1/(n-1)}$.

Remark 1: Effect of increasing the number of potential whistleblowers

- We may think that as more people observe wrongdoing the likelihood of a whistleblower increases
- The problem is that as the number of potential whistleblowers increases, each person becomes less likely to bear the cost
- We can see clearly that increasing n decreases the likelihood that any individual blows the whistle:
  - If n increases, $\frac{1}{n-1}$ decreases, so therefore $\left(\frac{c}{v}\right)^{1/(n-1)}$ increases and $1-\left(\frac{c}{v}\right)^{1/(n-1)}$ decreases.
- But what about the total probability of <u>any</u> whistleblower?

Remark 1: Continued

- We can determine the probability of any whistleblower by observing that:

$$\Pr[\text{no whistleblower}] = \Pr[\text{i does not blow whistle}] \Pr[\text{no other whistleblowers}]$$

- We know that $\Pr[\text{i does not blow whistle}]$ increases as n increases.

Remark 1: Continued

- What about $\Pr[\text{no other whistleblowers}]$? Recall our equilibrium condition was

$$\frac{c}{v} = \Pr[\text{no other whistleblowers}].$$

- Therefore the probability of no other whistleblowers is constantly $\frac{c}{v}$ regardless of the number of players.
- Therefore:

$$\Pr[\text{no whistleblower}] = \Pr[\text{i does not blow whistle}](c/v)$$

  which increases with n.
- Surprising: The more people that see wrongdoing, the less likely there is a whistleblower.

Remark 2: Public goods

This problem is a special case of a “threshhold public goods” game. These games have the following properties:

- The players value some public good (value v) but find it costly to contribute to it (cost c)
- The public good is provided if the number of people who contribute is greater than or equal to some threshhold $k \leq n$. Here $k = 1$.

In general these games are similar: there are “n choose k” PSNE in which exactly k people contribute, and a mixed equilibrium that sets

$$\frac{c}{v} = \Pr[\text{exactly k-1 others contribute}].$$

Paper, Rock, Scissors

		Player 2		
		P	R	S
Player1	P	$0, 0$	$1, -1$	$-1, 1$
	R	$-1, 1$	$0, 0$	$1, -1$
	S	$1, -1$	$-1, 1$	$0, 0$

We know that there is no PSNE. We also know that the players must randomize over ALL actions. For instance, if I never play Paper and randomize of Rock and Scissors then you can always play Rock and guarantee a win or a tie.

This means that both players must be indifferent over all three actions.

In some other 3x3 game we might have to check for mixed strategies over a lot more combinations of actions, but we are using what we know about the game to narrow things down.

Paper, Rock, Scissors

		Player 2 P	Player 2 R	Player 2 S
Player1	P	$0, 0$	$1, -1$	$-1, 1$
	R	$-1, 1$	$0, 0$	$1, -1$
	S	$1, -1$	$-1, 1$	$0, 0$

To solve, let p_R be the probability player 1 plays rock and p_S bet the probability of scissors. Then the probability of paper is $1 - p_R - p_S$. Likewise, q_R and q_S denote these same probabilities for player 2.

P1's strategy must make player 2 indifferent:

$$(1 - p_R - p_S)0 + p_R 1 + p_S(-1) = (1 - p_R - p_S)(-1) + p_R 0 + p_S 1 \qquad \text{(P vs R)}$$

$$(1 - p_R - p_S)(-1) + p_R 0 + p_S 1 = (1 - p_R - p_S)1 + p_R(-1) + p_S 0 \qquad \text{(R vs S)}$$

$$p_R - p_S = p_R + 2p_S - 1 \qquad \text{(P vs R)}$$

$$p_R + 2p_R - 1 = p_2 + 2p_R - 1 \qquad \text{(R vs S)}$$

$$p_S = \frac{1}{3} \qquad \text{(P vs R)}$$

Paper, Rock, Scissors

Obviously we can do the exact same calculation for Player 2 (I won't repeat it).

Therefore, the unique Nash equilibrium of paper rock scissors is one in which both players play each action with probabiltiy $\frac{1}{3}$.

This is the obvious prediction for this game: Each action has an action that clearly beats it, so if you can guess what your opponent will do you'll win. The whole idea of this game is to maximize uncertainty over what you will do, which is achieved by randomizing evenly over all the actions.

Mixed strategy dominance

		Player 2 P	Player 2 R
Player1	P	$3, -1$	$-1, 1$
	R	$0, 0$	$0, 0$
	S	$-1, 2$	$2, -1$

I want to solve this game by iteratively eliminating dominated strategies. Is any strategy dominated by any other pure strategy? No! Does this mean we are done and have to solve using best responses? Also no!

Dominance and mixed strategy

- If a strategy is dominated by a mixture between two or more other pure strategies then we can also eliminate that strategy

- Logic is the same: A rational player won't play a dominated pure strategy when they could instead play another strategy (pure or mixed that is always better)

Back to the game

		Player 2	
		L	R
Player1	U	$3,-1$	$-1,1$
	M	$0,0$	$0,0$
	D	$-1,2$	$2,-1$

I claim that M is dominated for Player 1 by a strategy $(1/2, 0, 1/2)$ which places probability $\frac{1}{2}$ on up and $\frac{1}{2}$ on down:

- If Player 2 plays L, then M yields 0 while $(1/2, 0, 1/2)$ yields $\frac{1}{2}3 + \frac{1}{2}(-1) = 1 > 0$.
- If Player 2 plays R, then M yields 0 while $(1/2, 0, 1/2)$ yields $\frac{1}{2}(-1) + \frac{1}{2}2 = \frac{1}{2} > 0$.

This shows that M is dominated, and we can eliminate it from the game!

Reduced game

		Player 2	
		L	R
Player1	U	$3, -1$	$-1, 1$
	D	$-1, 2$	$2, -1$

This game has no PSNE but we can solve for the MSNE more easily now.

- Solve for P1 strategy (p=Probability of U) by making P2 indifferent:

$$EU_L(p) = EU_R(p) \Rightarrow p(-1) + (1-p)2 = p(1) + (1-p)(-1) \Rightarrow p = \frac{3}{5}$$

- Solve for P2 (q = Probability of L) strategy by making P1 indifferent:

$$EU_U(q) = EU_D(q) \Rightarrow q3 + (1-q)(-1) = q(-1) + (1-q)(2) \Rightarrow q = \frac{3}{7}.$$

Therefore the MSNE of the original game is one in which P1 never plays M but player U with probability $\frac{3}{5}$ and D with probability $\frac{2}{5}$, and P2 plays L with probability $\frac{3}{7}$ and R with probability $\frac{4}{7}$.

Electoral Competition

Motivation

- What determines the platforms offered by political parties competing for votes?
- How do we explain the number of viable political parties?
- How do changes in voters' preferences affect electoral outcomes and chosen policies?

Today, we'll learn a game theoretic model that has helped provide some answers to all of these questions. We will only be able to scratch the surface of these models.

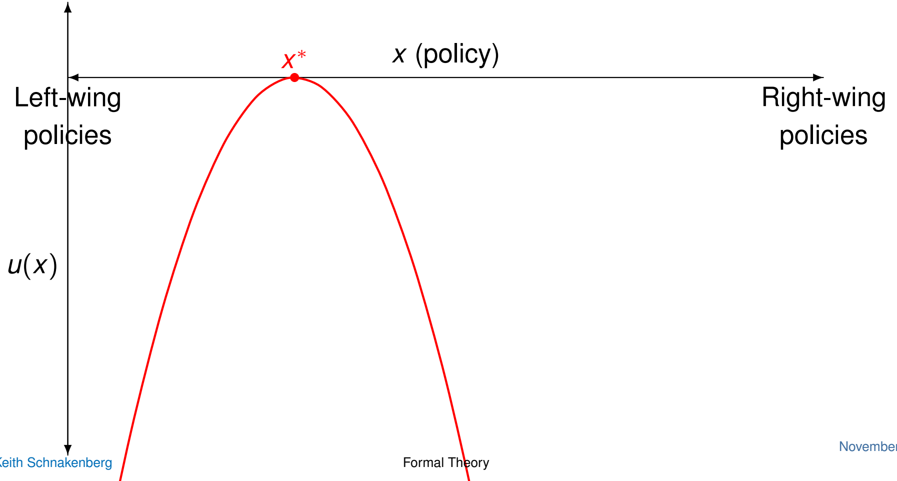
A spatial model of policy choice: Policies

The set of policies is equal to the real number line.

A spatial model of policy choice: Preferences

Each person is characterized by her ideal point (i.e. her favorite policy) x^* which is a point on the real line.

We assume a person's utility is decreasing in the distance between the policy and her ideal point.



The basic game

- Players: Two candidates (could be parties), a set of voters (for simplicity, odd number of voters). The voters each have an ideal point. A special significance is given to the median of the ideal points, which we will denote m.

- Actions: Each candidate's set of actions are the real number line. Each voter can decide which party to vote for.
  - The voters' choices are obvious: they choose the candidate closest to them, so we will analyze this as if it is a two player game, taking voters' choices to be mechanistic. We assume they randomize between candidates when they are indifferent between them.

- Candidates' payoffs: Candidates (for now) just want to wind the election: they get 1 if they win and 0 if they lose, regardless of chosen policies. Elections are determined by majority rule.

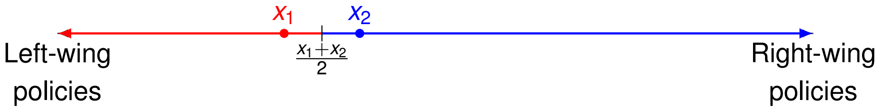
Why the median is important

Suppose party 1 chooses a position x_1 and party 2 chooses a distinct position x_2.

All voters to the left of x_1 vote for 1, and all parties to the right of x_2 vote for 2

Between x_1 and x_2, everyone closer to x_1 votes 1 and everyone closer to x_2 votes 2 (change point is the midpoint of x_1 and x_2)

The median is by definition the cutoff for a majority: If the set of A supporters includes the majority then it must be a majority. Otherwise it must not be.



$\Rightarrow$ When the candidates take distinct positions, the supporters of each candidate are divided according to left/right.

Analyzing Candidates' Best Responses

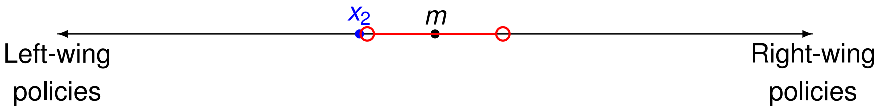
Fix a position x_2 of candidate 2. What are candidate 1's best responses to that position?

Any x_1 that causes candidate 1 to win is a best response. If 1 cannot guarantee a win, then a position causing a tie is a best response.

We can separately look at three cases:

- $x_2 < m$
- $x_2 = m$
- $x_2 > m$

Case 1: $x_2 < m$



Locating just barely to the right of x_2 would certainly cause 1 to win

Locating to the right of the median but closer to m than x_2 will also cause 1 to win

$\Rightarrow$ the set of best responses is $BR_1(x_2) = \{x_1 : x_2 < x_1 < 2m - x_2\}$

Note: $\frac{x_2+x_1}{2} = m \Rightarrow 2m = x_1 + x_2 \Rightarrow x_1 = 2m - x_2$

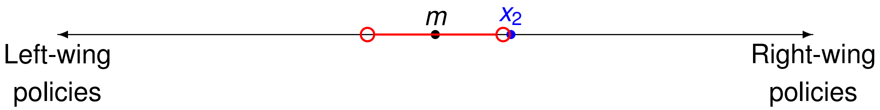
Case 2: $x_2 = m$

- Clearly no policy strictly beats $x_2 = m$ since m is the most preferred policy of the median.

- In fact, every $x_1 \neq m$ would cause 1 to lose for sure

- But $x_1 = m$ leads to winning with probability $\frac{1}{2}$, so this is a best response.

Case 3: $x_2 > m$



Same story as case 1, just reversed: $BR_1(x_2) = \{x_1 : 2m - x_2 < x_1 < x_2\}$

Visualizing best response functions

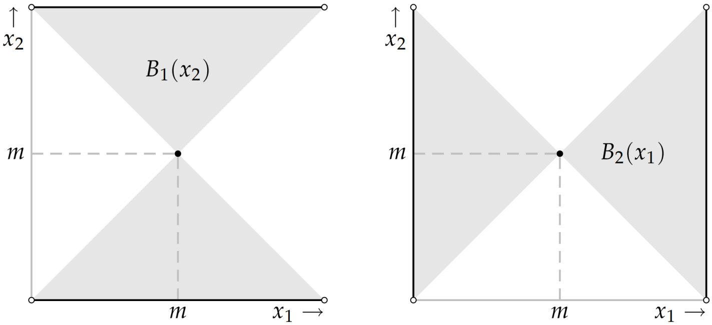


Figure 71.1 The candidates' best response functions in Hotelling's model of electoral competition with two candidates. Candidate 1's best response function is in the left panel; candidate 2's is in the right panel. (The edges of the shaded areas are excluded.)

The Hotelling-Downs Median Voter Result

The unique Nash equilibrium of the electoral competition game is one in which both candidates choose the same platform: the ideal point of the median voter.

Remark 1: Normative Theory Meets Positive Theory

- Even without the Hotelling-Downs result, the position of the median voter has a special normative status: It is the only policy we could choose for which no majority would prefer a different policy.
  - This is directly related to the concept of the core in economics
- Thus, we may think the median is the most desirable policy for this society to choose.

- Nash equilibrium is a positive criterion: it makes a prediction about what is going to happen. As we saw with the prisoner's dilemma, it is by no means guaranteed to select normatively desirable outcomes

- The Hotelling-Downs result shows us that these two things come together in this particular model of electoral competition: our best prediction (given the model) and our most desired outcome happily turn out to be the same.

Remark 2: Theory and Reality

- Okay but in reality political parties never converge to the same policy, so isn't this a useless theoretical result?

- If the point of theory is to make accurate predictions all the time, yes. But then we are doomed anyway.

- The result does a lot of useful things for us:
  - Provides a useful benchmark for developing more theory. If we want to explain why parties choose divergent positions, we can relax assumptions of the baseline model one at a time and see what gets us closer to the data. Decades of theory now do exactly that.
  - Illustrates one strategic force at play in elections. The purpose of a model is rarely to model every aspect of a problem. Models are representations used to analyze particular aspects of the problem by simplifying. A good model must always be a little bit wrong! (So must any model, really.)

Policy-motivated parties (the Calvert-Wittman model)

- Assume voters have single peaked policy preferences as before
- The players are two parties $P = \{L, R\}$ which have ideal points at 0 and 1, respectively.

$$u_L(x) = -|x|$$
$$u_R(x) = -|x - 1|$$

- We will also assume that $0 < m < 1$.

Policy-motivated parties (continued)

Letting $\pi(x_L, x_R)$ be the probability that L wins given the platforms, the parties maximize expected utilities:

$$\begin{aligned}
U_L(x_L, x_R) =& \pi(x_L, x_R)(-|x_L|) + (1 - \pi(x_L, x_R))(-|x_R|) \\
U_R(x_L, x_R) =& \pi(x_L, x_R)(-|x_L - 1|) + (1 - \pi(x_L, x_R))(-|x_R - 1|).
\end{aligned}$$

Convergence result

Proposition

This game has a unique Nash equilibrium in which both parties choose m, the position of the median voter.

Calvert-Wittman convergence proof

- Existence: Consider the strategy profile (m, m). We will show that neither party strictly benefits from deviating.
  - Consider a deviation by any party away from m toward its preferred policy
  - This causes that party to lose, yielding a policy of m. So the party is indifferent between m and deviating to any other policy.

Calvert-Wittman convergence proof (cont)

- Uniqueness: We have shown that m is a best response to m so we will consider situations where $0 \leq x_L < m < x_R \leq 1$, and show that none are equilibria
  - Case 1: $|x_L - m| > |x_R - m|$. L loses and the policy is x_R. Then L is not playing a best response since any policy in $(2m - x_R, x_R)$ cases L to win the election and implements a preferable policy.
  - Case 2: $|x_L - m| < |x_R - m|$. Same logic for L.
  - Case 3: $|x_L - m| = |x_R - m|$. Each party wins with probability $\frac{1}{2}$. However, neither player is playing a best response: each could move in the direction of m by some small amount $\epsilon > 0$ and win the election for sure.

Multiparty competition

So far we have considered models with only two parties.

There are a few ways to think about relaxing this:

- We could analyze a model with multiple parties
- We could analyze a model with two parties but assume that the parties believe another party <u>could</u> enter

A three party model

Consider a model with the same assumptions as the Hotelling-Downs model but with three parties $P = \{A, B, C\}$.

Do you think there is an equilibrium in which all three parties choose the median?

Non-convergence in three-party model

Proposition

There does not exist a convergent equilibrium in the three party model.

Proof of non-convergence with three parties

- Consider the strategy profile (m, m, m). In it, all three parties win with probabilityi $\frac{1}{3}$.

- Consider a deviation by party C: for some small number $\epsilon > 0$, C moves to $m - \epsilon$

- Party C now wins the votes of everyone to the left of m (just under 1/2 of the voters) and Parties A and B split the remaining voters. Thus, Party C wins for sure.

What is the equilibrium then?

We can find a Nash equilibrium in certain cases still. Consider the following example:

Example

Assume voter's ideal points are distributed uniformly on $[0, 1]$. There is an equilibrium in which $x_A = x_B = \frac{1}{3}$ and $x_C = \frac{2}{3}$.

Proof for three-player example

- Clearly C does not deviate from this profile since C wins the election for sure in this case. We will then consider each possible deviation for A (the same arguments would apply to B).
- $x_A < \frac{1}{3}$: C still wins with one half of the vote.
- $x_A \in \left(\frac{1}{3}, \frac{2}{3}\right)$. This gives A at most one sixth of the vote: everyone to the left of $\frac{1}{3}$ (one third of the voters) votes for B, everyone to the right of $\frac{2}{3}$ votes for B (another one third of the voters), and votes in the middle are split.
- $x_A = \frac{2}{3}$. This is essentially the same as the original profile, except B is the new winner.
- $x_A > \frac{2}{3}$: Essentially the same as $x_A < \frac{1}{3}$ but with B as the new winner.

This shows that nobody has a profitable deviation, so this is a Nash equilibrium.

Remarks on three party model

Though an interesting exercise, this equilibrium is a pretty silly prediction in several ways:

- The assumption of mechanistic voters seems not so innocuous now: if we treat the voters are actual strategic players they should coordinate on one of the two left-most candidates rather than splitting their votes on identical parties. We'll return to strategic voters soon.
- We also have parties entering the election know that they will lose. So we don't answer the question of why there are three parties. In fact, we usually expect not to have three major parties in an election like this one. We will also return to the question of entry.
- Real elections with multiple parties tend to be proportional representation, and in those we expect parties will maximize vote share. Those strategies don't work in this case (in fact we can show that there is not an equilibrium in the case above when parties maximize vote share).

Entry and strategic voting

Consider the following modifications of Hotelling-Downs:

- There are N potential candidates who may enter the elections (where N is very large)
- Candidates strategy sets: $x_P \in \mathbb{R} \cup \{\text{stay out}\}$ (i.e. they can enter and choose a position or they can not run)
- Cost of entry: δ. Benefit of winning: $v > \delta$
- The set of voters is finite and preferences are concave (i.e. voters are risk averse)
  - This would be satisfied for instance by $u_i(x) = -(x - x_i^*)^2$
- The voters are players in the game and behave strategically

Solution concept

- We restrict our attention to pure strategy equilibria
- We also only consider equilibria in which voters use weakly undominated strategies
- The latter is common in voting games. The purpose is to rule out trivial situations driven by voters choosing less-preferred candidates simply because they are not pivotal and therefore indifferent between all choices.

Preliminary results on entry

- Fact 1: Every candidate who enters must get the same number of votes.
  - In pure strategies, the mapping from candidate strategies to votes is deterministic
  - $\Rightarrow$ A candidate who loses should have expected to lose and stayed out since entry is costly
- Fact 2: If M is the number of candidates, we must have $M \leq \frac{v}{\delta}$
  - Probability of winning for each entrant must be $\frac{1}{M}$
  - Therefore expected utility is $\frac{v}{M} - \delta$, which must be greater than zero. Solving for M gives us the result.

Equilibria with entry and strategic voting

Proposition

In all equilibria, $1 \leq M \leq \frac{v}{\delta}$*, candidates enter and all of them choose the platform* m*. Furthermore, all voters vote sincerely in equilibrium.*

The proof involves some material from dynamic games that we haven't covered yet so I'll give the intuition without the technical details.

Since every candidate has the same number of votes, every voter is pivotal, hence sincere voting

Why don't we have equilibria like we described earlier in the multicandidate case? Strategic voters should coordinate on one of the preferred candidates rather than splitting votes.

Consider a situation where three candidates choose m and it's not profitable for a fourth candidate to enter. What happens if one candidate deviates to (say) the right? Everyone to the left coordinates on one of the median candidates, that candidate loses.

Citizen candidate model

Another class of models on electoral competition are citizen candidate models. The model:

- There are N citizens (the set of voters and the set of potential candidates is the same)
- Policy preferences: $u_i(x) = -|x - x_i|$
- Citizens pay a cost $\delta > 0$ of running for election and get a benefit $v \geq 0$ of winning (in addition to the policy benefit)
- Key departure from every model so far: Candidates cannot commit to policies. Therefore, any candidate who wins implements her ideal point.
- Citizens vote for one candidate or abstain. Winner determined by plurality rule with ties broken by lottery. Assume weakly dominant voting strategies
- If nobody enters, a status quo policy $\overline{x}$ is implemented

Median voter equilibrium

First question: When is there an equilibrium in which a citizen with ideal point m enters and nobody else enters?

First condition: The citizen at the median prefers to enter given that nobody else does:

$$-|m - m| + v - \delta \geq -|\overline{x} - m| \Rightarrow \delta \leq |\overline{x} - m| + v.$$

Second condition: Nobody else will enter. $\delta \geq \frac{v}{2}$. This guarantees that a second candidate at the median does not enter (a candidate located anywhere else should never enter anyway).

Two candidate equilibria

Let us restrict our attention to the case where $v = 0$ (there is policy motivation for entering but no other rewards to office)

Two necessary for a two-candidate equilibrium:

- Candidates must receive the same number of votes (since $\delta > 0$)
- ...but they must be at different positions (since $v = 0$)

These two conditions jointly imply that the candidates must be at two points equidistant from the median. Some candidate L locates at $x_L = m - \Delta$ and R locates at $x_R = m + \Delta$. Then we must also have $\delta \leq \Delta$.

Two candidate equilibria (continued)

The remaining condition is that nobody else wants to enter. Here the results depend whether we assume sincere or strategic voting.

- Sincere. With sincere voting, citizens with $x_i \leq x_L$ or $x_i \geq x_R$ do not enter because they would cause their least-preferred candidate to win. Centrist citizens may enter though, so x_L and x_R must be sufficiently close together that entry is not possible for these citizens. We have a "goldilocks" type of result: Δ must be large enough that both candidates want to enter, but small enough that nobody can profitably enter in between them.

- Strategic. The coordination aspect of strategic voting means that there are many more possibilities when a third candidate enters. As a result, we might have two extremist candidates entering with no entry by a moderate citizen simply because of coordination failures.

- Our analysis of entry decisions under strategic voting presumes that candidates correctly anticipate which candidates voters will coordinate on. This is a preview of solutions for sequential games, which we will get to soon.

