$\mathcal{RO}$ Dynamics Solutions Properties

$\mathcal{RO}$ dynamics (23) can be viewed as switched dynamical system with a discontinuous right-hand side. The conditions guaranteeing the existence and uniqueness of the solution and continuity w.r.t. initial conditions, for a general discontinuous dynamical system are provided in [45, Theorem 2.5]. In this section, we show that our $\mathcal{RO}$ dynamics (23) satisfies the refined conditions presented in [40].

To prove the existence and uniqueness of solutions for (23), and also the continuity of solutions w.r.t. the initial conditions, there are two main steps. The first step is showing that $\mathcal{RO}$ dynamics is a particular case of projected dynamical systems. The second step requires $\mathcal{RO}$ dynamics (23) satisfying the monotonicity property, which is our main result.

Definition 2. *(Projection operator) If $\mathcal{K}$ is a closed convex set, for any point $\bar{y} \in \mathbb{R}^q$, the point projection of $\bar{y}$ on the set $\mathcal{K}$ can be written as*

$$\mathit{proj}_{\mathcal{K}}(\bar{y}) = \mathit{argmin}_{y \in \mathcal{K}} \parallel y - \bar{y} \parallel \ .$$

For $\bar{y} \in \mathbb{R}^n$ and $y \in \mathcal{K}$, vector projection of $\bar{y}$ at y w.r.t. $\mathcal{K}$ is

$$\Pi_{\mathcal{K}}(y, \bar{y}) = \mathit{lim}_{\delta \to 0^+} \frac{\mathit{proj}_{\mathcal{K}}(y + \delta \bar{y}) - y}{\delta} \ . \tag{49}$$

Note that the map $\text{proj}_{\mathcal{K}}$ is Lipschitz on $\mathbb{R}^q$ with constant $L = 1$ [46, Proposition 2.4.1].

Definition 3. *[Projected dynamical system [47]] Considering a differential equation $\dot{y} = F(y)$ with $F : \mathbb{R}^q \to \mathbb{R}^q$, the associated projected dynamical system is defined as*

$$\dot{y} = \Pi_{\mathcal{K}}(y, F(y)), \ y(0) \in \mathcal{K} \ . \tag{50}$$

Lemma 5. *($\mathcal{RO}$ dynamics as a projected dynamics) $\mathcal{RO}$ dynamics (23) can be written as a projected dynamical system according to Definition 3.*

Proof. The proof of Lemma 5 follows along the lines of the construction outlined in [40]. Details omitted. □

Remark 12 (Projected Dynamics Foundation)**.** *The following proposition establishes existence despite discontinuities from projections, uniqueness via the Lipschitz property, continuous dependence on initial conditions, and thus the validity of Lyapunov analysis for Theorem 4.*

Proposition 6. *If F in the projected dynamical system (50) is Lipschitz on $\mathcal{K}$, we have the following existence, uniqueness, and continuity w.r.t. the initial condition results for the solutions of the projected dynamics (50):*

1) *For any $y_0 \in \mathcal{K}$, there exists a unique solution $t \to y(t)$ of the projected system (50) with $y(0) = y_0$ in $[0, \infty)$.*
2) *Consider a sequence of points $\{y_k\}_{k=1}^{\infty} \subset \mathcal{K}$ with $\lim_{k \to \infty} y_k = y$. Then, the sequence of solutions $\{t \to \gamma_k(t)\}_{k=1}^{\infty}$ of the projected dynamics (50) with $\gamma_k(0) = y_k$ for all k, converges to the solution $t \to \gamma(t)$ of (50) with $\gamma(0) = y$ uniformly on every compact set of $[0, \infty)$.*

The ability to write $\mathcal{RO}$ dynamics (23) as a projected dynamical system along with the monotonicity property is used in the proof of the existence, uniqueness and continuity of the solutions of the set $\mathbb{S}$.

