Proof. Note that $\nabla V(z)^\top$ equals to

$$
\begin{aligned}
&(x-x^\star)^\top(-\nabla_x f_0(x,u_0) - \sum_{i=1}^N (c_i+\lambda_i)\nabla_x f_i(x,u_i)) \\
&+\sum_{i=1}^N (\lambda_i-\lambda_i^\star)[f_i(x,u_i)-v_i^\top h_i(u_i)]^+_{\lambda_i} \\
&+(u_0-u_0^\star)^\top(\nabla_{u_0} f_0(x,u_0) - v_0^\top \nabla_{u_0} h_0(u_0)) \\
&+\sum_{i=1}^N (c_i+\lambda_i^\star)(u_i-u_i^\star)^\top(\nabla_{u_i} f_i(x,u_i) - v_i^\top \nabla_{u_i} h_i(u_i)) \\
&+(v_0-v_0^\star)^\top[h_0(u_0)]^+_{v_0} + \sum_{i=1}^N (v_i-v_i^\star)^\top[(c_i+\lambda_i)\ h_i(u_i)]^+_{v_i} \ .
\end{aligned} \tag{29}
$$

By convex and concave under-estimator properties according to Assumption 1, one can write [31, Section 3.1.3]

$$(x^\star - x)^\top \nabla_x f_i(x,u_i) \le f_i(x^\star,u_i) - f_i(x,u_i)\ , \tag{30}$$
$$(u_i - u_i^\star)^\top \nabla_{u_i} f_i(x,u_i) \le f_i(x,u_i) - f_i(x,u_i^\star)\ , \tag{31}$$
$$(u_i^\star - u_i)^\top \nabla_{u_i} h_{ij}(u_i) \le h_{ij}(u_i^\star) - h_{ij}(u_i)\ . \tag{32}$$

Moreover, using the fact that the projection operator is non-expansive, we have that

$$
\begin{aligned}
&(\lambda_i-\lambda_i^\star)[f_i(x,u_i)-v_i^\top h_i(u_i)]^+_{\lambda_i} \\
&\qquad \le (\lambda_i-\lambda_i^\star)(f_i(x,u_i)-v_i^\top h_i(u_i))\ ,
\end{aligned} \tag{33}
$$
$$(v_0-v_0^\star)^\top[h_0(u_0)]^+_{v_0} \le (v_0-v_0^\star)^\top h_0(u_0), \tag{34}$$
$$
\begin{aligned}
&(v_i-v_i^\star)^\top[(c_i+\lambda_i)\ h_i(u_i)]^+_{v_i} \\
&\qquad \le (v_i-v_i^\star)^\top(c_i+\lambda_i)\ h_i(u_i)\ .
\end{aligned} \tag{35}
$$

By substitution, we get

$$
\begin{aligned}
&\nabla V(z)^\top \dot z \le \\
&f_0(x^\star,u_0) - f_0(x,u_0) + \sum_{i=1}^N (c_i+\lambda_i)(f_i(x^\star,u_i) - f_i(x,u_i)) \\
&+\sum_{i=1}^N (c_i+\lambda_i) - (c+\lambda_i^\star))(f_i(x,u_i) - v_i^\top h_i(u_i)) \\
&+ f_0(x,u_0) - f_0(x,u_0^\star) + v_0^\top(h_0(u_0^\star) - h_0(u_0)) \\
&+\sum_{i=1}^N (c+\lambda_i^\star)(f_i(x,u_i) - f_i(x,u_i^\star) + v_i^\top(h_i(u_i^\star) - h_i(u_i)) \\
&+ (v_0-v_0^\star)^\top h_0(u_0) + \sum_{i=1}^N (v_i-v_i^\star)^\top(c_i+\lambda_i)h_i(u_i)\ .
\end{aligned} \tag{36}
$$

