Here, $h_{ij}(u_i)$ represents the j-th constraint function defining the i-th uncertainty set $\mathcal{U}_i$, and K_i denotes the total number of constraints that define the uncertainty set $\mathcal{U}_i$.

Remark 1. *Formulation (3) generalizes the standard form (2) by allowing uncertainty in the objective function itself and by explicitly representing uncertainty sets through inequality constraints, which facilitates the min-max-max-min structure needed for our dynamical system approach. Under the convexity assumptions stated below, the uncertainty sets $\mathcal{U}_i$ remain compact. Note that our framework accommodates nonlinear uncertainty constraints $h_{ij}(u_i) \leq 0$ through the gradient terms $\nabla_{u_i} h_i(u_i)$ appearing in the dynamics.*

The problem in (3) reduces to the one in (2) when $\mathcal{U}_0$ is a singleton. Following [30] and without loss of generality, we consider the $\mathcal{RO}$ problem dealing with constraint-wise uncertainties where each constraint f_i is only a function of u_i. The functions f_i and h_{ij} have scalar values with the following assumptions.

Assumption 1. *The functions $h_{ij}(u_i)$ are convex in u_i for $i \in [N]$ and $j \in [K_i]^+$. The function $f_0(x, u_0)$ is strictly convex in x for any $u_0 \in \mathcal{U}_0$, and concave in u_0 for any x. Also, for $i \in [N]^+$, $f_i(x, u_i)$ is convex in x for fixed u_i and concave in $u_i \in \mathcal{U}_i$, for fixed x. Finally, the functions f_i and h_i $i \in [N]$ are C^1 with local Lipschitz gradients.*

Remark 2. *Assumption 1 requires convexity in the decision variable and concavity in the uncertainty variable. These conditions hold for most practical $\mathcal{RO}$ problems, particularly when uncertainty enters affinely.*

Assumption 2. *Existence of optimal solutions and strong duality.*

1) *$\mathcal{RO}$ problem (4) is feasible. An optimal min-max solution $(x^\star, u^\star)$ exists and μ in the $\mathcal{RO}$ problem (4) is finite.*
2) *$\mathcal{RO}$ problem satisfies a regularity condition such as Slater condition, for both the upper level optimization and the lower level maximization problems [30], that is, for all $i \in [N]$ and all $j \in [K_i]^+$, there exist $u_i \in \mathcal{U}_i$ such that $h_{ij}(u_i) < 0,$ and for $i \in [N]^+$, there exists $x \in \mathbb{R}^n$ such that, $\mathcal{F}_i(x) < 0$.*

Remark 3. *The uncertainty is often parametrized affinely in $\mathcal{RO}$ problems, which automatically satisfies the concavity property of the f_i functions.*

Regularized Formulation

Finally, we introduce the problem we consider in this paper, a slight generalization of $\mathcal{RO}_0$ in (3).

$$\mathcal{RO}\begin{cases} \mu := \min_x \max_{u_0 \in \mathcal{U}_0} f_0(x, u_0) \\ \quad + \sum_{i=1}^N c_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \\ \\ \text{s.t.} \ \ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leq 0 \ , \ i \in [N]^+, \\ \\ \mathcal{U}_i := \{u_i \in \mathbb{R}^{m_i} \ : \ h_{ij}(u_i) \leq 0, \\ \qquad j \in [K_i]\} \ , \ i \in [N] \end{cases} \tag{4}$$

where $c_i \geq 0$ for $i \in [N]^+$. This setting can be seen as an elementary regularized version of more common formulations $\mathcal{RO}_0$ obtained for $c_i = 0$, $i \in [N]^+$. The role of c_i is further clarified below.

It is convenient to rewrite $\mathcal{RO}$ in the following form

$$\begin{aligned} \mu := \min_x \ & \mathcal{F}_0(x) + \sum_{i=1}^N c_i \mathcal{F}_i(x) \\ \text{s.t.} \ \ & \mathcal{F}_i(x) \leq 0 \ , \ i \in [N]^+ \ , \end{aligned} \tag{5}$$

where

$$\mathcal{F}_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \ , \ i \in [N] \ . \tag{6}$$

In this paper, we often call (6) the "lower optimization problems", and the minimization in (5) as the "upper optimization".

Remark 4. *Assumption [A2] guarantees that $\mathcal{RO}$ problem enjoys strong duality for upper and lower level optimization problems (5) and (6) [31, Section 5.2.3, 5.9.1]. Moreover, it enforces that saddle point and optimal dual solutions exist [30].*

Remark 5 (Problem Formulation and c_i Terms)**.** *Our formulation (4) adds regularization terms c_i to the classical $\mathcal{RO}$ problem to prevent singularity when constraints are inactive ($\lambda_i = 0$), improve numerical stability, and allow recovery of the classical problem as $c_i \to 0$. We maintain separate c_i and λ_i rather than a combined $\gamma_i = c_i + \lambda_i$ to preserve the dual variable interpretation and enable our Lyapunov construction.*

As already mentioned, our formulation includes the typical robust optimization formulation $\mathcal{RO}_0$

$$\begin{array}{ll} \mu = & \min_x f_0(x) \\ s.t. & \mathcal{F}_i(x) \leq 0, \quad i \in [N]^+ \end{array} \ , \tag{7}$$

or the form below that is popular in the machine learning context [32], [33]

$$\mu = \ \min_x \max_{u \in \mathcal{U}} f_0(x, u) \ . \tag{8}$$

We finally introduce the following assumption valid for most of this paper.

Assumption 3. *$c_i > 0$ for $i \in [N]^+$.*

Remark 6. *Assumption 3 provides regularization for inactive constraints. In practice, use small values (like $c_i = 10^{-6}$) for numerical stability. Section VI analyzes the $c_i \to 0$ limit rigorously.*

A. Robust feasible solution and robust counterpart

A meaningful solution to $\mathcal{RO}$ problem (4) has to be immune against the uncertainties in the sense that the solution vector x should satisfy the constraints for all u_i's within the uncertainty sets[3]. Such vector x is called a robust feasible solution (RFS). One approach to solving the problem (4) is to try to compute (6) in closed form.

[3]Similar to what is meant by feasibility in Robust Control [34].