EECE 5550
Mobile Robotics

Lecture 3: Introduction to Manifolds and Lie Theory

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Motivation

You are attempting to estimate the orientation of a robotic ground vehicle using measurements from a very noisy compass. You collect the following set of noisy measurements (in radian angle measure):

- $\tilde{\theta}_1$=0.0872,
- $\tilde{\theta}_2$=6.196,
- $\tilde{\theta}_3$=6.021,
- $\tilde{\theta}_4$=0.262

Question: Given this data, what is the best estimate $\hat{\theta}$ for the robot's heading?

Answer: $\hat{\theta} = 0$

Motivation

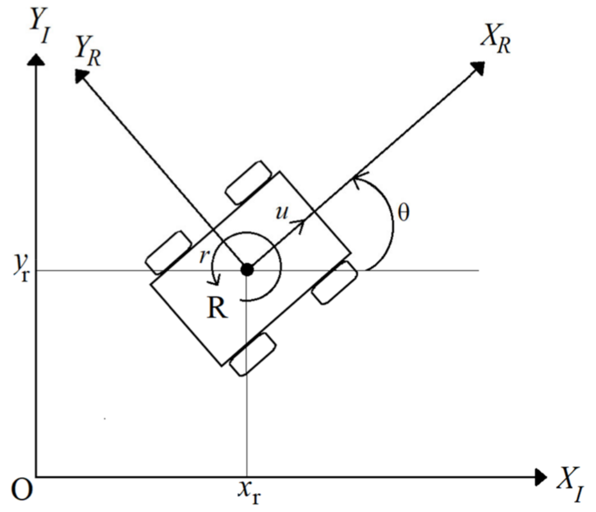
Goal: Estimate the orientation (heading angle) of the ground robot

Recall: SO(2) is the space of 2D rotations. Each element in SO(2) is in the form of:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$SO(d) \coloneqq \{R \in \mathbb{R}^{d \times d} | R^T R = I, \det(R) = 1\}$$

- Orthogonal:
  - each column is a unit length vector
  - Each column is orthogonal to all other columns
- Determinant is +1
  - The length of a vector is unchanged with rotation



How can we visualize SO(2)?

Motivation

Goal: Estimate the orientation (heading angle) of the ground robot

Recall: SO(2) is the space of 2D rotations. Each element in SO(2) is in the form of:

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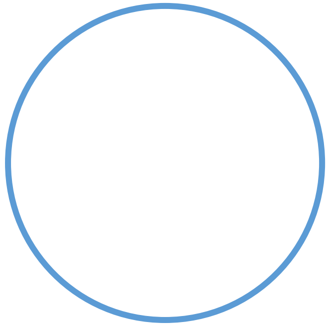
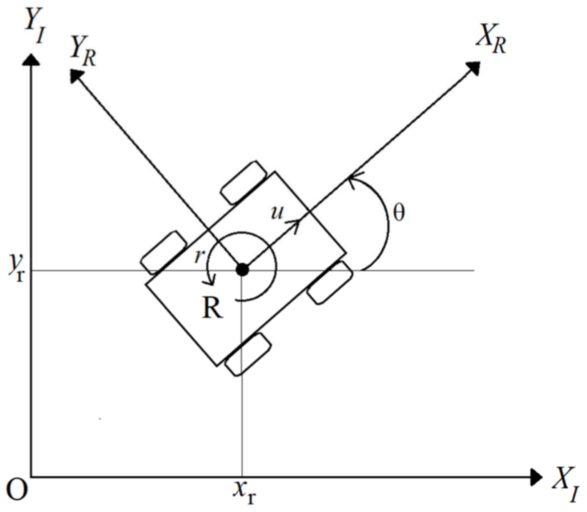
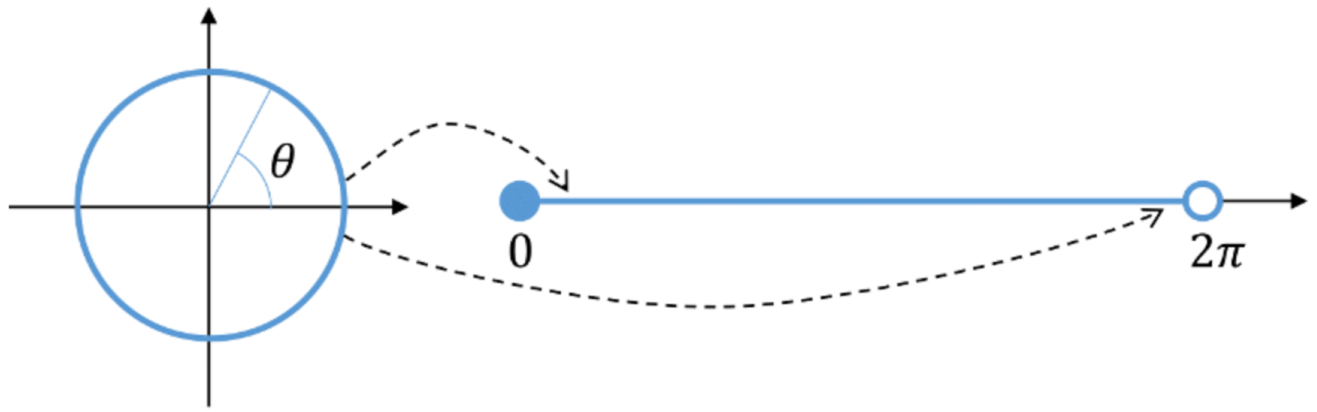


