Shuzhi Daishu Programming Homework 2

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Abstract

This report mainly discusses the implementation of three numerical methods: the bisection method, Newton's method, and the secant method. We implemented the three algorithms in C++ and used them to solve problems.

Problem 1

The interpolation formula is given by:

$$f(x) - p_1(f;x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1)$$

where $f(x) = \frac{1}{x}$, and the derivatives of $f(x)$ are:

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$$

Substituting these into the interpolation formula:

$$f(x) - \left(\frac{1}{x_1} + \frac{x - x_0}{x_1 - x_0}\left(\frac{1}{x_0} - \frac{1}{x_1}\right)\right) = \frac{2}{\xi(x)^3}\frac{(x - x_0)(x - x_1)}{2}$$

$$f(x) - \left(\frac{1}{x_1} + \frac{x - x_0}{x_1 - x_0}\left(\frac{1}{x_0} - \frac{1}{x_1}\right)\right) = \frac{(x - x_0)(x - x_1)}{\xi(x)^3}$$

2. Extending the Domain of $\xi(x)$ and Finding Maximum and Minimum Values The point $\xi(x)$ lies somewhere between x_0 and x_1. Since $f(x) = \frac{1}{x}$ is a monotonically decreasing function, and the behavior of its second derivative affects the distribution of $\xi(x)$, we can analyze the second derivative of $f(x)$:

$$f''(x) = \frac{2}{x^3}$$

Since $f''(x)$ is decreasing over the interval $(1, 2)$, the minimum value of $\xi(x)$ should be near $x_0 = 1$, and the maximum value should be near $x_1 = 2$. - The **maximum** of $\xi(x)$ occurs near $x_1 = 2$, - The **minimum** of $\xi(x)$ occurs near $x_0 = 1$. Thus, we can determine that the maximum and minimum values of $\xi(x)$ occur within the interval $[1, 2]$. Since $f''(x) = \frac{2}{x^3}$, and $f''(x)$ decreases as x increases, the maximum value of $f''(x)$ in the interval $[1, 2]$ occurs at $x = 1$, and the minimum value occurs at $x = 2$.

Maximum of $f''(\xi(x))$: At $x_0 = 1$, we have:

$$f''(1) = \frac{2}{1^3} = 2$$

Minimum of $f''(\xi(x))$: At $x_1 = 2$, we have:

$$f''(2) = \frac{2}{2^3} = \frac{2}{8} = 0.25$$

Problem 2

The Lagrange interpolation polynomial is given by:

$$p(x) = \sum_{i=0}^{n} f_i \cdot l_i(x)$$

where $l_i(x)$ is:

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

This construction ensures that $p(x_i) = f_i$ for each i. In order for the polynomial $p(x) = \sum_{i=0}^{n} f_i \cdot l_i(x)$ to be non-negative, we define:

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \left(\frac{(x - x_j)^2}{(x_i - x_j)^2} \right)^{1/2}$$

This modification ensures that the polynomial $p(x)$ satisfies the non-negativity requirement, as the squares of terms ensure non-negative values for all x. Thus, the polynomial $p(x)$ meets the given conditions.

Problem 3

We need to prove:

$$f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!} e^t$$

When $n = 0$, there is only one point t, and the divided difference at this point is $f[t] = e^t$. We can see that:

$$\frac{(e-1)^0}{0!} e^t = e^t$$

So, the base case holds. Induction hypothesis: Assume the statement is true for $n = k$, i.e.,

$$f[t, t+1, \ldots, t+k] = \frac{(e-1)^k}{k!} e^t$$

We need to prove that the statement holds for $n = k+1$. Using the recursive formula for divided differences, we have:

$$f[t, t+1, \ldots, t+k+1] = \frac{f[t+1, \ldots, t+k+1] - f[t, t+1, \ldots, t+k]}{(t+k+1) - t}$$

