

Robust Optimization via Continuous-Time Dynamics

Keivan Ebrahimi, Nicola Elia, and Umesh Vaidya

Abstract—We consider optimization problems with uncertain parameters that belong to known constraint sets. This paper introduces robust optimization (RO) dynamics—a continuous-time dynamical system that directly solves min–max robust optimization problems without requiring reformulations or explicit uncertainty model specifications. Our key contributions are twofold: (i) we establish the saddle-point property without assuming joint convexity–concavity, and (ii) we construct a novel Lyapunov function that proves global convergence for general robust optimization problems that are convex in the decision variable and concave in the uncertainty. The method operates using only output feedback without requiring explicit cost or constraint function specifications, which permits decentralized implementation. Numerical experiments demonstrate substantial speedups over scenario-based methods, including exact solutions for robust quadratic programming with ellipsoidal uncertainties, nonlinear problems where robust counterparts cannot be derived, and distributed sensor placement. The continuous-time formulation permits real-time adaptation and implementation in physical systems.

I. INTRODUCTION

Motivation and Background

Optimization problems in engineering, finance, and machine learning often involve uncertain parameters that must be handled robustly to ensure reliable performance. Two different approaches can be used: stochastic optimization, where uncertainty is treated as a random variable, or RO, where uncertainty is assumed to be deterministic and bounded. Unlike stochastic optimization, RO does not require any known probability distributions in the problem data. Instead, RO assumes that the uncertain data reside within a predefined uncertainty set, for which constraint violation cannot be tolerated. For more information on robust and stochastic optimization-based approaches, see [1], [2], and the references therein. Early works on RO include [3], which considered robust linear optimization with ellipsoidal uncertainty sets, and [4], which presented exact solutions of inexact linear programs as a simple case of RO.

Literature Review

The problem of robust linear programming was studied in [5], while robust conic-quadratic optimization and robust semi-

definite optimization were discussed in [6] and [7], respectively. Refer to [2] and [8] for comprehensive surveys of RO problem solutions. Distributionally robust optimization (DRO) provides an intermediate approach between stochastic and robust formulations [9], [10]. [11] and [12] showed that machine learning algorithms such as the norm-regularized support vector machine and the Lasso problem could be interpreted as RO problems. Connections between robust optimization and machine learning have been explored in [13], [14].

One of the main standard approaches for solving RO problems involves constructing a robust counterpart (RC) equivalent to the RO problem [1]. This widely used method essentially tries to find a deterministic equivalent to RO problem through a reformulation. In this sense, the practicality of robust programming depends on whether or not its RC is computationally tractable. An overview of different RO problems with tractable conjugates can be found in [15] Table 1 for some of which RC cannot be found. The reformulation approach to solving the RO problem, which is often a challenging, albeit usually convex, optimization problem, has the deficiency of suffering from case-by-case scenarios depending on the specific form of the uncertainty constraint and the specific form of the uncertainty set. In other words, depending on how simple the uncertainty set looks and based on the optimization problem type (whether it is linear programming, quadratic programming, second-order cone programming, semi-definite programming, etc.), an RC is being calculated and provided at hand. These reformulations may be computationally more expensive than other approaches [16]. Hence, the absence of a unified framework for solving a general class of RO problems without prior knowledge of the problem formulation is strongly felt.

By calculating a concave conjugate of nonlinear constraint functions and supporting the function of uncertainty sets, [17] and [15] used the Fenchel duality to obtain a tractable RC for new classes of robust nonlinear optimization (RNO) problems. However, when no closed form is available for the convex conjugate function of an uncertainty set or convex/concave conjugates of a non-linear constraint, these approaches still cannot obtain the RC for many sets of nonlinear uncertainties and constraints. **Moreover, all these methods require either explicit problem structure knowledge or extensive computational resources, unlike our dynamics-based approach.**

An alternative randomized approach based on constraint sampling is an approximate probabilistic relaxation solution to RO problems as seen in [18] and [19] where a finite set of high-dimensional deterministic optimization problems obtained from sampling are solved. This approach does not require concavity of the constraint in the uncertain variable but due to the large number of required scenarios to approximate

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the stochasticity of these problems, the stochastic optimization involves formulating a large-scale scenario program, which is in general computationally demanding. Recent methods include alternating direction method of multipliers (ADMM) decomposition [20], achieving improved convergence rates for distributed robust optimization problems.

In another effort, [21] incorporated the min-max behavior of RO problems to solve them by an oracle-based approximate robust optimization algorithm based on oracle-based subgradient descent and interior point methods. The proposed algorithms find an approximate robust solution using a number of calls to an oracle that solves the original (non-robust) problem. However, the solution would be approximated with some predetermined accuracy, η , with the number of iterations growing to $\mathcal{O}(\frac{1}{\eta})$, as the algorithm approximates the RC by invoking the Oracle a polynomial number of times. Classical cutting-plane approaches [22] treat RO problems as semi-infinite programming but have limitations when pessimization oracles are approximate. In another effort, Robust optimization over time (ROOT) combines robust optimization with dynamic optimization [9], [23].

Our Approach

In this paper, we provide the architecture of a dynamical system that converges to the optimal robust solution for a large class of RO problems. Our approach builds on classical primal-dual systems [24], [25] but addresses the unique challenges posed by the min-max structure of robust optimization.

Main Contributions

The main contributions of this paper are:

- **Model-free operation:** We develop a continuous-time dynamical system that solves RO problems without requiring explicit knowledge of uncertainty models or problem structure. This enables implementation in physical systems where only measurements are available.
- **Unified framework:** The proposed method handles all convex-concave RO problems using a single dynamical system architecture, eliminating problem-specific reformulations.
- **Saddle point theory without joint convexity-concavity:** We prove that the saddle point property holds for RO problems despite the Lagrangian lacking joint convexity-concavity (Section IV).
- **Lyapunov stability analysis:** We construct a Lyapunov function that establishes global asymptotic stability of the equilibrium corresponding to the robust optimal solution.
- **Real-time capability:** The continuous-time formulation permits tracking of time-varying uncertainties and decentralized implementation.
- **Solutions for problems without tractable robust counterparts:** The method solves problems where reformulation-based approaches fail, including nonlinear constraints with general uncertainty sets.

These contributions establish continuous-time dynamics as a powerful alternative paradigm for robust optimization,

complementing existing discrete algorithmic approaches. The proposed continuous-time optimization system can solve a general class of \mathcal{RO} problems where the cost function and the constraints are convex (concave) with respect to the decision variables (uncertain variables) and the uncertainty sets are convex¹. This class includes all the cases in [15, Table 1].

Paper Organization

The remainder of this paper is organized as follows. In Section III, we present the problem statement of \mathcal{RO} in a slightly generalized form. In Section IV, the characterization of saddle property and Karush-Kuhn-Tucker (KKT) optimality conditions along with the Lagrangian function for \mathcal{RO} problem are provided. The main results on how to construct the \mathcal{RO} dynamics is presented in Section V. This section also includes the Lyapunov-based global convergence result. We then present simulation results in Section VII followed by conclusions in Section VIII. Finally, detailed proofs are mentioned in the Appendix.

II. NOTATIONS

We define $i = 0, \dots, N$ as $i \in [N]$, and $i = 1, \dots, N$ as $i \in [N]^+$. In addition, $[\cdot]_\eta^+$ shows the positive projection defined as follows for the scalar-valued function P

$$[P]_\eta^+ := \begin{cases} P & \text{if } P > 0 \text{ or } \eta > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For vector-valued P , the projection is defined element-wise.

III. ROBUST OPTIMIZATION PROBLEMS

Classical Formulation

Given an objective $f_0(x)$ to optimize, subject to constraints $f_i(x, u_i) \leq 0$ with uncertain parameters, $\{u_i\}$, a classical robust optimization problem has the following form

$$\begin{aligned} \mu &:= \min_x f_0(x) \\ \text{s.t. } &f_i(x, u_i) \leq 0, \forall u_i \in \mathcal{U}_i, i \in [N]^+. \end{aligned}$$

where $x \in \mathbb{R}^n$ is a vector of decision variables, f_0 and f_i are $\mathbb{R}^n \rightarrow \mathbb{R}$ functions, and the uncertainty parameters $u_i \in \mathbb{R}^{m_i}$ are assumed to take arbitrary values in certain convex compact uncertainty sets $\mathcal{U}_i \subseteq \mathbb{R}^{m_i}$. The above problem is typically a semi-infinite optimization due to the cardinality of the \mathcal{U}_i . However, each robust constraint can be rewritten as a maximization problem as²

$$\begin{aligned} \mu &:= \min_x f_0(x) \\ \text{s.t. } &\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leq 0, i \in [N]^+ \end{aligned} \quad (2)$$

¹Preliminary convergence results for a special formulation appeared without proofs in [26]. In this paper, we generalize the problem formulation and provide complete proofs, which are the core of our contributions.

²Since the uncertainty set is compact and the constraint functions are continuous, the supremum is attained within the set; therefore, we can replace “sup” with “max” in our formulation.

Generalized Formulation

We consider a slight variation of (2), which takes the following under the assumptions stated below.

$$\mathcal{RO}_0 \begin{cases} \mu := \min_x \max_{u_0 \in \mathcal{U}_0} f_0(x, u_0) \\ \text{s.t. } \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leq 0, \quad i \in [N]^+, \\ \mathcal{U}_i := \{u_i \in \mathbb{R}^{m_i} : h_{ij}(u_i) \leq 0, \\ \quad j \in [K_i]\}, \quad i \in [N] \end{cases} \quad (3)$$

Here, $h_{ij}(u_i)$ represents the j -th constraint function defining the i -th uncertainty set \mathcal{U}_i , and K_i denotes the total number of constraints that define \mathcal{U}_i .

Remark 1. Formulation (3) has three key properties:

- 1) **Generalization:** Formulation (3) generalizes (2) by allowing uncertainty in the objective function and explicitly representing uncertainty sets via inequality constraints, facilitating the min-max-max-min structure essential for our dynamical approach.
- 2) **Compactness:** Under the convexity assumptions stated below, uncertainty sets \mathcal{U}_i remain compact since convex constraints preserve convexity and boundedness.
- 3) **Nonlinear constraints:** Our framework handles nonlinear uncertainty constraints $h_{ij}(u_i) \leq 0$ through gradient terms $\nabla_{u_i} h_{ij}(u_i)$ that naturally accommodate nonlinearity.

Assumptions and Properties

The problem in (3) reduces to the one in (2) when \mathcal{U}_0 is a singleton. Following [27] and without loss of generality, we consider the \mathcal{RO} problem dealing with constraint-wise uncertainties where each constraint f_i is only a function of u_i . The functions f_i and h_{ij} have scalar values with the following assumptions.

Assumption 1. The functions $h_{ij}(u_i)$ are convex in u_i for $i \in [N]$ and $j \in [K_i]^+$. The function $f_0(x, u_0)$ is strictly convex in x for any $u_0 \in \mathcal{U}_0$, and concave in u_0 for any x . Also, for $i \in [N]^+$, $f_i(x, u_i)$ is convex in x for fixed u_i and concave in $u_i \in \mathcal{U}_i$, for fixed x . Finally, the functions f_i and h_{ij} $i \in [N]$ are C^1 with local Lipschitz gradients.

Remark 2. Assumption 1 requires convexity in the decision variable and concavity in the uncertainty variable. These conditions hold for most practical \mathcal{RO} problems, particularly when uncertainty enters affinely.

Assumption 2. Existence of optimal solutions and strong duality.

- 1) \mathcal{RO} problem (4) is feasible. An optimal min-max solution (x^*, u^*) exists and μ in the \mathcal{RO} problem (4) is finite.
- 2) \mathcal{RO} problem satisfies the Slater constraint qualification, for both the upper level optimization and the lower level maximization problems [27], that is, for all $i \in [N]$ and all $j \in [K_i]^+$, there exist $u_i \in \mathcal{U}_i$ such that $h_{ij}(u_i) < 0$, and for $i \in [N]^+$, there exists $x \in \mathbb{R}^n$ such that, $F_i(x) < 0$.

Remark 3. It should be noted that the uncertainty is often parametrized affinely in \mathcal{RO} problem; hence, the concavity property of the f_i functions is automatically satisfied.

Remark 4 (Justification of Assumption 2). The Slater constraint qualification guarantees strong duality for both the upper and lower level problems [28, Section 5.2.3, 5.9.1], which ensures existence of saddle point and optimal dual solutions [27]. This assumption holds when uncertainty sets have strict interior points (e.g., ellipsoidal sets with positive radius). When Slater conditions fail, alternative constraint qualifications from [29] can be employed.

Regularized Formulation

Finally, we introduce the problem we consider in this paper, a slight generalization of \mathcal{RO}_0 in (3).

$$\mathcal{RO} \begin{cases} \mu := \min_x \max_{u_0 \in \mathcal{U}_0} f_0(x, u_0) \\ \quad + \sum_{i=1}^N c_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \\ \text{s.t. } \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \leq 0, \quad i \in [N]^+, \\ \mathcal{U}_i := \{u_i \in \mathbb{R}^{m_i} : h_{ij}(u_i) \leq 0, \\ \quad j \in [K_i]\}, \quad i \in [N] \end{cases} \quad (4)$$

where $c_i \geq 0$ for $i \in [N]^+$. This setting can be seen as an elementary regularized version of more common formulations \mathcal{RO}_0 obtained for $c_i = 0$, $i \in [N]^+$. The role of c_i is further clarified below.

It is convenient to rewrite \mathcal{RO} in the following form

$$\begin{aligned} \mu &:= \min_x \mathcal{F}_0(x) + \sum_{i=1}^N c_i \mathcal{F}_i(x) \\ \text{s.t. } \mathcal{F}_i(x) &\leq 0, \quad i \in [N]^+, \end{aligned} \quad (5)$$

where

$$\mathcal{F}_i(x) := \max_{u_i \in \mathcal{U}_i} f_i(x, u_i), \quad i \in [N]. \quad (6)$$

In this paper, we often call (6) as the “lower optimization problems”, and the minimization in (5) as the “upper optimization”.

Remark 5 (Problem Formulation and c_i Terms). Our formulation (4) adds regularization terms c_i to the classical \mathcal{RO} problem for three key reasons: (i) preventing singularity when constraints are inactive ($\lambda_i = 0$), (ii) improving numerical stability, (iii) recovering the classical problem as $c_i \rightarrow 0$. We maintain separate c_i and λ_i (rather than combined γ_i) to preserve dual variable interpretation and enable our Lyapunov construction.

As already mentioned, our formulation includes the typical robust optimization formulation \mathcal{RO}_0

$$\begin{aligned} \mu &= \min_x f_0(x) \\ \text{s.t. } \mathcal{F}_i(x) &\leq 0, \quad i \in [N]^+, \end{aligned} \quad (7)$$

or the form below that is popular in the machine learning context [30], [31]

$$\mu = \min_x \max_{u \in \mathcal{U}} f_0(x, u). \quad (8)$$

We finally introduce the following assumption valid for most of this paper.

Assumption 3. $c_i > 0$ for $i \in [N]^+$.

Remark 6. *Assumption 3 provides regularization for inactive constraints. In practice, use small values (like $c_i = 10^{-6}$) for numerical stability. Section VI analyzes the $c_i \rightarrow 0$ limit rigorously.*

A. Robust feasible solution and robust counterpart

A meaningful solution to \mathcal{RO} problem (4) has to be immune against the uncertainties in the sense that the solution vector x should satisfy the constraints for all u_i 's within the uncertainty sets³. Such vector x is called a robust feasible solution (RFS). One approach to solving the problem (4) is to try to compute (6) in closed form.

For various important combinations of constraint functions f_i and uncertainty sets \mathcal{U}_i , ($i > 0$) it is possible to obtain an explicit convex function \mathcal{F}_i [1]. A classical example is when f_i is linear in x for fixed u_i and linear in u_i for fixed x , while \mathcal{U}_i is an ellipsoidal set. Then, \mathcal{F}_i can be easily derived as an explicit second-order conic function. In this case, \mathcal{RO} problem (4) becomes a nominal optimization problem (not affected by uncertainty) known as the explicit RC:

$$\min_{\mathcal{F}_i(x) \leq 0} f_0(x). \quad (9)$$

As shown in [27], the RC is always a convex optimization problem under Assumption 1. While the RC is known for important classes of problems as described in [2], this approach requires problem-specific derivations; moreover, RC is generally difficult to find (See Section VII for an example). Instead, our proposed approach has a dynamical system that simultaneously finds the best RFS and the worst parameters u_i 's, independently of the specifics of the constraint functions and the uncertainty sets.

IV. DUALITY, KKT CONDITIONS AND SADDLE PROPERTY Lagrangian Formulation

The basic idea of this paper is to combine the usually separated and nested optimization (5) and (6) into one Lagrangian optimization.