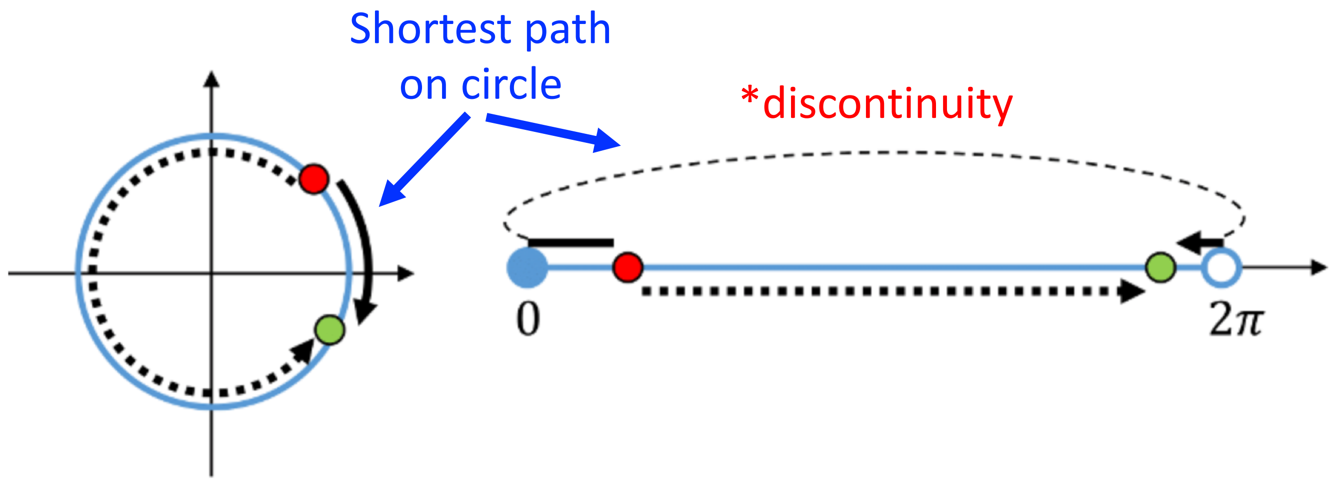
SO(2) [unit circle]



Motivation



One-to-one mapping between points on the circle and the interval [0,2π)



Operations on

$\mathbb{R}$ and SO(2) are

not equivalent

Motivation

Noisy measurements (in radian angle measure):

- $\tilde{\theta}_1$=0.0872,
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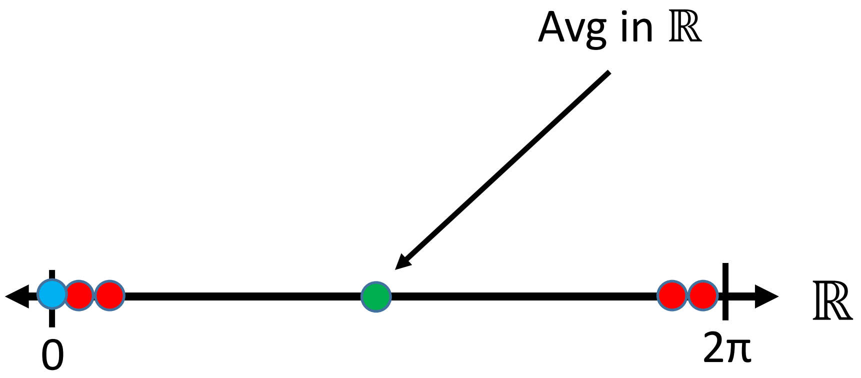
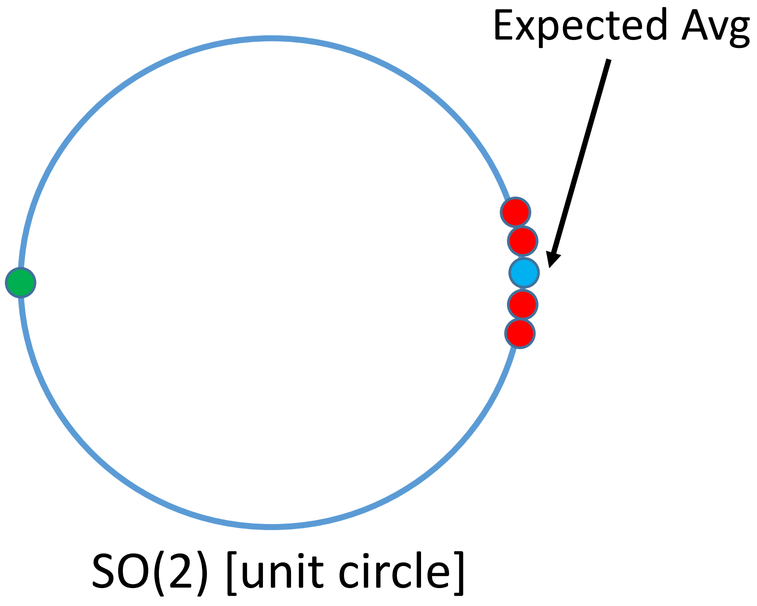
There is 180 degrees off!

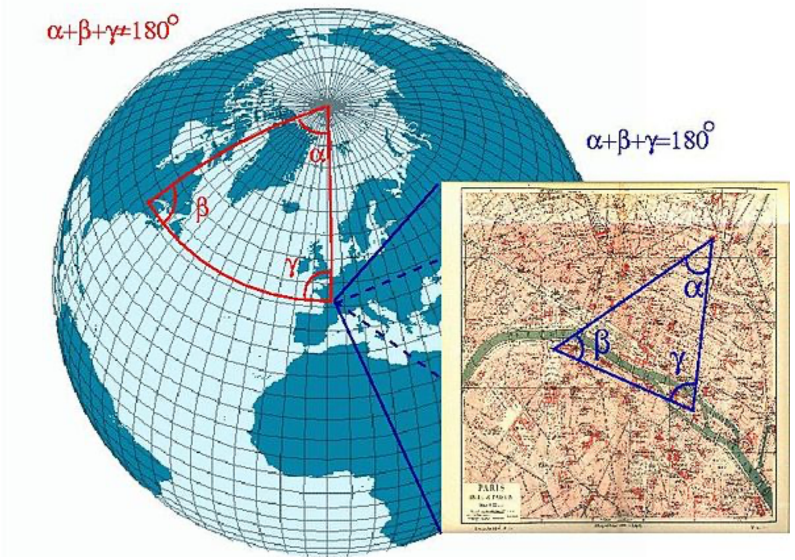
Punchline: We cannot "average" orientations in $\mathbb{R}$ because the (global) geometry of $\mathbb{R}$ does *not* reflect that of SO(2).





Motivation

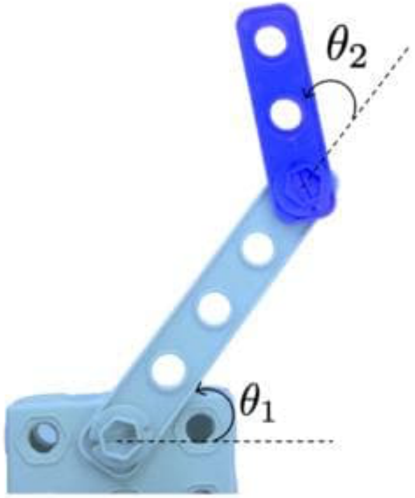
- Operations like distance, addition on Euclidian space do not work over curved spaces!