Substituting the induction hypothesis:

$$f[t, t+1, \ldots, t+k+1] = \frac{\frac{(e-1)^k}{k!} e^{t+1} - \frac{(e-1)^k}{k!} e^t}{k+1}$$

Since $e^{t+1} = e \cdot e^t$, we get:

$$f[t, t+1, \ldots, t+k+1] = \frac{\frac{(e-1)^k}{k!} e^t (e-1)}{k+1} = \frac{(e-1)^{k+1}}{(k+1)!} e^t$$

2. From Corollary 2.22, there exists a $\xi \in (0, n)$ such that:

$$f[0, 1, \ldots, n] = \frac{1}{n!} f^{(n)}(\xi)$$

For $f(x) = e^x$, its n-th derivative is:

$$f^{(n)}(x) = e^x$$

Thus, Corollary 2.22 gives:

$$\frac{(e-1)^n}{n!} = \frac{1}{n!} e^\xi$$

Multiplying both sides by $n!$, we get:

$$(e-1)^n = e^\xi$$

Taking the natural logarithm of both sides:

$$n \ln(e-1) = \xi$$

Hence, $\xi = n \ln(e-1)$.

3. Since $\ln(e-1)$ is a constant and $\ln(e-1) \approx 0.31$, we have:

$$\xi = n \cdot 0.31$$

Clearly, ξ is to the left of the midpoint $n/2$.

Problem 4

Newton Interpolation Polynomial $p_3(f;x)$ The Newton interpolation polynomial is constructed based on the given interpolation points. The general form of the interpolation polynomial is:

$$p_3(f;x) = f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

Given the values: $f(0) = 5, f(1) = 3, f(3) = 5, f(4) = 12$ We first calculate the divided differences:

$$f[0,1] = \frac{f(1)-f(0)}{1-0} = \frac{3-5}{1} = -2$$

$$f[1,3] = \frac{f(3)-f(1)}{3-1} = \frac{5-3}{2} = 1$$

$$f[3,4] = \frac{f(4)-f(3)}{4-3} = \frac{12-5}{1} = 7$$

$$f[0,1,3] = \frac{f[1,3]-f[0,1]}{3-0} = \frac{1-(-2)}{3} = \frac{3}{3} = 1$$

$$f[1,3,4] = \frac{f[3,4]-f[1,3]}{4-1} = \frac{7-1}{3} = \frac{6}{3} = 2$$

$$f[0,1,3,4] = \frac{f[1,3,4]-f[0,1,3]}{4-0} = \frac{2-1}{4} = \frac{1}{4}$$

Thus, we can write the polynomial as:

$$p_3(f;x) = f(0)+f[0,1](x-0)+f[0,1,3](x-0)(x-1)+f[0,1,3,4](x-0)(x-1)(x-3)$$

$$p_3(f;x) = 5-2(x-0)+1(x-0)(x-1)+\frac{1}{4}(x-0)(x-1)(x-3)$$

2.

$$p_3'(x) = -2+1(2x-1)+\frac{1}{4}(3x^2-8x+3)$$

Set $p_3'(x) = 0$ to find the critical points:

$$-2+1(2x-1)+\frac{1}{4}(3x^2-8x+3) = 0$$

$$-2+2x-1+\frac{3}{4}x^2-2x+\frac{3}{4} = 0$$

$$\frac{3}{4}x^2-2+\frac{3}{4} = 0$$

$$\frac{3}{4}x^2-\frac{5}{4} = 0$$

$$3x^2 = 5 \Rightarrow x^2 = \frac{5}{3} \Rightarrow x = \pm\sqrt{\frac{5}{3}} = \pm\frac{\sqrt{15}}{3}$$

Thus, the solutions are $x = \frac{\sqrt{15}}{3}$ and $x = -\frac{\sqrt{15}}{3}$.

