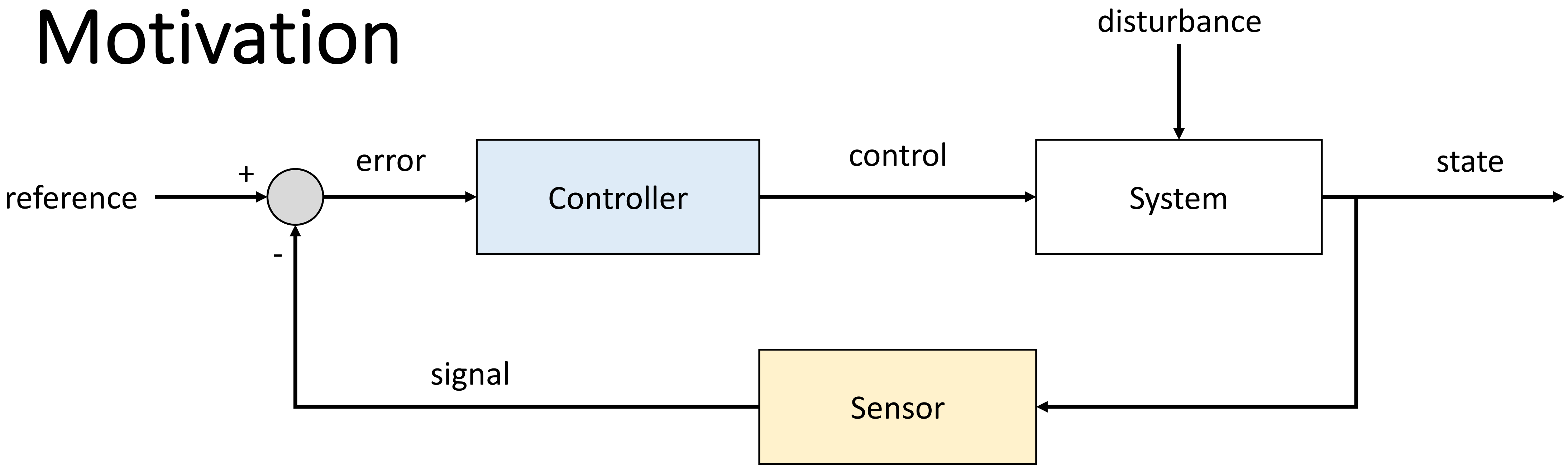
EECE 5550
Mobile Robotics

Lecture 20: Stability Analysis

Derya Aksaray
Assistant Professor
Department of Electrical and Computer Engineering
Northeastern University

Motivation



Recall: We want to devise controllers that will hold a system in a designated state *r*

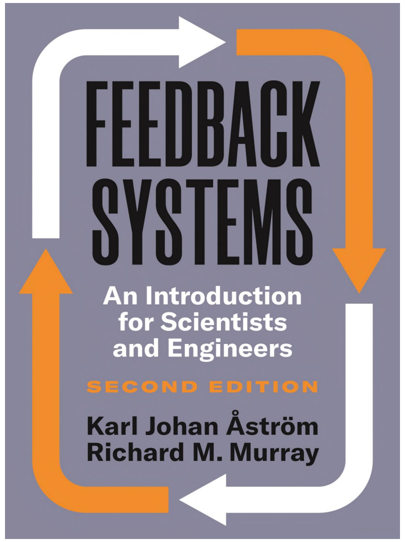
- In particular, we want these to counteract the effects of noise on the system (e.g. wind, etc.)
- **Intuitively:** We want our closed-loop system to be "stable" at *r*

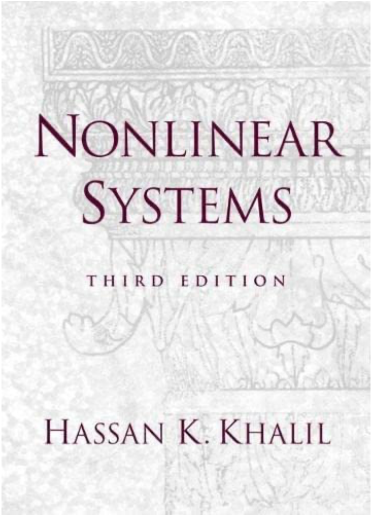
Key question: How can we determine whether a given controller design *achieves* this?

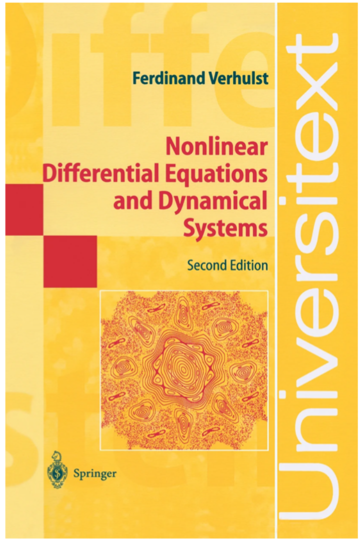
Plan of the day

- Notions of stability
- Stability analysis of linear systems
- Linearized stability analysis
- Lyapunov stability

Some references







Three important notions

Consider a scalar system, $\dot{x} = ax \quad \rightarrow \quad x(t) = e^{at}x(0)$

1. Asymptotically Stable (e.g., $a < 0$)

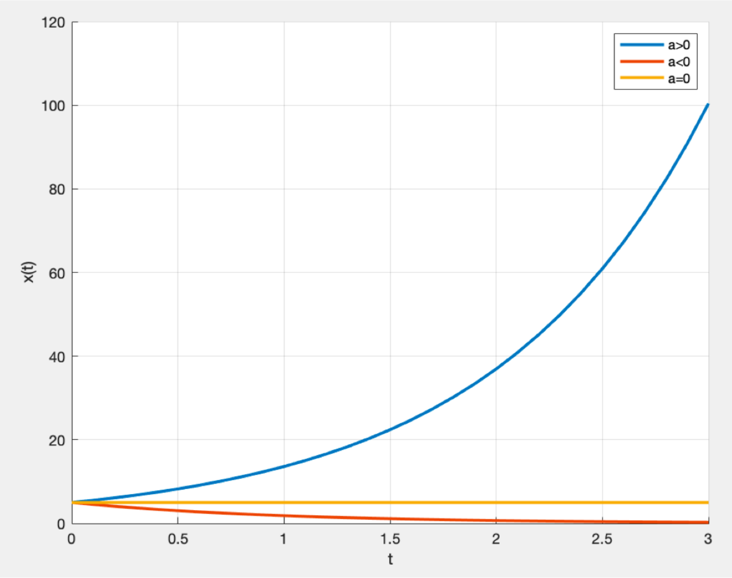
$$x(t) \rightarrow 0, \forall x(0)$$

2. Unstable (e.g., $a > 0$)

$$\exists x(0) \colon x(t) \rightarrow \infty$$

3. Critically Stable (e.g., $a = 0$)

Does not diverge but also does not go to zero



Linear time-invariant systems

Fix $A \in \mathbb{R}^{n \times n}$, and consider the first-order linear time invariant (LTI) system:

$$\dot{x} = Ax.$$

The (unique) trajectory $x(t; x_0)$ of this system determined by the initial condition $x(0) = x_0$ is:

$$x(t; x_0) = \exp(At)\, x_0$$

where:

$$\exp(X) \triangleq \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

is the matrix exponential.

