Rational Decisionmaking

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  - Agency head choosing between a number of projects
  - Mayor allocating disaster prevention resources (remember?)

Formal decision problem

A decision problem consists of:

- A set of possible *actions* A
- A set of possible *outcomes* X
- A preference relation $\succeq$ over outcomes, where:
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  - The decisionmaker strictly prefers x to x' if $x \succeq x'$ but NOT $x' \succeq x$. Then we can just write $x \succ x'$.

Example: Bar Patron

- Actions: $A = \{\text{whiskey}, \text{wine}, \text{beer}\}$
- Outcomes (in this example practically the same as actions): $X = \{wh, wi, b\}$
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We would expect to see this bar patron choose to drink wine.

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- Completeness: Any two outcomes x and x' in X can be ranked by the preferences relation, so that either $x \succeq x'$ or $x' \succeq x$
  - Does NOT mean there must be a strict preference between the two
- Transitivity: There are no cycles in preferences. For any three outcomes x, x', and x'' in X, if $x \succeq x'$ and $x' \succeq x''$ then $x \succeq x''$.

Violation of transitivity (bar example)

Bar patron strictly prefers outcome of ordering wine to beer, strictly prefers the outcome of beer to whisky, but strictly prefers the outcome of whisky to wine

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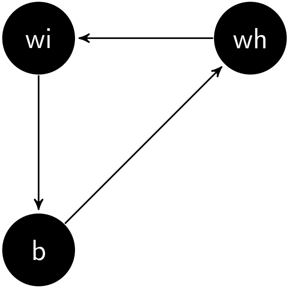
Bar patron strictly prefers outcome of ordering wine to beer, strictly prefers the outcome of beer to whisky, but strictly prefers the outcome of whisky to wine In a picture (arrows represent strict preference):



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- That is all it means! It does not mean:
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- However you have heard this term used, the *only* thing we mean is that people can rank outcomes.

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For example, the preference $wi \succ wh \sim b$ is represented by a utility function such as:

$$\begin{aligned} u(wi) =& 3 \\ u(wh) =& 2 \\ u(b) =& 2. \end{aligned}$$

Formally representing preferences with utility

A utility function *represents* the preference relation $\succeq$ if for any pair of outcomes x and x' in X, $u(x) \geq u(x')$ if and only if $x \succeq x'$.

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Fact: Any rational preference relation $\succeq$ can be represented by a payoff function.

Note: I did not say that such preferences can be represented by only one utility function. In fact, that is not true.

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What about

$$\begin{aligned} u(wi) &= 100 \\ u(wh) &= 0 \\ u(b) &= 0? \end{aligned}$$

What about

$$\begin{aligned} u(wi) &= 1 \\ u(wh) &= 1 \\ u(b) &= 1? \end{aligned}$$

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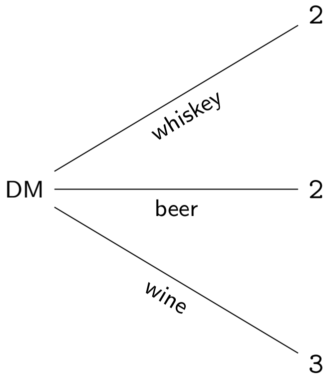
Whether a utility function represents a preference relation depends only on its *ordinal* properties.

Decision trees (simple example)



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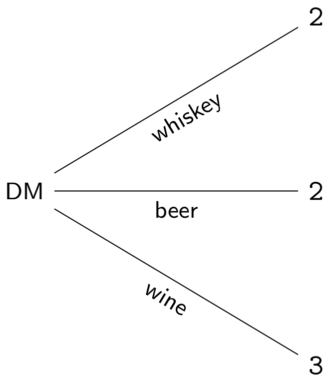

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Traditionally labeled with the player name, here "DM" for decision-maker

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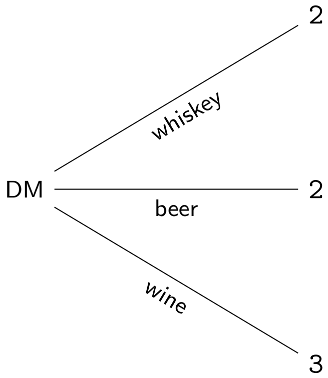