Shortest path on globe ≠ shortest path on map

Motivation

- We need another concept…

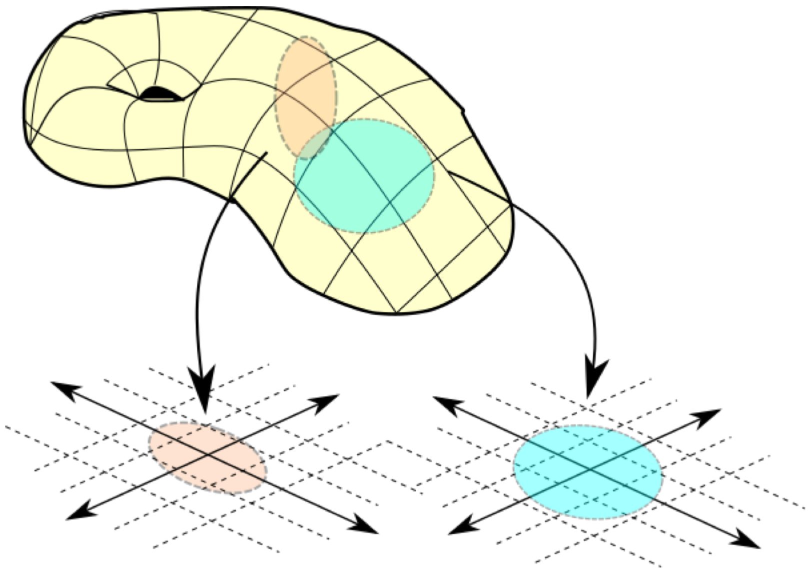


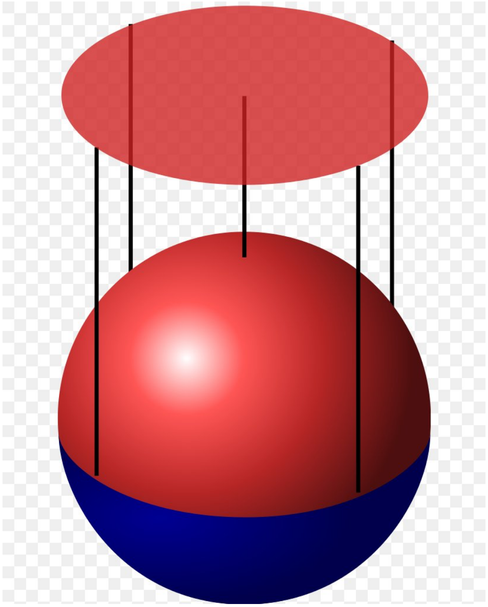
Robotic arm

Torus

Manifolds

Manifolds are spaces that *locally* “look like” $\mathbb{R}^n$, but may have a very different *global* geometry and topology.

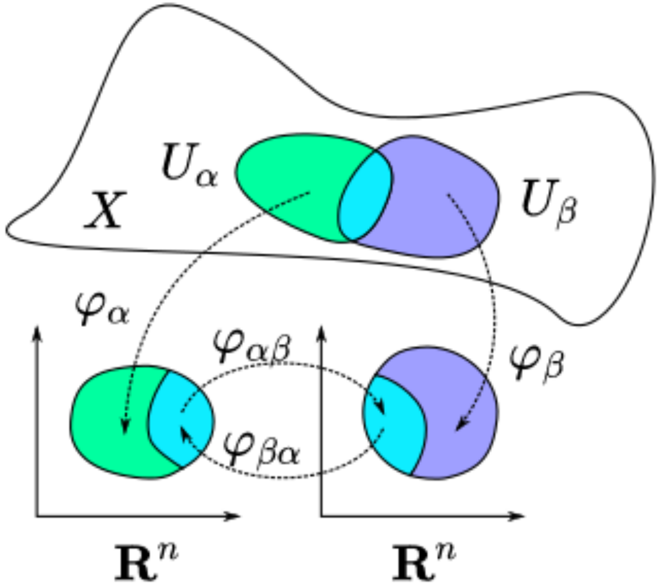




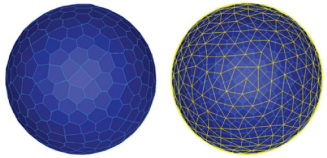
Manifolds

Def. : A manifold X is a topological space in which each point $x \in X$ has an open set (neighborhood) U and a continuous map $\varphi: U \to V \subseteq \mathbb{R}^n$ with a continuous inverse $\varphi^{-1}: V \to U$.

- The map φ is called a (*coordinate*) *chart*
  - ⇒This *identifies* a subset of X with a subset of $\mathbb{R}^n$
  - ⇒It lets us (*locally*!) define *coordinates* on X
  - ⇒n is called the *dimension* of X



- A set of *charts* that covers X is called an *atlas.*
  - ⇒Not unique, multiple ways to cover X



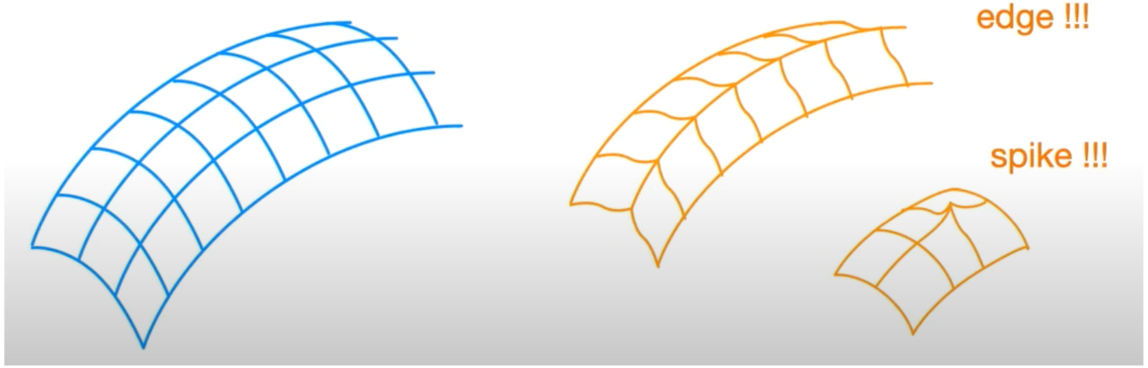
Smooth Manifold

Intersecting parts of U and V in their respective charts called a transition map $\varphi_{\alpha\beta} \triangleq \varphi_\beta \circ \varphi_\alpha^{-1}$