> If everything is scalar,
> $\dot{x} = ax, \quad x(0) = x_0 \quad \rightarrow \quad x(t) = e^{at} x_0$
>
> For higher-order systems:
> $\dot{x} = Ax, \quad x(0) = x_0 \quad \rightarrow \quad x(t) = e^{At} x_0$

Key fact: The eigenvalues of *A* completely characterize the stability of the stationary point x = 0!

Matrix exponentials

- The definition is similar to the scalar exponential:

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}, \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

- Let's take the derivative:

$$\frac{d}{dt}\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = 0 + \sum_{k=1}^{\infty} \frac{kA^k t^{k-1}}{k!} = A\sum_{k=1}^{\infty} \frac{A^{k-1} t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty} \frac{A^k t^k}{(k)!}$$

$$\boxed{\frac{d}{dt}e^{At} = Ae^{At}}$$

Special case: Symmetric coefficient matrix

Suppose that A is symmetric.
Then A admits a symmetric eigendecomposition of the form:

$$A = Q\Lambda Q^T$$

where Λ is diagonal and Q is orthogonal. Our differential equation can thus be written:

$$\dot{x} = Q\Lambda Q^T x.$$

Let's define $y \triangleq Q^T x$.
Then $\dot{y} = Q^T \dot{x}$, so (premultiplying by Q^T) the above differential equation is equivalent to:

$$\dot{y} = \Lambda y.$$

But now notice that since Λ is diagonal, its exponential is easy to compute:

$$\exp\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right) = \begin{pmatrix} \exp(\lambda_1) & & \\ & \ddots & \\ & & \exp(\lambda_n) \end{pmatrix}$$

If Λ is arbitrary, its exponential cannot be calculated by taking the exponential of its elements!

Special case: Symmetric coefficient matrix (cont'd)

It follows that the trajectory $y(t; y_0)$ determined by the initial data $y(0) = y_0$ is:

$$y(t; y_0) = \exp(\Lambda t)\, y_0 = \begin{pmatrix} \exp(t\lambda_1) & & \\ & \ddots & \\ & & \exp(t\lambda_n) \end{pmatrix} \begin{pmatrix} y_{0,1} \\ \vdots \\ y_{0,n} \end{pmatrix}$$

or equivalently:

$$y(t; y_0) = \sum_{k=1}^{n} y_{0,k} e^{t\lambda_k}$$

Key observations:

- The system trajectory $y(t; y_0)$ is a *superposition of n independent modes* in 1:1 correspondence with the eigenpairs (λ_k, q_k) of *A*.
- From this, we can immediately determine the stability of the origin:
  - Unstable if $\lambda_k > 0$ for *any* *k*.
  - Stable if $\lambda_k \leq 0$ for all *k*.
  - Asymptotically stable if $\lambda_k < 0$ for all *k*.

Note: Symmetric matrices have real eigenvalues.

General case

Now suppose only that $A \in \mathbb{R}^{n \times n}$ (not necessarily symmetric). Then *A* may not have a complete basis of eigenvectors (equivalently: *A* may not be diagonalizable).

However, it *is* always possible to bring *A* into an "almost" diagonal form called the *Jordan canonical form*:

$$A = TJT^{-1}$$

where $T \in GL(n)$ is a similarity transformation (not necessarily orthogonal), and *J* is a block-diagonal matrix:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_K \end{pmatrix}$$

consisting of *Jordan blocks*:

$$J_k \triangleq \begin{pmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{pmatrix}$$

General case (cont'd)

Since J is block-diagonal, its matrix exponential is also (relatively) easy to compute:

$$\exp(tJ) = \begin{pmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_K) \end{pmatrix}$$

where:

$$\exp(tJ_k) = \begin{pmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \frac{t^2 e^{\lambda_k t}}{2} & \cdots & \frac{t^{d_k-1} e^{\lambda_k t}}{(d_k-1)!} \\ 0 & e^{\lambda_k t} & te^{\lambda_k t} & & \frac{t^{d_k-2} e^{\lambda_k t}}{(d_k-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_k t} & te^{\lambda_k t} \\ 0 & \cdots & 0 & 0 & e^{\lambda_k t} \end{pmatrix}.$$

General case (cont'd)

Therefore, defining $y \triangleq T^{-1}x$, we can apply the same argument as in the symmetric case to conclude that the trajectory $y(t; y_0)$ determined by the initial data $y(0) = y_0$ is:

$$y(t; y_0) = \sum_{k=1}^{K} \exp(tJ_k)\, y_{0,k}$$

where here $y_{0,k}$ is the d_k-dimensional subvector of y_0 associated with the kth Jordan block J_k.

Key observation: Once again, the fact that the matrix exponential of a Jordan block has a simple closed-form enables us to determine the stability of the origin *y* = 0:

- Unstable if $\Re(\lambda_k) > 0$ for *any* *k*
- Asymptotically stable if $\Re(\lambda_k) < 0$ for *all* *k*.

$$\exp(tJ_k) = \begin{pmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \frac{t^2 e^{\lambda_k t}}{2} & \cdots & \frac{t^{d_k-1} e^{\lambda_k t}}{(d_k-1)!} \\ 0 & e^{\lambda_k t} & te^{\lambda_k t} & & \frac{t^{d_k-2} e^{\lambda_k t}}{(d_k-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_k t} & te^{\lambda_k t} \\ 0 & \cdots & 0 & 0 & e^{\lambda_k t} \end{pmatrix}$$

Linearized stability analysis

Recap: The eigenvalues of the coefficient matrix $A \in \mathbb{R}^{n\times n}$ completely determine the (*global*) stability properties of the stationary/equilibrium point $x^* = 0$ of the LTI system:

$$\dot{x} = Ax$$

But: What about *nonlinear* systems?

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz-continuous function, and suppose that $x^* \in \mathbb{R}^n$ is a stationary (equilibrium) point of the nonlinear ODE:

$$\dot{x} = f(x)$$

A continuous function limited in how fast it can change
$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$

Main idea: We can understand the *local* stability of this system in a neighborhood of x^* by studying the *linearization* of f at x^*.

Linearization of nonlinear ODEs

Let $x^* \in \mathbb{R}^n$ be a stationary point of the nonlinear ODE $\dot{x} = f(x)$, and consider the first-order Taylor series expansion of f at x^*:

$$f(x) = f(x^*) + \frac{\partial f}{\partial x}(x^*)(x - x^*) + O(\|x - x^*\|^2)$$

$$= \frac{\partial f}{\partial x}(x^*)(x - x^*) + O(\|x - x^*\|^2)$$

Now let $h \triangleq x - x^*$ (so that h is the *deviation* of the point x from the fixed point x^*).
Then $\dot{h} = \dot{x}$, and the above equation shows that:

$$\dot{h} = \frac{\partial f}{\partial x}(x^*)h + O(\|h\|^2)$$

Therefore, we can *approximate* the behavior of the system $\dot{x} = f(x)$ in a neighborhood of x^* using the *linearization*:

$$\dot{h} = Ah$$

for $A \triangleq \frac{\partial f}{\partial x}(x^*)$.

Stability analysis via local linearization

In fact, the stability of the *linearization* $\dot{h} = Ah$ actually provides information about the stability of the fixed point x^* for the *original nonlinear* system $\dot{x} = f(x)$.

