Formal Political Theory

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Pol Sci 505

Overview

Introduction

Motivating example: The Presidential veto

- In the United States and many other places, the President must sign legislation passed by the legislature (or be overridden) before it becomes law.
- Does the presence of a Presidential veto substantially shape the laws that are eventually implemented?

From Chuck Cameron's Veto Bargaining:

> *Consider the extreme rarity of vetoes. Between 1945 and 1992, Congress presented presidents with over 17,000 public general bills.... From this flood of bills, presidents vetoed only 434. In other words, presidents in the post-war period vetoed only 25 public general bills per 1,000 passed....How can a weapon that is hardly ever used shape the content of important legislation under frequently occurring circumstances? (9)*

A simple bargaining model

To specify a model we must:

- Name the relevant players
- Specify what choices those players can make and how those choices lead to different outcomes
- Specify players' preferences over outcomes

Bargaining model

Players Congress and the President.

Choices and order of moves Congress moves first and chooses to pass a left-wing bill (L) or a moderate bill (M). The President moves second, sees what bill is chosen by Congress, and chooses to veto or not.

Outcomes If a bill is passed and not vetoed then that bill becomes law. If a bill is vetoed then it does not become law and the outcome is a status quo policy.

Preferences President prefers the moderate bill to the status quo but prefers the status quo to the left-wing bill. Congress prefers the left-wing bill to the moderate bill and the moderate bill to the status quo.

When does the President veto the bill?

- President can unilaterally kill any bill and implement the status quo.
- $\Rightarrow$ President's choice is between Congress's bill and the status quo.
- President's preferences:

$$\text{Moderate bill} > \text{SQ} > \text{Left-wing bill}$$

- *Rightarrow* President will use the veto when the left-wing bill is passed and not when the Moderate bill is passed.

What bill will Congress pass?

- Congress prefers the left-wing policy to the moderate policy and prefers the moderate policy to the status quo.

- Congress would therefore LIKE to pass the left-wing bill.

- However, Congress knows that the left-wing bill would be vetoed resulting in the status quo but the moderate policy would not be vetoed.

- $\Rightarrow$ Congress passes the moderate bill.

Some quick take-aways

- We predict NO VETOES yet the presence of a veto changes the final policy
- The reason is that Congress is thinking strategically and should try to anticipate the actions of the President
- We could complicate this much further (Congress might make errors, there might be conflicts within Congress, we could allow veto overrides) but this simple model makes this point nicely
- Broader point: We need a theory of the interaction we are studying in order to even begin to think about the meaning of the data we observe.

What are formal models?

This is two questions: What is a model? And, what makes a model formal?

Start with the first question

Models

- Loosely speaking a model is a representation of some target system which the analyst wishes to understand

Some types of models:

- Physical representation (a globe and a map are different models of the earth)
- Verbal analogy (e.g. Wlezien thermostatic model of public opinion)
- Mathematical system (e.g. the models we'll study in this class)

Models and accuracy

- Every model differs from its target system in some deliberate way.
  - Frictionless environments in physics
  - Two-species models in biology
  - Perfectly rational agents in a social scientific model

- A good model should also resemble the target system in some key ways, which will depend on the aspects of the target system the analyst wants to study

What makes a model formal?

Formal model:

- System of symbols and axioms
- Set of rules for manipulating them to derive results

Why?

- Anyone who applies the rules will derive the same results
- This is good! We'd like to accumulate and share knowledge
- We will study mathematical models. This is one type of formal model but not the only type (e.g. computational models are formal models).

Game theory

Most (not all) formal models in political science are game theoretic.

Game theory is the study of mathematical models of strategic interaction among two or more decision-makers.

Questions a game theorist asks about a problem

- Who are the relevant players?
- What decisions can these players make?
- What does each player know at the time that he or she must make a decision? What are the possible outcomes?
- What are players' preferences over these outcomes?

The answers to these questions will be used to create a mathematical system which we call a <u>game</u>.

The rules for making predictions from these games will come in the form of <u>solution concepts</u> which will be the main focus of our technical discussion.

- Questions about syllabus?

- Break.

Individual decision theory

A decision problem

Three components

- Actions that are available to the player
- Outcomes (i.e., consequences as the result of actions and possibly unforeseen/random events). Sometimes this is also referred to as alternatives.
- Preferences (i.e., how the player ranks the outcomes).

Actions

- Actions available to the decisionmaker is a set denoted by A.
- Example: A vote choice model
  - Voter can choose between voting for the Democrat, voting for the Republican, voting for a third-party candidate, or abstaining.
  - $A = \{d, r, t, a\}$
- Example: How much water to drink from a 1 gallon jug
  - Decisionmaker can choose any proportion of the jug from 0 to 1
  - $A = [0, 1]$

Outcomes

- Set of outcomes denoted by X
- Vote choice example:
  - $A = \{d, r, t, a\}$
  - Outcomes: the Democrat wins, the Republican wins, a third party candidate wins: $X = \{D, R, T\}$.

Preferences

- Preferences are defined over outcomes.
- Preferences defined as a binary relation over elements of X.

Binary relation?

- Technically, a set of ordered pairs of elements of X
- Relates any two elements of a set according to some criterion.
- Familiar example: Weak inequality relation "$\geq$" defined on the set of integers, where the expression $x \geq y$ is interpreted as "integer x is at least as big as the integer y."
- ...but back to preferences.

Preference relation

- relates any two outcomes in X as more or less desirable to the agent.
- Our notation: xRy is interpreted as “x is at least as good as y for the agent.
- May say “weak preference relation” xRy does not rule out indifference between x and y.

Preferences: Strict and weak

We can define strict preference and indifference using R:

$$xPy \text{ if and only if } xRy \text{ and not} yRx$$
$$xIy \text{ if and only if } xRy \text{ and } yRx.$$

Preference notation review page

Concept	My notation	Alternative notation	Analogy to numbers
Weak preference	R	$\succeq$	$\geq$
Strict preference	P	$\succ$	$>$
Indifference	I	$\sim$	$=$

Rationality

So far we have placed no restrictions on preferences whatsoever

Unfortunately, without more restrictions, things can get weird

Example: Things get weird

Consider our voter. The relevant outcomes are $X = \{d, r, t, a\}$ (democrat, republican, third party, abstain).

Consider the following strict preference relation:

$$\begin{aligned} dPr \\ rPt \\ tPa \\ aPd. \end{aligned}$$

What should the agent choose?

$$dPr$$
$$rPt$$
$$tPa$$
$$aPd.$$

Consider each possibility:

- d? This cannot be best because aPd
- r? no, dPr
- t? no, rPt
- okay then, a! oh no! tPa!

There is no best choice!

Okay fine, let's make some assumptions

We want to make a sufficient number of assumptions to guarantee that we can find a maximal choice for the decision-maker

Otherwise we cannot say what the decision-maker would do

We will collectively refer to these assumptions as rationality

Maximal outcomes

Definition

An outcome $x \in X$ is a maximal element of X with respect to the binary relation R if and only if xRy for all $y \in X$.

Reflexivity and completeness

Definition

A binary relation R on X is

1. Reflexive if for all $x \in X$, xRx.
2. Complete if for all $x, y \in X$ such that $x \neq y$, either xRy or yRx (or both).

We need reflexivity and completeness

We want to find assumptions that guarantee that there is always a maximal choice (i.e. there is a maximal choice given any subset of X).

It turns out reflexivity and completeness are necessary.

Our example satisfied reflexivity and completeness, so obviously these two assumptions will not be sufficient.

Lemma

If R violates reflexivity or completeness then there exists some $S \subseteq X$ such that there is no maximal element of S with respect to R.

Proof.

Part 1:
Suppose R violates reflexivity.
Then there exists some $x \in X$ such that xRx does not hold.
This implies that x is not a maximal element of any set
Therefore there is no maximal element for the subset $S = \{x\}$ of X.
Part two: Now suppose that R violates completeness.
Then there exist some $x, y \in X$ such that $\neg xRy$ and $\neg yRx$.
This immediately implies that there is no maximal element of the subset $S = \{x, y\}$ with respect to R. □

We have not yet captured what was wrong with out voting example.

In that example the voter's preferences were inconsistent in a problematic way: I should not strictly prefer d to r and r to t but then strictly prefer t to d.

The next conditions deal with this problem

Consistency conditions

Definition

A binary relation R on X is:

1. Transitive if for all $x, y, z \in X$, xRy and yRz implies xRz.
2. Quasi-transitive if for all $x, y, z \in X$, xPy and yPz implies xPz.
3. Acyclic if for all $\{x, y, z, \ldots, u, v\} \in X$, $xPy \& yPz \ldots \& uPv$ implies xRv.

The are ordered from strong to weak: Transitivity implies quasi-transitivity and acyclicity but not the reverse, quasi-transitivity implies acyclicity but not the reverse

Example: Quasi-transitive but not transitive

Example

Let $X = \{x, y, z\}$ with xPy, yIz, zIx. Then R is quasitransitive (vacuuosly). However, it is not transitive: yRz and zRx but not yRx. $\square$

Example: Acyclic but not quasitransitive

Example

Let $X = \{x, y, z\}$ with xPy and yPz and xIz. Then R is acyclic (since xRz) but not quasitransitive (since zRx). $\square$

Theorem

Let R be reflexive and complete and let X be finite; then the set of maximal outcomes is nonempty for all $S \subseteq X$ if and only if R is acyclic.

To prove this we need to prove both directions of the “if and only if”: that is, we need to show that acyclicity is sufficent for a maximal choice and also that it is necessary.

Proof of sufficiency

For any $S \subseteq X$ choose $x \in S$. If, for all $s \in S$, xRs then x is a maximal element and we are done.

Otherwise (since R is reflexive and complete) there must exist $y \in S \backslash \{x\}$ such that yPx.

If for all $s \in S$, yRs then y is a maximal element and again the we are done. Otherwise, there exists $z \in S \backslash \{x, y\}$ such that zPy. By acyclicity, it must be that zRx as well.

If for all $s \in S$, zRs then again we are done. Otherwise there exists $w \in S \backslash \{x, y, z\}$ such that wPz. By acyclicity this means wRy and wRx as well.

Because X (and hence S) is finite, we can continue this logic to conclude that there must exist an alternative weakly preferred to all other alternatives in S. $\square$

Proof of necessity

Let $x_1, x_2, \ldots, x_n$ be elements of X and assume

$$x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n;$$

we wish to show that $x_1 R x_n$ if the set of maximal outcomes is nonempty.

Let $S = \{x_1, \ldots, x_n\} \subseteq X$ and suppose that the set of maximal elements of S with respect to R is nonempty.

Therefore. Because $x_{i-1} P x_i$ for $i = 2, \ldots, n$, we have that x_i is not maximal for any $i = 2, \ldots, n$.

Therefore, if the set of maximal elements is nonempty it must be the case that x_1 is maximal which implies in particular that $x_1 R x_i$ for all $i = 2, \ldots, n$; in particular $x_1 R x_n$ as required. $\square$

Rationality

- This result lays the foundation for a minimal set of requirements on preferences to guarantee that the agent has an optimal choice in any given situation.
- As shorthand, we will say an agent is rational if her preferences are reflexive, transitive, and complete.
- But we will typically make stronger assumptions to make the problem more tractable.

Utility

In practice, we would rather not work with binary relations all the time.

We therefore introduce utility, which lets us work with a mathematical function rather than an ordinal relation

Utility

A utility function is a function $u : X \to \mathbb{R}$ that assigns each outcome in X a real number

A utility function represents a preference relation R if the function assigns higher numbers to outcomes that are ranked higher under R

Definition

A utility function $u : X \to \mathbb{R}$ represents the preference relation R if for any pair $x, y \in X$, $u(x) \geq u(y)$ if and only if xRy.

Remarks on utility functions

1. No utility function uniquely represents a preference relation: since preference relations are only ordinal, any two functions preserving the same order represent the same preferences
   - e.g. if u represents the preference relation R, then so does any increasing transformation of u.

2. For the simplest settings with no uncertainty, the ordinal properties of the utility function will not matter for the solution.

3. We next turn to the question: When can preferences be represented by a utility function?

When can preferences be represented by a utility function?

Theorem

Let X be finite. Then any reflexive, complete, and transitive preference relation over X can be represented by a utility function.

Proof by construction

Let R be a reflexive, complete, and transitive preference relation over X.

Because the preference relation is complete and transitive, we can find a least-preferred outcome $\underline{x} \in X$ such that all other outcomes $y \in X$ are at least as good as $\underline{x}$.

Now define the “worst outcome equivalence set”, denote X_1, to include $\underline{x}$ and any other outcome y for which the player is indifferent between y and $\underline{x}$.

Proof by construction (continued)

Then, from the remaining elements $X \backslash X_1$, define the "second worst outcome equivalence set," X_2, and continue in this fashion until the "best outcome equivalence set" X_n, is created.

Because X is finite and R is reflexive, transitive, and complete, such a finite collection of n sets exists.