Problem 5

Given $f(x) = x^7$, we first calculate the fifth divided difference $f[0, 1, 1, 1, 2, 2]$. For $f(x) = x^7$, its fifth derivative is:

$$f^{(5)}(x) = 7 \times 6 \times 5 \times 4 \times 3 \times x^2 = 252x^2$$

We know that the divided difference can be expressed in terms of the fifth derivative:

$$f[0, 1, 1, 1, 2, 2] = \frac{1}{5!} f^{(5)}(\xi)$$

Where $\xi \in (0, 2)$, and since $f^{(5)}(x) = 252x^2$, we have:

$$f[0, 1, 1, 1, 2, 2] = \frac{1}{120} \times 252\xi^2 = \frac{21}{10}\xi^2$$

To determine the value of ξ:

$$C = \frac{21}{10}\xi^2$$

Solving for ξ:

$$\xi^2 = \frac{10C}{21} \quad \Rightarrow \quad \xi = \sqrt{\frac{10C}{21}}$$

This is the expression for ξ in terms of the known value C.

Problem 6

Given $f(0) = 1$, $f(1) = 2$, $f'(1) = -1$, $f(3) = 0$, and $f'(3) = 0$,

The Hermite interpolation polynomial is:

$$H(x) = f(1)h_0(x) + f'(1)h_1(x) + f(3)h_2(x) + f'(3)h_3(x) = 2 \cdot h_0(x) - 1 \cdot h_1(x) + 0 \cdot h_2(x) + 0 \cdot h_3(x) = 2 \cdot h_0(x) - h_1(x)$$

Next, we calculate $h_0(x)$ and $h_1(x)$:

$$h_0(x) = (1 - 2(x - x_0) \cdot l_0'(x)) \cdot l_0(x)^2$$

$$h_1(x) = (x - x_0) \cdot l_0(x)^2$$

Where

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Given $x_0 = 1$, and $x_1 = 3$, we have:

$$l_0(x) = \frac{x - 3}{1 - 3} = \frac{x - 3}{-2}, \quad l_1(x) = \frac{x - 1}{3 - 1} = \frac{x - 1}{2}$$

$$l_0(2) = \frac{2 - 3}{-2} = \frac{1}{2}, \quad l_0'(x) = \frac{1}{-2}$$

$$l_1(2) = \frac{2-1}{2} = \frac{1}{2}, \quad l_1'(x) = \frac{1}{2}$$

then

$$h_0(2) = (1 - 2(2-1) \cdot \frac{1}{-2}) \cdot \left(\frac{1}{2}\right)^2 = (1+1) \cdot \frac{1}{4} = \frac{1}{2}$$

$$h_1(2) = (2-1) \cdot \left(\frac{1}{2}\right)^2 = 1 \cdot \frac{1}{4} = \frac{1}{4}$$

Thus, the value of the interpolation polynomial at $x = 2$ is:

$$H(2) = 2 \cdot \frac{1}{2} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore,

$$f(2) \approx \frac{3}{4}$$

The error in Hermite interpolation can be expressed as:

$$R(x) = \frac{f^{(5)}(\xi)}{5!}(x - x_0)^2(x - x_1)^2$$

Where $\xi \in (x_0, x_1)$. Since we know $|f^{(5)}(x)| \leq M$ holds over the interval $[0, 3]$, the maximum error is:

$$|R(2)| \leq \frac{M}{5!}(2-0)^2(2-3)^2 = \frac{M}{120} \times 4 \times 1 = \frac{4M}{120} = \frac{M}{30}$$

The maximum possible error is $\frac{M}{30}$.

Problem 7

When $k = 1$, the forward difference is:

$$\Delta^1 f(x) = f(x+h) - f(x) = hf[x_0, x_1]$$

Assume that for $k = n$, the forward difference can be expressed as:

$$\Delta^n f(x) = n! h^n f[x_0, x_1, \ldots, x_n]$$

Now, we prove that the formula holds for $k = n + 1$:

$$\Delta^{n+1} f(x) = \Delta^n f(x+h) - \Delta^n f(x)$$

Using the inductive assumption:

$$\Delta^{n+1} f(x) = n! h^n f[x_1, x_2, \ldots, x_{n+1}] - n! h^n f[x_0, x_1, \ldots, x_n]$$

By applying the definition of divided differences, we get:

$$\Delta^{n+1} f(x) = (n+1)! h^{n+1} f[x_0, x_1, \ldots, x_{n+1}]$$

Thus, the recursive relationship for forward differences is established.

