Lecture 1
Matrices and
Linear Algebra

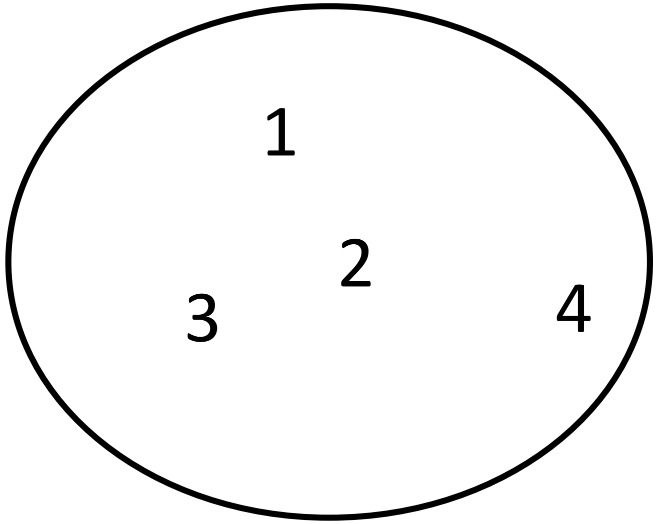
ref. Contemporary Linear Algebra, H Anton, RC Busby

1. Vectors and Matrices

1.1 What is a vector?

Matrix

- What is a matrix?



15	10	10	18	14
8		23	19	19
18	16	30	21	
13	19	29	20	25

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<tr><td>15</td><td>10</td><td>10</td><td>18</td><td>14</td></tr>
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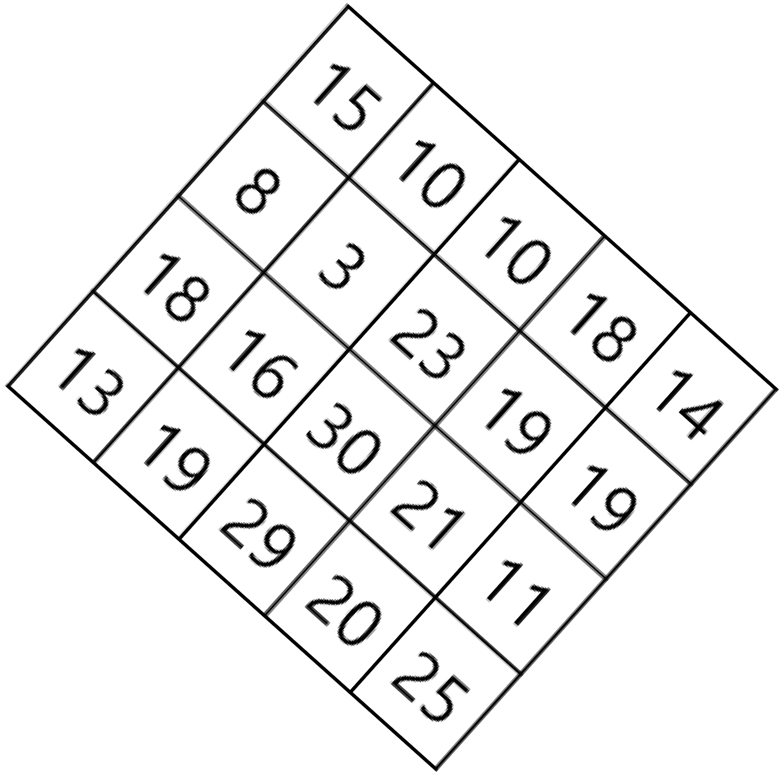
Matrix

- What is a matrix?

15	10	10	18	14
8	3	23	19	19
18	16	30	21	11
13	19	29	20	25



Vector

- What is a vector?
  - Something with magnitude and direction?
  - A tuple of numbers?
  - An arrow?

Vector

- Operations between vectors
  - Addition
  - Scalar multiplication

- What is a *linear* operation?

Vector space

Axiom	Statement
Associativity of vector addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of vector addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of vector addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of vector addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 3]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

https://en.wikipedia.org/wiki/Vector_space

Examples of vector spaces

- $\mathbb{R}^n$
  - Row vector / column vector
- A set of all functions from $\mathbb{R}$ to $\mathbb{R}$
- A set of all polynomials
- A set of all $m \times n$ matrices

What we can do with vectors

- Dot product and norm
  - *Length* of a vector
  - *Angle* between two vectors
  - *Orthogonality*

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norm
vecnorm
dot
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- Triangular inequality for vectors

$$\|u+v\| \le \|u\| + \|v\|$$

- Outer product

1.2 Linear systems and matrices

Matrix

- Five perspectives on viewing matrices
  - Collection of data (numeric)
  - Collection of vectors (geometric)
  - A system of linear equations (algebraic)
  - A linear operator (operational)
  - A tangent space of a function (differential)

Matrix is a collection of data

https://pixabay.com/

Matrix is a collection of vectors

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 1 & 9 & 8 & 9 \\ 7 & 4 & 2 & 7 \end{bmatrix} \xrightarrow[\text{vectors}]{\text{column}} \underset{(v_1)}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}}, \underset{(v_2)}{\begin{bmatrix} 4 \\ 9 \\ 4 \end{bmatrix}}, \underset{(v_3)}{\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}}, \underset{(v_4)}{\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & 4 & 5 & 6 \\ 1 & 9 & 8 & 9 \\ 7 & 4 & 2 & 7 \end{bmatrix} \xrightarrow[\text{vectors}]{\text{Row}} \begin{array}{l} \begin{bmatrix} 3 & 4 & 5 & 6 \end{bmatrix} \ (v_1) \\ \begin{bmatrix} 1 & 9 & 8 & 9 \end{bmatrix} \ (v_2) \\ \begin{bmatrix} 7 & 4 & 2 & 7 \end{bmatrix} \ (v_3) \end{array}$$

https://towardsdatascience.com/a-birds-eye-view-of-linear-algebra-the-basics-29ad2122d98f

Matrix represents a linear system

$$
\begin{aligned}
2x + 1y + 3z &= 9 \\
1x + 3y + 4z &= 12 \\
3x + 0y + 1z &= 5
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 5 \end{bmatrix}
$$

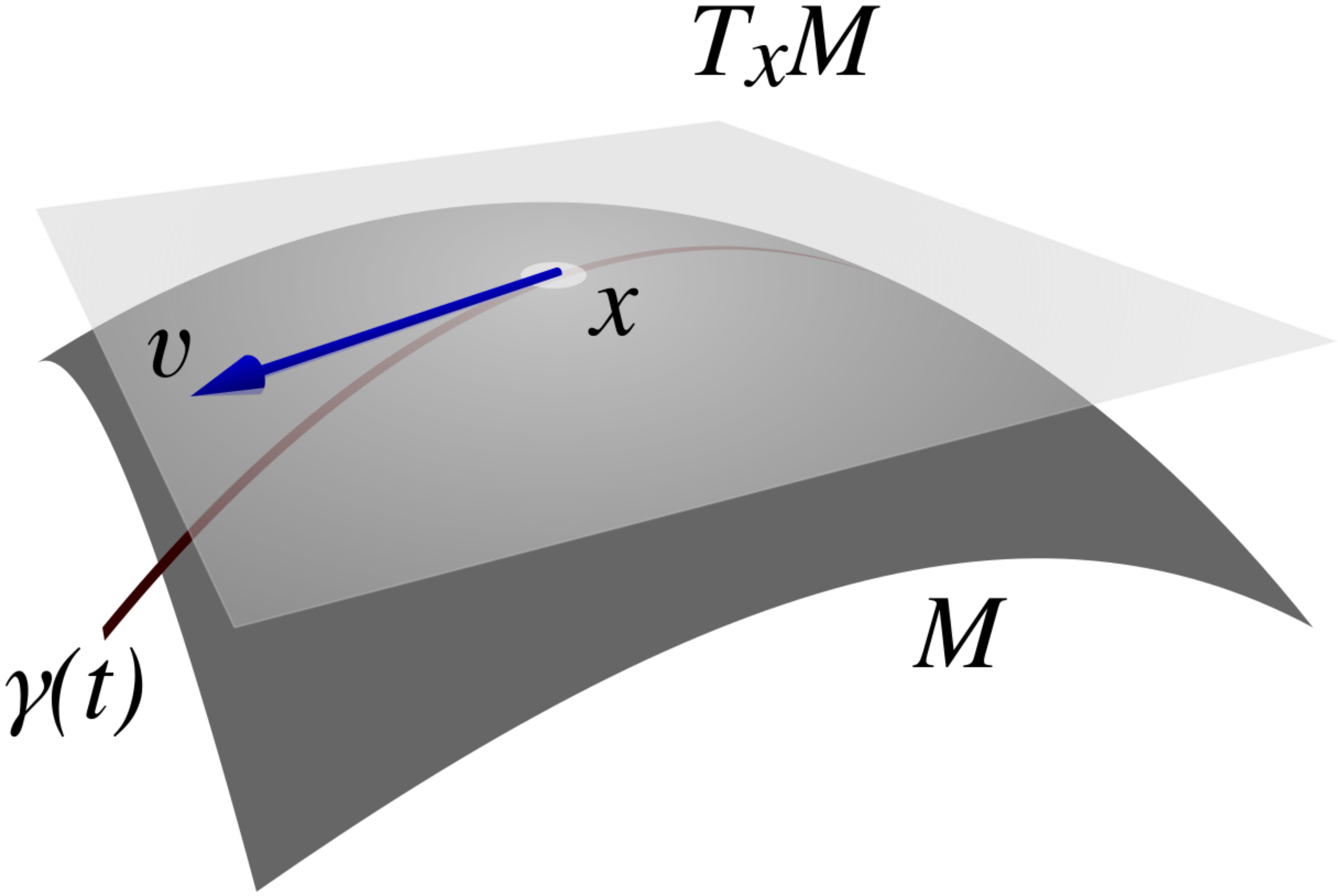
Matrix is a linear operator

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Q. Is the matrix multiplication a *linear* operator?

Matrix is a tangent space of a function



https://en.wikipedia.org/wiki/Tangent_space

Matrix

- Five perspectives on viewing matrices
  - Collection of data (numeric)
  - Collection of vectors (geometric)
  - A system of linear equations (algebraic)
  - A linear operator (operational)
  - A tangent space of a function (differential)

A system of linear equations

- System?
- Equation?
- Linear?

$$2x + 1y + 3z = 9$$
$$1x + 3y + 4z = 12$$
$$3x + 0y + 1z = 5$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 5 \end{bmatrix}$$

$$Ax = b$$

A linear system and a matrix

- Matrix multiplication is…
  - Associative?
  - Distributive?
  - Commutative?

$$
\begin{aligned}
2x + 1y + 3z &= 9 \qquad & 2u + 1v + 3w &= 7 \\
1x + 3y + 4z &= 12 \qquad & 1u + 3v + 4w &= 16 \\
3x + 0y + 1z &= 5 \qquad & 3u + 0v + 1w &= -1
\end{aligned}
$$

$$
\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}
=
\begin{bmatrix} 9 & 7 \\ 12 & 16 \\ 5 & -1 \end{bmatrix}
$$

Matrix multiplication is…

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 5 \end{bmatrix}$$

- A linear combination of column vectors
    - What is the column space of a matrix?
- Calculation of dot product between row and column vectors
    - What is the geometric meaning of the solution of a linear system?

Matrix multiplication is⋯

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 5 \end{bmatrix}$$

- A linear transformation: an $m \times n$ matrix $\cong$ a function from $\mathbb{R}^n$ to $\mathbb{R}^m$

- Quiz. Why is the matrix multiplication not commutative?

Manipulating matrices via Matlab

- Creation
- Operation
- Functions
- Indexing
- Editing

Solutions of a linear system

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rref
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- Elementary row operations and elemententrary matrices
- Gaussian elimination and row echelon form
- Gauss-Jordan elimination and reduced echelon form (rref)

$$\begin{matrix} 1 & 0 & 4 & 2 \\ 1 & 2 & 6 & 2 \\ 2 & 0 & 8 & 8 \\ 2 & 1 & 9 & 4 \end{matrix}$$

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Gaussian_elimination

Existence of solutions of a linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q. A linear system is consistent

if and only if…?

Particular solution and general solution

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1\\
5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 5\\
2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

$$Ax = b$$

Q. *Every* homogeneous linear system has a common solution. What is it?

$$Ax = 0$$

Q. How to get the *general* solution from a *particular* solution?

Particular solution and general solution

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}- 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 5 \\
2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

`mldivide`

$$Ax = b$$

$$Ax = 0$$

Q. How many solutions can a linear system have?

Q. A linear system cannot have only two distinct solutions. Why?

Q. How do computers solve a linear system?

Linear *Algebra*

- *Properties* of real numbers
  - Existence of identity element for addition and multiplication
  - Existence of inverse element for addition and multiplication
  - Associativity and commutativity for addition and multiplication
  - Distributivity

- What is an *operation*?
- What properties should an operation have?
- Do matrices satisfy the properties above?

Inverse of a matrix

- [Def] Let A be an $n \times n$ matrix. If there exists a matrix B such that

$$AB = BA = I_n$$

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inv
mldivide
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- then A is said to be invertible (or nonsingular).
- B is called an inverse of A.

- Quiz. Does $AB = I_n$ imply $BA = I_n$?
- Quiz. Can a non-square matrix have an inverse?
- Quiz. Is the inverse of a matrix unique?

Properties of inverse of a matrix

- If A and B are invertible with the same size, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A^n)^{-1} = (A^{-1})^n$
- $(kA)^{-1} = k^{-1}A^{-1}$ $(k \neq 0)$
- $(A^T)^{-1} = (A^{-1})^T$

Calculation of the inverse

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

- ERO (Elementary Row Operations)
- Why do EROs work?
- If a matrix is not invertible, how does it fail?
- How is the inverse actually calculated?

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lu
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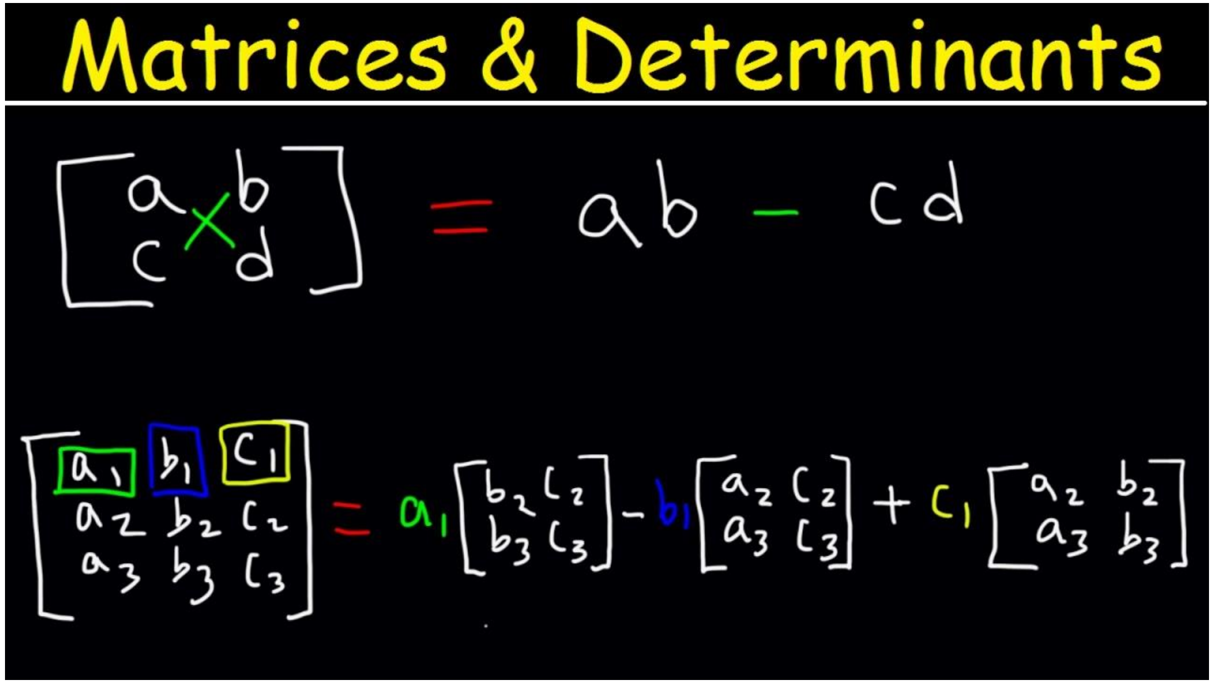
Condition number

cond

```
>> A = [ 400, -201
        -800,  401];

>> b = [200, -200]';

>> A\b
ans =
  -100
  -200
>>
```

```
>> A = [ 401, -201
        -800,  401];

>> b = [200, -200]';

>> A\b
ans =
          40000.0000000364
          79800.0000000726
>>
```

Determinant



det

- So what is the determinant?
- Fun fact: Determinant is a *signed* volume.
- Cofactor expansion

https://www.youtube.com/watch?v=3ROzG6n4yMc

Properties of determinant

- For $n \times n$ matrices A and B,
- $\det(AB) = \det(A)\det(B)$ ($\Rightarrow \det(A^{-1}) = 1/\det(A)$)
- $\det(A) = \det(A^T)$
- $\det(kA) = k^n \det(A)$
- Row multiplication by k: the determinant is multiplied by k. ($\det(E) = k$)
- Row addition: does not change the determinant. ($\det(E) = 1$)
- Row switching: change the sign of the determinant. ($\det(E) = -1$)
- Rows or columns are linearly dependent. $\Leftrightarrow \det(A) = 0$

2. Linear Algebra

2.1 Span, Subspace, Basis

Existence of an inverse of a matrix

- For an $n \times n$ matrix A, the followings are equivalent.