Dynamic Games of Complete Information

Example: Sequential BoS

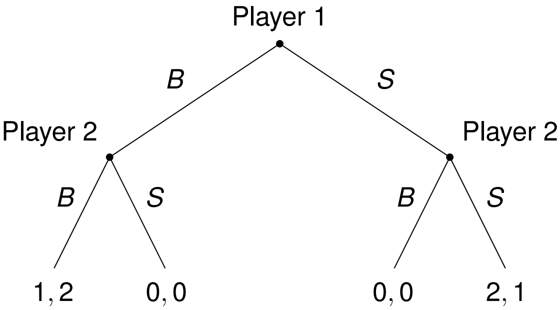
Recall the Bach or Stravinsky game:

		Player 2	
		Bach	Stravinsky
Player 1	Bach	$1, 2$	$0, 0$
	Stravinsky	$0, 0$	$2, 1$

$$NE = \{(B, B), (S, S), ((1/3, 2/3), (2/3, 1/3)\}$$

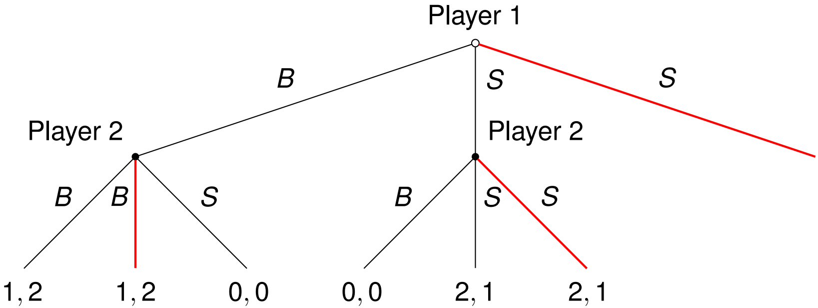
Example: Sequential BoS

Consider the following variant of the game: Player 1 moves first. Player 2 observes Player 1's choice and then moves second:



Question: Which Nash equilibria to BoS still seem reasonable in light of this additional information about sequence?

Example Sequential BoS: Continued



⇒ (S, S) seems like the only reasonable outcome given this sequence.

The plan for today

1. Formalize the idea of an extensive form game (i.e. a game with temporal information)

2. Generalize our intuition about the sequential BoS game to learn a better way of solving extensive form games: a refinement of Nash equilibrium which we'll call subgame perfect Nash equilibrium (SPNE)

3. Practice backward induction, the easiest way of solving for SPNE in finite-horizon extensive games with complete and perfect information. (We already used it to solve sequential BoS).

Extensive form games

In general, an extensive form game consists of:

1. A set of players N
2. A set of terminal histories (i.e. possibly end points of the game)
3. The order of play: a specification of when each player can move
4. A set of actions that each player can take each time that player moves
5. Preferences over terminal histories

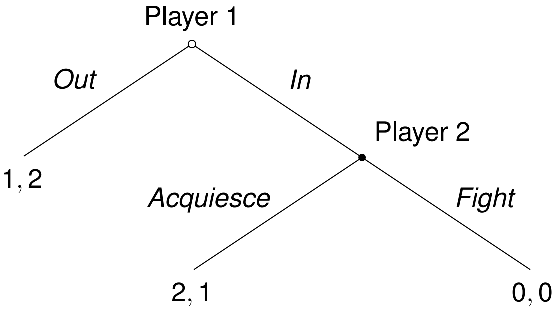
Intuition: An extensive form game has all of the information in a normal form game but also temporal information.

Game trees

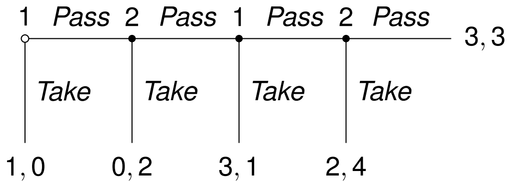
- A game tree is often a useful way of representing an extensive form game

- A game tree consists of:
  - Nodes with player labels: These communicate who the players are and when they move
  - Branches with action labels: These tell us what decisions each player can make when they move
  - Terminal nodes with payoffs: This tells us the terminal histories (outcomes of the game) and all players' preferences over them.

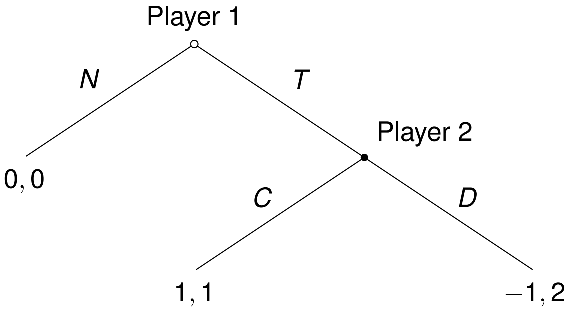
Example: Entry Game



Example: Centipede Game



Example: Trust game

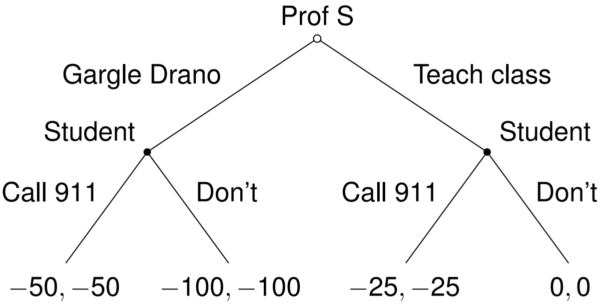


Pure strategies in extensive form games

- A pure strategy in an extensive form game is a complete plan of play describing what the player would do at any node where she can move
  - Example: In sequential BoS, a strategy for Player 1 might be “choose S” but a strategy for Player 2 would be “choose B if P1 chooses B, choose S if P1 chooses S”

- Important: A player’s strategy must say what she would do at every node where should could make a decision, even if she should normally arrive at that node

Example: Strategies as complete plans of play



It is obvious from the game that I should not gargle Drano, but your pure strategy must still specify what you would do in both scenarios.

Mixed strategies in extensive form games

Technically we can think of mixed strategies in extensive form games in two ways:

1. The literal definition: A probablity distribution over complete plans of play.
2. A “behavior strategy”: A probabilistic plan of action at each decision node

Kuhn (1953) showed that these two are equivalent for our purposes, and it turns out behavior strategy’s are must easier to deal with, so we will think of players as randomizing independently at each decision node.

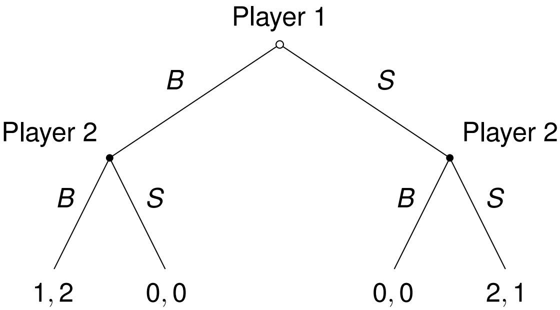
We will come back to this later but we’ll think about pure strategies today.

Solving extensive games

- Our solution concept for extensive form games is called subgame perfect Nash equilibria. I will introduce this concept in more detail next time. Today I want to give intuition and teach you the easiest solution method.

- Intuition: SPNE is meant to rule out equilibria that are based on non-credible threats

Back to sequential BoS



We determined that $(S,(B,S))$ was a good prediction to this game.
Why not $(B,(B,B))$?
We can think of this as a situation in which Player 2 says ”You may as well choose B because I am going to choose B no matter what you do.”
This is in fact a Nash equilibrium! Player 1 cannot do better by deviating to S, and, since the actual outcome is that both players choose B, Player 2 cannot do better by deviating to another pure strategy.

Back to sequential BoS

The problem is that this is a non-credible threat. Player 1 should say "I do not believe you. If I choose S you will clearly have an incentive to choose S and not B, so I will choose S."

SPNE rules out non-credible threats like this.

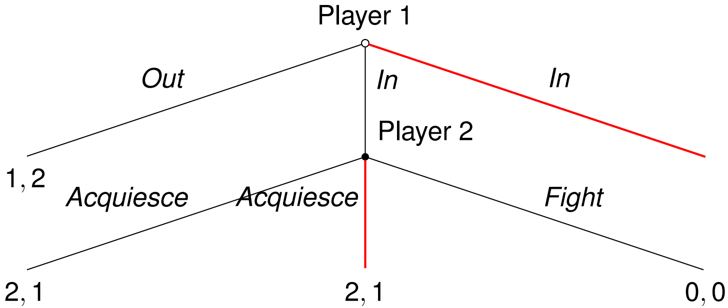
It does so by adding a sequential rationality requirement: Player 2 must be best responding at every node, even if that node is never actually reached in equilibrium.

Backward induction

We can solve for SPNE many extensive form games using backward induction:

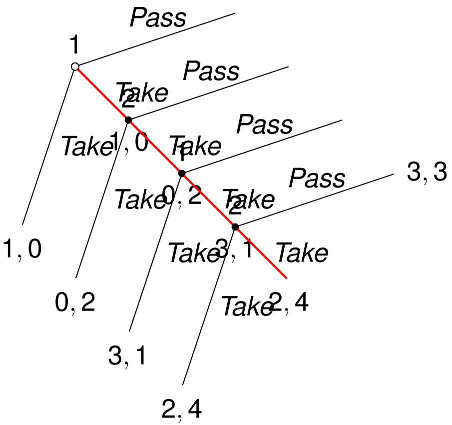
1. Starting with all of the last node, determine the player's best response at that node
2. Using this information, determine what the second-to-last player on each path should do assuming that the last player best responds
3. Repeat this process until you reach the first move(s) in the game

Example: Entry Game



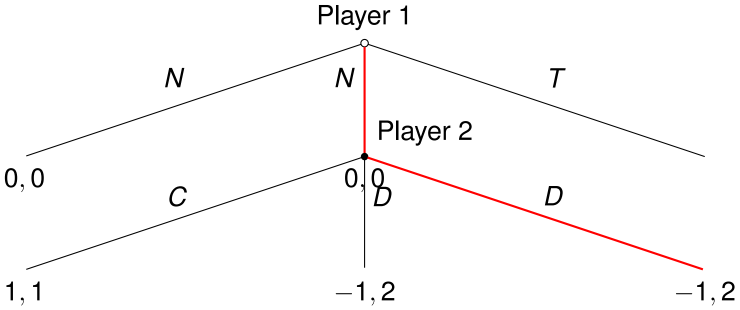
SPNE: (In, Acquiesce)

Example: Centipede Game



SPNE: (Take, Take), (Take, Take)

Example: Trust game



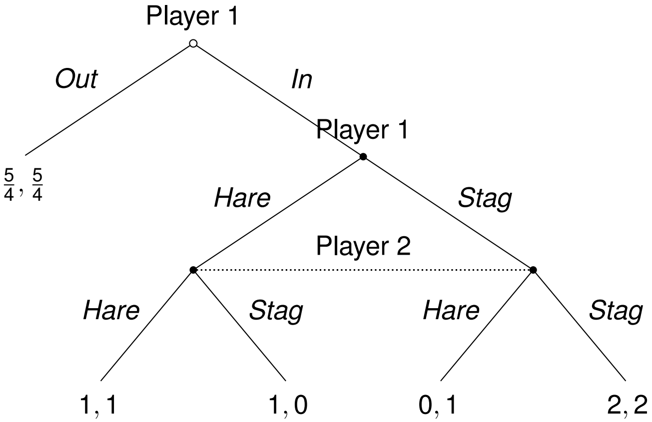
SPNE: (N, D)

Allowing for simultaneous moves: Voluntary Stag Hunt

Consider a different sequential Stag Hunt game: Player 1 chooses to play or not play. If Player 1 does not play then both players get a payoff of $\frac{5}{4}$. If she chooses to play then they play the simultaneous Stag Hunt game.

We can represent this as an extensive form game by recalling that simultaneous moves are equivalent to sequential moves in which the players do not see each others' actions.

This adds one new component to the extensive games we have studied so far: information sets showing what each player knows when she moves.



Plain old backward induction fails us here: there is no last move. We need a more general way to move forward.

Extensive form games with simultaneous moves

An extensive form game (generalized to allow simultaneous moves) is:

1. A set of players, N.
2. Players payoffs (u_i for all i in N) as a function of outcomes (terminal nodes)
3. Order of moves
4. Actions of players when they can move
5. What each player knows when they move
   - For this we use information sets, where two decision nodes x and y are in the same information set if the player does not know whether she is at x or y when she moves

A strategy in an extensive form games, like last week, is a complete plan of play: what each player would do every time she moves.

Some intuition about solving general extensive form games

A backward-induction-like method for solving the voluntary Stag Hunt game:

1. Solve the simultaneous-move Stag Hunt game as a normal form game
2. Assuming that Player 1 knows what equilibrium will be played after she chooses In, determine whether she should play in or out

The idea: Even though there is no last move, the game breaks up into two distinct games that let us solve backwards in a way that is similar to backward induction.

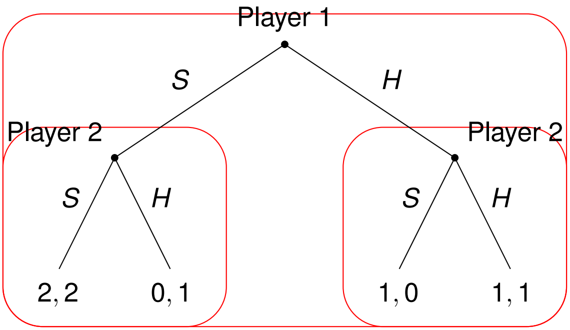
To apply this logic beyond this example, we need to formalize the sense in which the game breaks up into distinct games.

Subgames

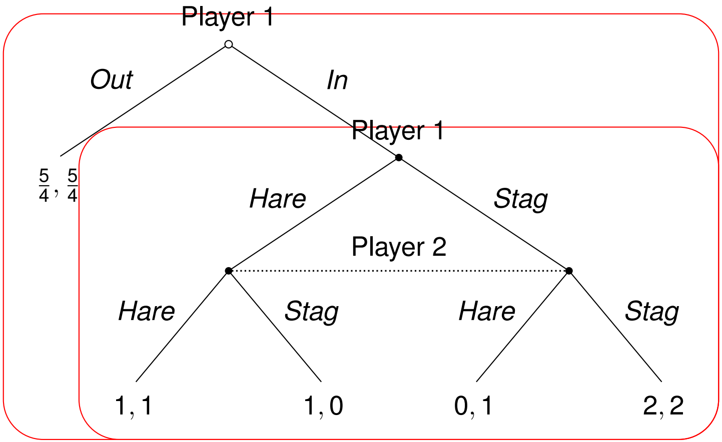
Definition

A subgame of an extensive form games consists of only a single node and all of its successors, with the property that any two nodes in the same information set are in the same subgame.

Subgames: Example 1



Subgames: Example 2



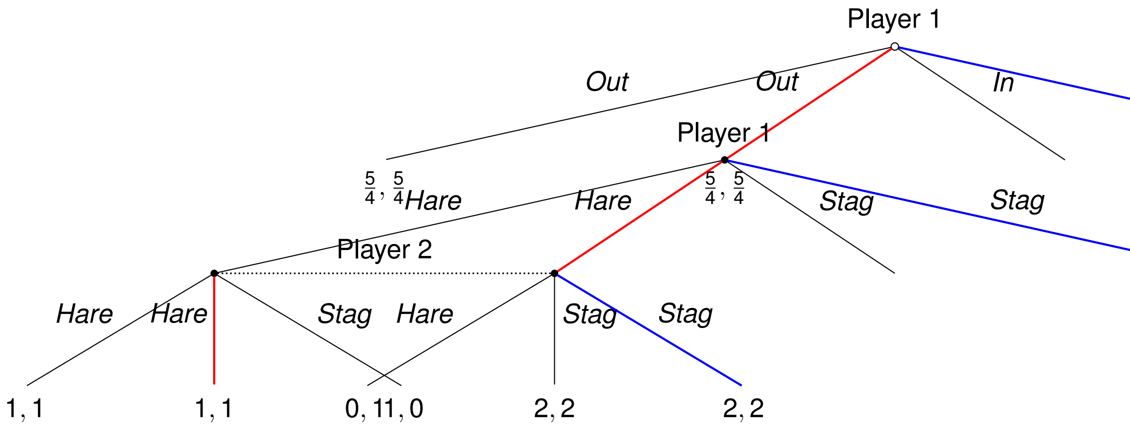
Subgame perfection

Definition

A strategy profile is a subgame-perfect Nash equilibrium to the extensive form game Γ if the strategies are a Nash equilibrium in each proper subgame of Γ.

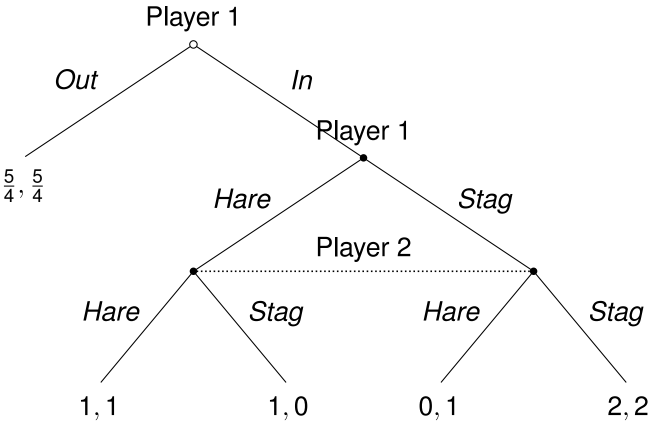
This is a direct generalization of our sequential rationality requirement from last week (a NE to a one-player subgame is just their optimal choice)

The voluntary Stag Hunt (pure strategies)



The PSNE to the Stag Hunt are (H, H) and (S, S).
P1 should choose Out if they will play the Hare equilibrium and In if they will play the Stag equilibrium
$\implies$ the pure strategy SPNE are ((Out, Hare), Hare) and ((In, Stag), Stag)

Voluntary Stag Hunt (mixed strategies)



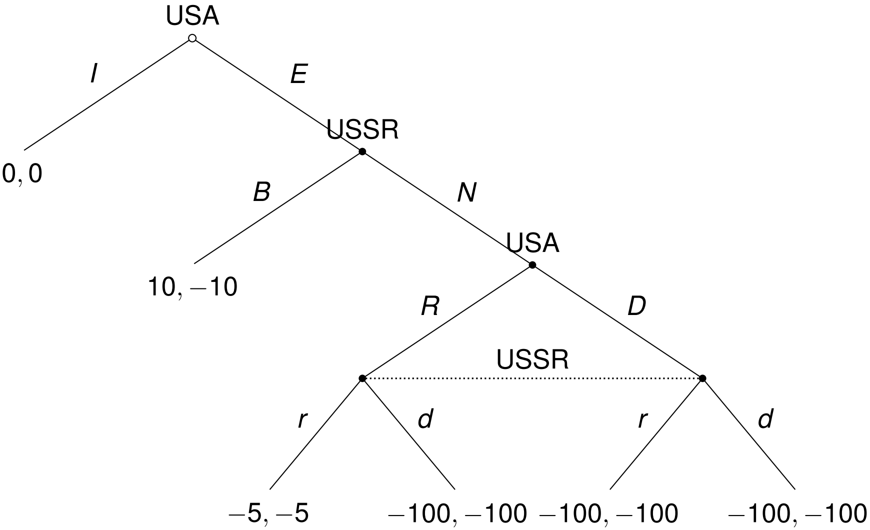
- The MSNE to the Stag Hunt is $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.
- This gives P1 an expected payoff of 1. Her payoff from choosing Out is greater than 1.
- $\Rightarrow$ there is also a mixed SPNE: $((\text{Out}, (\frac{1}{2}, \frac{1}{2})), (\frac{1}{2}, \frac{1}{2}))$.

Example: Mutually Assured Destruction

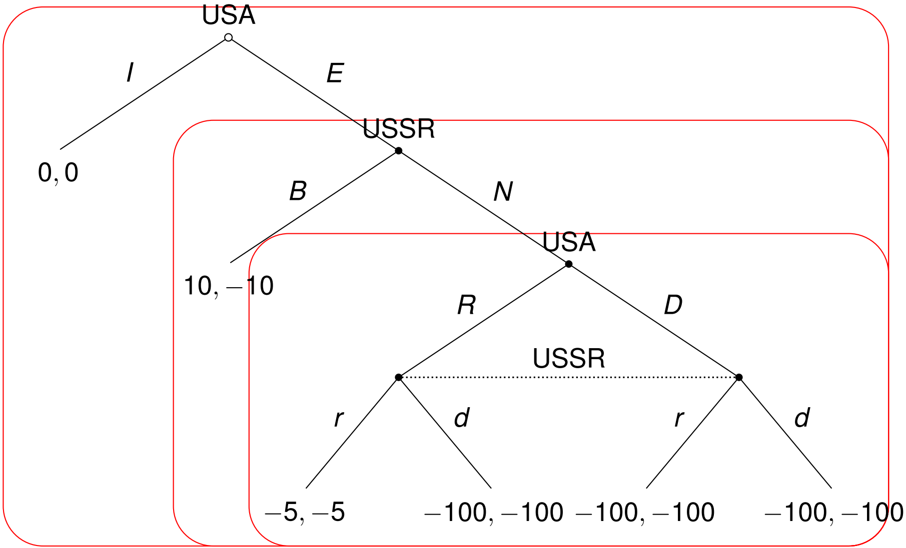
Light background: Cuban missile crisis

- The crisis started when the US discovered Soviet nuclear missiles in Cuba
- US escalated the crisis by quarantining Cuba
- The USSR back down, agreeing to remove its missiles from Cuba
- Suggests US had a credible threat: if you don't back down we both pay.
- Could this indeed be credible? Let's look at a stylized game.

Example: Mutually Assured Destruction



MAD Subgames



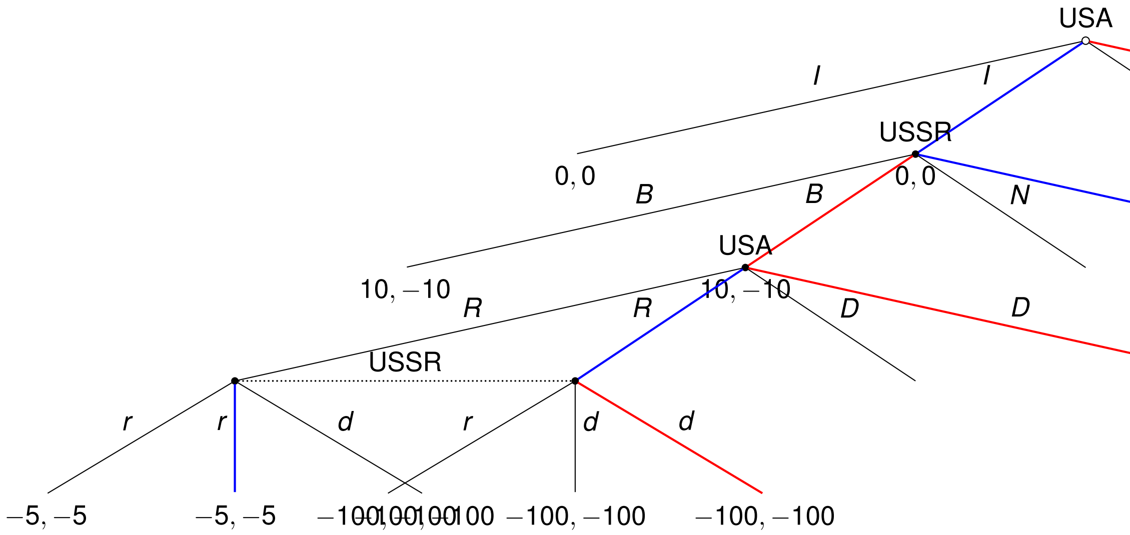
Solving the MAD game: Step 1

What first?

Last subgame:

		USSR r	USSR d
USA	R	$-5, -5$	$-100, -100$
	D	$-100, -100$	$-100, -100$

- PSNE?
  - (R, r), (D, d)
- MSNE?
  - No (rare occasion of an even number of NE).



⇒ we find two SPNE: ((I,R), (N,r)) and ((E,D), (B,d))

Prisoner's Dilemma, then Stag Hunt

- Game is simple: Two players play a simultaneous-move PD, see their final payoffs, then they play a simultaneous-move Stag Hunt. Players' payoffs are the sum of payoffs for the two simultaneous-move games.

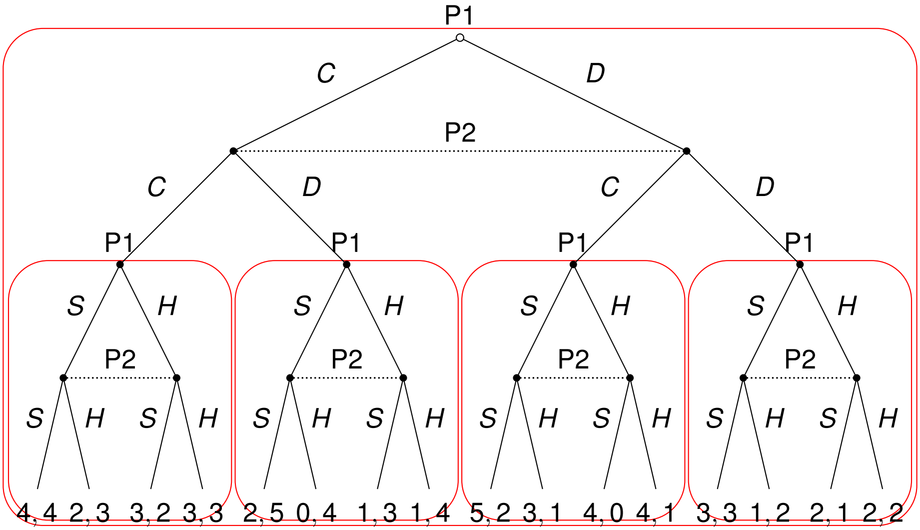
- There are many equilibria to this game so we'll focus on a narrower question: Is there some equilibrium in which both players cooperate in the PD stage?

