
Foreground/Background Segmentation with Probabilistic Graphical Models

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Abstract

In this paper, we experiment with six different inference and learning models on an image segmentation problem. The goal of the project is to segment each image into foreground pixels and background pixels. The foreground and background objects in each image come from a known library. In this case, there are five different foreground objects and seven different background objects. The six inference algorithm we look at in this paper include iterated conditional modes (ICM), exact expectation-maximization (EM), Gibbs sampling EM, variational (mean-field) EM, structured variational EM, and sum-product EM. The basic idea is to approximate the joint distribution of the visible and hidden variables with a Q -distribution. Different inference algorithms put slightly different restrictions on the Q -distribution. This Q -distribution can be found by minimizing the free-energy (KL-divergent of the posterior distribution and the Q -distribution).

1 Introduction

In recent years, there is increasing interest in solving machine learning problems with probabilistic graphical models. Probabilistic graphical models not only provide a simple way to visualize the interactions between different variables in a model, they also allow complex probabilistic computations to be expressed as graphical manipulations [1]. In a graphical model, two of the main issues we are trying to solve are inference (compute the expectation of some nodes given some observed nodes) and learning (setting the parameters of the model given some data). In this paper, we are going to follow the approaches in [2] to compare six different inference and learning algorithms: iterated conditional modes (ICM), exact expectation-maximization (EM), Gibbs sampling EM, variational (mean-field) EM, structured variational EM, and sum-product EM. In order to compare the above six algorithms quantitatively, those six algorithms were tested against a simple image segmentation problem where we are trying to describe pictures as a composition of foreground and background objects, selected from a pre-defined library. Segmenting an image into foreground and background can be helpful to solve many other computer vision problems such as object classification [2].

All six inference and learning algorithms mentioned above have been widely used in many different image segmentation problems. The basic idea is to approximate the joint distribution of the visible and hidden variables with a Q -distribution. ICM is initially introduced by Besag in 1983 [3] and

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since then, the original algorithm and its variants have been applied to many different image segmentation problems [4] [5] [6]. In general, ICM iterates between finding the optimal value of each hidden variable h_i and finding the optimal values for the model parameters [2]. On the other hand, the three EM algorithms accounts for uncertainty in the random variables in the update rules [2]. For the exact EM, there is no restrictions on the Q -distributions, for the Gibbs Sampling EM, we use sampling to update the hidden variables in the E-step, and mean-field EM assumes the Q -distribution can be fully factorized. The strucuted variational EM is an improved version of the variational EM where it takes some dependencies between hidden variables into account. The sum-product EM performs inference by message passing. EM algorithms are also widely used for image segmentation problems [7] [8] [9].

2 Algorithms

2.1 Data generation

A training image or testing image used in this paper is generated using the following procedure:

- Randomly select a foreground image. Each foreground image has a fixed mask which specifies the location of the foreground image within the background image.
- Randomly select a background image.
- Combine the foreground image and background image with the cut-out mask.
- Add some Gaussian noise. The Gaussian noise is added to model sensor noise and change in illumination so that we can have more data available in the training/testing stage. This is image that is shown to our model in the training and testing stage.

A graphical illustration of the above procedures can be seen in Fig 1.

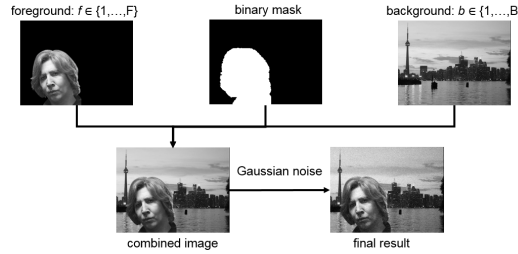


Figure 1: Overview of data generation

2.2 Model

In this project, we are using an occlusion model to describe each image generated in 2.1, where the foreground object and background object in each image come from the same library with J objects. If we assume each image has K pixels, then we can denote the pixel intensity in an image by z_1, \dots, z_K . The probability of choosing f^{th} foreground object is $P(f)$. Then depending on the foreground object, a binary mask is constructed $m = (m_1, \dots, m_K)$. m_i is either 0 (pixel z_i is a background pixel) or 1 (pixel z_i is a foreground pixel). The binary mask cut out the foreground object, but given the foreground object, the mask RVs are independent. Similarly, the probability of choosing b^{th} background object is $P(b)$. Lastly, when given the mask, the foreground object class and background object class, the intensities of the pixels in an image are independent. Therefore, the joint distribution of the model can be written as:

$$P(z, m, f, b) = P(b)P(f) \left(\prod_{i=1}^K P(m_i|f) \right) \left(\prod_{i=1}^K P(z_i|m_i, f, b) \right) \quad (1)$$