  (1) $\mathrm{rref}(A) = I_n$

  (2) A is a multiplication of elementary matrices.

  (3) A is invertible.

  (4) $Ax = 0$ has only a trivial.

  (5) $Ax = b$ is consistent for every vector $b \in \mathbb{R}^n$.

  (6) The solution of $Ax = b$ is unique for every vector $b \in \mathbb{R}^n$.

  (7) Column vectors of A are linearly independent.

  (8) Row vectors of A are linearly independent.

  (9) $\det(A) \neq 0$

Existence of an inverse of a matrix

- For an $n \times n$ matrix A, the followings are equivalent.

  (10) Column vectors of A span $\mathbb{R}^n$.

  (11) Row vectors of A span $\mathbb{R}^n$.

  (12) A set of column vectors of A forms a basis of $\mathbb{R}^n$.

  (13) A set of row vectors of A forms a basis of $\mathbb{R}^n$.

  (14) $\mathrm{rank}(A) = n$

  (15) $\mathrm{nullity}(A) = 0$

Linear independence

- Definition of a linear combination?

- Linearly independent $\leftrightarrow$ Linearly dependent
  - Any subset of linearly independent set is linearly independent.
  - Any superset of linearly dependent set is linearly dependent.

- Quiz. How can we check if two vectors with same size are linearly independent? What about three or more vectors?

Span

- [Thm] Given a vector space V,
  - If a set B spans V, any superset of B spans V.
  - If a set A does not span V, any subset of A does not span V.

- Quiz. What is the minimum size of a set that spans $\mathbb{R}^2$? What about $\mathbb{R}^3$ or $\mathbb{R}^n$?

Orthogonal complement and hyperplane

- A set of vectors (x_1, x_2) that satisfy below is a line in $\mathbb{R}^2$.

$$a_1x_1 + a_2x_2 = b \quad (a_i \neq 0 \text{ for some } n)$$

- The line is perpendicular to the vector (a_1, a_2).

- A set of vectors (x_1, x_2, x_3) that satisfy below is a plane in $\mathbb{R}^3$.

$$a_1x_1 + a_2x_2 + a_3x_3 = b \quad (a_i \neq 0 \text{ for some } n)$$

- The plane is perpendicular to the vector (a_1, a_2, a_3).

Orthogonal complement and hyperplane

- A set of vectors $(x_1, x_2, \ldots, x_n)$ that satisfy the below is called a hyperplane in $\mathbb{R}^n$.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (a_i \neq 0 \text{ for some } n)$$

- The hyperplane is perpendicular to the vector $(a_1, a_2, \ldots, x_n)$.
- If $b = 0$, the hyperplane passes through the origin of $\mathbb{R}^n$.
  - The hyperplane is the solution set of a linear system $\boldsymbol{ax} = \mathbf{0}$.
  - A set of vectors that are perpendicular to a vector $(a_1, a_2, \ldots, x_n)$: $\boldsymbol{a}^{\perp}$

Subspace

- [Def] A nonempty subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it is *closed under scalar multiplication and addition*, denoted by $V \leq \mathbb{R}^n$.

- [Thm] If A is an $m \times n$ matrix, the solution set of $Ax = 0$ is a subspace of $\mathbb{R}^n$.

- Quiz. Every subspace of $\mathbb{R}^n$ has a common element. What is it?
- Quiz. If V and W are subspaces of $\mathbb{R}^n$, then
  - Is $V \cap W$ also a subspace of $\mathbb{R}^n$?
  - Is $V \cup W$ also a subspace of $\mathbb{R}^n$?

Subspace – General solution of a linear system

- Pick a solution of $Ax = b$, say x_0.
- Let W be the solution space of $Ax = 0$.
- Then for every $x \in \mathrm{W}$, $x + x_0$ is also a solution of $Ax = b$.

- [Thm] If $Ax = b$ is consistent, letting W the solution space of $Ax = 0$, the solution space of $Ax = b$ is $x_0 + W$ where x_0 is *any* solution of $Ax = b$.

- Q. Why *any* solution of $Ax = b$?
- Q. Is the solution space of an *inhomogeneous* linear system a subspace?
- Q. What if $Ax = b$ has a unique solution?

Subspace – General solution of a linear system

- [Thm] Given an $m \times n$ matrix, the followings are equivalent.
  - (1) $Ax = 0$ has only the trivial solution.
  - (2) $Ax = b$ has at most one solution for every $b \in \mathbb{R}^n$.

- [Thm] If $Ax = b$ has more unknowns than equations, it is either inconsistent or has infinitely many soultions.
- [Thm] $Ax = b$ is consistent. $\Leftrightarrow b \in \mathrm{col}(A)$.
- [Thm] Given an $m \times n$ matrix A, the solution space of $Ax = 0$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row in A.

Basis

- [Def] A set of vectors in a subspace V of $\mathbb{R}^n$ is ***a*** basis for V if it is linearly independent and spans V.

- Obvious facts
  - Basis is not unique. (Exception?)
  - A basis is not a subspace.
  - B is a basis of $V \leq \mathbb{R}^n \Rightarrow$ Any proper subset of B does not span V.
  - B is a basis of $V \leq \mathbb{R}^n \Rightarrow$ Any proper superset of B spans V.
  - B is a basis of $V \leq \mathbb{R}^n \Rightarrow$ Any proper superset of B is linearly dependent.

Basis – Quizzes

orth

- What is the most *standard* basis of $\mathbb{R}^n$?
- What is the size of a basis of $\mathbb{R}^n$?
- What is the size of a basis of a plane through the origin in $\mathbb{R}^3$?
- What is the size of a basis of a line through the origin in $\mathbb{R}^3$?
- Why do we say a space is 3-D, a plane is 2-D, a line is 1-D?
- Can a set of more than n vectors be a basis of $\mathbb{R}^n$?

Basis - Theorems

- [Thm] Every vector space has a basis.
- [Thm] Any basis of $\mathbb{R}^n$ has exactly n elements. (*Definition* of dimension)
- [Thm] If a is a nonzero vector in $\mathbb{R}^n$, $\dim(a^{\perp}) = n - 1.$
- [Thm] If B is a basis of $V \leq \mathbb{R}^n$, every vector in $\mathbb{R}^n$ is expressed uniquely as a linear combination of vector in B.

2.2 Fundamental spaces, Rank, Nullity

Fundamental spaces of a matrix

- Given an $m \times \ n$ matrix A,
- $\text{row}(A)$: subspace of $\mathbb{R}^n$ spanned by the row vectors of A
- $\text{col}(A)$: subspace of $\mathbb{R}^n$ spanned by the column vectors of A
- $\text{null}(A)$: solution space of $Ax = 0$, which is a subspace of $\mathbb{R}^n$

- Fundamental spaces of a matrix A
  - $\text{row}(A)$
  - $\text{col}(A)$
  - $\text{null}(A)$
  - $\text{null}(A^T)$

Fundamental spaces of a matrix

- Quiz. Given an $m \times n$ matrix A and a vector $b \in \mathbb{R}^n$, how can we check if b is in $\mathrm{row}(A)$?

- Quiz. If we have a basis B of a subspace $V \leq \mathbb{R}^n$ and a vector $b \in \mathrm{V}$, how can we find the linear combination of the vector of B to make b?

Rank, Nullity

- [Def] For a matrix A, the dimension of $\mathrm{row}(A)$ is the rank of A, denoted by $\mathrm{rank}(A)$.
- [Def] For a matrix A, the dimension of the null space of A is the nullity of A, denoted by $\mathrm{nullity}(A)$.

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rank
null
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- [THE Rank Theorem] For a matrix A,

$$\mathrm{rank}(A) = \mathrm{rank}(A^T)$$

- Quiz. For an $m \times n$ matrix A, what is the largest possible value for $\mathrm{rank}(A)$?

Orthogonal complement of a set

- [Def] If S is a nonempty set in $\mathbb{R}^n$, then the orthogonal complement of S, denoted by $S^\perp$, is defined to be the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in S.

- [Thm] If S is a nonempty set in $\mathbb{R}^n$, then $S^\perp$ is a subspace of $\mathbb{R}^n$.
- [Thm] If W is a subspace of $\mathbb{R}^n$, then $W \cap \mathrm{W}^\perp = \{0\}$.
- [Thm] If S is a nonempty subset of $\mathbb{R}^n$, then $S^\perp = \mathrm{span}(S)^\perp$.
- [Thm] If W is a subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp = W$.

Orthogonal complements of fundamental spaces

- [Thm] For a matrix A,
  - $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are orthogonal complements.
  - $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$ are orthogonal complements.

- $\mathrm{row}(A)^\perp = \mathrm{null}(A)$
- $\mathrm{null}(A)^\perp = \mathrm{row}(A)$
- $\mathrm{col}(A)^\perp = \mathrm{null}(A^T)$
- $\mathrm{null}(A^T)^\perp = \mathrm{col}(A)$

ERO and fundamental spaces

- [Thm]
  - Elementary row operations does not change the row space of a matrix.
  - Elementary row operations does not change the null space of a matrix.
  - The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of the matrix.

- Quiz. Do the EROs change the *column space* of a matrix?
- Quiz. Given a matrix A, how can a basis of $\mathrm{row}(A)$ be found?

ERO and fundamental spaces

- [Thm] If A and B are matrices with the same number of columns, then the following statements are equivalent.
  - A and B have the same row space.
  - A and B have the same null space.
  - The row vectors of A are linear combinations of the row vector of B, and conversely.

ERO and fundamental spaces

- Quiz. Given a set of vectors $S \subset \mathbb{R}^m$, find conditions on the numbers $b_1, b_2, \dots, b_m$ under which $b = (b_1, b_2, \dots, b_m)$ will be in $\text{span}(S)$.

- Quiz. Given an $m \times n$ matrix A, find conditions on the numbers $b_1, b_2, \dots, b_m$ under which $b = (b_1, b_2, \dots, b_m)$ will be in $\text{col}(A)$.

- Quiz. Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, find conditions on the numbers $b_1, b_2, \dots, b_m$ under which $b = (b_1, b_2, \dots, b_m)$ will be in $\text{ran}(T)$.

Dimension Theorem

- Given an $m \times \mathrm{n}$ matrix A and a linear system $Ax = 0$ with n unknowns, assume the row echelon form of A has r nonzero rows.
  How many *free variables* does the system have?

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad\qquad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad\qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

```
>> A
A =
     1     3    -2     0     2     0
     2     6    -5    -2     4    -3
     0     0     5    10     0    15
     2     6     0     8     4    18
>> rref(A)
ans =
     1     3     0     4     2     0
     0     0     1     2     0     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

Dimension Theorem

- [Dimension Theorem for Homogeneous System]
  If a homogeneous system has n unknowns, and rref of the augmented matrix has r nonzero rows, then the system has $n - r$ free variables

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad\qquad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad\qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

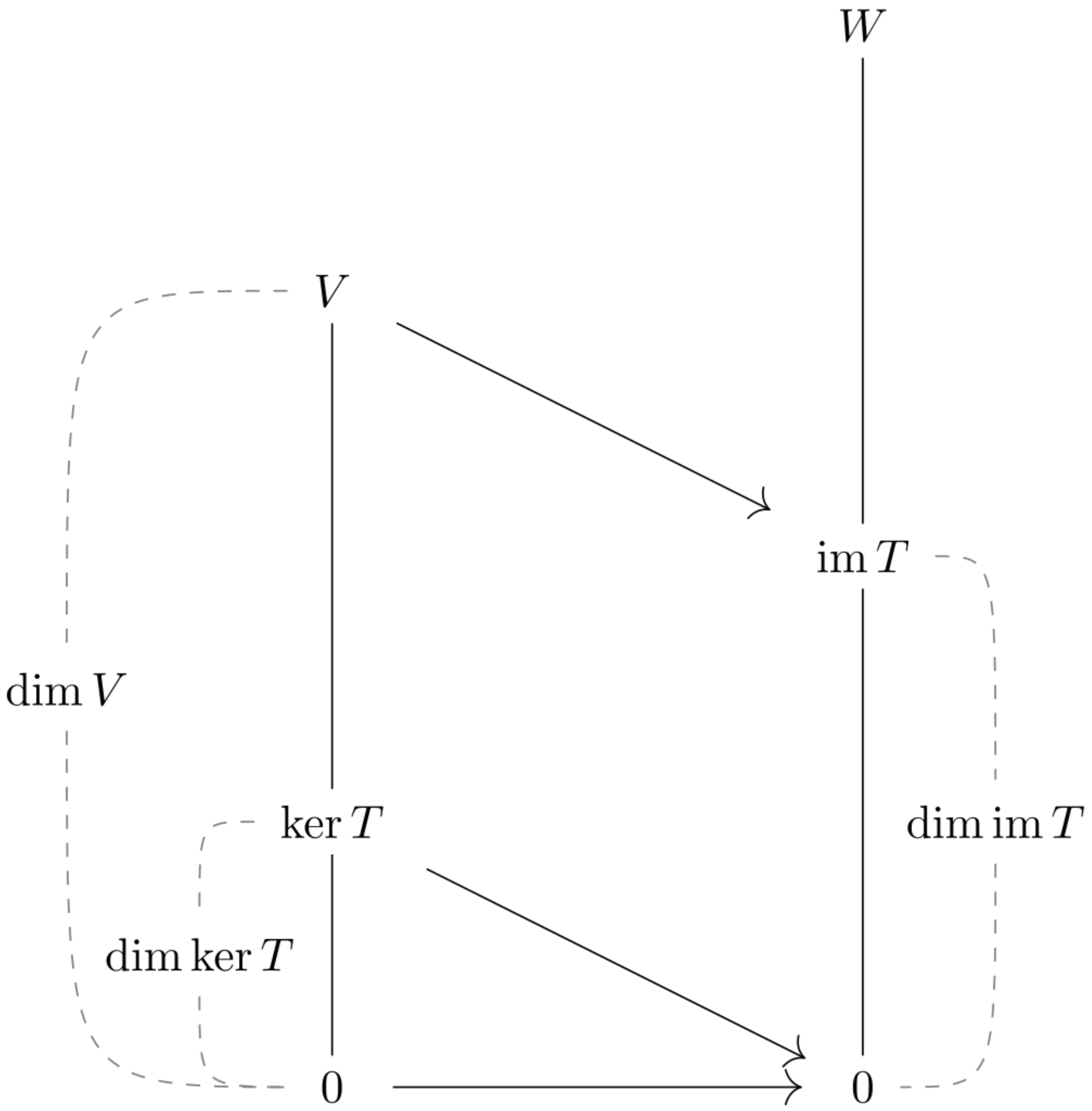
```
>> A
A =
     1     3    -2     0     2     0
     2     6    -5    -2     4    -3
     0     0     5    10     0    15
     2     6     0     8     4    18
>> rref(A)
ans =
     1     3     0     4     2     0
     0     0     1     2     0     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

THE Rank-Nullity Theorem

- [Rank-Nullity Theorem]
  If A is an $m \times n$ matrix, then

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n$$

- Quiz. How to check $\mathrm{nullity}(A)$ in Matlab?



https://en.wikipedia.org/wiki/Rank%E2%80%93nullity_theorem

THE Rank-Nullity Theorem

- [Thm] If an $m \times n$ matrix A has rank k, then
  - $\text{nullity}(A) = n - k$.
  - Every row echelon form of A has k nonzero rows.
  - Every row echelon form of A has $m - k$ zero rows.
  - The system $Ax = 0$ has k pivot variables (leading variables) and $n - k$ free variables.