The normal form games

		Player 2	
		C	D
Player 1	C	2, 2	0, 3
	D	3, 0	1, 1

		Player 2	
		Stag	Hare
Player 1	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

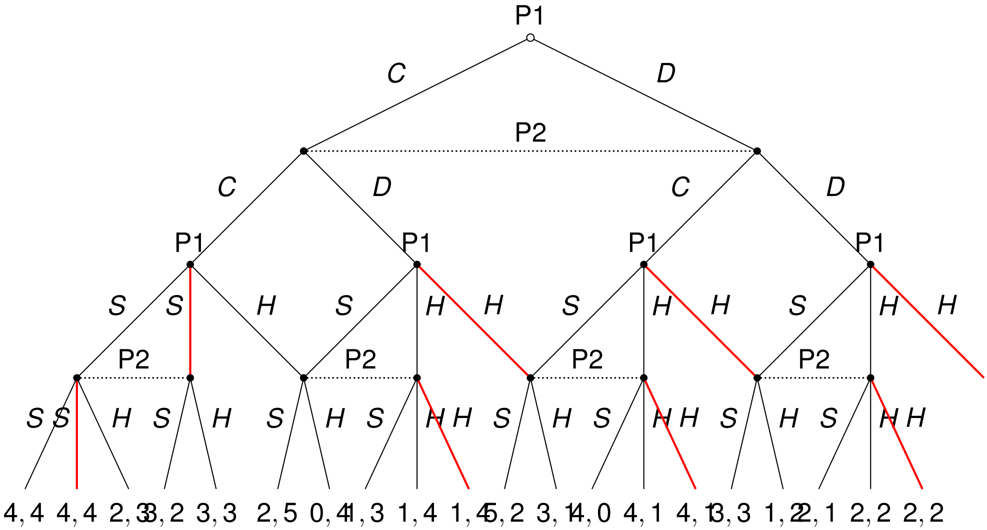
PD + SH Game Tree



Intuition: Cooperation and punishment

- The last stage is always a Stag Hunt so the players must play one of the equilibria to the stag hunt in any of the last subgames
- ...yet these constitute four different final subgames. They may play a different equilibrium to the Stag Hunt game in some of these final subgames than in others.
- This opens the door to a punishment strategy: If we both cooperated in the first period, we will play the (S, S) equilibrium in the second period. If anyone defected, we'll play the (H, H) equilibrium. Since both players would rather play the (S, S) equilibrium in the second period, perhaps this is enough to induce first-period cooperation.
- Let's see!

PD + SH Punishment Strategy



Induced first-period game

To see if this equilibrium indeed induces cooperation in the first period, we rewrite the PD game using the final payoffs induced by the punishment strategy:

		Player 2 C	Player 2 D
Player 1	C	$4, 4$	$1, 4$
	D	$4, 1$	$2, 2$

Is (C, C) an equilibrium to this game?

Remarks

- This problem is a little preview of repeated games: sometimes we can sustain cooperation when we otherwise could not due to the promise of future cooperation.

- Would this same approach work for a twice-repeated PD? Why or why not?

Policy bargaining and veto players

Theories of policymaking

- Sequential games are a useful tool for institutional analysis

- An institution tells us the rules of the game (who gets to move when, what can they do, etc.). If we combine this with information about the preferences of the players we can specify a game and make predictions.

- Questions we can answer:
  - What are the effects of institutions giving different actors agenda-setting or veto power?
  - What institutions tend to produce “good” policy outputs (however you want to define good policy)?
  - What can we infer about political power by observing the choices of the players?

Policy choice: Agenda setters and veto players

- Players: a committee chair (C) and a legislature (L)
  - L should be takent to represent the median legislator

- Policies: Numbers on the real line. x_0 denotes the status quo policy.

- Game: C proposes a new policy x, L chooses to accept or reject that policy. If the policy is accepted then the final policy is $y = x$. Otherwise $y = x_0$.

- Preferences: C and L have ideal points x_C and x_L, with preferences represented by

$$u_i(y) = -(x_i - y)^2$$

  for $i \in \{C, L\}$. Assume that $x_C < x_L$ (this is without loss of generality).

Remarks on the game

- The setup implies bills are considered under a closed rule (i.e. the committee recommends a bill and the floor must vote it up or down with no amendments). This is one possible institution but others are possible.

- The players could easily be re-labeled for other applications without changing the game. It could be Congress and the President, the Federal Reserve Chair and the FOMC, politicians designing an initiative for voters to approve, etc. The important thing is that one player has agenda-setting power and the other has veto power.

- This model is also very commonly used to analyze things like Supreme Court appointments: The President suggests a nominee with some ideology and the Senate must decide whether to approve the nominee.

Strategies

- L has an infinite number of decision nodes: one for each policy that x_C could possibly propose. L's strategy therefore prescribes an accept/reject decision for every possible policy. **Always remember and never forget: Strategies in sequential games are complete plans of action.**

- This is succinctly represented as an acceptance set $A_L \subset \mathbb{R}$ which is a set of policies that L will accept. That is, L chooses accept when $x \in A_L$ and reject otherwise.

- C has only one decision node, so C's strategy is simply a policy that she will propose.

General solution

- L's choice: for any proposal x, L's payoff is

$$\begin{cases} -(x - x_L)^2 & \text{if L accepts} \\ -(x_0 - x_L)^2 & \text{if L rejects.} \end{cases}$$

- Therefore L should accept if $|x - x_L| < |x_0 - x_L|$ and reject if $|x - x_L| > |x_0 - x_L|$.
- What should she do if she is indifferent? It is not clear from her preferences, but it will turn out to be useful to have her accept when she is indifferent. So we write her acceptance set

$$A_L = \{x \in \mathbb{R} : |x - x_L| \leq |x_0 - x_L|\}.$$

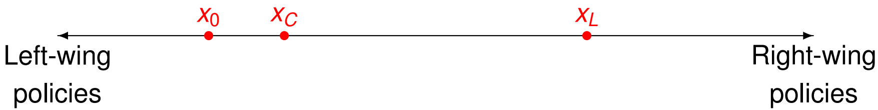
- Notice: The status quo is always an element of this set, so it's always an option for C not to change policy at all.

General solution

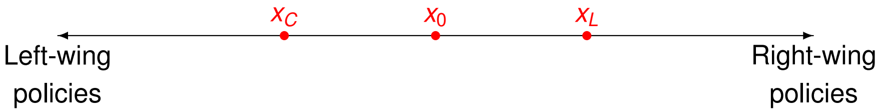
- Since having a proposal rejected is the same as proposing the status quo andhaving it accepted, we can limit C's choices to be in L's acceptance set.
- C's problem: Get as good of an outcome as possible subject to the constraint that L is willing to accept.
- In practice: if $x_C \in A_L$, propose ideal point. Otherwise, choose the nearest endpoint of A_L. (This will be clearer in the cases to follow.)

Analyzing the game: cases

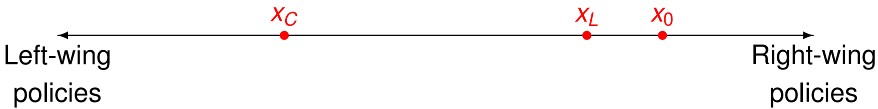
Case 1: $x_0 < x_C$



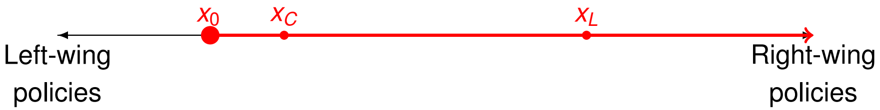
Case 2: $x_C < x_0 < x_L$



Case 3: $x_0 > x_L$

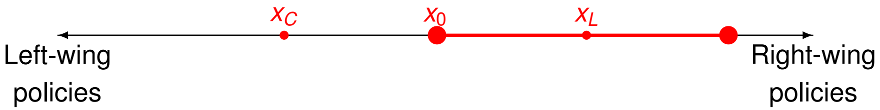


Case 1: $x_0 < x_C$



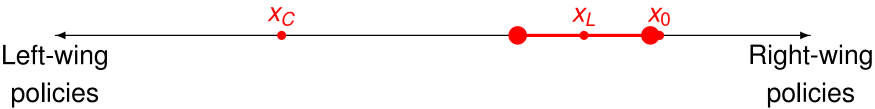
- L's acceptance set: all policies closer to x_L than the status quo: $[x_0, 2x_L - x_0]$.
- C can propose x_C and get it accepted, she does.
- The equilibrium (using our set notation for L's strategy) is therefore: $(x_C, [x_0, 2x_L - x_0])$.
- The equilibrium <u>outcome</u> is that x_C is proposed and accepted.

Case 2: $x_C < x_0 < x_L$



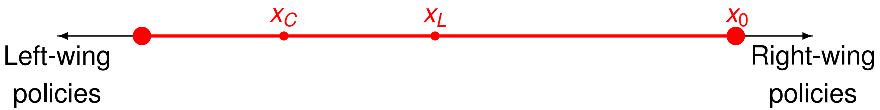
- L's acceptance set: still $[x_0, 2x_L - x_0]$ but this set is much smaller.
- What's the best policy C can get accepted in this case?
- x_0! So policy does not move in this case.
- The equilibrium is $(x_0, [x_0, 2x_L - x_0])$ and the equilibrium outcome is x_0.

Case 3: $x_0 > x_L$



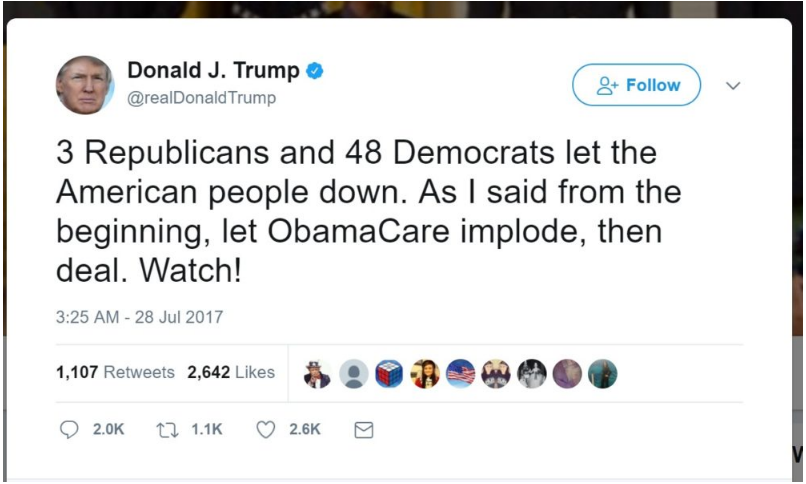
- L's acceptance set: $[2x_L - x_0, x_0]$
- What is the best C can do?
- The endpoint of the acceptance set: $2x_L - x_0$.
- The equilibrium here is $(2x_L - x_0, [2x_L - x_0, x_0])$. The equilibrium outcome is $2x_L - x_0$.

Case 3b: A more extreme status quo



Here the acceptance set expands so that *C* now gets his ideal point.

Oddly, a status quo that is much worse for *C* actually makes *C* much better off in this situation!


Donald J. Trump
@realDonaldTrump
Follow
3 Republicans and 48 Democrats let the American people down. As I said from the beginning, let ObamaCare implode, then deal. Watch!
3:25 AM - 28 Jul 2017
1,107 Retweets 2,642 Likes
2.0K
1.1K
2.6K

Remark 1: Agenda-setting power

- A puzzle: Our theories tell us that the median legislator must consent to any policy that passes, yet it seems that policies change considerably depending which party is in the majority (even if the median does not change that much)

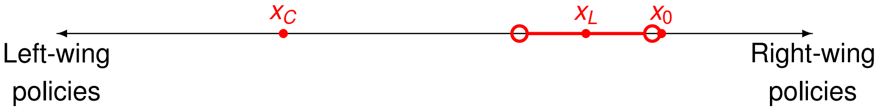
- An answer (Cox and McCubbins): The majority party has agenda-setting power. The power of the majority comes not from votes but from proposals. The majority party leadership controls what does and does not get voted on. As we saw, this can be quite significant.

- A question posed by Krehbiel but not answered here: Why would the majority choose a procedure that ends up skewing outcomes away from the median legislator?

Remark 2: The power of the veto

Remark 3: Tie-breaking rules

- We said that the veto player chooses to accept the proposed policy when she is indifferent. Of course, there is nothing making her do so. She could just as well reject when indifferent.
- So why did I say that in equilibrium she accepts when indifferent? Consider Case 3 but with a different decision rule



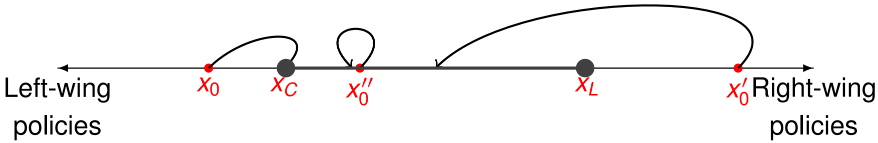
- Now C wants to get as close as possible to the left-most point in the acceptance set without quite reaching it. Maybe $2x_L - x_0 + \epsilon$ for some small ϵ.
- But this is not a best response because, for instance, $2x_L - x_0 + \epsilon/2$ is closer and still gets accepted. And this is true for any ϵ. So there cannot be an equilibrium like this because C does not have a best response!

Remark 4: Which policies are stable?

- A stable policy is a status quo from which there can be no policy movement.

- Intuition: Imagine repeating this game many times with short-sighted legislators. Each time, the policy outcome from the previous period is the status quo in the next period. Once you reach a stable policy, you will stay there forever.

Which policies are stable?



Policies move into the set $[x_C, x_L]$ in one step and never leave.

What about an open rule?

- We assumed before that policies were considered under a closed rule: The committee introduces a bill and it must be accepted or rejected by the floor, with no room for amendments.
- Perhaps this is why the agenda-setter has too much power, an we could limit this power by considering an open rule.
- This turns out to have some undesirable results.

The open rule game

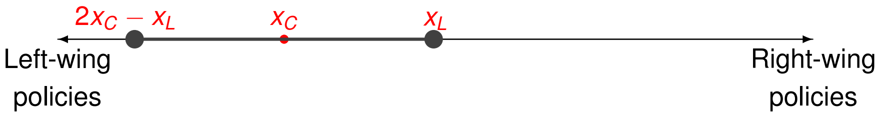
1. C decides whether or not to initiate any proposal. If C does not initiate a proposal, the game ends with the status quo x_0 being implemented.
2. If C initiates the proposal, L can amend it by changing the location to whatever she wants.
3. Payoffs are the same as before.

We have removed a lot of the agenda-setting power of C. However, C has gatekeeping power, deciding whether to initiate the consideration of a change to the status quo at all.

Solving the open rule game by backward induction

- If a proposal is intiated, L should clearly amend the bill to implement x_L.
- C's choice is therefore between x_0 and x_L. C initiates a proposal if x_L is closer to her ideal point than x_0.

Which policies are stable under an open rule?



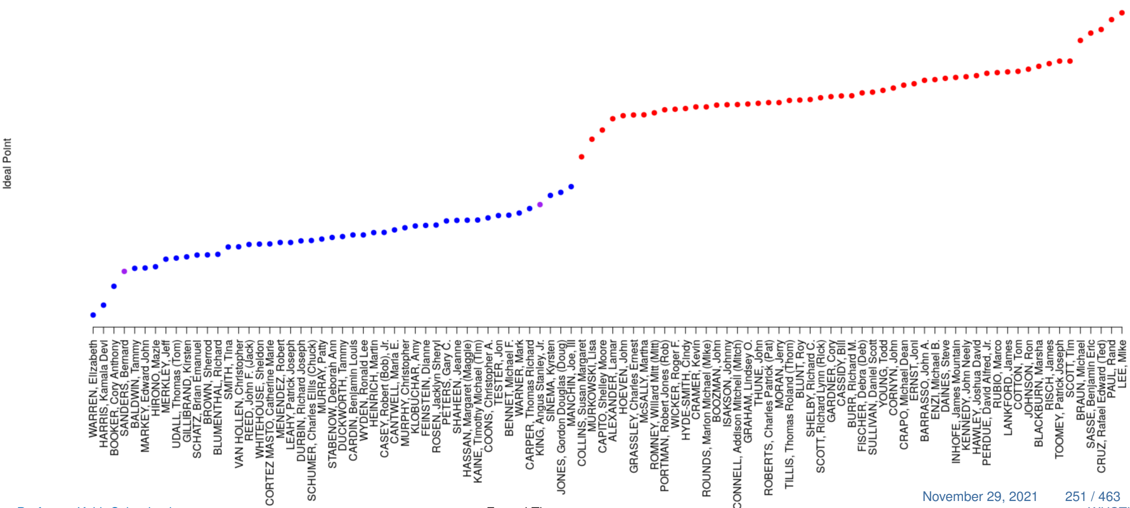
- Any policies closer to x_C than the median legislator's ideal point are now stable: in thise cases, C will choose not to propose any change and the status quo will remain. In all other cases, the policy will be x_L.
- But this is very pathological. Why?
- In the entire region from $2x_C - x_L$ to x_C, both players would prefer to move policy to the right, yet nothing happens.
- In these cases, L would love to commit to abstaining from amendments. So the median legislature may deliberately tie her hands and agree to consider the bill under a closed rule!

Supermajorities

- Example: The US Senate. The cloture rule requires 60 votes to end debate, so in practice 60 votes are usually required to conduct normal business in the Senate.

- Now there are two ways a policy could be blocked: a coalition of 40 people on the left might block it or a coalition of 40 people on the right might block it.

US Senate example



Supermajority game (closed rule)

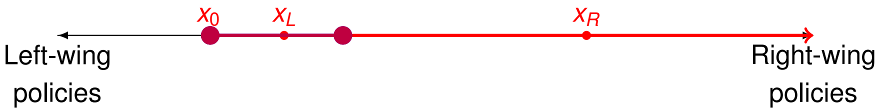
- Players: Committee chair C, then left and right pivots L and R
- C makes a proposal, both L and R must agree.
- Policy preferences are just like before, with the restriction that $x_L < x_R$.

Acceptance sets for L and R

As in the original game, the pivots will accept a policy if and only if it is weakly closer to their ideal point than the status quo.

Ignoring C for the moment, let's consider three different cases to learn what policies are feasible.

Case 1: $x_0 < x_L < x_R$

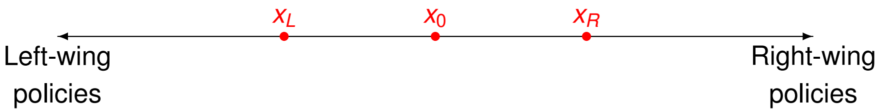


- R accepts anything to the right of the status quo until we get to very extreme right policies
- L accepts anything closer to her ideal point as well (a smaller set)
- The acceptance set is the overlapping portion of these two sets (here identical to L's acceptance set)

Case 2: $x_L < x_R < x_0$

The same story in reverse.

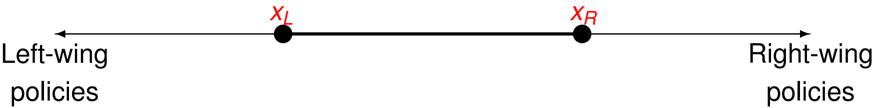
Case 3: $x_L < x_0 < x_R$



No policies are mutually acceptible to both pivots!

Here we have gridlock without even considering the preferences of C: there is a range of policies that <u>no</u> agenda-setter can ever change.

The gridlock interval



Krehbiel calls this the gridlock interval: no proposer can ever move a policy that is in this interval.

Current US Congress

What the theory implies for the current Congress:

No matter who the agenda-setter is (could be the House, for instance), if the status quo policy is somewhere in between the ideal points of Jon Tester and Mike Rounds, there will be no movement on that policy as long as there's a filibuster.

Agenda-setter strategies

The general solution for the agenda-setter strategy is similar to before: the agenda-setter gets the policy as close as possible to her ideal point subject to the constraint that the proposal can pass.

What does that entail here?

- If the status quo is in the gridlock interval, there is nothing the agenda-setter can do. She might propose the status quo, or just propose some other policy and get rejected.
- If the status quo is to the left of the proposer and to the left of the gridlock interval, the agenda-setter proposes the nearest policy inside the gridlock interval. Similar on the right-hand side.
- An extreme agenda-setter may still propose no change, if the status quo is between her and the nearest point of the gridlock interval. So there may be even more policy stability induced by the agenda-setter location.

Delegated Bargaining (Gailmard-Hammond)

A question in the literature on “legislative organization”: should committee chairs be representative of the chamber?

Gailmard and Hammond use a model of inter-cameral bargaining to think about this problem

The Gailmard-Hammond model

- Players:
  - S – the representative legislator in the Senate
  - H – the representative legislator in the House
  - C – a committee chair in the House
- Policy space: $x \in \mathbb{R}$
- Preferences:

$$u_j(x) = -(x - x_j)^2$$

(assume $x_H = 0 < x_S$)

G-H sequence of play

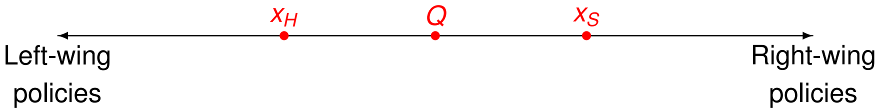
1. H chooses C's ideal point (interpret as, choosing a member of the legislature to head the committee)

2. Nature chooses bargaining roles: H makes a proposal to S with probability p and receives a proposal from S with probability $1 - p$

3. Bargaining process:
   - If S proposes: passage requires approval by C and H
   - If H proposes: C makes proposal which requires approval by S and H

4. If proposal passes it becomes policy. Otherwise the status quo Q is the final policy.

Case 1: $x_H < Q < x_S$



No change from the status quo is possible regardless of who is the proposer.

$\Rightarrow$ there is no strict benefit to choosing a biased Committee.

Case 2: $x_H < x_S < Q$

- C to the left:
  - When H proposes the outcome is no better: best outcome is still $2x_S - Q$
  - When S proposes, outcome is suboptimally (from H's perspective) far to the right
- C to the right:
  - When H proposes, the outcome is either the same or (if C is far enough to the right) further to the right
  - When S proposes, outcome is further to the right

$\Rightarrow$ can't do better than $C = x_H s$

Case 3: $Q < x_H < x_S$

- C to the left
  - When H proposes: The outcome is no better. Best outcome still $2x_S - Q$
  - When S proposes: Outcome may improve! Why?
- C to the right
  - Never optimal

$\Rightarrow$ The optimal committee is biased a little bit to the left

Repeated games

Prisoner's Dilemma: Big Picture

- Silly stories about prisoners aside, the PD is a fable about cooperation
- It captures situations in which: (1) the players would collectively benefit from cooperating, but (2) they individually have an incentive to cheat.
- Examples:
  - Exchange – players can deal honestly or cheat
  - An environmental agreement – players can honor their commitments or reneg and pollute

Repeated play

- In the one-shot PD we assume the players meet each other, choose whether to cooperate or defect, and then never see one another again. Perhaps the odds of cooperation are better with repeated play

- Players may think: “If I defect on this person today then she will defect on me tomorrow, but if I cooperate today she may cooperate tomorrow.”

Twice-repeated PD

Consider the following extensive form game: the players play a PD, observe the outcome, then play another PD. Recall the PD payoffs:

		P2 C	P2 D
P1	C	4, 4	-1, 5
	D	5, -1	1, 1

Note that this game has <u>FIVE</u> subgames: the whole game and then 4 subgames which are each a PD following each possible first-stage outcome.

