Homework 3

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1 Problem 1

$p(x)$ is a cubic polynomial, let:

$$p(x) = ax^3 + bx^2 + cx + d.$$

Since $s(0) = 0$, we have:

$$p(0) = a(0)^3 + b(0)^2 + c(0) + d = d = 0.$$

Thus:

$$p(x) = ax^3 + bx^2 + cx.$$

To ensure that $s(x)$ remains continuous and that its first and second derivatives are continuous at $x = 1$, we impose the following conditions: - $p(1) = (2-1)^3 = 1$, - $p'(1) = \frac{d}{dx}(2-x)^3\Big|_{x=1} = -3$, - $p''(1) = \frac{d^2}{dx^2}(2-x)^3\Big|_{x=1} = 6$.

This translates into the equations:

$$a(1)^3 + b(1)^2 + c(1) = a + b + c = 1,$$

$$3a(1)^2 + 2b(1) + c = 3a + 2b + c = -3,$$

$$6a(1) + 2b = 6a + 2b = 6 \Rightarrow 3a + b = 3.$$

$$\begin{cases} a + b + c = 1, \\ 3a + 2b + c = -3, \\ 3a + b = 3. \end{cases}$$

So we can get

$$a = 7, \quad b = -18, \quad c = 12.$$

Thus, we have:

$$p(x) = 7x^3 - 18x^2 + 12x.$$

Calculating the second derivative:

$$p''(x) = 42x - 36.$$

- $p''(0) = 42(0) - 36 = -36 \neq 0$, - $p''(2) = 42(2) - 36 = 84 - 36 = 48 \neq 0$.

Therefore, $s(x)$ is not a natural cubic spline.

2 Problem 2

2.1 Problem A

$$p_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i,$$

where there are $3(n-1)$ unknown coefficients a_i, b_i, c_i for $n-1$ subintervals.

At each interior point x_i, the spline must be continuous:

$$p_i(x_i) = p_{i+1}(x_i) \quad \text{for all } i = 2, 3, \ldots, n-1.$$

This provides $n-2$ equations.

At each interior point x_i, the first derivative of the spline must also be continuous:

$$p_i'(x_i) = p_{i+1}'(x_i) \quad \text{for all } i = 2, 3, \ldots, n-1.$$

This provides another $n-2$ equations.

- Total number of unknowns: $3(n-1)$. - Total number of equations: $2(n-2)$.

Since there are fewer equations than unknowns, an additional condition is required.

2.2 Problem B

$$p_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i.$$

Given:

$$p_i(x_i) = f_i, \quad p_i(x_{i+1}) = f_{i+1}, \quad p_i'(x_i) = m_i.$$

From $p_i(x_i) = a_i(x_i - x_i)^2 + b_i(x_i - x_i) + c_i = c_i = f_i$, we get:

$$c_i = f_i.$$

The first derivative:

$$p_i'(x) = 2a_i(x - x_i) + b_i.$$

At x_{i+1}:

$$p_i'(x_{i+1}) = 2a_i(x_{i+1} - x_i) + b_i = m_i.$$

Using:

$$a_i(x_{i+1} - x_i)^2 + m_i(x_{i+1} - x_i) + f_i = f_{i+1},$$

we find:

$$a_i = \frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}.$$

Thus:

$$p_i(x) = \frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)^2 + m_i(x - x_i) + f_i.$$

2.3 Problem C

Since:

$$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}),$$

we have a system of equations involving $m_2, m_3, \ldots, m_{n-1}$. Combining these equations with the known initial derivative m_1, we can solve for all m_i. This allows us to uniquely determine the quadratic spline s.

