which means that μ_ε converges to μ as $\varepsilon \to 0$.

To prove the second part of the theorem, that is, $x_\varepsilon^\star \to x^\star$ as $\varepsilon \to 0$, consider any sequence $\{\varepsilon_n\}$ converging to 0. Let $\{x_n^\star\}$ be the corresponding sequence of optimal solutions for $\mu(\varepsilon_n)$.

As $\mathcal{C}$ is compact and the same for both perturbed and original problem, $x_n^\star \in \mathcal{C}$ is bounded. Therefore, there exists a convergent sub-sequence $x_{n_t}^\star$ that converges to, say, $\hat{x} \in \mathcal{C}$, as $\varepsilon_{n_t} \to 0$, since $\mathcal{C}$ is closed by assumption.

This implies that $\mathcal{F}_0(x_{n_t}^\star) \to \mathcal{F}_0(\hat{x})$, since $\mathcal{F}_0(x)$ is continuous. Because $\hat{x}$ is feasible, $\mathcal{F}_0(\hat{x}) \geq \mu$. However, $\mathcal{F}_0(\hat{x}) > \mu$ is impossible since $\mu(\varepsilon_{n_t}) \to \mu$, from the first part of the proof, and $\mu(\varepsilon_{n_t}) = \mathcal{F}_0(x_{n_t}^\star) + \varepsilon_{n_t} \sum_{i=1}^{N} \mathcal{F}_i(x_{n_t}^\star) \to \mathcal{F}_0(\hat{x})$, since from (52) and (53), $\lim_{\varepsilon_{n_t} \to 0} \varepsilon_{n_t} \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_{n_t}}) = 0$.

Therefore, $\mathcal{F}_0(\hat{x}) = \mu = \mathcal{F}_0(x^\star)$. Since $f(x)$ is strictly convex, $\hat{x} = x^\star$. Since every convergent sub-sequence converges to $x^\star$, the whole sequence converges to it. Since the sequence was arbitrary, we have that $x_\varepsilon^\star \to x^\star$ as $\varepsilon \to 0$.

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