Theorem (Poincaré-Lyapunov)

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuously-differentiable, and $x^* \in \mathbb{R}^n$ a stationary point of the first-order nonlinear ODE:

$$\dot{x} = f(x)$$

Define $A \triangleq \frac{\partial f}{\partial x}(x^*)$, and suppose that $\Re(\lambda_k) < 0$ for all eigenvalues λ_k of A. Then the fixed point x^* is asymptotically stable.

Punchline: The Poincaré-Lyapunov Theorem enables us to show that stationary points of a *nonlinear* ODE are asymptotically stable by studying its *linearization*.

$$ml\ddot{\theta} = -b\dot{\theta} - mg\sin(\theta)$$

Damped pendulum redux

Recall the dynamics for the damped pendulum:

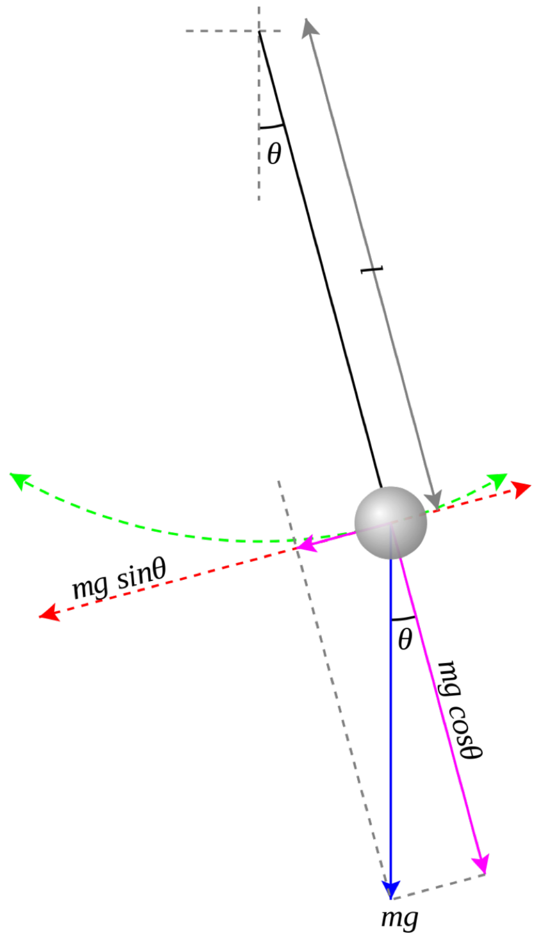
$$\ddot{\theta} = -\frac{\mu}{m}\dot{\theta} - \frac{g}{l}\sin(\theta)$$

Defining $x = (\theta, \dot{\theta}) \in \mathbb{R}^2$, we can express this system as the equivalent *nonlinear first-order* ODE:

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{m}x_2 \end{pmatrix} \triangleq f(x)$$

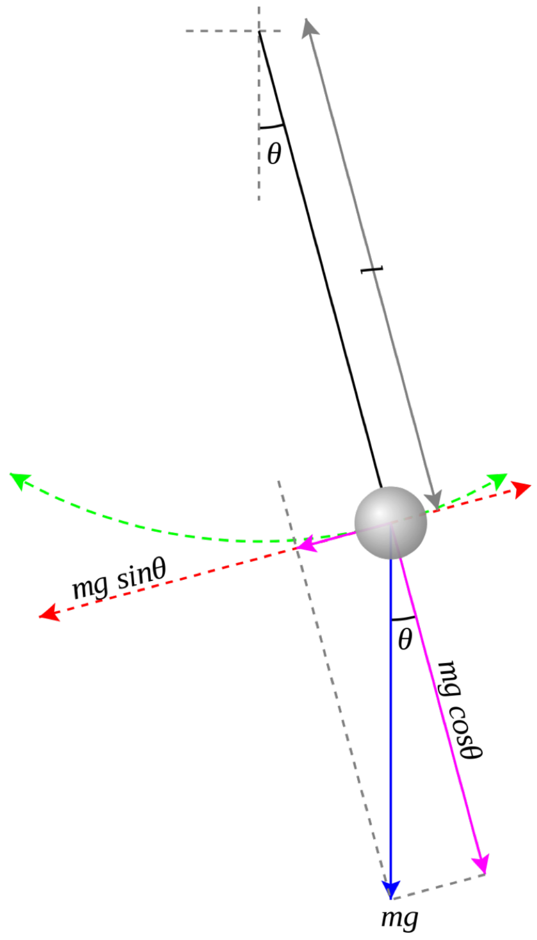
Note that $x = (0,0)$ is a stationary point of this system.

Exercise: Prove that this stationary point is asymptotically stable for $\mu > 0$.



Damped pendulum redux

$$\ddot{\theta} = -\frac{\mu}{m}\dot{\theta} - \frac{g}{l}\sin(\theta)$$



Define $x = (\theta, \dot{\theta}) \in \mathbb{R}^2$.
Then:

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{m}x_2 \end{pmatrix} \triangleq f(x)$$

Linearizing *f* at $x^* = (0,0)$, we obtain:

$$A \triangleq \frac{\partial f}{\partial x}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\mu}{m} \end{pmatrix}.$$

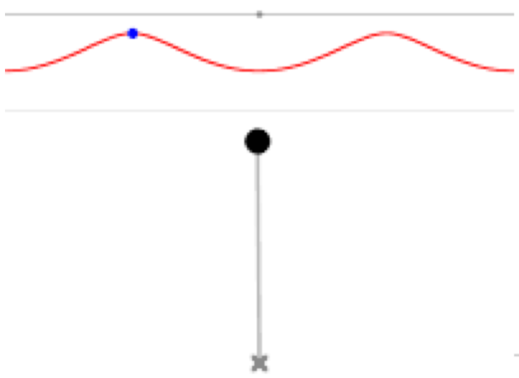
Eigenvalues of *A* are:

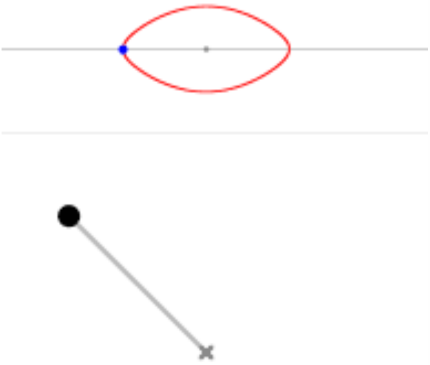
$$\lambda = -\frac{\mu}{m} \pm \sqrt{\frac{\mu^2}{m^2} - 4\frac{g}{l}}$$