- [The dimension theorem for subspaces]
  If W is a subspace of $\mathbb{R}^n$, then

$$\dim(W) + \dim(W^\perp) = n$$

Rank of orthogonal complement

- [Thm] If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = n - 1$, then there is a nonzero vector a such that $W = a^{\perp}$.

- [Thm] If u is a nonzero $m \times 1$ matrix and v is a nonzero $n \times 1$ matrix, then the outer product

$$A = uv^T$$

- has rank 1. Conversely, if A is an $m \times n$ matrix with rank 1, then A can be factored into a product of the above form.

Consistency of a linear system

- [The Consistency Theorem]
  If $Ax = b$ is a linear system of m equations with n unknowns, then the followings are equivalent.
  - $Ax = b$ is consistent.
  - $b \in \mathrm{col}(A)$.
  - The coefficient matrix A and the augmented matrix $[A \mid b]$ have the same rank.

- Quiz. What is the additional condition for $Ax = b$ to have a unique solution?

Full column rank, Full row rank

- [Def] An $m \times n$ matrix A is said to have
  - full column rank if its column vectors are linearly independent.
  - full row rank if its row vectors are linearly independent.

- [Thm] Let A be an $m \times n$ matrix.
  - A has full column rank $\Leftrightarrow$ Column vectors of A form a basis of $\mathrm{col}(A) \Leftrightarrow \mathrm{rank}(A) = n$
  - A has full row rank $\Leftrightarrow$ Row vectors of A form a basis of $\mathrm{row}(A) \Leftrightarrow \mathrm{rank}(A) = m$

- Quiz. If A is an $m \times n$ matrix with full column rank, what can you say about the relative sizes of m and n? What if A has full row rank?

Number of solutions of a linear system

- [Thm] If A is an $m \times n$ matrix, then the followings are equivalent.
  - $Ax = 0$ has only the trivial solution.
  - $Ax = b$ has at most one solution for every $b \in \mathbb{R}^m$.
  - A has full column rank.

- [Thm] Let A be an $m \times n$ matrix.
  - (Overdetermined case) If $m > n$, then the system $Ax = b$ is inconsistent for some vector $b \in \mathbb{R}^m$.
  - (Underdetermined case) If $m < n$, then for every vector $b \in \mathbb{R}^m$ the system $Ax = b$ is either inconsistent or has infinitely many soultions.
  - Quiz. What about $m = n$ case?
  - Quiz. Can an overdetermined case have infinitely many solutions?
  - Quiz. Can an underdetermined case have exactly one solution?

Existence of an inverse of a matrix

- For an $n \times n$ matrix A, the followings are equivalent.

  (1) $\mathrm{rref}(A) = I_n$

  (2) A is a multiplication of elementary matrices.

  (3) A is invertible.

  (4) $Ax = 0$ has only a trivial.

  (5) $Ax = b$ is consistent for every vector $b \in \mathbb{R}^n$.

  (6) The solution of $Ax = b$ is unique for every vector $b \in \mathbb{R}^n$.

  (7) Column vectors of A are linearly independent.

  (8) Row vectors of A are linearly independent.

  (9) $\det(A) \neq 0$

Existence of an inverse of a matrix

- For an $n \times n$ matrix A, the followings are equivalent.

  (10) Column vectors of A span $\mathbb{R}^n$.

  (11) Row vectors of A span $\mathbb{R}^n$.

  (12) A set of column vectors of A forms a basis of $\mathbb{R}^n$.

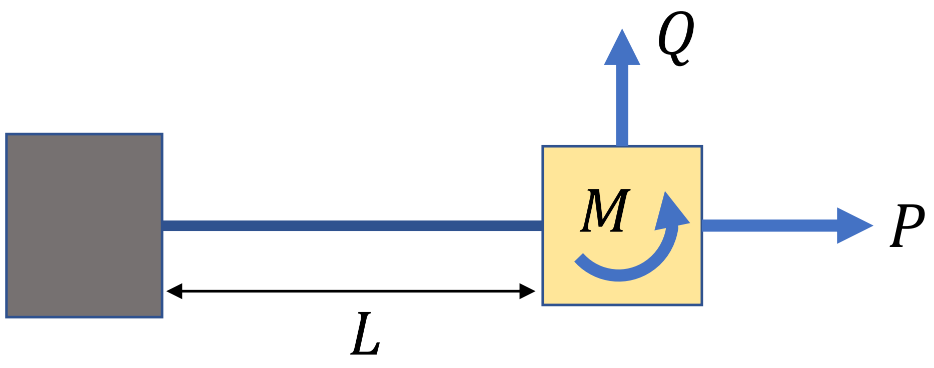
  (13) A set of row vectors of A forms a basis of $\mathbb{R}^n$.

  (14) $\mathrm{rank}(A) \ = \ n$

  (15) $\mathrm{nullity}(A) \ = \ 0$

2.3 Applications

Compliant mechanisms



$$\delta_x = \frac{PL}{EA}$$

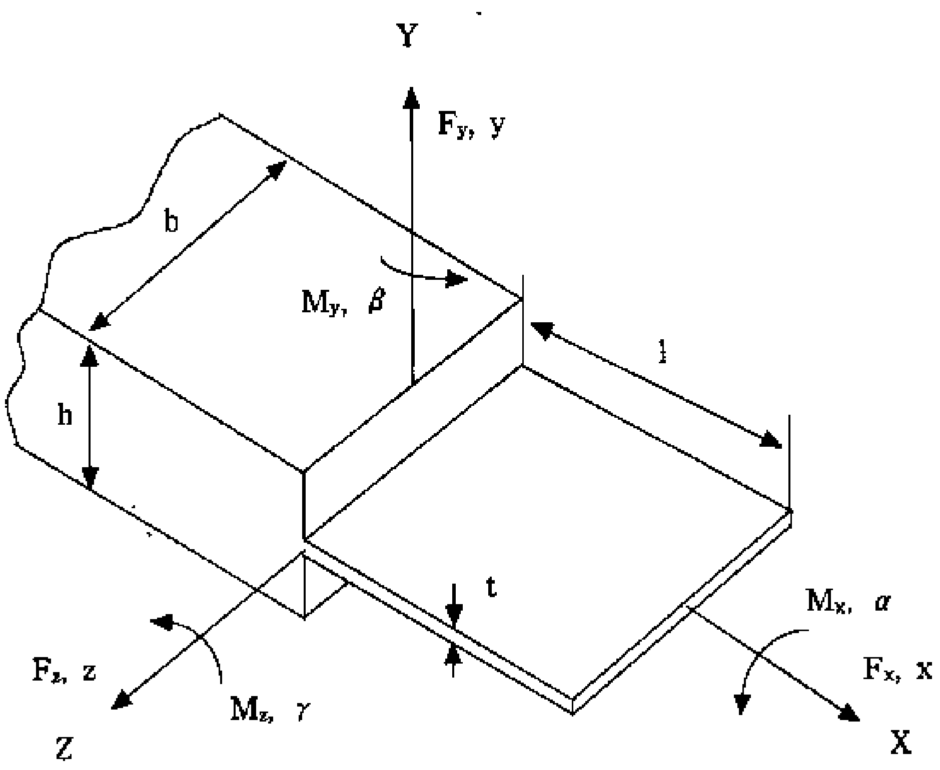
$$\delta_y = \frac{QL^3}{3EI} + \frac{ML^2}{2EI}$$

$$\theta_z = \frac{QL^2}{2EI} + \frac{ML}{EI}$$

$$\begin{bmatrix} \delta_x \\ \delta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{bmatrix} P \\ Q \\ M \end{bmatrix}$$

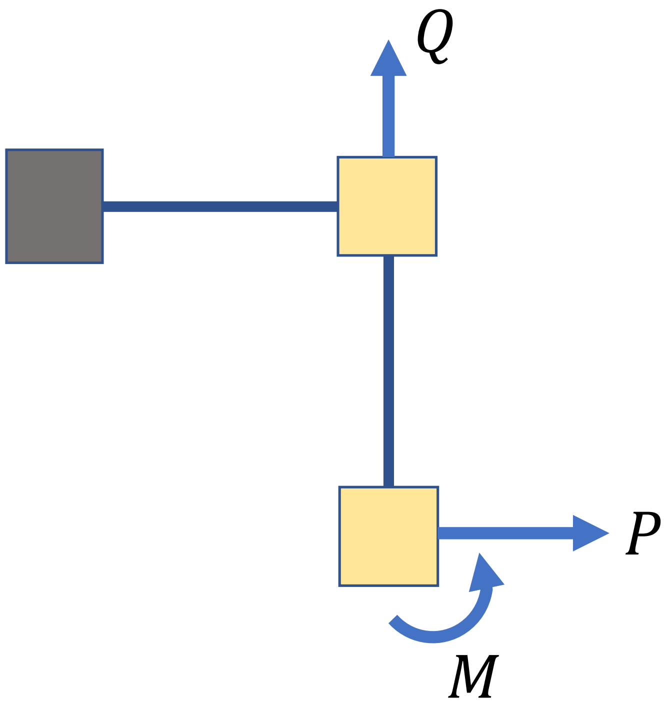
$$X = CF$$

Compliant mechanisms

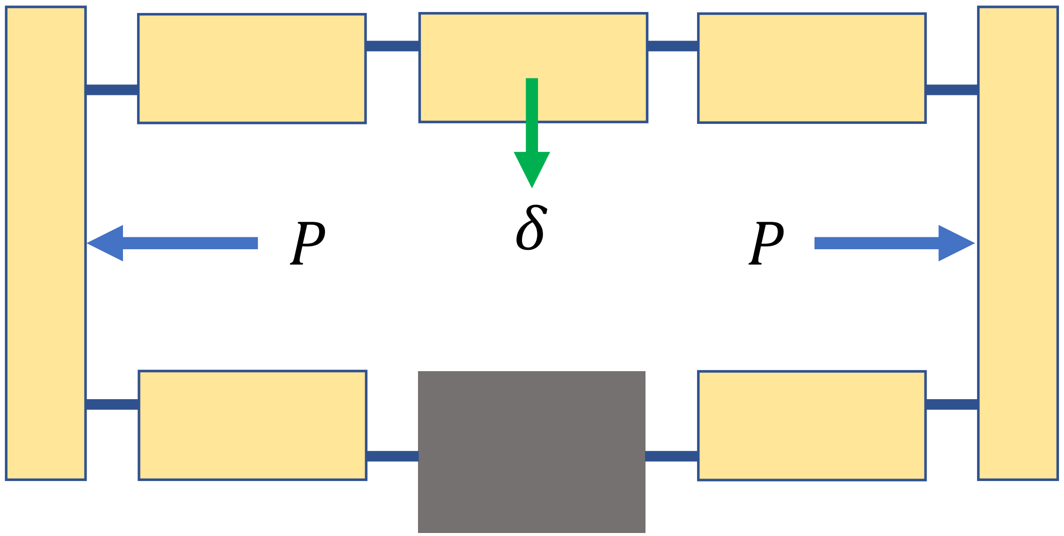


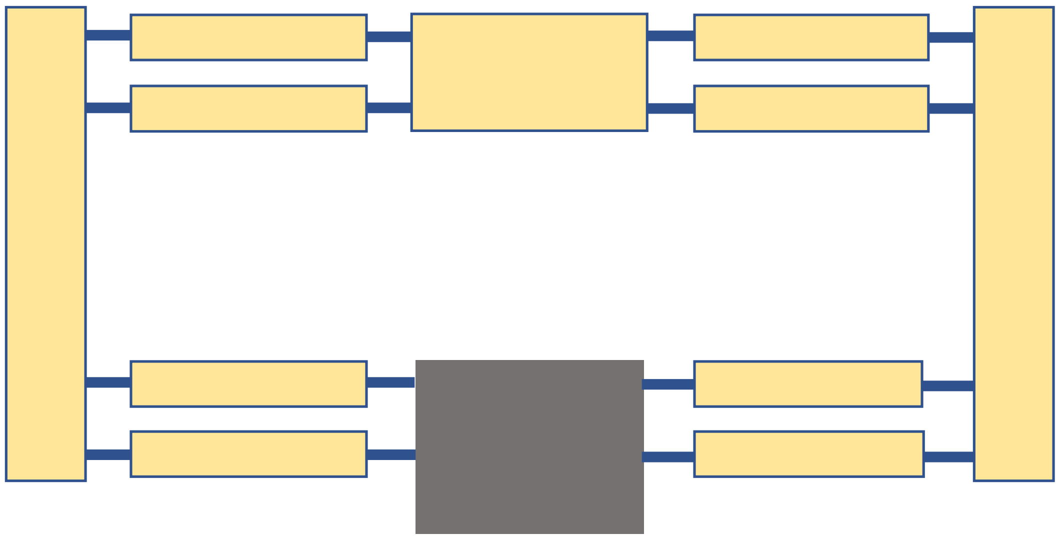
$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \alpha \\ \Delta \beta \\ \Delta \gamma \end{bmatrix} = \begin{bmatrix} \frac{\Delta x}{F_x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\Delta y}{F_y} & 0 & 0 & 0 & \frac{\Delta y}{M_z} \\ 0 & 0 & \frac{\Delta z}{F_z} & 0 & \frac{\Delta z}{M_y} & 0 \\ 0 & 0 & 0 & \frac{\Delta \alpha}{M_x} & 0 & 0 \\ 0 & 0 & \frac{\Delta \beta}{F_z} & 0 & \frac{\Delta \beta}{M_y} & 0 \\ 0 & \frac{\Delta \gamma}{F_y} & 0 & 0 & 0 & \frac{\Delta \gamma}{M_z} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