3 Problem 3

Given:

$$s_1(x) = 1 + c(x+1)^3,$$
$$s_1'(x) = 3c(x+1)^2,$$
$$s_1''(x) = 6c(x+1).$$

To satisfy $s_1''(-1) = 0$:

$$s_1''(-1) = 6c(-1+1) = 0 \implies c = 0.$$

Let $s_2(x)$ be a cubic polynomial:

$$s_2(x) = ax^3 + bx^2 + dx + e.$$

$s_1(0) = s_2(0)$:

$$s_1(0) = 1 + c(0+1)^3 = 1 + c \implies e = 1 + c = s_2(0).$$

$s_1'(0) = s_2'(0)$:

$$s_2'(x) = 3ax^2 + 2bx + d,$$
$$s_1'(0) = 3c(0+1)^2 = 3c \implies d = 3c = s_2'(0).$$

$s_1''(0) = s_2''(0)$:

$$s_2''(x) = 6ax + 2b,$$
$$s_1''(0) = 6c(0+1) = 6c \implies 2b = 6c \implies b = 3c.$$

$$s_2''(1) = 6a(1) + 2b = 6a + 2(3c) = 6a + 6c = 0 \implies a = -c.$$

Using the condition $s_2(1) = -1$:

$$s_2(1) = a(1)^3 + b(1)^2 + d(1) + e = -c + 3c + 3c + (1+c) = 6c + 1,$$

$$6c = -2 \implies c = -\frac{1}{3}.$$

The constant $c = -\frac{1}{3}$, and:

$$s_2(x) = -\frac{1}{3}x^3 + x^2 - x + \frac{2}{3}.$$

4 Problem 4

4.1 Problem a

We aim to find the natural cubic spline interpolation for the function $f(x) = \cos\left(\frac{\pi}{2}x\right)$ on the interval $[-1, 1]$:

- $f(-1) = \cos\left(-\frac{\pi}{2}\right) = 0,$
- $f(0) = \cos(0) = 1,$
- $f(1) = \cos\left(\frac{\pi}{2}\right) = 0.$

Define $s_1(x)$ on $[-1, 0]$ as:

$$s_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1.$$

Define $s_2(x)$ on $[0, 1]$ as:

$$s_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2.$$

$s_1''(-1) = 0$:

$s_1''(x) = 6a_1x+2b_1 \implies s_1''(-1) = 6a_1(-1)+2b_1 = -6a_1+2b_1 = 0 \implies b_1 = 3a_1.$

$s_2''(1) = 0$:

$s_2''(x) = 6a_2x + 2b_2 \implies s_2''(1) = 6a_2(1) + 2b_2 = 6a_2 + 2b_2 = 0 \implies b_2 = -3a_2.$

$s_1(0) = s_2(0) = 1$:

$$s_1(0) = a_1 \cdot 0^3 + b_1 \cdot 0^2 + c_1 \cdot 0 + d_1 = d_1 = 1,$$

$$s_2(0) = a_2 \cdot 0^3 + b_2 \cdot 0^2 + c_2 \cdot 0 + d_2 = d_2 = 1.$$

$s_1'(0) = s_2'(0)$:

$$s_1'(x) = 3a_1x^2 + 2b_1x + c_1 \implies s_1'(0) = 3a_1 \cdot 0^2 + 2b_1 \cdot 0 + c_1 = c_1,$$

$$s_2'(x) = 3a_2x^2 + 2b_2x + c_2 \implies s_2'(0) = 3a_2 \cdot 0^2 + 2b_2 \cdot 0 + c_2 = c_2.$$

Therefore, $c_1 = c_2$.
$s_1''(0) = s_2''(0)$:

$$s_1''(0) = 2b_1, \quad s_2''(0) = 2b_2 \implies b_1 = b_2.$$

Using these conditions, we obtain:

$$s_1(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2 + 1,$$

$$s_2(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

4.2 Problem b

1. $g(-1) = 0$:

$$a(-1)^2 + b(-1) + c = 0 \implies a - b + c = 0.$$

2. $g(0) = 1$:

$$a(0)^2 + b(0) + c = 1 \implies c = 1.$$

3. $g(1) = 0$:

$$a(1)^2 + b(1) + c = 0 \implies a + b + c = 0.$$

Substituting $c = 1$ into the first and third equations:

$$a - b + 1 = 0 \implies a - b = -1,$$

$$a + b + 1 = 0 \implies a + b = -1.$$

Solving this system:

$$2a = -2 \implies a = -1,$$

$$2b = 0 \implies b = 0.$$

Thus:

$$g(x) = -x^2 + 1.$$

$$\int_{-1}^{1} (g''(x))^2 \, dx = \frac{8}{3}.$$

Calculate $s''(x)$ and the bending energy:

$$\int_{-1}^{1} (s''(x))^2 \, dx = \int_{-1}^{0} (-3x - 3)^2 \, dx + \int_{0}^{1} (3x - 3)^2 \, dx > 6.$$

Verifying that the bending energy of the natural cubic spline is smaller, confirming that natural cubic splines minimize the total bending energy.