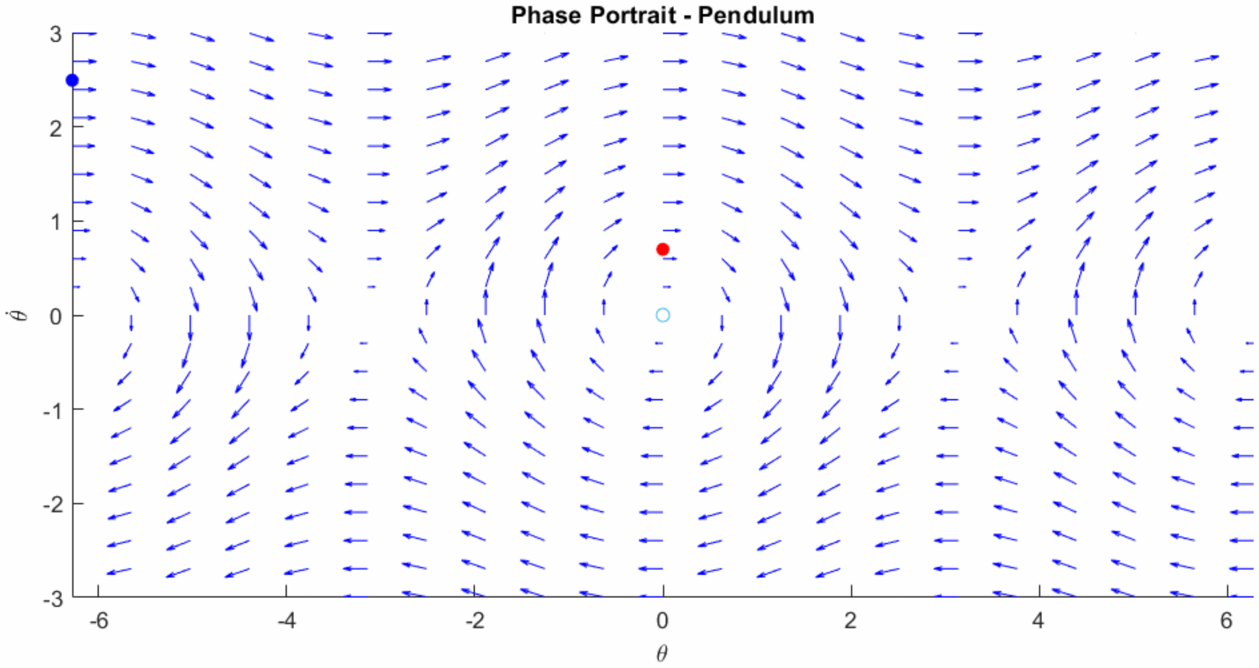
Therefore we conclude $x^* = (0,0)$ is asymptotically stable if $\mu > 0$.

Phase portrait of a pendulum

https://ocw.mit.edu/courses/18-03sc-differential-equations-fall-2011/pages/unit-iv-first-order-systems/qualitative-behavior-phase-portraits/