Now consider n arbitrary numbers such that $u_n > u_{n-1} > \cdots > u_2 > u_1$, and assign payoffs according the rule: for any $k \in \{1, \ldots, n\}$ and $x \in X_k$, $u(x) = u_k$. This payoff function represents the preference relation R and therefore we have proven that such a function exists. $\square$

Example: Two simple decision problems

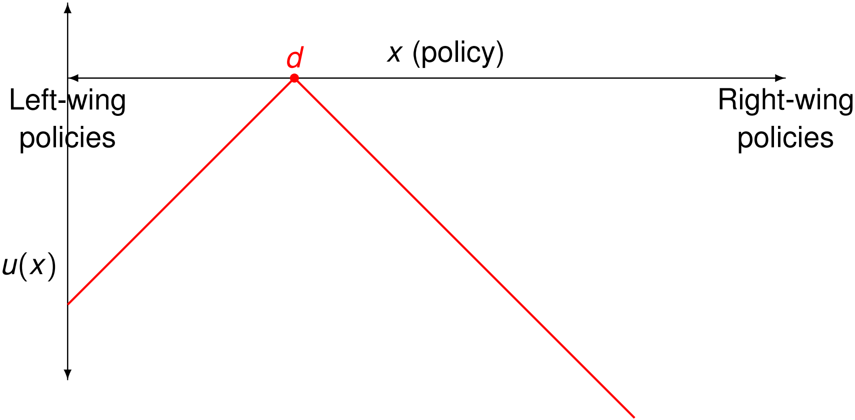
$X = A = \{a, b, c\}$

Utility function	$u(a) = 1, u(b) = 2, u(c) = 3$	$\tilde{u}(a) = 1, \tilde{u}(b) = \tilde{u}(c) = 3.$
Maximal elements	? $\{c\}$	? $\{b, c\}$

Example: Spatial policy choice

- Political scientists often represent policies and policy preferences as points on the real line.
- Let $x \in \mathbb{R}$ be a policy outcome. We think of smaller values of x as being more left-wing policies and larger values as being more right-wing policies.
- Let $d \in \mathbb{R}$ be the decisionmaker's ideal policy (often called an ideal point). This represents the policy the decisionmaker most prefers. The decisionmaker likes policies closer to d more than policies further away from d.
- We can represent these preferences with the utility function:

$$u(x) = -|x - d|$$



It's immediate that the decisionmaker's optimal policy choice is $x = d$.

Example: Allocating effort

- Consider a bureaucrat asked to allocate effort toward policy implementation
- The policy can be implemented poorly with little effort or implemented better with increasing effort
- The possible effort levels are the set of nonnegative real numbers (i.e. $X = [0, \infty)$.)
- The decisionmaker's preferences are represented by the following utility function:
$$u(x) = x - kx^2.$$

Example: Allocating effort (solution)

How do we solve for the maximal effort?

The first-order condition for a maximum is that the first derivative of the utility function with respect to x be equal to zero:

$$\frac{\partial u(x)}{\partial x} = 1 - 2kx = 0,$$

which is solved by $x = \frac{1}{2k}$.

The second-order condition to ensure that this is indeed a maximum is that the second derivative is negative:

$$\frac{\partial^2 u(x)}{\partial x^2} = -2k < 0,$$

which holds for all values of x since $k > 0$.

Next step: Decisionmaking under uncertainty

- So far we have only considered problems in which the agent faces no uncertainty

- In reality, decisionmakers are not always completely certain of the payoff consequences of their choices. We need to expand our framework to account for this.

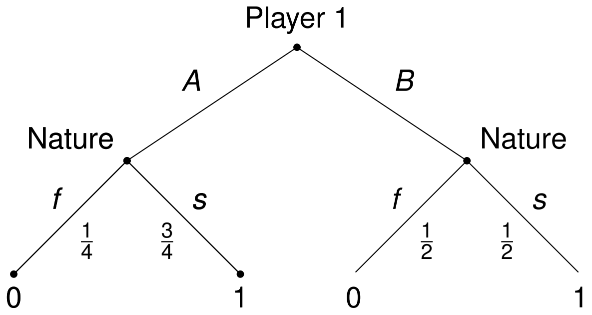
Motivating example: Choosing between two policies

- Two policies, A and B
- Policy A is successful with probability $\frac{3}{4}$ and unsuccessful with probability $\frac{1}{4}$. Policy B is successful with probability $\frac{1}{2}$ and unsuccessful with probability $\frac{1}{2}$.
- Payoff: 1 for a successful policy and 0 for an unsuccessful policy.

Motivating example: Choosing between two policies

- We can think of the decisionmaker as choosing between a lottery that offers a payoff of 0 with probability $\frac{1}{4}$ and of 1 with probability $\frac{3}{4}$ and another lottery that offers a payoff of 1 or 0 each with equal probabilities.

- We may think of this problem using a decision tree with moves by "Nature":


Player 1
A
B
Nature
Nature
f
s
f
s
1/4
3/4
1/2
1/2
0
1
0
1

Introducting lotteries

Definition (Lottery over finite outcomes)

A simple lottery over outcomes $X = \{x_1, x_2, \ldots, x_n\}$ is defined as a probability distribution $p = (p(x_1), p(x_2), \ldots, p(x_n))$, where $p(x_k) \geq 0$ is the probability that x_k occurs and $\sum_{k=1}^{n} p(x_k) = 1$.

Lotteries over infinite sets

Preferences over lotteries

- Conceptually there is no problem with simply defining preferences over lotteries and imposing the same rationality requirements as before
- In practice, it is more useful for us to think about preferences over lotteries as extensions of preferences over fundamental outcomes, combined with beliefs about probabilities
- To this end, we will think of agents as maximized <u>expected</u> utility

Expected utility

Definition

Let $u(x)$ be the player's utility function over outcomes $X = \{x_1, x_2, \ldots, x_n\}$, and let $p = (p_1, p_2, \ldots, p_n)$ be a lottery over X such that $p_k = \Pr[x = x_k]$. Then we define the player's expected utility from the lottery p as

$$\mathbb{E}[u(x)|p] = \sum_{k=1}^{n} p_k u(x_k).$$

Continuous version

Solving the two uncertain policies problem

- Expected utility from choosing A:

$$\mathbb{E}[u(x)|A] = \Pr[s|A]u(s) + \Pr[f|A]u(f) = \frac{3}{4}1 + \frac{1}{4}0 = \frac{3}{4}.$$

- Expected utility from choose B is

$$\mathbb{E}[u(x)|B] = \Pr[s|B]u(s) + \Pr[f|B]u(f) = \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}.$$

- $\Rightarrow$ A is the maximal choice

Allocating effort with uncertainty

- The bureaucrat allocates effort equal to $a \in [0, 1]$.
- The policy is successful ($x = 1$) with probability $\Pr[x = 1|a] = a$ and unsuccessful ($x = 0$) with probability $1 - a$.
- Bureaucrat's payoff:

$$u(x, a) = x - ka^2$$

Solving for optimal effort level

- Expected utility from any effort level a:

$$\begin{aligned}\mathbb{E}[u(x,a)|a] =& a(1-ka^2)+(1-a)(0-ka^2)\\ =& a-ka^2.\end{aligned}$$

- Differentiating and solving for the FOC gives: $a=\frac{1}{2k}$.

- Accounting for the fact that effort is constrained to be in [0, 1]: the agent gives maximum effort of $a=1$ for $k \leq \frac{1}{2}$ and gives $a=\frac{1}{2k}$ if $k > \frac{1}{2}$.

Remarks on expected utility

- The cardinal properties of the utility function matter now!
- The properties of u capture risk preferences. For continuous action sets, concavity of u implies risk aversion, convexity of u implies risk acceptance.
- This is not an innocuous assumption and there are other possible ways to represent preferences over lotteries. Real people sometimes do violate expected utility.

Dynamic decision problems

- We have analyzed only static problems
- Sometimes, we are interested in how a decisionmaker makes multiple interconnected choices at different points in time
- We will look in these cases for an entire plan that maximizes expected utility at every step.

Policy then effort choice

First, decisionmaker chooses between two policies:

- Safe policy: Implemented with no effort
- Risky policy: Better than the safe policy when it's successful, worse when it is not. Effort helps increase the likelihood of success.

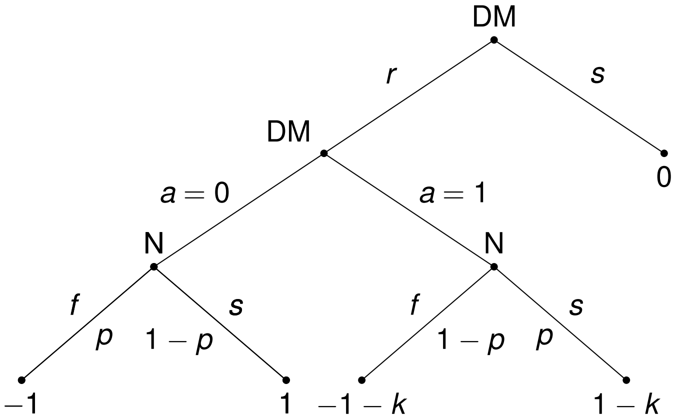
Second, decisionmaker chooses effort $a \in \{0, 1\}$.

The probability that the risky policy succeeds is $p > \frac{1}{2}$ if $a = 1$ and $1 - p$ if $a = 0$.

The policymaker's payoff is 0 from the safe policy and

$$u(r, a) = \begin{cases} 1 - ka & \text{if the policy succeeds} \\ -1 - ka & \text{if the policy fails} \end{cases}$$

from the risky policy, where $k > 0$ is a cost of effort.


DM
r
s
DM
0
a = 0
a = 1
N
N
f
s
f
s
p
1 − p
1 − p
p
−1
1
−1 − k
1 − k

Solving by backward induction

Start with the effort decision:

$$\mathbb{E}[u|r, a = 0] = p(-1) + (1 - p)1 = 1 - 2p.$$

$$\mathbb{E}[u|r, a = 1] = p(1 - k) + (1 - p)(-1 - k) = 2p - 1 - k.$$

$\Rightarrow$ policymaker makes an effort if

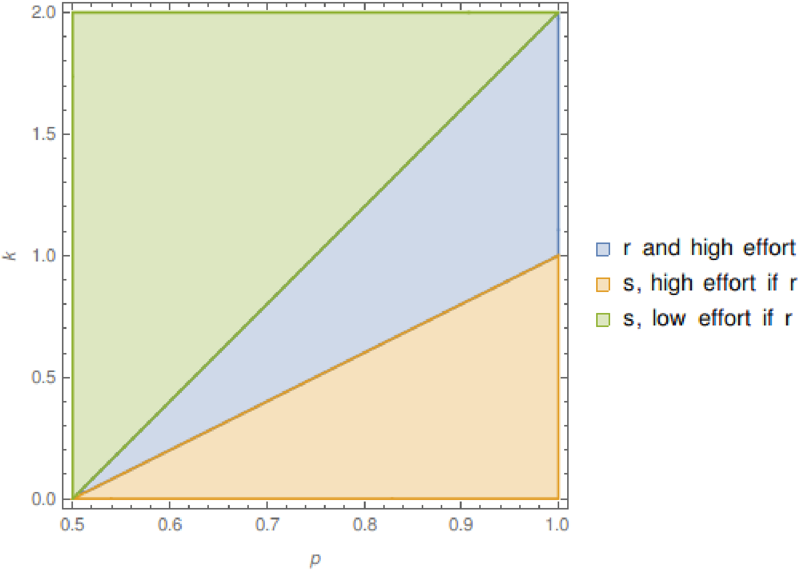
$$2p - 1 - k \geq 1 - 2p$$

Back one step: The policy decision

- Case 1: $k \leq 4p - 2$.
  - DM will make an effort if she chooses r, so her expected utility for choosing r is $2p - 1 - k$.
  - r is optimal in this case if $2p - 1 - k > 0$
  - $\Rightarrow$ if $2p - 1 < k \leq 4p - 2$ then the decisionmaker's optimal choice is s. If $k \leq 2p - 1$ then the optimal choice is r.

- Case 2: $k > 4p - 2$.
  - The agent would not make an effort after choosing r.
  - Then her expected utility for choosing r is $1 - 2p < 0$
  - Thus, the decisionmaker should always choose s in this case.

Solution pictured



Decision theory wrap up

- Rationality requirements: Acyclicity or transitivity, completeness, reflexivity
- Basic problems: Choose the maximal choice from a set
- Wrinkles: Uncertainty, dynamics

The rest of the class: Multiple decisionmakers (games)

Static games of Complete Information

A policy problem: Defending against natural disasters

Consider the mayor of a town: 

She must allocate resources to prepare for a potential natural disaster at two sites.

The sites:

		
	City Hall	The Lookout
Chances of disaster	75%	33%
Damages from disaster	50	200

Resource allocation lessens the damage in some way proportional to the amount of resources

- This is a decision-theoretic problem and is fairly easy to solve

- We know the probabilities and the outcomes, we just compute the expected damage and allocate funds to optimize the Mayor's objective function

For instance, if her objective is simply to minimize expected damages she should allocate all of the resources for the Lookout.

Related problem: Defending against a terrorist attack

Why is this problem so different?

Attacks are strategic!

The Mayor needs to think about what the bad kitties are going to do. Just as importantly...

The Mayor should assume that the cats are thinking about what she is going to do!

The solution must be different!

- Suppose Mayor Goodway devotes all of her resources to protecting the Lookout, meaning no damage will occur
- Then the Kitten Catastrophe Crew will for sure not attack the lookout and will instead target City Hall
- But then the original allocation of resources is no longer optimal!
- This is a game theoretic problem. Politics and policy is full of them. The rest of this class is about how to solve them.

Strategic form games

Definition

A game in strategic (or normal) form has three elements:

1. The set of players $N = \{1, 2, \ldots, n\}$, with a particular player denoted by $i \in N$.
2. A set of pure strategies S_i for each players, with the set of all pure strategies denoted $S = \{S_1, S_2, \ldots, S_n\}$.
3. A set of payoff functions $\{u_1, u_2, \ldots, u_n\}$ that give play i's von Neumann-Morgenstern utility $u_i(s)$ for every profile $(s_1, s_2, \ldots, s_n)$ of pure strategies.

Common knowledge

In a game of <u>complete information</u>, all aspects of the game are <u>common knowledge</u>

An event E is common knowledge if everyone knows E, everyone knows E, everyone knows that everyone knows E, and so on ad infinitum.

Important for how we understand games: I choose my actions anticipating that you understand the game, that you believe that I understand the game, that you believe that I believe that you understand the game, and so on.

Important example

A battle of wits

Example: Paper, Rock, Scissors

- Players: Player 1 and Player 2 ($N = \{1, 2\}$)
- $S_1 = S_2 = \{\text{Rock}, \text{Paper}, \text{Scissors}\}$
- Rock beats Scissors, Paper beats Rock, and Scissors beats Paper. Each player gets 1 for a win, -1 for a loss, and 0 for a draw.

