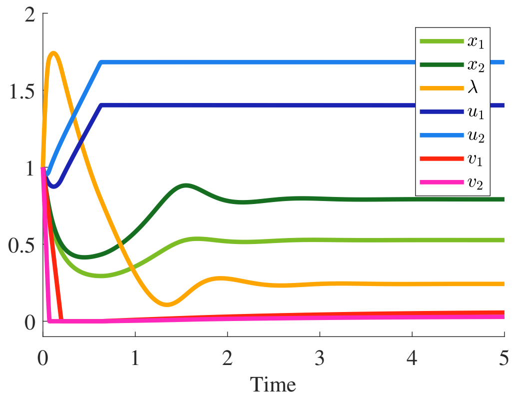

Fig. 3: Trajectory of $\mathcal{RO}$ dynamics for robust nonlinear optimization problem (45) in Example B.

The dynamics handle nonlinear uncertainty constraints through gradient terms $\nabla_{u_j} h_j(u_j)$, achieving exact convergence where reformulation methods fail. Scenario-based methods have complexity $O(N^3 n^3)$ for N scenarios, compared to $O(n^2)$ scaling for the dynamics.

Example C: Robust Dynamic Location Problem

We now consider the optimal cooperative and robust self-placement of autonomous vehicles, modeled as first-order kinematic points, to monitor multiple targets (initially assumed static). This is a dynamic generalization of a classical facility location problem [44] and here we provide a solution to the robust formulation of this problem using $\mathcal{RO}$ dynamics. The optimization problem is described by an undirected graph with N nodes and a set of edges $\mathbb{E}$. Among these nodes, the first N_1 nodes represent the fixed positions of the targets (also referred to as anchors), while the remaining $N_2 = N - N_1$ nodes represent the mobile agents. Each node $x_i \in \mathbb{R}^2$ denotes the location of an anchor or an agent. The anchors $x_1, \cdots, x_{N_1}$ have fixed positions and the agents $x_{N_1+1}, \cdots, x_N$ are mobile and can adjust their positions. The problem is to find the locations of the sensor nodes $x_{N_1+1}, \cdots, x_N$ to minimize the following cost

$$g(x_1, .., x_{N_1}) = \min_{x_{N_1+1}, \cdots, x_N} \sum_{(i,j)\in\mathbb{E}} f_{ij}(x_i, x_j) \ ,$$

which is the sum of some measure of "length" for each link. This problem and its generalizations have many applications [31], and can be solved efficiently and in a distributed fashion in continuous time [37]. Leaving the sensor nodes (agents) mobile and capable of computation in distributed mode, we obtain a distributed dynamical version *real-time* of the optimal placement. The sensor nodes, through local interactions, cooperate to find and move toward their globally optimal locations autonomously.

The problem can also be under a set of constraints for the position of agents $x_{N_1+1}, \cdots, x_{N_1}$ to be in a specified convex set. Specifically, we consider the following robust location and placement problem

$$\begin{aligned} &\min_{x_{N_1+1}, \cdots, x_N} \sum_{(i,j)\in\mathbb{E}} \frac{1}{2}\, w_{ij} \parallel x_i - x_j \parallel_2^2 \\ &\text{s.t.} \ \max_{u_i \in \mathcal{U}_i} (a_i + P_i u_i)^\top x_i \le b_i, \ i = N_1 + 1, \cdots, N \ , \end{aligned}$$

where the uncertainty u_i lies in $\mathcal{U}_i = \parallel u_i \parallel_2^2 \le \rho_i^2$ and $x_{N_1} + 1, \cdots, x_N$ are moving agents. The $\mathcal{RO}$ dynamics for this problem can be derived as follows for $i = N_1 + 1, \cdots, N$

$$\begin{cases} \dot{x}_i = \sum_j w_{ij}(x_i - x_j) - \hat{\lambda}_i\, (a_i + P_i u_i) \\ \dot{\hat{\lambda}}_i = [(a_i + u_i P_i)^\top x_i - P_i u_i^\top c_i - b_i - v_i\, \mathcal{U}_i]^{\varepsilon+}_{\hat{\lambda}_i} \\ \dot{u}_i = P_i^\top (x_i - c_i) - 2 v_i u_i \\ \dot{v}_i = [\hat{\lambda}_i\, \mathcal{U}_i] \end{cases} .$$

Note that this problem is naturally distributed and our dynamics reflects the distributed structure. We consider the setup shown in the figure below with five anchors and four agents. The figure also shows the (fixed) interconnection graph among all the elements of the problem. In this academic example, we consider the presence of a linear constraint defining the half-space where the agents can be. The constraint is simply

$$\mathbf{1}' x_i \le 2.5, \quad i = N_1 + 1, \cdots, N \ ,$$

for all the agents. In addition, we would like the agents to find robust locations based on the uncertainty on the nominal slope 45^o of the nominal constraint. Namely

$$\mathbf{1}' x_i + u_i P' x_i \le 2.5, \quad i = N_1 + 1, \cdots, N \ ,$$

where $P' = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\|u_i\|_2^2 \le \rho^2$. For example, when $\rho = 1$, the constraint can be any line passing through $x' = (1.25, 1.25)$, including the horizontal and vertical ones. The figure also shows the nominal linear constraint. The location of the shown agents is the optimal robust one w.r.t. the constraint being perturbed by the uncertainty u_i with $\|u_i\|_2^2 \le 0.1$.

First, agents start at their initial locations at the origin (white circle), and their paths converge to the optimal robust location for $\rho^2 = 0.1$. Such locations are indicated by full-colored circles. We see that agents 1 and 4 (blue and green) are on the boundary of the robust feasible set identifiable with the larger sector in the figure. The uncertain constraint is not active for the other agents. However, their optimal location is indirectly influenced by the robust constraint active in agents 1 and 4 as all agents and anchors are interconnected.

In the next phase of the simulation, we rotate the location of the anchors clockwise at constant speed. Although agents only react to their local neighbors (agents/anchors), the interconnected dynamical system shows the ability to globally track the coordinated motion of the anchors within the robust feasible set. It is interesting to notice that the boundary of the feasible set is not defined a priori or hard-coded in the simulation, but it emerges from the interactions built in the dynamical system. Finally, after some time, the uncertainty changes and increases with size $\rho^2 = 1$ at time $t = 300$. This implies that no location should be feasible above the horizontal line passing through $(1.25, 1.25)$ and on the right of the vertical