EECE 5550

Mobile Robotics

Lecture 22: Introduction to Control Lyapunov Functions and Control Barrier Functions

Derya Aksaray

Assistant Professor

Department of Electrical and Computer Engineering

Northeastern University

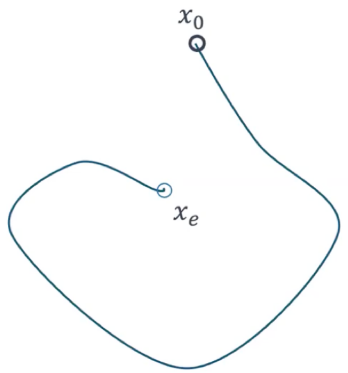
Lyapunov functions

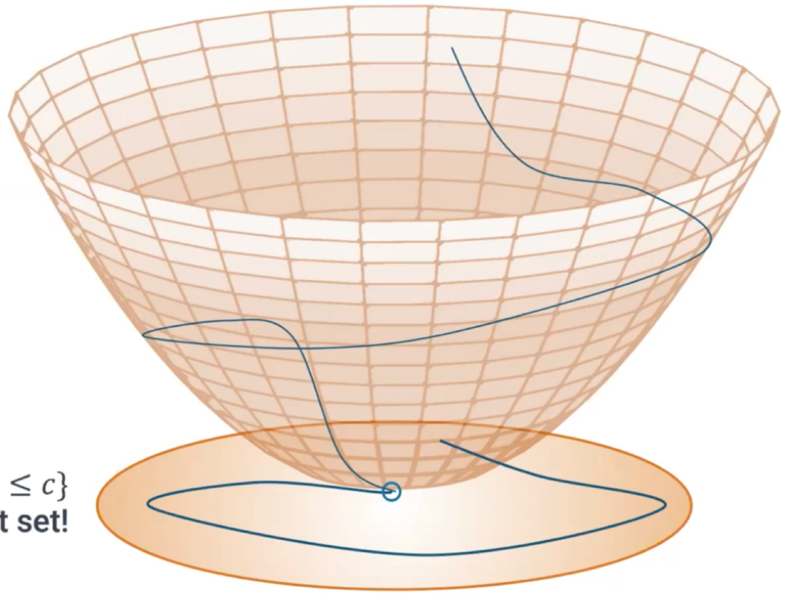
Stability analysis of nonlinear systems without solving them explicitly.

Lyapunov function candidate $\begin{cases} V(x_e) = 0 \\ V(x) > 0 \;\; for\ all\ x \neq x_e \end{cases}$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \;\; for\ x \neq x_e$$

$$\dot{x} = f(x)$$





$\Omega_c = \{x \mid V(x) \leq c\}$
is an invariant set!

If you start from c-level set, you will always stay in it due to the $\dot{V}(x) < 0$ property.

Control Lyapunov functions (CLF)

Formally,

Let $V(x): \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function.

If there exists a constant $c > 0$ such that

1. $\Omega_c \coloneqq \{x \in \mathbb{R}^n : V(x) \le c\}$, a sublevel set of $V(x)$ is bounded,
2. $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{x_e\}$ and $V(x_e) = 0$
3. $\min_{u \in U} \dot{V}(x) < 0$ for all $x \in \Omega_c \backslash \{x_e\}$