Solving the twice-repeated PD

I claim that the only SPNE to this game is one in which both players play D at every information set.

Proof:

- In each of the 4 final subgames, SPNE demands that the strategies restricted to that subgame are a NE of the game. The only NE to the PD game is (D, D).
- In the first stage, players know that they will play (D, D) in the final stage regardless of the outcome. Thus, game play today does not affect game play tomorrow, so the players will just play the NE to the PD in the first stage as well.

Note: This same reasoning holds if we repeat the PD <u>any</u> finite number of times. In the last stage the players must play (D, D), in the stage before that the players must also play (D, D) knowing that today's outcome does not affect game play tomorrow, and in the stage before that....(and on and on)

Infinitely repeated games

The problem with the finitely repeated PD is the following: at some point there is a time period in which all players know that they will never see one another again. This is typically unrealistic!

We fix this by repeating the game an infinite number of times.

We do not need to be too literal about this: obviously people eventually die. The point is that there is no period at which everyone <u>knows</u> that this is the last period.

Infinitely repeated games

Let G be some normal-form game. Let $t = 1, 2, \ldots$ index a countably infinite number of time periods. An infinitely repeated game is one in which the players play G in every time period, observing the outcome at the end of each period.

Players' payoffs in an infinitely repeated game are <u>discounted</u> sums of their payoffs from each stage game.

Discounting and discounted present value

Suppose a player i gets an infinite stream of payoffs

$$(v_i^1, v_i^2, \ldots, v_i^t, \ldots).$$

The discounted present value of her payoffs, given discount factor $\delta \in [0, 1)$, is:

$$v_i = \sum_{t=1}^{\infty} \delta^{t-1} v_i^t.$$

Elaboration on discount factor

We can interpret δ in two ways:

1. δ is a measure of patience: $\delta = 0$ means that a person only cares about the current time period, $\delta \approx 1$ means that the person cares about tomorrow approximately as much as today, and $\delta = .95$ means that the person cares about tomorrow 95% as much as today.

2. Alternative: δ is the probability that the game continues to the next period, and with probability $(1 - \delta)$ the game ends in any given period.

Using discount factors: A finite example

Suppose there are three periods and a person's stream of payoffs in those periods is $(1, 3, 2)$. Assume $\delta = .9$. The discounted present value of these payoffs is:

$$1\delta^0 + 3\delta^1 + 2\delta^2 = 1 + 3 \times .9 + (2 \times .9 \times .9) = 5.32.$$

If we start at time 2, the person's discounted present value is

$$3 + 2 \times .9 = 4.8.$$

And we can apply the discounted present value formula to this starting in time 1 to get back the same answer:

$$1 + 4.8 \times .9 = 5.32.$$

Using discount factors: geometric series

Assume a person gets a constant stream $(v, v, \dots)$ (i.e. the payoff is v in every period forever). The series

$$(v, \delta v, \delta^2 v, \dots, \delta^{t-1} v, \dots)$$

is a geometric series.

Some useful facts about geometric series (presented without proof):

- The sum of a geometric series over a finite time horizon T is $\sum_{t=1}^{T} \delta^{t-1} v = \frac{v(1-\delta^T)}{1-\delta}$
- The sum of an infinite series converges to $\sum_{t=1}^{\infty} \delta^{t-1} v = \frac{v}{1-\delta}$.
- The infinite sum over only the odd time periods is: $\sum_{k=1}^{\infty} \delta^{2k} v = \frac{v}{1-\delta^2}$
- The infinite sum over only even time periods is $\sum_{k=1}^{\infty} \delta^{2k+1} v = \frac{\delta v}{1-\delta^2}$

A warning: When you look this stuff up pay attention to whether they start time at 0 or 1.

Putting it together: the infinitely repeated PD

At each of an infinite number of time periods, 2 players play the stage game:

		P2	
		C	D
P1	C	4, 4	-1, 5
	D	5, -1	1, 1

Let v_i^t denote the payoff of player i in time t. Each player maximizes $\sum_{t=1}^{\infty} \delta^{t-1} v_i^t$.

Strategies

Strategies are potentially complicated in this game because they can be history dependent. A strategy speficies a complete plan of play: i.e. what will the player do at each time t given every possible history of the game up to time t.

This complicates analysis but opens up the door to the type of strategies we've discussed: history-dependent strategies allow us to punish a player in the future for defecting in the past and reward them for cooperating.

The one stage deviation principle

How can we possibly check a strategy against so many (potentially complicated) deviations?

The one-stage deviation principle turns out to make this task tractable.

Definition

A strategy is one-stage unimprovable for a player i if there is no information set at which i would change her action given her opponents' strategy and holding constant her own strategy at all other information sets.

Fact: A strategy is a best response if and only if it is one-stage unimprovable.

Implication: We don't have to check super complicated multi-stage deviations, we can just check each information set and see if the player would make a one-time deviation.

Supporting cooperation in the repeated PD: Grim trigger

Consider the following strategy profile, called the “grim trigger’ strategy:

- If nobody has ever defected, both players play cooperate
- If anybody has ever defected, both players play defect

Thus, on the path of play, both players always cooperate. Both players know that, if they ever defect (trigger), everyone will defect until the end of time (grim).

When is grim trigger a SPNE?

There are really two “states” of the game: the state in which nobody has defected and the state in which somebody has defected.

Using the one-stage deviation principle we only need to verify that the players should cooperate in the first state and defect in the second.

Defection state:

If someone has defected in the past, the strategy profile says that both players will always defect.

Since cooperating today cannot change this path, and defection is dominant in the static PD, each player is clearly willing to defect.

Cooperation state:

We check that the players should cooperate. Cooperating means that we continue cooperating forever (since we are only checking a one-stage deviation), which gives a discounted present value of

$$\frac{4}{1-\delta}.$$

Defecting means a higher payoff today (5 instead of 4) but both players defect in all future periods. The discounted present value is

$$5+\delta\frac{1}{1-\delta}.$$

Thus, the players cooperate if

$$\frac{4}{1-\delta} \geq 5+\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{1}{4}.$$

Will less severe punishments work? One period limited punishment.

Consider an alternative strategy profile:

- If nobody defected in the previous period we are in the cooperative state and choose cooperate
- If last period was a cooperative state and someone defected, move to a punishment state, and choose defect in this period
- If last period was a punishment state (regardless of what people did in the punishment state), move back to the cooperative state and choose cooperate

In other words, cooperate until someone defects, and then play defect for one period before returning to cooperation.

When is limited punishment a SPNE?

Punishment state: Since we return to a cooperative state no matter what either player does, it is clearly optimal for both players to choose defect in this state.

Cooperative state: Cooperate if

$$\underbrace{\frac{4}{1-\delta}}_{\text{value of cooperating}} \geq \underbrace{5 + \delta 1 + \delta^2 \frac{4}{1-\delta}}_{\text{value of defecting}}.$$

This holds if $\delta \geq \frac{1}{3}$. So a less severe punishment requires a bit more patience.

Review: Repeated PD

The players repeatedly played a PD with stage-game payoffs:

		P2	
		C	D
P1	C	4, 4	-1, 5
	D	5, -1	1, 1

The question: Can cooperation be sustained in a repeated setting?

Some strategies that work

- Grim trigger: We cooperate on the path of play. As soon as anyone defects, we defect in all future periods. We determined that this is a SPNE to the repeated game when $\delta \geq \frac{1}{4}$.

- One-period limited punishment: We cooperate on the path of play. As soon as anyone defects, we defect for one period, then return to cooperating. We determined that this is a SPNE when $\delta \geq \frac{1}{3}$.

Generalized limited punishment

We may consider a generalization of the two strategies above: we cooperate on the path of play, and when someone defects we defect for T periods before returning to cooperation.

$T = 1$ and $T = \infty$ are the versions we have already done.

When is T-period limited punishment a SPNE for given T?

Same calculations as before. Clearly players should defect when the strategy tells them to defect, so we can just check the cooperation conditions.

Payoff from defecting:

$$5 + \delta \frac{1 - \delta^T}{1 - \delta} + \delta^{T+1} \frac{4}{1 - \delta}.$$

Payoff from cooperating:

$$\frac{4}{1 - \delta} = 4 + 4\delta \frac{1 - \delta^T}{1 - \delta} + \delta^{T+1} \frac{4}{1 - \delta}.$$

When is T-period limited punishment a SPNE for given T?

Cooperate if

$$
\begin{aligned}
4 + 4\delta\frac{1-\delta^T}{1-\delta} + \delta^T\frac{4}{1-\delta} &\geq 5 + \delta\frac{1-\delta^T}{1-\delta} + \delta^T\frac{4}{1-\delta}\\
4 + 4\delta\frac{1-\delta^T}{1-\delta} &\geq 5 + \delta\frac{1-\delta^T}{1-\delta}\\
\frac{\delta(1-\delta^T)}{1-\delta} &\geq \frac{1}{3}.
\end{aligned}
$$

This is as far as we can simplify, but we get some immediate insights:

- This becomes easier to satisfy as T gets large (the LHS is increasing in T for constant δ), so more severe punishments work for lower levels of patience
- We can check our old answers with this:
  - $T = 1$: Condition becomes $\delta \geq 1/3$, our previous answer
  - $\lim_{T\to\infty} \frac{\delta(1-\delta^T)}{1-\delta} = \frac{\delta}{1-\delta}$, so the condition becomes $\delta \geq \frac{1}{4}$ as before.

Many equilibria in repeated games

It seems that a lot of strategy profiles work as SPNE to a repeated game.

In fact I have just characterized an infinite number of SPNE to the repeated PD when $\delta \geq \frac{1}{4}$. And in fact, that is just a subset of the SPNE.

And keep in mind: In addition to all these equilibria, it is of course still also an equilibrium for all players to defect at every history.

Some questions arise:

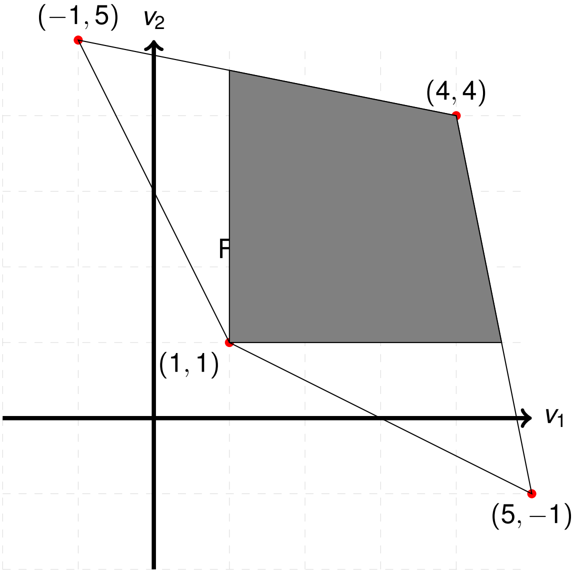
- How are we supposed to make predictions when there are so many equilibria?
- Can we at least characterize what is possible?

The folk theorem

I can't give you a unique prediction for a specific game. What I can do is tell you what payoffs are attainable for a high enough δ in any given game. I do this using the folk theorem.

The folk theorem says (informally): For any given stage game, any payoffs that are (1) feasible and (2) weakly better for each player than the worst equilibrium payoff are the expected payoffs of some SPNE to the repeated game for large δ.

Visualizing the folk theorem for the PD



Remark 1: Good news or bad news?

We can take the folk theorem as good news or bad news:

- Good news: Good stuff can happen through repetition even when the stage game equilibrium is suboptimal (as in the PD)
- Bad news: We cannot make solid predictions from infinitely repeated games. The model does not restrict outcomes all that much as $\delta \to 1$.

Remark 2: The role of institutions

One understanding of the role of institutions (i.e. from legal theory) is that they help us coordinate on good equilibria.

e.g. what side of the road should we drive on? There is a clear interest in coordination, so a law announcing that we all drive on the left-hand side is self-enforcing.

This is a way to understand solutions to the repeated PD. If an institution can create a common conjecture that we will all play (for example) the one-period limited punishment equilibrium, then nobody has an incentive to deviate.

Remark 2 (continued)

Consider an international agreement. The problem that many of these solve is something that looks like an n-player PD: there is an interest in global cooperation (on, say, climate) but individual countries have incentives to shirk.

Worse, international organizations typically don't have enforcement power. So what can they do?

One answer: The role of the IO is to announce a self-enforcing equilibrium. The IO announces, for instance, "If any country defects we will all defect on that country for a time before returning to cooperation" If the countries sign on to this it creates common knowledge that this is the equilibrium, and beyond that it is self-enforcing (by virtue of being an SPNE).

Dynamic bargaining

Two-player setup

- Examples:
  - Leaders of two countries settling a territorial dispute.
  - Two veto players negotiating over a policy

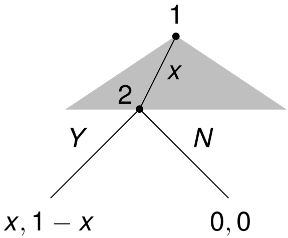
Relaxing assumptions from spatial games before: No one player has a monopoly over agenda-setting power.

Alternating offers bargaining

- $T \geq 1$ time periods. In odd-numbered time periods player 1 is the proposer. In even-numbered time periods player 2 is the proposer.
- The proposer suggests a division of a one-unit prize, where $x \in [0, 1]$ is the proportion that player 1 gets and $1 - x$ is the proportion that player 2 gets.
- As soon as a proposal is accepted the game ends.
- Players discount utility (so there is a cost for waiting)
- Payoffs (let T denote the period in which an offer is accepted):

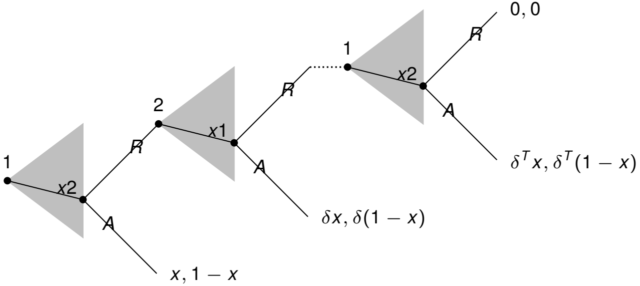
$$
\begin{aligned}
u_1(x, T) =& \delta^T x \\
u_2(x, T) =& \delta^T (1 - x)
\end{aligned}
$$

T=1: Ultimatum game



- This game is familiar (similar to spatial bargaining but simplified)
- What is the SPNE?
- 2 accepts any offer. 1 offers $x = 1$.
- This corresponds tot he case in which player 1 has a monopoly over agenda-setting power

$T > 1$ (finite)



Solving by backward induction

Assume T is odd (we can do a similar exercise for T even). Using backward induction, starting form the last period:

- $t = T$. P1 offers $x = 1$ and P2 accepts (ultimatum game). $v_1\ \delta^{T-1}$ and $v_2 = 0$.
- $t = T - 1$: P1 accepts any offer resulting in a payoff greater than or equal to δ^{T-1}. P2 offers $x = \delta$ and P1 accepts. Payoffs are $v_1 = \delta^{T-1}$ and $v_2 = \delta^{T-2}(1 - \delta)$
- $t = T - 2$: P2 accepts an x such that $\delta^{T-3}(1 - x) \geq \delta^{T-2}(1 - \delta)$ which implies $x \leq 1 - \delta + \delta^2$. P1 offers exactly $x = 1 - \delta + \delta^2$ and P2 accepts. Payoffs are $v_1 = \delta^{T-3}x = \delta^{T-3} - \delta^{T-2} + \delta^{T-1}$ and $v_2 = \delta^{T-2} - \delta^{T-1}$.
- $t = T - 3$: P2 offers $x = \delta - \delta^2 + \delta^3$ and P1 accepts. Payoffs are $v_1 = \delta^{T-3} - \delta^{T-2} + \delta^{T-1}$ and $v_2 = \delta^{T-4} - \delta^{T-3} + \delta^{T-2} - \delta^{T-1}$.

Solving by backward induction

The pattern is this:

In an odd period $T - s$ (with s even):

$$x_{T-s} = 1 - \delta + \delta^2 - \delta^3 + \cdots + \delta^s.$$

In an even period $T - s'$ (s' odd):

$$x_{T-s'} = \delta - \delta^2 + \delta^3 - \delta^4 + \cdots + \delta^{s'}.$$

Equilibrium outcomes

In the first period, by backward induction:

$$x_1 = 1 - \delta + \delta^2 - \delta^3 + \delta^4 + \cdots + \delta^{T-1}.$$

We can simplify by separating out the even and odd powers, using our gemoetric series knowledge, and gathering terms:

$$\begin{aligned}1 - \delta + \delta^2 - \delta^3 + \delta^4 + \cdots + \delta^{T-1} =& (1 + \delta^2 + \cdots + \delta^{T-1}) - (\delta + \delta^3 + \cdots + \delta^{T-2}) \\ =& \frac{1-\delta^{T+1}}{1-\delta^2} - \frac{\delta - \delta^T}{1-\delta^2} \\ =& \frac{1+\delta^T}{1+\delta}.\end{aligned}$$

P2 accepts this offer. Thus: $v_1^* = x_1 = \frac{1+\delta^T}{1+\delta}$ and $v_2^* = (1 - x_1) = \frac{\delta - \delta^T}{1+\delta}$.

Remark 1: T even

We can do a similar exercise assuming T is even. It is not worth repeating the calculations but we would get

$$x_1 = \frac{1 - \delta^T}{1 + \delta}.$$

Note: The difference is the $-\delta^T$ rather than $+\delta^T$ in the numerator.

Remark 2: Last-mover advantage

The difference between T odd and T even is which player is the last mover. That is, who would get to make an offer at time T?

The last mover advantage is worth $\frac{\delta^T}{1+\delta}$.

Remark 2: Last-mover advantage (continued)

The last-mover advantage reveals something artificial about this game: we rarely design bargaining protocols with a pre-specified last mover.

Happily, the infinite-horizon bargaining game that we'll analyze tomorrow does not have this limitation.

Remark 3: Limiting cases

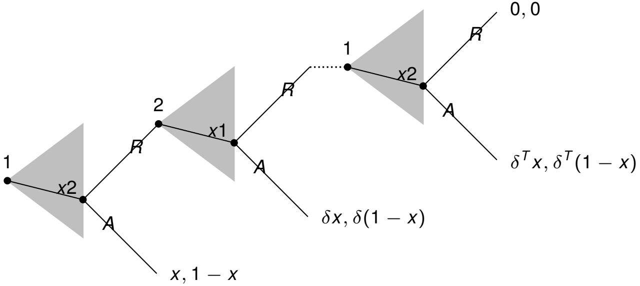
Recall that, for odd T, we have the following equilibrium payoffs:

$$v_1^* = \frac{1+\delta^T}{1+\delta}$$
$$v_2^* = \frac{\delta-\delta^T}{1+\delta}.$$

Consider what happens when δ gets large or small:

- As $\delta \to 0$, the payoffs approach that of the ultimatum game: player 1 takes the whole pie and player 2 gets nothing.
- As $\delta \to 1$ the payoffs approach $(\frac{1}{2}, \frac{1}{2})$. That is, foresight induces fairness!

Review: Finite Horizon Bargaining



Equilibrium outcomes

In the first period, by backward induction we found the first offer is:

$$x_1 = \frac{1+\delta^T}{1+\delta}.$$

P2 accepts this offer. Thus:

$$v_1^* = x_1 = \frac{1+\delta^T}{1+\delta}$$

and

$$v_2^* = (1 - x_1) = \frac{\delta - \delta^T}{1+\delta}.$$

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Infinite-horizon bargaining

We consider the same bargaining game except with no last period ($T = \infty$).

The most important difference: there is no “last mover.” The game keeps going as long as no offer has been accepted.

Note the difference from repeated games: this game still could end after one period, or any other number of periods, but we allow the possibility that it continues forever. (it won’t in equilibrium)

Difficulties and opportunities

- We cannot apply backward induction here: no last period!

- However, $T = \infty$ actually simplifies the game considerably once we know what to do. Why?
  - Stationarity: every odd period is essentially exactly the same as every other odd period (similar for even) so we can check just two states
  - Since there is no last mover, even and odd periods are really the same too, but with identities of the players switched
  - Implication from finite game holds: We should be able to reach an agreement at $T = 1$, since the first player can offer the second her continuation value

Solving the game

- Because of the stationarity property and the fact that there is no last mover, two things are true:
  - Player 1 should make the same offer every time she has a chance to propose
  - Player 2 should make the same offer as Player 1

- Using this fact, let $\overline{v}$ denote the value of ending the game when you are the proposer.

- The other player also gets $\overline{v}$ for ending the game as the proposer, so each player must make the other that gives a payoff equal to $\delta\overline{v}$.

- But this tells us that we must have $\overline{v} = 1 - \delta\overline{v}$ which implies $\overline{v} = \frac{1}{1+\delta}$.

- This completes our solution! Both players keep $1 - \delta\overline{v}$ and offer $\delta\overline{v}$ every time they propose. On the path of play the first offer is accepted.

Remark 1: Proposer advantage

Computing equilibrium utilities, we have

$$\begin{aligned} v_1^* &= \frac{1}{1+\delta} \\ v_2^* &= \frac{\delta}{1+\delta} \\ v_1^* - v_2^* &= \frac{1-\delta}{1+\delta}. \end{aligned}$$

The last line is the "proposer advantage."
Note:

$$\begin{aligned} \lim_{\delta \to 1} \frac{1-\delta}{1+\delta} &= 0 \\ \lim_{\delta \to 0} \frac{1-\delta}{1+\delta} &= 1 \end{aligned}$$

So we still have the insight that foresight induces fairness. This seems to be true in a very wide variety of bargaining games.

Remark 2: Limit of the finite game

The payoffs to the infinite-horizon bargaining game corresponse to the limit as $T \to \infty$ in the finite game.

This cannot always be counted on, but happens to be true here.

Legislative bargaining

The alternating offers bargaining game works nicely for certain two-player bargaining situations, such as an international dispute or an economic exchange.

We want to make this model work well for <u>policymaking</u> so for this we will introduce a few changes:

- More than two players
- Majority rule to accept a proposal
- Uncertain proposal rule: instead of a preset order we have uncertainty over who will get to propose in the future (say, generated by elections or sways in public opinion)

Legislative bargaining setup

- n (odd) players $N = \{1, \ldots, n\}$.
- An proposal requires a majority $k = \frac{n+1}{2}$ to vote in favor to accept
- Random recognition rule: In each period, every player proposes with probability $\frac{1}{n}$
  - We could allow different recognition probabilities to represent different levels of agenda-setting power, but let's work with the easiest version of the model.
- The proposer offers a distribution $(x_1, \ldots, x_n)$ such that $x_i \geq 0$ for all $i \in N$ and $\sum_{i=1}^{n} x_i \leq 1$.
- If the proposal x passes at time t, the game ends with payoff $\delta^{t-1} x_i$. Otherwise, continue to the next period and a new proposer is selected.

Refinements

- There are too many SPNE to this game, so we refine the solutions in two ways to focus on realistic equilibria.
  1. Stationarity:
     1.1 Proposer uses the same proposal strategy each time she is recognized
     1.2 Voters only consider the current proposal and expectations about future proposals when voting
     1.3 Rules out: History-dependent strategies as in the repeated PD, avoiding folk theorem types of results
  2. Weakly undominated strategies:
     2.1 Voters vote in favor of a proposal if and only if it gives them at least their continuation value
     2.2 Rules out: People voting for crazy stuff just because they aren't pivotal.
- Stationarity in particular gets us the stationarity property from the 2-player game, so the game essentially starts over every round.

Continuation values

- Stationarity implies that the vontinuation value $\overline{v}_i$ is simply i's expected value from the game (since the game essentially starts over each period)

- Here, all the players are symmetric, so we focus on symmetric equilibria such that $\overline{v}_i = \overline{v}$ for all $i \in N$.

Legislative bargaining strategies

- Take $\overline{v}$ as given for a moment.
- Then voters approve of a proposal if $x_i \geq \delta\overline{v}$.
- Therefore, what is the optimal proposal strategy?
- Offer exactly x_i $\delta\overline{v}$ to exactly $k-1$ other players and keep the rest (so keep $1-(k-1)\delta\overline{v}$.). (Assume for now that this proposal is good enough for the proposer that they want this proposal accepted, but we'll verify this at the end.)
- Caveat: Since we are driving toward a symmetric equilibrium, we also require here that coalition partners are chosen randomly.