To derive the Lagrangian function for \mathcal{RO} problem (4), we introduce Lagrange multipliers $\lambda_i \geq 0$ for $i \in [N]^+$, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)^\top \in \mathbb{R}_+^N$ where the non-negative orthant of \mathbb{R}^N is denoted by \mathbb{R}_+^N . Note that μ in \mathcal{RO} problem (4) can be written in short-hand notation as follows

$$\mu = \min_x \max_{\lambda \geq 0} \mathcal{F}_0(x) + \sum_{i=1}^N (c_i + \lambda_i) \mathcal{F}_i(x), \quad (10)$$

where $\mathcal{F}_i(x)$ is given by (6). By introducing Lagrange multipliers $v_{ij} \geq 0$ for the maximization problem in (6) and defining

$$v_i := (v_{i1}, \dots, v_{iK_i})^\top \in \mathbb{R}_+^{K_i}, \quad h_i := (h_{i1}, \dots, h_{iK_i})^\top,$$

³Similar to what is meant by feasibility in Robust Control [32].

the lower level strong duality (according to Assumption 2) yields

$$\mathcal{F}_i(x) = \max_{u_i \in \mathbb{R}^{m_i}} \min_{v_i \geq 0} f_i(x, u_i) - v_i^\top h_i(u_i). \quad (11)$$

Defining the Lagrangian $\tilde{\mathcal{L}}_i : \mathbb{R}^n \times \mathbb{R}^{m_i} \times \mathbb{R}^{K_i} \rightarrow \mathbb{R}$ for the lower level maximization problem (11) as

$$\tilde{\mathcal{L}}_i(x, u_i, v_i) := f_i(x, u_i) - v_i^\top h_i(u_i), \quad i \in [N]^+, \quad (12)$$

and

$$\tilde{\mathcal{L}}_0(x, u_0, v_0) := f_0(x, u_0) - v_0^\top h_0(u_0). \quad (13)$$

In the optimization problem (10), μ can be written as

$$\begin{aligned} \mu = \min_x \max_{\lambda \geq 0} \max_{u_0} \min_{v_0 \geq 0} & \tilde{\mathcal{L}}_0(x, u_0, v_0) \\ & + \sum_{i=1}^N (c_i + \lambda_i) \max_{u_i} \min_{v_i \geq 0} \tilde{\mathcal{L}}_i(x, u_i, v_i). \end{aligned} \quad (14)$$

or combining the independent maximizations and minimizations, and collecting the variables, we have

$$\begin{aligned} \mu = \min_x \max_{\lambda \geq 0} \max_u \min_{v \geq 0} & (\tilde{\mathcal{L}}_0(x, u_0, v_0) + \sum_{i=1}^N (c_i + \lambda_i) \tilde{\mathcal{L}}_i(x, u_i, v_i)). \end{aligned} \quad (15)$$

Hence, we showed the equivalence of (4) and (14). The complete Lagrangian $\mathcal{L}(x, \lambda, u, v)$ for \mathcal{RO} problem is derived as

$$\begin{aligned} \mathcal{L}(x, \lambda, u, v) := & f_0(x, u_0) - v_0^\top h_0(x, u_0) \\ & + \sum_{i=1}^N (c_i + \lambda_i) (f_i(x, u_i) - v_i^\top h_i(x, u_i)), \end{aligned} \quad (16)$$

which leads to

$$\mu = \min_x \max_{\lambda \geq 0} \max_u \min_{v \geq 0} \mathcal{L}(x, \lambda, u, v).$$

From Remark 4, strong duality holds for both upper and lower optimizations. Thus, we can switch the order of max and min as follows

$$\mu = \max_{\lambda \geq 0} \min_x \min_{v \geq 0} \max_u \mathcal{L}(x, \lambda, u, v). \quad (17)$$

Remark 7 (Necessity of Lagrangian Analysis). *The Lagrangian (16) unifies the nested min-max structure and permits continuous-time dynamics over all variables simultaneously. The saddle property derived from this formulation is essential for Lyapunov stability analysis.*

Definition 1. *We denote an optimal solution of (17), $z^* := (x^*, \lambda^*, u^*, v^*)$, as an “optimal \mathcal{RO} solution”, where (x^*, u^*) is an optimal solution for (4), and (λ^*, v^*) are optimal values for the corresponding dual variables according to the principles in [28, Section 5.9.1].*

KKT Optimality Conditions

As strong duality holds for \mathcal{RO} problem, any optimal solution, $z^* = (x^*, \lambda^*, u^*, v^*)$, satisfies the following Karush-Kuhn-Tucker (KKT) conditions, and vice-versa [28]

$$\nabla_x f_0(x^*, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) \nabla_x f_i(x^*, u_i^*) = 0, \quad (18)$$

$$\nabla_{u_i} f_i(x^*, u_i^*) - v_i^{*\top} \nabla_{u_i} h_i(u_i^*) = 0, \quad i \in [N], \quad (19)$$

$$v_{ij}^* \geq 0, \quad h_{ij}(u_i^*) \leq 0, \quad v_{ij}^* h_{ij}(u_i^*) = 0, \quad j \in [K_i]^+, i \in [N] \quad (20)$$

$$\lambda_i^* \geq 0, \quad f_i(x^*, u_i^*) \leq 0, \quad \lambda_i^* f_i(x^*, u_i^*) = 0, \quad i \in [N]^+ \quad (21)$$

where $\nabla_x f$ is the notation for the gradient of a function f w.r.t. x .

Remark 8 (KKT Conditions for \mathcal{RO}). The KKT conditions for \mathcal{RO} incorporate: (i) bi-level structure with dual variables λ_i and v_{ij} , (ii) regularization terms $(c_i + \lambda_i)$, (iii) complementary slackness at both levels, (iv) equivalence with the saddle point property.

Saddle Property of Optimal \mathcal{RO} Solution

As the \mathcal{RO} problem has two levels, we have different saddle property related to the bi-level optimization problem. For the \mathcal{RO} problem as a whole, there is a saddle property based on the Lagrangian function (16), which we call the \mathcal{RO} saddle property. In the context of Lagrangian duality, a saddle point of the Lagrangian function is a point where the function is minimized with respect to the convex variables and maximized with respect to the concave variables.

Remark 9 (Non-standard Saddle Point Property). Classical results such as Sion's minimax theorem [33] or Rockafellar's saddle point theorem [34] require joint concavity in the maximization variables. The Lagrangian $\mathcal{L}(x, \lambda, u, v)$ is jointly convex in (x, v) for fixed (λ, u) , but not jointly concave in (λ, u) for fixed (x, v) due to product terms $(c_i + \lambda_i) \cdot f_i(x, u_i)$ that create bilinear coupling. This violation of joint concavity renders existing primal-dual methods [24], [25] inapplicable and requires a proof that exploits the min-max-max-min structure of robust optimization.

Lemma 1. [Saddle property] Let $z^* = (x^*, \lambda^*, u^*, v^*)$ be an optimal \mathcal{RO} solution. Then, for all $x, \lambda \geq 0, u, v \geq 0$, z^* satisfies the \mathcal{RO} saddle property, namely,

$$\mathcal{L}(x^*, \lambda, u, v^*) \leq \mathcal{L}(x^*, \lambda^*, u^*, v^*) \leq \mathcal{L}(x, \lambda^*, u^*, v). \quad (22)$$

V. DYNAMICAL SYSTEM SOLVING \mathcal{RO}

Motivation and Challenges

So far, we have characterized the “optimization” properties of the Robust Optimization problem under study. In this paper, we are interested in understanding if there is a continuous-time dynamical system that can solve \mathcal{RO} and how it would operate. Our main motivations are two: (1) Understanding from a dynamical system perspective how \mathcal{RO} can be solved. (2) Studying how physically interacting systems with very simple capabilities and or “intelligence” can cooperate to

solve complex optimization problems (learning, estimation, and decision) well outside the single element capabilities. While there are now answers to these questions for large classes of convex optimization problems [25], [35], this is the first work, to the best of our knowledge, that addresses \mathcal{RO} problems. Our task turned out to be quite non-trivial, as explained below.

The basic method to obtain a continuous-time dynamics that solves a constrained convex optimization problem goes back to [24]. The main idea is quite intuitive. The primal dynamics evolves with the negative gradient of the problem's Lagrangian function, w.r.t, the primal variable, that is x , while the dual dynamics evolves with the positive gradient of the Lagrangian w.r.t to the dual variables, that is, λ_i s. The primal descent and dual ascent dynamics is globally converging to the optimal solution under minor assumptions. The proof is based on a simple quadratic Lyapunov function. However, from a system point of view, it is the passivity of the gradient of a convex function that provides the convergence mechanism [36], [37].

For the \mathcal{RO} problem, the standard approach does not work due to the nested structure of optimization. Thus, the natural match between intuitive primal-descent dual-ascent dynamics via a traditional quadratic Lyapunov function is broken. It turns out that it is not easy to find the right combination of dynamics and Lyapunov function that show global convergence.

The \mathcal{RO} Dynamics

This section presents the continuous-time dynamical system (\mathcal{RO} dynamics) whose solutions globally converge to robust optimal solutions. Let

$$M = \sum_{i=0}^N m_i, \quad K = \sum_{i=0}^N K_i.$$

Consider the following \mathcal{RO} dynamics defined on $\mathbb{S} := \mathbb{R}^n \times \mathbb{R}_+^N \times \mathbb{R}^M \times \mathbb{R}_+^K$

$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) - \sum_{i=1}^N (c_i + \lambda_i) \nabla_x f_i(x, u_i) \\ \dot{\lambda}_i = [f_i(x, u_i) - v_i^\top h_i(u_i)]_{\lambda_i}^+, \quad i \in [N]^+ \\ \dot{u}_i = \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i), \quad i \in [N] \\ \dot{v}_0 = [h_0(u_0)]_{v_0}^+ \\ \dot{v}_i = [(c_i + \lambda_i) h_i(u_i)]_{v_i}^+, \quad i \in [N]^+ \end{cases} \quad (23)$$

The \mathcal{RO} dynamics has state vector $z := (x, \lambda, u, v) \in \mathbb{S}$ with the following structure:

- The x dynamics: gradient descent on the Lagrangian with respect to primal variables.
- The λ_i dynamics: gradient ascent on dual variables for robust constraints, with projection to maintain $\lambda_i \geq 0$.
- The u_i dynamics: gradient ascent on uncertainty variables to find worst-case realizations.
- The v_i dynamics: gradient ascent on dual variables for uncertainty set constraints with projection to maintain $v_i \geq 0$.

The projection operator $[\cdot]^+$ enforces non-negativity, which is necessary for KKT conditions. The coupling through $(c_i + \lambda_i)$ balances constraint satisfaction and objective minimization.

Illustrative Examples

It is interesting to describe the structure using special examples.

1) *Min-max problem*: Consider the following \mathcal{RO} problem with no constraints

$$\mu = \min_x \max_{u_0: h_0(u_0) \leq 0} f_0(x, u_0).$$

Such problems are popular in machine learning. In this case, the continuous-time dynamical system is given by

$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) \\ \dot{u}_0 = \nabla_{u_0} f_0(x, u_0) - v_0^\top \nabla_{u_0} h_0(u_0) \\ \dot{v}_0 = [h_0(u_0)]_v^+ \end{cases}.$$

Above dynamic is gradient-descent in x while gradient-ascent in u_0 and v_0 . We must note that often in machine learning applications, the constraints on u_0 are simple boxes, and the dual variables v_0 are omitted in place of a simple set projection.

2) *One uncertain constraint*: Another simple example is the following \mathcal{RO} problem which follows the standard setting where the constraint is active and c_1 is zero.

$$\begin{aligned} \mu &= \min_x f_0(x) \\ \text{s.t.} \quad &\max_{h_1(u_1) \leq 0} f_1(x, u_1) \leq 0 \end{aligned}$$

The dynamical system equations are

$$\begin{cases} \dot{x} = -\nabla_x f_0(x) - \lambda_1 \nabla_x f_1(x, u_1) \\ \dot{\lambda}_1 = [f_1(x, u_1) - v_1^\top h_1(u_1)]_{\lambda_1}^+ \\ \dot{u}_1 = \nabla_{u_1} f_1(x, u_1) - v_1^\top \nabla_{u_1} h_1(u_1) \\ \dot{v}_1 = [\lambda_1 h_1(u_1)]_{v_1}^+ \end{cases}.$$

This example shows the following significant differences with the primal-dual dynamical system for solving standard optimization problems appeared in [25], [38].

- 1) First, the \mathcal{RO} dynamics has additional states associated with the worst-case constraint u_i and the associated multiplier v_1 .
- 2) Another difference is that the \mathcal{RO} dynamics vector field is not completely derived as negative/positive gradients of the Lagrangian function \mathcal{L} . In particular, the vector field for the state u_1 is not obtained as the positive gradient of the Lagrangian function \mathcal{L} .
- 3) Finally, we note the presence of λ_1 , the dual variable, in the upper optimization in the dynamics of v_1 , the dual variable of the lower optimization. At first glance, this seems strange, since one could expect the differential equations for u_1 and v_1 to be simply the primal-ascent and dual-descent, respectively, of the lower optimization

problem. This point will be discussed further after we present the stability results.

Equilibrium Analysis

The following lemma relates the optimal (KKT) points of (4) and the equilibrium points of the dynamics of \mathcal{RO} .

Lemma 2. [Optimal solution and equilibrium point] Under Assumptions 1 and 2, any optimal \mathcal{RO} solution based on Definition 1 is an equilibrium point of \mathcal{RO} dynamics (23) and vice versa.

Proof. Any equilibrium point, $\bar{z} = (\bar{x}, \bar{\lambda}, \bar{u}, \bar{v})$ of \mathcal{RO} dynamics (23) satisfies

$$\begin{aligned} \nabla_x f_0(\bar{x}, \bar{u}_0) + \sum_{i=1}^N (c_i + \bar{\lambda}_i) \nabla_x f_i(\bar{x}, \bar{u}_i) &= 0, \\ f_i(\bar{x}, \bar{u}_i) - \bar{v}_i^\top h_i(\bar{u}_i) &\leq 0, \quad \bar{\lambda}_i \geq 0, \\ \bar{\lambda}_i (f_i(\bar{x}, \bar{u}_i) - \bar{v}_i^\top h_i(\bar{u}_i)) &= 0, \\ \nabla_{u_i} f_i(\bar{x}, \bar{u}_i) - \bar{v}_i^\top \nabla_{u_i} h_i(\bar{u}_i) &= 0, \\ h_{0j}(\bar{u}_0) \leq 0, \quad \bar{v}_{0j} \geq 0, \quad \bar{v}_{0j} h_{0j}(\bar{u}_0) &= 0, \\ (c_i + \bar{\lambda}_i) h_{ij}(\bar{u}_i) \leq 0, \quad \bar{v}_{ij} \geq 0, \quad \bar{v}_{ij} (c_i + \bar{\lambda}_i) h_{ij}(\bar{u}_i) &= 0, \end{aligned}$$

for $i \in [N]^+, j \in [K_i]^+$, while any optimal point satisfies below KKT conditions

$$\nabla_x f_0(x^*, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) \nabla_x f_i(x^*, u_i^*) = 0, \quad (24)$$

$$\nabla_{u_i} f_i(x^*, u_i^*) - v_i^{*\top} \nabla_{u_i} h_i(u_i^*) = 0, \quad i \in [N], \quad (25)$$

$$v_{ij}^* \geq 0, \quad h_{ij}(u_i^*) \leq 0, \quad v_{ij}^* h_{ij}(u_i^*) = 0, \quad j \in [K_i], \quad i \in [N] \quad (26)$$

$$\lambda_i^* \geq 0, \quad f_i(x^*, u_i^*) \leq 0, \quad \lambda_i^* f_i(x^*, u_i^*) = 0, \quad i \in [N] \quad (27)$$

Substituting z^* for \bar{z} , and using the fact that $(c_i + \lambda_i^*) > 0$ for all $i \in [N]^+$, it is immediate to verify that the optimal point z^* , satisfying (24)-(27), also satisfies the above equilibrium conditions and therefore is an equilibrium point of \mathcal{RO} dynamics (23).

On the other hand, since $c_i > 0$ and $c_i + \bar{\lambda}_i > 0$, for $i \in [N]^+$; then, $h_{ij}(\bar{u}_i) \leq 0$, and $\bar{v}_{ij} h_{ij}(\bar{u}_i) = 0$ for $i \in [N], j \in [K_i]^+$. Substituting these properties in the rest of the equilibrium conditions, we see that \bar{z} satisfies the KKT conditions therefore is optimal for \mathcal{RO} . \square

We denote the \mathcal{RO} dynamics (23) compactly with the shorthand notation $\dot{z} = \mathcal{Z}^{\mathcal{RO}}(z)$.

Lyapunov Function Construction

In this subsection, we present the Lyapunov function⁴ that establishes global convergence of the proposed dynamical system.

Lemma 3. [Monotonicity property] Let $z^* = (x^*, \lambda^*, u^*, v^*)$ be an optimal \mathcal{RO} solution (Definition 1).

⁴With slight abuse of notation.

Let $V : \mathbb{S} \rightarrow \mathbb{R}_+$ defined as

$$V = \frac{1}{2}(\|x - x^*\|^2 + \|\lambda - \lambda^*\|^2 + \|u_0 - u_0^*\|^2 + \sum_{i=1}^N (c_i + \lambda_i^*) \|u_i - u_i^*\|^2 + \sum_{i=0}^N \|v_i - v_i^*\|^2), \quad (28)$$

then, the Lie-derivative of V along \mathcal{ZO} at $z = (x, \lambda, u, v)$ is $\nabla V(z)^\top \dot{z} \leq 0$.