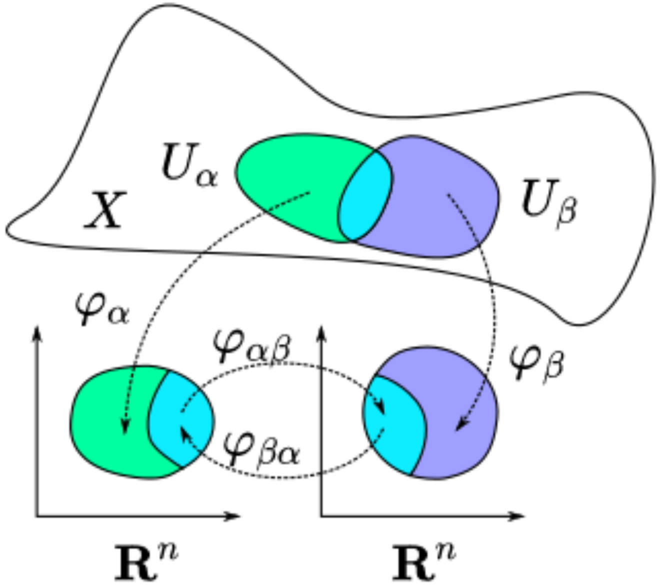
X is called a *smooth manifold* if additionally the *transition maps* are differentiable (in $\mathbb{R}^n$)

$\Rightarrow$ This lets us define *differentiation* on *X* (via $\mathbb{R}^n$)

$\Rightarrow$ We can do *calculus* on *X*



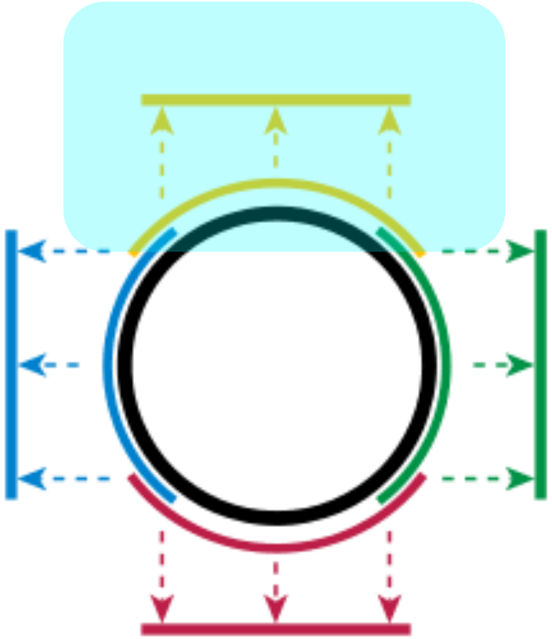
Smooth manifold vs. Non-smooth manifold



The intersection of the patches have a smooth 1-1 mapping in $\mathbb{R}^n$ Euclidean space

Some example of constructing charts

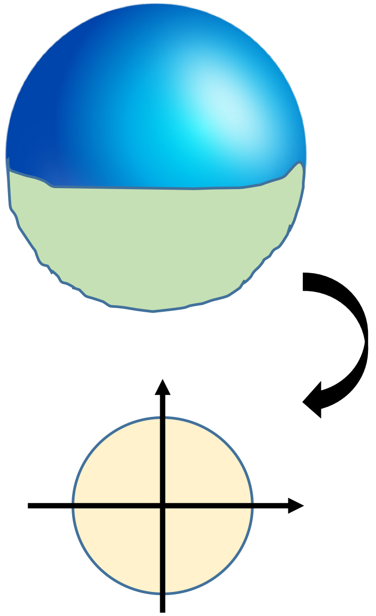
Treat as $\mathbb{R}$



The unit circle

The unit sphere

$$S^2 := \{x \in \mathbb{R}^3 | \|x\| = 1\}$$



e.g., south hemisphere

φ

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 | x_3 < 0 \right\}$$

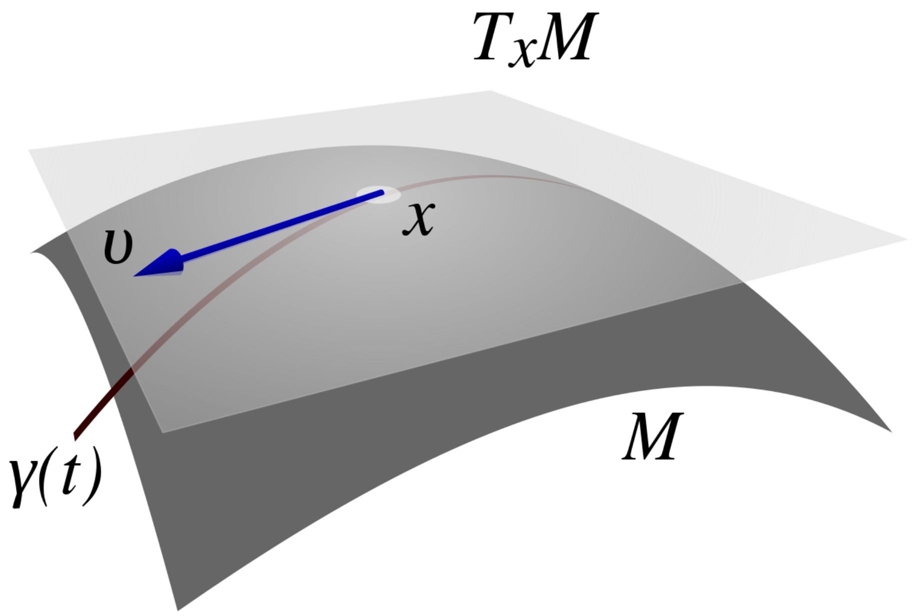
$$\{x' \in \mathbb{R}^2 | \|x'\| < 1\}$$

$$\varphi: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \varphi^{-1}: \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \to \begin{pmatrix} x'_1 \\ x'_2 \\ -\sqrt{1 - \|x'\|^2} \end{pmatrix}$$

*The combination of these charts that cover the sphere will form an atlas.