Finding the continuation value

We have the equilibrium strategies for a given continuation value. Now we just need to find the continuation value and we're done.

We have:

$$\overline{v} = \underbrace{\frac{1}{n}}_{\text{Pr[propose}} \left[\underbrace{1-(k-1)\delta\overline{v}}_{\text{value of proposing}}\right] + \underbrace{\frac{k-1}{n}}_{\text{Pr[in coalition]}} \underbrace{\delta\overline{v}}_{\text{offer from other proposer}}$$

$$= \frac{1}{n}\left[1-(k-1)\delta\overline{v} + (k-1)\delta\overline{v}\right]$$
$$= \frac{1}{n}.$$

Putting this all together: The equilibrium

The proposer chooses $k-1$ coalition partners at random, offers each of them $\frac{\delta}{n}$ and keeps

$$1-\delta\frac{k-1}{n}=1-\delta\frac{n-1}{2n}.$$

Note: This last expression is greater than $\frac{\delta}{n}$ so this proposal is incentive compatible.

Some remarks

- One implication of this model is that we should always expect minimal winning coalitions in settings like this.
- Another implication is that proposal power is valuable:

$$\left[1-\delta\frac{n-1}{2n}\right]-\frac{\delta}{n}=1-\delta\frac{n+1}{2n}>0.$$

- But also note that proposal power
  - Increases with n
  - Decreases with δ (foresight again induces fairness)

Extension: Supermajority rule

We can now slightly modify the model so that a supermajority $k^+ > \frac{n+1}{2}$ is needed to passa proposal.

- The proposers share is now
$$z = 1 - (k^+ - 1)\delta\overline{v}.$$
- The continuation values are then:
$$\overline{v} = \frac{z}{n} + \frac{k^+ - 1}{n}\delta\overline{v} = frac1n$$
(same as before).
- The proposer's share is now
$$1 - \delta\frac{k^+ - 1}{n}.$$
- $\Rightarrow$ the primary consequence of a supermajority rule is to decrease (but not eliminate) the proposer's advantage.

Asymmetric proposal power

- Until new we've assumed every player has the same probability of proposing. Now we'll relax this.

- A general distribution of proposal probabilities is possible but too tedious for today.

- Instead, consider a legislature with two parties, where the majority party has more agenda control.

Setup

- The players are now divided into two parties, A and B.
- Party A has $n_A = n - m \geq (n+1)/2$ members (i.e. A is the majority party)
- Party B has m members.
- Members of A are recognized with probability $p > \frac{1}{n}$.
- Members of B are recognized with probability $q < \frac{1}{n}$.
- $(n-m)p + mq = 1$, so that the model makes sense.
- Assume a more limited symmetry: Members of the same party must have the same strategy and continuation value.

Coninuation values and strategies

- Continuation values for the two types are v_A and v_B, respectively
- We will conjecture for now that $v_A > v_B$, but we will verify this at the end.
- Each player approves and proposal such that $x_i \geq \delta v_i$.

Proposal strategy

- Best strategy: Buy off the cheapest players first.

- Proposer from party A:
  - Gives δv_B to all m members of B
  - Gives δv_A to $k - m - 1$ randomly selected members of A
  - Keeps: $z_A = 1 - (k - m - 1)\delta v_A - m\delta v_B$.

- Proposer from party B:
  - Gives δv_B to $m - 1$ members of B
  - Gives δv_A to $k - m$ random members of A
  - Keeps: $z_B = 1 - (k - m)\delta v_A - (m - 1)\delta v_B$.

Continuation values

From these strategies, we get:

$$
\begin{aligned}
v_A =& p(1-(k-m-1)\delta v_A - m\delta v_B) + p(k-m-1)\delta v_A + \frac{qm(k-m)\delta v_A}{n-m} \\
v_B =& q(1-(k-m)\delta v_A - (m-1)\delta v_B) + (1-q)\delta v_B.
\end{aligned}
$$

This gives us 2 equations and 2 unknowns, which we can solve. I'll spare you the details. here is the solution for $n=3$ and $m=1$:

$$
\begin{aligned}
v_A =& \frac{(1-q)(1-\delta)}{2+q\delta-2\delta} \\
v_B =& \frac{q(2-\delta)}{2+q\delta-2\delta} \\
z_A =& 1-\delta v_B \\
z_B =& 1-\delta v_A.
\end{aligned}
$$

Last step

Finally, this is only an equilibrium when $v_A > v_B$, which we need to check. This turns out to be true when:

$$q < \frac{1-\delta}{3-2\delta} \leq \frac{1}{3}.$$

Intuition: B's proposal power disadvantage must be large enough to offset the effect of being included in every winning coalition.

Static games of incomplete information

Motivation

- We're heading toward games of incomplete information: so far the players know everything about the game, but we are going to need to relax this assumption in order to write realistic models of some situations
- Sometimes during the game the players will receive some information and have a chance to learn.
- Examples:
  - A policymaker does not know what policy will best stimulate the economy. An economist provides some advice to the policymaker in the form of an economic or empirical analysis. How should she update her beliefs?
  - Congress is bargaining over policy with the President but does not perfectly know the President's preferences. A bill is passed and the President vetoes the bill. What beliefs should Congress have about the President's preferences in light of this action?

Another example: A trip to the doctor

- During a trip to the doctor's office, the doctor orders a test for a particular illness
- The test comes back positive. You want to know the probability that you have the disease.
- Several ingredients required in order to form this belief:
  - The false positive rate of the test (how likely is it that the test comes back positive when you don't have the disease)
  - The false negative rate of the test (how likely is it that it comes back negative when you do have the disease)
  - Important but often overlooked: The prior probability that you had the disease. What was your belief that you had the disease before you got the test?

Bayesian conditional probability

- What we're trying to compute is a conditional probability: the probability that some even A occurs (e.g. you have a disease) given that another event B occurred (e.g. you got a positive test result)

- This is written $\Pr[A|B]$ which you should read as "probability of A given B."

- This is computed via Bayes' rule:

$$\Pr[A|B] = \frac{Pr[A \text{ AND } B]}{\Pr[B]} = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

Back to the doctor's office

- Suppose before I went to the doctor I thought I had a 60/40 chance of having the disease
- The doctor says that the false negative rate of the test is 5% so if I have the disease there is a 95% chance of a positive test result
- The false positive rate is 3%, so if I do not have the disease there is only a 3% chance of a positive test result.
- The test comes up positive. What is the probability I have the disease?

Computing Pr[disease] using Bayes' rule

$$\Pr[A|B] = \frac{Pr[A \text{ AND } B]}{\Pr[B]} = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

$$\Pr[\text{disease}] = .6$$

$$Pr[\text{positive}|\text{disease}] = .95$$

$$\begin{aligned}
\Pr[\text{positive}] &= \Pr[\text{disease}] Pr[\text{positive}|\text{disease}] + (1 - \Pr[\text{disease}]) \Pr[\text{positive}|\text{healthy}] \\
&= .6 \times .95 + .4 \times .03 \\
&= .582
\end{aligned}$$

$$\Pr[\text{disease}|\text{positive}] = \frac{\Pr[\text{disease}] Pr[\text{positive}|\text{disease}]}{\Pr[\text{positive}]} = \frac{.6 \times .95}{.582} \approx .98.$$

Where we are

	Complete Info	Incomplete Info
Static	Nash	Bayesian Nash
Dynamic	Subgame Perfect Nash	Perfect Bayesian (future)

Incomplete Info

- So far we assume complete information: everything about the game was common knowledge
- In many applications players may not know every aspect of the game. For instance, they may be uncertain about the preferences of another player.
- We will represent this idea by assigning players “types” which affect their preferences, assuming types are assigned by “Nature” and not revealed to other players

Example: Incomplete information Bach or Stravinsky

Consider the familiar Bach or Stravinsky game:

		P2	
		Bach	Stravinsky
P1	Bach	$1, 2$	$0, 0$
	Stravinsky	$0, 0$	$2, 1$

We will make the following modification. Player 1 does not know whether Player 2 is friendly or unfriendly. If Player 2 is friendly the payoffs are as above. If Player 2 is unfriendly then Player 2 prefers to go to the concert alone.

Let's suppose the prior probability that Player 2 is friendly is $\frac{1}{2}$.

Incomplete Info BoS

1

prob. $\frac{1}{2}$

2

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

2 wishes to meet 1

prob. $\frac{1}{2}$

2

	B	S
B	$2,0$	$0,2$
S	$0,1$	$1,0$

2 wishes to avoid 1

- Since Player 1 does not know Player 2's type, she only has one information set.
- Player 2 does know her own type, so she has two information sets.
- This is reflected in the strategies. P1's possible pure strategies are B and S as before. P2 has four pure strategies:
  - BB: Choose B whether or not she is friendly
  - BS: B if friendly, S if not
  - SB: S if friendly, B if not
  - SS: Choose S whether or not she is friendly

Incomplete Info BoS

1

prob. $\frac{1}{2}$ — 2

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

2 wishes to meet 1

prob. $\frac{1}{2}$ — 2

	B	S
B	$2, 0$	$0, 2$
S	$0, 1$	$1, 0$

2 wishes to avoid 1

We'll start by reducing the game to a normal form using these strategies:

P1 \ P2	BB	BS	SB	SS
B	$2, (1, 0)$	$1, (1, 2)$	$1, (0, 0)$	$0, (0, 2)$
S	$0, (0, 1)$	$\frac{1}{2}, (0, 0)$	$\frac{1}{2}, (2, 1)$	$1, (2, 0)$

Explanation: Player 1 does not know Player 2's type so we plug in her expected payoff for each strategy. Player 2 does know her type so we plug in 2 payoffs: one for each type.

Incomplete Info BoS

		P2			
		BB	BS	SB	SS
P1	B	$2, (1, 0)$	$1, (1, 2)$	$1, (0, 0)$	$0, (0, 2)$
	S	$0, (0, 1)$	$\frac{1}{2}, (0, 0)$	$\frac{1}{2}, (2, 1)$	$1, (2, 0)$

An equilibrium ought to have P1 and both types of P2 best responding to the other player's strategy. Let's check.

We can see that (B, BS) is an equilibrium.

A better way

		P2			
		BB	BS	SB	SS
P1	B	$2,(1,0)\frac{1}{2}$	$1,(1,2)\frac{3}{2}$	$1,(0,0)0$	$0,(0,2)1$
	S	$0,(0,1)\frac{1}{2}$	$\frac{1}{2},(0,0)0$	$\frac{1}{2},(2,1)\frac{3}{2}$	$1,(2,0)1$

Note: Similar to extensive games, the first part of P2's strategy only affects P2's payoff when she is actually friendly and the second part of P2's payoff only affects P2's payoff when she is actually unfriendly.

This fact lets us further reduce the game: Imagine that P2 had to choose her strategy before she knew what her actual type would be. The strategy that maximizes her expected payoff in this situation will also maximize the payoff of each type!

Note that we easily arrive at the same equilibrium by normal Nash equilibrium analysis now.

Bayesian Games in General

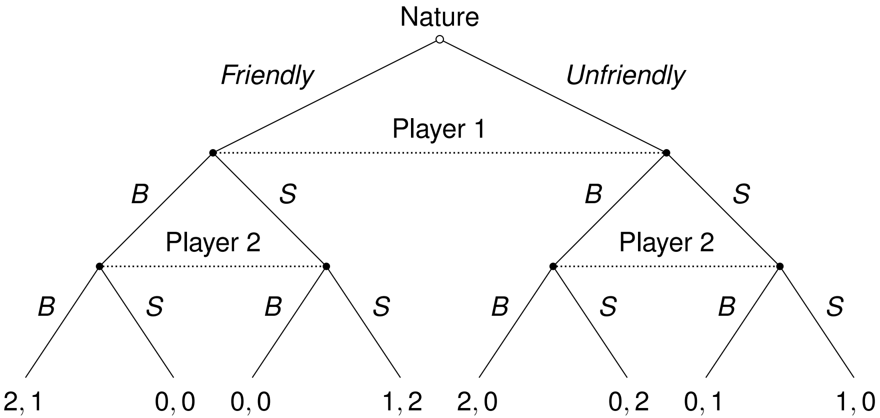
A Bayesian game is:

- A set of players $N = \{1, \ldots, n\}$.
- A set of actions A_i for each player $i \in N$.
- A set of types Θ_i for each player (player i's realized type is $\theta_i \in \Theta_i$)
- A payoff $u_i(a_i, a_{-i}; \theta_i)$ for each type of player i given her own actions and all other actions
- $p = (p_1, \ldots, p_n)$: Players' (common) beliefs about the distribution of all player's types

Implicit in our solutions: <u>conditional</u> beliefs are formed via Bayes rule.

Representing Bayesian games in extensive form

Even though our solution concept is (for now) static, it is often helpful to think of Bayesian games as extensive form games in which "Nature" moves first with exogenously given probabilities. Consider this for the BoS game:



Conditional beliefs in a Bayesian game

Suppose the probability distribution of types in a two player game is given by the following table:

		θ_2 c	θ_2 d
θ_1	a	1/6	1/3
	b	1/3	1/6

Types here are correlated, so when P1 learns her own type she learns something about P2 implicitly. Suppose P1's realized type is $\theta_1 = a$. What are her beliefs about θ_2?

$$p_1(\theta_2 = c|\theta_1) = \frac{p(\theta_1 = a \text{ and } \theta_2 = c)}{p(\theta_1 = a)} = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$p_1(\theta_2 = d|\theta_2) = \frac{p(\theta_1 = a \text{ and } \theta_2 = d)}{p(\theta_1 = a)} = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}$$

Bayesian Nash Equilibrium

- A pure strategy for player *i* is a function $s_i(\theta_i)$ telling us what action EACH TYPE of player i would take. In this sense, strategies are complete plans of play as in extensive games.
- A mixed strategy is a probability distribution over pure strategies.
  - Even though this is a static game, think of these more like behavioral strategies in extensive games.

- A Bayesian Nash equilibrium is one in which every type of every player is best responding to the other players' strategies given her beliefs about the other players' types.

Example: The Market for Lemons

Consider the following situation. P1 has a used car that P2 may wish to purchase. P1 has private information about the quality of the car, either low or medium or high: $\Theta_1 = \{L, M, H\}$. Assume each type has equal prior probability.

The value to P1 of owning the car is 10 if $\theta_1 = L$, 20 if $\theta_1 = M$, and 30 if $\theta_1 = H$. The value to P2 of owning the car is 14 if $\theta_1 = L$, 24 if $\theta_1 = M$, and 34 if $\theta_1 = H$. Notice that there should be gains to trade in every case.

Consider a simultaneous-move game in which P1 submits a minimum price at which she would sell the car and P2 submits the maximum she would pay. If P2 names a price higher than P1's reservation price, the trade is made, P1 gets a payoff equal to the price, and P2 gets a payoff equal to her value for the car minus the price. If no trade is made, P1 keeps the car and P2 gets a payoff of zero.

An efficient trade that is NOT an equilibrium

Since P2's expected value from owning the car is 24 and P1's is 20, our reasoning from the ultimatum game suggests that P2 should offer P=20 and P1 should accept. In fact, from an ex ante expected utility perspective this would be a Pareto improvement.

Why is it not an equilibrium?

P1 knows θ_1 and would reject this offer any time $\theta_1 = H$! But this means that P2's value of owning the car <u>conditional on an offer of 20 being accepted</u> is:

$$
\begin{aligned}
EU_2 =& \Pr[\theta_1 = L|\theta_1 \neq H] \times 14 + \Pr[\theta_1 = H|\theta_1 \neq H] \times 24 \\
=& \frac{\frac{1}{3}}{\frac{2}{3}} 14 + \frac{\frac{1}{3}}{\frac{2}{3}} 24 \\
=& \frac{1}{2} 14 + \frac{1}{2} 24 \\
=& 19.
\end{aligned}
$$

So maybe P2 should only offer 19?

Whoops! Now P1 will only accept when $\theta_L = L$, so P2's value of owning the car conditional on an offer of 19 being accepted is only 14.

Market for Lemons: Equilibrium

We therefore have the following:
In any BNE, P2 offers a price between 10 and 14. Each type of P1 only accepts offers that give her (weakly) more than her value from continuing to own the car, given her knowledge of θ_1.

This is a leading example of adverse selection which is important in economics and political science. Some related points:

- This shows that markets are inefficient with asymmetric information. Lots of people want to buy high quality used cars at prices that sellers would be happy to accept, but because of this asymmetric information problem the market is flooded with lemons (but see CarFax, etc.)
- We could have called P1 a patient and P2 an insurer offering a premium, and we would be explaining one of the main market failures in the market for private health insurance (see also flood insurance).
- Adverse selection problems also show up everywhere in politics: For instance, voters may want to delegate to an honest politician but at a given wage corrupt politicians are more attracted to public office. We'll think harder about this problem later in the class!

War Bargaining

A puzzle: Why would countries ever go to war?
Consider:

- Two countries in a dispute over the division of some resource
- Country 1 makes an offer to Country 2 which Country 2 accepts or rejects. If the offer is rejected the countries go to war, which is costly. Call the cost $k > 0$.
- In a war: Country 1 wins with probability p and Country 2 wins with probability $1 - p$. The winner keeps the whole resource. Suppose the value of keeping the whole resource is equal to 1
- This means Country 2's value for rejecting any offer is $(1 - p) - k$. Country 1's expected utility if an offer is rejected is $p - k$.
- But this means any bargain in which Country 1 offers country two a share between $(1 - p) - k$ and $(1 - p)$ is preferred by BOTH PARTIES to war. So countries should never go to war!

War bargaining with incomplete info

The leading theory for why countries go to war is based on including incomplete information in this game. Consider the game above but now Country 2 has private information about its probability of winning a war. Suppose Country 2 is either Weak or Strong (each with equal probability).

When Country 2 is weak its probability of winning a war is $\frac{1}{4}$ and when it is strong its probability of winning a war is $\frac{3}{4}$.

This means the weak type of Country 2 will accept any offer that gives it weakly more than $\frac{1}{4} - k$ and the strong type will only accept offers that give it weakly more than $\frac{3}{4} - k$.

War bargaining with incomplete info (continued)

What should Country 1 offer now?

- An offer of $\frac{3}{4} - k$ will be accepted by both types of Country 1. There is no reason to offer more than that.
- An offer of $\frac{1}{4} - k$ will be accepted by the weak type of Country 1 but not the strong type. There is no reason to offer strictly between $\frac{1}{4} - k$ and $\frac{3}{4} - k$ because it costs more without changing the probability of acceptance.
- What happens if Country 1 offers $\frac{1}{4} - k$?
  - With probability 1/2 Country 2 is weak and accepts, giving Country 1 a payoff of $\left(1 - \frac{1}{4} + k\right)$
  - With probability 1/2 Country 2 is strong and rejects, giving Country 1 a payoff of $\frac{1}{4} - k$.
  - The expected payoff from making this offer is then

$$\frac{1}{2}\left(1 - \frac{1}{4} + k\right) + \frac{1}{2}\left(\frac{1}{4} - k\right) = \frac{1}{2}.$$

- Compare this to the payoff of simply offering Country 2 $\frac{3}{4} - k$ with a guarantee of acceptance:

$$1 - \frac{3}{4} + k = \frac{1}{4} + k.$$

War bargaining with incomplete info (continued)

Conclusion: For some costs of war ($k \leq \frac{1}{4}$), the equilibrium is one in which the countries go to war with probability $\frac{1}{2}$.

Note the similarity to the adverse selection story from before: There is clearly a bargain that both sides prefer to fighting, but asymmetric information keeps them from reaching this bargain.

But why wouldn't Country 2 just reveal its type then?

Big problem! If Country 1 believes what Country 2 says about its type, then when Country 2 is weak it should say that it is strong to get a bigger offer. Knowing this, Country 1 should not believe anything Country 2 says. (See, cheap talk)

Big picture: Purposes of elections

- Preference aggregation: A procedure to generate a collective choice in the face of diverse preferences
  - Hotelling-Downs median voter result

- Information aggregation: An efficient method for figuring out what choice is best when information is distributed throughout the population
  - (Today)

- Selection and accountability: Coming soon!

History: Condorcet Jury Theorem

Condorcet Jury Theorem: Setup

- Population of voters $N = \{1, \ldots, n\}$.
- The voters face a choice between two policies A and B. One of these is the correct choice but the voters do not know which one.
- Each voter gets a signal $s_i \in \{a, b\}$, with a indicating that A is more likely to be the best policy and b indicating that B is more likely to be the best policy. Suppose each voter's signal is independent and is accurate with probability $p > \frac{1}{2}$ and that each voter votes according to their signal.
- Condorcet Jury Theorem: If the policy is chosen by majority rule, the probability of a correct decision goes to 1 as the population gets large.

Condorcet Jury Theorem: Proof

Today we would prove Condorcet's Jury Theorem using standard results from statistical theory. Let

$$S_n = \frac{\#\text{ correct votes}}{\#\text{ total votes}}$$

be a random variable denoting the proportion of correct votes given n voters. The expectation of S_n is p for any n.

The weak law of large numbers tells us that for all $\epsilon > 0$ we have:

$$\lim_{n\to\infty} \Pr[|S_n - p| \geq \epsilon] = 0.$$

Choosing $\epsilon = p - \frac{1}{2}$, this means that the probability of an incorrect majority decision goes to zero as the number of voters grows.

Jury Theorem: Broader meaning

- Big idea: Elections are good ways to aggregate information
- Brief tangent: Enlightenment and French Revolution:
  - Condorcet thought that science and philosophy would reveal the true moral and political order. The history of the human race was to be one of continual progress toward perfection.
  - The French Revolution was very tied up in Enlightenment philosophy: the idea was to sweep away to existing hierarchical structures to make way for a reason-based political order based on liberty and equality.
  - The Jury Theorem result is very much in line with this thinking: if there is a true moral and political order to be discovered, a polity is a good engine to achieve this.

Interrogating the assumptions of the jury theorem

- Individual citizens are more likely to be right than wrong ($p > 1/2$). This assumptions seems harmless: if $p < \frac{1}{2}$ then citizens should just interpret a signal of a to mean that B is better and a signal of b to mean that A is better!

- Citizens are equally competent. We could instead have a different p for each citizen. The big result still holds in this case (information is aggregated perfectly as $n \to \infty$), though majority rule might not be the optimal rule.
  - It turns out the optimal rule would be a weighted majority rule, where each citizens weight is $w_i = \frac{p_i}{1-p_i}$ (Shapley and Grofman 1984).

- Citizens vote sincerely. Should they? This is a game theoretic question and we'll explore it here.

Simplest case: One voter

Let π denote the prior probability that the correct choice is B. Let $\theta \in \{A, B\}$ denote the truth.

The voter receives a signal (accurate with probability p) and chooses either A or B. Her payoff is:

$$u(v_i, \theta) = \begin{cases} 0 & \text{if correct} \\ -z & \text{if } \theta = A \text{ and } v_i = B \\ -(1-z) & \text{if } \theta = B \text{ and } v_i = A. \end{cases}$$

$\Rightarrow$ B is the best choice if $\Pr[\theta = B] \geq z$.

When should the voter vote informatively (i.e. follow her signal)?

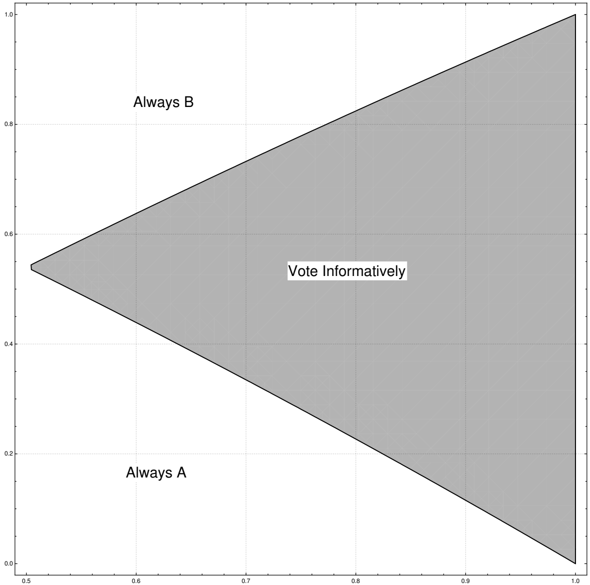
Beliefs:

$$\Pr[B|a] = \frac{\Pr[B]\Pr[a|B]}{\Pr[a]} = \frac{\pi(1-p)}{\pi(1-p)+(1-\pi)p}$$

$$\Pr[B|b] = \frac{\Pr[B]\Pr[a|B]}{\Pr[a]} = \frac{\pi p}{\pi p+(1-\pi)(1-p)}$$

The voter votes informatively as long as $\Pr[B|a] \leq z$ and $\Pr[B|b] \geq z$ (i.e. the signal needs to be strong enough relative to the prior to change the voter's mind

Illustration



Two voters

For illustration, consider two voters and a particular version of unanimity rule: the choice is A unless both players choose B. The outcomes are as follows:

		P2 A	P2 B
P1	A	*A*	*A*
	B	*A*	*B*

(This may resemble a jury where *B* is convict and *A* is not convict.)