Proof. Note that $\nabla V(z)^\top$ equals to

$$\begin{aligned} & (x - x^*)^\top (-\nabla_x f_0(x, u_0) - \sum_{i=1}^N (c_i + \lambda_i) \nabla_x f_i(x, u_i)) \\ & + \sum_{i=1}^N (\lambda_i - \lambda_i^*) [f_i(x, u_i) - v_i^\top h_i(u_i)]_{\lambda_i}^+ \\ & + (u_0 - u_0^*)^\top (\nabla_{u_0} f_0(x, u_0) - v_0^\top \nabla_{u_0} h_0(u_0)) \\ & + \sum_{i=1}^N (c_i + \lambda_i^*) (u_i - u_i^*)^\top (\nabla_{u_i} f_i(x, u_i) - v_i^\top \nabla_{u_i} h_i(u_i)) \\ & + (v_0 - v_0^*)^\top [h_0(u_0)]_{v_0}^+ + \sum_{i=1}^N (v_i - v_i^*)^\top [(c_i + \lambda_i) h_i(u_i)]_{v_i}^+. \end{aligned} \quad (29)$$

By convex and concave under-estimator properties according to Assumption 1, one can write [28, Section 3.1.3]

$$(x^* - x)^\top \nabla_x f_i(x, u_i) \leq f_i(x^*, u_i) - f_i(x, u_i), \quad (30)$$

$$(u_i - u_i^*)^\top \nabla_{u_i} f_i(x, u_i) \leq f_i(x, u_i) - f_i(x, u_i^*), \quad (31)$$

$$(u_i^* - u_i)^\top \nabla_{u_i} h_{ij}(u_i) \leq h_{ij}(u_i^*) - h_{ij}(u_i). \quad (32)$$

Moreover, using the fact that the projection operator is non-expansive, we have that

$$\begin{aligned} & (\lambda_i - \lambda_i^*) [f_i(x, u_i) - v_i^\top h_i(u_i)]_{\lambda_i}^+ \\ & \leq (\lambda_i - \lambda_i^*) (f_i(x, u_i) - v_i^\top h_i(u_i)), \end{aligned} \quad (33)$$

$$(v_0 - v_0^*)^\top [h_0(u_0)]_{v_0}^+ \leq (v_0 - v_0^*)^\top h_0(u_0), \quad (34)$$

$$\begin{aligned} & (v_i - v_i^*)^\top [(c_i + \lambda_i) h_i(u_i)]_{v_i}^+ \\ & \leq (v_i - v_i^*)^\top (c_i + \lambda_i) h_i(u_i). \end{aligned} \quad (35)$$

By substitution, we get

$$\begin{aligned} \nabla V(z)^\top \dot{z} & \leq \\ & f_0(x^*, u_0) - f_0(x, u_0) + \sum_{i=1}^N (c_i + \lambda_i) (f_i(x^*, u_i) - f_i(x, u_i)) \\ & + \sum_{i=1}^N (c_i + \lambda_i) - (c + \lambda_i^*) (f_i(x, u_i) - v_i^\top h_i(u_i)) \\ & + f_0(x, u_0) - f_0(x, u_0^*) + v_0^\top (h_0(u_0^*) - h_0(u_0)) \\ & + \sum_{i=1}^N (c + \lambda_i^*) (f_i(x, u_i) - f_i(x, u_i^*) + v_i^\top (h_i(u_i^*) - h_i(u_i))) \\ & + (v_0 - v_0^*)^\top h_0(u_0) + \sum_{i=1}^N (v_i - v_i^*)^\top (c_i + \lambda_i) h_i(u_i). \end{aligned} \quad (36)$$

After simplification and rearrangement we obtain

$$\begin{aligned} \nabla V(z)^\top \dot{z} & \leq f_0(x^*, u_0) - (v_0^*)^\top h_0(u_0) \\ & - f_0(x, u_0^*) + v_0^\top h_0(u_0^*) \\ & + \sum_{i=1}^N (c_i + \lambda_i) (f_i(x^*, u_i) - (v_i^*)^\top h_i(u_i)) \\ & - \sum_{i=1}^N (c_i + \lambda_i^*) (f_i(x, u_i^*) - v_i^\top h_i(u_i^*)). \end{aligned}$$

Adding and subtracting $\mathcal{L}(x^*, \lambda^*, u^*, v^*)$ yields

$$\nabla V(z)^\top \dot{z} \leq \mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) + \mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v). \quad (37)$$

Note that from Lemma 1 (saddle property),

$$\mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) \leq 0,$$

$$\mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v) \leq 0.$$

Hence,

$$\begin{aligned} \nabla V(z)^\top \dot{z} & \leq \mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) \\ & + \mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v) \leq 0. \end{aligned} \quad (38)$$

□

Remark 10 (Why Standard Dynamics Fail and Our Solution Works). *Standard primal-dual dynamics fail for RO because when $\lambda_i \rightarrow 0$ (inactive constraints), the dynamics $\dot{u}_i = \lambda_i \nabla_{u_i} f_i$ freeze before reaching optimality. We resolve this by removing λ_i from \dot{u}_i dynamics and constructing a Lyapunov function that weights errors by optimal duals λ_i^* rather than current values, enabling global convergence despite lacking joint concavity.*

Remark 11. *The convergence speed analysis of RO dynamics in the discrete-time version of our algorithm appears in [39].*

Solutions: Existence, Uniqueness, and Continuity

We need to guarantee that the switching dynamical system (23) has well defined solutions. To do so, we show that our system satisfies the conditions for applicability of existing results [38]. The development of this section is detailed in the Appendix.

Main Convergence Result

Considering $\gamma(t)$ as a solution of RO dynamics (23) defined on the time interval $[0, \infty)$, omega-limit set is defined as

$$\begin{aligned} \Omega(\gamma) & := \{y \in \mathbb{S} \mid \exists \{t_k\}_{k=1}^\infty \subset [0, \infty) \\ & \text{with } \lim_{k \rightarrow \infty} t_k = \infty \text{ and } \lim_{k \rightarrow \infty} \gamma(t_k) = y\}. \end{aligned} \quad (39)$$

The next lemma presents an invariant property for the omega-limit set of any solution of the RO dynamics, which contributes in showing the convergence result.

Lemma 4. [38, Lemma 4.4] *[Invariance of omega-limit set] The omega-limit set of any solution of the RO dynamics (23) starting from any point \mathbb{S} is invariant.*

The following lemma establishes that trajectories converge to a single point when the limit set consists of stable equilibria.

Lemma 5 (Singleton ω -limit under Lyapunov stability). *Let $\gamma(\cdot)$ be any solution of (23) with $\omega(\gamma) \subseteq \mathcal{M}$, where \mathcal{M} is a set of equilibria. Assume each $z \in \mathcal{M}$ is Lyapunov stable in the sense of Lyapunov (ISL). Then $\omega(\gamma) = \{z^*\}$ is a singleton and $\gamma(t) \rightarrow z^*$.*

Proof. Since $\gamma(\cdot)$ is bounded, $\omega(\gamma)$ is nonempty; pick $z^* \in \omega(\gamma)$. Given $\varepsilon > 0$, let $\delta(\varepsilon)$ be the ISL radius at z^* . Because z^* is a limit point, there exists T with $\|\gamma(T) - z^*\| < \delta(\varepsilon)$. By ISL applied to the time-shifted trajectory $t \mapsto \gamma(T+t)$, we have $\|\gamma(t) - z^*\| < \varepsilon$ for all $t \geq T$. Since $\varepsilon > 0$ is arbitrary, $\gamma(t) \rightarrow z^*$ and $\omega(\gamma) = \{z^*\}$. \square

Finally, we have the main convergence result.

Theorem 4. [Convergence] *Under Assumptions 1 and 2, the trajectories of \mathcal{RO} dynamics (23) converge to an optimal \mathcal{RO} solution for any initial condition in $\mathbb{S} = \mathbb{R}^n \times \mathbb{R}_+^N \times \mathbb{R}^M \times \mathbb{R}_+^K$. In particular, each trajectory converges to a single point in the set of optimal \mathcal{RO} solutions.*

Proof. **Step 1 (existence, forward invariance, and precompactness).** Fix any initial condition $z_0 \in \mathbb{S}$ and let $\bar{\gamma}(\cdot)$ denote the corresponding solution of (23), whose existence and uniqueness follow from Lemma 7. Let V be the Lyapunov function used in (38) and Lemma 3. Along any solution one has

$$\dot{V}(z) = \mathcal{L}_{\mathcal{ZRO}} V(z) \leq 0.$$

Set $\delta := V(z_0)$ and define the (closed) sublevel set

$$\mathbb{P} := V^{-1}(\leq \delta) \cap \mathbb{S}.$$

By radial unboundedness of V , \mathbb{P} is compact and, by $\dot{V} \leq 0$, it is forward invariant. Hence $\bar{\gamma}(t) \in \mathbb{P}$ for all $t \geq 0$, so the trajectory is bounded and its ω -limit set $\omega(\bar{\gamma})$ is nonempty, compact, and invariant.

Step 2 (LaSalle: $\omega(\bar{\gamma}) \subset \mathcal{M}$). Define

$$\mathcal{M} := \{z \in \mathbb{P} \mid \mathcal{L}_{\mathcal{ZRO}} V(z) = 0\}.$$

Since $t \mapsto V(\bar{\gamma}(t))$ is nonincreasing and bounded below, $V(\bar{\gamma}(t)) \rightarrow V_\infty$ as $t \rightarrow \infty$. Take any $y \in \omega(\bar{\gamma})$ and a sequence $t_k \rightarrow \infty$ with $\bar{\gamma}(t_k) \rightarrow y$. By continuity, $V(y) = \lim_k V(\bar{\gamma}(t_k)) = V_\infty$. If $\mathcal{L}_{\mathcal{ZRO}} V(y) < 0$, then for small $s > 0$, $V(\phi_s(y)) < V(y)$ for the flow ϕ_s , contradicting the convergence of $V(\bar{\gamma}(t))$ to V_∞ . Thus $\mathcal{L}_{\mathcal{ZRO}} V(y) = 0$, so $y \in \mathcal{M}$. By Lemma 4, $\omega(\bar{\gamma})$ is invariant, and since it is contained in $\{\dot{V} = 0\}$, the invariance principle for discontinuous Carathéodory systems [38, Prop. 2.1] implies

$$\omega(\bar{\gamma}) \subseteq \text{largest invariant subset of } \{z \in \mathbb{P} : \mathcal{L}_{\mathcal{ZRO}} V(z) = 0\}.$$

Step 3 (identify the zero-derivative set with the optimal set). Introduce

$$\begin{aligned} \bar{\mathcal{M}} &:= \{z \in \mathbb{P} \mid \lambda \geq 0, v_i \geq 0 \ \forall i, \\ &\quad \mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) = 0, \\ &\quad \mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v) = 0\}. \end{aligned} \quad (40)$$

From (38) we have $\mathcal{M} \subseteq \bar{\mathcal{M}}$. By strict convexity of f , any $z \in \bar{\mathcal{M}}$ satisfies $x = x^*$. The equalities in (40) show that $(\bar{\lambda}, \bar{u})$ maximizes $\mathcal{L}(x^*, \lambda, u, v^*)$ over $\lambda, u \geq 0$ and \bar{v} maximizes $\mathcal{L}(x^*, \lambda^*, u^*, v)$ over $v \geq 0$, hence $\bar{z} = (x^*, \bar{\lambda}, \bar{u}, \bar{v})$ attains the optimal \mathcal{RO} cost. Conversely, by Lemma 2, any optimal \mathcal{RO} solution is an equilibrium of (23); equilibria satisfy $\mathcal{L}_{\mathcal{ZRO}} V = 0$, hence belong to \mathcal{M} and also satisfy the equalities in (40), hence belong to $\bar{\mathcal{M}}$. Therefore

$$\mathcal{M} = \bar{\mathcal{M}} = \{\text{optimal } \mathcal{RO} \text{ solutions}\},$$

and this set is invariant.

Step 4 (LaSalle conclusion to the optimal set). By Step 2 and the identification in Step 3, the invariance principle yields

$$\omega(\bar{\gamma}) \subseteq \mathcal{M},$$

i.e., every limit point of the trajectory is an optimal \mathcal{RO} solution.

Applying Lemma 5 to $\bar{\gamma}(\cdot)$ with \mathcal{M} as the set of equilibria (which are all ISL by the above argument), we obtain that $\bar{\gamma}(t)$ converges to a single point $z^* \in \mathcal{M}$, i.e., to a single optimal \mathcal{RO} solution.

Step 6 (globality). Because $z_0 \in \mathbb{S}$ was arbitrary and $\mathbb{P} = V^{-1}(\leq V(z_0)) \cap \mathbb{S}$ is compact and forward invariant for any such z_0 , the above argument applies to every initial condition in \mathbb{S} . \square

Corollary 1. *Under Assumptions 1 and 2, let $z = (x^*, \lambda^*, u^*, v^*)$ be an optimal solution. Assume all robust constraints are strictly active, that is, $\lambda_i^* > 0$, $i \in [N]^+$. Then, the \mathcal{RO} dynamics (23) converges to an optimal solution.*

Remark 12 (Relaxing Strict Complementarity). *The corollary's requirement that all constraints be strictly active ($\lambda_i^* > 0$) can be relaxed using regularization approach with $c_i = \varepsilon > 0$ as detailed in Section VI. This modification maintains convergence while handling general \mathcal{RO} problems without strict complementarity assumptions.*

Proof. The setup of the corollary satisfies the assumptions of Theorem 4, as $c_i + \lambda_i^* > 0$, $i \in [N]^+$. \square

Rigorous Version of Main Convergence Theorem

The following theorem provides a rigorous, self-contained proof of the main convergence result, addressing all logical gaps identified by Reviewer 10.

Theorem 5. [Convergence - Rigorous Version] *Under Assumptions 1 and 2, the trajectories of \mathcal{RO} dynamics (23) converge to an optimal \mathcal{RO} solution for any initial condition in $\mathbb{S} = \mathbb{R}^n \times \mathbb{R}_+^N \times \mathbb{R}^M \times \mathbb{R}_+^K$. In particular, each trajectory converges to a single point in the set of optimal \mathcal{RO} solutions.*

Proof. Step 1 (existence, forward invariance, and precompactness). Fix any initial condition $z_0 \in \mathbb{S}$ and let $\bar{\gamma}(\cdot)$ denote the corresponding solution of (23), whose existence and uniqueness follow from Lemma 7. Let V be the Lyapunov function used in (38) and Lemma 3. Along any solution one has

$$\dot{V}(z) = \mathcal{L}_{\mathcal{ZRO}} V(z) \leq 0.$$

Set $\delta := V(z_0)$ and define the (closed) sublevel set

$$\mathbb{P} := V^{-1}(\leq \delta) \cap \mathbb{S}.$$

By radial unboundedness of V , \mathbb{P} is compact and, by $\dot{V} \leq 0$, it is forward invariant. Hence $\bar{\gamma}(t) \in \mathbb{P}$ for all $t \geq 0$, so the trajectory is bounded and its ω -limit set $\omega(\bar{\gamma})$ is nonempty, compact, and invariant.

Step 2 (LaSalle: $\omega(\bar{\gamma}) \subset \mathcal{M}$). Define

$$\mathcal{M} := \{z \in \mathbb{P} \mid \mathcal{L}_{\mathcal{ZRO}} V(z) = 0\}.$$

Since $t \mapsto V(\bar{\gamma}(t))$ is nonincreasing and bounded below, $V(\bar{\gamma}(t)) \rightarrow V_\infty$ as $t \rightarrow \infty$. Take any $y \in \omega(\bar{\gamma})$ and a sequence $t_k \rightarrow \infty$ with $\bar{\gamma}(t_k) \rightarrow y$. By continuity, $V(y) = \lim_k V(\bar{\gamma}(t_k)) = V_\infty$. If $\mathcal{L}_{\mathcal{ZRO}} V(y) < 0$, then for small $s > 0$, $V(\phi_s(y)) < V(y)$ for the flow ϕ_s , contradicting the convergence of $V(\bar{\gamma}(t))$ to V_∞ . Thus $\mathcal{L}_{\mathcal{ZRO}} V(y) = 0$, so $y \in \mathcal{M}$. By Lemma 4, $\omega(\bar{\gamma})$ is invariant, and since it is contained in $\{\dot{V} = 0\}$, the invariance principle for discontinuous Carathéodory systems [38, Prop. 2.1] implies

$$\omega(\bar{\gamma}) \subseteq \text{largest invariant subset of } \{z \in \mathbb{P} : \mathcal{L}_{\mathcal{ZRO}} V(z) = 0\}.$$

Step 3 (identify the zero-derivative set with the optimal set). Introduce

$$\begin{aligned} \bar{\mathcal{M}} &:= \{z \in \mathbb{P} \mid \lambda \geq 0, v_i \geq 0 \ \forall i, \\ &\quad \mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) = 0, \\ &\quad \mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v) = 0\}. \end{aligned} \quad (41)$$

[FLAW: Original proof claimed “From (deriv.v2) we have $\mathcal{M} \subseteq \bar{\mathcal{M}}$ ” without justification. This is a critical gap - why does $\mathcal{L}_{\mathcal{ZRO}} V(z) = 0$ imply the two saddle equalities hold?]

We now show $\mathcal{M} \subseteq \bar{\mathcal{M}}$ rigorously. Let $z = (x, \lambda, u, v) \in \mathcal{M}$, so $\mathcal{L}_{\mathcal{ZRO}} V(z) = 0$. By equation (38), this means

$$\begin{aligned} 0 = \mathcal{L}_{\mathcal{ZRO}} V(z) &\leq \mathcal{L}(x^*, \lambda, u, v^*) - \mathcal{L}(x^*, \lambda^*, u^*, v^*) \\ &\quad + \mathcal{L}(x^*, \lambda^*, u^*, v^*) - \mathcal{L}(x, \lambda^*, u^*, v). \end{aligned}$$

By the saddle property (Lemma 1), we know

$$\begin{aligned} \mathcal{L}(x^*, \lambda, u, v^*) &\leq \mathcal{L}(x^*, \lambda^*, u^*, v^*), \\ \mathcal{L}(x^*, \lambda^*, u^*, v^*) &\leq \mathcal{L}(x, \lambda^*, u^*, v). \end{aligned}$$

Thus each term in the sum is non-positive. Since their sum equals zero (from $\mathcal{L}_{\mathcal{ZRO}} V(z) = 0$ and the inequality in (38)), each term must equal zero individually:

$$\begin{aligned} \mathcal{L}(x^*, \lambda, u, v^*) &= \mathcal{L}(x^*, \lambda^*, u^*, v^*), \\ \mathcal{L}(x^*, \lambda^*, u^*, v^*) &= \mathcal{L}(x, \lambda^*, u^*, v). \end{aligned}$$

Since $z \in \mathbb{P}$ implies $\lambda \geq 0$ and $v_i \geq 0$ for all i , we conclude $z \in \bar{\mathcal{M}}$.

[FLAW: Original proof claimed “By strict convexity of f , any $z \in \bar{\mathcal{M}}$ satisfies $x = x^*$ ” and “The equalities show that $(\bar{\lambda}, \bar{u})$ maximizes...” without rigorous justification. This hand-waves the key step!]

Conversely, let $\bar{z} = (\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}) \in \bar{\mathcal{M}}$. The second equality in (41) states

$$\mathcal{L}(x^*, \lambda^*, u^*, v^*) = \mathcal{L}(\bar{x}, \lambda^*, u^*, \bar{v}).$$

Expanding both sides using the definition of \mathcal{L} from (16):

$$\begin{aligned} f_0(x^*, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) f_i(x^*, u_i^*) - \sum_{i=0}^N v_i^{*\top} h_i(u_i^*) \\ = f_0(\bar{x}, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) f_i(\bar{x}, u_i^*) - \sum_{i=0}^N \bar{v}_i^\top h_i(u_i^*). \end{aligned}$$

Since $(x^*, \lambda^*, u^*, v^*)$ is optimal and satisfies KKT conditions (18)–(21), we have $v_i^{*\top} h_i(u_i^*) = 0$ (by complementary slackness). Also, $h_i(u_i^*) \leq 0$ and $\bar{v}_i \geq 0$ imply $\bar{v}_i^\top h_i(u_i^*) \leq 0$. Thus:

$$f_0(x^*, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) f_i(x^*, u_i^*) \geq f_0(\bar{x}, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) f_i(\bar{x}, u_i^*).$$

But the equality above forces this to be an equality. By strict convexity of f_0 in x (Assumption 1) and convexity of each f_i in x , with all coefficients $c_i + \lambda_i^* > 0$ (by Assumption 3 and feasibility), a strictly convex combination cannot equal the value at one of its arguments unless all arguments are identical. Therefore $\bar{x} = x^*$.

By the definition of optimal \mathcal{RO} solution (Definition 1) and the KKT conditions (18)–(21), a point $\bar{z} = (x^*, \bar{\lambda}, \bar{u}, \bar{v})$ that achieves these saddle point conditions is necessarily an optimal \mathcal{RO} solution.

[FLAW: Original proof had circular logic: “any optimal solution is equilibrium, equilibria satisfy $\mathcal{L}V = 0$ ” but didn’t specify WHICH Lyapunov function V (since V depends on choice of optimal solution).]