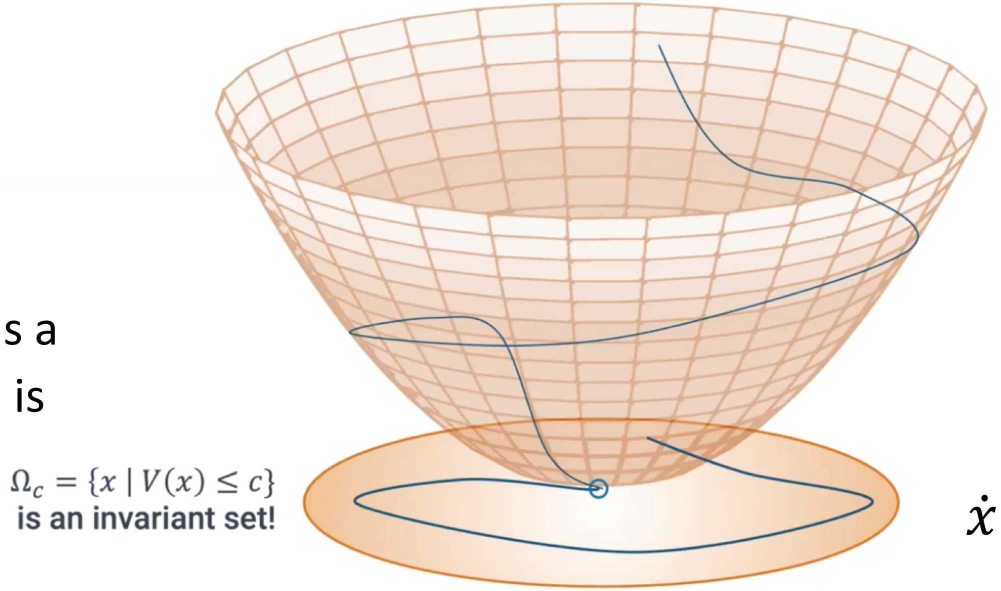
Then,

$V(x)$ is a (local) Control Lyapunov function and Ω_c is a region of attraction, meaning that every state in Ω_c is asymptotically stabilizable to x_e.

$V(x_e) = 0$

$V(x) > 0 \;\; for\ all\ x \neq x_e$

$\exists u \;\; s.t. \dot{V}(x) = \frac{\partial V}{\partial x} f(x,u) < 0 \;\; for\ x \neq x_e$



$\dot{x} = f(x,u)$

Control Lyapunov functions

Differential drive robot

$$\begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta}_r \end{pmatrix} = \begin{pmatrix} \frac{r}{2}(\dot{\varphi}_r + \dot{\varphi}_l) \\ 0 \\ \frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) \end{pmatrix}$$

$$= 0 + \begin{pmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ \frac{r}{w} & -\frac{r}{w} \end{pmatrix} \begin{pmatrix} \dot{\varphi}_r \\ \dot{\varphi}_l \end{pmatrix}$$

Suppose that we have a nonlinear affine control system:

$$\dot{x} = f(x) + g(x)u$$

*This is a quite generic model, most mechanical systems can be written in this form.

Now, let's consider the derivative of V along the dynamics:

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}(x)\right)^T \dot{x}$$

$$= \left(\frac{\partial V}{\partial x}(x)\right)^T f(x) + \left(\frac{\partial V}{\partial x}(x)\right)^T g(x)u$$

$$= L_f\, V(x) + L_g V(x) u$$

Def: Lie derivative operator

$$L_f V(x) \triangleq \left(\frac{\partial V}{\partial x}(x)\right)^T f(x)$$

If we can find u such that $\dot{V} < 0$, then we can stabilize the system.

Control Lyapunov functions

Suppose that we enforce the following:

$$\dot{V}(x) \leq -\gamma(V(x))$$

where $\gamma: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a class κ function defined on the entire real line (i.e., $\gamma(0) = 0$ and it is strictly monotonic: for all $r_1, r_2 \in \mathbb{R}_{\geq 0}$, $r_1 < \mathrm{r}_2$ implies $\gamma(r_1) < \gamma(r_2)$. Why?

This constraint helps us to analyze the decay of $\dot{V}$ (finite-time convergence)

$$argmin_u \left(u - u_{ref}\right)^T H\left(u - u_{ref}\right)$$

subject to: $\quad L_f V(x) + L_g V(x)u + \gamma(V(x)) \leq 0$

$\quad u \in U$

Minimally adjusting u_{ref} to ensure asymptotic stability.

Quadratic program which can be solved very quickly (e.g., in real-time).