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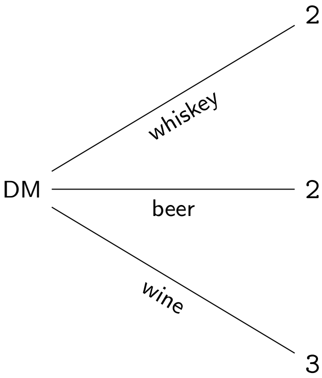

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In this case it makes very clear that DM should choose wine

Continuous action set example

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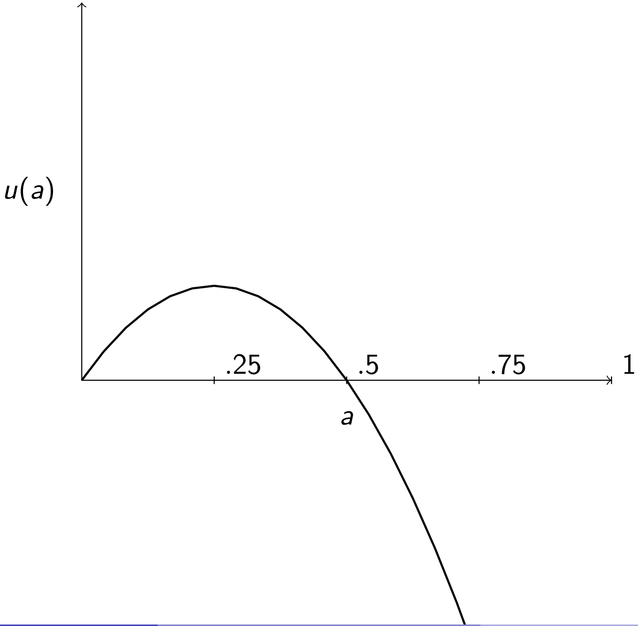
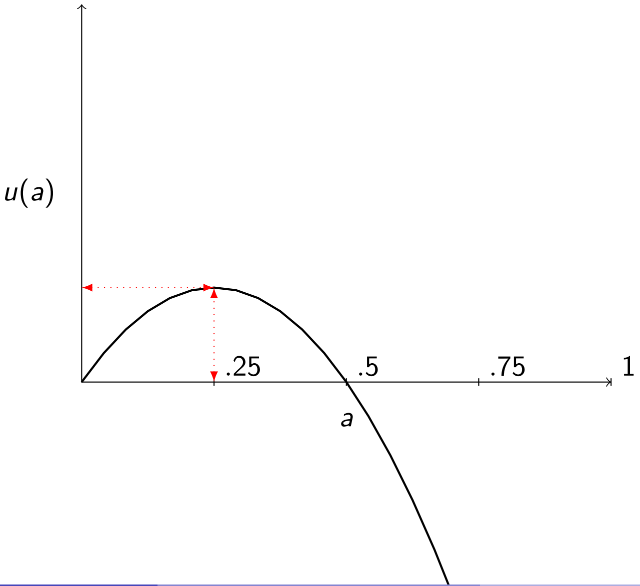
We can think of the first term as the decision-maker's enjoyment of wine and the second term as representing the negative effects of too much wine.

Utility from wine drinking (picture)



Utility from wine drinking (picture)

Optimal consumption is at .25



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Check second-order condition: second derivative is -8, which is indeed less than zero.

Rational decision-making under uncertainty

- Decision-makers might be uncertain about the exact mapping from choices to outcomes
- Examples:
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- We will assume that, although decision-makers do not know the exact outcomes, they know which outcomes are possible and have beliefs that assign probabilities to the different events
- As a convenient modeling device we think of the probabilities as being generated by moves by "Nature" (sometimes Chance, not the Rapper)

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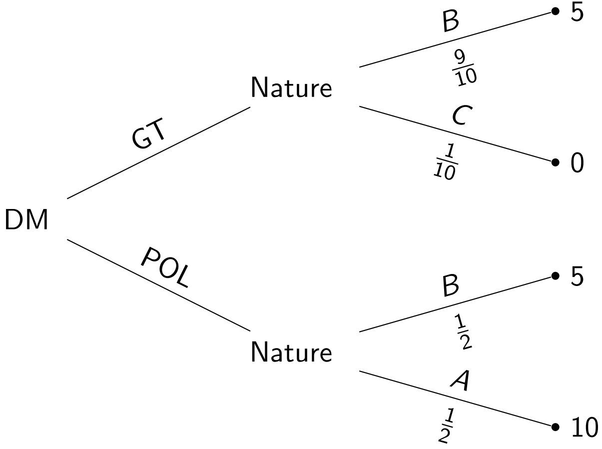
- Her decision turns on her expected grade. She knows the grade distribution in each class and figures that her likelihood of each grade is as follows:
    - In game theory, nobody gets an A but it is fairly easy to get a B (this is just an example – it's not true!), so she estimates that she is 90% likely to get a B and 10% likely to get a C
    - In policy analysis everyone gets an A or a B but the likelihood of each outcome is about the same, so she estimates a 50-50 chance of an A or B