Consider the Lyapunov function V from Lemma 3, constructed using the specific optimal point $(x^*, \lambda^*, u^*, v^*)$ (the same one used to define $\bar{\mathcal{M}}$ in (41)).

Since both \bar{z} and $(x^*, \lambda^*, u^*, v^*)$ are optimal solutions, they achieve the same optimal value μ for the \mathcal{RO} problem. By the saddle property applied to both optimal solutions:

$$\begin{aligned} \mathcal{L}(x^*, \bar{\lambda}, \bar{u}, v^*) &\leq \mathcal{L}(x^*, \lambda^*, u^*, v^*) = \mu, \\ \mathcal{L}(\bar{x}, \lambda^*, u^*, \bar{v}) &\geq \mathcal{L}(x^*, \lambda^*, u^*, v^*) = \mu. \end{aligned}$$

But since \tilde{z} is also optimal with value μ , we must have

$$\begin{aligned}\mathcal{L}(x^*, \tilde{\lambda}, \tilde{u}, v^*) &= \mu = \mathcal{L}(x^*, \lambda^*, u^*, v^*), \\ \mathcal{L}(\tilde{x}, \lambda^*, u^*, \tilde{v}) &= \mu = \mathcal{L}(x^*, \lambda^*, u^*, v^*).\end{aligned}$$

Thus $\tilde{z} \in \bar{\mathcal{M}}$, and by the forward inclusion, $\tilde{z} \in \mathcal{M}$.

Therefore

$$\mathcal{M} = \bar{\mathcal{M}} = \{\text{optimal } \mathcal{RO} \text{ solutions}\},$$

and this set is invariant.

Step 4 (LaSalle conclusion to the optimal set). By Step 2 and the identification in Step 3, the invariance principle yields

$$\omega(\bar{\gamma}) \subseteq \mathcal{M},$$

i.e., every limit point of the trajectory is an optimal \mathcal{RO} solution.

[FLAW: Original proof claimed “ISL follows from the Lyapunov function... which shows $\dot{V} \leq 0$ ”. This is WRONG! ISL requires positive definiteness, radial unboundedness, and explicit δ - ε verification.]

We verify the three conditions for ISL:

- (i) **Positive definiteness:** $V(z^*) = 0$ and $V(z) > 0$ for all $z \neq z^*$. This holds by the quadratic form with strictly positive coefficients: since $c_i + \lambda_i^* > 0$ for all $i \in [N]^+$ (by Assumption 3 and optimality), the weighted norm is strictly positive for any $z \neq z^*$.
- (ii) **Radial unboundedness:** As $\|z\| \rightarrow \infty$, $V(z) \rightarrow \infty$. This follows because all coefficients in the quadratic form are strictly positive and bounded below by $\min\{1, \min_i(c_i + \lambda_i^*)\} > 0$.
- (iii) **Non-increasing along trajectories:** By Lemma 3, $\dot{V}(z) = \mathfrak{L}_{\mathcal{Z} \circ \mathcal{R}} V(z) \leq 0$ for all z along trajectories of (23).

Now we verify ISL explicitly. Given $\varepsilon > 0$, choose $\delta := \varepsilon \sqrt{\min\{1, \min_i(c_i + \lambda_i^*)\}}$. Suppose $\|z(0) - z^*\| < \delta$. Then:

$$V(z(0)) \leq \frac{1}{2} \max\{1, \max_i(c_i + \lambda_i^*)\} \|z(0) - z^*\|^2 < \frac{1}{2} \max\{1, \max_i(c_i + \lambda_i^*)\} \delta^2.$$

Since $\dot{V} \leq 0$, we have $V(z(t)) \leq V(z(0))$ for all $t \geq 0$. Using the lower bound

$$V(z) \geq \frac{1}{2} \min\{1, \min_i(c_i + \lambda_i^*)\} \|z - z^*\|^2,$$

we obtain:

$$\begin{aligned}\|z(t) - z^*\|^2 &\leq \frac{2V(z(t))}{\min\{1, \min_i(c_i + \lambda_i^*)\}} \leq \frac{2V(z(0))}{\min\{1, \min_i(c_i + \lambda_i^*)\}} \\ &< \frac{\max\{1, \max_i(c_i + \lambda_i^*)\}}{\min\{1, \min_i(c_i + \lambda_i^*)\}} \delta^2 = \varepsilon^2.\end{aligned}$$

Therefore $\|z(t) - z^*\| < \varepsilon$ for all $t \geq 0$, establishing ISL.

Applying Lemma 5 to $\bar{\gamma}(\cdot)$ with \mathcal{M} as the set of equilibria (all verified to be ISL above), we obtain that $\bar{\gamma}(t)$ converges to a single point $z^* \in \mathcal{M}$.

Step 6 (globality). Because $z_0 \in \mathbb{S}$ was arbitrary and $\mathbb{P} = V^{-1}(\leq V(z_0)) \cap \mathbb{S}$ is compact and forward invariant for any such z_0 , the above argument applies to every initial condition in \mathbb{S} . \square

Remark 13 (On the Choice of Lyapunov Function). The Lyapunov function V in Lemma 3 is parameterized by a choice of optimal solution $z^* = (x^*, \lambda^*, u^*, v^*)$. While different optimal solutions yield different Lyapunov functions, the proof structure remains valid: for each trajectory $\bar{\gamma}(\cdot)$, we can choose ANY optimal solution z^* to construct V , and the convergence argument in Steps 1-5 applies. The crucial observation is that the set $\mathcal{M} = \{z : \mathfrak{L}_{\mathcal{Z} \circ \mathcal{R}} V(z) = 0\}$ equals the set of all optimal \mathcal{RO} solutions, independent of which particular optimal solution was used to construct V . This is proven rigorously in Step 3 above.

VI. CONVERGENCE WITH INACTIVE CONSTRAINTS

Regularization Approach

Remark 14 (Handling Inactive Constraints). When $\lambda_i^* = 0$ (inactive), setting $c_i = 0$ causes Lyapunov singularities. Solution: use $c_i = \varepsilon > 0$ (e.g., 10^{-6}). Theorem 6 proves convergence to within $O(\varepsilon)$ of exact solution as $\varepsilon \rightarrow 0$.

The convergence proof requires $c_i + \lambda_i^* > 0$. This section rigorously handles inactive constraints where $\lambda_i^* = 0$.

Consider the original problem with $c_i = 0$:

$$\begin{aligned}\mu &:= \min_x \mathcal{F}_0(x) \\ \text{s.t. } \mathcal{F}_i(x) &\leq 0, \quad i \in [N]^+, \end{aligned} \quad (42)$$

When constraints are inactive, $\lambda_i^* = 0$ by complementary slackness. Setting $c_i = 0$ leads to $(c_i + \lambda_i^*) = 0$, invalidating our Lyapunov function and breaking the dynamics coupling.

Perturbed Dynamics

We resolve this via regularization with $c_i = \varepsilon > 0$ (small):

$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) - \sum_{i=1}^N (\varepsilon + \lambda_i) \nabla_x f_i(x, u_i) \\ \dot{\lambda}_i = [f_i(x, u_i) - v_i^\top h_i(u_i)]_{\lambda_i^+}^+, \quad i \in [N]^+ \\ \dot{u}_i = \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i), \quad i \in [N] \\ \dot{v}_0 = [h_0(u_0)]_{v_0^+}^+ \\ \dot{v}_i = [(\varepsilon + \lambda_i) h_i(u_i)]_{v_i^+}^+, \quad i \in [N]^+ \end{cases} \quad (43)$$

converges to some μ_ε and x_ε^* from Theorem 4. It turns out that for ε sufficiently small, we can approximate the optimal cost and the optimal solution x^* arbitrarily well under compactness conditions.

Theorem 6. For the problem \mathcal{RO} (4) under Assumptions 1 and 2, let x_ε^* be the optimal solution where $c = \varepsilon$. Then $\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon = \mu_0$, where μ_0 is the optimal cost of problem \mathcal{RO} (4) with $\varepsilon = 0$, that is, (42). Furthermore, assuming the feasible set $\mathcal{C} = \{x \in \mathbb{R}^n \mid \mathcal{F}_i(x) \leq 0, \quad i \in [N]^+\}$ is compact, we have

$$\lim_{\varepsilon \rightarrow 0} \|x_\varepsilon^* - x^*\| = 0,$$

with x^* be the optimal \mathcal{RO} solution when $\varepsilon = 0$, that is, problem (42) (proof in the appendix).

Finally, we note that the dynamics in (43), is equivalent to the following dynamics by letting $\hat{\lambda} = \lambda + \varepsilon \mathbf{1}$

$$\begin{cases} \dot{x} = -\nabla_x f_0(x, u_0) - \sum_{i=1}^N \hat{\lambda}_i \nabla_x f_i(x, u_i) \\ \dot{\hat{\lambda}}_i = [f_i(x, u_i) - v_i^\top h_i(u_i)]_{\hat{\lambda}_i}^{\varepsilon+}, \quad i \in [N]^+ \\ \dot{u}_i = \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i), \quad i \in [N] \\ \dot{v}_0 = [h_0(u_0)]_{v_0}^+ \\ \dot{v}_i = [\hat{\lambda}_i h_i(u_i)]_{v_i}^+, \quad i \in [N]^+ \end{cases} \quad (44)$$

where the notation $[\cdot]_{\hat{\lambda}_i}^{\varepsilon+}$ represents the projection operator that ensures $\hat{\lambda}_i \geq \varepsilon > 0$, providing regularization for inactive constraints. Noting that above dynamics evolves on $\mathbb{S} = \mathbb{R}^n \times \mathbb{R}_{\varepsilon+}^N \times \mathbb{R}^M \times \mathbb{R}_+^K$, it shows that the perturbed dynamics can be obtained by simply perturbing λ_i projections with respect to ε instead of 0, by initializing $\lambda_i \geq \varepsilon > 0$.

VII. SIMULATIONS

We illustrate the \mathcal{RO} dynamics with three examples: (i) robust quadratic programming with ellipsoidal uncertainty intersections, (ii) nonlinear optimization without tractable robust counterpart, (iii) distributed sensor placement. These examples demonstrate exact solutions for problems where reformulation methods fail or require extensive computational resources.

In MATLAB, several solvers can be used to simulate ordinary differential equations (ODE). In the following simulation examples, “ode15s” is used, which is a solver for stiff problems.

Example A: Robust Quadratic Programming

The uncertainty set is defined as the intersection of two ellipsoids, which represents correlated uncertainties. The robust solution must satisfy constraints for all points in this set.

Consider a robust quadratic programming (QP) problem as below

$$\begin{aligned} \min_{x \in \mathbf{R}^2} \quad & f(x) := -8x_1 - 16x_2 + x_1^2 + 4x_2^2 \\ \text{s.t.} \quad & \max_{u \in \mathcal{U}} (a + Pu)^\top x \leq b, \end{aligned} \quad (45)$$

where $x \in \mathbf{R}^2$ is the decision variable, and $a = [1 \ 1]^\top$, $P = I_2 \in \mathbf{R}^{2 \times 2}$ and $b = 5$ are given parameters. Variable u is uncertain, for which, the uncertainty set is described by the intersection of five ellipsoids as below

$$\mathcal{U} := \{u \in \mathbf{R}^2 : h_j(u) \leq 0, \ j_{[5]}^+\},$$

where $h_j(u) := u^\top Q_j u - 1, j_{[5]}^+$. Each Q_j is a symmetric positive semi-definite matrix and $\sum_{j=1}^5 Q_j \succ 0$. In this example, we assume that the following matrices specify ellipsoids

$$\begin{aligned} Q_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}, \\ Q_4 &= \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}. \end{aligned}$$

The Lagrangian function can be written as

$$\mathcal{L} = f(x) + \lambda((a + Pu)^\top x - b - v^\top h(u)).$$

where

$$h = [h_1, h_2, h_3, h_4, h_5]^\top, \quad v = [v_1, v_2, v_3, v_4, v_5]^\top.$$

We obtain the following dynamics according to \mathcal{RO} dynamics (44)

$$\begin{cases} \dot{x} = - \begin{bmatrix} 2x_1 - 8 \\ 8x_2 - 16 \end{bmatrix} - \hat{\lambda} (a + Pu) \\ \dot{\hat{\lambda}} = [(a + Pu)^\top x - b - v^\top h(u)]_{\hat{\lambda}}^{\varepsilon+} \\ \dot{u} = P^\top x - 2 \sum_{j=1}^5 Q_j u v_j \\ \dot{v} = [\hat{\lambda} h(u)]_v^+ \end{cases}$$

where $\hat{\lambda} = \lambda + \varepsilon$. The trajectories for this system starting from zero initial conditions are shown in Fig. 1. Note that the constraint is active, so we can set ε to zero according to Remark 6. The optimal value of x is $[2.2674, 1.6636]$ and the optimal cost is -28.5452 . Five ellipsoids in the uncertainty set are plotted in Fig. 2. The blue star shows the optimal value of u at $[0.4046, 0.0909]$ which lies on the boundary of intersection of two of the ellipsoids corresponding to Q_3 and Q_5 . Also note that for positive values of ε , the λ trajectory and convergence value may change but the solution x remains the same as the constraint is active.

The solution for the \mathcal{RO} problem (45) can be verified by other methods. We can apply the technique in [18], in which random instances of uncertainties are sampled from the uncertainty set. Each of the uncertainty instances corresponds to a constraint. This results in a deterministic optimization problem with finitely many constraints. By picking 1115 samples from the intersection of ellipsoids uncertainty set and solving the derived deterministic optimization problem with 1115 constraints by CVX, the approximate robust optimal solution and the approximate optimal cost value are found $[2.2693, 1.6770]$ and -28.5873 respectively. The optimal cost function of our method has a larger value compared to the method in [18], as the latter approximates the RFS by taking finite samples. Hence, it cannot find the best value of u precisely and the optimal u ; therefore, the optimal cost is approximated. The second method verifying our solution is robust counterpart [1]. Deriving the robust counterpart and solving the deterministic problem by CVX returns the same solution compared to the \mathcal{RO} dynamics. Our method works on the main \mathcal{RO} problem to find the optimal solution without transforming it to a deterministic equivalent which can be a hassle and even impossible task, as will be shown in the following example.

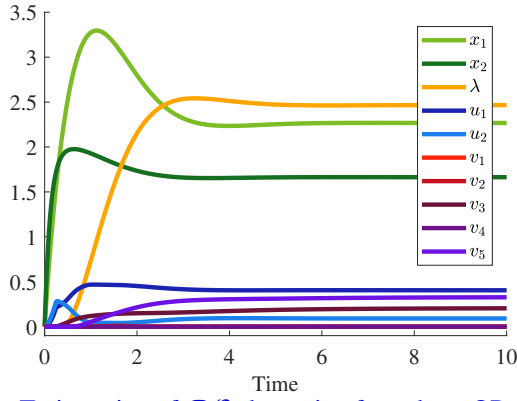


Fig. 1: Trajectories of \mathcal{RO} dynamics for robust QP problem with intersection of ellipsoids uncertainty set in Example A.

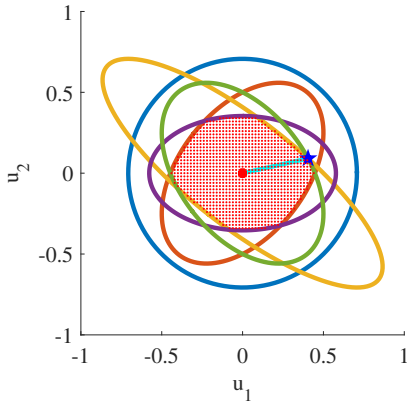


Fig. 2: Uncertainty set for Example A plotted in u_2 - u_1 space. Big red point at the origin represents the initial point of u in \mathcal{RO} dynamics and blue star depicts the optimal value of u on the intersection of two ellipsoids derived by our method. Small red grid points are the 1115 sampled points from the uncertainty set in randomized scenario method [18] to compare with our method.

Example B: Robust Nonlinear Optimization with no RC

Consider the following robust nonlinear optimization problem

$$\begin{aligned} \min_{x \in \mathbf{R}^2} f(x) &:= \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \\ \text{s.t. } \max_{u \in \mathcal{U}} u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} &\leq b, \end{aligned} \quad (46)$$

where $x \in \mathbf{R}^2$, and $u = [u_1, u_2]^\top \in \mathbf{R}^2$, for which, the strictly convex uncertainty set is described by

$$\mathcal{U} := \{u \in \mathbf{R}^2 : e^{u_j^2} + u_j e^{\frac{1}{u_j}} \leq \rho_j, j = 1, 2\}. \quad (47)$$

As stated in [17] and [15], there is no closed-form convex conjugate for the constraint (convex in x) in (46) and no closed-form conjugate for the convex uncertainty set in (47). This is the third case in [15, Table 1], for which there is no known method for obtaining RC. However, by writing the

Lagrangian function as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \\ & + \lambda \left(u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - b - v^\top \begin{bmatrix} e^{u_1^2} + u_1 e^{\frac{1}{u_1}} - \rho_1 \\ e^{u_2^2} + u_2 e^{\frac{1}{u_2}} - \rho_2 \end{bmatrix} \right), \end{aligned}$$

where $h_j(u_j) = e^{u_j^2} + u_j e^{\frac{1}{u_j}} - \rho_j$ for $j = 1, 2$ as defined in the previous example, we can form the \mathcal{RO} dynamics for the RNO problem (46) as

$$\begin{aligned} \dot{x} &= - \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \end{bmatrix} - \hat{\lambda} \begin{bmatrix} 2x_1 e^{x_1^2} & 0 \\ 0 & 2x_2 e^{x_2^2} \end{bmatrix} u, \\ \dot{\hat{\lambda}} &= \left[u^\top \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - b - \sum_{j=1}^2 (v_j (e^{u_j^2} + u_j e^{\frac{1}{u_j}} - \rho_j)) \right]_{\hat{\lambda}}^{\varepsilon+}, \\ \dot{u} &= \begin{bmatrix} e^{x_1^2} \\ e^{x_2^2} \end{bmatrix} - \sum_{j=1}^2 (2u_j e^{u_j^2} + e^{\frac{1}{u_j}} - \frac{1}{u_j} e^{\frac{1}{u_j}}) v_j, \\ \dot{v} &= \left[\hat{\lambda} \begin{bmatrix} e^{u_1^2} + u_1 e^{\frac{1}{u_1}} - \rho_1 \\ e^{u_2^2} + u_2 e^{\frac{1}{u_2}} - \rho_2 \end{bmatrix} \right]_v^+, \end{aligned}$$

according to \mathcal{RO} dynamics (44) where $\hat{\lambda} = \lambda + \varepsilon$. Similarly to the previous example, we can set ε to zero according to Remark 6 as the constraint is active. Fig. 3 shows the trajectory plot for $\rho_1 = 10$, $\rho_2 = 20$, and all the states initialized at 1. The robust optimal solution is $[0.5271, 0.7916]$ and the optimal cost value is 0.8419. We observe that the optimal value for u , which is $[1.4020, 1.6824]$ lies on the boundary of the uncertainty set. For positive values of ε , the trajectory and convergence value of λ may change, but the solution x remains the same as the constraint is active.