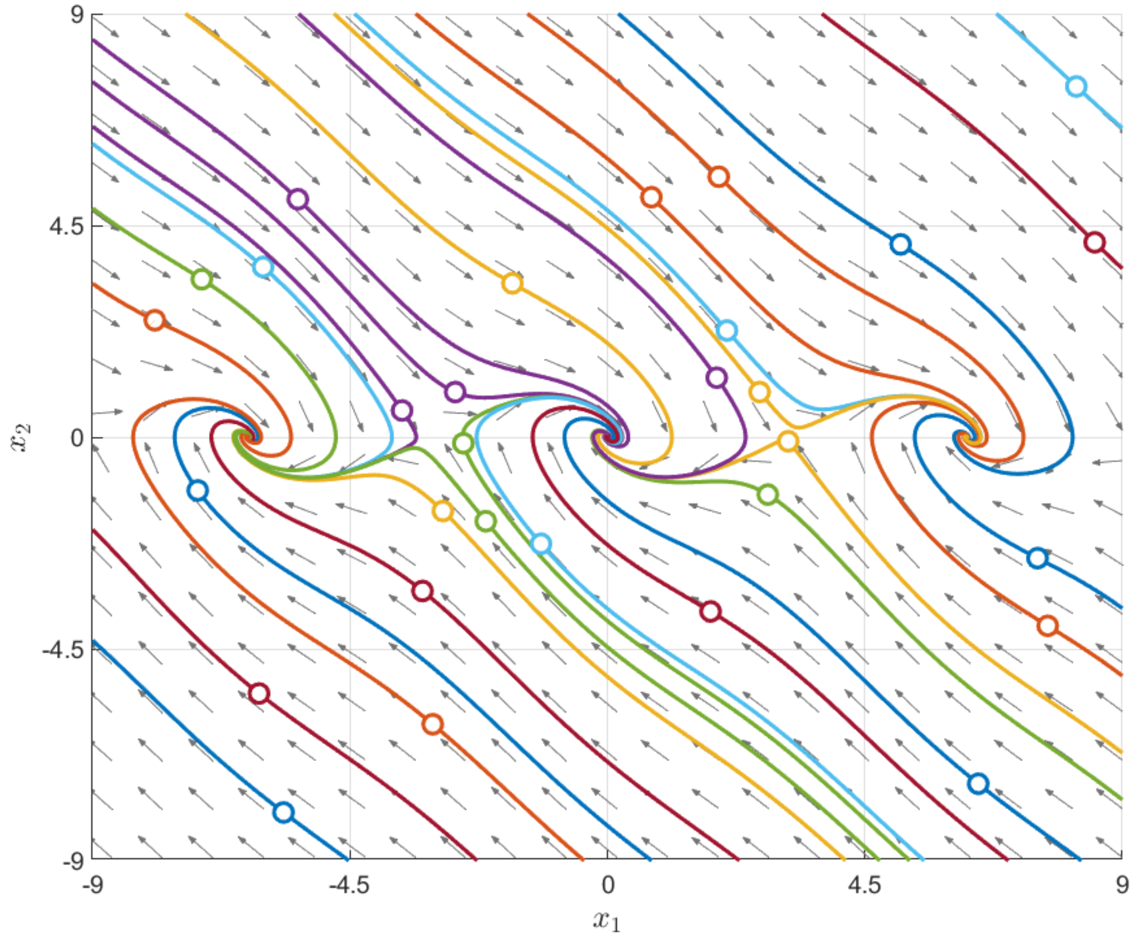




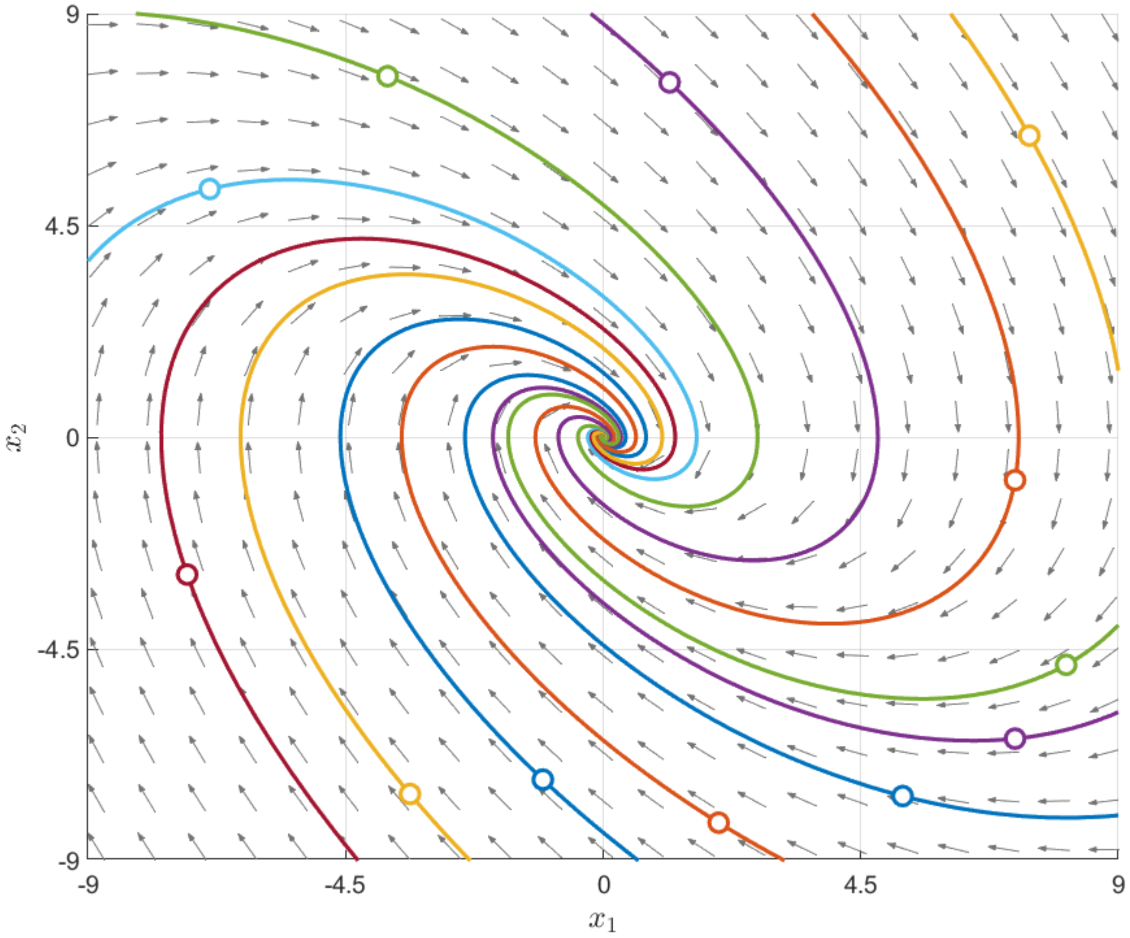


Picture of all potential dynamical behaviors

Damped pendulum redux

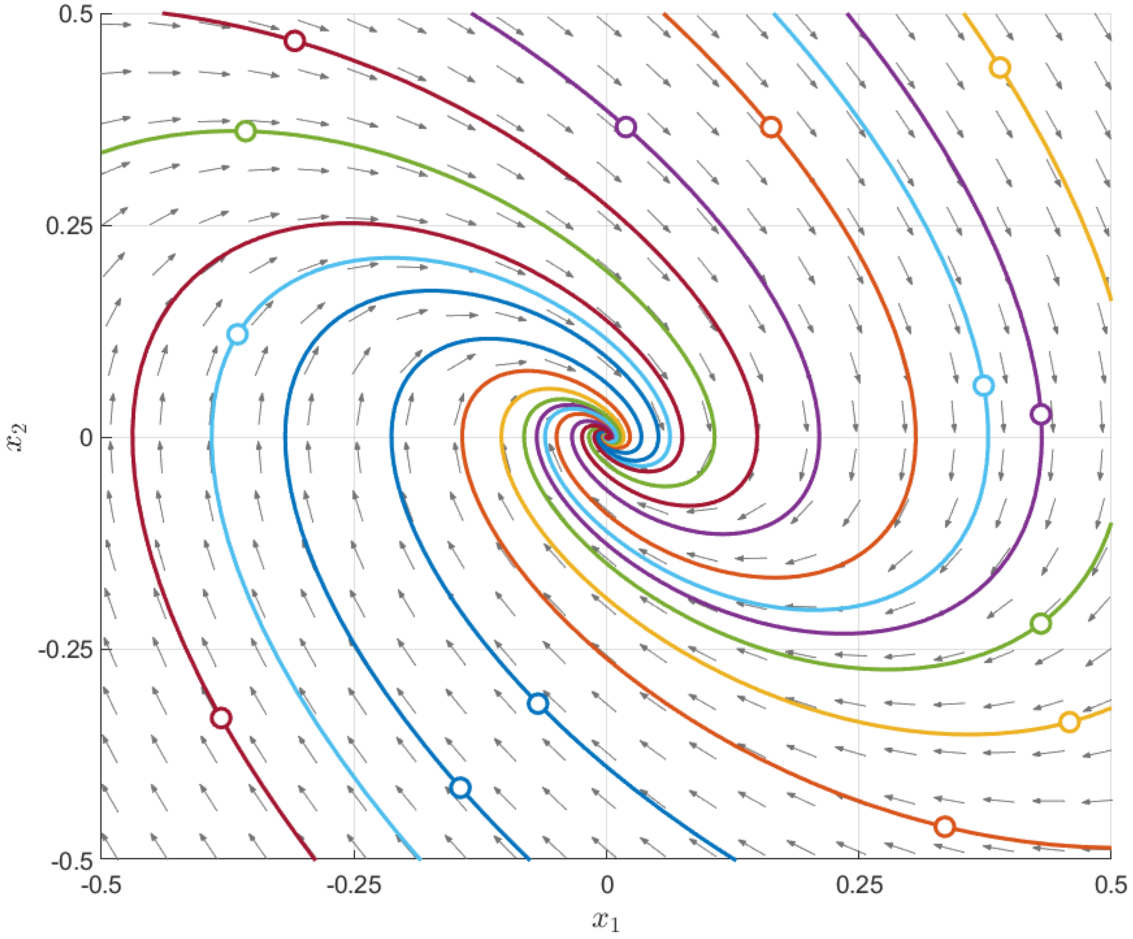


Nonlinear system $\dot{x} = f(x)$

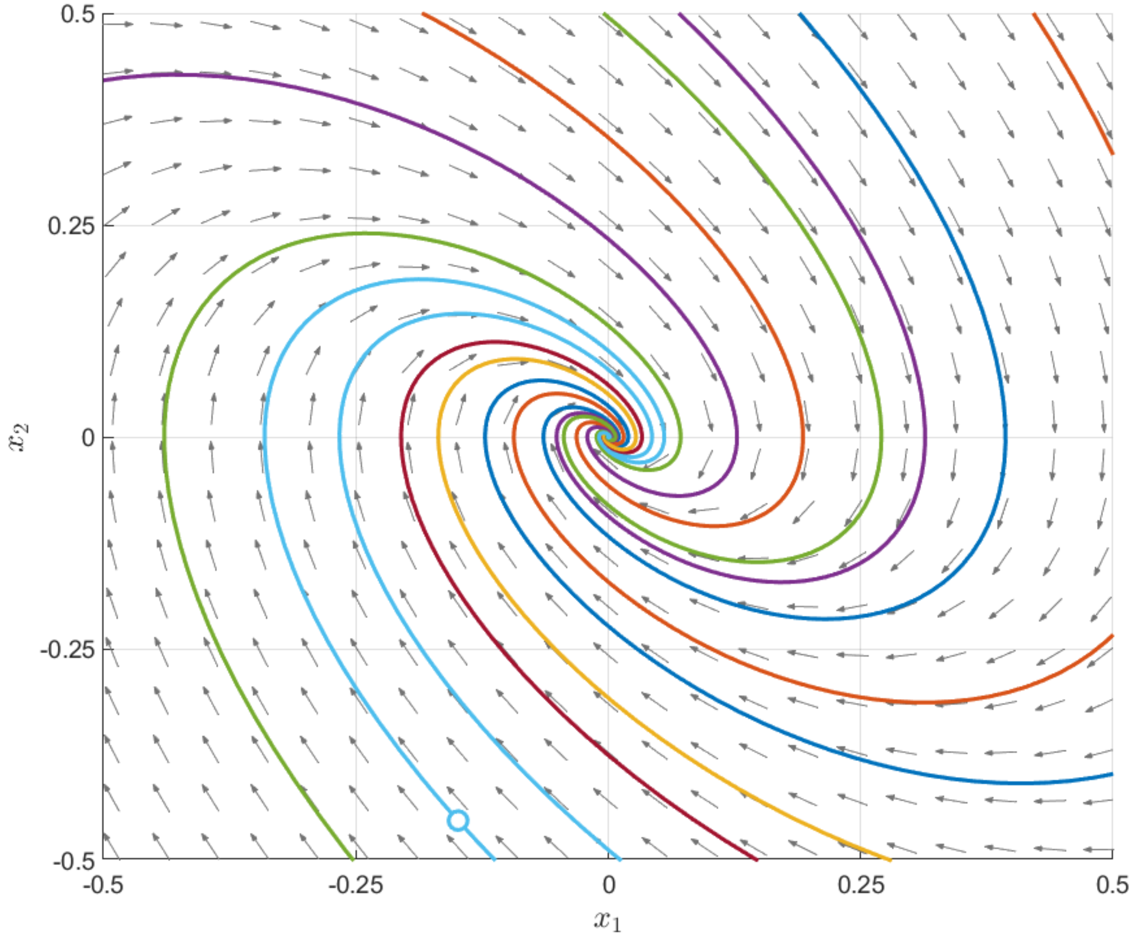


Linearization $h = Ah$

Damped pendulum redux



Nonlinear system $\dot{x} = f(x)$



Linearization $\dot{h} = Ah$

Beyond local stability analysis

Recap: We can analyze the *local* stability of a stationary point x^* of the nonlinear ODE $\dot{x} = f(x)$ by studying its *linearization* $\dot{h} = \frac{\partial f}{\partial x}(x^*)h$ about x^*.

⇒ This can tell us that points starting sufficiently close to x^* will converge to x^*.

But: How close is "sufficiently close"??

Problem: Linearized stability analysis is based on the *derivative* of f. This only tells us information about f's behavior in an *infinitesimally small neighborhood* of x^*.

⇒ We can't conclude anything about *how large* the region of attraction around x^* is.