5 Problem 5

5.1 Problem A

The first-degree B-spline $B_i^1(x)$ is defined as:

$$B_i^1(x) = \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}} & \text{if } x \in [t_{i-1}, t_i], \\ \frac{t_{i+1} - x}{t_{i+1} - t_i} & \text{if } x \in [t_i, t_{i+1}], \\ 0 & \text{otherwise.} \end{cases}$$

The next B-spline $B^1_{i+1}(x)$ is given by:

$$B^1_{i+1}(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i} & \text{if } x \in [t_i, t_{i+1}], \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & \text{if } x \in [t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise.} \end{cases}$$

Using the recursive formula, the second-degree B-spline $B^2_i(x)$ is:

$$B^2_i(x) = \frac{x - t_i}{t_{i+2} - t_i} B^1_i(x) + \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} B^1_{i+1}(x),$$

which gives:

$$B^2_i(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & \text{if } x \in [t_{i-1}, t_i], \\ \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & \text{if } x \in [t_i, t_{i+1}], \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & \text{if } x \in [t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise.} \end{cases}$$

5.2 Problem B

We compute $\frac{d}{dx} B^2_i(x)$ piecewise for the intervals $[t_{i-1}, t_i], [t_i, t_{i+1}], [t_{i+1}, t_{i+2}]$:

1. At $x = t_i$:

$$p_1(t_i) = \frac{2(t_i - t_{i-1})}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = \frac{2}{t_{i+1} - t_{i-1}} = p_2(t_i).$$

2. At $x = t_{i+1}$, ensure $p_2(t_{i+1}) = p_3(t_{i+1})$:

$$p_3(t_{i+1}) = \frac{2(t_{i+2} - t_{i+1})}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} = \frac{-2}{t_{i+2} - t_i} = p_2(t_{i+1}).$$

Hence, $\frac{d}{dx} B^2_i(x)$ is continuous at $x = t_i$ and $x = t_{i+1}$.

5.3 Problem C

The derivative $\frac{d}{dx} B^2_i(x)$ is positive on (t_{i-1}, t_i) and negative on (t_i, t_{i+1}), indicating a unique extremum point x^* within (t_i, t_{i+1}).

We solve for x^* such that:

$$\frac{t_{i+1} + t_{i-1} - 2x^*}{t_{i+1} - t_{i-1}} + \frac{t_{i+2} + t_i - 2x^*}{t_{i+2} - t_i} = 0.$$

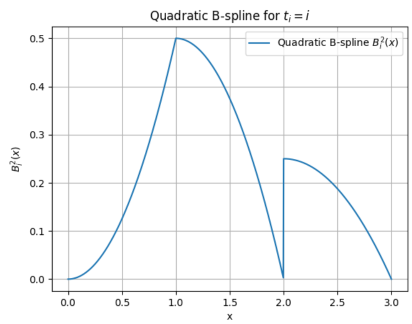
Solving for x^*:

$$x^* = \frac{t_i t_{i+2} + t_i t_{i+1} - t_{i-1} t_i}{(t_{i+2} + t_{i+1}) - (t_i + t_{i-1})}.$$

5.4 Problem D

The function $B_i^2(x)$ is zero at t_{i-1} and t_{i+2}, and it achieves a maximum value between these points. The peak value at x^* is less than 1, ensuring $B_i^2(x) \in [0, 1]$.

5.5 Problem E



6 Problem 6

Given:

$$[t_{i-1}, t_i, t_{i+1}, t_{i+2}]f = \frac{[t_i, t_{i+1}, t_{i+2}]f - [t_{i-1}, t_i, t_{i+1}]f}{t_{i+2} - t_{i-1}},$$

for $f(x) = (t - x)_+^2$.