Note: If P2 votes A then it doesn't matter what P1 does (and vice versa). Therefore, both players' optimal actions are to do whatever would be optimal whenever the other player votes B.

However, conditioning on this event may contain some information that changes what each player should do. This basic insight comes from Feddersen and Pessendorfer (1996) and Austen-Smith and Banks (1996).

Is there an equilibrium in which both players vote informatively?

We approach this question by assuming that P2 votes informatively (*A* when her signal is *a* and *B* when her signal is *b*) and seeing if P1 would like to deviate.

Remember, P1's optimal vote is whatever would be optimal when P2 votes B. Since P2 is voting informatively, she will only vote B when her signal is b. This means player 1 should <u>assume that P2's signal is b when deciding how to vote!</u>

Is there an equilibrium in which both players vote informatively?

Updating belifs in this way (letting s_1 denote 1's signal and s_2 denote 2's signal):

$$\Pr[B|s_1 = a \text{ and } s_2 = b] = \frac{\pi(1-p)p}{\pi(1-p)p + (1-\pi)(1-p)p} = \pi$$

$$\Pr[B|s_1 = b \text{ and } s_2 = b] = \frac{\pi p^2}{\pi p^2 + (1-\pi)(1-p)^2}.$$

Therefore, P1 acts as if an *a* signal is completely uninformative and a *b* signal is twice as informative. If $\pi > z$ then this means P1 should ignore her signal completely and vote B no matter what.

If $\pi < z$ and signals are sufficiently informative then we may have an equilibrium with informative voting, but in general we cannot make this claim.

Effects of increasing number of voters

As the number of voters gets larger it becomes impossible to support voting informatively under unanimity rule.

Why?

If everyone else were voting informatively, then being pivotal when I get a signal of *a* means that <u>everyone other than me got a signal of b</u>. It is much more likely that my signal is wrong than it is that 8 or 9 or 1000 other signals are wrong. Thus, conditional on being pivotal, I believe the state is very likely to be *b* so I should vote *B*.

Equilibrium when $\pi > z$.

e. The player's types are the signals, *a* and *b*, so each player has four possible pure strategies, representing what to do when her signal is *a* and what to do when her signal is *b*:

		P2			
		AA	AB	BA	BB
P1	AA				
	AB				
	BA				
	BB				

Formulating the Bayesian game ($\pi = .55, z = .5, p = .6$)

P1 \ P2	AA	AB	BA	BB
AA				
AB				
BA				
BB				

P1 \ P2	AA	AB	BB
AA	$-.275, -.275$	$-.275, -.275$	$-.275, -.275$
AB	$-.275, -.275$	$-.212, -.212$	$-.2, -.2$
BB	$-.275, -.275$	$-.2, -.2$	$-.225, -.225$

P1 \ P2	AB	BB
AB	$-.212, -.212$	$-.2, -.2$
BB	$-.2, -.2$	$-.225, -.225$

The strategy "BA" is anti-informative voting (voting the opposite of your signal). This is clearly weakly dominated: when the player is pivotal it will lead to a lower payoff. So we can eliminate this from the game. Next, we fill in the payoffs for each player as if she must choose her strategy before she learns her signal. If any player chooses AA, the outcome is A no matter what. The players' expected payoffs are:

$$.45 \times 0 + .55 \times -.5 = -.275$$

If both players choose BB, the outcome is B no matter what. The players' expected payoffs are:

Back to Condorcet

What does this say about majority rule? Feddersen and Pessendorfer (1997) analyze this question.

In a majority rule election, given the hypothesis of informative voting by everyone else, any individual is pivotal only if the same number of people got each signal.

$\Rightarrow$ being pivotal means I should be less certain of the right choice. This leads to the swing voter's curse, which I won't discuss today because it's in your homework.

Back to Condorcet

This work shows that the behavior assumed by Condorcet is not always incentive compatible.

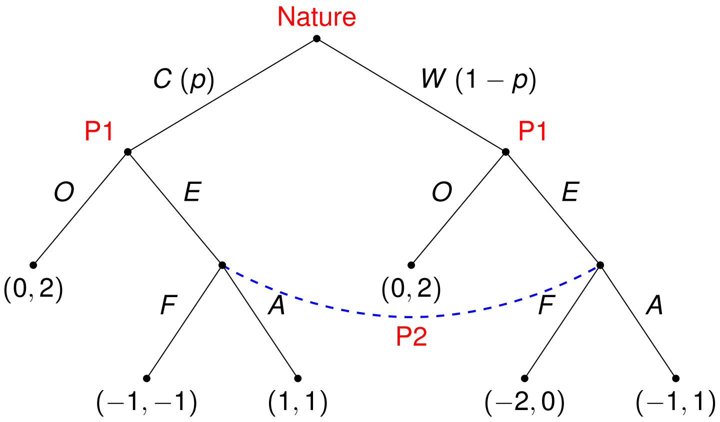
However, the most important conclusion from Condorcet is still true in a BNE of the majority rule game. As n gets large, majority rule still aggregates information perfectly.

Dynamic games of incomplete information

Where we are

	Complete Info	Incomplete Info
Static	Nash	Bayesian Nash
Dynamic	Subgame Perfect Nash	**Perfect Bayesian**

Dynamic example: Entry



- What are the subgames?
- SPNE does not help us at all (equivalent to Nash equilibrium since there is only one subgame)

Dynamic example: Entry

Okay fine, let's look at the normal form game then and just solve the BNE:

		P2	
		F	A
P1	OO	**0, 2**	0, 2
	OE	$-1, 1$	$-1/2, 3/2$
	EO	$-1/2, 1/2$	**1/2, 3/2**
	EE	$-3/2, -1/2$	0, 1

We can see that (OO, F) and (EO, A) are each pure strategy BNE of the Bayesian game.

But the (OO, F) equilibrium is pathological in a way that reminds us of why we needed subgame perfection: P2 threatens to fight and therefore never has to do so, but if he actually reached this information set it would not be rational for him to fight. This is not a credible threat!

Enter: Perfect Bayesian Equilibrium

Perfect Bayesian equilibrium (PBE) is our next solution concept and it is designed to capture this credible threat logic in games of incomplete information.

Obviously we will have to use some concept that does not rely on subgames, for the reasons you saw above.

Shifting our thinking to understand PBE

We need to change our conception of dynamic games in a couple of ways to think about PBE

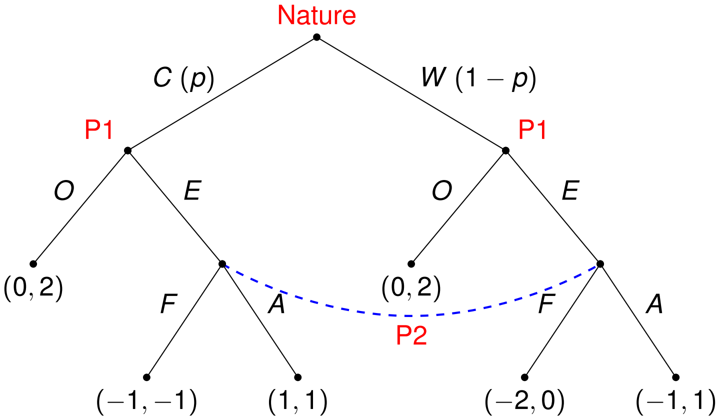
- Instead of just talking about strategy profiles, we will talk about a strategy profile AND a system of beliefs (i.e. beliefs at every information set). A perfect Bayesian equilibrium is going to be not just a strategy profile but a profile of strategies AND BELIEFS for all players. Sometimes this is referred to as an assessment.
  - So to reiterate: BELIEFS WILL BE PART OF THE SOLUTION TO THE GAME WHEN USING PERFECT BAYESIAN EQUILIBRIUM
- We need to define a notion of sequential rationality that does not rely on subgames. Instead, sequential rationality will be imposed at every information set given beliefs
- Another challenge with beliefs: We want players to form beliefs using Bayes rule, and like with strategies we want to specify them at every information set (even those that are not reached), but what should they believe to be true at information sets that they should never reach?

Concept 1: Paths of Play

Definition

Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Bayesian Nash equilibrium strategy profile in a game of incomplete information. We say that an information set is **on the equilibrium path** if given σ^* it is reached with positive probability. Otherwise that information set is **off the equilibrium path** (i.e. if it is reached with zero probability).

Paths of Play Example



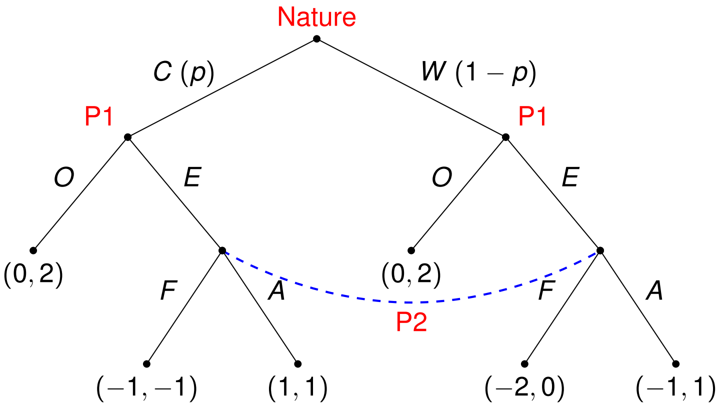
- In the BNE profile (EO, A), player 2's information set is on the path of play.
- In the BNE profile (OO, F), player 2's information set is off the path of play.

Concept 2: A System of Beliefs

Definition

A system of beliefs μ of an extensive form game assigns a probability distribution over decision nodes to every information set. i.e. the system of beliefs tells us, for every information set, the probability the player associates with being at each node in that information set.

System of Beliefs Example



- P1 always knows what decision node she is at so her system of beliefs places probability 1 on the correct node at both of her information sets.
- P2 has one information set with two nodes, which correspond to P1's type. P2's system of beliefs must tell us the probability she assigns to P1's type being C at this information set.

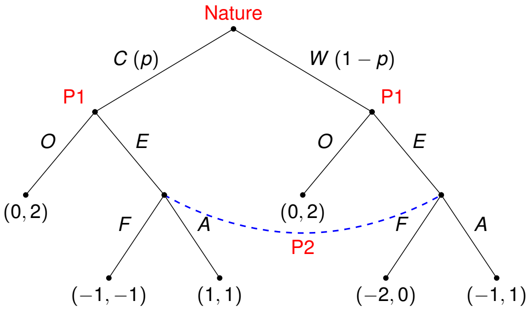
What is the right system of beliefs?

- We have to distinguish between beliefs ON the equilibrium path and beliefs OFF the equilibrium path. We will place the following requirements on beliefs:

- Let σ^* be a BNE profile of strategies. We require that beliefs at all information sets <u>on the equilibrium path</u> must be consistent with Bayes rule given σ^* and the prior beliefs defined by the game.

- We do not place any requirements on beliefs OFF the equilibrium path (more on this soon).

System of Beliefs Example

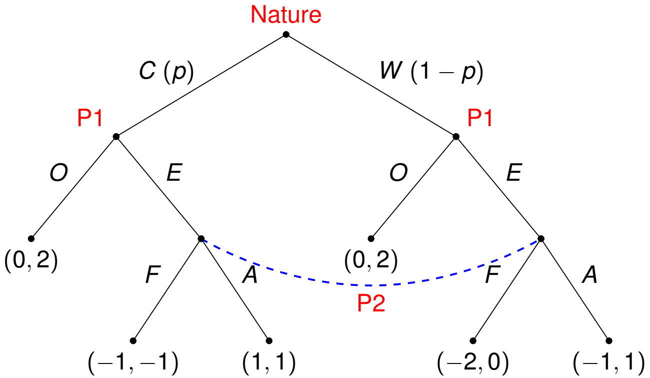


- Let $\sigma_1(E|C)$ be the probability P1 chooses E when she is type C and $\sigma_1(E|W)$ be the probability that P1 chooses E when she is type W. If either probability is positive, then P2's information set is on the path of play and her belief that P1 is type C must be:

$$\mu(C|E) = \frac{\Pr[C]\Pr[E|C]}{\Pr[E]} = \frac{p\sigma_1(E|C)}{p\sigma_1(E|C) + (1-p)\sigma_1(E|W)}.$$

- In the BNE profile (EO, A), we have $\sigma_1(E|C) = 1$ and $\sigma_1(E|W) = 0$ so $\mu(C|E) = 1$.

OFF-PATH Beliefs Example



- What about for the BNE (OO, F)?
- In this case $\sigma_1(E|C) = 0$ and $\sigma_1(E|W) = 0$

$$\mu(C|E) = \frac{\Pr[C]\Pr[E|C]}{\Pr[E]} = \frac{p\sigma_1(E|C)}{p\sigma_1(E|C) + (1-p)\sigma_1(E|W)} = \frac{0}{0}.$$

- **Undefined!** Bayes rule does not apply when the probability of the data is zero. We cannot say what is rational to believe conditional on something impossible.
- This is why perfect Bayesian equilibrium will not restrict such beliefs. To reiterate: WE CAN ASSIGN ANY BELIEFS AT INFORMATION SETS OFF THE EQUILIBRIUM PATH OF PLAY.

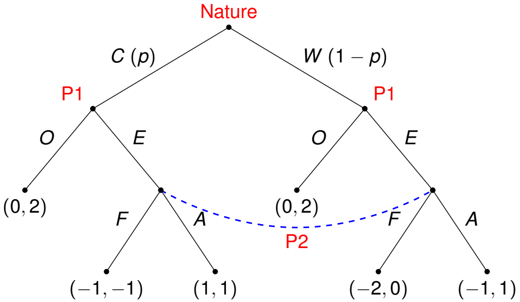
Putting it together: Perfect Bayesian Equilibrium

Definition

A Bayesian Nash Equilibrium profile σ^* together with a system of beliefs μ constitutes a perfect Bayesian equilibrium if it satisfies the following requirements:

1. Every player has a well-defined belief over where she is in each of her information sets. That is, the system of beliefs is defined for every information set.
2. The system of beliefs is consistent with Bayes rule given the prior type probabilities and the strategies for all information sets on the equilibrium path of play
3. Given their beliefs, players' strategies are sequentially rational: in <u>every information set</u> players will play a best response to their beliefs.

Applying PBE to the entry game



Let's check each BNE profile and see if we can support a PBE:

- First we check (EO, A). Given this strategy profile, P2's Bayesian belief after P1 enters is that she is type C with probability 1. The best response to this belief is indeed to choose A. Thus, there exists a PBE with strategies (EO, A) and beliefs $\mu(C|E) = 1$. (P1 is type C if she picked E.)
- Next, check (OO, F). Given this strategy profile, P2's belief can be $\mu(C|E) = q$ for ANY $q \in [0, 1]$. However, F is not a best response to any belief, so this strategy profile cannot be part of a PBE.

Signaling games: key elements

- Players: A Sender and Receiver
- Types: Sender type θ_S, known to the Sender but not the Receiver
- Actions: Sender moves first and chooses a message. Receiver observes the message and chooses an action. The Receiver's optimal action depends on her beliefs about the Sender's type.

A few common applications

- Economics: job market signaling
- Evolutionary biology: color adaptations
- Political science: campaign finance (today), political agency models (later)

Key concepts

We classify equilibria by how well they transmit information about players' types. There are basically three kinds of equilibria:

- Pooling equilibria: All types of Sender use the same message, so the message does not reveal any information about the Sender's type.
  - This happens when one type has too strong of a temptation to imitate another type
  - Example: A job application asks "Are you going to steal from us or not?" Thieves and non-thieves both have an incentive to say they they will not.
- Separating equilibrium: All types of Sender use different messages, so the message reveals the Sender's type perfectly.
  - For a message to be a credible signal that someone's type is θ, it must be the case that type-θ senders have an incentive to send that message AND that all other types do NOT have an incentive to send that message.
  - Usually, the messages must be costly and the cost of the message varies by type.
- Semi-separating: The Sender's strategy is correlated with type so that message reveal some information, but that information is not perfect.
  - Either two types send the same message but some other types send a different message, or the players use a mixed strategy

Example: Campaign war chests and challenger entry

- Puzzle: Incumbents expend time and effort to raise campaign funds and often seem to raise way more than they need in order to run their campaigns. Why don't they spend their time on something else after a point?

- One explanation: Big campaign war chests show that an Incumbent is hard to beat and therefore deters entry by challengers.

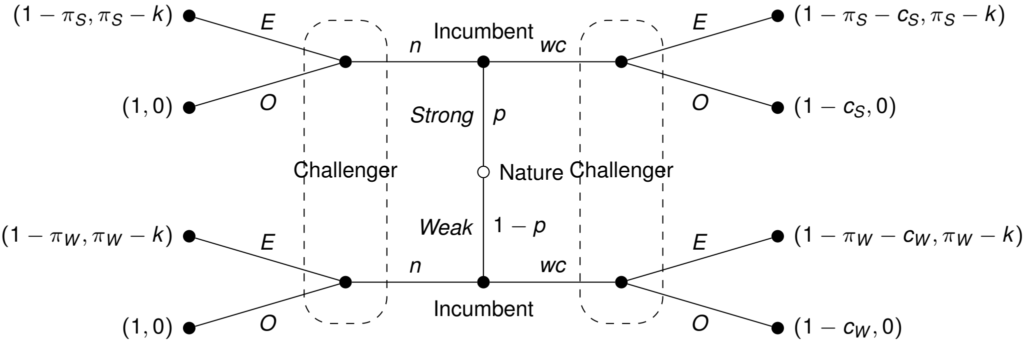
- Interestingly, this explanation may make sense even if the campaign money does not directly increase the likelihood that the Incumbent wins.

Campaign war chests game: Setup

- The Sender is an Incumbent and the Receiver is a Challenger
- Incumbent types: Weak or Strong. Types indicate electoral strength. The prior probability of a strong type is $p \in (0, 1)$.
- The "message" is a level of fundraising. We will simplify this decision to be binary: either the Incumbent builds a war chest (i.e. raises an excessive amount of money) or does not build a war chest (i.e. raises only the amount needed to run her campaign).
- The Challenger observes the fundraising choice and decides whether to enter the election or not.

Campaign war chests game: payoffs

- Incumbent
  - Values being in office (utility from being in office is normalized to 1)
  - Pays a cost for builiding a war chest. The cost depends on type: weak types pays c_w and strong type pays c_s with $c_w > c_s$.
  - THIS IS CRITICAL. The “better” type also has a lower cost of sending the signal.
- Challenger
  - Value of 1 for winning office
  - Pays a cost k for entering the election
- Election outcomes:
  - The probability Challenger wins against a weak types is π_w and the probability the Challenger wins against a strong type is π_s. Naturally, $\pi_w > \pi_s$.
  - To make things interesting, assume $\pi_w > k > \pi_s$, so that the Challenger's entry decision can change based on her belief.


$(1-\pi_S, \pi_S - k)$
E
$(1,0)$
O
Challenger
n
Incumbent
wc
Strong
p
Nature
Weak
$1-p$
E
$(1-\pi_S - c_S, \pi_S - k)$
O
$(1 - c_S, 0)$
Challenger
$(1-\pi_W, \pi_W - k)$
E
$(1,0)$
O
n
wc
Incumbent
E
$(1-\pi_W - c_W, \pi_W - k)$
O
$(1 - c_W, 0)$

Solving this game

- Though it is of course possible to solve for each of the BNE and then check each one, the more efficient method for solving signaling games is guess and verify

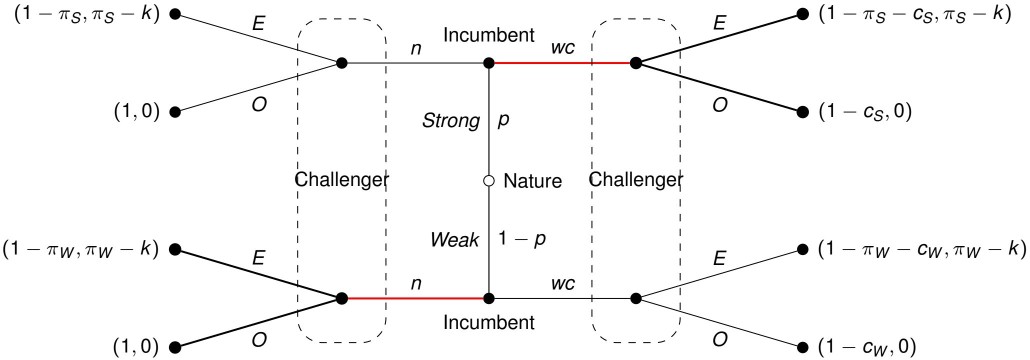
- My “guess and verify” procedure:
  1. Start with a strategy profile for the Sender (pooling, separating, semi-separating)
  2. Compute beliefs for the Receiver at each information set given that Sender strategy
  3. Using these beliefs, compute Receiver’s optimal action at each information set
  4. Return to your proposed Sender strategy and check to see if it is indeed a best response to the Receiver’s strategy

- The categories of signaling equilibria structure our guesses: we’ll try separating, pooling, and semi-separating.

Separating equilibrium

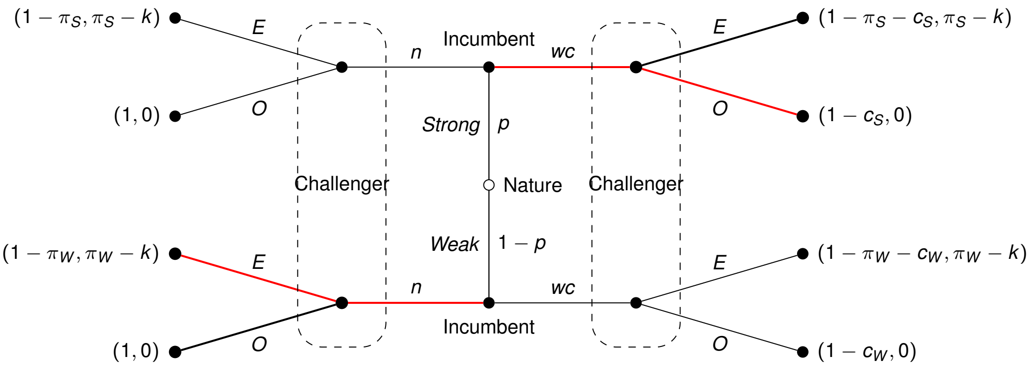
- Recall: A separating equilibrium is one in which each type of player takes a different action
- The game provides a hint as to which separating profile to check: The Strong type has a lower cost for building a war chest so if there is a separating equilibrium it must be one in which the Strong type builds a war chest and the Weak type does not.
- Let's take this guess and see if we can verify it!

Separating equilibrium



1. Proposed Incumbent strategy: (wc, n) [war chest if strong, not if weak]
2. Challenger beliefs: $\Pr[S|wc] = 1$, $\Pr[S|n] = 0$.
3. Challenger actions: O if wc ($\pi_s - k < 0$), E if n ($\pi_w - k > 0$)

Separating equilibrium



4 Is this separating strategy optimal for the Incumbent? Two conditions must be satisfied:

- The Strong type must want to play *wc*: we must therefore have $1 - c_s > 1 - \pi_s \Rightarrow c_s < \pi_s$.
- The Weak type must NOT want to imitate by playing *wc*: we must have $c_w > \pi_w$.

Separating equilibrium: Summing up

We have proven the following:

If $c_s < \pi_s$ and $c_w > \pi_w$ then the following assessment constitutes a perfect Bayesian equilibrium:

- Strong incumbent plays *wc*, weak incumbent plays *n*
- Challenger plays *E* when she sees *n* and *O* when she sees *wc*
- Challenger's beliefs are: $\Pr[S|wc] = 1$, $\Pr[S|n] = 0$.