Compliant mechanisms

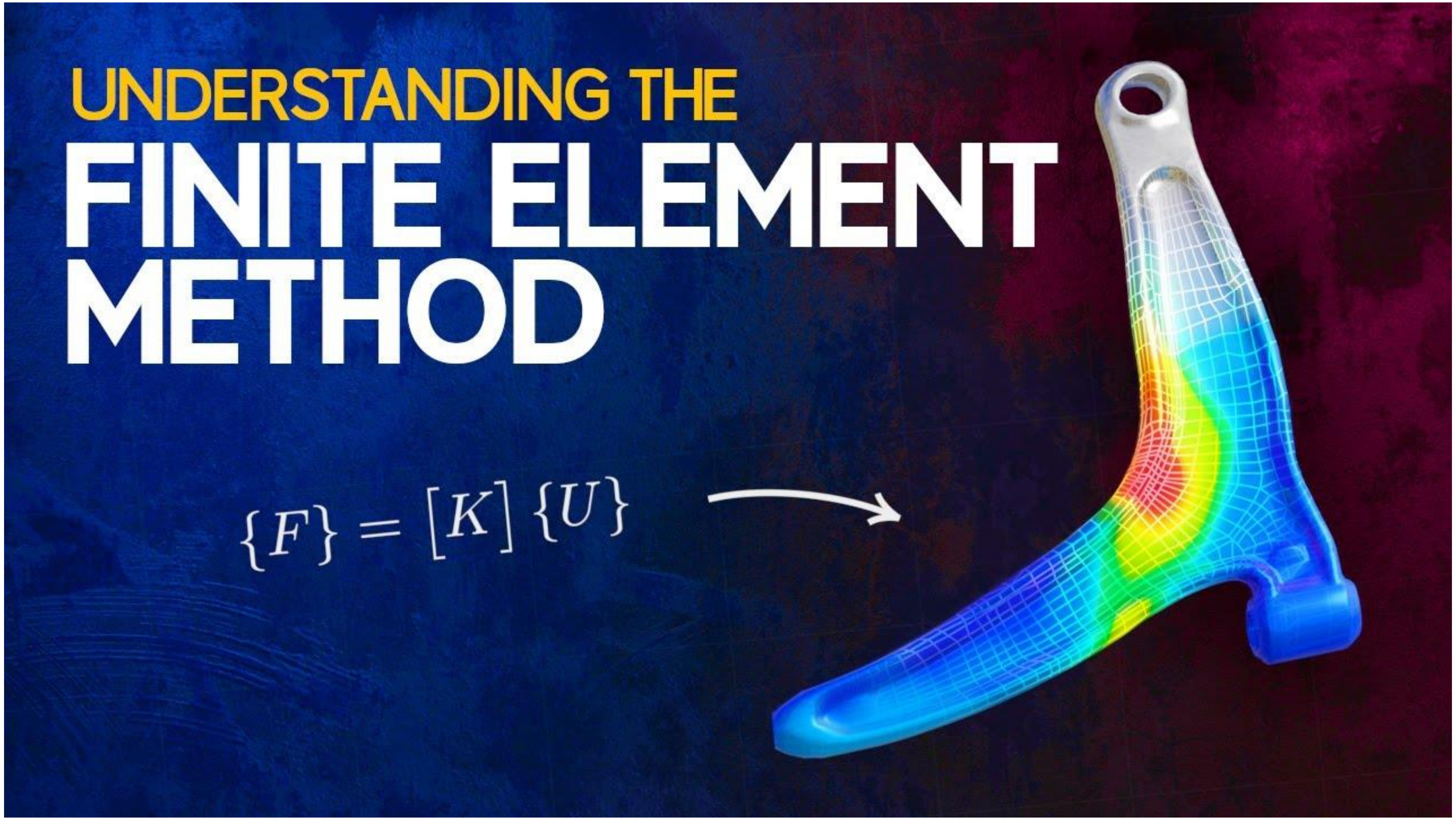


$$X = CF$$



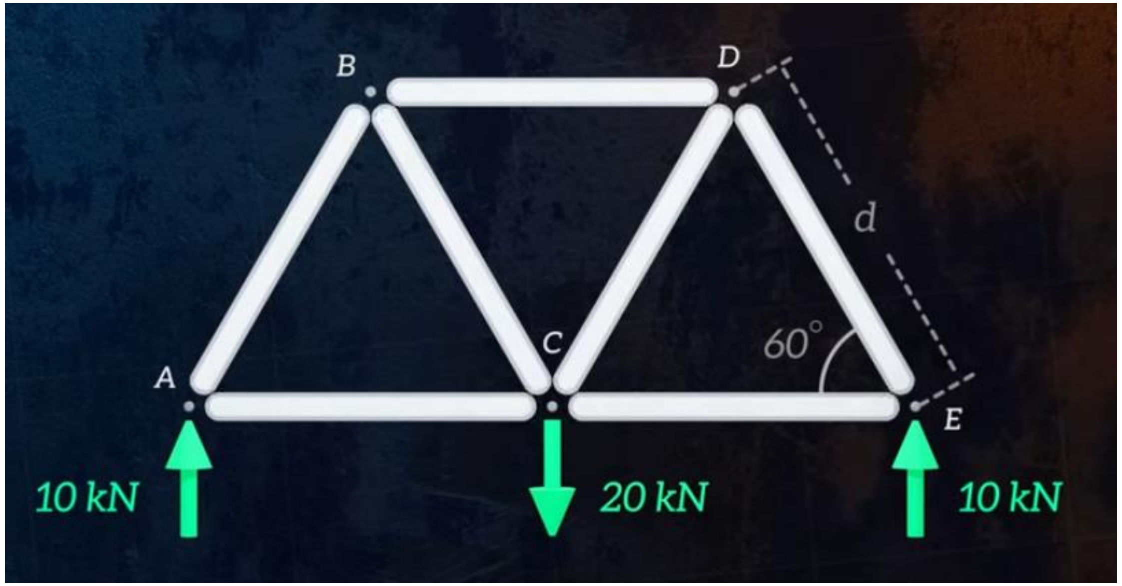


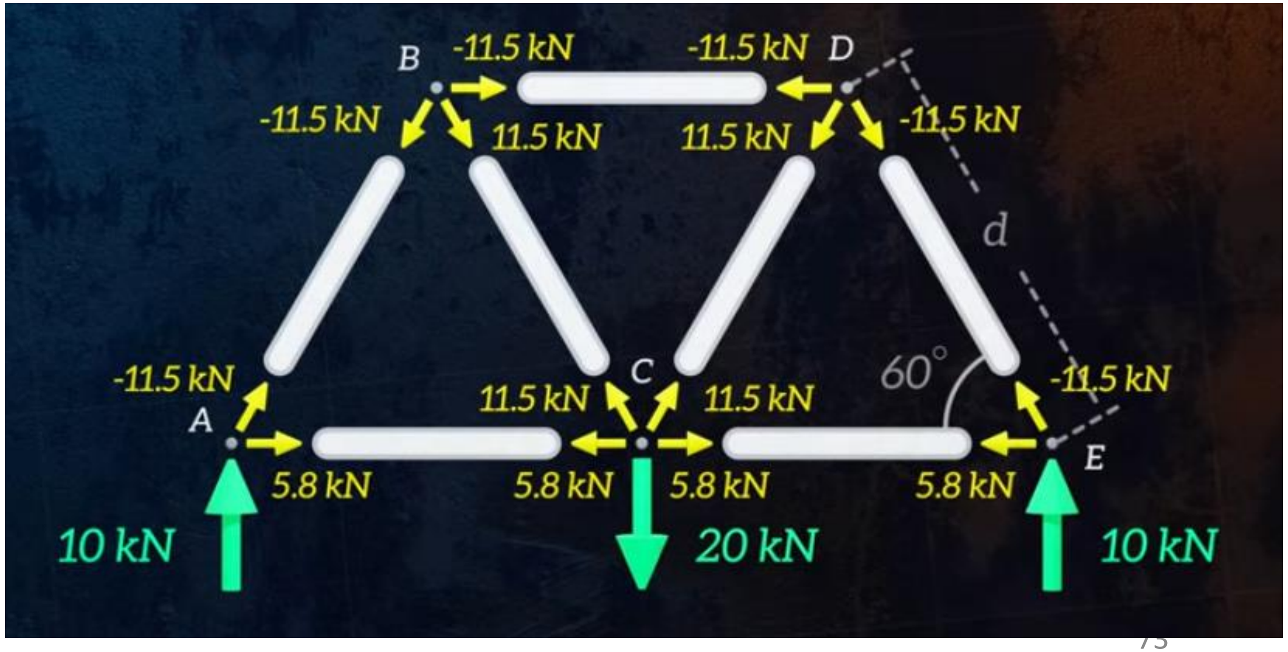
Finite Element Analysis



https://www.youtube.com/watch?v=GHjopp47vvQ

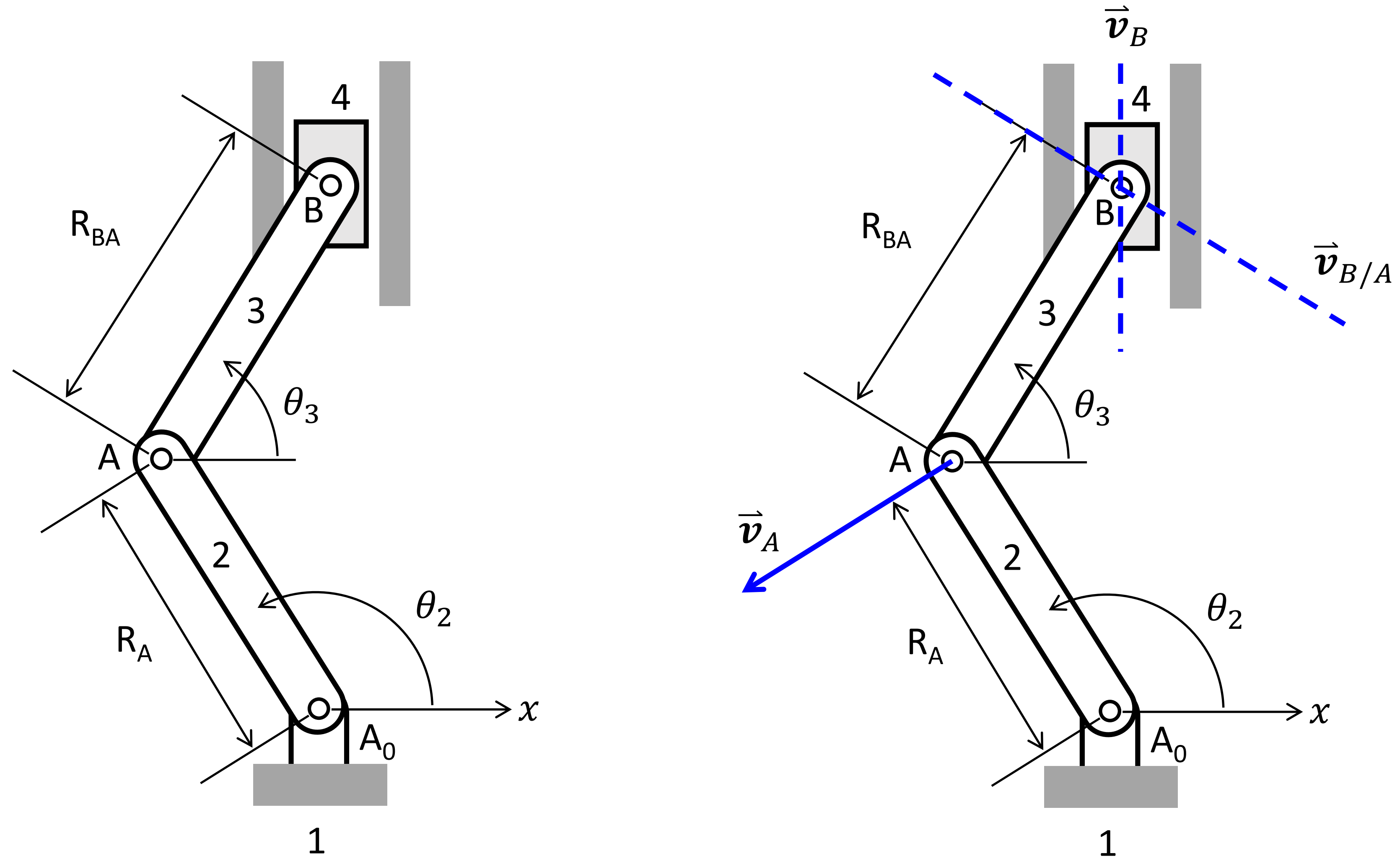
Truss analysis



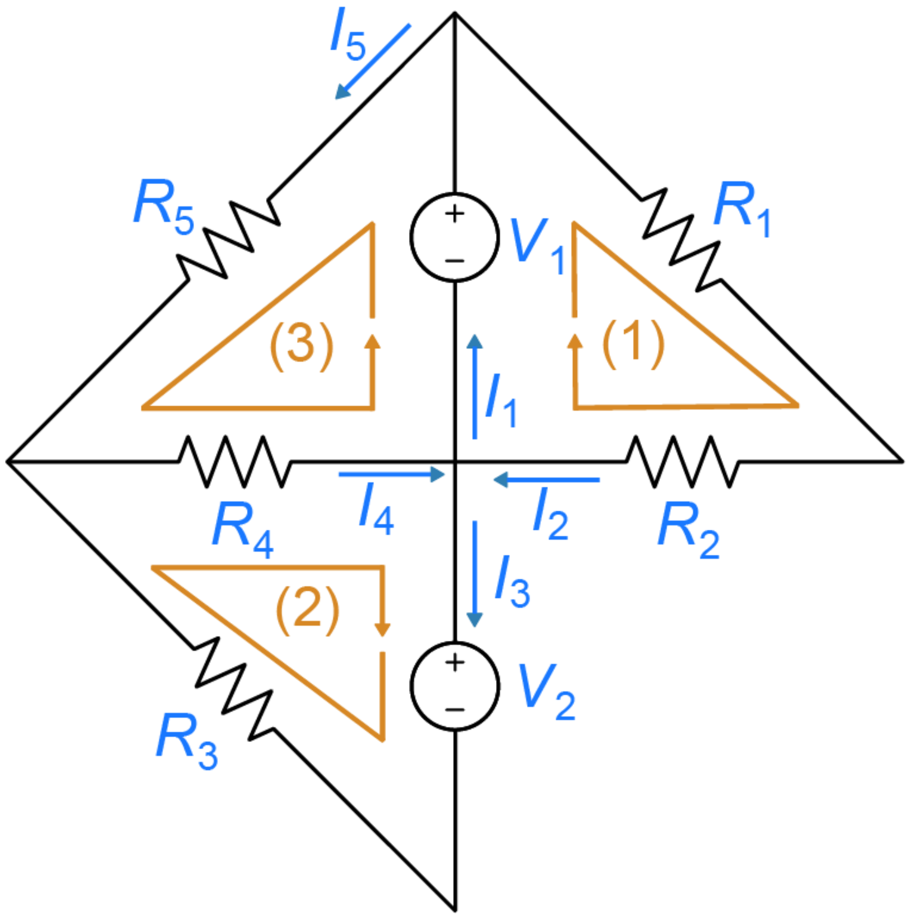


https://www.youtube.com/watch?v=Hn_iozUo9m4

Mechanism analysis



Circuit analysis



Applying **Kirchhoff's current law** to the central node yields:

$$I_2 + I_4 = I_1 + I_3$$

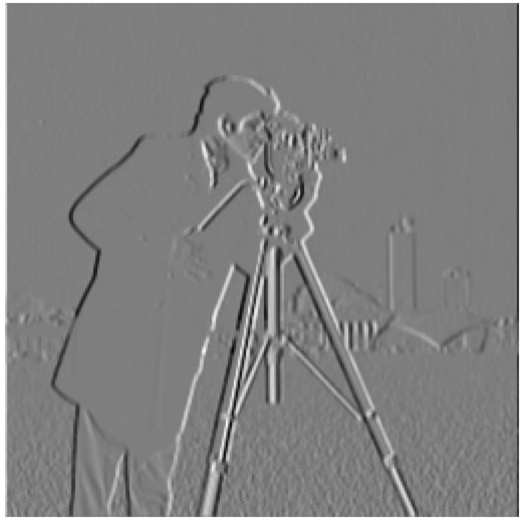
Applying **Kirchhoff's voltage law** to the three loops adds three additional equations:

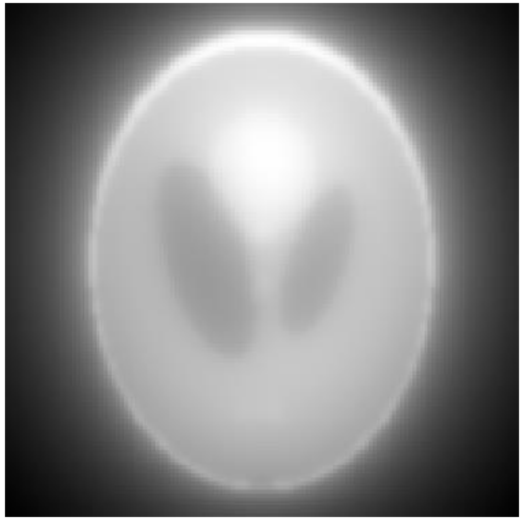
$$\begin{aligned} V_1 - I_2R_1 - I_2R_2 &= 0 \\ -V_2 - I_3R_3 - I_4R_4 &= 0 \\ V_1 - I_5R_5 - I_4R_4 &= 0 \end{aligned}$$

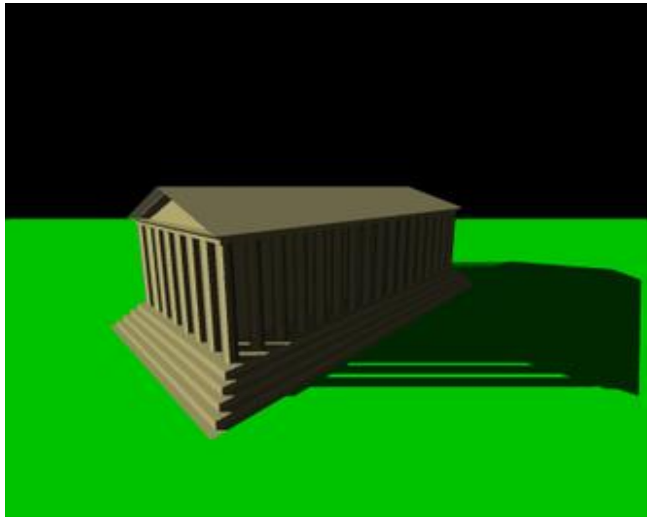
https://www.mathworks.com/matlabcentral/fileexchange/94730-matrix-methods-of-linear-algebra