Moreover, if the linearized model has eigenvalues that do not have a positive real part but zero real part, the linearized model's stability analysis does not conclude anything about the nonlinear system.

Therefore: We're going to need some more powerful tools …

Lyapunov stability

Consider an autonomous (time-invariant) system:

$$\dot{x} = f(x) \tag{1}$$

where $f: D \to \mathbb{R}^n$ be locally Lipschitz map from a domain $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$ and $x^* \in D$ be an equilibrium point (i.e., $f(x^*)$=0).

Typically, we analyze systems when the equilibrium point is at the origin of $\mathbb{R}^n$ (i.e., $x^* = 0$).

W.L.O.G, any equilibrium can be shifted to the origin by change of variables. Suppose $x^* \neq 0$ and consider $y = x - x^*$. Then,

$$\dot{y} = \dot{x} = f(x) = f(y + x^*) \stackrel{\text{def}}{=} g(y), \qquad \text{where } g(0) = 0$$

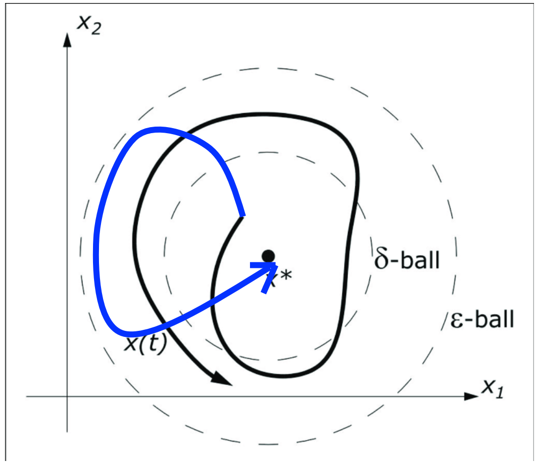
Def: The equilibrium point $x = 0$ of (1) is

- Stable (in the sense of Lyapunov) if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

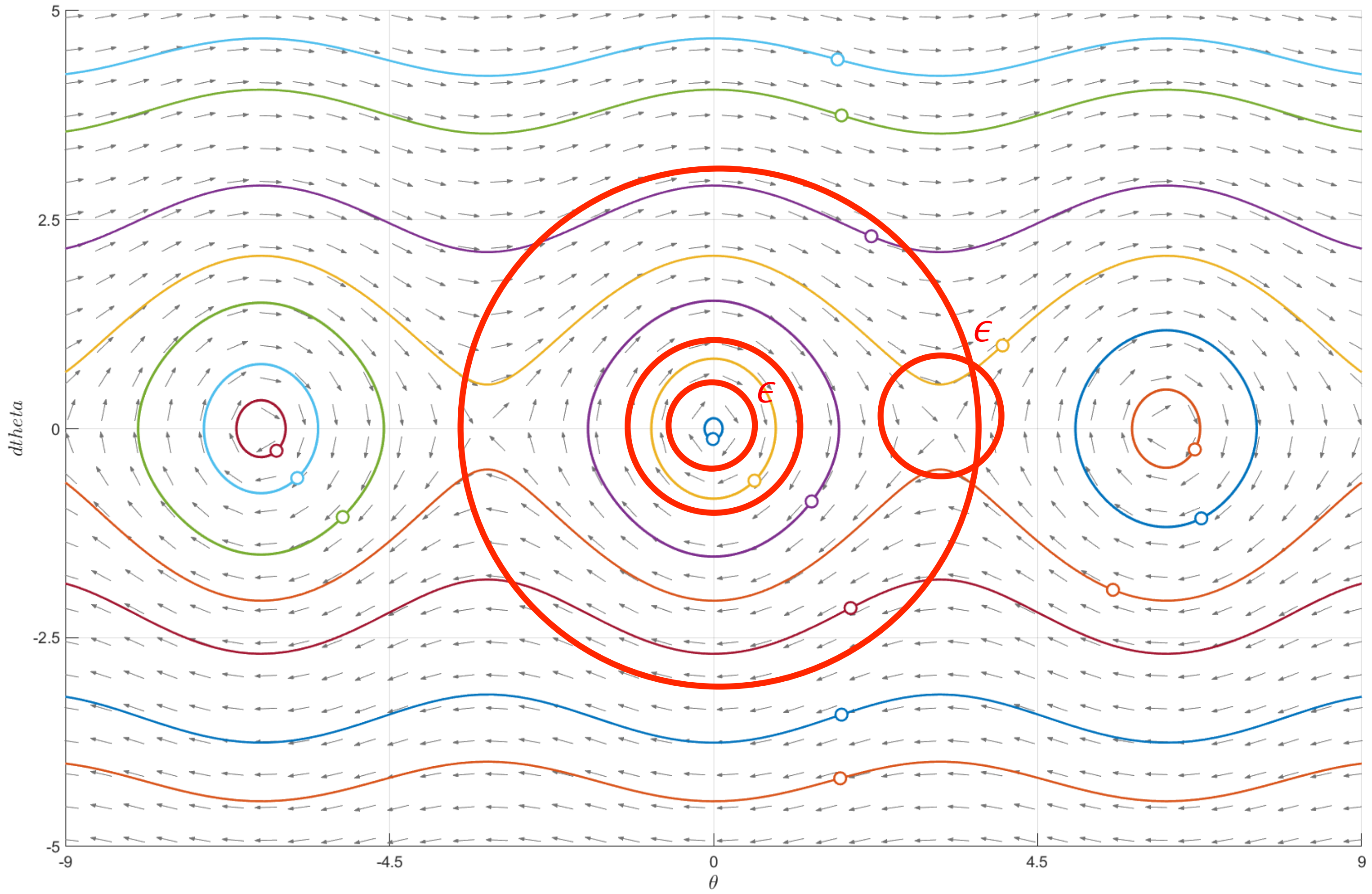
$$\|x(0)\| < \delta \quad \to \quad \|x(t)\| < \epsilon \qquad \text{for all } t \geq 0.$$