Matrix Form

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

Strategies

- A mixed strategy σ_i for player $i \in N$ is a probability distribution over i's pure strategies S_i.
- i.e. $\sigma_i(s_i)$ is the probability that i's strategy assigns to s_i.
- Let Σ_i be the set of all of player i's mixed strategies

- Each pure strategy can be represented as a degenerate case of a mixed strategy: a strategy in which i always plays some action s is simply a mixed strategy assigning $\sigma_i(s) = 1$ and $\sigma_i(s') = 0$ for all $s' \neq s$.

Back to Rock, Paper, Scissors

Examples of mixed strategy profiles:

$$\sigma_1(\text{Rock}) = \frac{1}{3}, \sigma_1(\text{Paper}) = \frac{1}{3}, \sigma_1(\text{Scissors}) = \frac{1}{3}$$
$$\sigma_2(\text{Rock}) = \frac{1}{2}, \sigma_2(\text{Paper}) = \frac{1}{4}, \sigma_2(\text{Scissors}) = \frac{1}{4}.$$

$$\sigma_1(\text{Rock}) = 1, \sigma_1(\text{Paper}) = 0, \sigma_1(\text{Scissors}) = 0$$
$$\sigma_2(\text{Rock}) = 0, \sigma_2(\text{Paper}) = 0, \sigma_2(\text{Scissors}) = 1,$$

(ok just to write $s_1 = \text{Rock}, s_2 = \text{Scissors}$.)

Payoffs and expected payoffs given strategies

- $u_i(s)$ is player i's payoff when every player's action follows the strategy profile s.
  - e.g. $s = (\text{Rock}, \text{Scissors})$ is a particular strategy profile in the RPS game, for which we have $u_1(s) = 1$ and $u_2(s) = -1$
- Convention: For a profile s, let $s = (s_i, s_{-i})$ for some player i, where $s_{-i} \in S_{-i}$ is the strategy used by every player other than i
- Hence, $u_i(s) = u_i(s_i, s_{-i})$
- Similarly, (σ_i, σ_{-i}) can denote a particular mixed strategy profile

Expected payoffs from arbitrary mixed strategies

Player i's expected utility from choosing the pure strategy $s_i \in S_i$ when her opponents choose the mixed strategy $\sigma_{-i} \in \Sigma_{-i}$ is

$$U_i(s_i, \sigma_{-i}) = \sum_{s \in S_{-i}} \Pr[s|\sigma_{-i}] u_i(s_i, s_{-i})$$

$$= \sum_{s \in S_{-i}} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s_i, s_{-i}).$$

Expected payoffs from arbitrary mixed strategies (cont)

Player i's expected utility from playing the pure strategy σ_i when her opponents play σ_{-i} is therefore

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) U(_i s_i, \sigma_{-i}), \tag{1}$$

by the law of iterated expectations.

Back to rock, paper scissors

Consider this strategy:

$$\sigma_1(\text{Rock}) = \frac{1}{3}, \sigma_1(\text{Paper}) = \frac{1}{3}, \sigma_1(\text{Scissors}) = \frac{1}{3}$$
$$\sigma_2(\text{Rock}) = \frac{1}{2}, \sigma_2(\text{Paper}) = \frac{1}{4}, \sigma_2(\text{Scissors}) = \frac{1}{4}.$$

$$\begin{aligned} U_1(\text{Rock}, \sigma_2) =& \sigma_2(\text{Rock})0 + \sigma_2(\text{Paper})(-1) + \sigma_2(\text{Scissors})1 \\ =& \frac{1}{2}0 + \frac{1}{4}(-1) + \frac{1}{4}(1) \\ =& 0, \end{aligned}$$

$$\begin{aligned} U_1(\text{Paper}, \sigma_2) =& \sigma_2(\text{Rock})1 + \sigma_2(\text{Paper})0 + \sigma_2(\text{Scissors})(-1) \\ =& \frac{1}{2}1 + \frac{1}{4}(0) + \frac{1}{4}(-1) \end{aligned}$$

Which strategies are better?

We now need to develop a notion of how agents should choose strategies

The choice is not obvious because we have to take into account beliefs about what other players will do, knowing that these other players are also trying to choose the best strategies

A set of conditions that allow us to choose the best strategy of every player will be called a solution concept. We will work through several.

Dominance

- We will start with a very strong way in which one strategy might be better than another
- We do this by assuming nothing about what i believes the other players will do: can we still say that some strategies are better than others?
- Sometimes, yes: we will say a strategy <u>dominates</u> another one if that strategy is better no matter what the other players do

Dominance in pure strategies

Definition

Let $s_i \in S_i$ and $s_i' \in S_i$ be possible strategies for player i. s_i' is strictly dominated by s_i if

$$u_i(s_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Note: We may also refer to weak dominance, which occurs when the inequality above holds weakly for all $s_{-i} \in S_{-i}$ and strictly for some particular s_{-i}.

Remarks on dominance

- Dominance avoids the difficulties associated with beliefs about other players' strategies: a rational player should never play a dominated strategy, regardless of what they believe the other players will do

- Disadvantage: Often we will have a pair of strategies for which no strategy strictly dominates the other (consider rock, paper, scissors), in which case dominance does not make a prediction

Strictly dominant strategies

Definition

$s_i \in S_i$ is a <u>strictly dominant strategy</u> if every other strategy of i is strictly dominated by it:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$.

In words: Regardless of what the other players do, s_i is the best choice

Dominant strategy equilibrium

This leads us to our first solution concept for games:

Definition

The strategy profile $s^D \in S$ is a strict dominant strategy equilibrium if $s_i^D \in S_i$ is a strict dominant strategy for all $i \in N$.

Example: Prisoner's Dilemma

The story:

- Two people are taken in for questioning
- Each can stay quiet ("cooperate" with the other person) or provide evidence against the other person ("defect")
- There is not enough evidence to convince on the principal charge, but enough to convict each on a lesser charge
- Both people offered a deal: defect on the other and avoid jail
- If both defect, the testimony is no longer needed so both are convicted, perhaps with some slight leniency in sentencing for helping prosecutors
- If one defects and the other cooperates, the cooperator gets the maximum sentence and the other gets off

A matrix form for the prisoner's dilemma

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	$-1, -1$	$-3, 0$
	Defect	$0, -3$	$-2, -2$

Dominance in the PD

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	$-1, -1$	$-3, 0$
	Defect	$0, -3$	$-2, -2$

Does Player 1 have a dominant strategy?
If Player 2 cooperates, Player 1 is better off defecting.
If player 2 defects, Player 1 is better off defecting.
That is, Player 1 is better off defecting no matter what Player 2 does. A dominant strategy!
Notice that the same is true for Player 2.
If Player 1 cooperates, Player 2 is better of defecting.
If Player 1 cooperates, player 2 is better off defecting
Therefore, Player 2 also has a dominant strategy to defect.
Thus, (Defect, Defect) is a strict dominant strategy equilibrium: for both players, Defect is a strictly dominant strategy.

Mixed strategy dominance

- We defined dominance for pure strategies but we should extend the idea to mixed strategies
- Fact 1: If s_i' is strictly dominated by s_i then any mixed strategy that assigns positive probability to s_i' is also strictly dominated.
- Fact 2: A strategy s_i' may be strictly dominated by some mixed strategy even if it is not strictly dominated by any pure strategy.

Mixed strategy dominance example

Player 2

		L	C	R
Player 1	U	$5,1$	$1,4$	$1,0$
	M	$3,2$	$0,0$	$3,5$
	D	$4,3$	$4,4$	$0,3$

No pure strategy is dominated by any other pure strategy (check)
For player 2, I claim that L is dominated by $(0, \frac{1}{2}, \frac{1}{2})$ (randomizing evenly between C and R).
We can check this for each pure strategy of player 1:

	L	$(0,1/2,1/2)$	
U	1	$\frac{1}{2}4 + \frac{1}{2}0 =$	2
M	2	$\frac{1}{2}0 + \frac{1}{2}5 =$	2.5
D	3	$\frac{1}{2}4 + \frac{1}{2}3 =$	3.5

Dominated strategy

Definition

Let $\sigma_i \in \Delta S_i$ and $s_i' \in S_i$ be possible strategies for player i. We say s' is strictly dominated by σ_i if

$$u_i(\sigma_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.
s_i' is a strictly dominated strategy if there exists a strategy $\sigma_i \in \Delta S_i$ such that σ_i strictly dominates s_i'.

Iterated Elimination of Strictly Dominated Strategies

- Strict dominated strategy equilibrium has the same strength and weakness: it relies only on player rationality
- This is a strength because we do not have to make strong assumptions about what players believe
- This is a weakness because it very often means we cannot make a prediction
- Furthermore: if players are rational, we might think that players should believe that other players are rational. This lets us take one step further: players will not play dominated strategies AND they will not believe that other players will play dominated strategies.

Iterated Elimination of Strictly Dominated Strategies

- Adding common knowledge of rationality suggests an iterative procedure that we can use to eliminate strategies (called IESDS):
  1. Eliminate strictly dominated strategies from the original game
  2. Consider the new game formed after eliminating those strategies: are there any strictly dominated strategies? If so delete them.
  3. Continue this process until there are no strictly dominate strategies remaining.

- Any strategy profile that survives IESDS is called an iterated-elimination equilibrium.

IESDS example

Player 1 (rows) vs. Player 2 (columns):

	Left	Middle	Right
Up	1, 0	1, 2	0, 1
Down	0, 3	0, 1	2, 0

Player 1 (rows) vs. Player 2 (columns):

	Left	Middle
Up	1, 0	1, 2
Down	0, 3	0, 1

Player 1 (rows) vs. Player 2 (columns):

	Left	Middle
Up	1, 0	1, 2

Player 1 (rows) vs. Player 2 (columns):

	Middle
Up	1, 2

Best response

- Both of the solution concepts we have used so far are based on eliminating actions that players should never play
- Alternatively we might ask: What strategies might players choose to play and under what conditions?
- To move in that direction we ask: What strategy should a player choose if she believes the other players are playing some strategy σ_{-i}? This is the concept of best response.

Best response: Definition

Definition

A strategy σ_i is a best response to an opponent strategy profile σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$$

for all $s_i \in S_i$.

Nash equilibrium

Definition

A mixed strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Sigma_i$.

Nash equilibrium: Remarks

- One way we think about Nash equilibrium. We require:
  1. People best respond to their beliefs about what others will do
  2. Their beliefs are correct

- Mathematically: A Nash equilibrium is a fixed point of the best response correspondence
  1. Fixed point of a function: x such that $f(x) = x$.
  2. Significant for studying stability in different systems

- Relatedly, we can reach Nash equilibria as the stable state of simple dynamic learning processes

Nash's theorem

Theorem

Every game in which each player has finitely many actions has a mixed strategy Nash equilibrium.

Remarks:

- Recall a pure strategy equilibrium is a mixed strategy equilibrium in which all the probabilities are zero or one, so this includes games with only PSNE
- Finitely many actions is important: for games with a continuum of actions we need more assumptions to guarantee existence of Nash equilibria.
- This is an important result: Shows that Nash equilibrium is a complete solution, at least for finite games.

Nash equilibrium in pure strategies

A pure strategy profile is a Nash equilibrium if for all i we have $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

These do not always exist but when they do we can find them by analyzing the best response correspondences of all the players.

Example: Bach or Stravinsky

		Player 2	
		Bach	Stravinsky
Player 1	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

Two pure strategy Nash equilibria: (Bach, Bach), (Stravinsky, Stravinsky).

Example: 3x3 Game

			Player 2	
		L	C	R
Player 1	U	7, 7	4, 2	1, 8
	M	2, 4	5, 5	2, 3
	D	8, 1	3, 2	0, 0

One pure strategy Nash equilibrium: (M, C).

Tragedy of the commons (2 players)

- Two individuals share a common pool resource (say, a fishery). $K > 0$ represents the total amount of resource available.
- Each decides how much of the resource to consume.
- Each player gets a private benefit from consumption but also cares about conservation:

$$u_i(s_i, s_{-i}) = \underbrace{\log(s_i)}_{\text{consumption}} + \underbrace{\log(K - s_i - s_{-i})}_{\text{conservation}}$$

Tragedy of the commons: Best responses

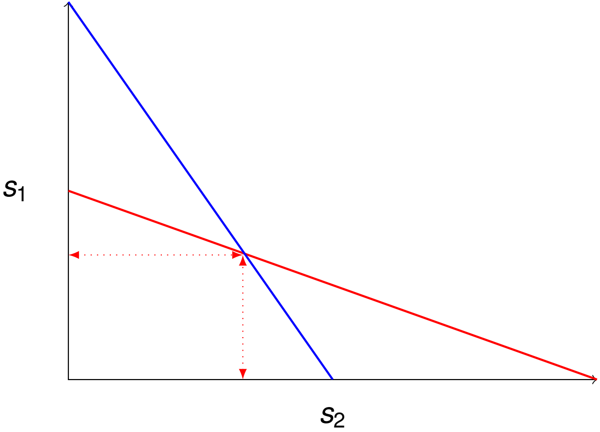
The best response of player i so a consumption level s_{-i} is found using some calculus. First-order condition:

$$\frac{\partial u(s_i, s_{-i})}{\partial s_i} = \frac{1}{s_i} - \frac{1}{K - s_i - s_{-i}} = 0$$

Solving for s:

$$s_i^*(s_{-i}) = \frac{K - s_{-i}}{2}$$

Visualizing best responses



Player 1's best response: $s_1 = \frac{K - s_2}{2}$

Player 2's best response: $s_2 = \frac{K - s_1}{2}$

Equilibrium is at intersection of these best response functions: this is the point at which both players are playing best responses

Solving mathematically

Given our best responses, a Nash Equilibrium occurs when both of the following are true:

$$s_1 = \frac{K - s_2}{2}$$
$$s_2 = \frac{K - s_1}{2}.$$

We can solve this system for s_1 and s_2. Plugging in the formula for s_2 into the first equation:

$$s_1 = \frac{K - \frac{K - s_1}{2}}{2} = \frac{K + s_1}{4}$$
$$\frac{3}{4}s_1 = \frac{K}{4}$$
$$\Rightarrow s_1 = \frac{K}{3}.$$

Solving mathematically (continued)

Plugging in our solution to P2's best response function:

$$
\begin{aligned}
s_2 &= \frac{K - \frac{K}{3}}{2} \\
&= \frac{\frac{2}{3}K}{2} \\
&= \frac{K}{3}.
\end{aligned}
$$

Therefore, the Nash equilibrium to this game is one in which both players consume $\frac{K}{3}$.