After simplification and rearrangement we obtain

$$
\begin{aligned}
\nabla V(z)^\top \dot z \le\ & f_0(x^\star,u_0) - (v_0^\star)^\top h_0(u_0) \\
& - f_0(x,u_0^\star) + v_0^\top h_0(u_0^\star) \\
& + \sum_{i=1}^N (c_i+\lambda_i)\Big(f_i(x^\star,u_i) - (v_i^\star)^\top h_i(u_i)\Big) \\
& - \sum_{i=1}^N (c_i+\lambda_i^\star)\Big(f_i(x,u_i^\star) - v_i^\top h_i(u_i^\star)\Big)\ .
\end{aligned}
$$

Adding and subtracting $\mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star)$ yields

$$
\begin{aligned}
\nabla V(z)^\top \dot z \le\ & \mathcal{L}(x^\star,\lambda,u,v^\star) - \mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) \\
& + \mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) - \mathcal{L}(x,\lambda^\star,u^\star,v)\ .
\end{aligned} \tag{37}
$$

Note that from Lemma 1 (saddle property),

$$
\begin{aligned}
&\mathcal{L}(x^\star,\lambda,u,v^\star) - \mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) \le 0\ , \\
&\mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) - \mathcal{L}(x,\lambda^\star,u^\star,v) \le 0\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\nabla V(z)^\top \dot z \le\ & \mathcal{L}(x^\star,\lambda,u,v^\star) - \mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) \\
& + \mathcal{L}(x^\star,\lambda^\star,u^\star,v^\star) - \mathcal{L}(x,\lambda^\star,u^\star,v) \le 0\ .
\end{aligned} \tag{38}
$$

□

Remark 10 (Why Standard Dynamics Fail and Our Solution Works)**.** *Standard primal-dual dynamics fail for RO because when $\lambda_i \to 0$ (inactive constraints), the dynamics $\dot u_i = \lambda_i \nabla_{u_i} f_i$ freeze before reaching optimality. We resolve this by removing λ_i from $\dot u_i$ dynamics and constructing a Lyapunov function that weights errors by optimal duals $\lambda_i^\star$ rather than current values, enabling global convergence despite lacking joint concavity.*

Solutions: Existence, Uniqueness, and Continuity

We need to guarantee that the switching dynamical system (23) has well defined solutions. To do so, we show that our system satisfies the conditions for applicability of existing results [40]. The development of this section is detailed in the Appendix.

Main Convergence Result

Considering $\gamma(t)$ as a solution of $\mathcal{RO}$ dynamics (23) defined on the time interval $[0,\infty)$, omega-limit set is defined as

$$
\begin{aligned}
&\Omega(\gamma) := \{y \in \mathbb{S} \mid \exists \{t_k\}_{k=1}^\infty \subset [0,\infty) \\
&\qquad \text{with } \lim_{k\to\infty} t_k = \infty \text{ and } \lim_{k\to\infty} \gamma(t_k) = y\}\ .
\end{aligned} \tag{39}
$$

The next lemma presents an invariant property for the omega-limit set of any solution of the $\mathcal{RO}$ dynamics, which contributes in showing the convergence result.

Lemma 4. *[40, Lemma 4.4][Invariance of omega-limit set] The omega-limit set of any solution of the $\mathcal{RO}$ dynamics (23) starting from any point $\mathbb{S}$ is invariant.*

Finally, we have the main convergence result.

Theorem 4. *[Convergence] Under Assumptions 1 and 2, the trajectories of $\mathcal{RO}$ dynamics* (23) *converge to an optimal $\mathcal{RO}$ solution for any initial condition in $\mathbb{S} = \mathbb{R}^n \times \mathbb{R}^N_+ \times \mathbb{R}^M \times \mathbb{R}^K_+$. In particular, each trajectory converges to a single point in the set of optimal $\mathcal{RO}$ solutions.*

Proof. Let $z^\star = (x^\star,\lambda^\star,u^\star,v^\star)$ be an optimal $\mathcal{RO}$ solution under Assumptions 1 and 2. Lemma 6 implies that a unique solution of $\mathcal{RO}$ dynamics $\mathcal{Z}^{\mathcal{RO}}$ exists starting from any point in the compact set $\mathbb{P} = V^{-1}(\le \delta) \cap \mathbb{S}$ for any $\delta > 0$ [5]. We now call this solution $\bar\gamma(t)$.

[5]As V is radially unbounded, the set $\mathbb{P}$ is always compact.