PLEASE NOTE BELIEFS ARE PART OF MY DESCRIPTION OF THE EQUILIBRIUM.

Pooling equilibrium

A pooling equilibrium is one in which both types choose the same action.

The game does not provide an obvious clue as to which pooling action is most likely so we will check both.

- Pooling on n: no type builds a war chest, so incumbents do not engage in excessive fundraising
- Pooling on wc: all types engage in excessive fundraising because not doing so is taken as a signal of weakness.

Beliefs in a pooling equilibrium

On the path of play, beliefs in a pooling equilibrium should be equal to the prior.

- Intuitive explanation: if both types choose the same action then actions do not reveal anything about types, so Challenger's beliefs should not change
- Mathematical explanation: Apply Bayes' rule:

$$\begin{aligned}\Pr[S|\text{message}] &= \frac{p\Pr[\text{message}|S]}{p\Pr[\text{message}|S] + (1-p)\Pr[\text{message}|W]} \\ &= \frac{p}{p+1-p} \\ &= \frac{p}{1} \\ &= p.\end{aligned}$$

- Off-the-path-of-play: Bayes rule doesn't apply, which will give us an extra degree of freedom to support our "guess"

Pooling on n

Guess and verify for pooling equilibria:

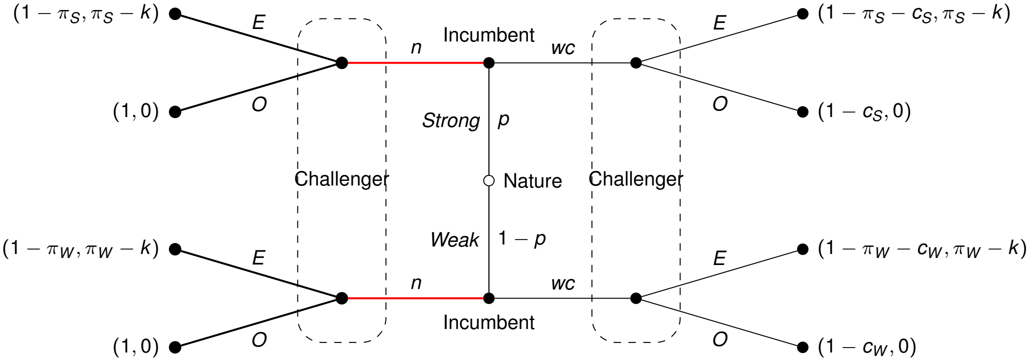
1. Proposed strategy: (n, n)

2. Challenger beliefs: $\Pr[S|n] = p$, $\Pr[S|wc] =???$,
   - *wc* is off the path of play so this belief is undefined
   - ...but we are going to check to see if there is ANY belief that supports this strategy profile as an equilibrium

3. Challenger actions:
   - Following n, Challenger must take whatever action is optimal given her prior.
   - Under some priors Challenger will Enter and under others Challenger will stay Out. We will consider each case separately

4. Return to check whether (n, n) was indeed optimal

Pooling on n: actions under prior

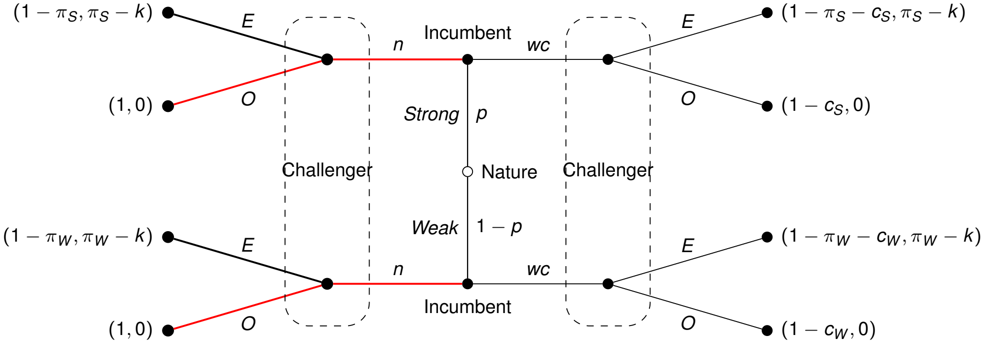


Challenger optimal action given prior:

$$EU_C(E|n) = p(\pi_s - k) + (1-p)(\pi_w - k)$$
$$EU_C(O|n) = 0.$$

Challenger stays out given prior if $p\pi_s + (1-p)\pi_w \leq k$ which holds for $p \geq \frac{k-\pi_w}{\pi_s-\pi_w}$.

Case 1: Challenger stays out given prior $\left(p \geq \frac{k-\pi_w}{\pi_s-\pi_w}\right)$



- We still must specify what C would believe and do off the path of play.
- Let $\mu = \Pr[S|wc]$ denote C's off-path beliefs. Recall again that this is not restricted by Bayes' rule!
- However, we know that C should stay out if $\mu \geq \frac{k-\pi_w}{\pi_s-\pi_w}$ and enter otherwise.

Notice that EITHER action would indeed cause I to choose (n, n): C is already staying out given no war chest, so building a war chest at best incurs a cost with no benefit.

Case 1: Challenger stays out given prior $\left(p \geq \frac{k-\pi_w}{\pi_s-\pi_w}\right)$

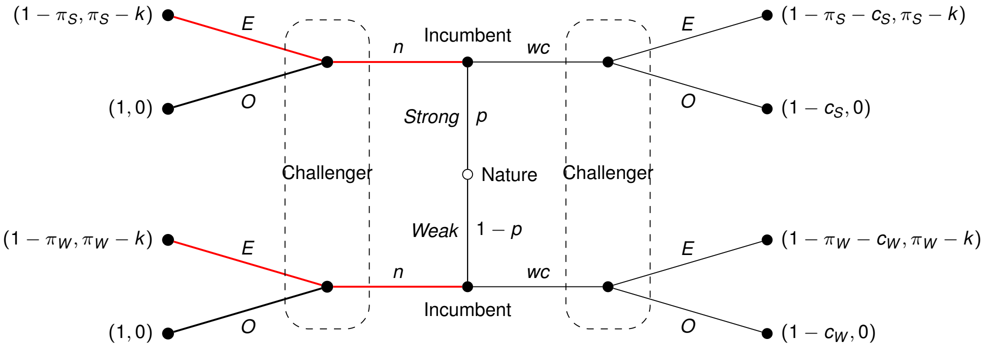
Putting this together, we have the following result:

If $p \geq \frac{k-\pi_w}{\pi_s-\pi_w}$ then all of the following assessments represent a pooling perfect Bayesian equilibrium:

- Both types of Incumbent choose n, Challenger stays out following either wc or n, $\mu(n) = p$ and $\mu(wc) \in [\frac{k-\pi_w}{\pi_s-\pi_w}, 1]$.
- Both types of Incumbent choose n, Challenger stays out following n and enters following wc, $\mu(n) = p$ and $\mu(wc) \in [0, \frac{k-\pi_w}{\pi_s-\pi_w}]$.

Note: I have listed a continuum of assessments even though there are only two strategy profiles, because a range of possible off-path beliefs are consistent with PBE here.

Case 2: Challenger enters given prior $\left(p \leq \frac{k-\pi_w}{\pi_s-\pi_w}\right)$



We will again consider both possible off-path actions for Challenger:

- What if Challenger enters following *wc*? This is optimal if $\mu(wc) \leq \frac{k-\pi_w}{\pi_s-\pi_w}$. This would clearly cause (n, n) to be optimal for Incumbent.
- What if Challenger stays out following *wc*? This is optimal if $\mu(wc) \geq \frac{k-\pi_w}{\pi_s-\pi_w}$. But this may cause Incumbent to choose *wc* so this only supports pooling on *n* if we also have $c_w > \pi_w$ and $c_s > \pi_s$.

Case 2: Challenger enters given prior $\left(p \leq \frac{k-\pi_w}{\pi_s-\pi_w}\right)$

Putting this together, we have the following result:

If $p \leq \frac{k-\pi_w}{\pi_s-\pi_w}$ then the following assessment is a pooling perfect Bayesian equilibrium: both types of Incumbent choose n, Challenger enters following n and wc, and beliefs are $\mu(n) = p$ and $\mu(wc) \in [0, \frac{k-\pi_w}{\pi_s-\pi_w}]$. If additionally $c_w > \pi_w$ and $c_s > \pi_s$ then there is also a pooling perfect Bayesian equilibrium in which both types of Incumbent choose n, Challenger enters following n and stays out following wc, and beliefs are $\mu(n) = p$ and $\mu(wc) \in [\frac{k-\pi_w}{\pi_s-\pi_w}, 1]$.

What about pooling on *wc*?

There is a narrower scope for pooling on *wc*. Because *wc* is the costly action, it must provide some benefit for either type to choose it. This tells us that we will only have pooling on *wc* if Challenger stays out given her prior and enters the race off the path of play if Incumbent were to choose *n*. It must also be the case that both types are willing to pay to build the war chest in order to deter entry.

This gives us the following:

If $p \geq \frac{k-\pi_w}{\pi_s-\pi_w}$ (Challenger enters given prior) and $c_w < \pi_w$ and $c_s < \pi_s$ (both types of Incumbent willing to pay for war chest if it deters entry), then there is a pooling perfect Bayesian equilibrium in which both types of Incumbent choose *wc*, Challenger stays out given *wc* and enters given *n*, and beliefs are $\mu(wc) = p$ and $\mu(n) \in [0, \frac{k-\pi_w}{\pi_s-\pi_w}]$.

Wrapping up

We worked through a model of campaign war chests as a signal of Incumbent strength. Qualitatively, we learned:

- If Strong incumbents are sufficiently different from Weak ones (in terms of both victory probability and fundraising ability) then war chests may perfectly reveal Incumbent strength to potential Challengers, scaring them off when there is a strong Incumbent
- Even when Weak incumbents are tempted to imitate, there may still be equilibria in which war chests reveal some information about Incumbent quality, causing some degree of scare off (this is below in the extra slides about semi-separating equilibrium)
- We can never completely rule out pooling equilibria though, so this prediction may not hold in every election – common expectations matter!

Partially separating equilibrium

There is also a chance of a partially separating equilibrium: that means that war chests convey some information about incumbent strength but that information is not perfect.

A reason to characterize partially separating equilibrium is that we might be trying to answer the question: Can signaling strength be the motivation for building a war chest? Sometimes we found that we can support a separating equilibrium, but the answer may still be yes even if we do not find a full separating equilibrium. For that reason we may want to look for the most informative equilibrium.

A partially separating equilibrium to this game

There are a few possibilities for partially separating equilibria but I will illustrate just one: Strong incumbents always build a war chest, while Weak incumbents sometimes build one in order to look strong. This occurs with probability q. Challenger always enters the race upon seeing n and enters with probability r if a war chest is built.

Now we will repeat our "guess and verify" procedure while finding the values of q and r along the way.

Guess and verify for partially separating equilibrium

1. Guess: Strong Incumbent choose *wc* with probability 1, Weak Incumbent chooses *wc* with probability q. Challenger mixes following *wc*, entering with probability r.

2. Challenger beliefs:

$$\mu(n) = \Pr[S|n] = 0$$
$$\mu(wc) = \Pr[S|wc] = \frac{p}{p + (1-p)q}.$$

Guess and verify for partially separating equilibrium

3 Challenger best response: Enter following n is clearly optimal. We guessed that Challenger is using a mixed strategy so Challenger must be indifferent over entering and staying out. Incumbent's mixed strategy is the one that sets

$$\frac{p}{p+(1-p)q}\pi_s + \left(1 - \frac{p}{p+(1-p)q}\right)\pi_w = k$$

and solving for q gives

$$q^* = \frac{p(k-\pi_s)}{(1-p)(\pi_w - k)}.$$

The requirement that $q \leq 1$ places some restriction on the parameters of the game, which are satisfied when

$$p \geq \frac{k-\pi_w}{\pi_s - \pi_w}$$

(notice these are the same as the conditions under which Challenger stays out given the prior.)

Guess and verify for partially separating equilibrium

4. For Incumbent's strategy to be a best response she must prefer *wc* when Strong and be indifferent between *wc* and *n* when weak.

   For Weak Incumbent to be indifferent we must have

   $$1 - \pi_w = r(1 - \pi_w) + (1 - r) - c_w.$$

   Solving for r gives $r^* = \frac{\pi_w - c_w}{\pi_w}$. This must be positive so we must have $\pi_w > c_w$ to support this equilibrium.

   The strong type must also prefer *wc*. This holds here because the weak type is indifferent and we have $c_w > c_s$ and $\pi_s < \pi_w$.

Partially separating equilibrium

We now have shown the following:

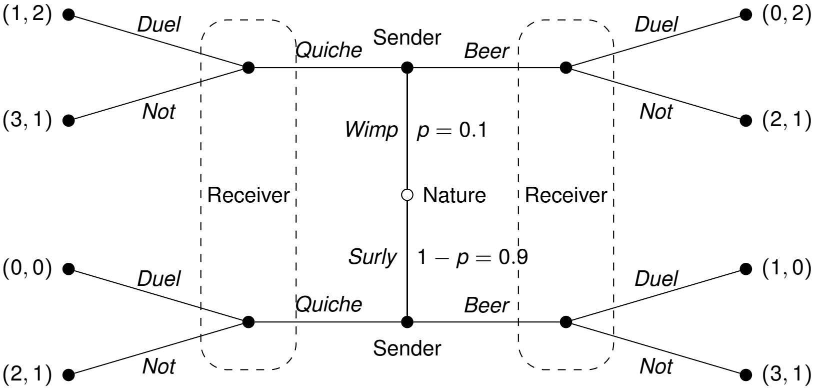
Let $\pi_w \geq c_w$ and $p \geq \frac{k-\pi_w}{\pi_s-\pi_w}$. Then there exists a partially separating perfect Bayesian equilibrium with the following assessment:

- Strong Incumbents always choose *wc* and Weak Incumbents choose *wc* with probability $\frac{p(k-\pi_s)}{(1-p)(\pi_w-k)}$.
- Challenger always enters following *n* and enters following *wc* with probability $\frac{\pi_w-c_w}{\pi_w}$.
- Beliefs are $\mu(n) = 0$ and $\mu(wc) = \frac{k-\pi_w}{\pi_s-\pi_w}$.

Example 1: The famous beer-quiche game (Cho and Kreps (1987))

- Two players: Sender and Receiver
- Sender is either a *Wimp* or *Surly*
  - $Pr[W] = p = 0.1$ and $Pr[S] = 1 - p = 0.9$
- Nature selects Sender type then Sender chooses what to eat for breakfast: Beer or Quiche
- Receiver observes what Sender ate for breakfast, but not Sender's type, then chooses to Duel Sender or Not
- Surly Senders prefer Beer and Wimp Senders prefer Quiche for breakfast and all types prefer to avoid duels
  - Sender gets 1 for eating preferred breakfast (0 otherwise) and 2 for avoiding a duel (and 0 if has to duel)
  - e.g., If Surly Sender drinks Beer and avoids duel then gets total payoff of 3
- Receiver prefers to Duel if and only if Sender is a Wimp
  - If Sender is Surly then Receiver gets 0 from D and 1 from N. If Sender is Wimp then Receiver gets 2 from D and 1 from N.

The famous beer-quiche game (Cho and Kreps (1987))



Reminder: “Guess-and-verify”

- My “guess and verify” procedure:

  1. Start with a strategy profile for the Sender (e.g., pooling, separating)
  2. Compute beliefs for the Receiver at each information set given that Sender strategy
  3. Using these beliefs, compute Receiver’s optimal action at each information set
  4. Return to your proposed Sender strategy and check to see if it is indeed a best response to the Receiver’s strategy

- The categories of signaling equilibria structure our guesses: we’ll try separating and pooling.

Separating equilibrium

- Recall: A separating equilibrium is one in which each type of player takes a different action

- Let's try an equilibrium where only Surly Senders have Beer for breakfast...

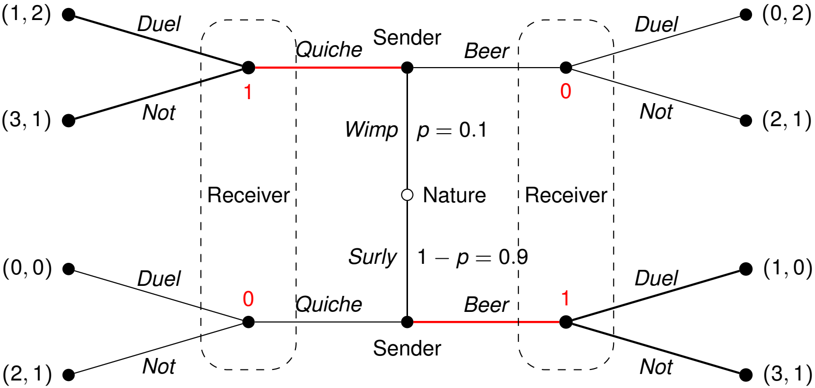
Separating equilibrium

1. Sender plays *Beer* if Surly and *Quiche* if Wimp
2. That means Receiver's beliefs at each information set are as follows:
   - If Receiver observes *Beer* then $Pr[Surly|Beer] = 1$ since Sender is separating:

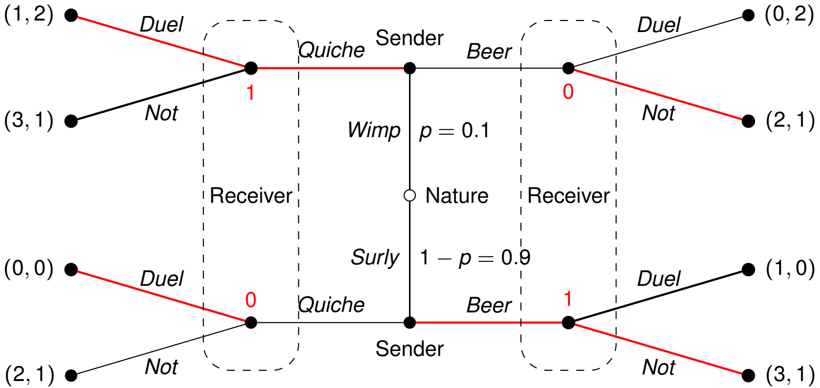
$$
\begin{aligned}
Pr[Surly|Beer] &= \frac{\overbrace{Pr[Beer|Surly]}^{\text{From } S \text{ strategy}}\ \overbrace{Pr[Surly]}^{\text{Prior}}}{\underbrace{Pr[Beer|Surly]}_{\text{From } S \text{ strategy}}\ \underbrace{Pr[Surly]}_{\text{Prior}} + \underbrace{Pr[Beer|Wimp]}_{\text{From } S \text{ strategy}}\ \underbrace{Pr[Wimp]}_{\text{Prior}}} \\
&= \frac{1 \times (1-p)}{1 \times (1-p) + 0 \times p} \\
&= \frac{1-p}{1-p} = 1.
\end{aligned}
$$

   - Same can be done to show $Pr[Surly|Quiche] = 0$: Note that $Pr[Quiche|Surly] = 0$ given strategy from Step 1, which means numerator in Bayes rule calculation is zero
   - So we have Receiver's beliefs for both of her information sets: she believes that Sender is Surly for sure after Beer and Wimp for sure after Quiche

The famous beer-quiche game (Cho and Kreps (1987))

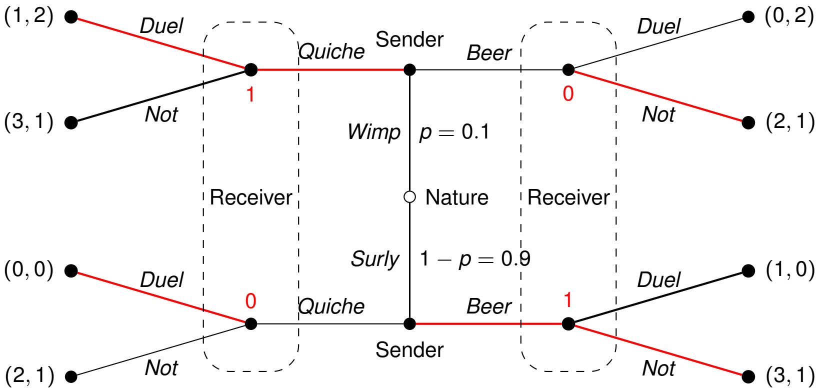


Separating equilibrium



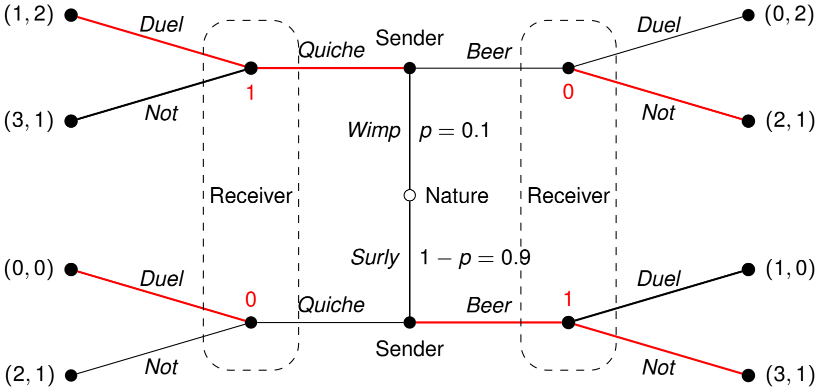
3. Given Receiver's beliefs what are her optimal actions?
   Duel after Quiche and Not after Beer

Separating equilibrium



4. Now we need to check if any type of Sender would deviate
   - *How about Surly Senders?* Nope, Surly Sender gets 3 from Beer (given Receiver plays Not) and 0 from Quiche (given Receiver will play Duel in that case)
   - *How about Wimp Senders?* Yes! Wimp Senders do want to deviate: gets 1 from Quiche (given Receivers plays Duel) and 2 from Beer (given Receiver plays Not)

Separating equilibrium



- Thus, we cannot support this equilibrium!
- Check for yourself whether you can support a separating equilibrium in which Weak Senders play Beer and Surly Senders play Quiche...

Pooling equilibrium: $(Beer, Beer)$

- Let's check if we can support a pooling equilibrium in which all types of Senders have Beer for breakfast

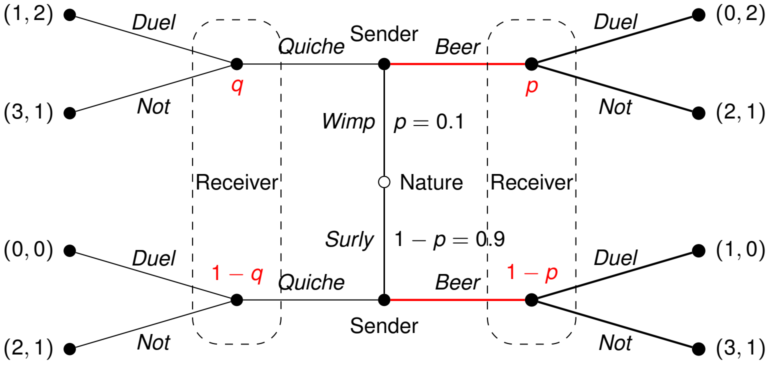
1. Sender plays $(Beer, Beer)$.
2. Receiver retains her prior about Sender's type in the information set following *Beer*:

$$
\begin{aligned}
Pr[Surly|Beer] &= \frac{\overbrace{Pr[Beer|Surly]}^{\text{From } S \text{ strategy}}\ \overbrace{Pr[Surly]}^{\text{Prior}}}{\underbrace{Pr[Beer|Surly]}_{\text{From } S \text{ strategy}}\ \underbrace{Pr[Surly]}_{\text{Prior}} + \underbrace{Pr[Beer|Wimp]}_{\text{From } S \text{ strategy}}\ \underbrace{Pr[Wimp]}_{\text{Prior}}} \\
&= \frac{1 \times (1-p)}{1 \times (1-p) + 1 \times p} \\
&= \frac{1-p}{1} = 1-p.
\end{aligned}
$$

Obviously, this also means that $Pr[Wimp|Beer] = p$.