Tangent vector

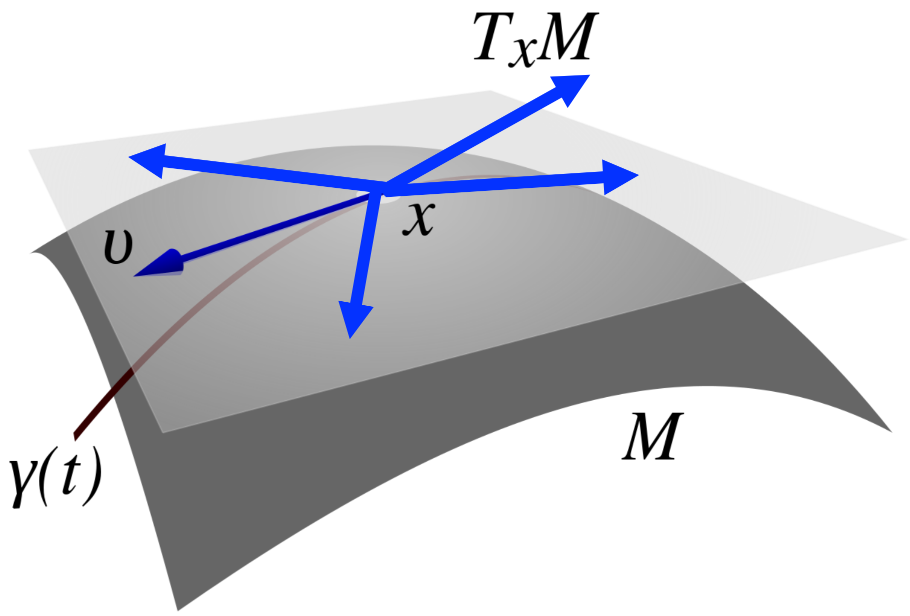
Let γ be a curve (smooth trajectory) on manifold M. Then, its (directional) derivative $\dot{\gamma}$ is a tangent vector.



How many tangent vectors exist at point x?

Tangent vector

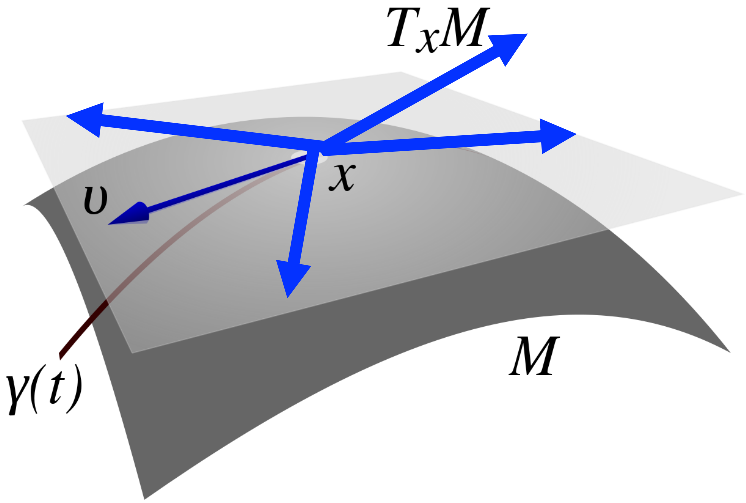
Let γ be a curve (smooth trajectory) on manifold M. Then, its (directional) derivative $\dot{\gamma}$ is a tangent vector.



Consider all curves (smooth trajectories) passing through x

Tangent space

The *tangent space at point x of the manifold M is denoted by* $T_x(M)$, that is the set of **all** tangent vectors at *x*.

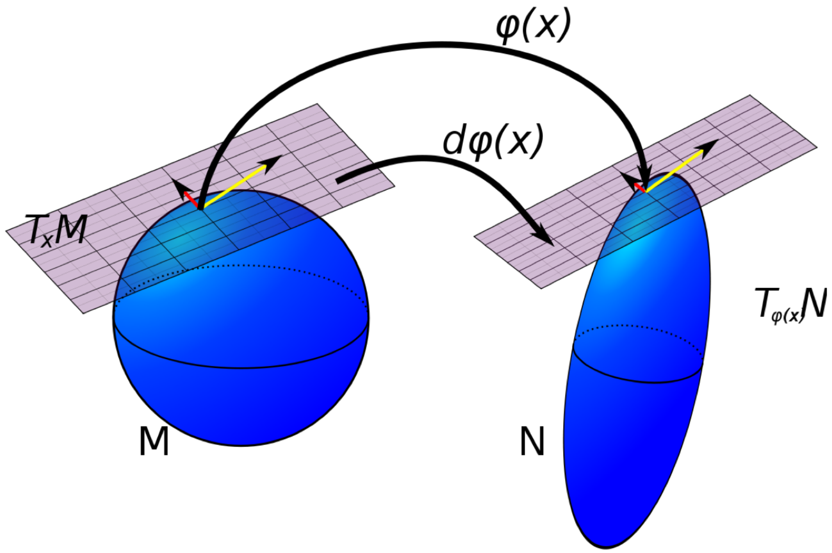


Key facts:

- The tangent space is a *linear space* of dimension dim(*M*).
- Intuitively, provides a *linear approximation* of *M* near *x*.

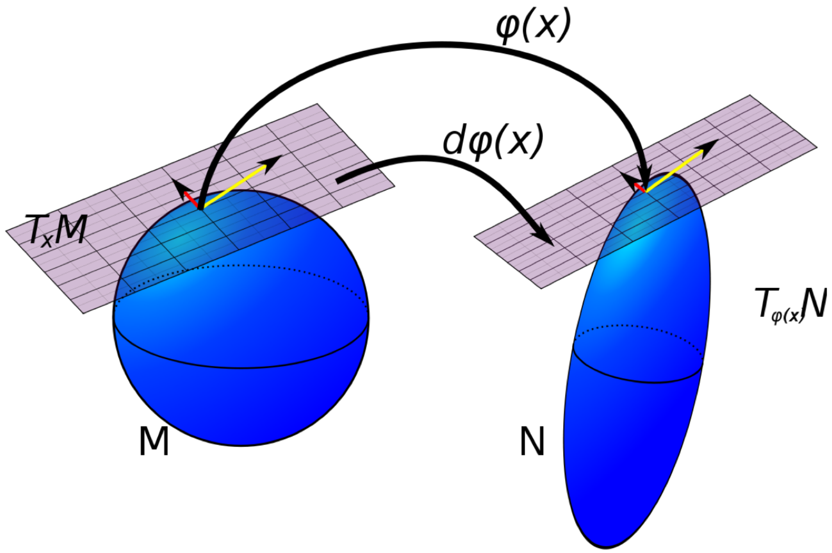
Calculus on manifolds

- Let *M* and *N* be smooth manifolds, and $\varphi: M \to N$ a smooth map.
- At each *x* in *M*, the map φ induces a map $d\varphi_x: T_x(M) \longrightarrow T_{\varphi(x)}(N)$ between the tangent spaces of *M* and *N*, called the *derivative* of φ.
- $d\varphi_x$ describes the infinitesimal change in $\varphi(x)$ under infinitesimal changes in *x*.