Image processing



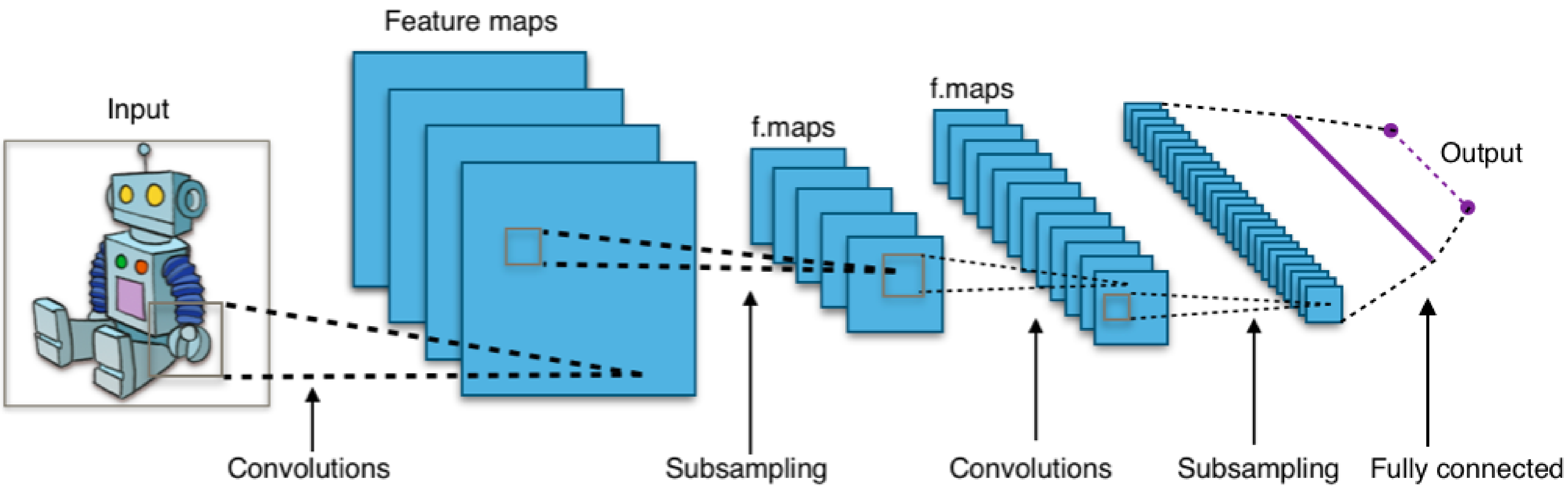




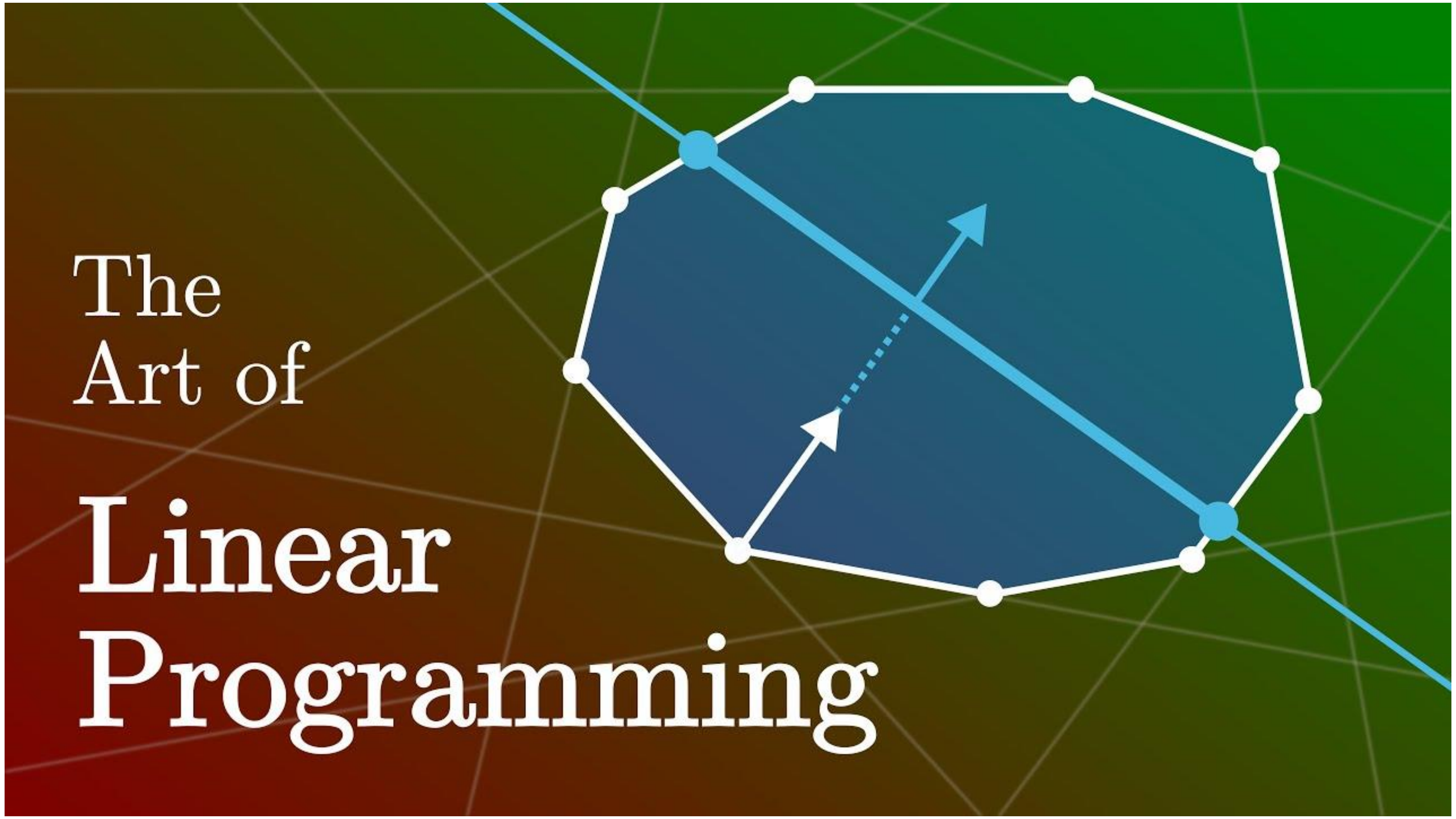
https://en.wikipedia.org/wiki/Shadow_mapping

Convolutional Neural Network



https://en.wikipedia.org/wiki/Convolutional_neural_network

Linear programming



https://www.youtube.com/watch?v=E72DWgKP_1Y

Markov chain

날씨 예측		내일날씨	
		맑음	강우
오늘날씨	맑음	0.6	0.4
	강우	0.7	0.3

$$\begin{array}{cc|c} & & \text{내일날씨} \\ & & \begin{array}{cc} \text{맑음} & \text{강우} \end{array} \\ \text{오늘날씨} & \begin{array}{c} \text{맑음} \\ \\ \text{강우} \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ \text{1일 후} & \\ 0.3 & 0.7 \end{bmatrix} \end{array} \times \begin{bmatrix} 0.6 & 0.4 \\ \text{2일 후} & \\ 0.3 & 0.7 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ \text{3일 후} & \\ 0.3 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 0.6\times0.6+0.4\times0.3 & 0.6\times0.4+0.4\times0.7 \\ 0.48 & 0.52 \\ 0.3\times0.6+0.7\times0.3 & 0.3\times0.4+0.7\times0.7 \\ 0.39 & 0.61 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 0.48\times0.6+0.52\times0.3 & 0.48\times0.4+0.52\times0.7 \\ 0.444 & 0.556 \\ 0.39\times0.6+0.61\times0.3 & 0.39\times0.4+0.61\times0.7 \\ 0.417 & 0.583 \end{bmatrix}$$

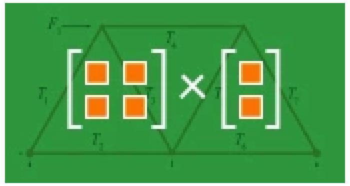
프로그래머를 위한 강화학습, 멀티코어, 프리렉, 2021.

Online sources

MathWorks

Self-Paced Online Courses

Online Courses Home | My Courses | Online Training Suite



Introduction to Linear Algebra with MATLAB

Course available through the **Online Training Suite** **View options**

공돌이의 수학정리노트 (Angelo's Math Notes)

3Blue1Brown

@3blue1brown · 6.6M subscribers · 185 videos

My name is Grant Sanderson. Videos here cover a variety of topics in math, or adjacent fi ...**more**

3blue1brown.com **and 7 more links**

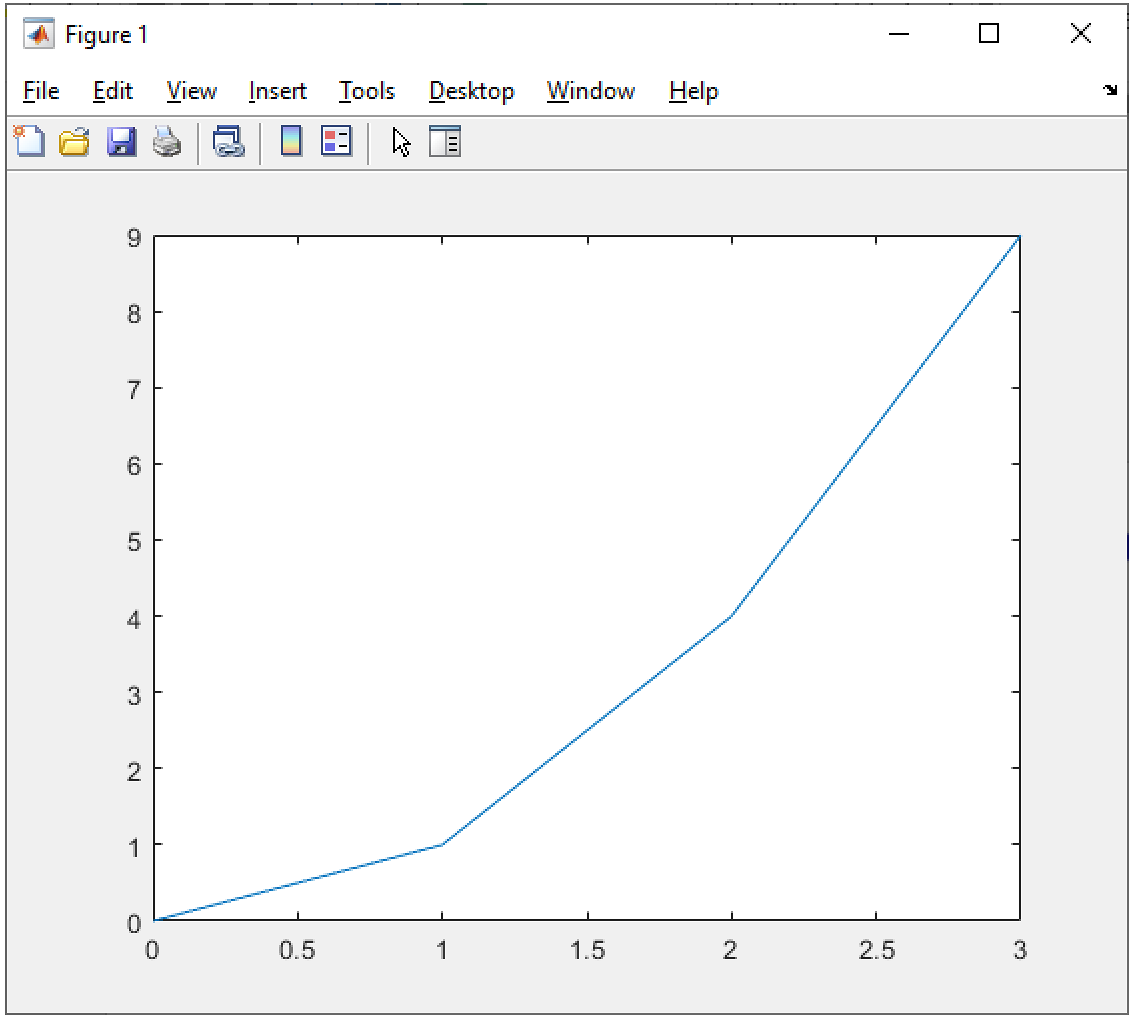
Subscribed

3. Visualization

3.1 2D plot

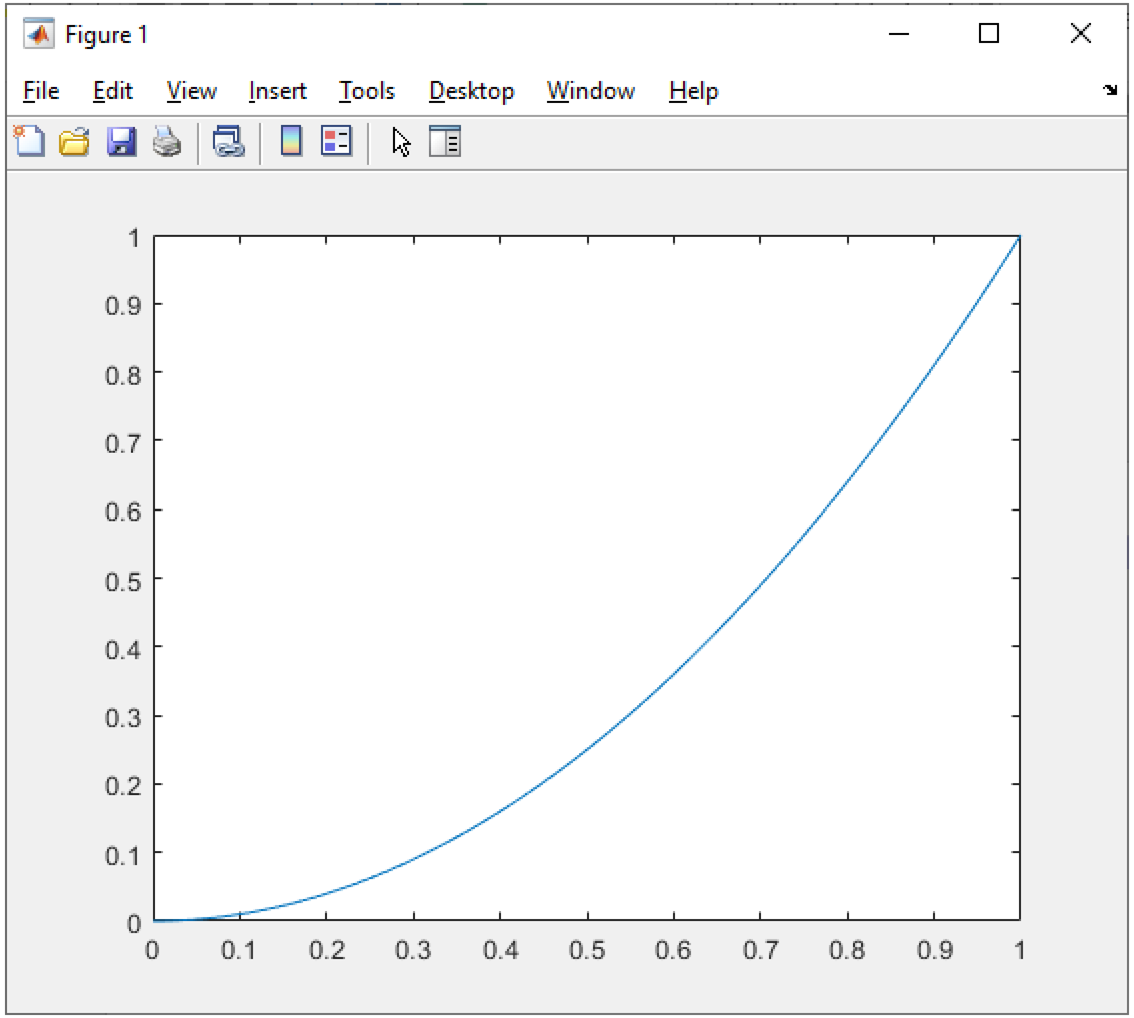
2-D graph is a collection of line segments

```
x = [0, 1, 2, 3];
y = [0, 1, 4, 9];

plot(x, y)
```