Remember, off-path beliefs are not restricted. So, define $q = Pr[Wimp|Quiche]$ be the belief that Sender is a Wimp following an observation of *Quiche* by Receiver

Pooling equilibrium: ($Beer$, $Beer$)



Pooling equilibrium: (*Beer*, *Beer*)

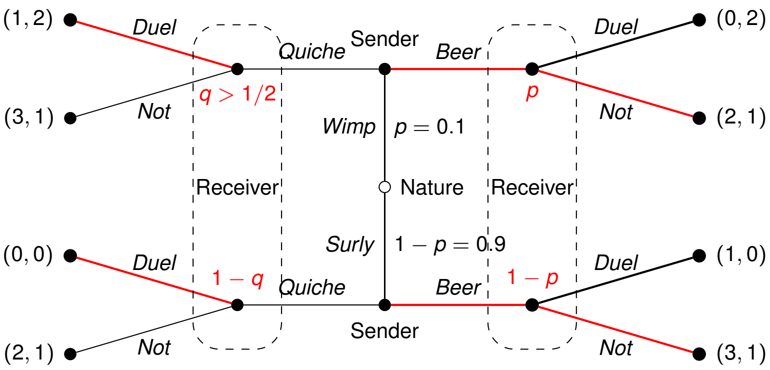
Since q captures Receiver's off-path beliefs following *Quiche* we can figure out how those beliefs affect Receiver's actions for step 3...

3. Following Beer Receiver retained her prior so we compute optimal actions by comparing her expected utilities given her prior beliefs about Sender's type:
   - If Receiver plays *Duel* after *Beer* then, $EU_R(D|B,p) = 2p + (1-p)0 = 2p = 2(0.1) = 0.2$.
   - If Receiver plays *Not* after *Beer* then, $EU_R(N|B,p) = (1)p + (1-p)1 = 1$.
   - Thus, following *Beer*, given Receiver's prior beliefs about Sender's type, it is optimal to *Not*.

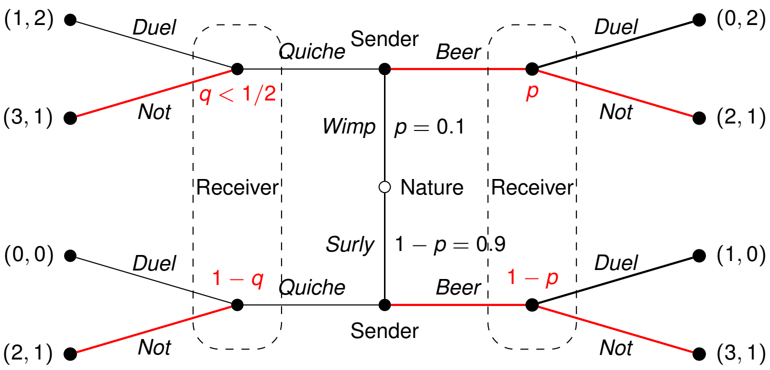
We can also compute Receiver's optimal action given off-path beliefs $q = Pr[Wimp|Quiche] \in [0,1]$.

- **Note:** one quick check is to set $q = 0$ and $q = 1$ and see what Receiver would do in those cases. We will compute the threshold below.
- When would Receiver play *Duel* given q? When $EU_R(D|Q,q) > EU_R(N|Q,q)$:
- $EU_R(D|Q) = 2q + (1-q)0 = 2q$
- $EU_R(N|Q) = (1)q + (1-q)1 = 1$
- Thus, Receiver plays *Duel* if and only if $q > \frac{1}{2}$ and *Not* if $q < \frac{1}{2}$ (we can break indifference however we want here)

Pooling equilibrium: (*Beer*, *Beer*)



Pooling equilibrium: (*Beer*, *Beer*)



Pooling equilibrium: (*Beer*, *Beer*)

4. Now we go back and check whether either type of Sender would deviate:

- If Sender is a Wimp and plays Beer then he gets 2 since Receiver plays N after B. If he instead played *Quiche* and Receiver played *Duel* then he'd get 1 and if Receiver played *Not* after *Quiche* then he'd get 3. So, a Wimp Sender will stick with *Beer* only when Receiver will play *Duel* following *Quiche*.

- If Sender is Surly and plays Beer then he gets 3 since Receiver plays N after B. If he instead chose *Quiche* and Receiver played *Duel* then he'd get 0 and if Receiver played *Not* then he'd get 2. In either case Surly Senders will stick with playing *Beer*.

- Thus, we can support this pooling equilibrium as long as Receiver plays *Duel* following *Quiche*, which we know requires $q > \frac{1}{2}$. Luckily, we are not restricted for what beliefs we assign following *Quiche* so we can just set $q > \frac{1}{2}$ and we have the pooling PBE!

Pooling equilibrium: $(Beer, Beer)$

Result. The following collection of strategies and beliefs constitute a pooling PBE of the beer-quiche game:

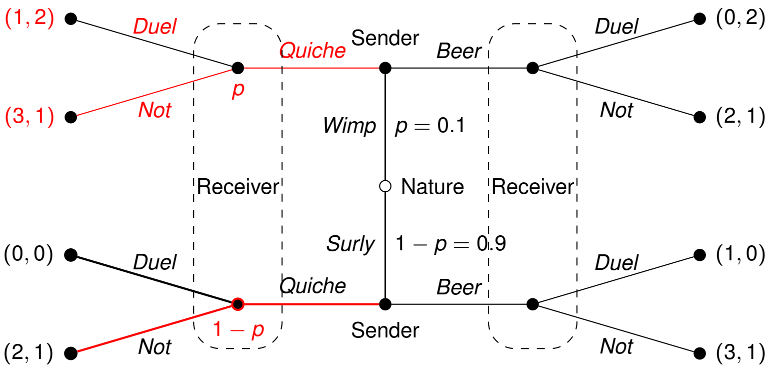
- Both *Wimp* and *Surly* Senders have *Beer* for breakfast.
- Receiver plays *Not* following *Beer* and *Duel* following *Quiche*.
- Receiver's beliefs are given by $Pr[Wimp|Beer] = p = 0.1$ and $Pr[Wimp|Quiche] = q > \frac{1}{2}$.

Pooling equilibrium: (*Quiche*, *Quiche*)

Likewise:

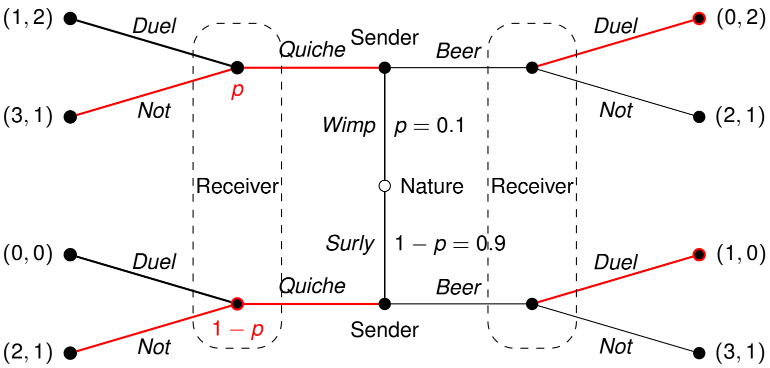
- Assuming (*Quiche*, *Quiche*), $Pr[Wimp|Quiche] = p \Rightarrow$ Receiver does NOT duel

Pooling equilibrium: (*Quiche*, *Quiche*)



If Receiver chooses Not following Beer, Strong type will deviate to Beer. Thus, to pool on Quiche we must have Duel after Beer.

Pooling equilibrium: (*Quiche*, *Quiche*)



Belief requirement: $Pr[Wimp|Beer] = q > \frac{1}{2}$.

Refining signaling equilibria

Two pooling equilibria to beer-quiche:

- Pooling on beer: Receiver assumes wimpy is more likely following quiche, so both types choose beer to avoid looking wimpy
- Pooling on quiche: Receiver assumes wimpy is more likely following beer, so both types choose quiche to avoid looking wimpy

Do both of these equilibria seem equally reasonable? If not, why not?

Refining signaling equilibria

The qualitative difference between the two equilibria boils down to how we set off-path beliefs.

Both pooling equilibria are sustained by the idea that the Receiver would believe the Sender to be more likely to be Wimpy after observing the off-path action

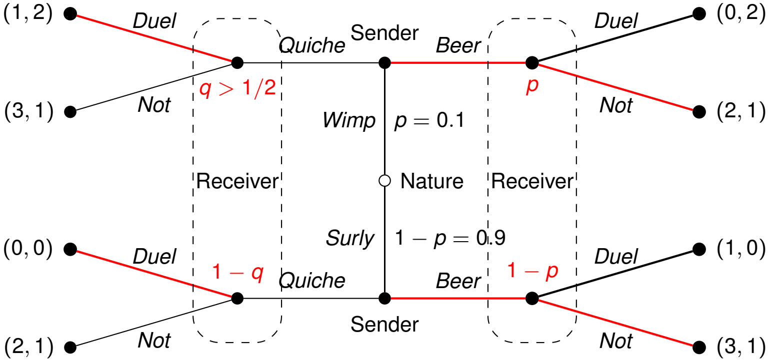
The difference: Wimpy types are more inclined to choose Quiche. We might think the Receiver's off-path beliefs should somehow account for the incentives of different types of the Sender. This is motivation for the Intuitive Criterion (and many similar refinements which we will not learn)

Intuitive Criterion: Informal idea

Imagine that Receiver's reflect a bit more deeply on the situation when they receive an unexpected signal. They may think:

> *Let me think about which types would possibly deviate to the unexpected action. If neither or both type could benefit, I will leave my off-path beliefs as before. If one type of worker can benefit and the other cannot, then I will update my beliefs that the deviation must have come from a type that could potentially benefit from doing so.*

Applying the logic to (*Beer*, *Beer*)

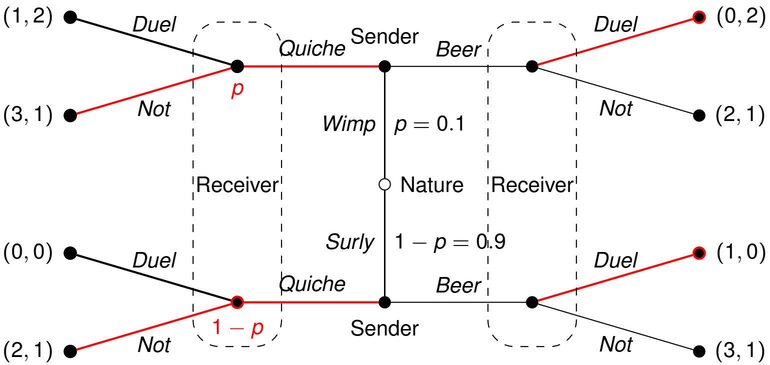


If Receiver switched to Not after Quiche, surly types would choose Beer but weak types would still choose Quiche

Therefore, off the path of play the Receiver should think that senders who deviate to Quiche are wimpy

Duel is optimal under this belief so pooling on Beer with q=1 survives this criterion.

Applying the logic to (Quiche, Quiche)



If Receiver switched to Not after beer, Surly types would switch to beer but wimpy types would still choose quiche

Therefore, off the path of play Receiver may assume that a Sender who chose Beer is Surly

It is not optimal to duel given this belief, so (Quiche, Quiche) does not survive the application of this criterion.

Intuitive criterion: Formalities

Consider a signaling game in which Player 1's types is $\theta \in \Theta$ and chooses an action $a_1 \in A_1$, then player 2 observes a_1 and chooses $a_2 \in A_2$.

For any subset $\hat{\Theta} \subset \Theta$, let $BR_2(\hat{\Theta}, a_1)$ be the set of best response actions of player 2 to a_1 any time player 2's beliefs place positive probability only on types in $\hat{\Theta}$

Formally:

$$BR_2(\hat{\Theta}, a_1) = \bigcup_{\mu \in \Delta(\Theta)} \arg\max_{a_2 \in A_2} \sum_{\theta \in \hat{\Theta}} v_2(a_1, a_2; \theta)\mu(\theta)$$

A PBE σ^* fails the intuitive criterion if there exist $a_1 \in A_1$, $\theta \in \Theta$, and $\hat{\Theta} \subset \Theta$ such that:

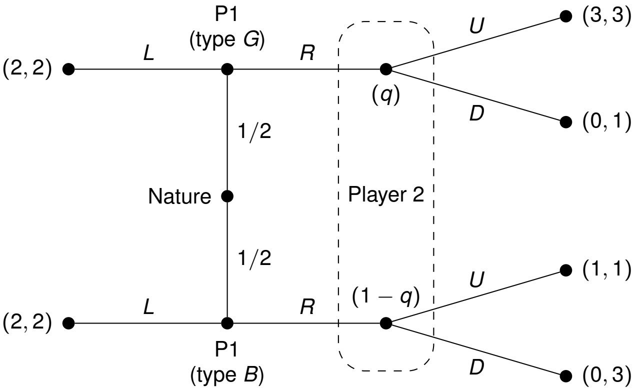
1. No type in $\hat{\Theta}$ would ever choose a_1 regardless of what player 2 believes: $v_1(\sigma^*, \theta) > \max_{a_2 \in BR_2(\Theta, a_1)} v_1(a_1, a_2, \theta)$ for all $\theta \in \hat{\Theta}$
2. Type θ will do strictly better than the equilibrium by playing a_1 as long as it convinces P2 that his type is not in $\hat{\Theta}$: $v_1(\sigma^*, \theta) < \min_{a_2 \in BR_2(\Theta \setminus \hat{\Theta}, a_1)} v_2(a_1, a_2, \theta)$

Back to the quiche pooling equilibrium

Let $\hat{\Theta} = \{\text{wimpy}\}$. The Quiche, Quiche pooling equilibrium fails because for the action Beer and type Surly:

- Wimpy types will never deviate from the equilibrium to choose beer even if it causes P2 to believe they are Surly
- The Surly type can do strictly better by deviating to beer as long as it causes the Receiver to believe that she is Surly

Additional examples:



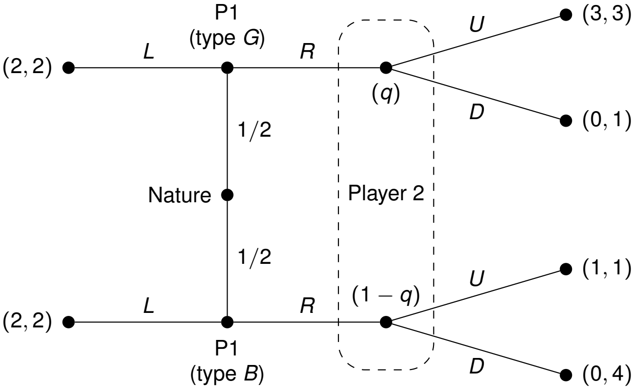
- What are Player 1's strategies?
- They consist of what she does when she is each type, so 4 strategies: *LL*, *LR*, *RL*, *RR*
- What are Player 2's strategies?
- Just U and D. COUNT INFORMATION SETS. P2 has only one.

Types of pure strategy equilibria

Recall: A pure strategy equilibrium is either pooling or separating. THIS CLASSIFICATION USES ONLY PLAYER 1's STRATEGY. If both types of Player 1 take the same action it is pooling. If the types of Player 1 take different actions, it is separating.

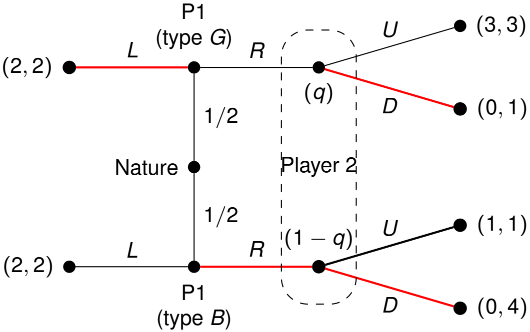
- LL: Pooling
- LR: Separating
- RL: Separating
- RR: Pooling

Is there a separating equilibrium?



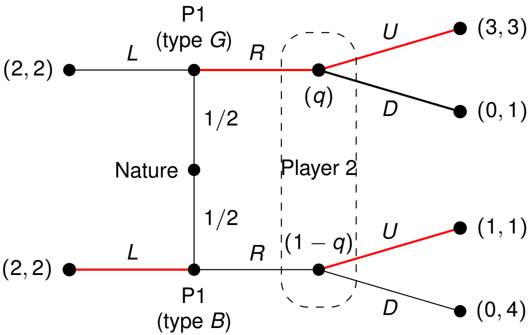
To answer the question, we'll just take the two SEPARATING P1 strategies (LR and RL) and perform our guess and verify procedure. Let's check LR first:

Is there a separating equilibrium?



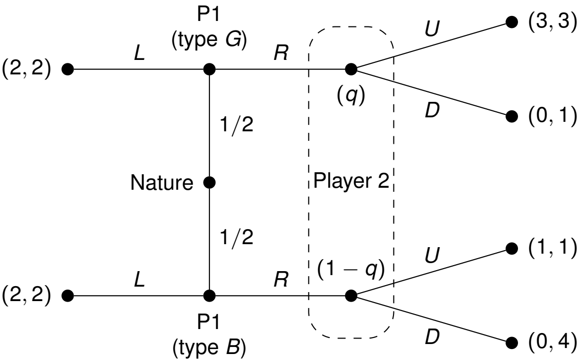
1. Guess: LR
2. P2 beliefs at information set: When she sees R she infers P1 must be type B because ony type B chooses R. So $q = 0$.
3. P2 action: Given $q = 0$ she will choose D.
4. Verify: Does P1 want to play LR? No! The B type would deviate to L. Not an equilibrium!

Is there a separating equilibrium?



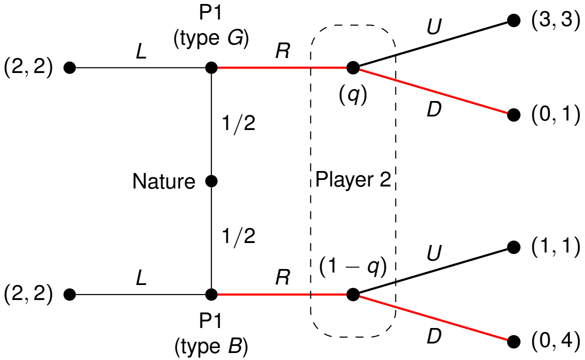
1. Second guess: RL
2. P2 beliefs at information set: When she sees R she infers P1 must be type G because ony type G chooses R. So $q = 1$.
3. P2 action: Given $q = 1$ she will choose U.
4. Verify: Does P1 want to play RL? Yes! Type G benefits from R ($3 > 2$) and B benefits from L ($2 > 1$).

What about pooling equilibria?



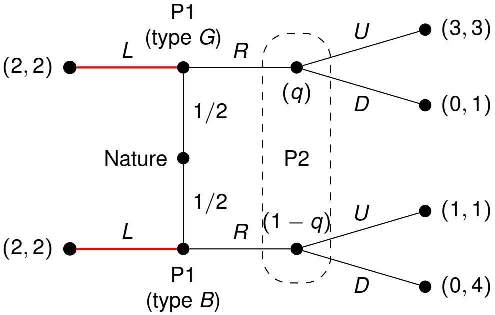
To check for pooling equilibria we can just repeat our guess and verify procedure for *LL* and *RR*. Start with *RR*:

What about pooling equilibria?



1. Guess: RR
2. P2 beliefs: ? THE PRIOR. $q = \frac{1}{2}$
3. P2 action: $EU_2(U|RR) = \frac{1}{2}(3) + \frac{1}{2}(1) = 2$, $EU_2(D|RR) = \frac{1}{2}(1) + \frac{1}{2}(4) = \frac{5}{2}$, so P2 chooses D.
4. Verify: Will P1 play RR? No! Both types want to deviate to L!

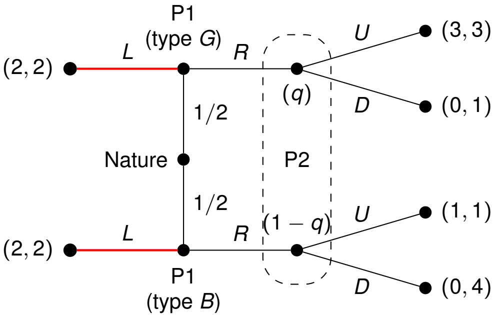
Checking the other pooling equilibrium



1. Guess: LL
2. P2 beliefs: ? OFF-PATH. Not pinned down by Bayes' rule
3. P2 action: Depends. We ask: What action would support LL? The answer is clearly D. Would SOME belief cause P2 to play D? Yes! For instance, $q = 0$. (Any $q < \frac{3}{5}$ will lead P2 to play D: $EU_2(U|q) < EU_2(D|q) \Rightarrow 3q + (1-q)1 < 1q + (1-q)4 \Rightarrow q < \frac{3}{5}$)
4. Verify: If P2 plays D then P1 would indeed play LL. So $(LL; D, q = 0)$ is a pooling PBE. (And there are a range of pooling PBE with $(LL; D, q < 3/5)$).

Evaluating the LL equilibrium

Does the pooling equilibrium with LL, D and $\mu(G|R) = q < \frac{3}{5}$ survive the intuitive criterion?



If P2 switched to U then type G would switch to R and type B would stay with L. $\Rightarrow$ P2 should think that P1 is type G after seeing R, in which case she would play U. $\Rightarrow$ the pooling equilibrium does not survive the application of the intuitive criterion

Supplemental slides

Lotteries for infinite sets

Definition

A simple lottery over an interval $X = [\underline{x}, \overline{x}]$ is given by a cumulative distribution function $F : X \to [0, 1]$ where $F(\hat{x}) = \Pr[x \leq \hat{x}]$ is the probability that the outcome is less than or equal to $\hat{x}$.

Back to finite lotteries

Continuous expected utility

Statistics review:

- A density function f for a continuous random variable is a function satisfying

$$\Pr[a \leq x \leq b] = \int_a^b f(x)dx. \tag{2}$$

  That is, the probability that a random variable x falls into the interval $[a, b]$ is found by finding the area under the density curve between points a and b.

- $\Rightarrow$ CDF can be defined as

$$F(\hat{x}) = \int_{\underline{x}}^{\hat{x}} f(x)dx \tag{3}$$

  if $\underline{x}$ is the smallest possible value of x.

- $\Rightarrow$ if F is differentiable then

$$f(x) = \frac{dF(x)}{dx}. \tag{4}$$

Continuous expected utility

Definition

Let $u(x)$ be the player's payoff function over outcomes in the interval $X = [\underline{x}, \overline{x}]$ with a lottery given by the cumulative distribution $F(x)$, with density $f(x)$. Then we define the player's expected payoff as

$$\mathbb{E}[u(x)] = \int_{\underline{x}}^{\overline{x}} f(x)u(x)dx.$$

Back to finite expected utility

Example: Two policies with continuous uncertainty

- Decisionmaker must again choose between two policies A and B.
- The set of possible outcomes from each policy choice is $X = [0, 1]$.
- Preferences: $u(x) = x$.
- The lotteries induced by each policy are CDFs: $F_A(x) = x^2$ and $F_B(x) = x$ for $0 \leq x \leq 1$.

Expected utility for policy A

- First we can derive the PDF:

$$f_A(x) = \frac{dF_A(x)}{dx} = 2x.$$

- The expected utility of choosing policy A is therefore:

$$\mathbb{E}[u(x)|A] = \int_0^1 f_A(x)xdx = \int_0^1 2xxdx = \frac{2}{3}.$$

Expected utility for policy B

$$f_B(x) = \frac{dF_B(x)}{dx} = 1.$$

$$\mathbb{E}[u(x)|B] = \int_0^1 f_B(x)xdx = \int_0^1 1xdx = \frac{1}{2}.$$

$\Rightarrow$ A maximizes expected utility.

Example: Choosing effort with continuous uncertainty

- We will think of a politician choosing effort to produce good outcomes in order to gain reelection

- The politician chooses a level of effort $a \in A = [0, 1]$.

- The politician's vote share in the next election is drawn from a uniform distribution on the interval $[0, a]$. (Thus, the pdf of outcomes is $f(x) = 1/a$ for $x \in [0, a]$)

- Payoff is:

$$u(x, a) = \sqrt{x} - a.$$

  where x is vote share

Solving the politician's problem

Expected utility for an effort level a:

$$\mathbb{E}[u(x,a)|a] = \int_0^a \frac{\sqrt{x}}{a} dx - a \tag{5}$$

$$= \frac{2}{3}\sqrt{a} - a. \tag{6}$$

Maximizing with respect to a. FOC:

$$\frac{1}{3\sqrt{a}} = 1$$

$\Rightarrow a = \frac{1}{9}$.

SOC: $-\frac{1}{6a^{\frac{3}{2}}} < 0$ ✓