2. Proof of Backward Difference When $k = 1$, the backward difference is:

$$\nabla^1 f(x) = f(x) - f(x - h)$$

Using the divided difference:

$$\nabla^1 f(x) = hf[x_0, x_{-1}]$$

Assume that for $k = n$, the backward difference can be expressed as:

$$\nabla^n f(x) = n!h^n f[x_0, x_{-1}, \dots, x_{-n}]$$

Now, we prove that the formula holds for $k = n + 1$:

$$\nabla^{n+1} f(x) = \nabla^n f(x) - \nabla^n f(x - h)$$

Using the inductive assumption:

$$\nabla^{n+1} f(x) = n!h^n f[x_{-1}, x_{-2}, \dots, x_{-(n+1)}] - n!h^n f[x_0, x_{-1}, \dots, x_{-n}]$$

By applying the definition of divided differences, we get:

$$\nabla^{n+1} f(x) = (n + 1)!h^{n+1} f[x_0, x_{-1}, \dots, x_{-(n+1)}]$$

Thus, the recursive relationship for backward differences is also established.

Problem 8

$$f[x_0, x_1, \dots, x_n] = \frac{f(x_1, \dots, x_n) - f(x_0)}{x_1 - x_0}$$

Now, we differentiate with respect to x_0:

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] = \lim_{h \to 0} \frac{f[x_0 + h, x_1, \dots, x_n] - f[x_0, x_1, \dots, x_n]}{h}$$

Since f is differentiable at x_0, we can reuse x_0 in the divided difference expression:

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] = f[x_0, x_0, x_1, \dots, x_n]$$

Thus, the result is proven.

Problem 9

To minimize the maximum of the given polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ on the interval $[a, b]$, we solve:

$$\max_{x \in [a,b]} |P(x)|$$

Then, we choose appropriate coefficients $a_1, a_2, \ldots, a_n$ to minimize this maximum.

The Chebyshev polynomial $T_n(x)$ is the optimal polynomial approximation on the interval $[-1, 1]$. It minimizes the maximum absolute value of the oscillation on $[-1, 1]$.

$$x' = \frac{2x - (b + a)}{b - a}$$

This maps the interval $[a, b]$ to $[-1, 1]$. Therefore, the optimal solution is in the form of a polynomial that is a linear combination of Chebyshev polynomials:

$$P(x) = cT_n\left(\frac{2x - (b + a)}{b - a}\right)$$

where c is a constant, and $T_n(x)$ is the Chebyshev polynomial of degree n.

Problem 10

The polynomial $\hat{p}_n(x)$ is defined as:

$$\hat{p}_n(x) = \frac{T_n(x)}{T_n(a)}$$

where $T_n(x)$ is the Chebyshev polynomial of degree n. Since $T_n(a)$ is the value of the Chebyshev polynomial at a, the polynomial $\hat{p}_n(x)$ satisfies the normalization condition:

$$\hat{p}_n(a) = \frac{T_n(a)}{T_n(a)} = 1$$

This ensures that $\hat{p}_n(x) \in P_n^a$. The Chebyshev polynomial $T_n(x)$ attains its absolute maximum on the interval $[-1, 1]$ at several points where the value is ± 1. Therefore, we have:

$$\|\hat{p}_n(x)\|_\infty = \max_{x \in [-1,1]} \left| \frac{T_n(x)}{T_n(a)} \right| = \frac{1}{|T_n(a)|}$$

Since $T_n(x)$ minimizes the maximum oscillation on the interval $[-1, 1]$, and $\hat{p}_n(x)$ is the normalized form of $T_n(x)$, for any $p \in P_n^a$, we have:

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty$$

This holds because $\hat{p}_n(x)$ already minimizes the maximum oscillation on the interval $[-1, 1]$.