Since the mask RVs (m_i) are binary, the last term in equation 1 can be factorized. When $m_i = 0$, the pixel intensity (z_i) only depends on the background object index b , whereas when $m_i = 1$, z_i only depends on the foreground object index f . Therefore, equation 1 can be re-written as:

$$P(z, m, f, b) = P(b)P(f) \left(\prod_{i=1}^K P(m_i|f) \right) \left(\prod_{i=1}^K P(z_i|f)^{m_i} \right) \left(\prod_{i=1}^K P(z_i|b)^{1-m_i} \right) \quad (2)$$

To reduce the computation complexity, it is often useful to parametrize the model. In this case, we assume the pixel intensities in each image follows a Gaussian distribution, where given a foreground or background object index k , the pixel intensity z_i equals to μ_{ki} plus zero-mean Gaussian noise with a variance of φ_{ki} . This added noise would account for the noise we added to each of the images. We also denote the probability of class k by π_k . Lastly, we denote the probability of $m_i = 1$, given the foreground class is f be α_{fi} . Using this parametrization, equation 2 can be written as:

$$P(z, m, f, b) = \pi_b \pi_f \left(\prod_{i=1}^K (\alpha_{fi})^{m_i} (1 - \alpha_{fi})^{1-m_i} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi})^{m_i} \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi})^{1-m_i} \right) \quad (3)$$

where \mathcal{N} represent the Gaussian distribution.

2.3 Finding model parameters by minimizing the free energy

The model parameters are found based on each training image. In each training image, we can group the variables into hidden RVs, visible RVs and parameters. The visible RVs in this case are the pixel intensities: $v^{(t)} = z^{(t)}$. The hidden RVs are $h^{(t)} = (f^{(t)}, b^{(t)}, m^{(t)})$. The model parameters are $\theta = (\mu, \varphi, \pi, \alpha)$. Then the joint distribution of the visible and hidden RVs can be written as:

$$P(h, v) = P(\theta) \prod_{t=1}^T P(h^{(t)}, v^{(t)} | \theta) \quad (4)$$

where (t) represents the index of the training cases.

Usually, it is very hard to find the posterior distribution $P(h|v)$ directly since the calculation for the partition function is intractable. Therefore, we often use approximate inference techniques instead. The basic idea of those techniques is that we are trying to use a simpler Q -distribution $Q(h)$ approximate the true posterior distribution $P(h|v)$. The free energy is a measurement of similarity between the true posterior distribution and the Q -distribution we have currently. The free energy can be defined as:

$$\begin{aligned} F(Q, P) &= KL(Q, P) - \ln P(v) \\ &= \int_h Q(h) \ln \frac{Q(h)}{P(h|v)} - \int_h Q(h) \ln P(v) \\ &= \int_h Q(h) \ln \frac{Q(h)}{P(h, v)} \end{aligned} \quad (5)$$

where $KL(Q, P)$ is the KL-divergent between the Q -distribution and the posterior distribution $P(h|v)$. The term $\ln P(v)$ is added to make the calculation tractable (since we do not have a tractable form of $P(h|v)$), but it does not have a direct impact final solution for the Q -distribution (since Pv does not depend on the Q -distribution). Since the KL divergence is a similarity measure of two distributions, minimizing the free energy $F(Q, P)$ makes the Q -distribution more similar to the the posterior distribution $P(h|v)$. Therefore, the goal of the inference problem is to search through the space to find $Q(h)$ that resembles the true posterior distribution closely.