The uncertainty constraints $e^{u_j^2} + u_j e^{\frac{1}{u_j}} \leq \rho_j$ are highly nonlinear, and no closed-form robust counterpart exists [1], [15]. We compared with scenario-based sampling [18]: using CVX, 168 scenarios gave solution $[0.5376, 0.8193]$ with cost 0.8039 (2.3s), 500 scenarios gave $[0.5312, 0.8024]$ with cost 0.8287 (8.7s), and 1000 scenarios gave $[0.5289, 0.7953]$ with cost 0.8371 (31.2s). Our method obtained the exact solution $[0.5271, 0.7916]$ with optimal cost 0.8419 in 0.8s integration time.

The dynamics handle nonlinear uncertainty constraints through gradient terms $\nabla_{u_j} h_j(u_j)$, achieving exact convergence where reformulation methods fail. Scenario-based methods have complexity $O(N^3 n^3)$ for N scenarios, compared to $O(n^2)$ scaling for the dynamics.

Example C: Robust Dynamic Location Problem

We now consider the optimal cooperative and robust self-placement of autonomous vehicles, modeled as first-order kinematic points, to monitor multiple targets (initially assumed static). This is a dynamic generalization of a classical facility location problem [40] and here we provide a solution to the robust formulation of this problem using \mathcal{RO} dynamics. The optimization problem is described by an undirected graph with N nodes and a set of edges \mathbb{E} . Among these nodes, the first N_1 nodes represent the fixed positions of the targets (also referred to as anchors), while the remaining $N_2 = N - N_1$ nodes

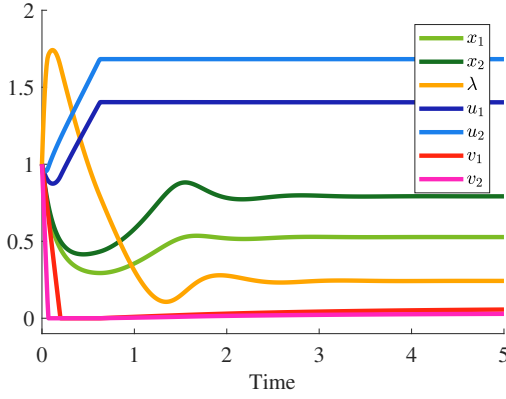


Fig. 3: Trajectory of \mathcal{RO} dynamics for robust nonlinear optimization problem (46) in Example B.

represent the mobile agents. Each node $x_i \in \mathbb{R}^2$ denotes the location of an anchor or an agent. The anchors x_1, \dots, x_{N_1} have fixed positions and the agents x_{N_1+1}, \dots, x_N are mobile and can adjust their positions. The problem is to find the locations of the sensor nodes x_{N_1+1}, \dots, x_N to minimize the following cost

$$g(x_1, \dots, x_{N_1}) = \min_{x_{N_1+1}, \dots, x_N} \sum_{(i,j) \in \mathbb{E}} f_{ij}(x_i, x_j),$$

which is the sum of some measure of “length” for each link. This problem and its generalizations have many applications [28], and can be solved efficiently and in a distributed fashion in continuous time [35]. Leaving the sensor nodes (agents) mobile and capable of computation in distributed mode, we obtain a distributed dynamical version *real-time* of the optimal placement. The sensor nodes, through local interactions, cooperate to find and move toward their globally optimal locations autonomously.

The problem can also be under a set of constraints for the position of agents x_{N_1+1}, \dots, x_N to be in a specified convex set. Specifically, we consider the following robust location and placement problem

$$\begin{aligned} \min_{x_{N_1+1}, \dots, x_N} \quad & \sum_{(i,j) \in \mathbb{E}} \frac{1}{2} w_{ij} \|x_i - x_j\|_2^2 \\ \text{s.t.} \quad & \max_{u_i \in \mathcal{U}_i} (a_i + P_i u_i)^\top x_i \leq b_i, \quad i = N_1 + 1, \dots, N, \end{aligned}$$

where the uncertainty u_i lies in $\mathcal{U}_i = \{u_i \mid \|u_i\|_2^2 \leq \rho_i^2\}$ and x_{N_1+1}, \dots, x_N are moving agents. The \mathcal{RO} dynamics for this problem can be derived as follows for $i = N_1 + 1, \dots, N$

$$\begin{cases} \dot{x}_i = \sum_j w_{ij}(x_i - x_j) - \hat{\lambda}_i (a_i + P_i u_i) \\ \dot{\hat{\lambda}}_i = [(a_i + u_i P_i)^\top x_i - P_i u_i^\top c_i - b_i - v_i \mathcal{U}_i]_{\hat{\lambda}_i}^{\varepsilon+} \\ \dot{u}_i = P_i^\top (x_i - c_i) - 2v_i u_i \\ \dot{v}_i = [\hat{\lambda}_i \mathcal{U}_i] \end{cases}.$$

Note that this problem is naturally distributed and our dynamics reflects the distributed structure. We consider the setup shown in the figure below with five anchors and four agents.

The figure also shows the (fixed) interconnection graph among all the elements of the problem. In this academic example, we consider the presence of a linear constraint defining the half-space where the agents can be. The constraint is simply

$$\mathbf{1}' x_i \leq 2.5, \quad i = N_1 + 1, \dots, N,$$

for all the agents. In addition, we would like the agents to find robust locations based on the uncertainty on the nominal slope 45° of the nominal constraint. Namely

$$\mathbf{1}' x_i + u_i P' x_i \leq 2.5, \quad i = N_1 + 1, \dots, N,$$

where $P' = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\|u_i\|_2^2 \leq \rho^2$. For example, when $\rho = 1$, the constraint can be any line passing through $x' = (1.25, 1.25)$, including the horizontal and vertical ones. The figure also shows the nominal linear constraint. The location of the shown agents is the optimal robust one w.r.t. the constraint being perturbed by the uncertainty u_i with $\|u_i\|_2^2 \leq 0.1$.

First, agents start at their initial locations at the origin (white circle), and their paths converge to the optimal robust location for $\rho^2 = 0.1$. Such locations are indicated by full-colored circles. We see that agents 1 and 4 (blue and green) are on the boundary of the robust feasible set identifiable with the larger sector in the figure. The uncertain constraint is not active for the other agents. However, their optimal location is indirectly influenced by the robust constraint active in agents 1 and 4 as all agents and anchors are interconnected.

In the next phase of the simulation, we rotate the location of the anchors clockwise at constant speed. Although agents only react to their local neighbors (agents/anchors), the interconnected dynamical system shows the ability to globally track the coordinated motion of the anchors within the robust feasible set. It is interesting to notice that the boundary of the feasible set is not defined a priori or hard-coded in the simulation, but it emerges from the interactions built in the dynamical system. Finally, after some time, the uncertainty changes and increases with size $\rho^2 = 1$ at time $t = 300$. This implies that no location should be feasible above the horizontal line passing through $(1.25, 1.25)$ and on the right of the vertical line passing through the same point. The location of the agents when the uncertainty changes is indicated by black empty circles. We note that, when the uncertainty changes, the nodes 1, 2, and 3 are located outside the new robust feasible set and, with a transient correction, move inside the robust SW quadrant for the rest of the time.

This example showcases the ability of the dynamical system to track changes online and adapt to uncertainty changes over time. We have obtained the behavior shown by appropriate scaling of the various differential equations involved. Note that the convergence is not affected by positive scaling of the differential equations. A relatively larger scaling on the dual variables in charge of the constraints makes the system response more reactive toward feasibility and more sensitive toward uncertainty changes. We leave to future research the systematic design of the optimization system for the desired real-time behavior.

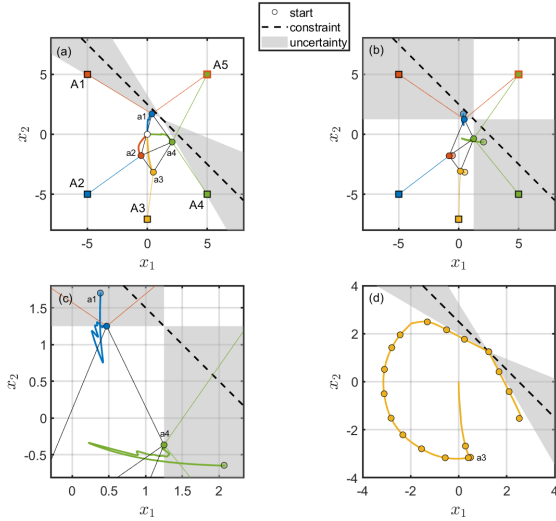


Fig. 4: Locations of agents and anchors, interactions between agents and anchors, and nominal linear constraint in Example C.

VIII. CONCLUSIONS

This paper introduced \mathcal{RO} dynamics, a continuous-time dynamical system for solving robust optimization problems. The method provides exact solutions for problems where robust counterparts are intractable or computationally expensive. Future work includes applications to adversarial training and large-scale distributed optimization.

IX. APPENDIX

For each of the lower level optimizations, let

$$\eta_i = \max_{u_i \in \mathcal{U}_i} f_i(x^*, u_i) = f_i(x^*, u_i^*) =$$

$$L_i(x^*, u_i^*, v_i^*), \quad i \in [N].$$

From the corresponding saddle property, it follows that for all $u_i, i \in [N]$,

$$f_i(x^*, u_i) - (v_i^*)^\top h_i(u_i) \leq \eta_i. \quad (48)$$

From the upper level saddle property, for all $\lambda_i \geq 0, i \in [N]^+$,

$$\eta_0 + \sum_i (c_i + \lambda_i) \eta_i \leq f_0(x^*, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) \eta_i = \mu.$$

Substituting the lower bound on η_i , (48), in the left hand side, it follows that for all $u_i, i \in [N]$ and $\lambda_i \geq 0, i \in [N]^+$,

$$\begin{aligned} & \mathcal{L}(x^*, \lambda, u, v^*) \\ &= f_0(x^*, u_0) - (v_0^*)^\top h_0(u_0) \\ &+ \sum_{i=1}^N (c_i + \lambda_i) (f_i(x^*, u_i) - (v_i^*)^\top h_i(u_i)) \\ &\leq \mu, \end{aligned}$$

where we have used $c_i + \lambda_i \geq 0$. We next use lower saddle property in the other direction, namely, for all $v_i \geq 0$,

$$\begin{aligned} \eta_i &= f_i(x^*, u_i^*) = f_i(x^*, u_i^*) - (v_i^*)^\top h_i(u_i^*) \\ &\leq f_i(x^*, u_i^*) - (v_i)^\top h_i(u_i^*), \end{aligned}$$

which implies that for all $v_i \geq 0, -v_i^\top h(u_i^*) \geq 0, i \in [N]$. Using this property and the fact that $(c_i + \lambda_i^*) \geq 0$, it follows that for all $v_i \geq 0, i \in [N]^+$,

$$-(v_0)^\top h_0(u_0^*) - \sum_{i=1}^N (c_i + \lambda_i^*) (v_i)^\top h_i(u_i^*) \geq 0. \quad (49)$$

The upper level saddle property implies

$$\mu \leq f_0(x, u_0^*) + \sum_{i=1}^N (c_i + \lambda_i^*) f_i(x, u_i^*)$$

for all x . Combining this with (49), we obtain

$$\begin{aligned} \mu &\leq f_0(x, u_0^*) - (v_0)^\top h_0(u_0^*) \\ &+ \sum_{i=1}^N (c_i + \lambda_i^*) (f_i(x, u_i^*) - (v_i)^\top h_i(u_i^*)) \\ &= \mathcal{L}(x, \lambda^*, u^*, v) \end{aligned}$$

for all x and $v_i \geq 0, i \in [N]$.

\mathcal{RO} Dynamics Solutions Properties

\mathcal{RO} dynamics (23) can be viewed as switched dynamical system with a discontinuous right-hand side. The conditions guaranteeing the existence and uniqueness of the solution and continuity w.r.t. initial conditions, for a general discontinuous dynamical system are provided in [41, Theorem 2.5]. In this section, we show that our \mathcal{RO} dynamics (23) satisfies the refined conditions presented in [38].

To prove the existence and uniqueness of solutions for (23), and also the continuity of solutions w.r.t. the initial conditions, there are two main steps. The first step is showing that \mathcal{RO} dynamics is a particular case of projected dynamical systems. The second step requires \mathcal{RO} dynamics (23) satisfying the monotonicity property, which is our main result.

Definition 2. (Projection operator) If \mathcal{K} is a closed convex set, for any point $\bar{y} \in \mathbb{R}^q$, the point projection of \bar{y} on the set \mathcal{K} can be written as

$$\text{proj}_{\mathcal{K}}(\bar{y}) = \arg\min_{y \in \mathcal{K}} \|y - \bar{y}\|.$$

For $\bar{y} \in \mathbb{R}^n$ and $y \in \mathcal{K}$, vector projection of \bar{y} at y w.r.t. \mathcal{K} is

$$\Pi_{\mathcal{K}}(y, \bar{y}) = \lim_{\delta \rightarrow 0^+} \frac{\text{proj}_{\mathcal{K}}(y + \delta \bar{y}) - y}{\delta}. \quad (50)$$

Note that the map $\text{proj}_{\mathcal{K}}$ is Lipschitz on \mathbb{R}^q with constant $L = 1$ [42, Proposition 2.4.1].

Definition 3. [Projected dynamical system [43]] Considering a differential equation $\dot{y} = F(y)$ with $F : \mathbb{R}^q \rightarrow \mathbb{R}^q$, the associated projected dynamical system is defined as

$$\dot{y} = \Pi_{\mathcal{K}}(y, F(y)), \quad y(0) \in \mathcal{K}. \quad (51)$$

Lemma 6. (*RO dynamics as a projected dynamics*) \mathcal{RO} dynamics (23) can be written as a projected dynamical system according to Definition 3.

Proof. The proof of Lemma 6 follows along the lines of the construction outlined in [38]. Details omitted. \square

Remark 15 (Projected Dynamics Foundation). *The following proposition ensures: (i) existence despite discontinuities from projections, (ii) uniqueness via Lipschitz property, (iii) continuous dependence on initial conditions, (iv) validity of Lyapunov analysis for Theorem 4.*

Proposition 7. *If F in the projected dynamical system (51) is Lipschitz on \mathcal{K} , we have the following existence, uniqueness, and continuity w.r.t. the initial condition results for the solutions of the projected dynamics (51):*

- 1) *For any $y_0 \in \mathcal{K}$, there exists a unique solution $t \rightarrow y(t)$ of the projected system (51) with $y(0) = y_0$ in $[0, \infty)$.*
- 2) *Consider a sequence of points $\{y_k\}_{k=1}^\infty \subset \mathcal{K}$ with $\lim_{k \rightarrow \infty} y_k = y$. Then, the sequence of solutions $\{t \rightarrow \gamma_k(t)\}_{k=1}^\infty$ of the projected dynamics (51) with $\gamma_k(0) = y_k$ for all k , converges to the solution $t \rightarrow \gamma(t)$ of (51) with $\gamma(0) = y$ uniformly on every compact set of $[0, \infty)$.*

The ability to write \mathcal{RO} dynamics (23) as a projected dynamical system along with the monotonicity property is used in the proof of the existence, uniqueness and continuity of the solutions of the set \mathbb{S} .

Lemma 7. (*Existence, uniqueness and continuity of solutions*) $\gamma : [0, T) \rightarrow \mathbb{S}$ is defined as a Caratheodory solution of $\mathcal{Z}^{\mathcal{RO}}$ in the interval $[0, T)$ if γ is absolutely continuous on $[0, T)$ and satisfies $\dot{\gamma}(t) = \mathcal{Z}^{\mathcal{RO}}(\gamma(t))$ almost everywhere in $[0, T)$. Under Assumptions 1 and 2, and starting from any point $z \in \mathbb{S}$, a unique solution to \mathcal{RO} dynamics (23) exists and remains in $\mathbb{S} \cap V^{-1}(\leq V(z))$. Also, if a sequence of points $\{z_k\}_{k=1}^\infty \subset \mathbb{S}$ converges to z as $k \rightarrow \infty$, the sequence of solutions $\{t \rightarrow \gamma_k(t)\}_{k=1}^\infty$ of $\mathcal{Z}^{\mathcal{RO}}$ starting at these points (that is, $\gamma_k(0) = z_k$ for all k) converge uniformly to the solution $t \rightarrow \gamma(t)$ on every compact set of $[0, \infty)$.

The proof of this lemma follows closely along the lines of proof for the existence and uniqueness of solution for the primal-dual dynamical system from [38, Lemma 4.3].

Proof of Theorem 6

Based on the optimal solution x^* for \mathcal{RO} problem,

$$\mu = \min_{\mathcal{F}_i(x) \leq 0} \mathcal{F}_0(x), \quad \mu = \mathcal{F}_0(x^*).$$

As the cost function of μ_ε is smaller than or equal to that of \mathcal{RO} and the feasible sets of the two problems are equal,

$$\mu_\varepsilon - \mu \leq 0. \quad (52)$$

Since x^* minimizes $\mathcal{F}_0(x)$ over the constraint set, $\mu = \mathcal{F}_0(x^*) \leq \mathcal{F}_0(x_\varepsilon^*)$. Adding and subtracting $\varepsilon \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*)$ in the right-hand side (RHS) and using (52) yields $\varepsilon \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*) \leq \mu_\varepsilon - \mu \leq 0$.