Calculus on manifolds

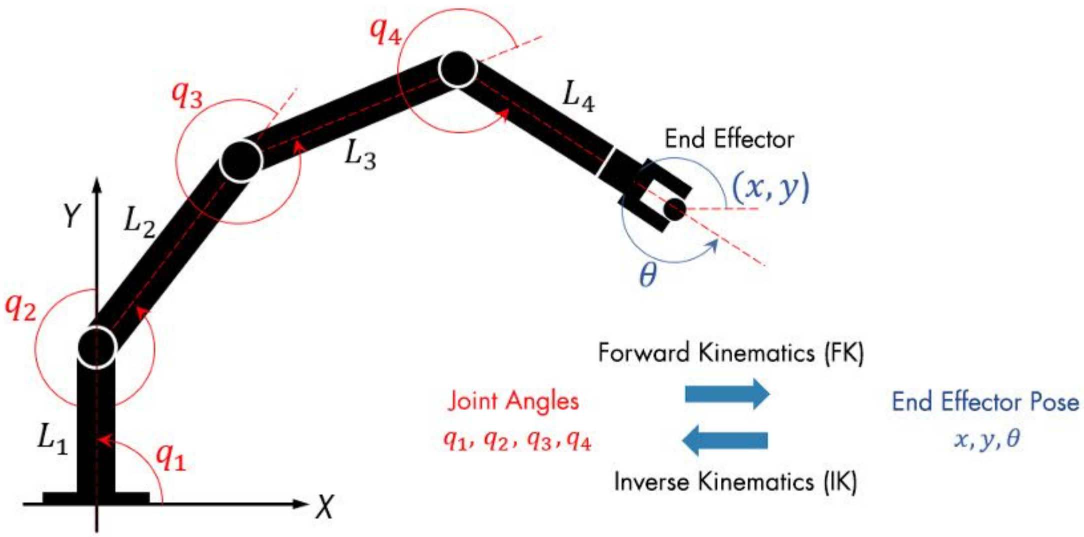
- The derivative $d\varphi_x\colon T_x(M) \longrightarrow T_{\varphi(x)}(N)$ is a *linear map** between the tangent spaces.
- If $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ are *submanifolds* of Euclidean spaces, then the derivative $d\varphi_x$ is just the usual Jacobian $\frac{\partial\varphi}{\partial x}$.



*Linear map is a mapping between two vector spaces that preserves the operation of vector addition and scalar multiplication.

Example

- Calculus on smooth manifolds will be essential in rigid body motion



Lie Groups

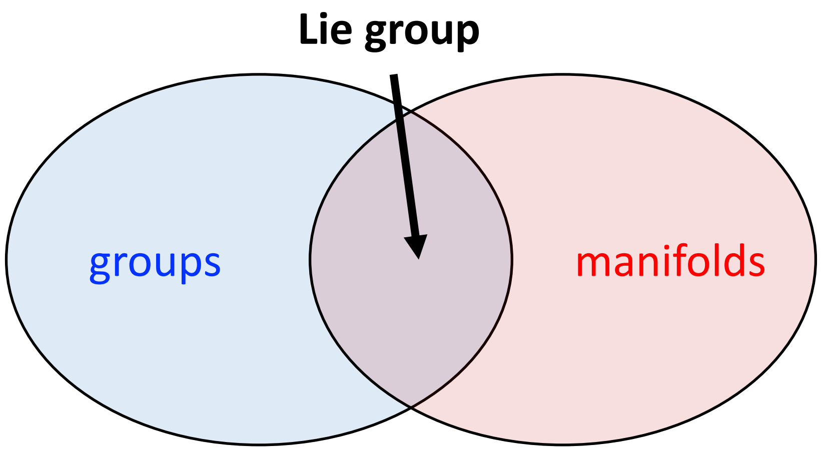
A group G is a set of elements $\{X, Y, Z, \dots\}$ with an operation "$*$" such that:

- Composition stays in the group: $X * Y$ is in G
- Identity element is in the group: $X * E = E * X = X$
- Inverse element is in the group: $X^{-1} * X = X * X^{-1} = E$
- Operation is associative: $X * (Y * Z) = (X * Y) * Z$

- A Lie group is a group that is also a smooth manifold.

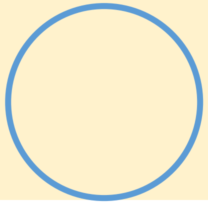
Robotics intuition: This means that no single small change in a joint configuration will cause a large change in the end effector position which means smooth and stable motions.

- SO(d), O(d), SE(d) are matrix Lie groups.



Ex: SO(2)

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$



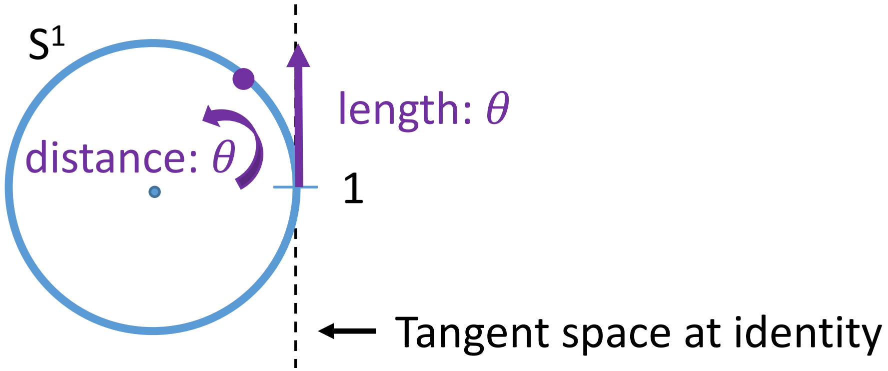
SO(2) [unit circle]

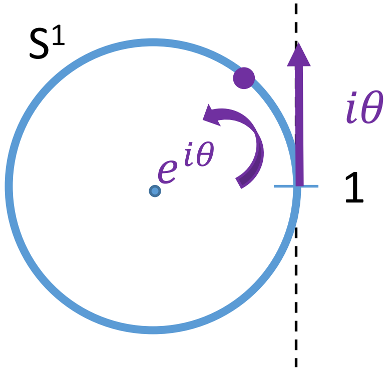
Lie algebra

Computations on curved spaces are harder than the ones on flat spaces.

Curved space: Lie group
Flat space: Lie algebra

- Tangent space $T_e(G)$ at the identity is called Lie Algebra.



