Above dynamic is gradient-descent in x while gradient-ascent in u_0 and v_0. We must note that often in machine learning applications, the constraints on u_0 are simple boxes, and the dual variables v_0 are omitted in place of a simple set projection.

2) One uncertain constraint: Another simple example is the following $\mathcal{RO}$ problem which follows the standard setting where the constraint is active and c_1 is zero.

$$\begin{aligned} \mu \quad &= \quad \min_{x} f_0(x) \\ s.t. \quad & \max_{h_1(u_1)\le 0} f_1(x,u_1) \le 0 \end{aligned}$$

The dynamical system equations are

$$\begin{cases} \dot{x} = -\nabla_x f_0(x) - \lambda_1 \nabla_x f_1(x,u_1) \\ \dot{\lambda}_1 = [f_1(x,u_1) - v_1^\top h_1(u_1)]^+_{\lambda_1} \\ \dot{u}_1 = \nabla_{u_1} f_1(x,u_1) - v_1^\top \nabla_{u_1} h_1(u_1) \\ \dot{v}_1 = [\lambda_1 h_1(u_1)]^+_{v_1} \end{cases} .$$

This example shows the following significant differences with the primal-dual dynamical system for solving standard optimization problems appeared in [28], [40].

1) First, the $\mathcal{RO}$ dynamics has additional states associated with the worst-case constraint u_1 and the associated multiplier v_1.
2) Another difference is that the $\mathcal{RO}$ dynamics vector field is not completely derived as negative/positive gradients of the Lagrangian function $\mathcal{L}$. In particular, the vector field for the state u_1 is not obtained as the positive gradient of the Lagrangian function $\mathcal{L}$.
3) Finally, we note the presence of λ_1, the dual variable, in the upper optimization in the dynamics of v_1, the dual variable of the lower optimization. At first glance, this seems strange, since one could expect the differential equations for u_1 and v_1 to be simply the primal-ascent and dual-descent, respectively, of the lower optimization problem. This point will be discussed further after we present the stability results.

Equilibrium Analysis

The following lemma relates the optimal (KKT) points of (4) and the equilibrium points of the dynamics of $\mathcal{RO}$.

Lemma 2. *[Optimal solution and equilibrium point] Under Assumptions 1 and 2, any optimal $\mathcal{RO}$ solution based on Definition 1 is an equilibrium point of $\mathcal{RO}$ dynamics (23) and vice versa.*

Proof. Any equilibrium point, $\bar{z} = (\bar{x}, \bar{\lambda}, \bar{u}, \bar{v})$ of $\mathcal{RO}$ dynamics (23) satisfies

$$\begin{aligned} &\nabla_x f_0(\bar{x},\bar{u}_0) + \sum_{i=1}^{N} (c_i + \bar{\lambda}_i) \nabla_x f_i(\bar{x},\bar{u}_i) = 0 , \\ &f_i(\bar{x},\bar{u}_i) - \bar{v}_i^\top h_i(\bar{u}_i) \le 0,\ \bar{\lambda}_i \ge 0 , \\ &\bar{\lambda}_i (f_i(\bar{x},\bar{u}_i) - \bar{v}_i^\top h_i(\bar{u}_i)) = 0 , \\ &\nabla_{u_i} f_i(\bar{x},\bar{u}_i) - \bar{v}_i^\top \nabla_{u_i} h_i(\bar{u}_i) = 0 , \\ &h_{0j}(\bar{u}_0) \le 0,\ \bar{v}_{0j} \ge 0,\ \bar{v}_{0j} h_{0j}(\bar{u}_0) = 0 , \\ &(c_i + \bar{\lambda}_i) h_{ij}(\bar{u}_i) \le 0,\ \bar{v}_{ij} \ge 0,\ \bar{v}_{ij}(c_i + \bar{\lambda}_i) h_{ij}(\bar{u}_i) = 0 , \end{aligned}$$

for $i \in [N]^+, j \in [K_i]^+$, while any optimal point satisfies below KKT conditions

$$\nabla_x f_0(x^\star, u_0^\star) + \sum_{i=1}^{N} (c_i + \lambda_i^\star) \nabla_x f_i(x^\star, u_i^\star) = 0, \tag{24}$$

$$\nabla_{u_i} f_i(x^\star, u_i^\star) - v_i^{\star\top} \nabla_{u_i} h_i(u_i^\star) = 0, \quad i \in [N], \tag{25}$$

$$v_{ij}^\star \ge 0,\ h_{ij}(u_i^\star) \le 0,\ v_{ij}^\star h_{ij}(u_i^\star) = 0,\ j \in [K_i],\ i \in [N] \tag{26}$$

$$\lambda_i^\star \ge 0,\ f_i(x^\star, u_i^\star) \le 0,\ \lambda_i^\star f_i(x^\star, u_i^\star) = 0,\ i \in [N] \tag{27}$$

Substituting $z^\star$ for $\bar{z}$, and using the fact that $(c_i + \lambda_i^\star) > 0$ for all $i \in [N]^+$, it is immediate to verify that the optimal point $z^\star$, satisfying (24)-(27), also satisfies the above equilibrium conditions and therefore is an equilibrium point of $\mathcal{RO}$ dynamics (23).

On the other hand, since $c_i > 0$ and $c_i + \bar{\lambda}_i > 0$, for $i \in [N]^+$; then, $h_{ij}(\bar{u}_i) \le 0$, and $\bar{v}_{ij} h(\bar{u}_{ij}) = 0$ for $i \in [N], j \in [K_i]^+$. Substituting these properties in the rest of the equilibrium conditions, we see that $\bar{z}$ satisfies the KKT conditions therefore is optimal for $\mathcal{RO}$. s ☐

We denote the $\mathcal{RO}$ dynamics (23) compactly with the shorthand notation $\dot{z} = \mathcal{Z}^{\mathcal{RO}}(z)$.

Lyapunov Function Construction

In this subsection, we present the Lyapunov function[4] that establishes global convergence of the proposed dynamical system.

Lemma 3. *[Monotonicity property] Let $z^\star = (x^\star, \lambda^\star, u^\star, v^\star)$ be an optimal $\mathcal{RO}$ solution (Definition 1).*

Let $V : \mathbb{S} \to \mathbb{R}_+$ defined as

$$\begin{aligned} V = \frac{1}{2}\big(&\|x - x^\star\|^2 + \|\lambda - \lambda^\star\|^2 + \|u_0 - u_0^\star\|^2 + \\ &\sum_{i=1}^{N} (c_i + \lambda_i^\star)\|u_i - u_i^\star\|^2 + \sum_{i=0}^{N} \|v_i - v_i^\star\|^2\big) , \end{aligned} \tag{28}$$

then, the Lie-derivative of V along $\mathcal{Z}^{\mathcal{RO}}$ at $z = (x, \lambda, u, v)$ is $\nabla V(z)^\top \dot{z} \le 0$.

[4]With slight abuse of notation.