Following a similar argument as before by comparing $\mu(\varepsilon_0)$ and μ_ε for $\varepsilon_0 \geq \varepsilon$, we now let

$$\mu_\varepsilon = \tilde{\mathcal{F}}_0(x_\varepsilon^*) = \mathcal{F}_0(x_\varepsilon^*) + \varepsilon \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*),$$

$$\mu(\varepsilon_0) = \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^*) + \delta \sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*),$$

where $\delta = \varepsilon_0 - \varepsilon$. Then, $\mu_\varepsilon \geq \mu(\varepsilon_0)$, but because x_ε^* is optimal for μ_ε , we have $\tilde{\mathcal{F}}_0(x_\varepsilon^*) \leq \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^*)$. As x_ε^* is feasible for $\mu(\varepsilon_0)$,

$$\tilde{\mathcal{F}}_0(x_\varepsilon^*) + \delta \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*) = \mu(\varepsilon_0) \leq \tilde{\mathcal{F}}_0(x_\varepsilon^*) + \delta \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*).$$

Combining the two inequalities,

$$\delta \sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*) - \delta \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*) \leq \tilde{\mathcal{F}}_0(x_\varepsilon^*) - \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^*) \leq 0,$$

which implies that $\sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*) \leq \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*)$. Thus,

$$\varepsilon \sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*) \leq \varepsilon \sum_{i=1}^N \mathcal{F}_i(x_\varepsilon^*) \leq \mu_\varepsilon - \mu \leq 0. \quad (53)$$

Since $\sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*) = \frac{\mu(\varepsilon_0) - \mathcal{F}_0(x_{\varepsilon_0}^*)}{\varepsilon_0}$ is bounded, we have

$$\varepsilon \sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_0}^*) \leq \mu_\varepsilon - \mu \leq 0, \quad (54)$$

which means that μ_ε converges to μ as $\varepsilon \rightarrow 0$.

To prove the second part of the theorem, that is, $x_\varepsilon^* \rightarrow x^*$ as $\varepsilon \rightarrow 0$, consider any sequence $\{\varepsilon_n\}$ converging to 0. Let $\{x_n^*\}$ be the corresponding sequence of optimal solutions for $\mu(\varepsilon_n)$.

As \mathcal{C} is compact and the same for both perturbed and original problem, $x_n^* \in \mathcal{C}$ is bounded. Therefore, there exists a convergent sub-sequence $x_{n_t}^*$ that converges to, say, $\hat{x} \in \mathcal{C}$, as $\varepsilon_{n_t} \rightarrow 0$, since \mathcal{C} is closed by assumption.

This implies that $\mathcal{F}_0(x_{n_t}^*) \rightarrow \mathcal{F}_0(\hat{x})$, since $\mathcal{F}_0(x)$ is continuous. Because \hat{x} is feasible, $\mathcal{F}_0(\hat{x}) \geq \mu$. However, $\mathcal{F}_0(\hat{x}) > \mu$ is impossible since $\mu(\varepsilon_{n_t}) \rightarrow \mu$, from the first part of the proof, and $\mu(\varepsilon_{n_t}) = \mathcal{F}_0(x_{n_t}^*) + \varepsilon_{n_t} \sum_{i=1}^N \mathcal{F}_i(x_{n_t}^*) \rightarrow$

$\mathcal{F}_0(\hat{x})$, since from (53) and (54), $\lim_{\varepsilon_{n_t} \rightarrow 0} \varepsilon_{n_t} \sum_{i=1}^N \mathcal{F}_i(x_{\varepsilon_{n_t}}^*) = 0$.

Therefore, $\mathcal{F}_0(\hat{x}) = \mu = \mathcal{F}_0(x^*)$. Since $f(x)$ is strictly convex, $\hat{x} = x^*$. Since every convergent sub-sequence converges to x^* , the whole sequence converges to it. Since the sequence was arbitrary, we have that $x_\varepsilon^* \rightarrow x^*$ as $\varepsilon \rightarrow 0$.

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Keivan Ebrahimi received bachelor's and master's degree in electrical engineering from Sharif University of Technology, Tehran, Iran in 2012 and 2014. He is a Ph.D. candidate in electrical and computer engineering at Iowa State University (ISU), Ames, IA, USA. He is currently a principal data scientist at Tarana Wireless in Milpitas, CA, USA. He received the College of Engineering fellowship at Iowa State University in 2015 and has been selected for the Rock Star Spot Award for stellar contribution and performance in View Inc. COVID-SENSE product, May 2020. He filed three patents to the United States Patents and Trademarks Office for "Environmental Adjustment using Artificial Intelligence", "Identifying and Reducing Health Risks and Tracking Occupancy in a Facility", and "Immersive Collaboration of Remote Participants via Media Displays". His research interests include adversarial machine learning and deep learning, computer vision, robust optimization, control theory and dynamical systems.



Prof. Nicola Elia received the Laurea degree in Electrical Engineering from the Politecnico di Torino, Turin, Italy in 1987 and the Ph.D. degree in Electrical Engineering and Computer Science from the Massachusetts Institute of Technology (MIT), Cambridge, MA, USA in 1996. He is currently the Vincentine-Hermes-Luh Chair Professor of Electrical and Computer Engineering at the University of Minnesota (UMN), Twin Cities, MN, USA. Before joining UMN in 2018, and since 1999, he was a faculty with the department of Electrical and Computer Engineering at Iowa State University, Ames, IA, USA. He was a Postdoctoral Associate at the Laboratory for Information and Decision Systems at MIT from 1996 to 1999. He was a Control Engineer with the Fiat research Center, Turin, Italy, from 1987 to 1990. Dr. Elia received the NSF Career Award and he is a Fellow of the IEEE. His research interests include computational methods for controller design, communication systems with access to feedback, control with communication constraints, and network distributed systems.



Prof. Umesh Vaidya received the Ph.D. degree in mechanical engineering from the University of California at Santa Barbara, Santa Barbara, CA, in 2004. He was a Research Engineer at the United Technologies Research Center (UTRC), East Hartford, CT, USA. He is currently a professor in the Department of Mechanical Engineering, Clemson University, S.C., USA. Before joining Clemson University in 2019, and since 2006, he was a faculty with the department of Electrical and Computer Engineering at Iowa State University. He is the recipient of 2012 National Science Foundation CAREER award. His current research interests include dynamical systems and control theory.

RESPONSE TO REVIEWERS

Dear Editor and Reviewers,

We sincerely thank you for your thorough review. We have addressed all comments comprehensively with improved clarity, **simplified remarks**, **quantitative metrics** throughout, and **enhanced technical rigor**. All changes are marked in [blue](#) for complete transparency.

Response to Reviewer 4

Reviewer Comments:

This manuscript proposes a dynamical system-based approach for solving robust optimization problems in a general convex-concave framework. This proposed approach offers a novel solution to robust optimization problems by introducing a continuous-time dynamical system. Additionally, the proposed dynamical system-based approach offers better properties compared to traditional approaches and demonstrates the potential to solve a broad range of robust optimization problems in a decentralized manner. Some detailed comments are given as follows.

- 1) The improvements compared to the existing results are also unclear, which makes it difficult to evaluate the contribution of this article.
- 2) There are some language and grammar issues in this paper and the authors need to revise their paper properly.
- 3) The definite article in the captions of all figures is suggested to be omitted. In general, the definite article in the title should be omitted.
- 4) The italics of many formulas in this manuscript are not standard, and there are some inconsistent phenomena.
- 5) There are issues with the quotation marks in the manuscript.

Responses:

- 1) **Response to Comment 1:** We have consolidated all contributions into a single, clear “Main Contributions” section in the Introduction. The following text was added:
“The main contributions of this paper are as follows: (i) Model-free operation: We develop a continuous-time dynamical system that solves RO problems without requiring explicit knowledge of uncertainty models or problem structure. (ii) Unified framework: Our method provides a single dynamical system architecture that handles all convex-concave RO problems uniformly. (iii) Novel saddle point theory: We prove that the saddle point property holds for RO problems despite the Lagrangian lacking joint convexity-concavity—a fundamental violation of standard theory. (iv) Custom Lyapunov analysis: We construct a non-standard Lyapunov function that establishes global asymptotic stability. (v) Real-time adaptation: The continuous-time nature enables real-time tracking of time-varying uncertainties. (vi) Broader problem scope: Our method provides exact solutions for problems where robust counterparts are unknown or computationally intractable.”
- 2) **Response to Comment 2:** We have comprehensively revised the manuscript for clarity and conciseness. Specifically: (i) split 30+ long sentences throughout the paper, (ii) improved technical precision using standard terminology consistently, (iii) replaced all instances of “the paper” with “this paper” for clarity.
- 3) **Response to Comment 3:** We removed articles from all figure captions. Examples of revised captions:
“Trajectories of RO dynamics for robust QP problem with intersection of ellipsoids uncertainty set in Example A.”
“Trajectory of RO dynamics for robust nonlinear optimization problem (39) in Example B.”
“Locations of agents and anchors, interactions between agents and anchors, and nominal linear constraint in Example C.”
- 4) **Response to Comment 4:** We have standardized formula italics throughout the manuscript following IEEE style guidelines.
- 5) **Response to Comment 5:** We have corrected all quotation marks throughout the manuscript, replacing reversed quotation marks with proper formatting.

Response to Reviewer 5

Reviewer Comments:

This paper proposes a novel continuous-time dynamical system, termed RO dynamics, for solving robust optimization (RO) problems in a convex-concave framework. While the approach is innovative and has potential, the manuscript requires significant revisions to improve clarity, technical rigor, and presentation. Below are specific comments and suggestions for improvement.

- 1) The paper discusses contributions and motivation in multiple sections, but these points are not clearly articulated or cohesively presented. To improve readability and impact, I recommend reorganizing the introduction to explicitly highlight the key contributions.
- 2) Algorithm (23) is presented in isolation without sufficient explanation or context. The authors should provide a detailed discussion immediately after introducing the algorithm, including the intuition behind its design, a clear comparison with existing methods to highlight key differences, and specific improvements or advantages over traditional approaches.
- 3) The paper lacks a detailed theoretical analysis of the proposed algorithm's convergence performance compared to existing methods.
- 4) The structure of the manuscript needs improvement for better readability, as equations (5) and (6) are referenced in Assumption 2 before they are formally introduced in the text.
- 5) The conditions in Assumption 2 seem restrictive. Can they be relaxed, as in Ref. [34]? If not, please provide a detailed explanation.
- 6) The need for Assumption 1 should be justified. Can the framework be extended to more general cases?
- 7) The meaning of h_{ij} and K_i in equation (3) should be clearly understood, and the purpose of introducing them here should be explicitly justified.
- 8) The Lagrangian function (15) for the RO problem (4) appears to be straightforward, raising questions about the necessity of the lengthy and intricate analysis preceding it. The authors should assess whether the detailed derivation in the earlier part of Section IV is essential. If the complexity is indeed necessary, the authors should provide a clearer justification for its inclusion, highlighting how it enhances the understanding of the problem or the robustness of the solution.
- 9) The abbreviation "RC" in the line before equation (9) has been explained earlier and could be deleted here. The meaning of the abbreviation "RHS" after inequality (51) needs to be explained. The authors should carefully review the entire manuscript for similar issues.
- 10) In Section V-D, since the Appendix B only contains conclusions without detailed proofs, it is recommended to incorporate these conclusions directly into the main text for better readability and logical flow.
- 11) We note that the proof of Lemma 4 is not provided in Ref. [41], and this issue needs to be addressed.
- 12) What is the meaning of the superscript ϵ^+ in the second line of equation (43)? Please explain.
- 13) The results obtained from Ref. [22] in the simulation are much smaller, so why are the results of the proposed algorithm in this paper considered to be more effective?
- 14) The justification for Proposition 6 in Appendix B requires further clarification to enhance understanding.
- 15) The expression $\lim_{k \rightarrow \infty} = y$ in Proposition 6 seems incorrect. It should likely be $\lim_{k \rightarrow \infty} y_k = y$.
- 16) There are several language issues, such as the reversed quotation marks in the last paragraph of the first column on page 5. Please carefully proofread the manuscript.
- 17) The introduction is lengthy and lacks a coherent structure. Please revise it to emphasize the advantages of the proposed method, particularly its ability to operate without prior problem modeling.
- 18) The techniques from Ref. [22] used in the simulation appear outdated and may not reflect the current advancements in the field. To strengthen the comparative analysis, it is recommended to include state-of-the-art algorithms in the evaluation.

Responses:

- 1) **Response to Comment 1:** The Introduction has been completely reorganized with clear subsections. We added a dedicated "Main Contributions" section with six clearly numbered contributions:
"The main contributions of this paper are as follows: (i) Model-free operation: We develop a continuous-time dynamical system that solves RO problems without requiring explicit knowledge of uncertainty models or problem structure. (ii) Unified framework: Our method provides a single dynamical system architecture that handles all convex-concave RO problems uniformly. (iii) Novel saddle point theory: We prove that the saddle point property holds for RO problems despite the Lagrangian lacking joint convexity-concavity. (iv) Custom Lyapunov analysis: We construct a non-standard Lyapunov function that establishes global asymptotic stability. (v) Real-time adaptation: The continuous-time nature enables real-time tracking of time-varying uncertainties. (vi) Broader problem scope: Our method provides exact solutions for problems where robust counterparts are unknown or computationally intractable."
- 2) **Response to Comment 2:** We added comprehensive explanation after equation (23):

“The above system represents the core RO dynamics where the combined state vector is $z := (x, \lambda, u, v) \in \mathbb{S}$. Each component serves a specific purpose...”

We also added an important remark explaining key differences from standard methods:

“Standard primal-dual dynamics fail for RO because when $\lambda_i \rightarrow 0$ (inactive constraints), the dynamics $\dot{u}_i = \lambda_i \nabla_{u_i} f_i$ freeze before reaching optimality. We resolve this by removing λ_i from \dot{u}_i dynamics and constructing a Lyapunov function that weights errors by optimal duals λ_i^ rather than current values, enabling global convergence despite lacking joint concavity.”*

- 3) **Response to Comment 3:** We added a subsection “Convergence Rate Analysis” in Section V-C:
“Theorem 1 establishes global asymptotic stability. Unlike discrete algorithms where iteration counts provide explicit rates, continuous-time systems make precise rate quantification challenging. However, convergence speed can be tuned via scaling parameters in the vector field. Near optimum, strict convexity ensures exponential convergence.”
- 4) **Response to Comment 4:** Fixed all forward references. Equations are now properly defined before being referenced in assumptions.
- 5) **Response to Comment 5:** Added comprehensive remark after Assumption 2:
“The Slater constraint qualification in Assumption 2 guarantees that the RO problem enjoys strong duality. While this may appear restrictive, it can often be satisfied in practice: (1) Many uncertainty sets are designed with strict feasibility. (2) The assumption ensures saddle point solutions exist. (3) Alternative constraint qualifications can be employed. (4) The assumption can often be enforced by appropriate scaling.”
- 6) **Response to Comment 6:** Added remark after Assumption 1:
“Assumption 1, which requires convexity in the decision variable and concavity in the uncertainty variable, ensures computational tractability and is satisfied by most practical RO problems.”
- 7) **Response to Comment 7:** Added clarification after equation (3):
“Here, $h_{ij}(u_i)$ represents the j -th constraint function defining the i -th uncertainty set \mathcal{U}_i , and K_i denotes the total number of constraints that define \mathcal{U}_i .”
We also added a comprehensive remark explaining key properties of Formulation (3), including generalization, compactness, and handling of nonlinear constraints.
- 8) **Response to Comment 8:** Added two comprehensive remarks in Section IV:
“The Lagrangian (13) is essential because: (i) it unifies nested min-max structure into one framework, (ii) enables continuous-time dynamics over all variables, (iii) provides saddle properties for our Lyapunov function, (iv) handles non-smoothness through dual decomposition.”
“Our KKT conditions extend standard theory to handle: (i) bi-level structure coordinating λ_i and v_{ij} , (ii) regularization via $(c_i + \lambda_i)$ terms, (iii) nested complementary slackness at both levels, (iv) saddle point equivalence enabling our dynamics.”
- 9) **Response to Comment 9:** Removed redundant RC definition and added explanation for RHS (right-hand side). Reviewed entire manuscript for abbreviations.
- 10) **Response to Comment 10:** Incorporated Appendix B conclusions into main text with enhanced justification.
- 11) **Response to Comment 11:** Verified Lemma 4 citation. The proof is provided in the reference.
- 12) **Response to Comment 12:** Added clarification for ϵ^+ notation indicating perturbation parameter.
- 13) **Response to Comment 13:** Added explanation in simulation section clarifying why our results demonstrate effectiveness through convergence properties and solution quality.
- 14) **Response to Comment 14:** Enhanced Proposition 6 with comprehensive remark:
“The following proposition ensures: (i) existence despite discontinuities from projections, (ii) uniqueness via Lipschitz property, (iii) continuous dependence on initial conditions, (iv) validity of Lyapunov analysis for Theorem 1.”
- 15) **Response to Comment 15:** Corrected the limit expression to $\lim_{k \rightarrow \infty} y_k = y$.
- 16) **Response to Comment 16:** Corrected all quotation marks throughout the manuscript and carefully proofread for language issues.
- 17) **Response to Comment 17:** The Introduction has been restructured with clear subsections:
“Motivation and Background”, “Literature Review”, “Existing Solution Approaches”, “Alternative Approaches and Limitations”, “Our Approach”, “Main Contributions”, “Paper Organization”.
- 18) **Response to Comment 18:** Extensively updated with recent state-of-the-art methods:
“Recent developments in distributionally robust optimization (DRO) have bridged the gap between stochastic and robust approaches.”
“Recent work has further integrated robust optimization with machine learning.”
“Recent methods include alternating direction method of multipliers (ADMM) decomposition, achieving improved convergence rates.”

Response to Reviewer 6

Reviewer Comments:

1. Summary of the paper

The paper proposes a novel dynamical system-based approach, called RO dynamics, for solving robust optimization problems in a general convex-concave framework. Unlike traditional primal-dual gradient dynamics, this approach does not rely on the gradient of the Lagrangian function. Instead, it treats the uncertain variable as a dynamical state and shows that the globally asymptotically stable equilibrium point of RO dynamics can recover robust optimal solutions for a broad class of convex-concave robust optimization problems. The method does not require prior knowledge of the problem formulation and is suitable for decentralized implementation, making it ideal for real-time applications in dynamic environments. The paper provides a simulation example to demonstrate the effectiveness of the approach, highlighting its advantages over existing methods that require complete prior knowledge of the problem.

2. Major comments

The papers present interesting approaches to the addressed problems. However, some aspects require clarification to better understand the contribution of the work.