Note that the equilibrium is suboptimal: Their payoffs would both be higher if they could commit to only consuming $\frac{K}{4}$ (you can check this later). The "tragedy" is that they cannot coordinate on a better outcome because of externalities from consumption.

Example: Rock, Paper, Scissors

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

Uh oh. No PSNE! We need to look for mixed strategy Nash equilibria. Coming right up.

Mixed strategy in games

Imagine $\sigma_1(T) = p$ and $\sigma_2(L) = q$. Knowing this, we can compute the likelihood of each outcome:

P2

		L (q)	R ($1-q$)
P1	T (p)	pq	$p(1-q)$
	B ($1-p$)	$(1-p)q$	$(1-p)(1-q)$

Figure: Probabilities of outcomes

- Outcome (T, L) occurs with probability $p \times q$
- Outcome (B, L) occurs with probability $(1-p) \times q$
- Outcome (T, R) occurs with probability $p \times (1-q)$
- Outcome (B, R) occurs with probability $(1-p) \times (1-q)$

An example: American football

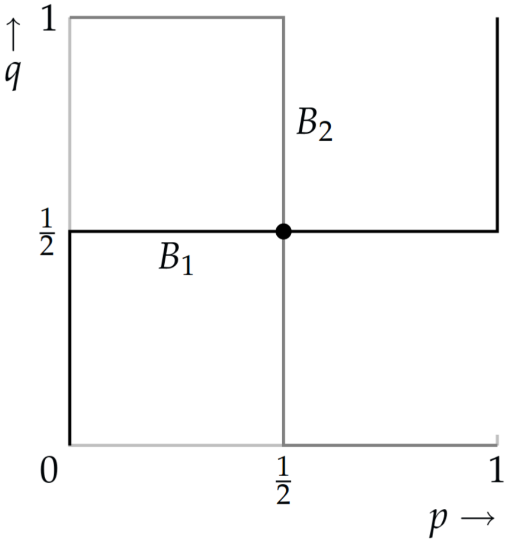
		Defense	
		Pass	Run
Offense	Pass	$-1, 1$	$1, -1$
	Run	$1, -1$	$-1, 1$

- Is there a PSNE? No.
- So what do we do? Look for MSNE.
- In this game, the MSNE is given by $\left((\frac{1}{2}\text{Pass}, \frac{1}{2}\text{Run})\,;\,(\frac{1}{2}\text{Pass}, \frac{1}{2}\text{Run}) \right)$

See also: Techmo Bowl

See also: Techmo Bowl

Visualizing best responses and equilibrium



Solving the football game

			Defense	
			Pass (q)	Run ($1-q$)
Offense	Pass (p)		$-1, 1$	$1, -1$
	Run ($1-p$)		$1, -1$	$-1, 1$

- Offense chooses p to make Defense indifferent (so it can't be taken advantage of) and Defense does the same
- So, to get optimal p we need to think about Defense's payoffs:

$$
\begin{aligned}
EU_D(\text{Pass}) &= p(1) + (1-p)(-1) = p - (1-p) = 2p - 1 \\
EU_D(\text{Run}) &= p(-1) + (1-p)(1) = (1-p) - p = 1 - 2p \\
EU_D(\text{Pass}) &= EU_D(\text{Run}) \text{ [INDIFFERENCE]} \\
2p - 1 &= 1 - 2p \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}
\end{aligned}
$$

- We can do the same solving for q

Mixed strategy Nash equilibrium: the weird part

- Notice that we made a strange move to solve for the mixed Nash Equilibrium: to compute one player's strategy, we used the OTHER player's payoff
- The reasoning:
  - For Player 1's best response to be a mixed strategy, she must be indifferent between the actions over which she is mixing. Otherwise she should choose the action she strictly prefers rather than randomizing
  - Player 1's indifference depends on Player 2's strategy, so we choose a randomization probability for Player 2 that would make Player 1 indifferent between her actions
  - But for Player 2 to randomize, she also must be indifferent, so then we must choose a strategy for Player 1 that makes this so.

Remark 1: Intepreting mixed strategies

- Interpretation 1: The players literally randomize their actions.
  - This probably works well for something like rock paper scissors but not so well for other games
- Interpretation 2: There are a distribution of players and each player is uncertain what type of player they are facing.
  - Here we can interpret a mixed strategy as a proportion playing a particular pure strategy in the population
  - This interpretation is sometimes preferred in biology and might make sense for games describing day-to-day interations
- Purification (Harsanyi 1973): Shows that mixed strategies are the limit of pure strategy equilibria to games with small amounts of noise in the payoffs.

Another example: Regulation

		Warren G: Regulate	Warren G: Do Nothing
Nate Dogg	Regulate	$2, 2$	$0, 1$
	Do nothing	$1, 0$	$1, 1$

- Do you recognize this game?
- What are the PURE strategy Nash equilibria?
- HINT: Almost all games have an odd number of Nash equilibria, so if you find two PSNE keep looking.
- Finding the mixed Nash equilibrium. Let $p = \Pr[\text{Nate Dogg regulates}]$ and $q = \Pr[\text{Warren G regulates}]$:
  1. Find Nate Dogg's strategy by making Warren G indifferent: $p2 + (1-p)0 = p1 + (1-p)1 \Rightarrow p2 = 1 \Rightarrow p = \frac{1}{2}$
  2. Find Warren G's strategy by making Nate Dogg indifferent: $q2 + (1-q)0 = 1 \Rightarrow q = \frac{1}{2}$
- $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is a MSNE.

Bach or Stravinsky

		Player 2	
		Bach	Stravinsky
Player 1	Bach	$1, 2$	$0, 0$
	Stravinsky	$0, 0$	$2, 1$

- What are the pure strategy Nash equilibria?
- Computing the mixed strategy equilibrium, letting $p = \Pr[\text{P1 plays Bach}]$ and $q = \Pr[\text{P2 plays Bach}]$:
  1. Find P1's strategy by making P2 indifferent:

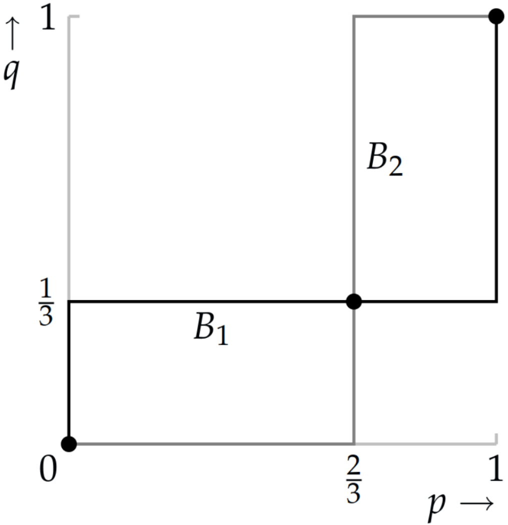
$$p2 + (1-p)0 = p0 + (1-p)1 \Rightarrow 2p = (1-p) \Rightarrow p = \frac{1}{3}$$

  2. Find P2's strategy by making P1 indifferent:

$$q1 + (1-q)0 = q0 + (1-q)2 \Rightarrow q = 2(1-q) \Rightarrow q = \frac{2}{3}$$

- So $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$ is a MSNE.

Visualizing best responses and equilibrium



Whistleblowers

Suppose some number of White House and DOJ staff observe what they see as wrongdoing by the President.

Federal whistleblower protections provide a way for these staff to notify Congress while maintaining anonymity. But blowing the whistle is still costly. And also...

Amy Fiscus @amyfiscus · Sep 26

NEW: The whistleblower is a CIA officer who was detailed to the White House at one point. His complaint suggests he's an analyst. @adamgoldmanNYT @nytmike @julianbarnes

White House Knew of Whistle-Blower's Allegations Soon After Trum…

The whistle-blower, a C.I.A. officer detailed to the White House at one point, first expressed his concerns anonymously to the agency's top lawyer.

nytimes.com

174 171 327

Amy Fiscus
@amyfiscus

Follow

And yes, it's a he.

Whistleblowers

So even among people who agree that Congress should be made aware of wrongdoing, it is better if someone else does it.

The game:
Players: n potential whistleblowers
Actions: Each player can blow the whistle or not.
Payoffs: Each player gets:

- v - c for blowing the whistle
- 0 if nobody blows the whistle
- v if somebody else blows the whistle

Assume $v > c > 0$, so each person is in principle willing to blow the whistle.

Whistleblowers: PSNE

Claim 1: There is no PSNE in which nobody blows the whistle.

Proof: Consider a strategy profile in which all players choose "don't blow the whistle." All players get a payoff of 0 in this profile. However, any player could get a payoff of $v - c > 0$ for switching to "blow the whistle". Therefore, this profile cannot be a Nash equilibrium.

Big idea: Any player would blow the whistle if she knew she was pivotal for whether or not the crime was reported at all. If nobody is blowing the whistle then every player is pivotal, so any player would be willing to blow the whistle.

Whistleblowers: PSNE (cont)

Claim 2: There is no PSNE in which everybody blows the whistle.

Proof: Consider a strategy profile in which all players choose "blow the whistle." All players get $v - c$ in this profile. However, if some player unilaterally changed to "don't blow the whistle" then the crime would still be reported, so that person would get $v > v - c$. Therefore, this profile cannot be a Nash equilibrium.

Intuition: Since everyone is blowing the whistle and only one whistleblower is needed, NO player is pivotal. Therefore, any player would switch to "don't blow the whistle" and free-ride off of the others.

Whistleblowers: PSNE (cont)

Claim 3: There is no PSNE in which more than one player blows the whistle.

Proof: This is the exact same reasoning as before. The whistleblowers get $v - c$ and the players who do not blow the whistle get v. Since more than one person is blowing the whistle, one of the whistleblowers could switch to “don't blow the whistle” and get $v > v - c$.

The intuition once again relies on the idea of pivotality: if I can avoid the cost of whistleblowing without changing the outcome (i.e. I am not pivotal) then I will do so.

Whistleblowers: PSNE (cont)

Claim 4: There are n PSNE in which exactly one person blows the whistle.

Proof: Consider any profile in which one players chooses "blow the whistle" and the rest choose "don't blow the whistle." In such a profile, the whistleblower gets a payoff of $v - c$ and the others get a payoff of v. If any non-whistleblower switches to blowing the whistle, they get $v - c < v$, so this deviation is not profitable. If the whistleblower switches to "don't blow the whistle" he (the NYT tells me it's a he) would get 0 rather than $v - c$, so this deviation also is not profitable. Therefore any such is a Nash equilibrium. Since the whistleblower could be any of the players, there are n such PSNE.

Intuition from pivotality: We can only have one person blow the whistle because that guarantees that the person reporting is a pivotal player so they are willing to report.

Whistleblowers: MSNE

The MSNE of this game also builds on the idea of pivotality.

Each player blows the whistle with some probability, with the key equilibrium condition being that the probability of being pivotal (i.e. that nobody else blows the whistle) is just large enough to make every player indifferent between reporting and not reporting.

We will look for symmetric equilibria, meaning that every player reports with the same probability p.

Whistleblowers: MSNE

The key equilibrium condition:

$$
\begin{aligned}
EU(\text{report}) =& EU(\text{ don't report}) \\
v - c =& 0 \Pr[\text{no other whistleblowers}] + v \Pr[\text{at least one other whistleblower}] \\
v - c =& v(1 - \Pr[\text{no other whistleblowers}]) \\
\frac{c}{v} =& \Pr[\text{no other whistleblowers}].
\end{aligned}
$$

To find the equilibrium, we just find $\Pr[\text{no other whistleblowers}]$ as a function of p and then solve for p.

Whistleblowers: MSNE

Consider computing this probability from the perspective of player 1:

$$\begin{aligned}\Pr[\text{no other whistleblowers}] =& \Pr[\text{player 2 abstains}] \times \cdots \times \Pr[\text{player n abstains}] \\ =&(1-p) \times (1-p) \times \cdots \times (1-p) \\ =&(1-p)^{n-1}.\end{aligned}$$

The equilibrium condition is now

$$\frac{c}{v} = (1-p)^{n-1}.$$

Whistleblowers: MSNE

Solving for the mixed strategies:

$$
\begin{aligned}
\frac{c}{v} &= (1-p)^{n-1} && \text{(eqbm condition)}\\
\left(\frac{c}{v}\right)^{1/(n-1)} &= 1-p && \text{(raise both sides to } \tfrac{1}{n-1}\text{)}\\
1-\left(\frac{c}{v}\right)^{1/(n-1)} &= p. && \text{(rearrange)}
\end{aligned}
$$

$\Rightarrow$ There is a MSNE in which all players blow the whistle with probability $1-\left(\frac{c}{v}\right)^{1/(n-1)}$.

Remark 1: Effect of increasing the number of potential whistleblowers

- We may think that as more people observe wrongdoing the likelihood of a whistleblower increases

- The problem is that as the number of potential whistleblowers increases, each person becomes less likely to bear the cost

- We can see clearly that increasing n decreases the likelihood that any individual blows the whistle:
  - If n increases, $\frac{1}{n-1}$ decreases, so therefore $\left(\frac{c}{v}\right)^{1/(n-1)}$ increases and $1-\left(\frac{c}{v}\right)^{1/(n-1)}$ decreases.
- But what about the total probability of <u>any</u> whistleblower?