If we assume the training data is i.i.d, we can de-couple the free energy so that we have one term for each training case. In this case, we can substitute equation 4 into equation 5 and obtain:

$$F(Q, P) = \int_h Q(h) \ln Q(h) - \int_{\theta} Q(\theta) \ln P(\theta) - \sum_{t=1}^T \int_{h^{(t)} \theta} Q(h^{(t)}, \theta) \ln P(h^{(t)}, v^{(t)} | \theta) \quad (6)$$

2.4 Iterated Conditional Modes (ICM)

The ICM algorithm alternates between updating the hidden variables and the model parameters. In the ICM update procedure, we are only considering local information. We update one variable/parameter at a time by setting it to its Maximum a posterior (MAP) value while holding other variables/parameters constant. Since we are using the MAP estimation where there is a hard-assignment for each pixel (it either belongs to the foreground or the background class), the Q -distribution is a product of delta functions:

$$Q = \left(\prod_k \delta(\pi_k - \hat{\pi}_k) \right) \left(\prod_{k,i} \delta(\mu_{ki} - \hat{\mu}_{ki}) \right) \left(\prod_{k,i} \delta(\varphi_{ki} - \hat{\varphi}_{ki}) \right) \left(\prod_{k,i} \delta(\alpha_{ki} - \hat{\alpha}_{ki}) \right) \left(\prod_t [b^{(t)} = \hat{b}^{(t)}] \right) \left(\prod_t [f^{(t)} = \hat{f}^{(t)}] \right) \left(\prod_t \prod_i [m_i^{(t)} = \hat{m}_i^{(t)}] \right) \quad (7)$$

Substituting equation 7 and 3 into equation 6, taking the derivative with respect to each parameter/variable and set to zero, we can find the update rule for the ICM model as follows (Repeat until convergence):

E-step (hidden variable update)

for $t = 1, \dots, T$:

$$f \leftarrow \operatorname{argmax}_f \left(\pi_f \prod_{i=1}^K (\alpha_{fi})^{m_i} (1 - \alpha_{fi})^{1-m_i} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi})^{m_i} \right)$$

for $i = 1, \dots, K$:

$$m_i \leftarrow \operatorname{argmax}_{m_i} = \begin{cases} \alpha_{fi} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi}), & \text{if } m_i = 1 \\ (1 - \alpha_{fi}) \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi}), & \text{if } m_i = 0 \end{cases}$$

$$b \leftarrow \operatorname{argmax}_b \left(\pi_b \prod_{i=1}^K \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi})^{m_i} \right)$$

M-step (model parameter update)

For $j = 1, \dots, J$:

$$\pi_j \leftarrow \frac{1}{2T} \sum_{t=1}^T ([f^{(t)} = j] + [b^{(t)} = j])$$

For $j = 1, \dots, J$ and For $i = 1, \dots, K$:

$$\alpha_{ji} \leftarrow \frac{\sum_{t=1}^T [f^{(t)} = j] m_i^{(t)}}{\sum_{t=1}^T [f^{(t)} = j]} \quad \mu_{ji} \leftarrow \frac{\sum_{t=1}^T [f^{(t)} = j \text{ or } b^{(t)} = j] z_i^{(t)}}{\sum_{t=1}^T [f^{(t)} = j \text{ or } b^{(t)} = j]} \\ \varphi_{ji} \leftarrow \frac{\sum_{t=1}^T [f^{(t)} = j \text{ or } b^{(t)} = j] (z_i^{(t)} - \mu_{ji})^2}{\sum_{t=1}^T [f^{(t)} = j \text{ or } b^{(t)} = j]}$$

The ICM algorithm is really easy to implement, but it can get stuck at local optima very easily. Thus, this algorithm does not work well in many cases in practice.

2.5 Exact Expectation-Maximization (EM)

Unlike ICM, the EM algorithm takes into account for uncertainty in the RVs in the update process. The Q -distribution for the i.i.d training cases can be written as:

$$Q(h) = \delta(\theta - \hat{\theta}) \prod_{t=1}^T Q(h^{(t)}) \quad (8)$$

Again, substituting equation 8 into equation 6, taking the derivative with respect to each parameter/ variable and set to zero, we can find the update rule for the ICM model as follows (Repeat until convergence):