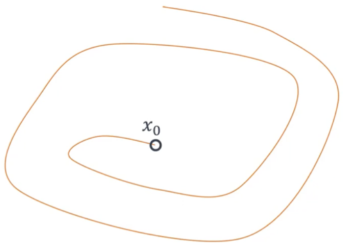
Nagumo's invariance principle

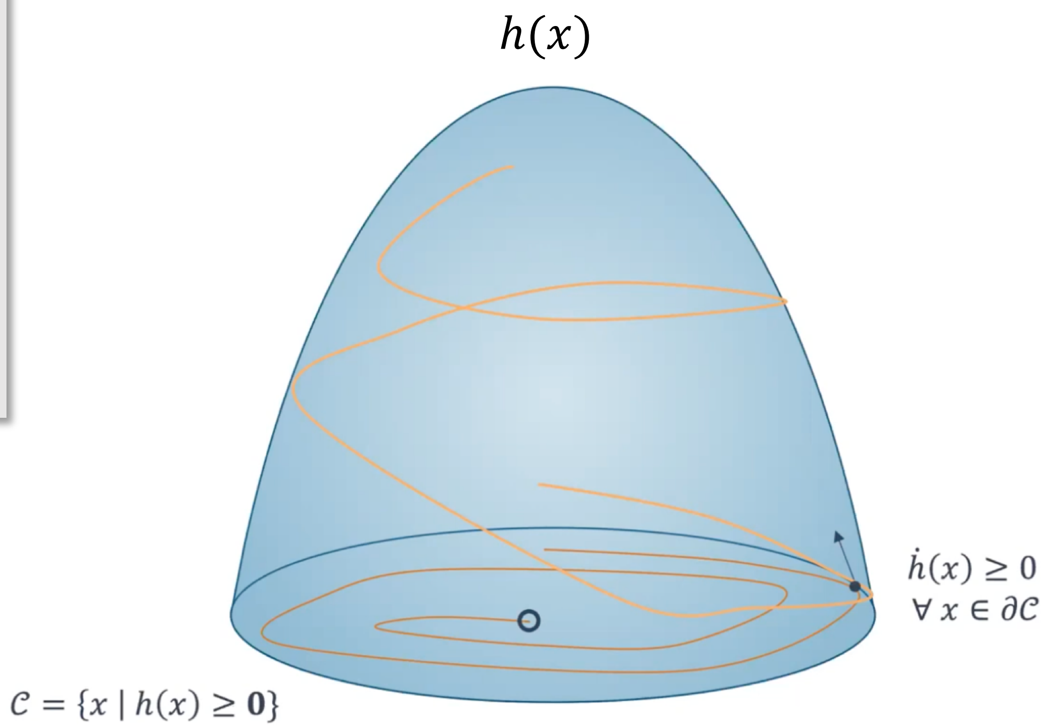
- The goal is staying inside of the safe set C.

$$C = \{x|h(x) \geq 0\}$$
$$\partial C = \{x|h(x) = 0\}$$
$$Int(C) = \{x|h(x) > 0\}$$

- Unlike Lyapunov functions, the system may not converge to the center.
- $h(x)$ is like the flipped version of $V(x)$.

$$\dot{x} = f(x)$$





Zero-level set is an invariant set.

Control barrier function (CBF)

Similarly, we will consider a control affine system:

$$\dot{x} = f(x) + g(x)u$$

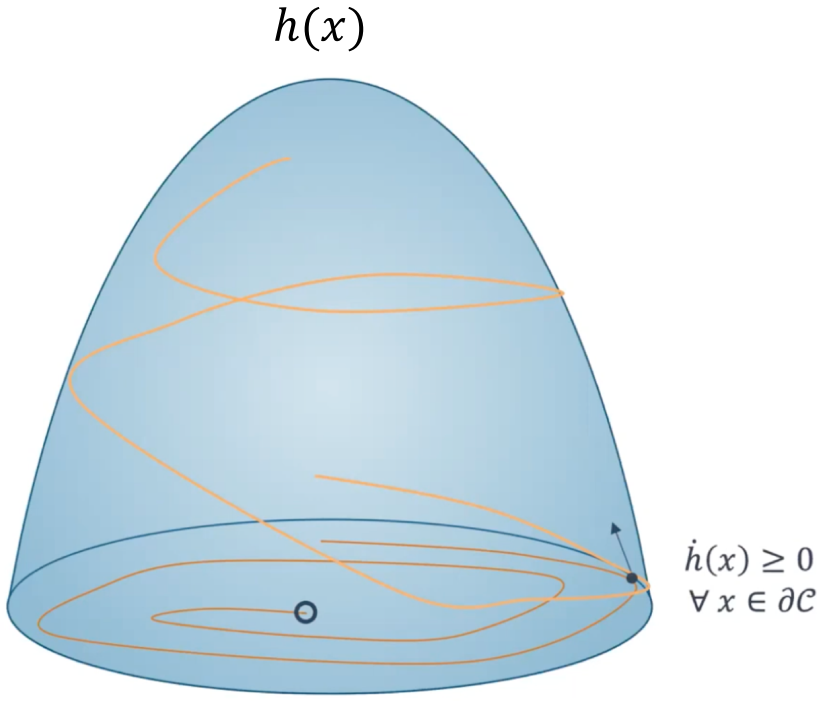
Let C be a desired safe set.