Remark 1: Continued

- We can determine the probability of any whistleblower by observing that:

$$\Pr[\text{no whistleblower}] = \Pr[\text{i does not blow whistle}]\Pr[\text{no other whistleblowers}]$$

- We know that $\Pr[\text{i does not blow whistle}]$ increases as n increases.

Remark 1: Continued

- What about $\Pr$[no other whistleblowers]? Recall our equilibrium condition was

$$\frac{c}{v} = \Pr[\text{no other whistleblowers}].$$

- Therefore the probability of no other whistleblowers is constantly $\frac{c}{v}$ regardless of the number of players.
- Therefore:

$$\Pr[\text{no whistleblower}] = \Pr[\text{i does not blow whistle}](c/v)$$

  which increases with n.
- Surprising: The more people that see wrongdoing, the less likely there is a whistleblower.

Remark 2: Public goods

This problem is a special case of a “threshhold public goods” game. These games have the following properties:

- The players value some public good (value v) but find it costly to contribute to it (cost c)
- The public good is provided if the number of people who contribute is greater than or equal to some threshhold $k \leq n$. Here $k = 1$.

In general these games are similar: there are “n choose k” PSNE in which exactly k people contribute, and a mixed equilibrium that sets

$$\frac{c}{v} = \Pr[\text{exactly k-1 others contribute}].$$

Paper, Rock, Scissors

		Player 2 P	R	S
Player1	P	$0, 0$	$1, -1$	$-1, 1$
	R	$-1, 1$	$0, 0$	$1, -1$
	S	$1, -1$	$-1, 1$	$0, 0$

We know that there is no PSNE. We also know that the players must randomize over ALL actions. For instance, if I never play Paper and randomize of Rock and Scissors then you can always play Rock and guarantee a win or a tie.

This means that both players must be indifferent over all three actions.

In some other 3x3 game we might have to check for mixed strategies over a lot more combinations of actions, but we are using what we know about the game to narrow things down.

Paper, Rock, Scissors

		Player 2		
		P	R	S
Player1	P	$0,0$	$1,-1$	$-1,1$
	R	$-1,1$	$0,0$	$1,-1$
	S	$1,-1$	$-1,1$	$0,0$

To solve, let p_R be the probability player 1 plays rock and p_S bet the probability of scissors. Then the probability of paper is $1-p_R-p_S$. Likewise, q_R and q_S denote these same probabilities for player 2.

P1's strategy must make player 2 indifferent:

$$(1-p_R-p_S)0+p_R1+p_S(-1)=(1-p_R-p_S)(-1)+p_R0+p_S1 \qquad \text{(P vs R)}$$

$$(1-p_R-p_S)(-1)+p_R0+p_S1=(1-p_R-p_S)1+p_R(-1)+p_S0 \qquad \text{(R vs S)}$$

$$p_R-p_S=p_R+2p_S-1 \qquad \text{(P vs R)}$$

$$p_R+2p_R-1=p_2+2p_R-1 \qquad \text{(R vs S)}$$

Paper, Rock, Scissors

Obviously we can do the exact same calculation for Player 2 (I won't repeat it).

Therefore, the unique Nash equilibrium of paper rock scissors is one in which both players play each action with probabiltiy $\frac{1}{3}$.

This is the obvious prediction for this game: Each action has an action that clearly beats it, so if you can guess what your opponent will do you'll win. The whole idea of this game is to maximize uncertainty over what you will do, which is achieved by randomizing evenly over all the actions.

Mixed strategy dominance

		Player 2 P	Player 2 R
Player1	P	$3, -1$	$-1, 1$
	R	$0, 0$	$0, 0$
	S	$-1, 2$	$2, -1$

I want to solve this game by iteratively eliminating dominated strategies. Is any strategy dominated by any other pure strategy? No! Does this mean we are done and have to solve using best responses? Also no!

Dominance and mixed strategy

- If a strategy is dominated by a mixture between two or more other pure strategies then we can also eliminate that strategy

- Logic is the same: A rational player won't play a dominated pure strategy when they could instead play another strategy (pure or mixed that is always better)

Back to the game

		Player 2 L	Player 2 R
Player1	U	$3, -1$	$-1, 1$
	M	$0, 0$	$0, 0$
	D	$-1, 2$	$2, -1$

I claim that M is dominated for Player 1 by a strategy $(1/2, 0, 1/2)$ which places probability $\frac{1}{2}$ on up and $\frac{1}{2}$ on down:

- If Player 2 plays L, then M yields 0 while $(1/2, 0, 1/2)$ yields $\frac{1}{2}3 + \frac{1}{2}(-1) = 1 > 0$.
- If Player 2 plays R, then M yields 0 while $(1/2, 0, 1/2)$ yields $\frac{1}{2}(-1) + \frac{1}{2}2 = \frac{1}{2} > 0$.

This shows that M is dominated, and we can eliminate it from the game!

Reduced game

		Player 2	
		L	R
Player1	U	$3, -1$	$-1, 1$
	D	$-1, 2$	$2, -1$

This game has no PSNE but we can solve for the MSNE more easily now.

- Solve for P1 strategy (p=Probability of U) by making P2 indifferent:

$$EU_L(p) = EU_R(p) \Rightarrow p(-1) + (1-p)2 = p(1) + (1-p)(-1) \Rightarrow p = \frac{3}{5}$$

- Solve for P2 (q = Probability of L) strategy by making P1 indifferent:

$$EU_U(q) = EU_D(q) \Rightarrow q3 + (1-q)(-1) = q(-1) + (1-q)(2) \Rightarrow q = \frac{3}{7}.$$

Therefore the MSNE of the original game is one in which P1 never plays M but player U with probability $\frac{3}{5}$ and D with probability $\frac{2}{5}$, and P2 plays L with probability $\frac{3}{7}$ and R with probability $\frac{4}{7}$.

Electoral Competition

Motivation

- What determines the platforms offered by political parties competing for votes?
- How do we explain the number of viable political parties?
- How do changes in voters' preferences affect electoral outcomes and chosen policies?

Today, we'll learn a game theoretic model that has helped provide some answers to all of these questions. We will only be able to scratch the surface of these models.

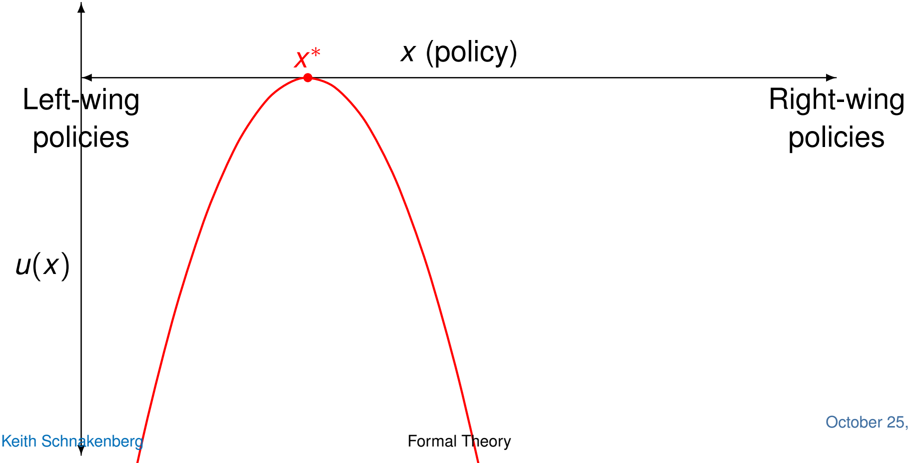
A spatial model of policy choice: Policies

The set of policies is equal to the real number line.

A spatial model of policy choice: Preferences

Each person is characterized by her ideal point (i.e. her favorite policy) x^* which is a point on the real line.

We assume a person's utility is decreasing in the distance between the policy and her ideal point.



The basic game

- Players: Two candidates (could be parties), a set of voters (for simplicity, odd number of voters). The voters each have an ideal point. A special significance is given to the median of the ideal points, which we will denote m.
- Actions: Each candidate's set of actions are the real number line. Each voter can decide which party to vote for.
  - The voters' choices are obvious: they choose the candidate closest to them, so we will analyze this as if it is a two player game, taking voters' choices to be mechanistic. We assume they randomize between candidates when they are indifferent between them.
- Candidates' payoffs: Candidates (for now) just want to wind the election: they get 1 if they win and 0 if they lose, regardless of chosen policies. Elections are determined by majority rule.

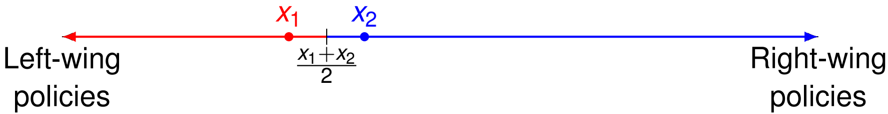
Why the median is important

Suppose party 1 chooses a position x_1 and party 2 chooses a distinct position x_2.

All voters to the left of x_1 vote for 1, and all parties to the right of x_2 vote for 2

Between x_1 and x_2, everyone closer to x_1 votes 1 and everyone closer to x_2 votes 2 (change point is the midpoint of x_1 and x_2)

The median is by definition the cutoff for a majority: If the set of A supporters includes the majority then it must be a majority. Otherwise it must not be.



$\Rightarrow$ When the candidates take distinct positions, the supporters of each candidate are divided according to left/right.

Analyzing Candidates' Best Responses

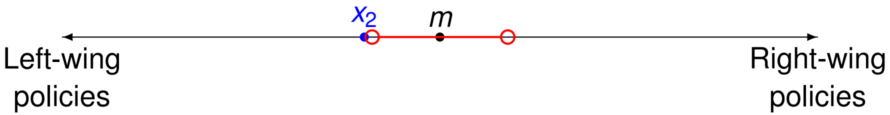
Fix a position x_2 of candidate 2. What are candidate 1's best responses to that position?

Any x_1 that causes candidate 1 to win is a best response. If 1 cannot guarantee a win, then a position causing a tie is a best response.

We can separately look at three cases:

- $x_2 < m$
- $x_2 = m$
- $x_2 > m$

Case 1: $x_2 < m$



Locating just barely to the right of x_2 would certainly cause 1 to win

Locating to the right of the median but closer to m than x_2 will also cause 1 to win

$\Rightarrow$ the set of best responses is $BR_1(x_2) = \{x_1 : x_2 < x_1 < 2m - x_2\}$

Note: $\frac{x_2+x_1}{2} = m \Rightarrow 2m = x_1 + x_2 \Rightarrow x_1 = 2m - x_2$

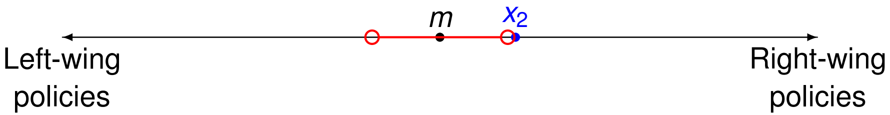
Case 2: $x_2 = m$

- Clearly no policy strictly beats $x_2 = m$ since m is the most preferred policy of the median.

- In fact, every $x_1 \neq m$ would cause 1 to lose for sure

- But $x_1 = m$ leads to winning with probability $\frac{1}{2}$, so this is a best response.

Case 3: $x_2 > m$



Same story as case 1, just reversed: $BR_1(x_2) = \{x_1 : 2m - x_2 < x_1 < x_2\}$

Visualizing best response functions

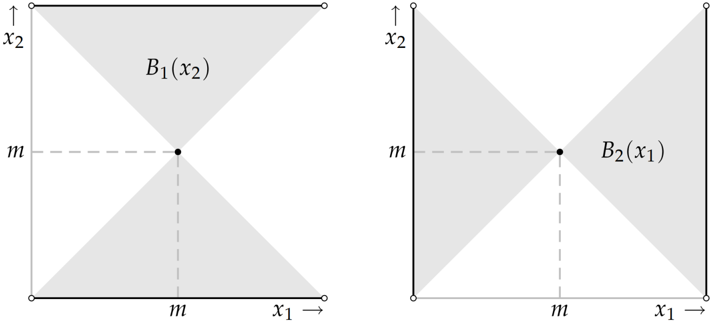


Figure 71.1 The candidates' best response functions in Hotelling's model of electoral competition with two candidates. Candidate 1's best response function is in the left panel; candidate 2's is in the right panel. (The edges of the shaded areas are excluded.)

The Hotelling-Downs Median Voter Result

The unique Nash equilibrium of the electoral competition game is one in which both candidates choose the same platform: the ideal point of the median voter.

Remark 1: Normative Theory Meets Positive Theory

- Even without the Hotelling-Downs result, the position of the median voter has a special normative status: It is the only policy we could choose for which no majority would prefer a different policy.
  - This is directly related to the concept of the core in economics
- Thus, we may think the median is the most desirable policy for this society to choose.
- Nash equilibrium is a positive criterion: it makes a prediction about what is going to happen. As we saw with the prisoner's dilemma, it is by no means guaranteed to select normatively desirable outcomes
- The Hotelling-Downs result shows us that these two things come together in this particular model of electoral competition: our best prediction (given the model) and our most desired outcome happily turn out to be the same.

Remark 2: Theory and Reality

- Okay but in reality political parties never converge to the same policy, so isn't this a useless theoretical result?

- If the point of theory is to make accurate predictions all the time, yes. But then we are doomed anyway.

- The result does a lot of useful things for us:
  - Provides a useful benchmark for developing more theory. If we want to explain why parties choose divergent positions, we can relax assumptions of the baseline model one at a time and see what gets us closer to the data. Decades of theory now do exactly that.
  - Illustrates one strategic force at play in elections. The purpose of a model is rarely to model every aspect of a problem. Models are representations used to analyze particular aspects of the problem by <u>simplifying</u>. A good model must always be a little bit wrong! (So must any model, really.)

Policy-motivated parties (the Calvert-Wittman model)

- Assume voters have single peaked policy preferences as before
- The players are two parties $P = \{L, R\}$ which have ideal points at 0 and 1, respectively.

$$u_L(x) = -|x|$$
$$u_R(x) = -|x-1|$$

- We will also assume that $0 < m < 1$.

