- **Theorem 6.14**: Let G be a *connected planar graph* with n vertices, m edges, and r faces. Then, $n-m+r=2$.
- **Proof**: (by induction on m).
  - **Base case**: When $m=0$, G is a trivial graph and $n-0+r=2$ holds.
  - **Inductive Hypothesis:** Assume the formula holds for all graphs with $m=k$ edges, i.e., $n-k+r=2$.
  - **Inductive step**: We now prove it for $m=k+1$ by considering two cases:

  ①If G contains *no cycle*, then G must have a vertex v of degree 1. By deleting v and its incident edge, we get a new graph G', which is connected, has $n-1$ vertices, k edges, and r faces. By the inductive hypothesis, $(n-1)-k+r=2$, so $n-(k+1)+r=2$, which proves the conclusion for $m=k+1$.

  ②If G contains a *cycle*, we can delete one edge from the cycle, resulting in a new graph G' with n vertices, k edges, and $r-1$ faces. Again, by the inductive hypothesis, $n-k+(r-1)=2$, so $n-(k+1)+r=2$, proving the conclusion for $m=k+1$.
- Thus, by induction, the theorem holds.

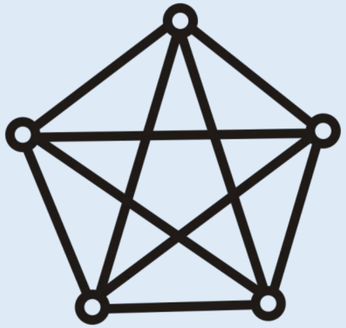
↳Corollary: Euler's Formula for Disconnected Planar Graphs

- **Corollary**: Let G be a planar graph with p connected components ($p \geq 2$). Then, $n-m+r=p+1$ where n, m, and r are the number of vertices, edges, and faces of G, respectively.
- **Proof**: Let the ***i-th*** connected component have n_i vertices, m_i edges, and r_i faces.
  - By Euler's formula for each connected component, we have: $n_i - m_i + r_i = 2$, $i = 1, 2, \ldots, p$.
  - Summing these equations gives: $(n_1 + n_2 + \cdots + n_p) - (m_1 + m_2 + \cdots + m_p) + (r_1 + r_2 + \cdots + r_p) = 2p$.
  - Note that the total number of faces $r = r_1 + \ldots + r_p - p + 1$, so we obtain: $n-m+r=p+1$.
- Thus, the *corollary is proven*.

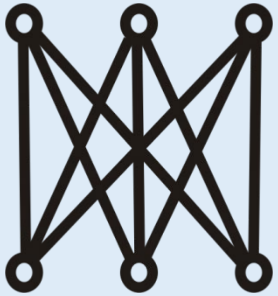
- **Theorem 6.15**: Let ***G*** be a *connected planar graph* with ***n*** vertices and ***m*** edges, where the degree of each face is at least ***l*** **(*l* ≥3)**, Then $m \le \frac{l}{l-2}(n-2)$.
- **Proof**: In the planar graph ***G***, the sum of the degrees of all faces is **2*m***. Let the number of faces be ***r***.
  - Since the degree of each face is at least ***l***, we have **r·*l*≤2*m***, By Euler's formula, ***n−m+r*=2**, thus ***r* =2+*m-n*** , Substituting this into the inequality: $2m \ge l(2+m-n)$, $2m - lm \ge 2l - ln$, $m(2-l) \ge 2l - ln$.
  - Since $l{\ge}3, 2{-}l{<}0$, thus dividing by $2-l$ reverses the inequality: $m \le \frac{l}{l-2}(n-2)$.
- Thus, the inequality is proven.

↳Edge Bound for Connected Planar Graphs（e.g.）

- **Example:** Prove that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar graphs.



- **Proof**: We use proof by contradiction. Assume that they are planar graphs.
  - For K_5 : n=5, m=10, l=3， Assuming the graph satisfies Theorem 6.15 , $m \le \frac{l}{l-2}(n-2)$， **10 <= 9.**
  - For $K_{3,3}$: n=6, m=9, l=4 ， Similarly, we get: **9 <= 8.**
  - This also leads to a contradiction. Therefore, the assumption is incorrect, meaning K_5 **and** $K_{3,3}$ are ***not planar graphs***.