Let C be the super-level set of a continuously differentiable function $h(x): \mathbb{R}^n \to \mathbb{R}$:

$$C = \{x | h(x) \geq 0\}$$

Then, h is a control barrier function if there exists an extended class κ function $\alpha: \mathbb{R} \to \mathbb{R}$ (strictly increasing and $\alpha(0) = 0$) s.t.

$$\sup_u L_f h(x) + L_g h(x) u \geq -\alpha\big(h(x)\big) \quad \text{for all } x.$$



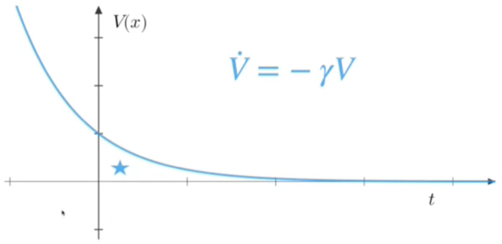
Safety critical control

- Suppose that we are given a feedback controller $u_{ref} = k(x)$.
- The controller $k(x)$ may not guarantee safety for some x.
- In order to minimally modify the control:

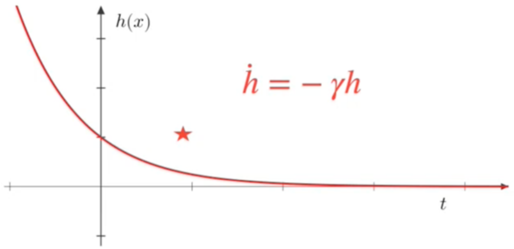
$$u(x) = argmin_u \frac{1}{2} \|u - k(x)\|^2$$

Subject to: $L_f h(x) + L_g h(x) u \geq -\alpha\big(h(x)\big)$

Quadratic program which can be solved very quickly (e.g., in real-time).



$\dot{V} = L_f V + L_g V u \leq -\gamma V$



$\dot{h} = L_f h + L_g h u \geq -\gamma h$

CBF vs. CLF constraints

$\dot{V} < 0$

$\dot{h} \geq -\beta$ where $\beta > 0$

Safety critical control

A. Ames, X. Xu, J.W. Grizzle, and P. Tabuada. "Control barrier function based quadratic programs for safety critical systems." *IEEE Transactions on Automatic Control* 62, no. 8 (2016): 3861-3876.

A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada. "Control barrier functions: Theory and applications." *18th European control conference (ECC)*, pp. 3420-3431, 2019.

- Suppose that we are given a feedback controller $u_{ref} = k(x)$.
- The controller $k(x)$ may not guarantee safety for some x.
- In order to minimally modify the control:

$$\left.\begin{aligned} & u(x) = argmin_u \frac{1}{2}\|u - k(x)\|^2 \\ & \text{Subject to:} \quad L_f h(x) + L_g h(x) u \geq -\alpha\big(h(x)\big) \end{aligned}\right\}$$

Quadratic program which can be solved very quickly (e.g., in real-time).

- One can also combine CLF and CBF, then solve the QP for the control design.

Ex: Avoiding collision avoidance between two robots by ensuring a safety distance $\delta > 0$:

$$h(x_i, x_j) = \|x_i - x_j\|^2 - \delta^2 \geq 0$$

Safety critical control