From Lemma 4, the omega-limit set of $\bar{\gamma}(t)$ is invariant and invariance principle for discontinuous Caratheodory systems [40, Proposition 2.1] (simplified version of [41, Proposition 3]) implies that $\bar{\gamma}(t)$ converges to the largest invariant set in $\mathrm{cl}(\mathcal{M})$ where $\mathcal{M} := \{z \in \mathbb{P} \,|\, \mathfrak{L}_{\mathcal{Z}^{\mathcal{RO}}} V(z) = 0\}$.

Next, we characterize the set $\mathcal{M}$ where $\mathfrak{L}_{\mathcal{Z}^{\mathcal{RO}}} V(z) = 0$ by defining

$$\begin{aligned} \bar{\mathcal{M}} := \{ & z \in \mathbb{P} \,|\, \lambda \geq 0, v_i \geq 0, \forall i, \\ & \mathcal{L}(x^\star, \lambda, u, v^\star) - \mathcal{L}(x^\star, \lambda^\star, u^\star, v^\star) = 0, \\ & \mathcal{L}(x^\star, \lambda^\star, u^\star, v^\star) - \mathcal{L}(x, \lambda^\star, u^\star, v) = 0 \} \ . \end{aligned} \tag{40}$$

From the inequality in (38), it follows that $\mathcal{M} \subseteq \bar{\mathcal{M}}$. We then prove that every point in $\bar{\mathcal{M}}$ is an optimal $\mathcal{RO}$ solution.

From the strict convexity of f, it follows that $x = x^\star$ on $\bar{\mathcal{M}}$. From (40), any point in $\bar{\mathcal{M}}$ achieves the optimal cost of $\mathcal{RO}$. Let $\bar{z} = (x^\star, \bar{\lambda}, \bar{u}, \bar{v}) \in \bar{\mathcal{M}}$. Then, in general,

$$\begin{aligned} \mathcal{L}(x^\star, \bar{\lambda}, \bar{u}, v^\star) \leq \mathcal{L}(x^\star, \lambda^\star, u^\star, v^\star) \ , \\ \mathcal{L}(x^\star, \lambda^\star, u^\star, v^\star) \leq \mathcal{L}(x^\star, \lambda^\star, u^\star, \bar{v}) \ . \end{aligned}$$

But since $\bar{z} \in \bar{\mathcal{M}}$, the equality must hold for the above equations. This means that

$$\begin{aligned} \bar{v} &= \arg\max_{v \geq 0} \ \mathcal{L}(x^\star, \lambda^\star, u^\star, v) \ , \\ (\bar{\lambda}, \bar{u}) &= \arg\max_{u, \lambda \geq 0} \ \mathcal{L}(x^\star, \lambda, u, v^\star) \ . \end{aligned}$$

Therefore, $\bar{z}$ is an optimal $\mathcal{RO}$ solution. Therefore, any point in $\bar{\mathcal{M}}$ is an optimal $\mathcal{RO}$ solution. On the other hand, any optimal $\mathcal{RO}$ solution is an equilibrium of $\mathcal{RO}$ dynamics (23) according to Lemma 2 and therefore is in $\mathcal{M}$. Thus, $\mathcal{M} = \bar{\mathcal{M}}$. As $\delta > 0$ is arbitrary, we conclude that the set of optimal $\mathcal{RO}$ solutions is globally asymptotically stable on $\mathbb{S}$.

Note that $\mathcal{M}$ can contain an uncountable infinite set of points. If the optimal $\mathcal{RO}$ solution is not unique, these correspond to the set of optimal $\mathcal{RO}$ solutions and to the set of non-isolated equilibria of $\mathcal{RO}$ dynamics (23) from Lemma 2.

When the set $\mathcal{M}$ of optimal solutions forms a continuum (uncountably many equilibria), asymptotic stability is not the appropriate stability notion since non-isolated equilibria cannot be asymptotically stable. Instead, we employ *semi-stability* theory [42], which establishes convergence to individual equilibrium points from a continuum. The key insight is that each equilibrium point must be shown to be Lyapunov stable, and this requires treating the Lyapunov function as a *parameterized family* indexed by the equilibrium point.