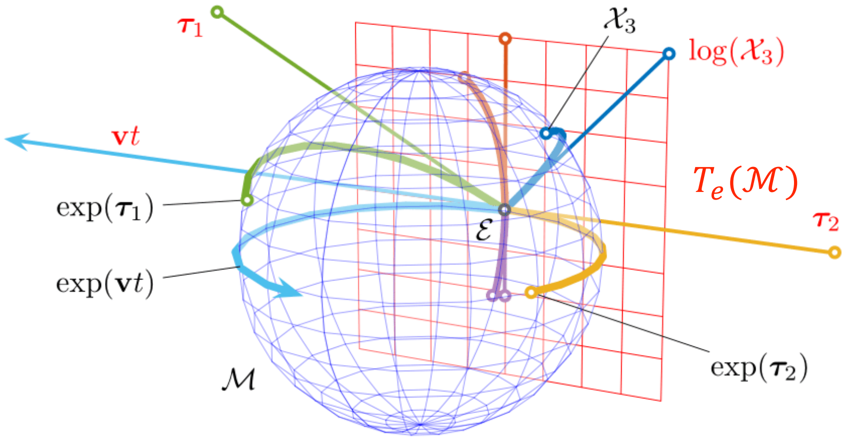


$$i\theta \rightarrow \exp(i\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Lie Algebra and Cartesian Tangent Spaces

Two isomorphic representations of the tangent space

- Lie algebra: $T_e(M)$
- Cartesian: $\mathbb{R}^m$



Example: $SO(3)$

- Lie algebra: $\mathfrak{so}(3)$

$$w = [\omega]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Cartesian: $\mathbb{R}^3$

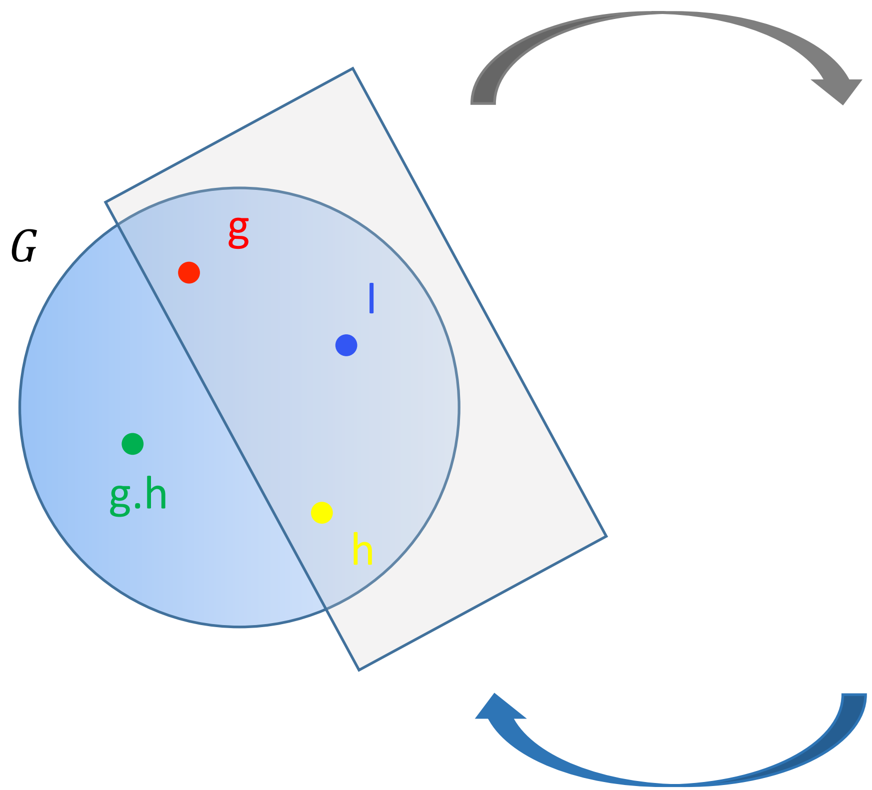
$$\omega = (\omega_x, \omega_y, \omega_z)$$

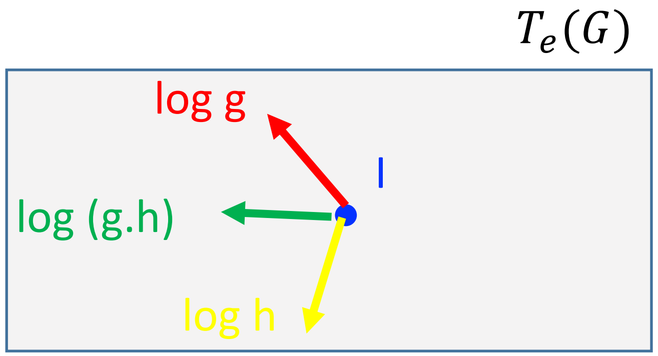
$$[\omega]_\times = \omega_x \mathrm{E_x} + \omega_y \mathrm{E_y} + \omega_z \mathrm{E_z}$$

$$\mathrm{E_x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathrm{E_y} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \mathrm{E_z} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exponential / Logarithm Map

Logarithm map: $G \rightarrow T_e\ (G)$





Exponential map: $T_e(G) \rightarrow G$

- The operations on Lie algebra may not necessarily be simple vector additions.
- Lie brackets

Example (cont.)

Noisy measurements (in radian angle measure):

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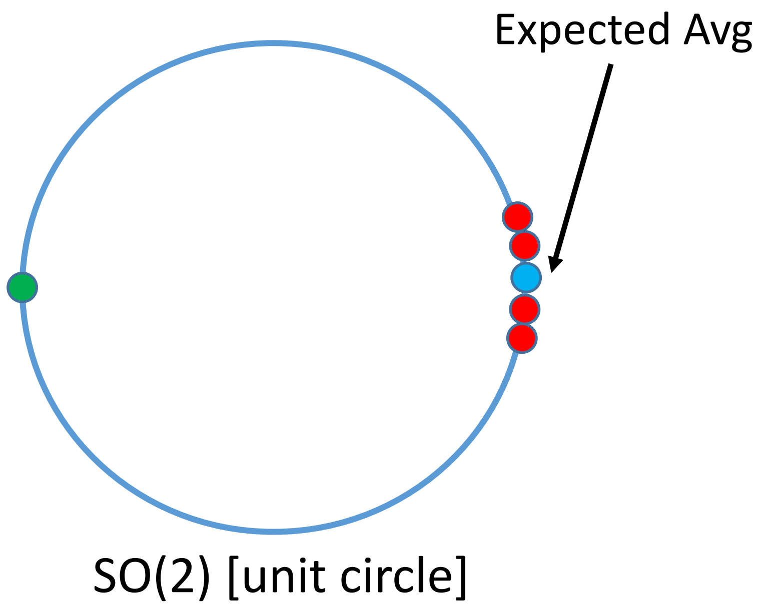


The Lie group SO(2)

$$R(\theta) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \in SO(2)$$

The Lie algebra of SO(2): The tangent space of SO(2) at the identity element.

$$R(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Identity element of SO(2)

Identity element is in the group:

$$X * E = E * X = X$$

Example (cont.)