Policy-motivated parties (continued)

Letting $\pi(x_L, x_R)$ be the probability that L wins given the platforms, the parties maximize expected utilities:

$$U_L(x_L, x_R) = \pi(x_L, x_R)(-|x_L|) + (1 - \pi(x_L, x_R))(-|x_R|)$$
$$U_R(x_L, x_R) = \pi(x_L, x_R)(-|x_L - 1|) + (1 - \pi(x_L, x_R))(-|x_R - 1|).$$

Convergence result

Proposition

This game has a unique Nash equilibrium in which both parties choose m, the position of the median voter.

Calvert-Wittman convergence proof

- Existence: Consider the strategy profile (m, m). We will show that neither party strictly benefits from deviating.
  - Consider a deviation by any party away from m toward its preferred policy
  - This causes that party to lose, yielding a policy of m. So the party is indifferent between m and deviating to any other policy.

Calvert-Wittman convergence proof (cont)

- Uniqueness: We have shown that m is a best response to m so we will consider situations where $0 \leq x_L < m < x_R \leq 1$, and show that none are equilibria
  - Case 1: $|x_L - m| > |x_R - m|$. L loses and the policy is x_R. Then L is not playing a best response since any policy in $(2m - x_R, x_R)$ cases L to win the election and implements a preferable policy.
  - Case 2: $|x_L - m| < |x_R - m|$. Same logic for L.
  - Case 3: $|x_L - m| = |x_R - m|$. Each party wins with probability $\frac{1}{2}$. However, neither player is playing a best response: each could move in the direction of m by some small amount $\epsilon > 0$ and win the election for sure.

Multiparty competition

So far we have considered models with only two parties.

There are a few ways to think about relaxing this:

- We could analyze a model with multiple parties
- We could analyze a model with two parties but assume that the parties believe another party <u>could</u> enter

A three party model

Consider a model with the same assumptions as the Hotelling-Downs model but with three parties $P = \{A, B, C\}$.

Do you think there is an equilibrium in which all three parties choose the median?

Non-convergence in three-party model

Proposition

There does not exist a convergent equilibrium in the three party model.

Proof of non-convergence with three parties

- Consider the strategy profile (m, m, m). In it, all three parties win with probabilityi $\frac{1}{3}$.
- Consider a deviation by party C: for some small number $\epsilon > 0$, C moves to $m - \epsilon$
- Party C now wins the votes of everyone to the left of m (just under 1/2 of the voters) and Parties A and B split the remaining voters. Thus, Party C wins for sure.

What is the equilibrium then?

We can find a Nash equilibrium in certain cases still. Consider the following example:

Example

Assume voter's ideal points are distributed uniformly on $[0, 1]$. There is an equilibrium in which $x_A = x_B = \frac{1}{3}$ and $x_C = \frac{2}{3}$.

Proof for three-player example

- Clearly C does not deviate from this profile since C wins the election for sure in this case. We will then consider each possible deviation for A (the same arguments would apply to B).
- $x_A < \frac{1}{3}$: C still wins with one half of the vote.
- $x_A \in \left(\frac{1}{3}, \frac{2}{3}\right)$. This gives A at most one sixth of the vote: everyone to the left of $\frac{1}{3}$ (one third of the voters) votes for B, everyone to the right of $\frac{2}{3}$ votes for B (another one third of the voters), and votes in the middle are split.
- $x_A = \frac{2}{3}$. This is essentially the same as the original profile, except B is the new winner.
- $x_A > \frac{2}{3}$: Essentially the same as $x_A < \frac{1}{3}$ but with B as the new winner.

This shows that nobody has a profitable deviation, so this is a Nash equilibrium.

Remarks on three party model

Though an interesting exercise, this equilibrium is a pretty silly prediction in several ways:

- The assumption of mechanistic voters seems not so innocuous now: if we treat the voters are actual strategic players they should coordinate on one of the two left-most candidates rather than splitting their votes on identical parties. We'll return to strategic voters soon.
- We also have parties entering the election know that they will lose. So we don't answer the question of why there are three parties. In fact, we usually expect not to have three major parties in an election like this one. We will also return to the question of entry.
- Real elections with multiple parties tend to be proportional representation, and in those we expect parties will maximize <u>vote share</u>. Those strategies don't work in this case (in fact we can show that there is not an equilibrium in the case above when parties maximize vote share).

Entry and strategic voting

Consider the following modifications of Hotelling-Downs:

- There are N potential candidates who may enter the elections (where N is very large)
- Candidates strategy sets: $x_P \in \mathbb{R} \cup \{\text{stay out}\}$ (i.e. they can enter and choose a position or they can not run)
- Cost of entry: δ. Benefit of winning: $v > \delta$
- The set of voters is finite and preferences are concave (i.e. voters are risk averse)
    - This would be satisfied for instance by $u_i(x) = -(x - x_i^*)^2$
- The voters are players in the game and behave strategically

Solution concept

- We restrict our attention to pure strategy equilibria
- We also only consider equilibria in which voters use weakly undominated strategies
- The latter is common in voting games. The purpose is to rule out trivial situations driven by voters choosing less-preferred candidates simply because they are not pivotal and therefore indifferent between all choices.

Preliminary results on entry

- Fact 1: Every candidate who enters must get the same number of votes.
  - In pure strategies, the mapping from candidate strategies to votes is deterministic
  - $\Rightarrow$ A candidate who loses should have expected to lose and stayed out since entry is costly
- Fact 2: If M is the number of candidates, we must have $M \leq \frac{v}{\delta}$
  - Probability of winning for each entrant must be $\frac{1}{M}$
  - Therefore expected utility is $\frac{v}{M} - \delta$, which must be greater than zero. Solving for M gives us the result.

Equilibria with entry and strategic voting

Proposition

In all equilibria, $1 \leq M \leq \frac{v}{\delta}$, candidates enter and all of them choose the platform m. Furthermore, all voters vote sincerely in equilibrium.

The proof involves some material from dynamic games that we haven't covered yet so I'll give the intuition without the technical details.

Since every candidate has the same number of votes, every voter is pivotal, hence sincere voting

Why don't we have equilibria like we described earlier in the multicandidate case? Strategic voters should coordinate on one of the preferred candidates rather than splitting votes.

Consider a situation where three candidates choose m and it's not profitable for a fourth candidate to enter. What happens if one candidate deviates to (say) the right? Everyone to the left coordinates on one of the median candidates, that candidate loses.

Citizen candidate model

Another class of models on electoral competition are citizen candidate models.
The model:

- There are N citizens (the set of voters and the set of potential candidates is the same)
- Policy preferences: $u_i(x) = -|x - x_i|$
- Citizens pay a cost $\delta > 0$ of running for election and get a benefit $v \geq 0$ of winning (in addition to the policy benefit)
- Key departure from every model so far: Candidates cannot commit to policies. Therefore, any candidate who wins implements her ideal point.
- Citizens vote for one candidate or abstain. Winner determined by plurality rule with ties broken by lottery. Assume weakly dominant voting strategies
- If nobody enters, a status quo policy $\overline{x}$ is implemented

Median voter equilibrium

First question: When is there an equilibrium in which a citizen with ideal point m enters and nobody else enters?

First condition: The citizen at the median prefers to enter given that nobody else does:

$$-|m - m| + v - \delta \geq -|\overline{x} - m| \Rightarrow \delta \leq |\overline{x} - m| + v.$$

Second condition: Nobody else will enter. $\delta \geq \frac{v}{2}$. This guarantees that a second candidate at the median does not enter (a candidate located anywhere else should never enter anyway).

Two candidate equilibria

Let us restrict our attention to the case where $v = 0$ (there is policy motivation for entering but no other rewards to office)

Two necessary for a two-candidate equilibrium:

- Candidates must receive the same number of votes (since $\delta > 0$)
- ...but they must be at different positions (since $v = 0$)

These two conditions jointly imply that the candidates must be at two points equidistant from the median. Some candidate L locates at $x_L = m - \Delta$ and R locates at $x_R = m + \Delta$. Then we must also have $\delta \leq \Delta$.

Two candidate equilibria (continued)

The remaining condition is that nobody else wants to enter. Here the results depend whether we assume sincere or strategic voting.

- Sincere. With sincere voting, citizens with $x_i \leq x_L$ or $x_i \geq x_R$ do not enter because they would cause their least-preferred candidate to win. Centrist citizens may enter though, so x_L and x_R must be sufficiently close together that entry is not possible for these citizens. We have a "goldilocks" type of result: Δ must be large enough that both candidates want to enter, but small enough that nobody can profitably enter in between them.
- Strategic. The coordination aspect of strategic voting means that there are many more possibilities when a third candidate enters. As a result, we might have two extremist candidates entering with no entry by a moderate citizen simply because of coordination failures.
- Our analysis of entry decisions under strategic voting presumes that candidates correctly anticipate which candidates voters will coordinate on. This is a preview of solutions for sequential games, which we will get to soon.

Dynamic Games of Complete Information

Example: Sequential BoS

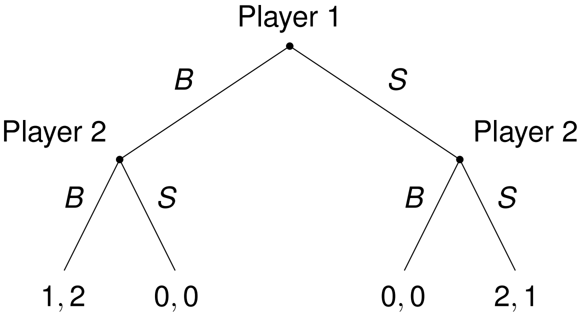
Recall the Bach or Stravinsky game:

		Player 2	
		Bach	Stravinsky
Player 1	Bach	$1, 2$	$0, 0$
	Stravinsky	$0, 0$	$2, 1$

$$NE = \{(B, B), (S, S), ((1/3, 2/3), (2/3, 1/3)\}$$

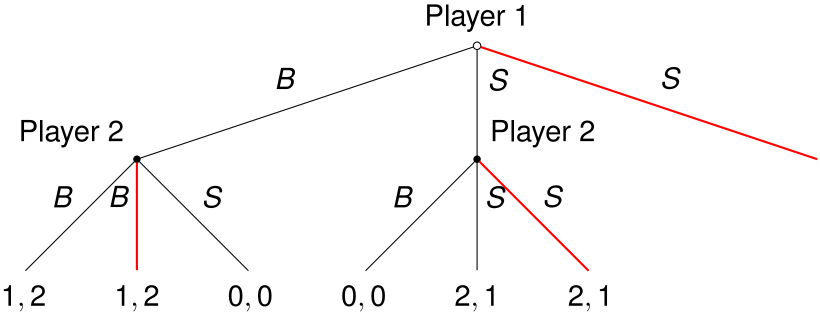
Example: Sequential BoS

Consider the following variant of the game: Player 1 moves first. Player 2 observes Player 1's choice and then moves second:



Question: Which Nash equilibria to BoS still seem reasonable in light of this additional information about sequence?

Example Sequential BoS: Continued



⇒ (S, S) seems like the only reasonable outcome given this sequence.

The plan for today

1. Formalize the idea of an extensive form game (i.e. a game with temporal information)

2. Generalize our intuition about the sequential BoS game to learn a better way of solving extensive form games: a refinement of Nash equilibrium which we'll call subgame perfect Nash equilibrium (SPNE)

3. Practice backward induction, the easiest way of solving for SPNE in finite-horizon extensive games with complete and perfect information. (We already used it to solve sequential BoS).

Extensive form games

In general, an extensive form game consists of:

1. A set of players N
2. A set of terminal histories (i.e. possibly end points of the game)
3. The order of play: a specification of when each player can move
4. A set of actions that each player can take each time that player moves
5. Preferences over terminal histories

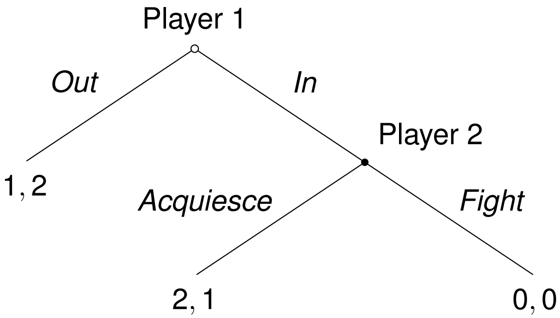
Intuition: An extensive form game has all of the information in a normal form game but also temporal information.

Game trees

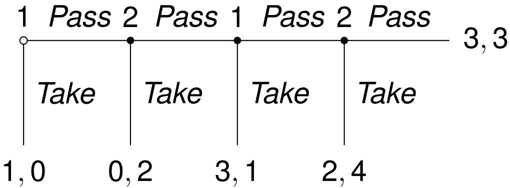
- A game tree is often a useful way of representing an extensive form game

- A game tree consists of:
  - Nodes with player labels: These communicate who the players are and when they move
  - Branches with action labels: These tell us what decisions each player can make when they move
  - Terminal nodes with payoffs: This tells us the terminal histories (outcomes of the game) and all players' preferences over them.