- Her assignment of payoffs to outcomes is $u(A) = 10, u(B) = 5, u(C) = 0$.

A decision tree



The optimal class

The decision is made by computing the *expected utility* of each option:

$$\begin{aligned} EU(GT) =& u(A)\Pr[A|GT] + u(B)\Pr[B|GT] + u(C)\Pr[C|GT] \\ =& 0 + .9 * 5 + .1 * 0 = 4.5 \end{aligned}$$

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$\Rightarrow$ this student would prefer to take policy analysis. (Note: $\Pr[A|GT]$ is read "probability of the outcome A given that GT was chosen.")

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Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of outcomes and let $(p_1, p_2, \ldots, p_n)$ be the probability of each outcome, so that p_1 is the probability of x_1 and so on.

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The expected utility from the lottery p is then

$$E[u(x)|p] = \sum_{k=1}^{n} p_k u(x_k) = p_1 u(x_1) + p_2 u(x_2) + \cdots + p_n u(x_n).$$

Gambling example

I offer you the following gamble: I flip a fair coin. If it lands heads, I give you \$3. If it lands heads, you owe me \$2. If you choose not to play, you get nothing but pay nothing. Assume your utility is equal to the amount of money in your pocket. Should you play?

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$$EU(\text{play}) = \frac{1}{2}3 + \frac{1}{2}(-2) = \frac{1}{2}.$$

This is greater than zero, so you should play.

Gambling example, continued

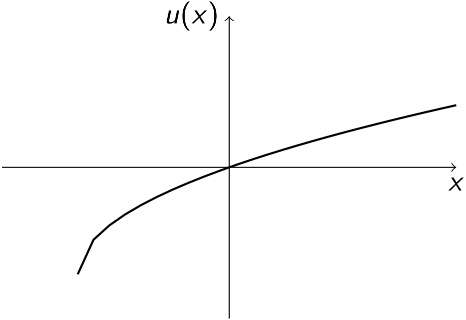
Suppose you have a different utility function: $u(x) = \sqrt{2 + x} - \sqrt{2}$, where x is the amount of money.

Gambling example, continued

Suppose you have a different utility function: $u(x) = \sqrt{2+x} - \sqrt{2}$, where x is the amount of money.

The expected utility of not playing is still zero. But now your have decreasing marginal returns to wealth:

Gambling example, decreasing marginal returns



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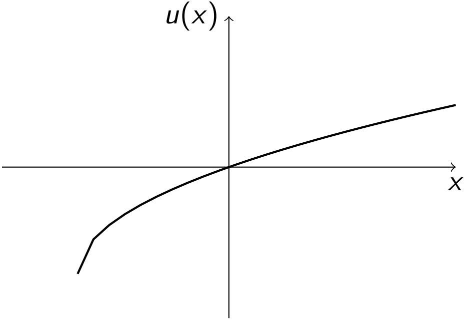

Decreasing marginal returns generate risk aversion.

Computing expected utility:

$$\begin{aligned} EU(\text{play}) =& \frac{1}{2}(\sqrt{5} - \sqrt{2}) + \frac{1}{2}(\sqrt{0} - \sqrt{2}) \\ =& \frac{1}{2}\left(\sqrt{5} - 2\sqrt{2}\right) \\ \approx& -.296. \end{aligned}$$

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This is not greater than 0 so this decision-maker should not play!

Summing up

- Rationality means that a decision-maker can rank all of the relevant outcomes in a sensible way
  - When uncertainty is involved we also assume that they rank lotteries by comparing expected utility

- The benefit of assuming that agents are rational is that we can analyze their decisions by assuming that they maximize a utility function, which gives us a nice analytical framework to use for theorizing

- We still have not discussed how to analyze multiple decisionmakers, but that is what we are going to do for the rest of the class.