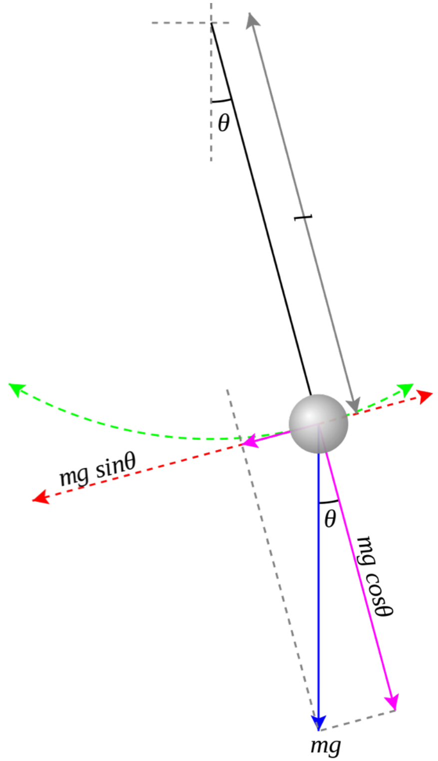
- Unstable if it is not stable
- Asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \quad \to \quad \lim_{t\to\infty} \|x(t)\| = 0$$



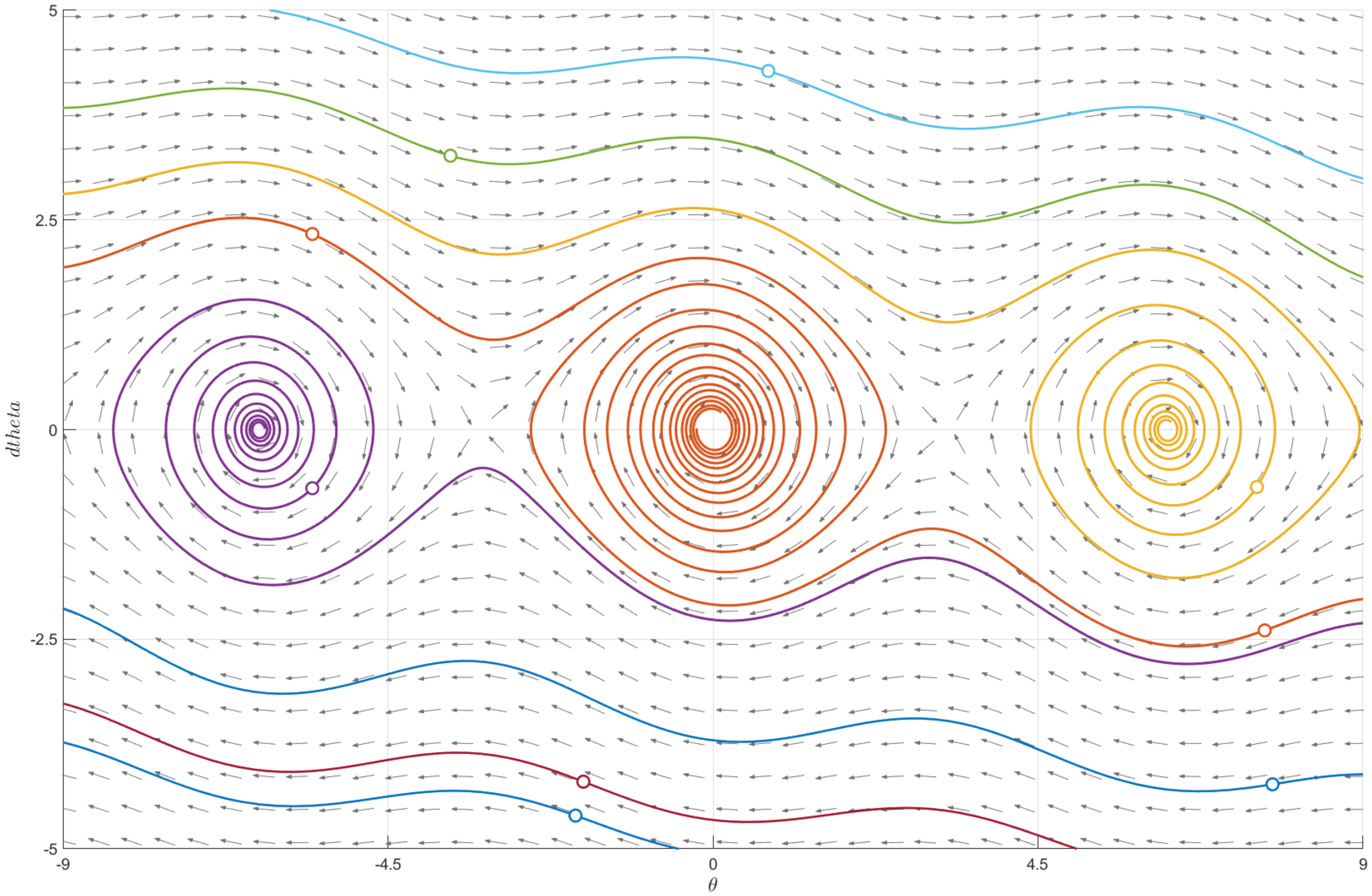
Example: Simple pendulum (frictionless)

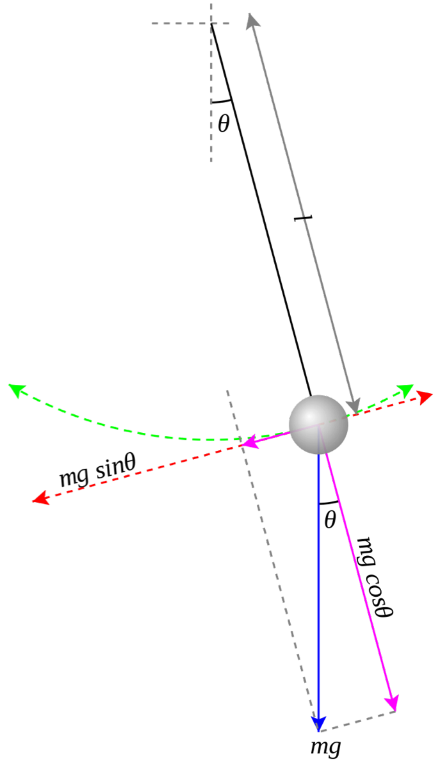




$$\ddot{\theta} = -\frac{g}{l}\sin(\theta)$$

Example: Damped pendulum





$$\ddot{\theta} = -\frac{\mu}{m}\dot{\theta} - \frac{g}{l}\sin(\theta)$$

Lyapunov's Direct Method

Lyapunov's direct method is an approach to stability analysis that works directly with a nonlinear ODE $\dot{x} = f(x)$; it does *not* employ any linearization or approximations.