Lemma 6. *(Existence, uniqueness and continuity of solutions) $\gamma : [0, T) \to \mathbb{S}$ is defined as a Caratheodory solution of $\mathcal{Z}^{\mathcal{RO}}$ in the interval $[0, T)$ if γ is absolutely continuous on $[0, T)$ and satisfies $\dot{\gamma}(t) = \mathcal{Z}^{\mathcal{RO}}(\gamma(t))$ almost everywhere in $[0, T)$. Under Assumptions 1 and 2, and starting from any point $z \in \mathbb{S}$, a unique solution to $\mathcal{RO}$ dynamics (23) exists and remains in $\mathbb{S} \cap V^{-1}(\leq V(z))$. Also, if a sequence of points $\{z_k\}_{k=1}^{\infty} \subset \mathbb{S}$ converges to z as $k \to \infty$, the sequence of solutions $\{t \to \gamma_k(t)\}_{k=1}^{\infty}$ of $\mathcal{Z}^{\mathcal{RO}}$ starting at these points (that is, $\gamma_k(0) = z_k$ for all k) converge uniformly to the solution $t \to \gamma(t)$ on every compact set of $[0, \infty)$.*

The proof of this lemma follows closely along the lines of proof for the existence and uniqueness of solution for the primal-dual dynamical system from [40, Lemma 4.3].

Proof of Theorem 5

Based on the optimal solution $x^\star$ for $\mathcal{RO}$ problem,

$$\mu = \min_{\mathcal{F}_i(x) \leq 0} \mathcal{F}_0(x) \ , \ \mu = \mathcal{F}_0(x^\star \ .$$

As the cost function of μ_ε is smaller than or equal to that of $\mathcal{RO}$ and the feasible sets of the two problems are equal,

$$\mu_\varepsilon - \mu \leq 0 \ . \tag{51}$$

Since $x^\star$ minimizes $\mathcal{F}_0(x)$ over the constraint set, $\mu = \mathcal{F}_0(x^\star) \leq \mathcal{F}_0(x_\varepsilon^\star)$. Adding and subtracting $\varepsilon \sum_{i=1}^{n} \mathcal{F}_i(x_\varepsilon^\star)$ in the right-hand side (RHS) and using (51) yields $\varepsilon \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star) \leq \mu_\varepsilon - \mu \leq 0$.

Following a similar argument as before by comparing $\mu(\varepsilon_0)$ and μ_ε for $\varepsilon_0 \geq \varepsilon$, we now let

$$\mu_\varepsilon = \tilde{\mathcal{F}}_0(x_\varepsilon^\star) = \mathcal{F}_0(x_\varepsilon^\star) + \varepsilon \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star) \ ,$$

$$\mu(\varepsilon_0) = \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^\star) + \delta \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) \ ,$$

where $\delta = \varepsilon_0 - \varepsilon$. Then, $\mu_\varepsilon \geq \mu(\varepsilon_0)$, but because $x_\varepsilon^\star$ is optimal for μ_ε , we have $\tilde{\mathcal{F}}_0(x_\varepsilon^\star) \leq \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^\star)$. As $x_\varepsilon^\star$ is feasible for $\mu(\varepsilon_0)$,

$$\tilde{\mathcal{F}}_0(x_{\varepsilon_0}^\star) + \delta \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) = \mu(\varepsilon_0) \leq \tilde{\mathcal{F}}_0(x_\varepsilon^\star) + \delta \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star) \ .$$

Combining the two inequalities,

$$\delta \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) - \delta \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star) \leq \tilde{\mathcal{F}}_0(x_\varepsilon^\star) - \tilde{\mathcal{F}}_0(x_{\varepsilon_0}^\star) \leq 0 \ ,$$

which implies that $\sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) \leq \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star)$. Thus,

$$\varepsilon \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) \leq \varepsilon \sum_{i=1}^{N} \mathcal{F}_i(x_\varepsilon^\star) \leq \mu_\varepsilon - \mu \leq 0 \ . \tag{52}$$

Since $\sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) = \frac{\mu(\varepsilon_0) - \mathcal{F}_0(x_{\varepsilon_0}^\star)}{\varepsilon_0}$ is bounded, we have

$$\varepsilon \sum_{i=1}^{N} \mathcal{F}_i(x_{\varepsilon_0}^\star) \leq \mu_\varepsilon - \mu \leq 0 \ , \tag{53}$$