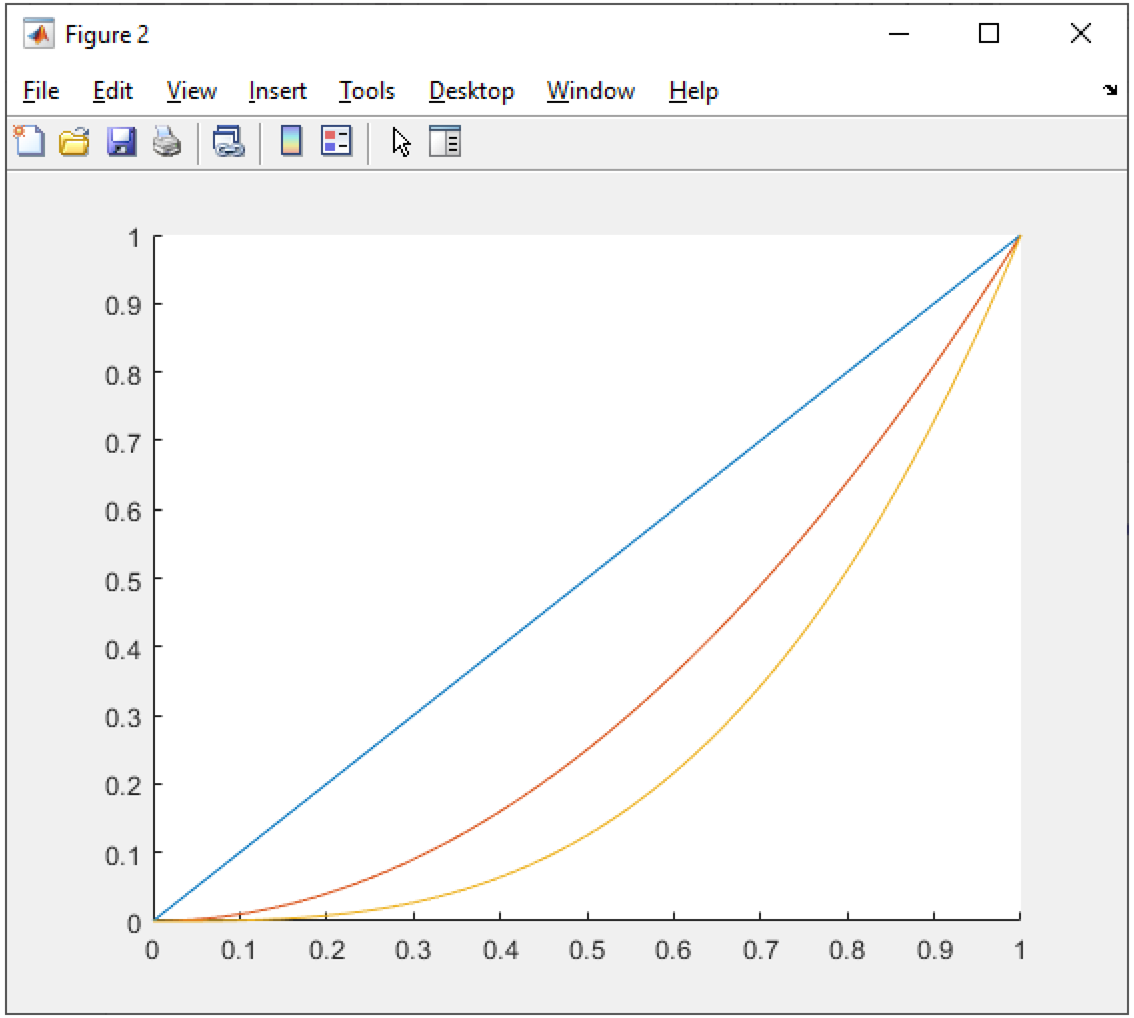
Lots of line segments ≒ a smooth curve

```
x = linspace(0, 1);
y = x.^2;

plot(x, y)
```



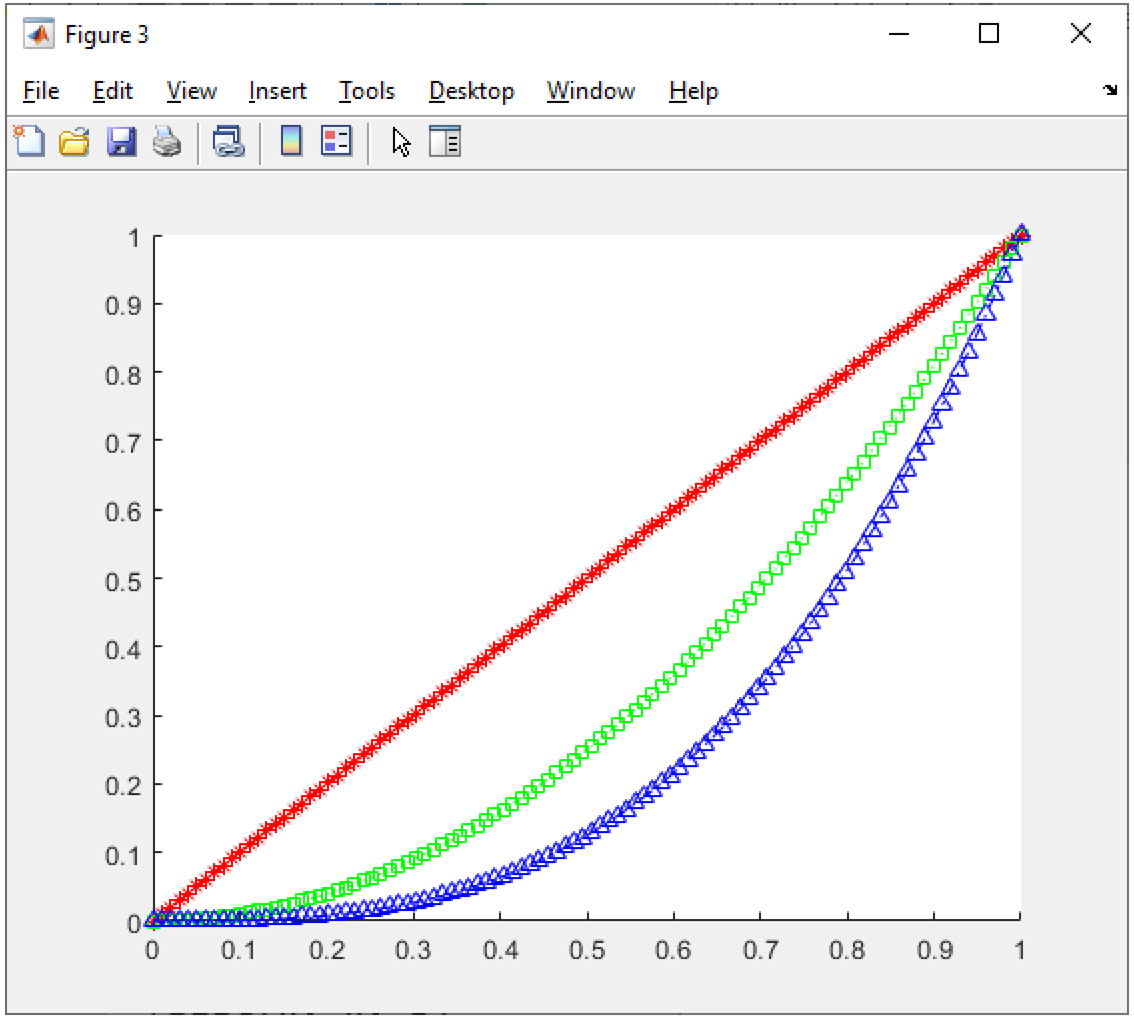
Drawing multiple lines on an *axes*

```
x = linspace(0, 1);

figure, hold on,
plot(x, x.^1)
plot(x, x.^2)
plot(x, x.^3)
```



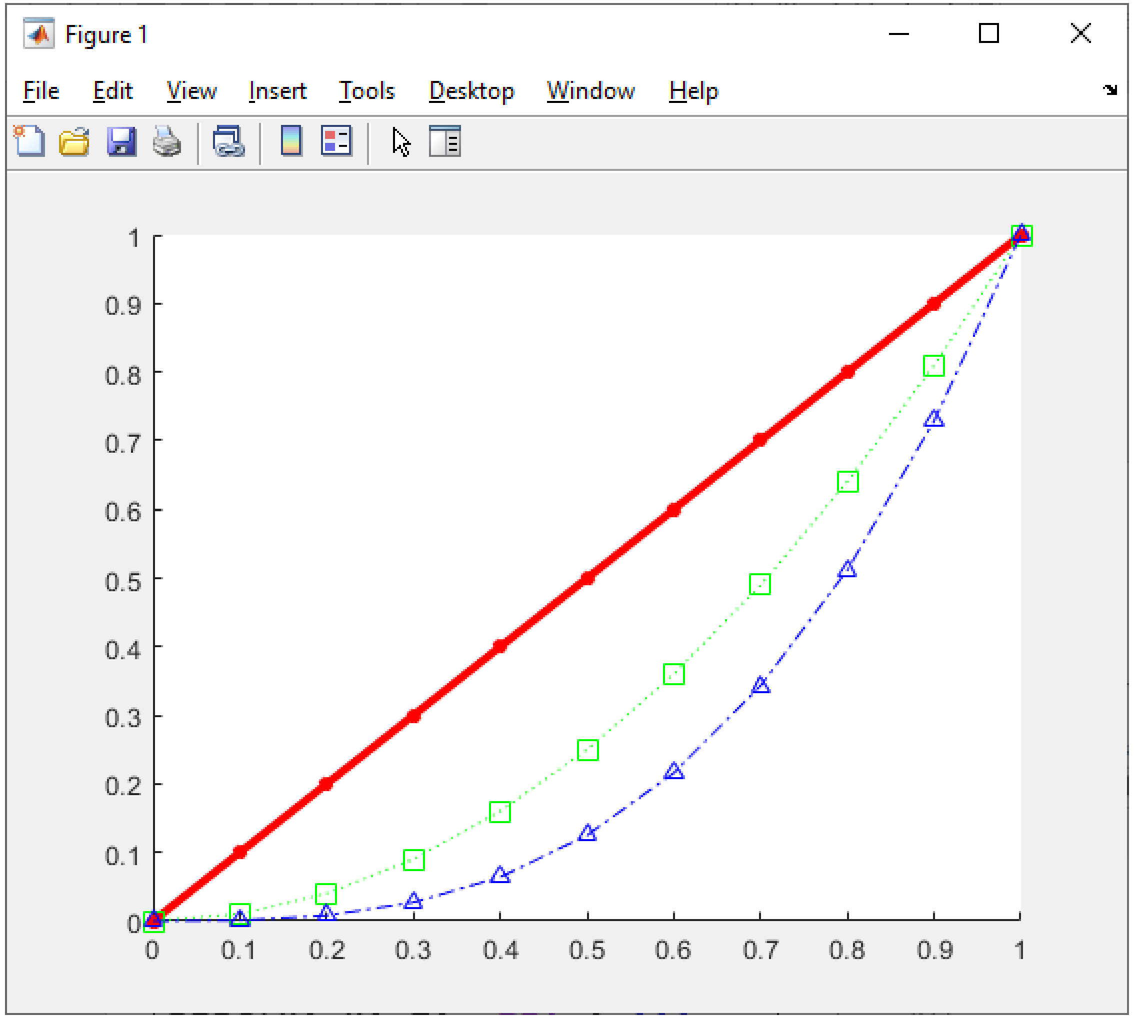
Line specifier

```
x = linspace(0, 1);

figure, hold on,
plot(x, x.^1, 'r*-')
plot(x, x.^2, 'gs:')
plot(x, x.^3, 'b^-.')
```



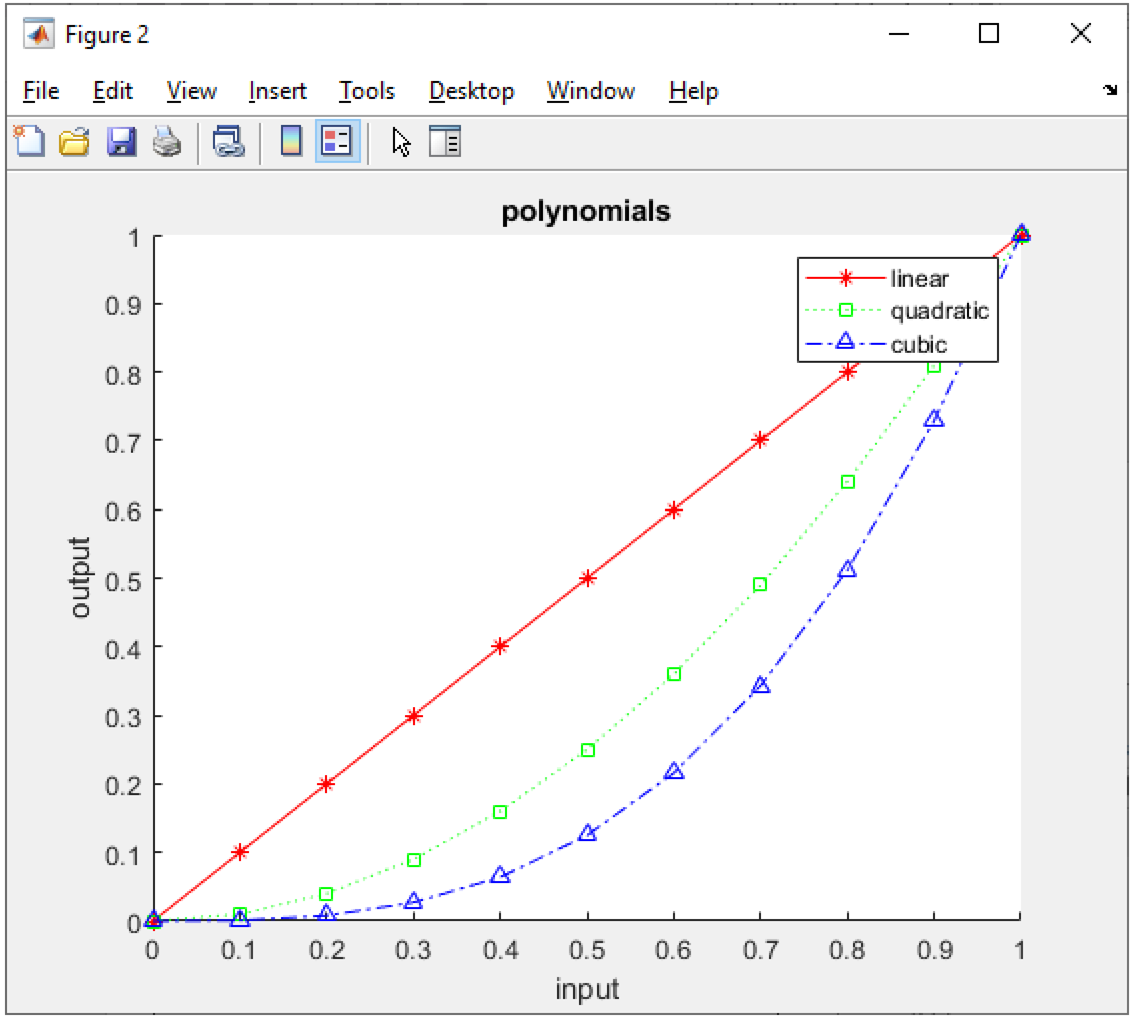
Linewidth, MarkerSize

```
x = linspace(0, 1, 11);

figure, hold on,
plot(x, x.^1, 'r*-', ...
    LineWidth=3, ...
    MarkerSize=5)
plot(x, x.^2, 'gs:', ...
    MarkerSize=10)
plot(x, x.^3, 'b^-.')
```



