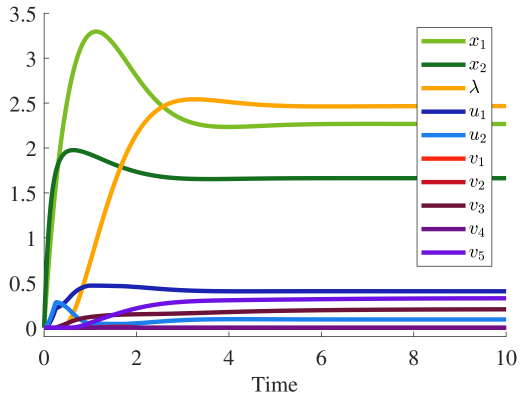

Fig. 1: Trajectories of $\mathcal{RO}$ dynamics for robust QP problem with intersection of ellipsoids uncertainty set in Example A.

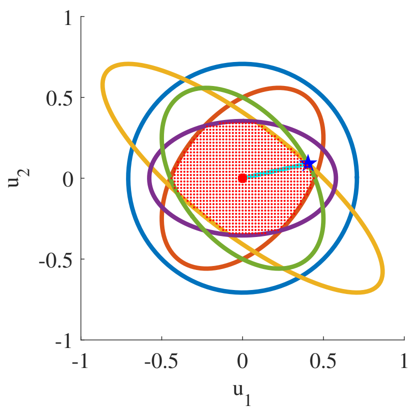

Fig. 2: Uncertainty set for Example A plotted in u_2-u_1 space. Big red point at the origin represents the initial point of u in $\mathcal{RO}$ dynamics and blue star depicts the optimal value of u on the intersection of two ellipsoids derived by our method. Small red grid points are the 1115 sampled points from the uncertainty set in randomized scenario method [22] to compare with our method.

samples from the intersection of ellipsoids uncertainty set and solving the derived deterministic optimization problem with 1115 constraints by CVX, the approximate robust optimal solution and the approximate optimal cost value are found $[2.2693, 1.6770]$ and -28.5873 respectively. The optimal cost function of our method has a larger value compared to the method in [22], as the latter approximates the RFS by taking finite samples. Hence, it cannot find the best value of u precisely and the optimal u; therefore, the optimal cost is approximated. The second method verifying our solution is robust counterpart [2]. Deriving the robust counterpart and solving the deterministic problem by CVX returns the same solution compared to the $\mathcal{RO}$ dynamics. Our method works on the main $\mathcal{RO}$ problem to find the optimal solution without transforming it to a deterministic equivalent which can be a hassle and even impossible task, as will be shown in the following example.

Example B: Robust Nonlinear Optimization with no RC

Consider the following robust nonlinear optimization problem

$$\begin{aligned} \min_{x\in\mathbf{R}^2} \ & f(x) := \frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-2)^2 \\ \text{s.t.} \ & \max_{u\in\mathcal{U}} u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} \le b \ , \end{aligned} \tag{45}$$

where $x \in \mathbf{R}^2$, and $u = [u_1, u_2]^\top \in \mathbf{R}^2$, for which, the strictly convex uncertainty set is described by

$$\mathcal{U} := \{u \in \mathbb{R}^2 : e^{u_j^2} + u_j e^{\frac{1}{u_j}} \le \rho_j \ , \ j = 1, 2\} \ . \tag{46}$$

As stated in [21] and [19], there is no closed-form convex conjugate for the constraint (convex in x) in (45) and no closed-form conjugate for the convex uncertainty set in (46). This is the third case in [19, Table 1], for which there is no known method for obtaining RC. However, by writing the Lagrangian function as

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-2)^2 \\ &+ \lambda\Big(u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - b - v^\top \begin{bmatrix} e^{u_1^2} + u_1 e^{\frac{1}{u_1}} - \rho_1 \\ e^{u_2^2} + u_2 e^{\frac{1}{u_2}} - \rho_2 \end{bmatrix}\Big) \ , \end{aligned}$$

where $h_j(u_j) = e^{u_j^2} + u_j e^{\frac{1}{u_j}} - \rho_j$ for $j = 1, 2$ as defined in the previous example, we can form the $\mathcal{RO}$ dynamics for the RNO problem (45) as

$$\begin{aligned} \dot{x} &= -\begin{bmatrix} x_1 - 1 \\ x_2 - 2 \end{bmatrix} - \hat{\lambda} \begin{bmatrix} 2x_1 e^{x_1^2} & 0 \\ 0 & 2x_2 e^{x_2^2} \end{bmatrix} u \ , \\ \dot{\hat{\lambda}} &= \Big[u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - b - \sum_{j=1}^{2} \big(v_j (e^{u_j^2} + u_j e^{\frac{1}{u_j}} - \rho_j)\big)\Big]_{\hat{\lambda}}^{\varepsilon+} \ , \\ \dot{u} &= \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - \sum_{j=1}^{2} (2u_j e^{u_j^2} + e^{\frac{1}{u_j}} - \frac{1}{u_j} e^{\frac{1}{u_j}}) v_j \ , \\ \dot{v} &= \Big[\hat{\lambda} \begin{bmatrix} e^{u_1^2} + u_1 e^{\frac{1}{u_1}} - \rho_1 \\ e^{u_2^2} + u_2 e^{\frac{1}{u_2}} - \rho_2 \end{bmatrix}\Big]_v^+ \ , \end{aligned}$$

according to $\mathcal{RO}$ dynamics (43) where $\hat{\lambda} = \lambda + \varepsilon$. Similarly to the previous example, we can set ε to zero according to Remark 6 as the constraint is active. Fig. 3 shows the trajectory plot for $\rho_1 = 10$, $\rho_2 = 20$, and all the states initialized at 1. The robust optimal solution is $[0.5271, 0.7916]$ and the optimal cost value is 0.8419. We observe that the optimal value for u, which is $[1.4020, 1.6824]$ lies on the boundary of the uncertainty set. For positive values of ε, the trajectory and convergence value of λ may change, but the solution x remains the same as the constraint is active.

The uncertainty constraints $e^{u_j^2} + u_j e^{1/u_j} \le \rho_j$ are highly nonlinear, and no closed-form robust counterpart exists [2], [19]. We compared with scenario-based sampling [22]: using CVX, 168 scenarios gave solution $[0.5376, 0.8193]$ with cost 0.8039 (2.3s), 500 scenarios gave $[0.5312, 0.8024]$ with cost 0.8287 (8.7s), and 1000 scenarios gave $[0.5289, 0.7953]$ with cost 0.8371 (31.2s). Our method obtained the exact solution $[0.5271, 0.7916]$ with optimal cost 0.8419 in 0.8s integration time.