Let $x = 0$ be an equilibrium point and $D \subseteq \mathbb{R}^n$ be a domain containing $x = 0$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that:

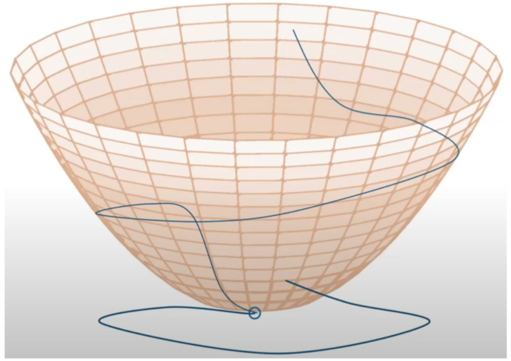
- $V(0) = 0$.
- $V(x) > 0$ if and only if $x \neq 0$.
- The function $V(x)$ is *nonincreasing* along the trajectories of the system:

$$\dot{V}(x) = \frac{d}{dt}[V(x(t))] = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x) \leq 0 \quad \text{in } D$$

Then $x = 0$ is stable in the sense of Lyapunov. Moreover, if

$$\dot{V}(x) = \nabla V(x) \cdot f(x) < 0 \qquad \text{in } D \backslash \{0\}$$

then $x = 0$ is asymptotically stable.



Analyze stability without solving the ODE.

Lyapunov functions

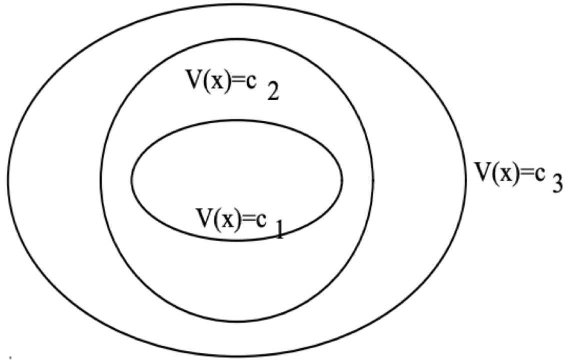
The function $V: D \to \mathbb{R}$ is for $D \subseteq \mathbb{R}^n$ called a *Lyapunov function* for f.

If the Lyapunov function is also radially unbounded $V(x) \to \infty$ as $\|x\| \to \infty$, the previous conditions can help to show global stability.

Main idea: A Lyapunov function *V* can be thought of as measuring an (abstract) notion of "energy" in the dynamical system $\dot{x} = f(x)$.

The conditions on *V* can then be interpreted as:

- The energy of the system is always nonnegative;
- The energy is zero *only* at the equilibrium point $x = 0$;
- The energy is *nonincreasing* along the trajectories of the system.



Main takeaways:

- Lyapunov functions are great! They permit direct analysis of nonlinear systems.
- But: No general technique for producing these – they can be hard to find for a particular system *f*.

Example

Suppose $x \in \mathbb{R}$

$$\dot{x} = -x^3$$

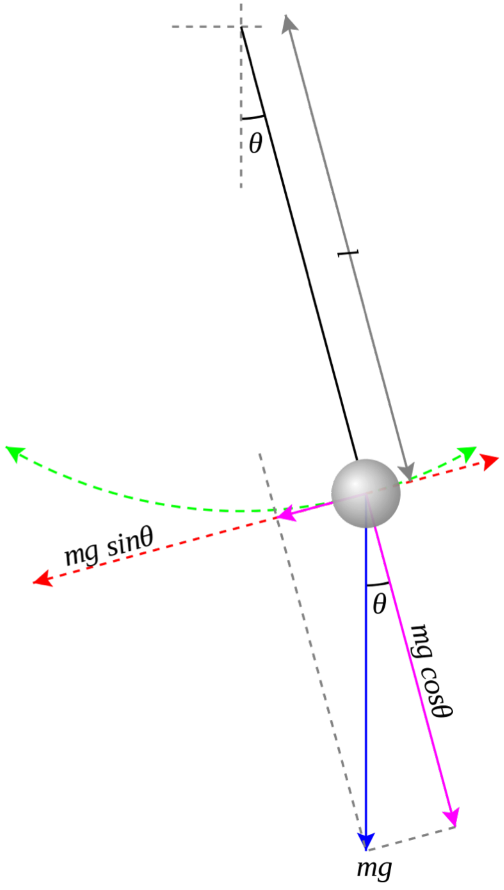
What is the equilibrium point?

Select $V(x) = x^2$ (check Lyapunov candidate criteria)

$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = -2x^4 < 0$ for $x \neq 0$

Damped pendulum (reprise)

$$\ddot{\theta} = -\frac{\mu}{m}\dot{\theta} - \frac{g}{l}\sin(\theta)$$



Recall the first-order ODE description for the damped pendulum:

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{m}x_2 \end{pmatrix} \triangleq f(x)$$

How might we build a Lyapunov function for this system?

Idea: Since Lyapunov functions are meant to model the "energy" of a system, let's try the (physical) energy of this mechanical system:

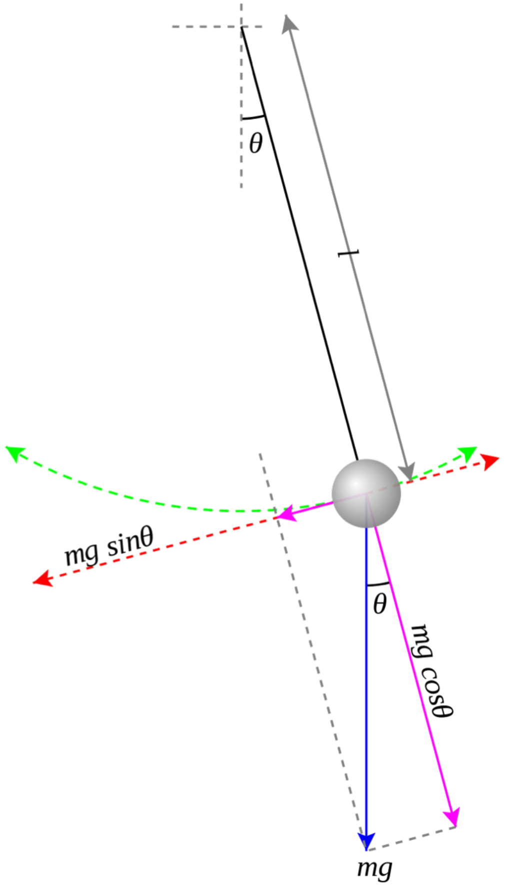
$$V(x) = \underbrace{\frac{1}{2}ml^2x_2^2}_{\text{Kinetic}} + \underbrace{mgl(1-\cos(x_1))}_{\text{Gravitational potential}}$$

Damped pendulum (reprise)

$$\ddot{\theta} = -\frac{\mu}{m}\dot{\theta} - \frac{g}{l}\sin(\theta)$$

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{m}x_2 \end{pmatrix} \triangleq f(x)$$

$$V(x) = \frac{1}{2}ml^2x_2^2 + mgl(1 - \cos(x_1))$$



For this choice of *V*:

- $V(x) \geq 0$.
- $V(x) = 0$ if and only if $x = (2\pi k, 0)$.
- $\nabla V(x) = (mgl\sin(x_1),\ ml^2x_2)$, and therefore:

$$\nabla V(x) \cdot f(x) = -\mu l^2 x_2^2 \leq 0$$

Damped pendulum (reprise)

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{m}x_2 \end{pmatrix} \qquad V(x) = \frac{1}{2}ml^2x_2^2 + mgl(1-\cos(x_1))$$