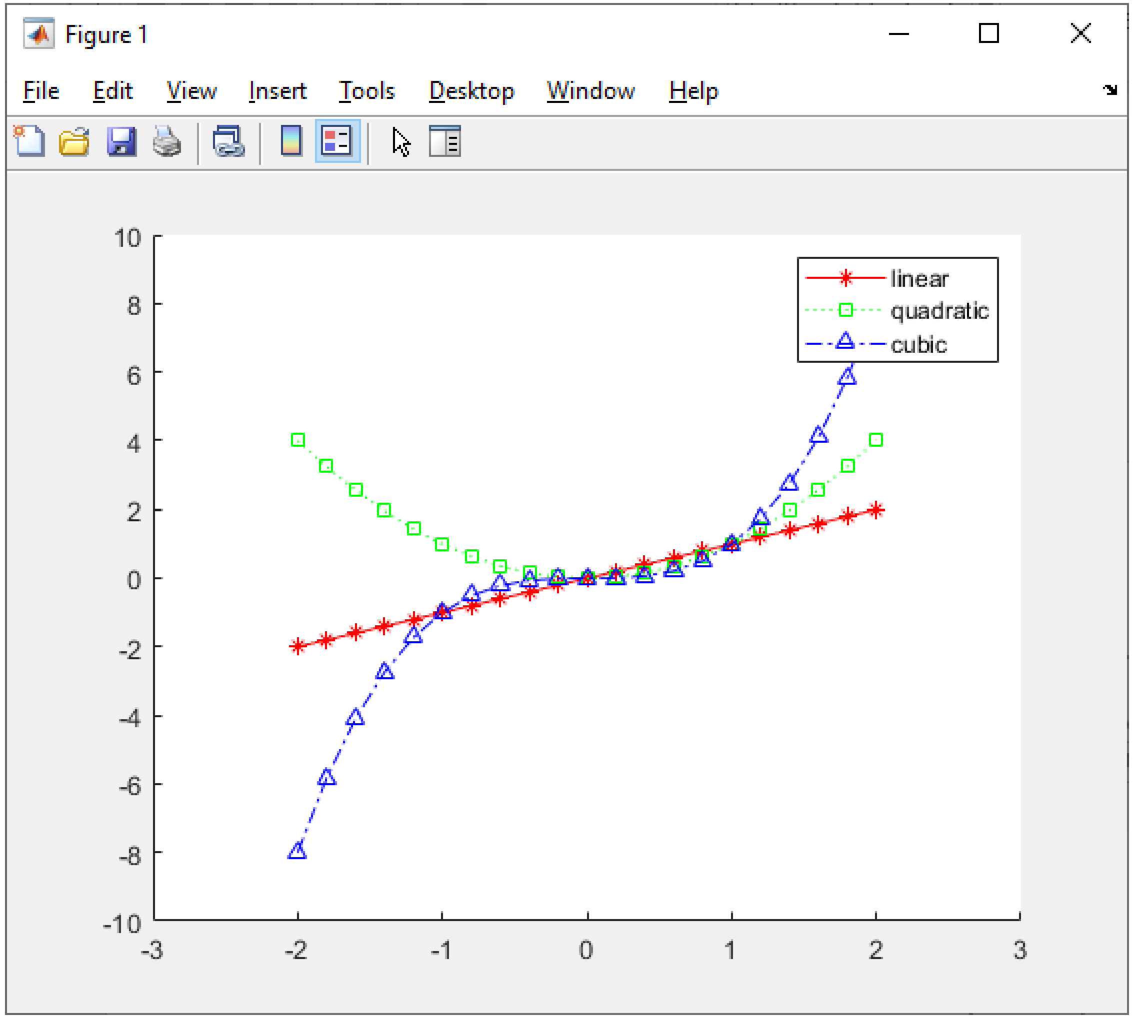
xlabel, ylabel, title, legend

```
x = linspace(0, 1, 11);

figure, hold on,
plot(x, x.^1, 'r*-')
plot(x, x.^2, 'gs:')
plot(x, x.^3, 'b^-.')

xlabel('input')
ylabel('output')
title('polynomials')
legend('linear', ...
       'quadratic', ...
       'cubic')
```



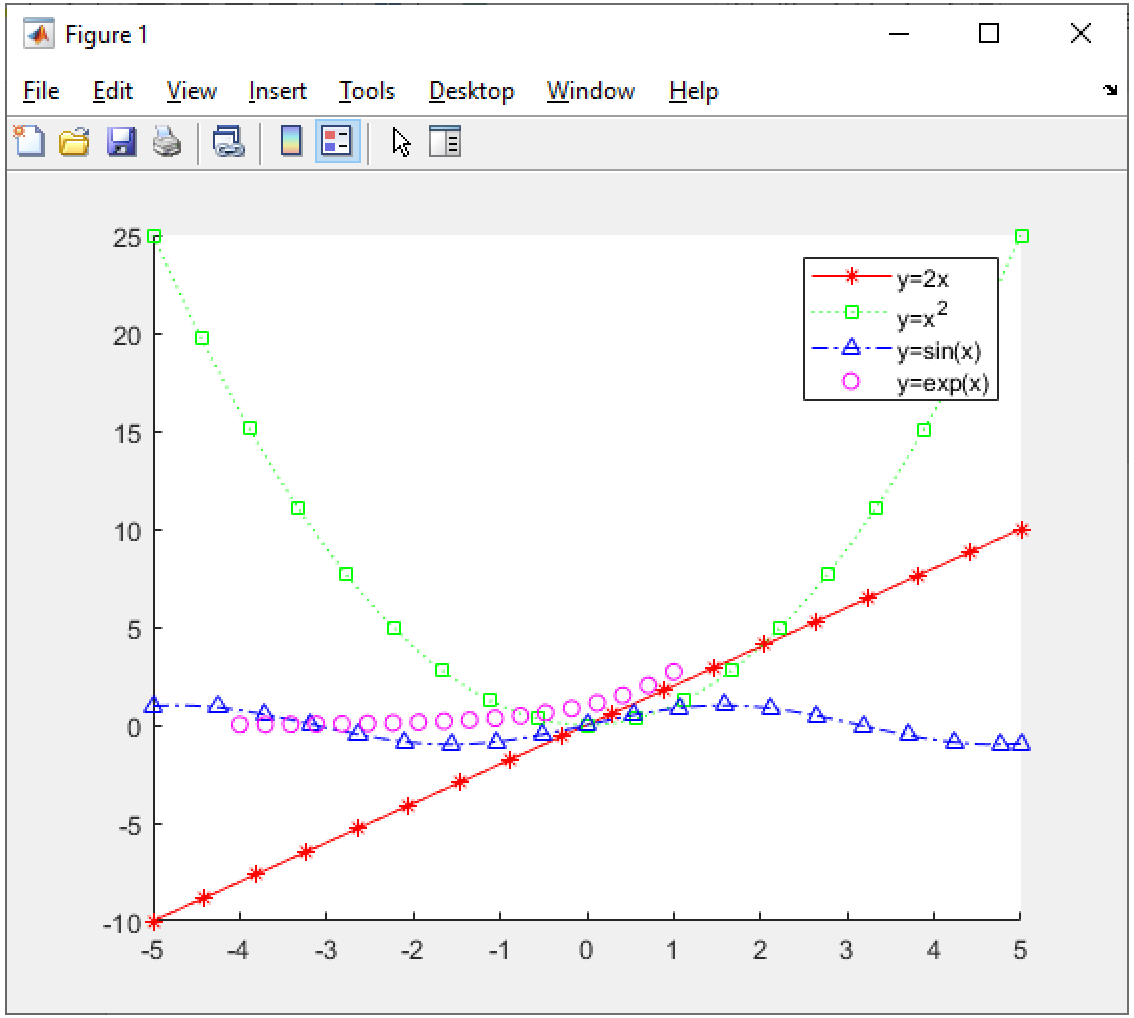
axis

```
x = linspace(-2, 2, 21);

figure, hold on,
plot(x, x.^1, 'r*-')
plot(x, x.^2, 'gs:')
plot(x, x.^3, 'b^-.')

legend('linear', ...
       'quadratic', ...
       'cubic')

xlim([-3, 3])
ylim([-10, 10])
```



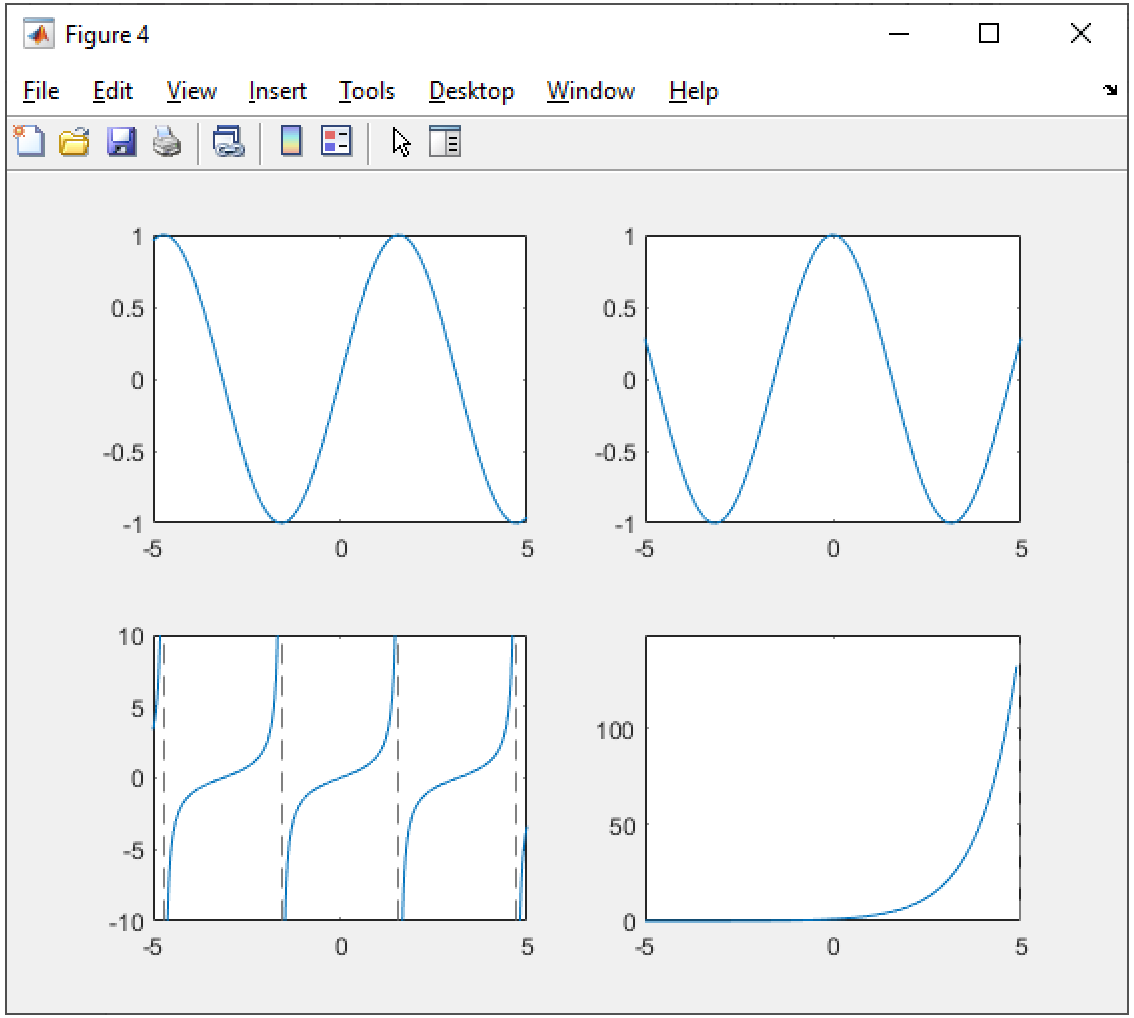
fplot

```
figure, hold on,
fplot(@(x) 2*x, 'r*-')
fplot(@(x) x.^2, 'gs:')
fplot(@sin, 'b^-.')
fplot(@exp, [-4, 1], 'mo')

legend('y=2x', 'y=x^2', ...
       'y=sin(x)', 'y=exp(x)')
```



subplot

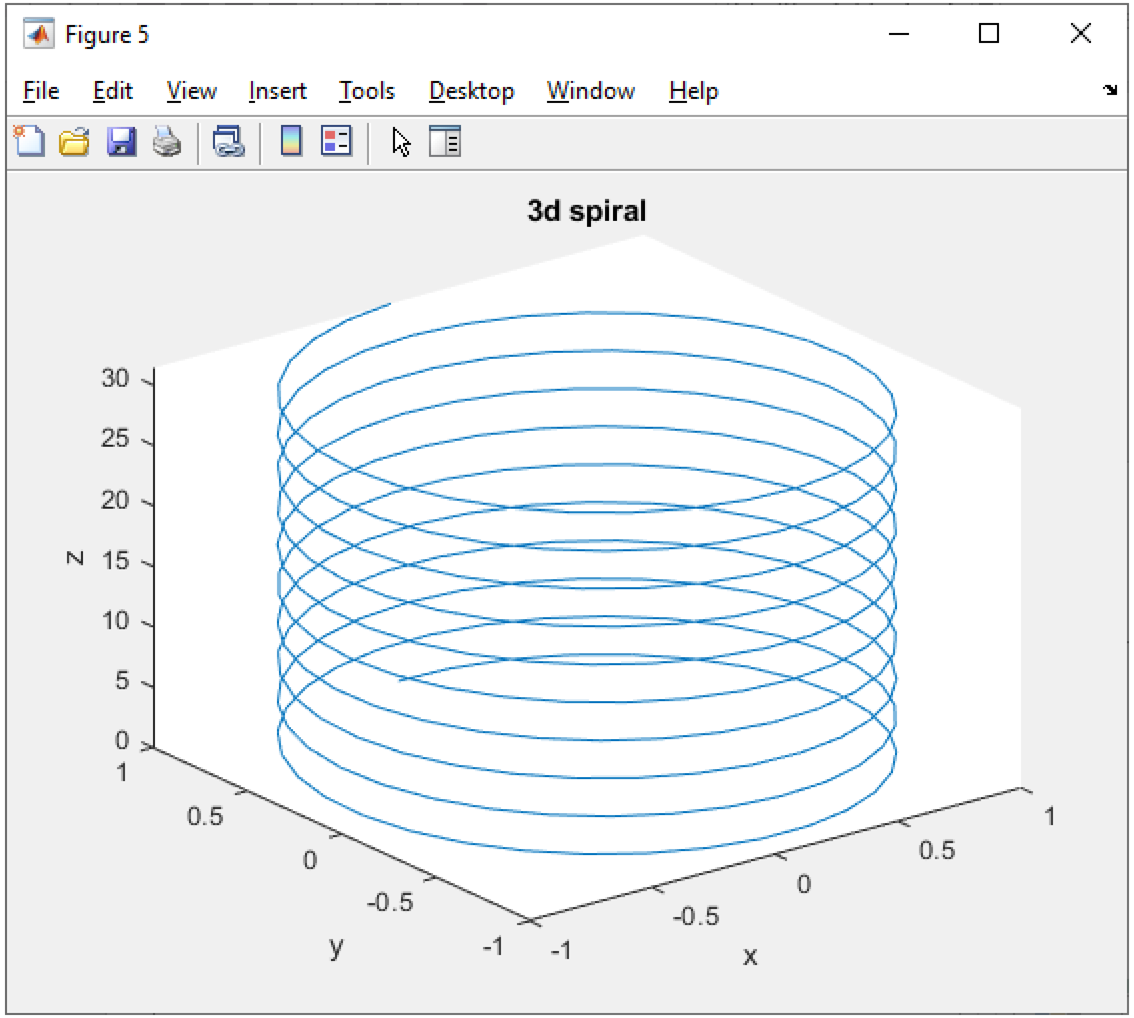
```
figure,
subplot(2, 2, 1), fplot(@sin)
subplot(2, 2, 2), fplot(@cos)
subplot(2, 2, 3), fplot(@tan)
subplot(2, 2, 4), fplot(@exp)
```



3.2 3D plot

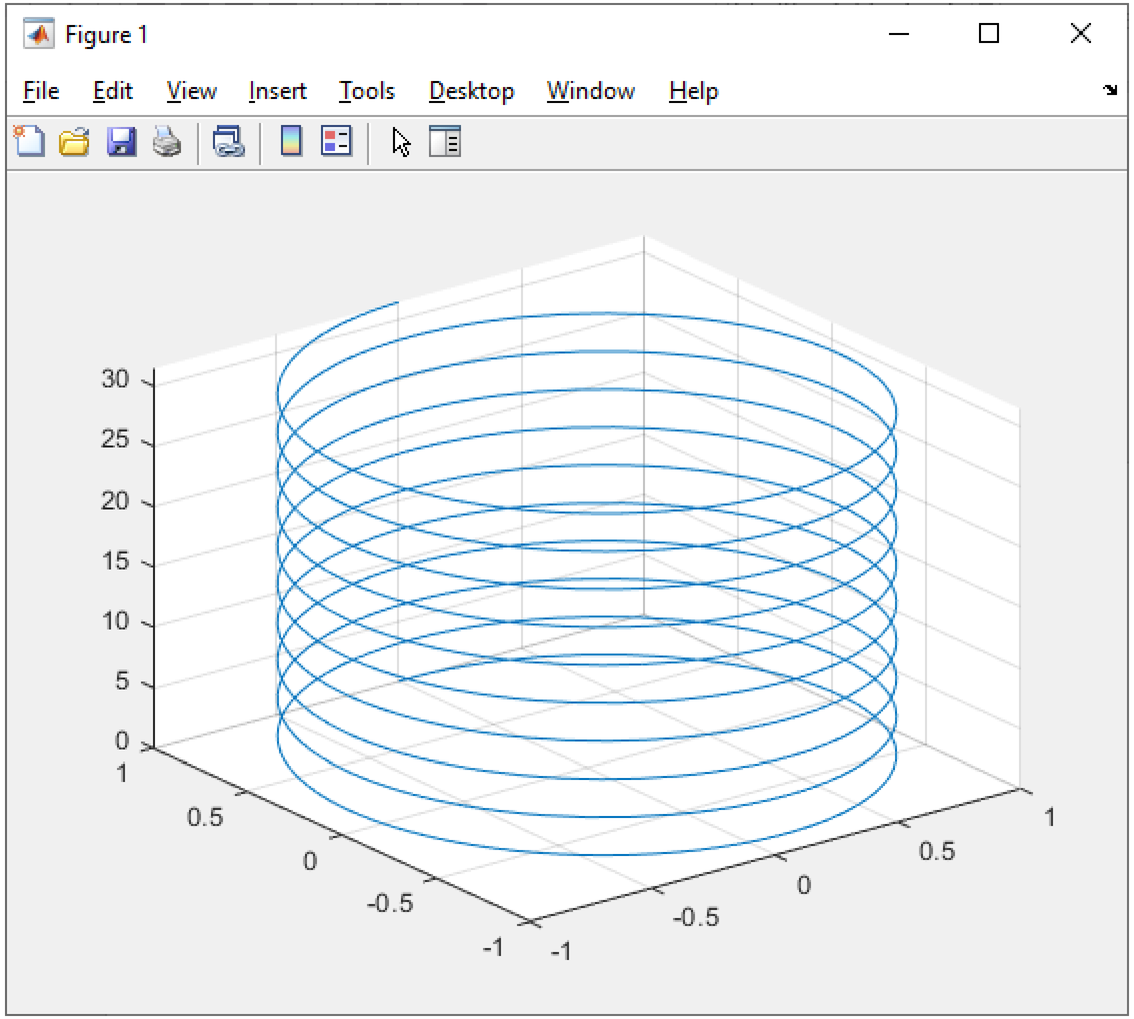
plot3

```
t = 0:0.1:10*pi;

x = sin(2*t);
y = cos(2*t);
z = t;

figure,
plot3(x, y, z)
xlabel('x'), ylabel('y'),
zlabel('z')
title('3d spiral')
```