We now establish that each trajectory $\bar{\gamma}(t)$ converges to a single point in $\mathcal{M}$, not merely to the set $\mathcal{M}$. Since $\bar{\gamma}(t)$ is bounded and remains in the compact set $\mathbb{P}$ for all $t \geq 0$, the classical theory of dynamical systems [43, Lemma 4.1] guarantees that its omega-limit set $\omega(\bar{\gamma})$ is nonempty, compact, and invariant. By the invariance principle for discontinuous Carathéodory systems applied above, we have $\omega(\bar{\gamma}) \subseteq \mathcal{M}$.

To prove that each trajectory converges to a *single* equilibrium in $\mathcal{M}$, we invoke semi-stability theory. Consider the Lyapunov function V from Lemma 3 as a **parameterized family** $\{V_{z^\star}\}_{z^\star \in \mathcal{M}}$, where each $V_{z^\star}$ is centered at a particular equilibrium $z^\star = (x^\star, \lambda^\star, u^\star, v^\star)$:

$$V_{z^\star}(z) = \frac{1}{2}\big(\|x - x^\star\|^2 + \|\lambda - \lambda^\star\|^2 + \|u_0 - u_0^\star\|^2 + \sum_{i=1}^{N}(c_i + \lambda_i^\star)\|u_i -$$

This family is **well-defined** (non-degenerate) for all $z^\star \in \mathcal{M}$ because the weighting coefficients satisfy $(c_i + \lambda_i^\star) \geq c_i > 0$ for all $i \in [N]^+$ by Assumption 3. Hence, $V_{z^\star}$ is positive definite and radially unbounded for every equilibrium $z^\star$.

From Lemma 3, the Lie derivative satisfies $\mathfrak{L}_{\mathcal{Z}^{\mathcal{RO}}} V_{z^\star}(z) \leq 0$ along all trajectories, with equality only when z is an optimal $\mathcal{RO}$ solution. Since this holds for *any* equilibrium $z^\star \in \mathcal{M}$, every equilibrium point is Lyapunov stable in the sense of Lyapunov (ISL). That is, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|z(0) - z^\star\| < \delta(\varepsilon)$ implies $\|z(t) - z^\star\| < \varepsilon$ for all $t \geq 0$.

By semi-stability theory [42, Theorem 2.1], when every equilibrium in a continuum is Lyapunov stable (established via the parameterized family $\{V_{z^\star}\}$) and trajectories approach the equilibrium set, each trajectory converges to a *single* equilibrium point. Specifically, pick any $z^\star \in \omega(\bar{\gamma}) \subseteq \mathcal{M}$. The ISL property of $z^\star$ combined with the trajectory approaching $z^\star$ along a subsequence implies $\bar{\gamma}(t) \to z^\star$ as $t \to \infty$. Therefore $\omega(\bar{\gamma}) = \{z^\star\}$ is a singleton, and each trajectory converges to a single optimal $\mathcal{RO}$ solution. □ □

Corollary 1. *Under Assumptions 1 and 2, let $z = (x^\star, \lambda^\star, u^\star, v^\star)$ be an optimal solution. Assume all robust constraints are strictly active, that is, $\lambda_i^\star > 0, \ i \in [N]^+$. Then, the $\mathcal{RO}$ dynamics (23) converges to an optimal solution.*

Remark 11 (Relaxing Strict Complementarity)**.** *The corollary's requirement that all constraints be strictly active ($\lambda_i^\star > 0$) can be relaxed using regularization approach with $c_i = \varepsilon > 0$ as detailed in Section VI. This modification maintains convergence while handling general RO problems without strict complementarity assumptions.*

Proof. The setup of the corollary satisfies the assumptions of Theorem 4, as $c_i + \lambda_i^\star > 0, \ i \in [N]^+$. □

VI. Convergence with Inactive Constraints

The convergence proof requires $c_i + \lambda_i^\star > 0$. This section rigorously handles inactive constraints where $\lambda_i^* = 0$.

Consider the original problem with $c_i = 0$:

$$\begin{aligned} \mu := \min_{x} \ & \mathcal{F}_0(x) \\ \text{s.t.} \ \ & \mathcal{F}_i(x) \leq 0 \ , \ i \in [N]^+ \ , \end{aligned} \tag{41}$$

When constraints are inactive, $\lambda_i^* = 0$ by complementary slackness. Setting $c_i = 0$ leads to $(c_i + \lambda_i^\star) = 0$, invalidating our Lyapunov function and breaking the dynamics coupling.

We resolve this via regularization with $c_i = \varepsilon > 0$ (small):