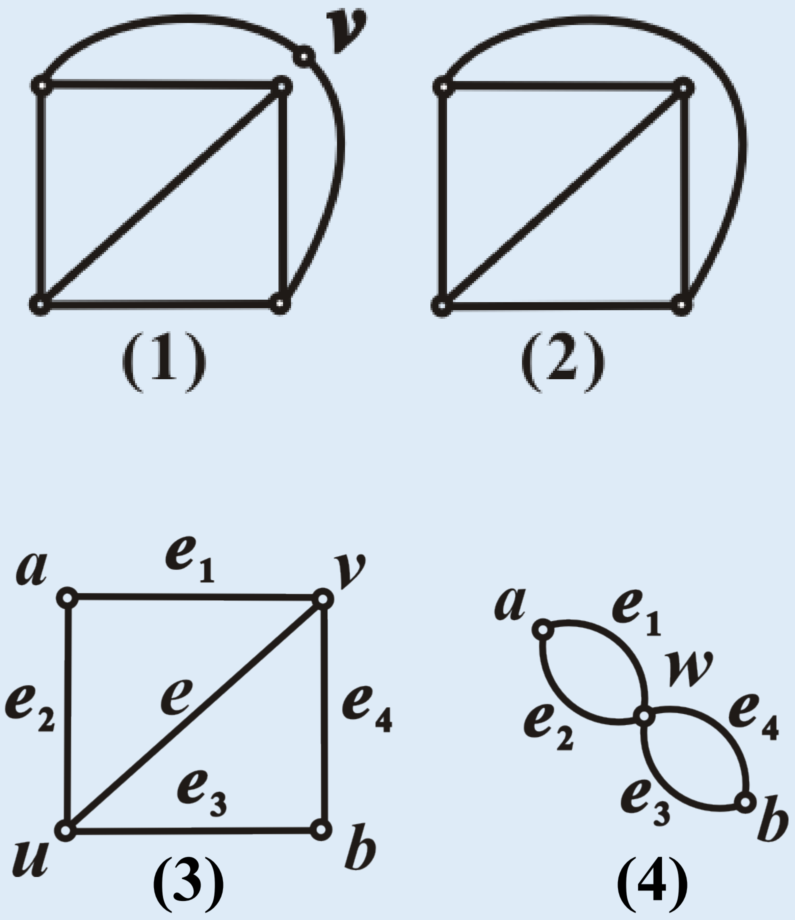


- **Note**： $K_{3,3}$ there are no simple cycles of length 1 or 2. Any closed path must pass through an even number of edges, so each face is surrounded by at least 4 boundary edges, which means the degree of each face l ≥4。

- ***Homotopy***: Homotopy focuses on the isomorphism between two graphs after inserting or **removing 2-degree vertices**.
  - Homotopy helps in understanding whether two different graphs are "***essentially the same***" and aids in recognizing the fundamental similarities or equivalences between different structures.
  - *Homotopy transformations* are typically a concept in topology, and graph transformations are considered homotopy transformations in the graph's topological structure (such as inserting or removing 2-degree vertices).

- *Contraction*: Contraction simplifies a graph by removing an edge and replacing the two original vertices with a new vertex.
  - Contraction helps reduce the complexity of a problem, making it easier to analyze and solve. It can assist in solving complex optimization problems such as finding the minimum cut, network flow, and graph coloring.
  - Contraction is one of the graph ***transformation operations*** that simplifies a graph by merging edges while maintaining its topological structure.

- *Delete a 2-degree vertex* **v**: As shown, from **(1) to (2).**
- *Insert a 2-degree vertex* **v** : As shown, from **(2) to (1).**
- G_1 **and** G_2 **are** *homotopic*: G_1 and G_2 are **isomorphic**, or they become isomorphic after repeatedly **inserting or removing 2-degree vertices.**
- *Edge contraction* ***e***: As shown, from**(3) to (4)**



- **Theorem 6.16**: A graph is planar if and only if it contains neither a subgraph **homeomorphic to K_5** nor a subgraph **homeomorphic to $K_{3,3}$** .
- **Theorem 6.17**: A graph is planar if and only if it contains neither a subgraph that can be **contracted to K_5** nor a subgraph that can be **contracted to $K_{3,3}$**.
- **Note:** K_5 (the complete graph with five vertices) and $K_{3,3}$ (the complete bipartite graph with two sets of three vertices) are **typical non-planar graphs**.

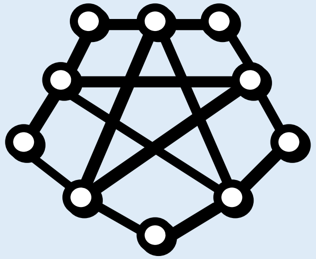
■ **Explanation:**

①A subgraph homeomorphic to K_5 or $K_{3,3}$ refers to a graph obtained by performing ***Homotopy*** transformations (adding or deleting 2-degree vertices) on K_5 or $K_{3,3}$. These *transformations do not change the non-planarity or bipartiteness* of K_5 or $K_{3,3}$.

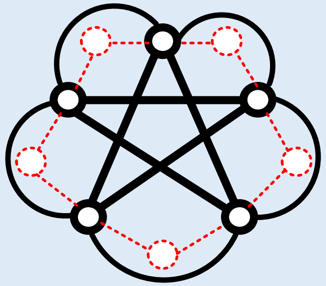
②Theorem 6.17 emphasizes that if, after any ***edge contraction*** (deleting edges, merging vertices), the graph ***cannot be simplified to*** K_5 ***or*** $K_{3,3}$, then the graph is planar.

③Homotopy focuses on ***edge subdivision*** (inserting a 2-degree vertex) and the removal of 2-degree vertices, while contraction focuses on ***edge merging*** and vertex merging. These two operations are equivalent when determining the planarity of a graph.