The Lie algebra of SO(2): The tangent space of SO(2) at the identity element.

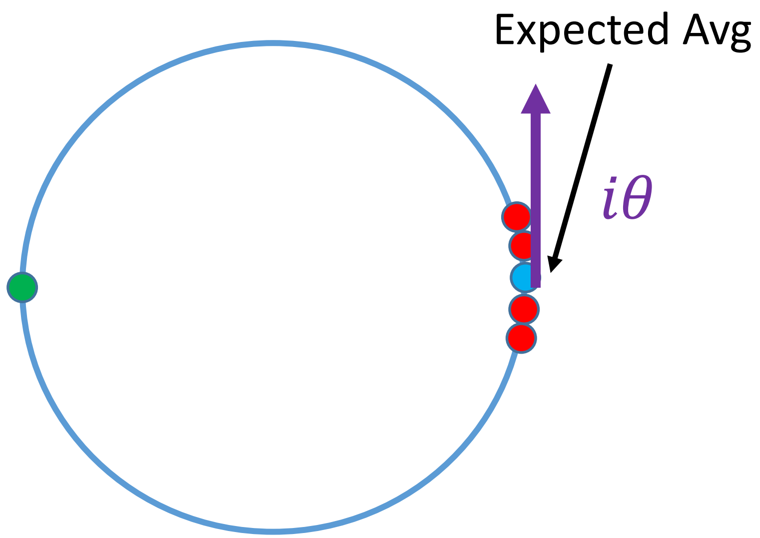
$$R(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Identity element of SO(2)

$$\left.\frac{dR(\theta)}{d\theta}\right|_{\theta=0} = \left.\frac{d}{d\theta}\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}\right|_{\theta=0}$$

$$= \left.\begin{bmatrix} -sin\theta & -cos\theta \\ cos\theta & -sin\theta \end{bmatrix}\right|_{\theta=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Any element in Lie algebra can be written as

$$A = \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$



SO(2) [unit circle]

$$a + ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$0 + i\theta = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

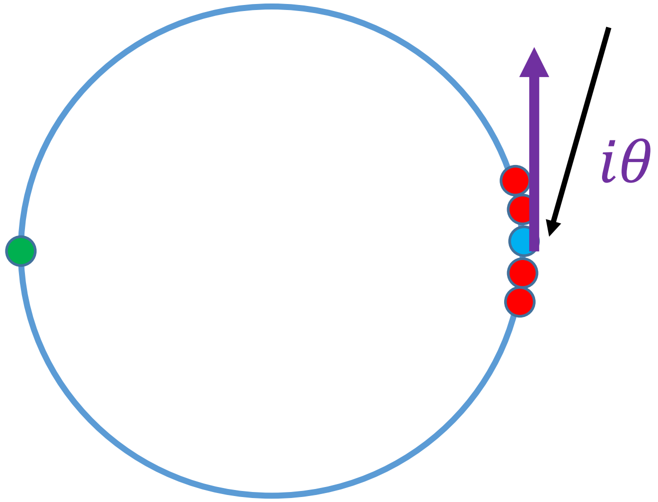
Example (cont.)

The elements of Lie algebra for the noisy measurements are

$$i0.0872, i6.196, i6.021, i0.262$$

Let's find the corresponding group elements:

- $\exp(i0.0872) = e^{i0.0872} = \cos(0.0872) + \mathrm{i}\sin(0.0872) = 0.996 + \mathrm{i}0.087$
- $\exp(i6.196) = e^{i6.196} = \cos(6.196) + \mathrm{i}\sin(6.196) = 0.996 - \mathrm{i}0.087$
- $\exp(i6.021) = e^{i6.021} = \cos(6.021) + \mathrm{i}\sin(6.021) = 0.965 - \mathrm{i}0.259$
- $\exp(i0.262) = e^{i0.262} = \cos(0.262) + \mathrm{i}\sin(0.262) = 0.965 + \mathrm{i}0.259$



Let's find the average of the group elements (averaging complex numbers)

$$0.980 + i0 \quad \rightarrow \exp(i0)$$

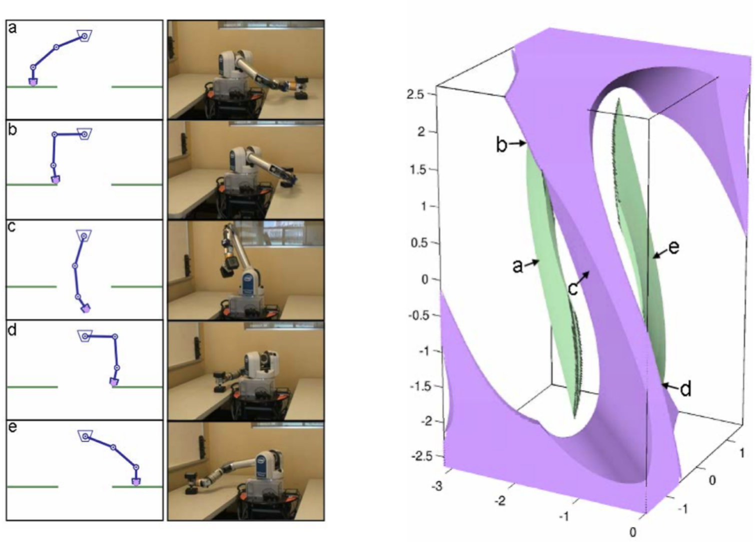
This corresponds to $\theta_{avg} = 0\ rad$

Summary

Manifolds: Since they are *locally* Euclidean, we can still (locally) stick coordinates on them and do calculus.

Lie groups: Manifold + group

- Exp ↔ Log maps : Lie algebra ↔ Lie group
- A unified theory to solve high dimensional problems



Dmitry Berenson et al, "Manipulation planning on constraint manifolds," *ICRA 2009.*