Problem 11

The Bernstein basis polynomial is given by:

$$\begin{pmatrix} b_{n,k}(t) = n \\ kt^k(1-t)^{n-k}. \end{pmatrix}$$

According to the definition:

$$\begin{pmatrix} \frac{n-k}{n} b_{n,k}(t) = \frac{n-k}{n} n \\ kt^k(1-t)^{n-k}, \end{pmatrix}$$

$$\begin{pmatrix} \frac{k+1}{n} b_{n,k+1}(t) = \frac{k+1}{n} n \\ k+1 t^{k+1}(1-t)^{n-k-1}. \end{pmatrix}$$

For the first term:

$$\begin{pmatrix} \frac{n-k}{n} n \\ kt^k(1-t)^{n-k}. \end{pmatrix}$$

We use the binomial coefficient:

$$\begin{pmatrix} n \\ k = \frac{n!}{k!(n-k)!} \end{pmatrix}$$

Thus:

$$\begin{pmatrix} \frac{n-k}{n} n \\ k = \frac{n-k}{n} \cdot \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-k-1)!}. \end{pmatrix}$$

So:

$$\begin{pmatrix} \frac{n-k}{n} b_{n,k}(t) = n-1 \\ kt^k(1-t)^{n-k-1} \end{pmatrix}$$

For the second term:

$$\begin{pmatrix} \frac{k+1}{n} n \\ k+1 t^{k+1}(1-t)^{n-k-1}. \end{pmatrix}$$

Again, using the binomial coefficient:

$$\begin{pmatrix} n \\ k+1 = \frac{n!}{(k+1)!(n-k-1)!}. \end{pmatrix}$$

Thus:

$$\begin{pmatrix} \frac{k+1}{n} n \\ k+1 = \frac{k+1}{n} \cdot \frac{n!}{(k+1)!(n-k-1)!} = \frac{(n-1)!}{(k+1)!(n-k-1)!}. \end{pmatrix}$$

So, the second term simplifies to:

$$\begin{pmatrix} \frac{k+1}{n} b_{n,k+1}(t) = n-1 \\ k+1 t^{k+1}(1-t)^{n-k-1}. \end{pmatrix}$$

We obtain:

$$\begin{pmatrix} b_{n-1,k}(t) = n-1 \\ kt^k(1-t)^{n-k-1} + n-1 k+1 t^{k+1}(1-t)^{n-k-1} \end{pmatrix}$$

Thus, the proof is complete.

Problem 12

$$\binom{\int_0^1 b_{n,k}(t)\,dt = n}{k\int_0^1 t^k(1-t)^{n-k}\,dt.}$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

For our case, we set $x = k + 1$ and $y = n - k + 1$, so we have:

$$\int_0^1 t^k(1-t)^{n-k}\,dt = B(k+1, n-k+1).$$

The Beta function is related to the Gamma function through the following relationship

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$\Gamma(n+1) = n!$

$$B(k+1, n-k+1) = \frac{k!(n-k)!}{(n+1)!}.$$

$$\binom{\int_0^1 b_{n,k}(t)\,dt = n}{k \cdot \frac{k!(n-k)!}{(n+1)!}.}$$

Since $\binom{n}{k=\frac{n!}{k!(n-k)!}}$, we can simplify this expression to:

$$\int_0^1 b_{n,k}(t)\,dt = \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}.$$

We have proven that for all $k = 0, 1, \ldots, n$, the following holds:

$$\int_0^1 b_{n,k}(t)\,dt = \frac{1}{n+1}.$$

This shows that the integral of the Bernstein basis polynomial depends only on its degree n.