E-step

for $t = 1, \dots, T$:

$$Q(b, f) \leftarrow c_2 \pi_b \pi_f \prod_{i=1}^K \left(\alpha_{fi} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi}) + (1 - \alpha_{fi}) \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi}) \right)$$

$$Q(b) \leftarrow \sum_f Q(b, f) \quad Q(f) \leftarrow \sum_b Q(b, f)$$

for $i = 1, \dots, K$:

$$Q(m_i = 1|b, f) \leftarrow c_1 \alpha_{fi} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi}) \quad Q(m_i = 0|b, f) \leftarrow c_1 \alpha_{fi} \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi})$$

for $i = 1, \dots, K$:

$$Q(m_i, b) \leftarrow \sum_f Q(m_i|b, f) Q(b, f) \quad Q(m_i, f) \leftarrow \sum_b Q(m_i|b, f) Q(b, f)$$

M-step

For $j = 1, \dots, J$:

$$\pi_j \leftarrow \frac{1}{2T} \sum_{t=1}^T \left(Q(f^{(t)} = j) + Q(b^{(t)} = j) \right)$$

For $j = 1, \dots, J$ and For $i = 1, \dots, K$:

$$\alpha_{ji} \leftarrow \frac{\sum_{t=1}^T Q(m_i^{(t)} = 1, f^{(t)} = j)}{\sum_{t=1}^T Q(f^{(t)} = j)}$$

$$\mu_{ji} \leftarrow \frac{\sum_{t=1}^T \left(Q(m_i^{(t)} = 1, f^{(t)} = j) + Q(m_i^{(t)} = 0, b^{(t)} = j) \right) z_i^{(t)}}{\sum_{t=1}^T \left(Q(m_i^{(t)} = 1, f^{(t)} = j) + Q(m_i^{(t)} = 0, b^{(t)} = j) \right)}$$

$$\varphi_{ji} \leftarrow \frac{\sum_{t=1}^T \left(Q(m_i^{(t)} = 1, f^{(t)} = j) + Q(m_i^{(t)} = 0, b^{(t)} = j) \right) (z_i^{(t)} - \mu_{ji})^2}{\sum_{t=1}^T \left(Q(m_i^{(t)} = 1, f^{(t)} = j) + Q(m_i^{(t)} = 0, b^{(t)} = j) \right)}$$

2.6 Gibbs Sampling EM

Gibbs sampling EM is very similar to ICM except it uses Gibbs sampling to update the hidden variables in the E-step (The M-step in this case is the same as ICM, but we use Gibbs sampling method to update the hidden variables instead of using the MAP value). Since this sampling approach is a stochastic process, it helps the algorithm to jump out from local optima and thus improving the overall performance of the algorithm.

2.7 Variational (Mean field) EM

In the variational EM, we assume the hidden variables in the Q -distribution factorizes completely:

$$Q(h) = \prod_{i=1}^L Q(h_i) \tag{9}$$

In this case, the M-step is exactly the same as the exact EM algorithm, but the E-step can be simplified to:

E-step

for $t = 1, \dots, T$:

$$Q(f) \leftarrow c_3 \pi_f \prod_{i=1}^K \left(\left(\alpha_{fi} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi}) \right)^{Q(m_i=1)} (1 - \alpha_{fi})^{Q(m_i=0)} \right)$$

for $i = 1, \dots, K$:

$$Q(m_i = 1) \leftarrow c_1 \prod_f \left(\alpha_{fi} \mathcal{N}(z_i; \mu_{fi}, \varphi_{fi}) \right)^{Q(f)}$$

$$Q(m_i = 1) \leftarrow c_1 \left(\prod_f (1 - \alpha_{fi})^{Q(f)} \right) \left(\prod_b \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi})^{Q(b)} \right)$$

$$Q(b) \leftarrow c_2 \pi_b \prod_{i=1}^K \mathcal{N}(z_i; \mu_{bi}, \varphi_{bi})^{Q(m_i=0)}$$

2.8 Structured Variational EM

Unlike mean-field approximation, the structured variational EM takes into account some dependencies between the hidden RVs. In general, if more dependencies are included in the model, we will have more exact inference, but the computational complexity also increases. The details of the update equations can be found in [2].

2.9 Sum-Product EM

The sum-product algorithm performs inference by message passing. If the graphical model is a tree, all the posterior marginal distributions will be exact once we have passed the messages from the leaves to the root and then from the root back to the leaves. However, in our case, the graphical model constructed for this problem has cycles. The messages need to be passed through the model several times until convergence. The details of the update equations can be found in [2].

3 References

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