We have:

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x),$$

where $B_i^1(x)$ and $B_{i+1}^1(x)$ are the first-degree B-splines. The explicit expression for a first-degree B-spline is:

$$B_i^1(x) = \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}}, & \text{if } t_{i-1} \le x < t_i, \\ \frac{t_{i+1} - x}{t_{i+1} - t_i}, & \text{if } t_i \le x < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Using the recursive definition, we have:

$$\begin{aligned} B_i^2(x) &= (x - t_{i-1})[t_{i-1}, t_i, t_{i+1}](t - x)_+^1 + (t_{i+2} - x)[t_i, t_{i+1}, t_{i+2}](t - x)_+^1 \\ &= -[t_{i-1}, t_i, t_{i+1}](t - x)_+^2 + [t_i, t_{i+1}, t_{i+2}](t - x)_+^2 \\ &= (t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t - x)_+^2. \end{aligned}$$

Thus, we obtain:

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x).$$

Q.E.D.

7 Problem 7

Given:

$$\frac{d}{dx} B_i^{n+1}(x) = (n+1)\left(\frac{B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i}\right), \quad n \geq 1.$$

Integrate both sides over the support interval $[t_{i-1}, t_{i+n+1}]$:

$$\int_{t_{i-1}}^{t_{i+n+1}} \frac{d}{dx} B_i^{n+1}(x)\, dx = \int_{t_{i-1}}^{t_{i+n+1}} \left(\frac{(n+1)B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1} - t_i}\right) dx.$$

This simplifies to:

$$B_i^{n+1}(t_{i+n+1}) - B_i^{n+1}(t_{i-1}) = (n+1)\left(\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n} - t_{i-1}}\, dx - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i}\, dx\right).$$

Simplifying further:

$$0 = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1} - t_i}\, dx.$$

Hence, we conclude that the scaled integral of the B-spline $B_i^n(x)$ over its support interval is independent of the index i.

8 Problem 8

8.1 Problem A

Let:

$$e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

The divided difference is defined as:

$$[x_i, x_j]f = \frac{f(x_i) - f(x_j)}{x_i - x_j}.$$

Here, $f(x) = x$, and we express the following using binomial terms:

- $[x_1, x_2]$: Represents the product x_1x_2,

- $[x_1, x_3]$: Represents the product x_1x_3,
- $[x_1, x_4]$: Represents the product x_1x_4,
- $[x_2, x_3]$: Represents the product x_2x_3,
- $[x_2, x_4]$: Represents the product x_2x_4,
- $[x_3, x_4]$: Represents the product x_3x_4.

Thus:

$$e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

By computing the above divided differences and adding these terms, we observe that all pairwise products appear in the complete symmetric polynomial. Hence, the theorem holds for $m = 4$ and $n = 2$.

8.2 Problem B

$$(x_{i+n+1} - x_i)\sigma_{m-n-1}(x_i, \ldots, x_{i+n+1})$$
$$= \sigma_{m-n}(x_i, \ldots, x_{i+n+1}) - \sigma_{m-n}(x_i, \ldots, x_{i+n}) - x_i\sigma_{m-n-1}(x_i, \ldots, x_{i+n+1}).$$

This simplifies to:

$$= \sigma_{m-n}(x_{i+1}, \ldots, x_{i+n+1}) - \sigma_{m-n}(x_i, \ldots, x_{i+n}).$$

For $n = 0$:

$$\sigma_m(x_i) = [x_i]x_i^m = x_i^m.$$

Assume it holds for $k = n$, and consider the case for $k = n + 1$:

$$\sigma_{m-n-1}(x_i, \ldots, x_{i+n+1}) = \frac{\sigma_{m-n}(x_i, \ldots, x_{i+n+1}) - \sigma_{m-n}(x_i, \ldots, x_{i+n})}{x_{i+n+1} - x_i}.$$

This can be rewritten as:

$$= \frac{[x_{i+1}, \ldots, x_{i+n+1}]x^m - [x_i, \ldots, x_{i+n}]x^m}{x_{i+n+1} - x_i} = [x_i, x_{i+1}, \ldots, x_{i+n+1}]x^m.$$

Thus, the proof is complete.