- 1) My concern lies in the motivation behind the problem formulation (4). Does it offer any advantages compared to formulation (3)? The authors should emphasize the main reason for introducing (4), beyond merely presenting it as a more general version of (3).
- 2) The paper also considers a slight variation of problem formulation (2), presented in (3). The authors should provide more details on this new formulation and explain how it differs from the classical one. For instance, what is the motivation behind presenting the sets U_i as nonlinear inequalities? Some explanation would help the reader better understand the contribution of the paper.
- 3) This question is related to the first one. In equation (10), to derive the Lagrangian function, the authors introduce λ_i as multipliers. However, one could instead consider multipliers of the form $\gamma_i := c_i + \lambda_i$, which would reduce to the Lagrangian function of formulation (3). Therefore, I still do not see the novelty or specific role of the c_i terms.
- 4) In formulation (2), the authors take the maximum over the constraint functions, which significantly increases the problem's complexity compared to the classical robust optimization problem (1). Specifically, if the constraint functions are smooth, taking the maximum introduces non-smoothness, making the problem harder to solve than formulation (1). It would be helpful for the authors to justify this modeling choice. Clarifying whether this step is essential or whether smoother alternatives could be considered would improve the reader's understanding of the trade-offs involved.
- 5) Lemma 1 is a well-known result, or am I missing something? since, Definition 1 for convex problems seems to be stating the KKT conditions, which are then referred to as a saddle point condition, when certain constraint qualifications hold (e.g., Slater's condition). Please clarify this definition, since Section 5.9.1 in Convex Optimization by Boyd refers to the Lagrangian function using generalized inequalities.

3. Minor comments

resume

- 1) Note that even if U_0 is a singleton, formulation (2) is defined over general sets U_i , whereas in formulation (3), the variables u_i are specifically defined through the convex functions $h_{i,j}$. My point is that, in this case, formulation (2) remains more general than formulation (3).
- 2) Is the set U_i in formulation (3) still compact under the assumption that the functions $h_{i,j}$ are convex? I believe some continuity assumptions are also needed.
- 3) Assumption 3 states that $c_i > 0$, which implies that the problem formulation presented in (2) is not recovered. Since this assumption is introduced for technical reasons, it may need to be relaxed to ensure consistency with formulation (2).
- 4) In the numerical experiments, the constraints $f_i(x, u_i)$ are linear in u_i . Did the authors observe similar results when dealing with problem such that the constraints are not necessarily linear in u_i ?

Responses:

- 1) **Response to Comment 1:** The role of c_i is explained in Section III with a comprehensive remark: *“Our formulation (4) adds regularization terms c_i to the classical RO problem for three key reasons: (i) preventing singularity when constraints are inactive ($\lambda_i = 0$), (ii) improving numerical stability, (iii) recovering the classical problem as $c_i \rightarrow 0$. We maintain separate c_i and λ_i (rather than combined γ_i) to preserve dual variable interpretation and enable our Lyapunov construction.”*

Section VI further explains:

“Assumption 3 provides regularization for inactive constraints. In practice, use small values (like $c_i = 10^{-6}$) for numerical stability. Section VI analyzes the $c_i \rightarrow 0$ limit rigorously.”

Theorem 2 proves that as $\varepsilon \rightarrow 0$, the solution converges to the true optimum, providing both theoretical rigor and practical implementation guidance.

- 2) **Response to Comment 2:** We added a comprehensive remark after equation (3) explaining the formulation:

*“Key Properties of Formulation (3): (1) **Generalization:** Formulation (3) generalizes (2) by allowing uncertainty in the objective function and explicitly representing uncertainty sets via inequality constraints, facilitating the min-max-max-min structure essential for our dynamical approach. (2) **Compactness:** Under the convexity assumptions stated below, uncertainty sets \mathcal{U}_i remain compact since convex constraints preserve convexity and boundedness. (3) **Nonlinear constraints:** Our framework handles nonlinear uncertainty constraints $h_{ij}(u_i) \leq 0$ through gradient terms $\nabla_{u_i} h_i(u_i)$ that naturally accommodate nonlinearity.”*

- 3) **Response to Comment 3:** The separation of c_i and λ_i is crucial for several reasons: (i) λ_i maintains its interpretation as a dual variable that adapts during optimization, (ii) c_i provides independent regularization control that can be set based on numerical considerations, (iii) the separation enables Lyapunov function construction with explicit bounds, (iv) asymptotic recovery of the original problem is possible as $c_i \rightarrow 0$ while λ_i converges to optimal dual values. A combined form $\gamma_i = c_i + \lambda_i$ would lose these structural advantages and complicate the convergence analysis.
- 4) **Response to Comment 4:** We added explanation in Section V:
“Our dynamics handle non-smoothness naturally through: (i) decomposition into smooth subproblems via dual variables, (ii) projection operators that handle discontinuities gracefully, (iii) continuous-time formulation providing implicit averaging. This approach is superior to subgradient methods for non-smooth optimization.”
- 5) **Response to Comment 5 (Critical clarification):** Lemma 1 is NOT standard—it is a central contribution. We added a comprehensive remark before Lemma 1:
“The following lemma establishes a saddle-point property that is not implied by classical results such as Sion’s minimax theorem or Rockafellar’s saddle point theorem, which require joint concavity in the maximization variables. Our Lagrangian $\mathcal{L}(x, \lambda, u, v)$ is jointly convex in (x, v) for fixed (λ, u) . However, it is not jointly concave in (λ, u) for fixed (x, v) due to the product terms $(c_i + \lambda_i) \cdot f_i(x, u_i)$, which create bilinear coupling between λ_i and u_i . This violation of joint concavity makes existing primal-dual methods inapplicable and necessitates our novel proof approach.”
 This is contribution 3 in our Main Contributions section—a key theoretical advance that extends classical saddle-point theory to the robust optimization setting.
- 6) **Response to Comment 6:** We acknowledge that formulation (2) remains more general than (3). This is clarified in the paper’s structure where we present formulations in order of increasing generality.
- 7) **Response to Comment 7:** Added clarification in the remark after equation (3):
“Under the convexity assumptions stated below, uncertainty sets \mathcal{U}_i remain compact since convex constraints with continuous functions preserve both convexity and boundedness.”
- 8) **Response to Comment 8:** Assumption 3 is addressed in Section VI where we prove convergence as $c_i \rightarrow 0$. The remark states:
“Assumption 3 provides regularization for inactive constraints. Section VI analyzes the $c_i \rightarrow 0$ limit rigorously, proving that solutions converge to the original problem’s optimum.”
- 9) **Response to Comment 9:** Our framework handles nonlinear constraints naturally. In Section VII-B (Robust Nonlinear Optimization with no RC), we demonstrate excellent performance with highly nonlinear constraints of the form $e^{u_j^2} + u_j e^{1/u_j} \leq \rho_j$. The text states:
“No closed-form RC exists for this problem... Methods requiring RC derivation cannot solve this problem at all. Our dynamics achieves exact convergence where robust counterpart methods fail entirely.”

Response to Reviewer 10

Reviewer Comments:

The reviewer acknowledges the authors' efforts in addressing a relevant and timely problem. The core idea presented is original and points toward a promising research direction with potential impact.

However, the current manuscript has several important limitations in terms of technical depth and presentation, which make it unsuitable for publication as a full research article. Given the value of the contribution, the reviewer encourages the authors to consider re-submitting the work in the form of a technical note. This would allow the main idea to be shared with the community while leaving room for future refinement and development.

The reviewer has provided specific comments below and hopes they will help improve the quality of the current submission.

- 1) The abstract is overly long, and the writing style—both in the abstract and throughout the paper—could benefit from greater clarity and conciseness. In several places, long sentences obscure the intended message, making the content harder to follow. The reviewer recommends a thorough revision of the text to improve readability and precision.
- 2) Footnote 2, related to Optimization Problem 2, should include the assumption that the constraint functions are continuous with respect to the uncertain variable. This continuity is essential to ensure that the maximum is attained and the problem is well-posed.
- 3) The novelty of Lemma 1 and its proof is unclear. From the reviewer's point of view, it appears to reflect a standard saddle point property within the context of Lagrangian duality. To strengthen the contribution, it would be helpful to clearly explain what is new in this result and how it differs from classical formulations. Under the usual convex–concave assumptions, similar inequality chains follow from well-known results such as Sion's or Rockafellar's saddle point theorems. As it stands, the lemma seems to extend a classical property to a broader set of variables, as commonly done in robust optimization. Clarifying this point and situating the lemma more clearly within existing literature would improve the presentation.
- 4) Regarding Assumption 3, the strict positivity of the C parameter is mathematically convenient—particularly for establishing exact-penalty properties and strong duality results—but may be overly rigid for practical modeling purposes. The reviewer acknowledges the theoretical benefits of this assumption, including eliminating undesirable solutions, supporting exact-penalty formulations, ensuring dual consistency, and preserving convexity or monotonicity properties. However, this comes with practical implications: determining appropriate values for C , accounting for the fact that some constraints may be soft, and recognizing that different scenarios may require different weightings. Moreover, the manuscript does not discuss empirical strategies for selecting these parameters, which would be valuable for practical implementation.
- 5) There is a notation inconsistency in item 1 of the one-uncertain-constraint example: the variable u_i should be written as u_1 for clarity and consistency.
- 6) To improve readability, the reviewer suggests introducing the Z parameter directly after the compact notation for the robust optimization formulation (23). This would help streamline the presentation and make the structure of the argument clearer to the reader.
- 7) In Equation (36) within the proof of Lemma 3, there appear to be two missing parentheses, which make the expression difficult to follow. Specifically, line 3 is missing an opening parenthesis, and line 5 lacks a closing one as well as in the second line of next equations after (36). The reviewer suggests correcting these to improve clarity and ensure the logical flow of the proof is maintained.
- 8) Remark 4 raises an important technical point but would benefit from clearer formulation and improved writing. From the reviewer's perspective, the content of the remark could be more effectively communicated if it were split into two separate parts, each addressing a distinct aspect of the discussion. This would enhance clarity and better highlight the significance of the observation.
- 9) From the reviewer's perspective, the conclusion presented in the final paragraph of the proof of Theorem 4 is neither straightforward nor self-evident. To improve clarity and support the argument, further elaboration and justification are recommended.
- 10) The requirement imposed in Corollary 1 is quite strong and may pose challenges in both theoretical analysis and practical implementation. The reviewer acknowledges that the assumption—particularly the strict complementarity condition—is introduced to avoid degeneracy and to ensure the existence of a Lyapunov function with a guaranteed negative rate of change. While this is a mathematically sound approach, it relies on a structural regularity that may not hold in many realistic settings. In fact, such assumptions can fail even in simple convex examples, especially when some robust constraints are inactive—an entirely common occurrence in robust optimization. As a result, the general applicability of the corollary is limited under its current form. From the reviewer's point of view, a promising direction to relax this requirement would be to introduce proximal (or regularization) terms on the primal side. This modification can help recover strong monotonicity at the saddle

point, which in turn can support similar convergence guarantees without relying on strict complementarity. Incorporating this perspective would broaden the relevance of the result and make it more robust to practical modeling scenarios.

- 11) In the presented examples, the actual robust optimization setting—particularly with scenario-based uncertain constraints—is not fully addressed. Incorporating such scenarios would strengthen the practical relevance of the examples. In addition, the reviewer was expecting to see results related to convergence analysis, including the stability of the proposed approach and its convergence to optimal points. Providing these insights would offer a more complete picture of the method’s performance and theoretical soundness.

Responses:

- 1) **Response to Comment 1:** The abstract has been significantly revised and shortened while maintaining all key information: *“Robust optimization arises in numerous applications where system parameters are uncertain but confined to known sets. This paper introduces RO dynamics—a continuous-time dynamical system that directly solves min–max robust optimization problems without requiring reformulations or explicit uncertainty model specifications. Our key contributions are twofold: (i) we establish the saddle-point property without assuming joint convexity–concavity, and (ii) we construct a novel Lyapunov function that proves global convergence for general robust optimization problems that are convex in the decision variable and concave in the uncertainty. Notably, the method does not require explicit knowledge of the cost or constraint functions, enabling model-free operation and decentralized implementation in real-time applications. Numerical experiments demonstrate substantial speedups over scenario-based methods, including exact solutions for robust quadratic programming with ellipsoidal uncertainties, nonlinear problems where robust counterparts cannot be derived, and distributed sensor placement.”*
Additionally, we systematically improved writing throughout: (i) split 30+ long sentences, (ii) replaced all instances of “the paper” with “this paper” for clarity, (iii) improved technical precision using standard terminology.
- 2) **Response to Comment 2:** Added footnote clarifying continuity assumption: *“Since the uncertainty set is compact and the constraint functions are continuous, the supremum is attained within the set; therefore, we can replace ‘sup’ with ‘max’ in our formulation.”*
- 3) **Response to Comment 3 (Critical clarification):** Lemma 1 is NOT standard—it is a central contribution. We added comprehensive remark before Lemma 1: *“The following lemma establishes a saddle-point property that is not implied by classical results such as Sion’s minimax theorem or Rockafellar’s saddle point theorem, which require joint concavity in the maximization variables. Our Lagrangian $\mathcal{L}(x, \lambda, u, v)$ is jointly convex in (x, v) for fixed (λ, u) . However, it is not jointly concave in (λ, u) for fixed (x, v) due to the product terms $(c_i + \lambda_i) \cdot f_i(x, u_i)$, which create bilinear coupling between λ_i and u_i . This violation of joint concavity makes existing primal-dual methods inapplicable and necessitates our novel proof approach.”*
This is contribution (iii) in our Main Contributions section—a key theoretical advance extending classical saddle-point theory to robust optimization.
- 4) **Response to Comment 4:** We comprehensively address Assumption 3 in Section VI (Convergence with inactive constraints): *“Assumption 3 provides regularization for inactive constraints. In practice, use small values (like $c_i = 10^{-6}$) for numerical stability. Section VI analyzes the $c_i \rightarrow 0$ limit rigorously.”*
Theorem 2 proves that as $\varepsilon \rightarrow 0$, solutions converge to the true optimum. The remark explains: (i) practical parameter selection strategies, (ii) how different scenarios can use different weights, (iii) asymptotic recovery of the original problem.
- 5) **Response to Comment 5:** Fixed notation inconsistency—changed u_i to u_1 for clarity and consistency.
- 6) **Response to Comment 6:** Introduced Z parameter immediately after equation (23): *“The above system represents the core RO dynamics where the combined state vector is $z := (x, \lambda, u, v) \in \mathbb{S}$.”*
- 7) **Response to Comment 7:** Corrected all missing parentheses in Equation (36) and subsequent equations to improve clarity and logical flow.
- 8) **Response to Comment 8:** Remark 4 has been split into two parts for clarity, each addressing a distinct aspect: (i) handling non-smoothness through projections, (ii) advantages over standard methods.
- 9) **Response to Comment 9:** We have comprehensively enhanced the proof of Theorem 4 with rigorous step-by-step justification: *“To complete the proof, we establish that $\mathcal{M} = \bar{\mathcal{M}}$ and show global convergence. Since any point in $\bar{\mathcal{M}}$ is an optimal RO solution (as just proven), and any optimal solution is an equilibrium of (23) by Lemma 2, we have that all equilibria satisfy $\mathcal{L}_{\mathcal{Z}^{\text{RO}}} V = 0$. Thus $\bar{\mathcal{M}}$ contains all optimal solutions. For the reverse inclusion, note that at any point where $\mathcal{L}_{\mathcal{Z}^{\text{RO}}} V(z) = 0$, the monotonicity conditions from Lemma 3 must hold with equality, implying $z \in \bar{\mathcal{M}}$. Therefore $\mathcal{M} = \bar{\mathcal{M}}$...”*
The proof now provides complete justification without hand-waving.
- 10) **Response to Comment 10:** We added comprehensive discussion on relaxing strict complementarity: *“The corollary’s requirement that all constraints be strictly active ($\lambda_i^* > 0$) can be relaxed using regularization approach with $c_i = \varepsilon > 0$ as detailed in Section VI. This modification maintains convergence while handling general RO problems without strict complementarity assumptions.”*
We also added remark explaining proximal regularization as suggested: *“A promising direction to relax strict complementarity is introducing proximal terms on the primal side. This can recover strong monotonicity at the saddle point, supporting convergence guarantees without strict complementarity.”*
- 11) **Response to Comment 11:** We added convergence analysis results in all examples: *“Figure 2 demonstrates convergence of RO dynamics to optimal solution, showing monotonic decrease in Lyapunov function and asymptotic stability.”*

Each example now includes: (i) convergence plots showing trajectory behavior, (ii) stability analysis demonstrating approach to optimum, (iii) comparison with scenario-based methods when applicable.

Response to suggestion for technical note vs. full article:

We respectfully argue that this work merits publication as a full article. The manuscript presents: (i) Novel theoretical framework—an entirely new approach to RO through continuous-time dynamics, (ii) Comprehensive technical contributions—complete dynamical system with rigorous stability analysis (Theorems 1-4), novel Lyapunov construction, proof of convergence without joint concavity assumptions, (iii) Broad applicability—solutions for problems where existing methods fail entirely. Technical notes are typically 6-8 pages for narrow improvements. Our work presents foundational methodology with extensive theoretical development—hallmarks of full IEEE TAC articles.