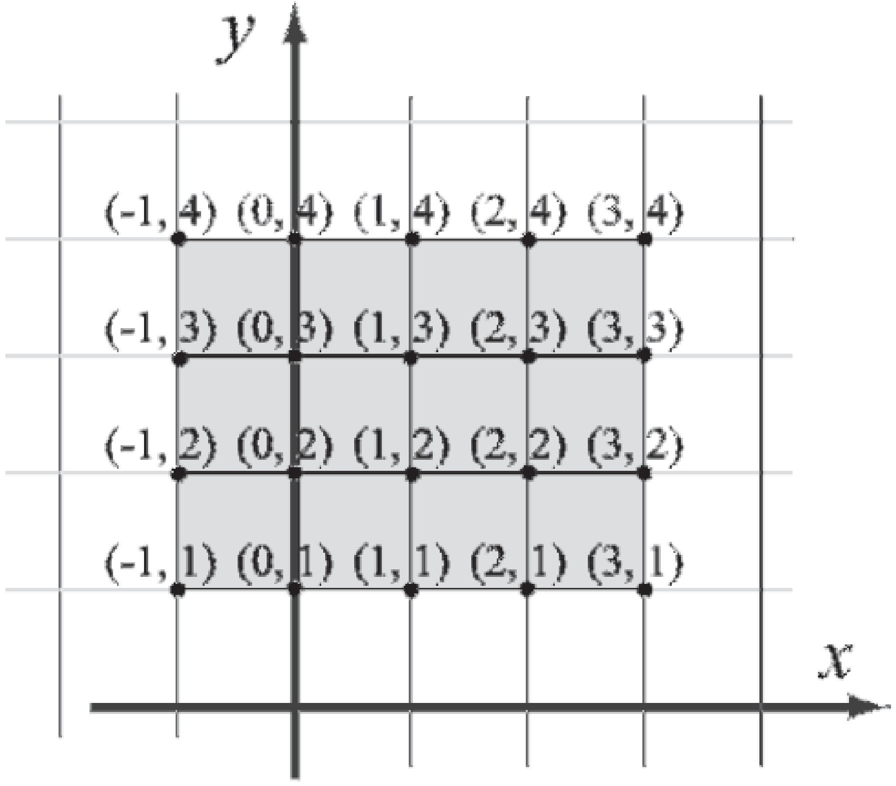
fplot3

```
fplot3(@(t) sin(2*t), ...
       @(t) cos(2*t), ...
       @(t) t, ...
[0, 10*pi])
```



meshgrid

```
x = -1:3;
y = 1:4;
[xx, yy] = meshgrid(x, y)
```




```
xx =
    -1     0     1     2     3
    -1     0     1     2     3
    -1     0     1     2     3
    -1     0     1     2     3
yy =
     1     1     1     1     1
     2     2     2     2     2
     3     3     3     3     3
     4     4     4     4     4
```

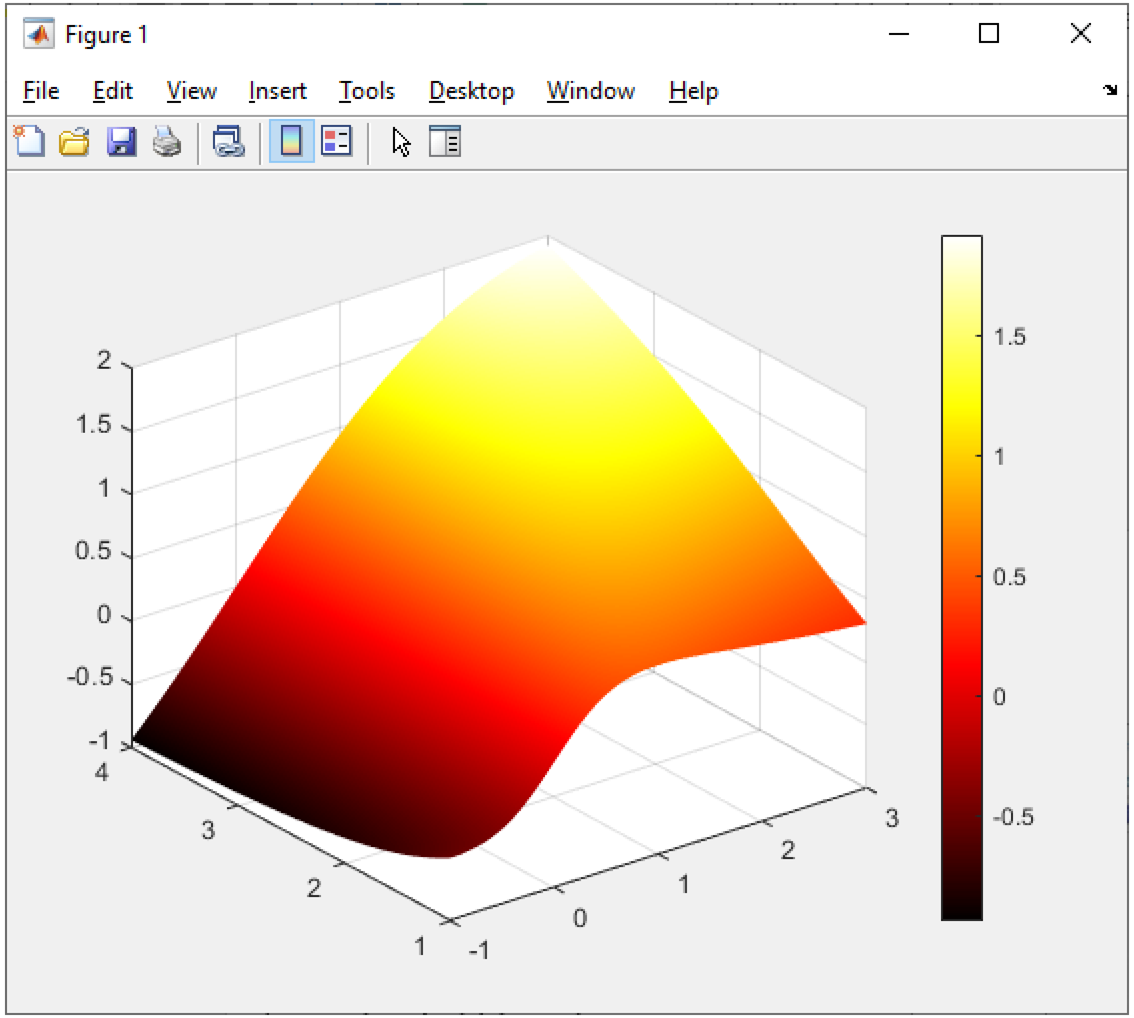
surf

```
x = -1:3;
y = 1:4;
[xx, yy] = meshgrid(x, y);

zz = xx.*yy.^2./(xx.^2+yy.^2);

surf(xx, yy, zz)
```

$$z = \frac{xy^2}{x^2 + y^2}$$

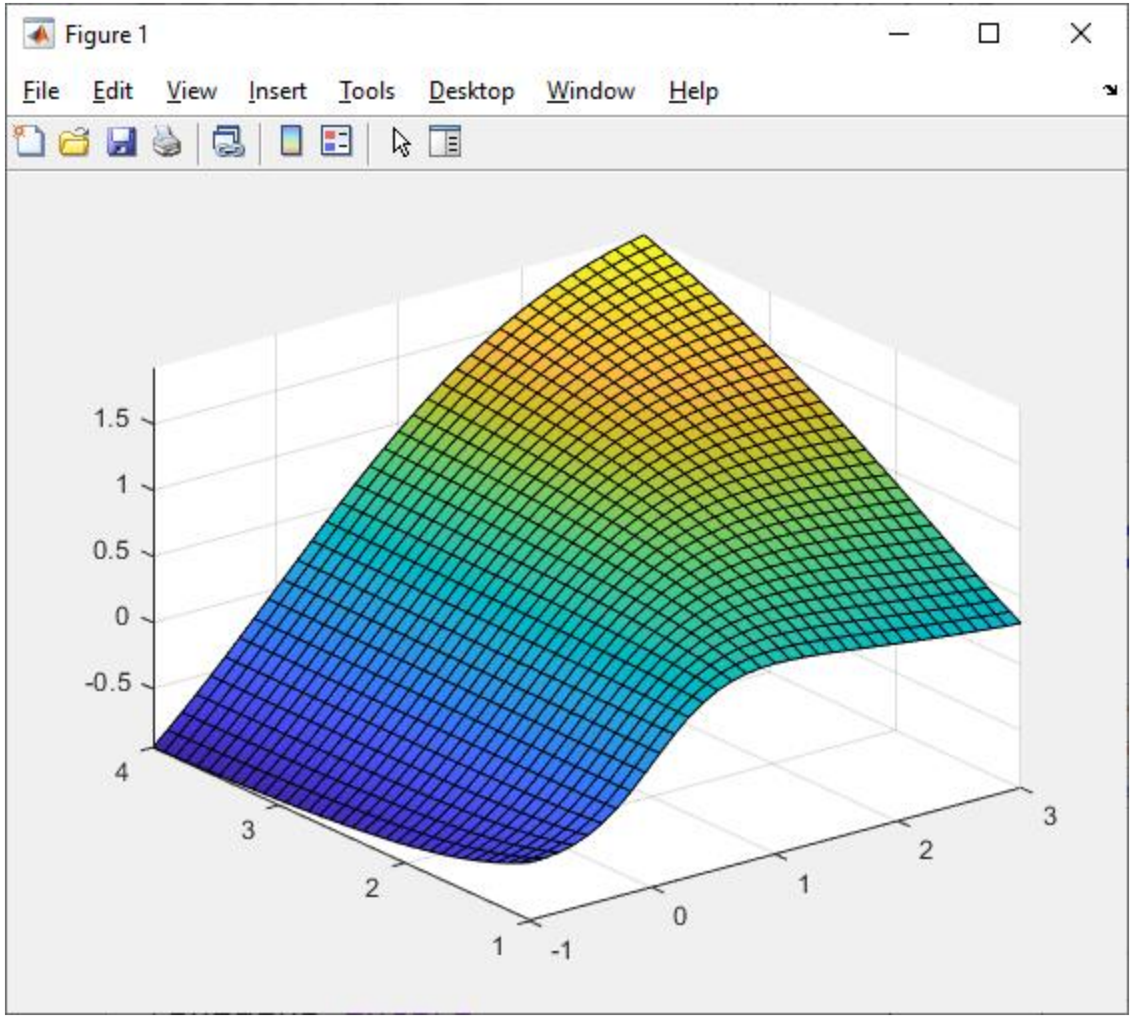
Drawing a smooth surface / colormap

```
x = linspace(-1, 3);
y = linspace(1, 4);
[xx, yy] = meshgrid(x, y);
zz = xx.*yy.^2./(xx.^2+yy.^2);

surf(xx, yy, zz)
shading interp

colormap hot
colorbar
```

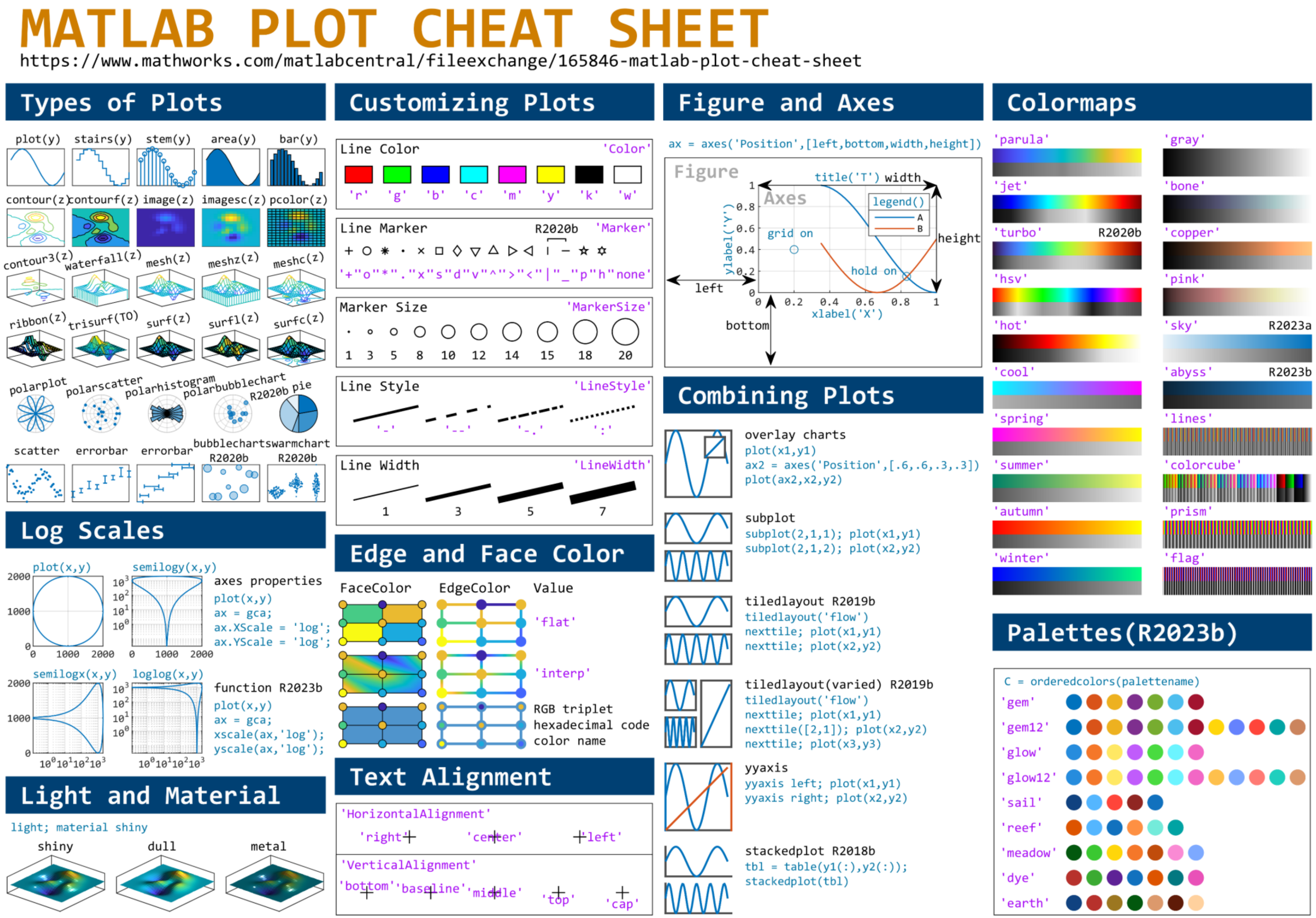


fsurf

```
fsurf(@(x, y) x.*y.^2./ ...
      (x.^2 + y.^2), ...
      [-1, 3, 1, 4])
```



Matlab Plot Cheat Sheet




QUESTIONS?