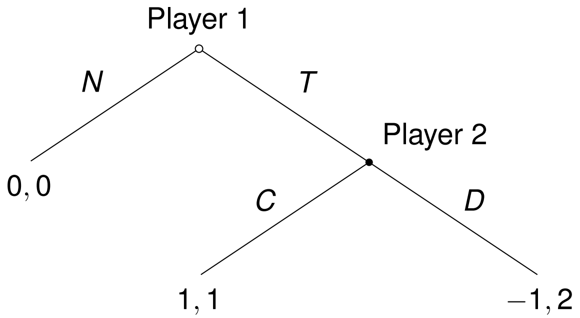
Example: Entry Game



Example: Centipede Game



Example: Trust game

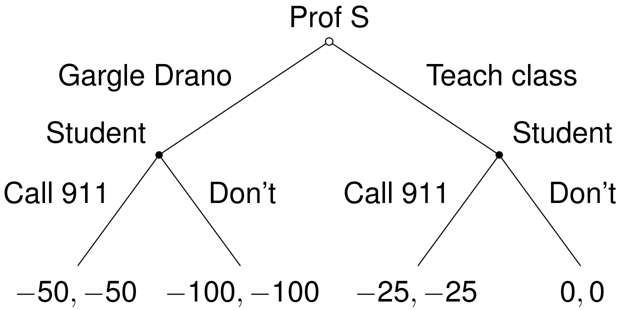


Pure strategies in extensive form games

- A pure strategy in an extensive form game is a complete plan of play describing what the player would do at any node where she can move
  - Example: In sequential BoS, a strategy for Player 1 might be “choose S” but a strategy for Player 2 would be “choose B if P1 chooses B, choose S if P1 chooses S”

- Important: A player’s strategy must say what she would do at every node where should could make a decision, even if she should normally arrive at that node

Example: Strategies as complete plans of play



It is obvious from the game that I should not gargle Drano, but your pure strategy must still specify what you would do in both scenarios.

Mixed strategies in extensive form games

Technically we can think of mixed strategies in extensive form games in two ways:

1. The literal definition: A probablity distribution over complete plans of play.
2. A "behavior strategy": A probabilistic plan of action at each decision node

Kuhn (1953) showed that these two are equivalent for our purposes, and it turns out behavior strategy's are must easier to deal with, so we will think of players as randomizing independently at each decision node.

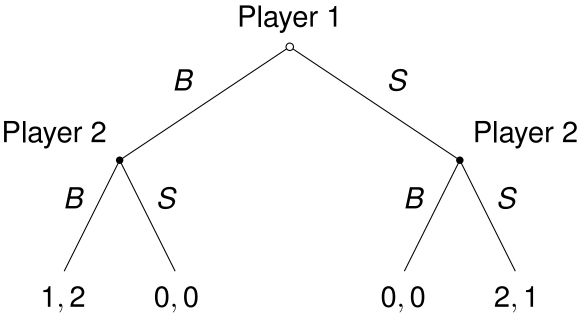
We will come back to this later but we'll think about pure strategies today.

Solving extensive games

- Our solution concept for extensive form games is called subgame perfect Nash equilibria. I will introduce this concept in more detail next time. Today I want to give intuition and teach you the easiest solution method.

- Intuition: SPNE is meant to rule out equilibria that are based on non-credible threats

Back to sequential BoS



We determined that $(S, (B, S))$ was a good prediction to this game.

Why not $(B, (B, B))$?

We can think of this as a situation in which Player 2 says ”You may as well choose B because I am going to choose B no matter what you do.”

This is in fact a Nash equilibrium! Player 1 cannot do better by deviating to S, and, since the actual outcome is that both players choose B, Player 2 cannot do better by deviating to another pure strategy.

Back to sequential BoS

The problem is that this is a non-credible threat. Player 1 should say “I do not believe you. If I choose S you will clearly have an incentive to choose S and not B, so I will choose S.”

SPNE rules out non-credible threats like this.

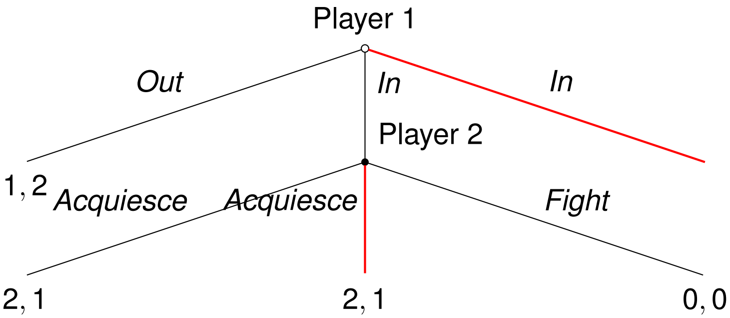
It does so by adding a sequential rationality requirement: Player 2 must be best responding at every node, even if that node is never actually reached in equilibrium.

Backward induction

We can solve for SPNE many extensive form games using backward induction:

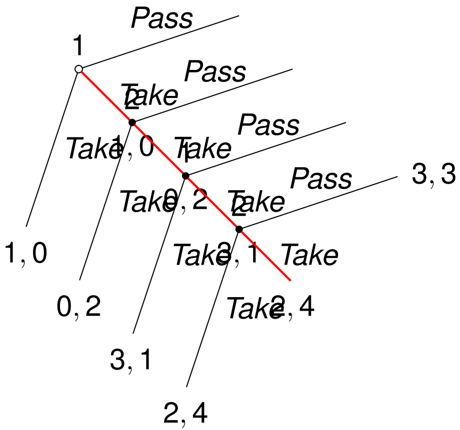
1. Starting with all of the last node, determine the player's best response at that node
2. Using this information, determine what the second-to-last player on each path should do assuming that the last player best responds
3. Repeat this process until you reach the first move(s) in the game

Example: Entry Game



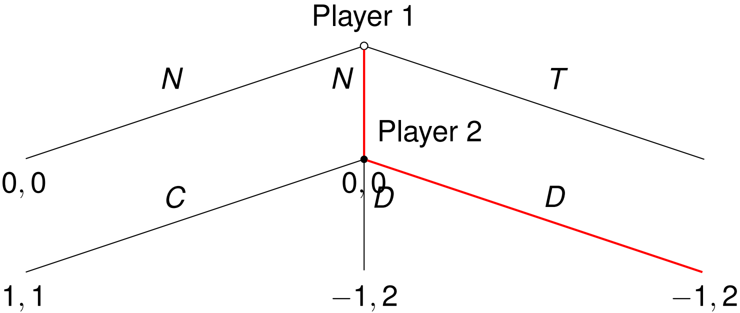
SPNE: (In, Acquiesce)

Example: Centipede Game



SPNE: (Take, Take), (Take, Take)

Example: Trust game



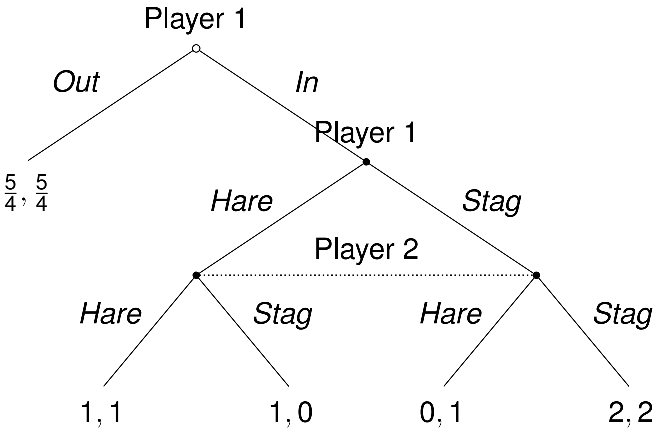
SPNE: (N, D)

Allowing for simultaneous moves: Voluntary Stag Hunt

Consider a different sequential Stag Hunt game: Player 1 chooses to play or not play. If Player 1 does not play then both players get a payoff of $\frac{5}{4}$. If she chooses to play then they play the simultaneous Stag Hunt game.

We can represent this as an extensive form game by recalling that simultaneous moves are equivalent to sequential moves in which the players do not see each others' actions.

This adds one new component to the extensive games we have studied so far: information sets showing what each player knows when she moves.



Plain old backward induction fails us here: there is no last move. We need a more general way to move forward.

Extensive form games with simultaneous moves

An extensive form game (generalized to allow simultaneous moves) is:

1. A set of players, N.
2. Players payoffs (u_i for all i in N) as a function of outcomes (terminal nodes)
3. Order of moves
4. Actions of players when they can move
5. What each player knows when they move
    - For this we use information sets, where two decision nodes x and y are in the same information set if the player does not know whether she is at x or y when she moves

A strategy in an extensive form games, like last week, is a complete plan of play: what each player would do every time she moves.

Some intuition about solving general extensive form games

A backward-induction-like method for solving the voluntary Stag Hunt game:

1. Solve the simultaneous-move Stag Hunt game as a normal form game
2. Assuming that Player 1 knows what equilibrium will be played after she chooses In, determine whether she should play in or out

The idea: Even though there is no last move, the game breaks up into two distinct games that let us solve backwards in a way that is similar to backward induction.

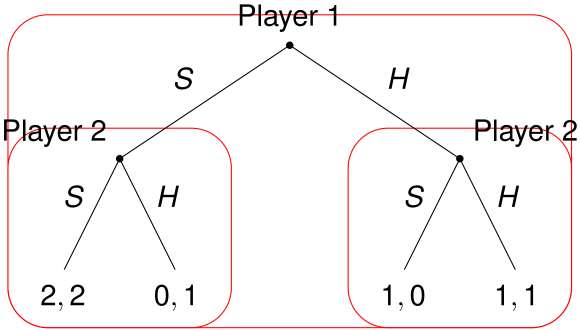
To apply this logic beyond this example, we need to formalize the sense in which the game breaks up into distinct games.

Subgames

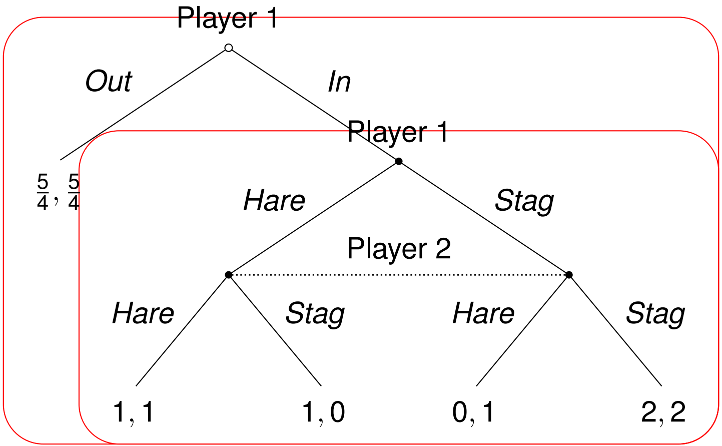
Definition

A <u>subgame</u> of an extensive form games consists of only a single node and all of its successors, with the property that any two nodes in the same information set are in the same subgame.

Subgames: Example 1



Subgames: Example 2



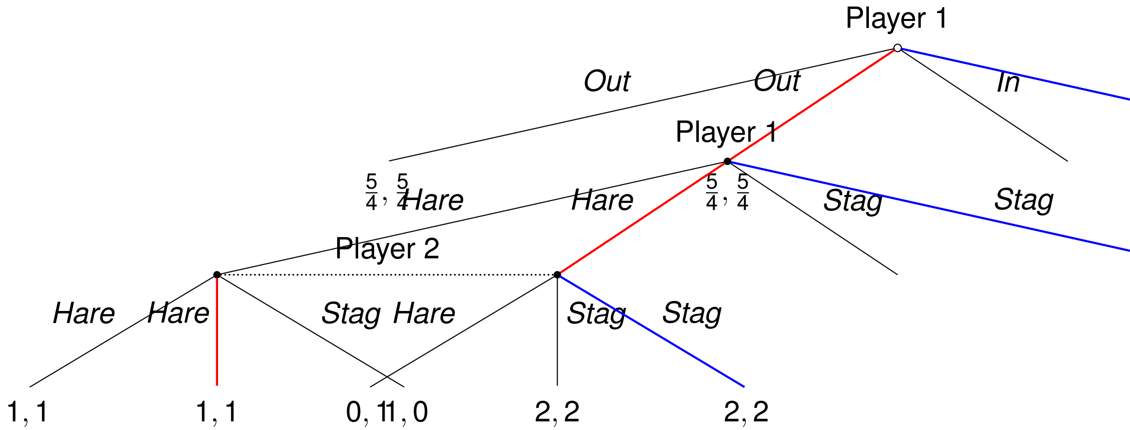
Subgame perfection

Definition

A strategy profile is a subgame-perfect Nash equilibrium to the extensive form game Γ if the strategies are a Nash equilibrium in each proper subgame of Γ.

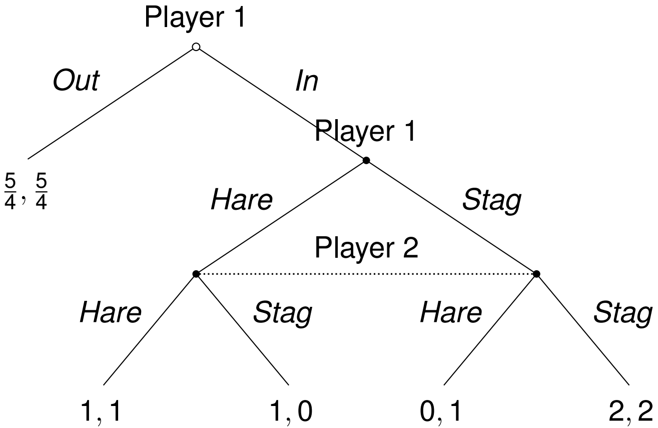
This is a direct generalization of our sequential rationality requirement from last week (a NE to a one-player subgame is just their optimal choice)

The voluntary Stag Hunt (pure strategies)



The PSNE to the Stag Hunt are (H, H) and (S, S).
P1 should choose Out if they will play the Hare equilibrium and In if they will play the Stag equilibrium

Voluntary Stag Hunt (mixed strategies)



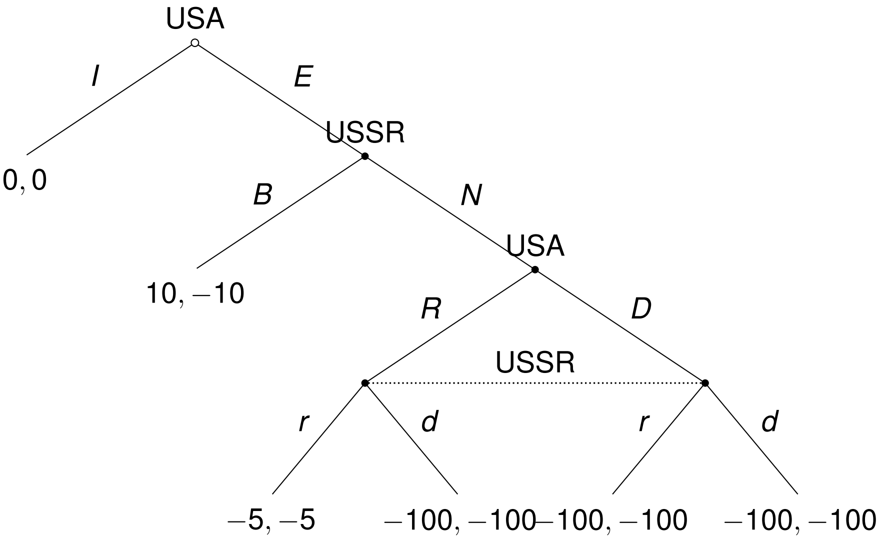
- The MSNE to the Stag Hunt is $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.
- This gives P1 an expected payoff of 1. Her payoff from choosing Out is greater than 1.
- $\Rightarrow$ there is also a mixed SPNE: $((Out, (\frac{1}{2}, \frac{1}{2})), (\frac{1}{2}, \frac{1}{2}))$.