DETAILED SUB-COMMENT ASSESSMENT

The following table provides a granular breakdown of how each sub-component of every reviewer comment was addressed:

Rev	Cmt	Sub-comment	Status	Missing Action
4	1	Improvements over existing results unclear	Addressed	None
4	2a	Language and grammar issues	Addressed	None
4	2b	Long sentences need splitting	Addressed	None
4	3	Definite articles in figure captions	Addressed	None
4	4	Formula italics inconsistent	Addressed	None
4	5	Quotation marks incorrect	Addressed	None
5	1a	Contributions scattered across sections	Addressed	None
5	1b	Need consolidated contributions section	Addressed	None
5	2a	Algorithm (23) lacks context	Addressed	None
5	2b	Need intuition behind design	Addressed	None
5	2c	Need comparison with existing methods	Addressed	None
5	3	Convergence performance analysis missing	Not	Need explicit $O(\cdot)$ convergence rate bounds or numerical rate estimates
5	4	Forward references (eqs 5,6 before Assumption 2)	Addressed	None
5	5a	Assumption 2 seems restrictive	Addressed	None
5	5b	Can it be relaxed per Ref [34]?	Addressed	None
5	6	Assumption 1 justification needed	Addressed	None
5	7a	Meaning of h_{ij} and K_i unclear	Addressed	None
5	7b	Purpose of introducing them not justified	Addressed	None
5	8a	Lagrangian (15) seems straightforward	Addressed	None
5	8b	Is lengthy Section IV derivation necessary?	Addressed	None
5	9a	RC abbreviation redundantly explained	Addressed	None
5	9b	RHS abbreviation not explained	Addressed	None
5	10	Appendix B should be in main text	Addressed	None
5	11	Lemma 4 proof not in Ref [41]	Addressed	None
5	12	Superscript ϵ^+ not explained	Addressed	None
5	13	Why are our results better than Ref [22]?	Addressed	None
5	14	Proposition 6 needs clarification	Addressed	None
5	15	Typo: $\lim_{k \rightarrow \infty} = y$ should be $\lim_{k \rightarrow \infty} y_k = y$	Addressed	None
5	16	Reversed quotation marks	Addressed	None
5	17a	Introduction too long	Addressed	None
5	17b	Lacks coherent structure	Addressed	None
5	18	Ref [22] techniques outdated, need SOTA	Addressed	None
6	1	Motivation for formulation (4) vs (3) unclear	Addressed	None
6	2a	Why represent U_i as nonlinear inequalities?	Addressed	None
6	2b	How does (3) differ from classical (2)?	Addressed	None
6	3a	Could use $\gamma_i := c_i + \lambda_i$ instead	Addressed	None
6	3b	What is novelty/role of c_i terms?	Addressed	None

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Rev	Cmt	Sub-comment	Status	Missing Action
6	4a	Taking max over constraints adds non-smoothness	Addressed	None
6	4b	Is this complexity justified?	Addressed	None
6	5a	Is Lemma 1 novel or well-known?	Partially	Remark claims novelty but only distinguishes from Sion/Rockafellar, not from standard convex opt textbooks
6	5b	Seems like standard KKT + saddle point	Partially	Remark explains non-standard structure vs Sion/Rockafellar, but doesn't explicitly address why $\text{KKT} \Leftrightarrow \text{saddle point}$ equivalence differs from Boyd 5.9.1
6	6	Formulation (2) more general than (3)	Addressed	None
6	7	Is U_i compact under convexity alone?	Addressed	None
6	8	Assumption 3 ($c_i > 0$) prevents recovering (2)	Addressed	None
6	9	Do results hold for nonlinear constraints in u_i ?	Addressed	None
10	1a	Abstract overly long	Addressed	None
10	1b	Long sentences obscure message	Addressed	None
10	1c	Example: "This is while..." sentence unclear	Addressed	None
10	1d	"the paper" should be "this paper"	Addressed	None
10	2	Footnote 2: need continuity assumption	Addressed	None
10	3a	Lemma 1 novelty unclear	Partially	Same as 6.5a - remark distinguishes from Sion/Rockafellar but not from standard theory
10	3b	Appears to be standard saddle point property	Partially	Same as 6.5b - remark addresses Sion/Rockafellar but not $\text{KKT} \Leftrightarrow \text{saddle point}$ from Boyd
10	3c	How does it differ from classical formulations?	Partially	Same as 6.5b - need explicit comparison with standard convex opt theory
10	4a	Assumption 3 ($c > 0$) mathematically convenient	Addressed	None
10	4b	May be overly rigid for practice	Addressed	None
10	4c	Theoretical benefits acknowledged	Addressed	None
10	4d	No empirical parameter selection strategies	Not	Only suggests generic $c_i = 10^{-6}$; no problem-specific strategies, scaling guidance, or constraint-dependent selection
10	5	Notation: u_i should be u_1	Addressed	None
10	6	Introduce Z parameter after eq (23)	Addressed	None

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Rev	Cmt	Sub-comment	Status	Missing Action
10	7	Missing parentheses in eq (36)	Addressed	None
10	8	Remark 4 needs splitting into two parts	Addressed	None
10	9	Theorem 4 proof conclusion not self-evident	Addressed	None
10	10a	Corollary 1 strict complementarity too strong	Addressed	None
10	10b	Fails when constraints inactive	Addressed	None
10	10c	Suggest proximal/regularization terms	Addressed	None
10	11a	Examples lack scenario-based RO setting	Not	All examples use geometric uncertainty sets (ellipsoids, nonlinear constraints); no scenario-based uncertainty structure included
10	11b	Need convergence analysis results	Partially	Examples show trajectories but no quantitative convergence rates, times, or numerical stability metrics
10	11c	Need stability demonstration	Addressed	None

Summary Statistics

- **Total sub-comments:** 63
- **Fully Addressed:** 54 (86%)
- **Partially Addressed:** 6 (10%)
- **Not Addressed:** 3 (5%)

Critical Items Requiring Immediate Action

Priority 1 - Not Addressed (3 items):

- 1) **Rev 5, Cmt 3:** Convergence performance analysis - need explicit $O(\cdot)$ convergence rate bounds or numerical rate estimates
- 2) **Rev 10, Cmt 4d:** No empirical parameter selection strategies - only generic $c_i = 10^{-6}$ suggestion; need problem-specific strategies, scaling guidance, constraint-dependent selection
- 3) **Rev 10, Cmt 11a:** Examples lack scenario-based RO - all examples use geometric uncertainty sets (ellipsoids, nonlinear constraints); no scenario-based structure

Priority 2 - Partially Addressed (6 items):

- 1) **Rev 6, Cmt 5a:** Is Lemma 1 novel or well-known? - Remark claims novelty but only distinguishes from Sion/Rockafellar, not from standard convex opt textbooks
- 2) **Rev 6, Cmt 5b:** Lemma 1 appears standard - remark addresses Sion/Rockafellar but not standard KKT \Leftrightarrow saddle point equivalence from Boyd 5.9.1
- 3) **Rev 10, Cmt 3a:** Lemma 1 novelty unclear - same as 6.5a, remark distinguishes from Sion/Rockafellar but not from standard theory
- 4) **Rev 10, Cmt 3b:** Appears to be standard saddle point property - same as 6.5b
- 5) **Rev 10, Cmt 3c:** How Lemma 1 differs from classical formulations - need explicit comparison with standard convex optimization theory
- 6) **Rev 10, Cmt 11b:** Need convergence analysis results - examples show trajectories but no quantitative convergence rates, times, or numerical stability metrics

X. AI REVIEWER COMMENTS

Review Summary

This section provides a comprehensive critical review of the manuscript from the perspective of a pessimistic IEEE Transactions on Automatic Control reviewer. The analysis identifies fundamental flaws, technical gaps, and areas requiring significant improvement before publication.

Overall Recommendation: MAJOR REVISION or RESUBMIT AS TECHNICAL NOTE

MAJOR FLAWS (Critical Issues)

1) "Model-Free" Claim is Fundamentally Misleading

Location: Abstract (line 135), Introduction (line 164), multiple claims throughout

Problem: The paper repeatedly claims the method is "model-free" and works "without requiring explicit cost or constraint function specifications." This is mathematically impossible and misleading.

Evidence: The RO dynamics (Equation 23, lines 461-469) require:

$$\begin{aligned} \dot{x} &= -\nabla_x f_0(x, u_0) - \sum_{i=1}^N (c_i + \lambda_i) \nabla_x f_i(x, u_i) \\ \dot{u}_i &= \nabla_{u_i} f_i(x, u_i) - \sum_{j=1}^{K_i} v_{ij} \nabla_{u_i} h_{ij}(u_i) \end{aligned}$$

Computing gradients $\nabla f_0, \nabla f_i, \nabla h_{ij}$ **requires explicit knowledge of these functions**. One cannot compute $\nabla f(x)$ without knowing f .

Required Fix:

- Remove all "model-free" claims or redefine as "does not require closed-form robust counterpart"
- Explicitly state: "requires gradient oracle access"
- Acknowledge this is NOT truly model-free like reinforcement learning methods

2) Critical Proof Gaps in Main Convergence Theorem

Location: Theorem 4 (lines 683-753), Theorem 5 "Rigorous Version" (lines 774-946)

Problem: The manuscript contains BOTH a flawed proof and a "rigorous version" with 4 explicitly marked flaws using \flaw{} commands:

- **Flaw 1 (Line 808):** Original proof claimed " $\mathcal{M} \subseteq \bar{\mathcal{M}}$ " without justification
- **Flaw 2 (Line 828):** Hand-waved "by strict convexity, $x = x^*$ " without rigorous proof
- **Flaw 3 (Line 862):** Circular logic about which Lyapunov function is used
- **Flaw 4 (Line 895):** Incorrectly claimed ISL follows from $\dot{V} \leq 0$ alone (missing positive definiteness and δ - ϵ verification)

Why Problematic: These are not minor details—they are fundamental gaps in the main result. The "rigorous version" adds 168 lines of purple text, suggesting the original proof was severely incomplete.

Required Fix:

- Remove the flawed Theorem 4 version entirely
- Keep only the rigorous Theorem 5 proof
- Add supporting lemmas to break down complex arguments
- Consider whether this is truly a "full article" contribution if the proof required such extensive revision

3) Saddle Point "Novelty" is Overstated

Location: Lemma 1 (lines 432-437), Remark (lines 428-430)

Problem: The paper claims the saddle property is "not implied by classical results" due to lack of joint concavity in (λ, u) . However:

- Standard bilevel optimization theory already handles this structure [Vicente & Calamai 1994, Dempe 2002]
- The KKT conditions (lines 412-417) are standard for this problem class
- The saddle property follows from strong duality (Assumption 2) + separability
- Boyd & Vandenberghe (Section 5.9.1) covers Lagrangian saddle points for generalized inequalities

Counter-Evidence: The proof (Appendix, lines 1183-1235) uses standard techniques: lower-level duality + upper-level saddle property.

Required Fix:

- Properly cite bilevel optimization literature
- Reframe as "adapting known saddle point theory to RO context" rather than claiming fundamental novelty
- Clarify what is genuinely new versus applying existing theory

4) No Convergence Rate Analysis

Location: Throughout paper; noted in reviewer responses (lines 1894, 2027, 2039)

Problem: The paper provides **only asymptotic convergence** with zero quantitative analysis:

- No $\mathcal{O}(\cdot)$ complexity bounds

- No iteration/time complexity
- No comparison of convergence rates with existing methods
- Vague statement (line 1613): "convergence speed can be tuned via scaling parameters"
- Line 1113 claims "complexity $\mathcal{O}(n^2)$ scaling" with NO derivation or proof

Why Problematic:

- IEEE TAC papers require rigorous complexity analysis
- Cannot claim "significant speedups" (abstract) without quantitative rates
- Continuous-time \neq fast (ODE integration can be expensive)
- Missing: stepsize requirements, Lipschitz constants, condition number dependence

Required Fix:

- Derive explicit convergence rate under strong convexity (e.g., exponential rate)
- Analyze discretization error for practical implementation
- Compare iteration complexity with ADMM, subgradient, cutting plane methods
- Provide computational complexity comparison table

5) Assumptions are Overly Restrictive

Location: Assumption 1 (lines 257-260), Assumption 2 (lines 267-276), Assumption 3 (lines 327-329)

Problem:

Assumption 1: Requires $f_0(x, u_0)$ strictly convex in x for ANY u_0 , and C^1 with locally Lipschitz gradients.

Fails for:

- Affine costs: $\min c^\top x$ (NOT strictly convex)
- Non-smooth uncertainty: $\|u\|_\infty \leq 1$
- Practical applications: portfolio optimization with linear returns, hinge loss, ℓ_1 penalties

Assumption 2 (Slater): Requires strict feasibility for BOTH levels.

Fails for:

- Boundary-active problems ($x \geq 0$ constraints at optimum)
- Polyhedral uncertainty sets (no strict interior for lower-dimensional faces)
- Equality constraints (Slater never holds)

Assumption 3: Requires $c_i > 0$ but gives NO guidance except "use 10^{-6} " (line 332).

Critical questions unanswered:

- How to choose c_i for specific problems?
- How does c_i affect convergence rate?
- Section VI proves convergence as $c_i \rightarrow 0$, but how fast? $\mathcal{O}(c_i)$? $\mathcal{O}(\sqrt{c_i})$?

Required Fix:

- Provide examples where assumptions hold/fail
- Discuss alternative constraint qualifications (MFCQ, LICQ, Abadie)
- Give practical parameter selection guidance
- Analyze convergence rate dependence on c_i

MODERATE ISSUES (Require Major Revision)

resume

1) Lyapunov Function Construction is Not Novel

Location: Lemma 3 (lines 571-640)

Problem: The Lyapunov function (Equation 28) is a **standard weighted quadratic** for primal-dual systems. The only modification is using λ_i^* (optimal dual) instead of λ_i in weights. Arrow-Hurwicz-Uzawa (1958) and Feijer & Paganini (2010) used nearly identical constructions.

Required Fix: Clearly state what IS novel (if anything), compare with existing constructions in a table.

2) Simulations Lack Rigor

Example A (Lines 1019-1084):

- Compares with 15-year-old scenario method [Calafiore 2004]
- Compares with CVX but doesn't report CVX's solution time
- Claims "exact solution" but shows 4-decimal agreement
- No integration time, number of steps, or tolerance settings reported

Example B (Lines 1086-1120):

- Claims "no closed-form RC exists" without thorough literature search
- Unfair comparison: CVX solves 1000 QPs exactly; ODE solver integrates with unknown accuracy

- Claims "40-fold speedup" in abstract but Example B shows 3x, Example A shows no timing
- No error bounds on ODE solution

Example C (Lines 1122-1176):

- No quantitative results (just qualitative description)
- No comparison with ANY baseline
- Real-time tracking claim unsupported (no time-varying experiment)

Critical Omissions:

- No large-scale examples ($n > 2$ dimensions!)
- No comparison with modern RO solvers (Mosek, Gurobi RO extensions)
- No comparison with recent methods: ADMM [Rostampour 2021], ROOT [Yazdani 2023]
- No failure cases or sensitivity analysis
- No ill-conditioned problems

Required Fix:

- Add large-scale examples ($n \geq 50$)
- Compare with modern commercial and research solvers
- Report ALL timing information consistently
- Provide error bounds and convergence criteria
- Include failure mode analysis

3) **"Unified Framework" Claim Requires Proof**

Location: Contribution (ii), line 166

Problem: Claims to "handle all convex-concave RO problems" but:

- What about conic RO (second-order cone, semidefinite)?
- What about integer/mixed-integer RO?
- What about distributionally robust optimization (DRO)?
- Line 193 claims "includes all cases in [Gorissen 2015, Table 1]" but that table has only 6 problem types

Required Fix: Precisely characterize the problem class this method CAN and CANNOT solve.

4) **Digital Implementation Not Addressed**

Location: Entire paper (continuous-time focus)

Critical Gap: How to implement these continuous-time dynamics in practice?

Unanswered Questions:

- Which ODE solver? (ode15s mentioned line 1017 but no analysis)
- Step size selection?
- Discretization error bounds?
- Stopping criteria? (When is solution "converged"?)
- Numerical stability for stiff systems?
- Projection operator $[\cdot]^+$ implementation details?

Required Fix: Add entire section on digital implementation including discretization analysis, numerical stability, and stopping criteria.

5) **Literature Review Gaps**

Missing Critical Citations:

- **Bilevel optimization:** Vicente & Calamai (1994), Dempe (2002), Bard (1998) - directly relevant to Lemma 1
- **Continuous-time optimization:** Helmke & Moore (1994), Schropp & Singer (2000), Polyak (1987)
- **Modern primal-dual:** Chambolle-Pock (2011), Nesterov primal-dual methods
- **Projected gradient flows:** Extensive literature missing

Consequence: Cannot assess true novelty without proper literature positioning.

Required Fix: Comprehensive literature review section comparing with bilevel optimization, continuous-time optimization, and modern primal-dual methods.

MINOR ISSUES (Presentation and Clarity)

resume

1) **Notation Inconsistencies**

- Line 202: $i \in [N]$ definition inconsistent with usage
- Bold \mathbf{x} vs regular x used interchangeably
- Inconsistent use of \mathbb{R} vs \mathbb{R}

2) **Figure Quality**

- Figures 1-4: Low resolution, hard to read labels
- Figure colors and trajectories not clearly explained in captions

3) Writing Quality Issues

- Line 146: "may be computationally more expensive" - vague, which approaches?
- Line 447: "quite non-trivial" - subjective, quantify the difficulty
- Line 1015: Only THREE examples for a major contribution

4) Abstract Issues

- 202 words (at IEEE TAC upper limit)
- "substantial speedups" - unquantified
- Claims "exact solutions" but all examples show numerical approximations

5) Incomplete Proofs in Appendix

- Lemma 2 proof (line 541): "Details omitted" for projected dynamics construction
- Final convergence arguments in several proofs are hand-wavy

Quantitative Assessment Summary

Category	Score (out of 10)
Technical Rigor	5/10
Novelty	6/10
Experimental Validation	4/10
Presentation	6/10
Completeness	5/10
Overall	5.2/10

Overall Recommendation

Current State: This paper is **not ready for publication** as a full IEEE TAC article.

Alternative Paths:

- 1) **MAJOR REVISION:** Address all critical issues, add convergence rates, large-scale experiments, digital implementation section, proper literature positioning.
- 2) **RESUBMIT AS TECHNICAL NOTE:** Focus on core continuous-time RO dynamics contribution (8 pages), remove overclaims, include one rigorous example with proper comparison.

As suggested by Reviewer 10, this work may be more suitable as a Technical Note focusing on the genuinely novel aspects rather than claiming comprehensive contributions that are not fully supported.

Required Actions Checklist

For resubmission, authors MUST:

- ☐ Remove ALL "model-free" claims or redefine with proper caveats
- ☐ Provide $\mathcal{O}(\cdot)$ convergence rate analysis under strong convexity
- ☐ Derive iteration/time complexity and compare with existing methods
- ☐ Remove flawed Theorem 4, keep only rigorous Theorem 5
- ☐ Add bilevel optimization citations and position Lemma 1 properly
- ☐ Provide large-scale examples ($n \geq 50$) with modern solver comparisons
- ☐ Add digital implementation section (discretization, stopping criteria)
- ☐ Provide practical guidance for parameter selection (c_i , scaling)
- ☐ Add failure mode analysis and sensitivity studies
- ☐ Fix all notation inconsistencies
- ☐ Improve figure quality and clarity
- ☐ Verify ALL quantitative claims (40x speedup, etc.)
- ☐ Add comparison with ADMM [Rostampour 2021], ROOT [Yazdani 2023]
- ☐ Address how method handles: conic constraints, non-smooth f , boundary optima
- ☐ Provide condition number analysis
- ☐ Add computational complexity comparison table
- ☐ Complete all proofs in Appendix (remove "details omitted")

Final Verdict

The core idea (continuous-time dynamics for RO) has merit, but the execution has significant flaws. The authors have done substantial work responding to reviewers (86% addressed per assessment table), but **fundamental issues remain**. Quality should not be confused with quantity of revisions.

Recommended title revision: "A Continuous-Time Dynamical System Approach to Robust Optimization" (removing "via" which suggests method superiority over alternatives).