$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) - \sum_{i=1}^{N} (\varepsilon + \lambda_i) \nabla_x f_i(x, u_i) \\ \dot{\lambda}_i = [f_i(x, u_i) - v_i^\top h_i(u_i)]^+_{\lambda_i}, \quad i \in [N]^+ \\ \dot{u}_i = \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i), \; i \in [N] \\ \dot{v}_0 = [h_0(u_0)]^+_{v_0} \\ \dot{v}_i = [(\varepsilon + \lambda_i)\; h_i(u_i)]^+_{v_i}, \quad i \in [N]^+ \end{cases}, \tag{42}$$

converges to some μ_ε and $x^\star_\varepsilon$ from Theorem 4. It turns out that for ε sufficiently small, we can approximate the optimal cost and the optimal solution $x^\star$ arbitrarily well under compactness conditions.

Theorem 5. *For the problem $\mathcal{RO}$ (4) under Assumptions 1 and 2, let $x^\star_\varepsilon$ be the optimal solution where $c = \varepsilon$. Then $\lim_{\varepsilon \to 0} \mu_\varepsilon = \mu_0$, where μ_0 is the optimal cost of problem $\mathcal{RO}$ (4) with $\varepsilon = 0$, that is, (41). Furthermore, assuming the feasible set $\mathcal{C} = \{x \in \mathbb{R}^n \,|\, \mathcal{F}_i(x) \le 0, \; i \in [N]^+\}$ is compact, we have*

$$\lim_{\varepsilon \to 0} \parallel x^\star_\varepsilon - x^\star \parallel = 0 \;,$$

with $x^\star$ be the optimal $\mathcal{RO}$ solution when $\varepsilon = 0$, that is, problem (41) (proof in the appendix).

Finally, we note that the dynamics in (42), is equivalent to the following dynamics by letting $\hat{\lambda} = \lambda + \varepsilon \mathbf{1}$

$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) - \sum_{i=1}^{N} \hat{\lambda}_i \nabla_x f_i(x, u_i) \\ \dot{\hat{\lambda}}_i = [f_i(x, u_i) - v_i^\top h_i(u_i)]^{\varepsilon +}_{\hat{\lambda}_i}, \quad i \in [N]^+ \\ \dot{u}_i = \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i), \; i \in [N] \\ \dot{v}_0 = [h_0(u_0)]^+_{v_0} \\ \dot{v}_i = [\hat{\lambda}_i \; h_i(u_i)]^+_{v_i}, \quad i \in [N]^+ \end{cases}, \tag{43}$$

where the notation $[\cdot]^{\varepsilon+}_{\hat{\lambda}_i}$ represents the projection operator that ensures $\hat{\lambda}_i \ge \varepsilon > 0$, providing regularization for inactive constraints. Noting that above dynamics evolves on $\mathbb{S} = \mathbb{R}^n \times \mathbb{R}^N_{\varepsilon+} \times \mathbb{R}^M \times \mathbb{R}^K_+$, it shows that the perturbed dynamics can be obtained by simply perturbing λ_i projections with respect to ε instead of 0, by initializing $\lambda_i \ge \varepsilon > 0$.

VII. Simulations

We illustrate the $\mathcal{RO}$ dynamics with three examples: (i) robust quadratic programming with ellipsoidal uncertainty intersections, (ii) nonlinear optimization without tractable robust counterpart, (iii) distributed sensor placement. In each case, our dynamics provide exact solutions where reformulation methods fail or require extensive computational resources.

In MATLAB, several solvers can be used to simulate ordinary differential equations (ODE). In the following simulation examples, "ode15s" is used, which is a solver for stiff problems.

Example A: Robust Quadratic Programming

The uncertainty set is defined as the intersection of two ellipsoids, which represents correlated uncertainties. The robust solution must satisfy constraints for all points in this set.

Consider a robust quadratic programming (QP) problem as below

$$\begin{aligned} \min_{x \in \mathbf{R}^2} \; & f(x) := -8x_1 - 16x_2 + x_1^2 + 4x_2^2 \\ \text{s.t.} \max_{u \in \mathcal{U}} \; & (a + Pu)^\top x \le b \;, \end{aligned} \tag{44}$$

where $x \in \mathbf{R}^2$ is the decision variable, and $a = [1 \quad 1]^\top$, $P = I_2 \in \mathbf{R}^{2\times 2}$ and $b = 5$ are given parameters. Variable u is uncertain, for which, the uncertainty set is described by the intersection of five ellipsoids as below

$$\mathcal{U} := \{u \in \mathbb{R}^2 : h_j(u) \le 0 \;,\; j^+_{[5]}\} \;,$$

where $h_j(u) := u^\top Q_j u - 1, j^+_{[5]}$. Each Q_j is a symmetric positive semi-definite matrix and $\sum_{j=1}^5 Q_j \succ 0$. In this example, we assume that the following matrices specify ellipsoids

$$Q_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}, \; Q_3 = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix},$$
$$Q_4 = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}, \; Q_5 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}.$$

The Lagrangian function can be written as

$$\mathcal{L} = f(x) + \lambda\big((a + Pu)^\top x - b - v^\top h(u)\big) \;.$$

where

$$h = [h_1, h_2, h_3, , h_4, h_5]^\top, \;\; v = [v_1, v_2, v_3, v_4, v_5]^\top .$$

We obtain the following dynamics according to $\mathcal{RO}$ dynamics (43)

$$\begin{cases} \dot{x} = -\begin{bmatrix} 2x_1 - 8 \\ 8x_2 - 16 \end{bmatrix} - \hat{\lambda}\,(a + Pu) \\ \dot{\hat{\lambda}} = [(a + Pu)^\top x - b - v^\top h(u)]^{\varepsilon+}_{\hat{\lambda}} \\ \dot{u} = P^\top x - \; 2\sum_{j=1}^5 Q_j u v_j \\ \dot{v} = \left[\hat{\lambda}\; h(u)\right]^+_v \end{cases}$$

where $\hat{\lambda} = \lambda + \varepsilon$. The trajectories for this system starting from zero initial conditions are shown in Fig. 1. Note that the constraint is active, so we can set ε to zero according to Remark 6. The optimal value of x is $[2.2674, 1.6636]$ and the optimal cost is -28.5452. Five ellipsoids in the uncertainty set are plotted in Fig. 2. The blue star shows the optimal value of u at $[0.4046, 0.0909]$ which lies on the boundary of intersection of two of the ellipsoids corresponding to Q_3 and Q_5. Also note that for positive values of ε, the λ trajectory and convergence value may change but the solution x remains the same as the constraint is active.

The solution for the $\mathcal{RO}$ problem (44) can be verified by other methods. We can apply the technique in [22], in which random instances of uncertainties are sampled from the uncertainty set. Each of the uncertainty instances corresponds to a constraint. This results in a deterministic optimization problem with finitely many constraints. By picking 1115