Example: Mutually Assured Destruction

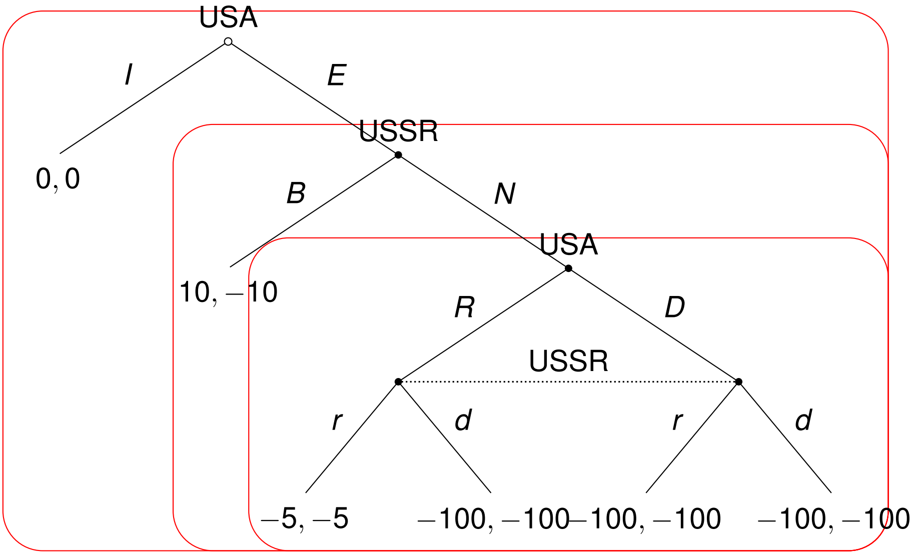
Light background: Cuban missile crisis

- The crisis started when the US discovered Soviet nuclear missiles in Cuba
- US escalated the crisis by quarantining Cuba
- The USSR back down, agreeing to remove its missiles from Cuba
- Suggests US had a credible threat: if you don't back down we both pay.
- Could this indeed be credible? Let's look at a stylized game.

Example: Mutually Assured Destruction



MAD Subgames



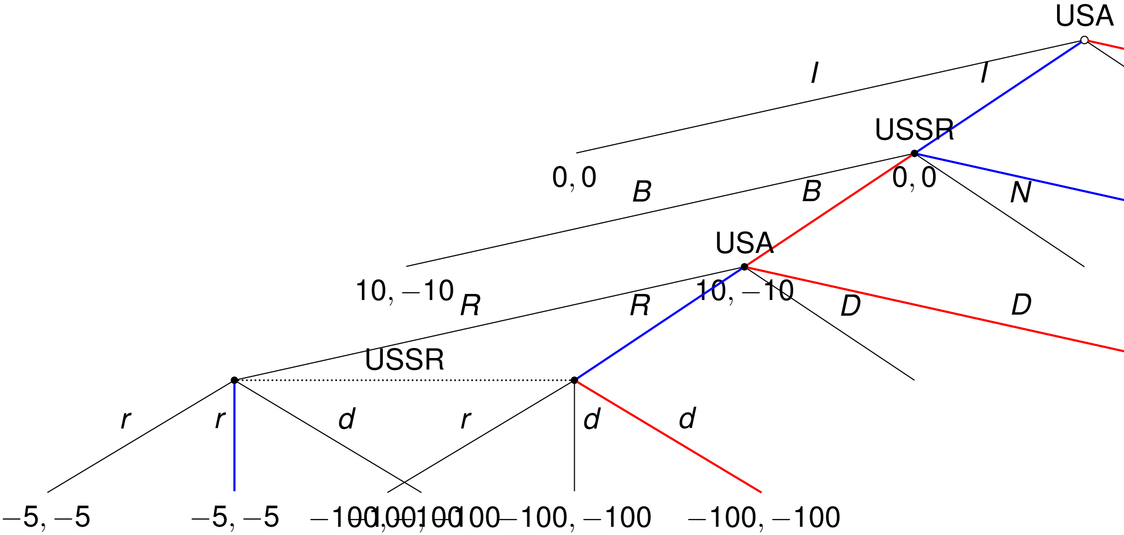
Solving the MAD game: Step 1

What first?

Last subgame:

		USSR: r	USSR: d
USA	R	$-5, -5$	$-100, -100$
	D	$-100, -100$	$-100, -100$

- PSNE?
  - (R, r), (D, d)
- MSNE?
  - No (rare occasion of an even number of NE).



⇒ we find two SPNE: ((I,R), (N,r)) and ((E,D), (B,d))

Prisoner's Dilemma, then Stag Hunt

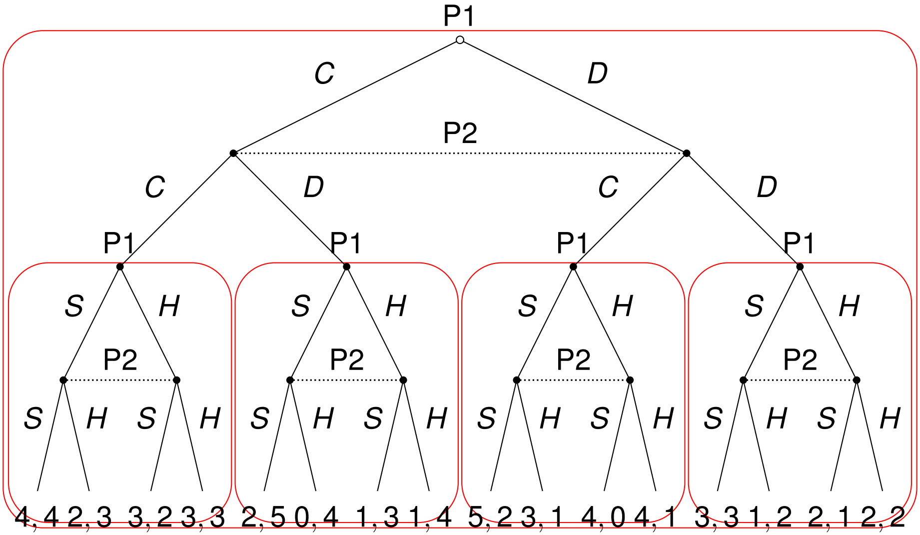
- Game is simple: Two players play a simultaneous-move PD, see their final payoffs, then they play a simultaneous-move Stag Hunt. Players' payoffs are the sum of payoffs for the two simultaneous-move games.
- There are many equilibria to this game so we'll focus on a narrower question: Is there some equilibrium in which both players cooperate in the PD stage?

The normal form games

		Player 2	
		C	D
Player 1	C	2, 2	0, 3
	D	3, 0	1, 1

		Player 2	
		Stag	Hare
Player 1	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

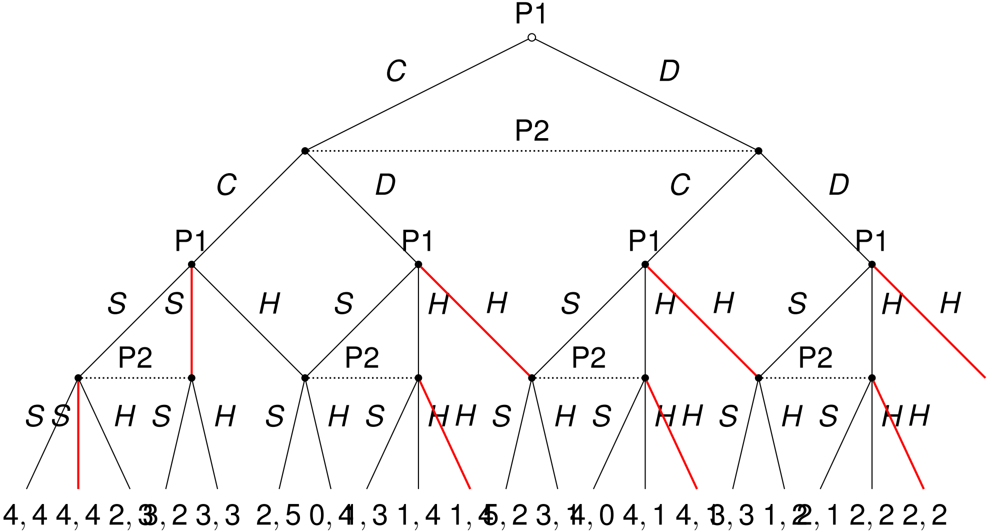
PD + SH Game Tree



Intuition: Cooperation and punishment

- The last stage is always a Stag Hunt so the players must play one of the equilibria to the stag hunt in any of the last subgames
- ...yet these constitute four different final subgames. They may play a <u>different</u> equilibrium to the Stag Hunt game in some of these final subgames than in others.
- This opens the door to a punishment strategy: If we both cooperated in the first period, we will play the (S, S) equilibrium in the second period. If anyone defected, we'll play the (H, H) equilibrium. Since both players would rather play the (S, S) equilibrium in the second period, perhaps this is enough to induce first-period cooperation.
- Let's see!

PD + SH Punishment Strategy



Induced first-period game

To see if this equilibrium indeed induces cooperation in the first period, we rewrite the PD game using the final payoffs induced by the punishment strategy:

		Player 2	
		C	D
Player 1	C	$4, 4$	$1, 4$
	D	$4, 1$	$2, 2$

Is (C, C) an equilibrium to this game?

Remarks

- This problem is a little preview of repeated games: sometimes we can sustain cooperation when we otherwise could not due to the promise of future cooperation.

- Would this same approach work for a twice-repeated PD? Why or why not?

Supplemental slides

Lotteries for infinite sets

Definition

A simple lottery over an interval $X = [\underline{x}, \overline{x}]$ is given by a cumulative distribution function $F : X \rightarrow [0, 1]$ where $F(\hat{x}) = \Pr[x \leq \hat{x}]$ is the probability that the outcome is less than or equal to $\hat{x}$.

Back to finite lotteries

Continuous expected utility

Statistics review:

- A density function f for a continuous random variable is a function satisfying

$$\Pr[a \leq x \leq b] = \int_a^b f(x)dx. \tag{2}$$

  That is, the probability that a random variable x falls into the interval $[a, b]$ is found by finding the area under the density curve between points a and b.
- $\Rightarrow$ CDF can be defined as

$$F(\hat{x}) = \int_{\underline{x}}^{\hat{x}} f(x)dx \tag{3}$$

  if $\underline{x}$ is the smallest possible value of x.
- $\Rightarrow$ if F is differentiable then

$$f(x) = \frac{dF(x)}{dx}. \tag{4}$$

Continuous expected utility

Definition

Let $u(x)$ be the player's payoff function over outcomes in the interval $X = [\underline{x}, \overline{x}]$ with a lottery given by the cumulative distribution $F(x)$, with density $f(x)$. Then we define the player's expected payoff as

$$\mathbb{E}[u(x)] = \int_{\underline{x}}^{\overline{x}} f(x)u(x)dx.$$

Back to finite expected utility

Example: Two policies with continuous uncertainty

- Decisionmaker must again choose between two policies A and B.
- The set of possible outcomes from each policy choice is $X = [0, 1]$.
- Preferences: $u(x) = x$.
- The lotteries induced by each policy are CDFs: $F_A(x) = x^2$ and $F_B(x) = x$ for $0 \leq x \leq 1$.

Expected utility for policy A

- First we can derive the PDF:

$$f_A(x) = \frac{dF_A(x)}{dx} = 2x.$$

- The expected utility of choosing policy A is therefore:

$$\mathbb{E}[u(x)|A] = \int_0^1 f_A(x)xdx = \int_0^1 2xxdx = \frac{2}{3}.$$

Expected utility for policy B

$$f_B(x) = \frac{dF_B(x)}{dx} = 1.$$

$$\mathbb{E}[u(x)|B] = \int_0^1 f_B(x)xdx = \int_0^1 1xdx = \frac{1}{2}.$$

$\Rightarrow$ A maximizes expected utility.

Example: Choosing effort with continuous uncertainty

- We will think of a politician choosing effort to produce good outcomes in order to gain reelection
- The politician chooses a level of effort $a \in A = [0, 1]$.
- The politician's vote share in the next election is drawn from a uniform distribution on the interval $[0, a]$. (Thus, the pdf of outcomes is $f(x) = 1/a$ for $x \in [0, a]$)
- Payoff is:

$$u(x, a) = \sqrt{x} - a.$$

  where x is vote share

Solving the politician's problem

Expected utility for an effort level a:

$$\mathbb{E}[u(x,a)|a] = \int_0^a \frac{\sqrt{x}}{a} dx - a \tag{5}$$

$$= \frac{2}{3}\sqrt{a} - a. \tag{6}$$

Maximizing with respect to a. FOC:

$$\frac{1}{3\sqrt{a}} = 1$$

$\Rightarrow a = \frac{1}{9}$.

SOC: $-\frac{1}{6a^{\frac{3}{2}}} < 0$ ✓