For various important combinations of constraint functions f_i and uncertainty sets $\mathcal{U}_i$, $(i > 0)$ it is possible to obtain an explicit convex function $\mathcal{F}_i$ [2]. A classical example is when f_i is linear in x for fixed u_i and linear in u_i for fixed x, while $\mathcal{U}_i$ is an ellipsoidal set. Then, $\mathcal{F}_i$ can be easily derived as an explicit second-order conic function. In this case, $\mathcal{RO}$ problem (4) becomes a nominal optimization problem (not affected by uncertainty) known as the explicit RC:

$$\min_{\mathcal{F}_i(x)\leq 0} f_0(x) \; . \tag{9}$$

As shown in [30], the RC is always a convex optimization problem under Assumption 1. While the RC is known for important classes of problems as described in [3], this approach requires problem-specific derivations; moreover, RC is generally difficult to find (See Section VII for an example). Instead, our proposed approach has a dynamical system that simultaneously finds the best RFS and the worst parameters u_i' s, independently of the specifics of the constraint functions and the uncertainty sets.

IV. Duality, KKT conditions and saddle property

Lagrangian Formulation

The basic idea of this paper is to combine the usually separated and nested optimization (5) and (6) into one Lagrangian optimization.

To derive the Lagrangian function for $\mathcal{RO}$ problem (4), we introduce Lagrange multipliers $\lambda_i \geq 0$ for $i \in [N]^+$, and let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)^\top \in \mathbb{R}_+^N$ where the non-negative orthant of $\mathbb{R}^N$ is denoted by $\mathbb{R}_+^N$. Note that μ in $\mathcal{RO}$ problem (4) can be written in short-hand notation as follows

$$\mu = \min_x \max_{\lambda\geq 0} \mathcal{F}_0(x) + \sum_{i=1}^N (c_i + \lambda_i)\, \mathcal{F}_i(x) \; , \tag{10}$$

where $\mathcal{F}_i(x)$ is given by (6). By introducing Lagrange multipliers $v_{ij} \geq 0$ for the maximization problem in (6) and defining

$$v_i := (v_{i1}, \cdots, v_{iK_i})^\top \in \mathbb{R}_+^{K_i},\; h_i := (h_{i1}, \cdots, h_{iK_i})^\top \; ,$$

the lower level strong duality (according to Assumption 2) yields

$$\mathcal{F}_i(x) = \max_{u_i\in\mathbb{R}^{m_i}} \min_{v_i\geq 0} f_i(x, u_i) - v_i^\top h_i(u_i) \; . \tag{11}$$

Defining the Lagrangian $\tilde{\mathcal{L}}_i : \mathbb{R}^n \times \mathbb{R}^{m_i} \times \mathbb{R}^{K_i} \to \mathbb{R}$ for the lower level maximization problem (11) as

$$\tilde{\mathcal{L}}_i(x, u_i, v_i) := f_i(x, u_i) - v_i^\top h_i(u_i),\; i \in [N]^+, \tag{12}$$

and

$$\tilde{\mathcal{L}}_0(x, u_0, v_0) := f_0(x, u_0) - v_0^\top h_0(u_0) \; . \tag{13}$$

In the optimization problem (10), μ can be written as

$$\begin{aligned}\mu = \min_x \max_{\lambda\geq 0} \max_{u_0} \min_{v_0\geq 0} \; & \tilde{\mathcal{L}}_0(x, u_0, v_0) \\ & + \sum_{i=1}^N (c_i + \lambda_i) \max_{u_i} \min_{v_i\geq 0} \; \tilde{\mathcal{L}}_i(x, u_i, v_i) \; .\end{aligned} \tag{14}$$

or combining the independent maximizations and minimizations, and collecting the variables, we have

$$\begin{aligned}\mu = & \min_x \max_{\lambda\geq 0} \max_u \min_{v\geq 0} \\ & \Big(\tilde{\mathcal{L}}_0(x, u_0, v_0) + \sum_{i=1}^N (c_i + \lambda_i)\, \tilde{\mathcal{L}}_i(x, u_i, v_i)\Big) \; .\end{aligned} \tag{15}$$

Hence, we showed the equivalence of (4) and (14). The complete Lagrangian $\mathcal{L}(x, \lambda, u, v)$ for $\mathcal{RO}$ problem is derived as

$$\begin{aligned}\mathcal{L}(x, \lambda, u, v) \; := \; & f_0(x, u_0) - v_0^\top h_0(x, u_0) \\ & + \sum_{i=1}^N (c_i + \lambda_i)\big(f_i(x, u_i) - v_i^\top h_i(x, u_i)\big) \; ,\end{aligned} \tag{16}$$

which leads to

$$\mu = \min_x \max_{\lambda\geq 0} \max_u \min_{v\geq 0} \; \mathcal{L}(x, \lambda, u, v) \; .$$

From Remark 4, strong duality holds for both upper and lower optimizations. Thus, we can switch the order of max and min as follows

$$\mu = \max_{\lambda\geq 0} \min_x \min_{v\geq 0} \max_u \; \mathcal{L}(x, \lambda, u, v) \; . \tag{17}$$

Remark 7 (Necessity of Lagrangian Analysis)**.** *The Lagrangian (16) unifies the nested min-max structure and permits continuous-time dynamics over all variables simultaneously. The saddle property derived from this formulation is essential for Lyapunov stability analysis.*

Definition 1. *We denote an optimal solution of (17), $z^\star := (x^\star, \lambda^\star, u^\star, v^\star)$, as an "optimal $\mathcal{RO}$ solution", where $(x^\star, u^\star)$ is an optimal solution for (4), and $(\lambda^\star, v^\star)$ are optimal values for the corresponding dual variables according to the principles in [31, Section 5.9.1].*

KKT Optimality Conditions

As strong duality holds for $\mathcal{RO}$ problem, any optimal solution, $z^\star = (x^\star, \lambda^\star, u^\star, v^\star)$, satisfies the following Karush-Kuhn-Tucker (KKT) conditions, and vice-versa [31]

$$\nabla_x f_0(x^\star, u_0^\star) + \sum_{i=1}^N (c_i + \lambda_i^\star)\nabla_x f_i(x^\star, u_i^\star) = 0, \tag{18}$$

$$\nabla_{u_i} f_i(x^\star, u_i^\star) - {v_i^\star}^\top \nabla_{u_i} h_i(u_i^\star) = 0,\; i \in [N], \tag{19}$$

$$v_{ij}^\star \geq 0,\; h_{ij}(u_i^\star) \leq 0,\; v_{ij}^\star h_{ij}(u_i^\star) = 0,\; j \in [K_i]^+, i \in [N] \tag{20}$$

$$\lambda_i^\star \geq 0,\; f_i(x^\star, u_i^\star) \leq 0,\; \lambda_i^\star f_i(x^\star, u_i^\star) = 0,\; i \in [N]^+ \tag{21}$$

where $\nabla_x f$ is the notation for the gradient of a function f w.r.t. x.

Remark 8 (KKT Conditions for $\mathcal{RO}$)**.** *The KKT conditions for $\mathcal{RO}$ incorporate: (i) bi-level structure with dual variables λ_i and v_{ij}, (ii) regularization terms $(c_i + \lambda_i)$, (iii) complementary slackness at both levels, (iv) equivalence with the saddle point property.*