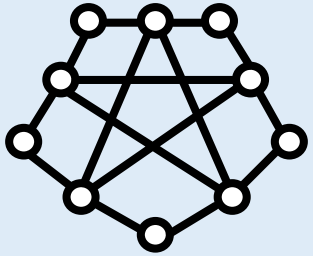
- **Example:** Prove that the following graph is non-planar.



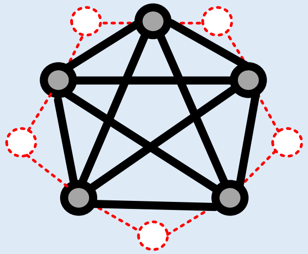
Removing five 2-degree vertices results in K_5 ⇒



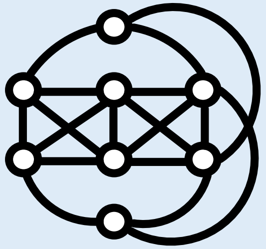
⇒ **Homeomorphic to K_5** , a **non-planar graph.**



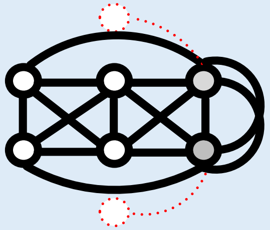
Contracting 5 edges results in K_5 ⇒



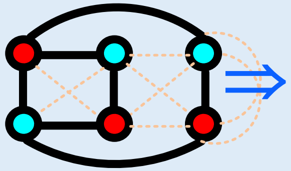
⇒ **Contracting to K_5** , a **non-planar graph.**



Contracting 2 edges ⇒



Extract $K_{3,3}$ subgraph ⇒



⇒ A **non-planar graph** that is **homeomorphic to $K_{3,3}$**

Definition 6.14:

Let G be a planar graph with n vertices, m edges, and r faces. The *dual graph* $G^*=<V^*,E^*>$ is constructed as follows:

- For each face R_i of G, choose an arbitrary point v_i^* within R_i to serve as a vertex of G^*, $V^*=\{ v_i^* \mid i=1,2,\ldots,r \}$.
- For each edge e_k in G:

  If e_k lies *on the common boundary* of faces R_i and R_j, create an edge $e_k^*=(v_i^*,v_j^*)$, in G*, such that e_k^* intersects e_k .

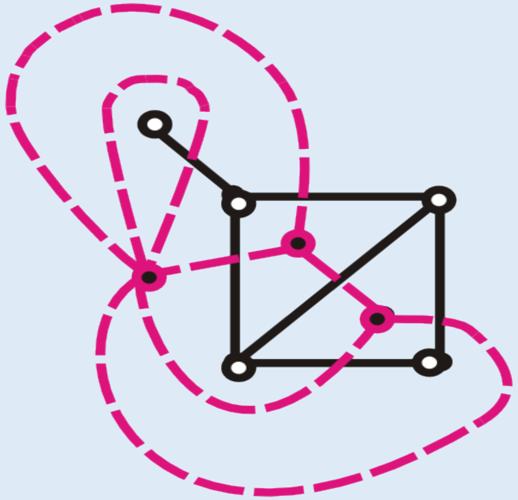
  If e_k lies only *on the boundary of a single face* R_i, create a loop $e_k^*=(v_i^*,v_i^*)$. $E^*=\{ e_k^* \mid k=1,2, \ldots,m \}$.

↳ Planar Graph ⇒ Dual Graph: Lost Information

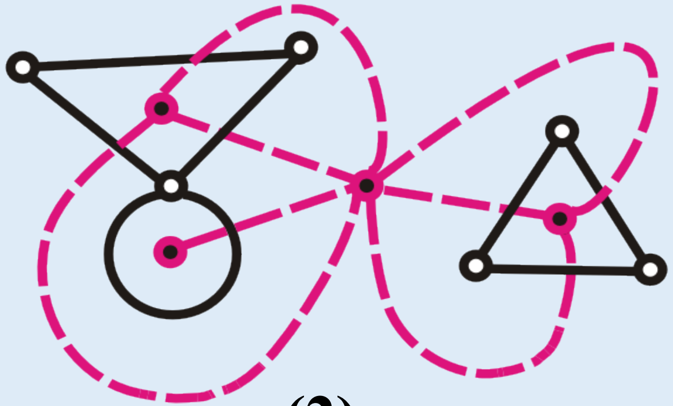
- **Details of the planar graph *lost* in the dual graph:**
  - **Original layout of vertices and edges:**
    In the dual graph, faces of the original planar graph become vertices, and adjacency between faces becomes edges. However, the original layout, including the positions, placements, and relative distances of vertices and edges, is no longer directly preserved in the dual graph。
  - **Vertex degrees and edge connections (such as edge crossings or winding patterns):**
    The degree of vertices and the specific ways edges connect (e.g., crossings or how edges wrap around certain regions) are also not directly reflected in the dual graph.

- **Properties of the planar graph *preserved* in the dual graph :**
  - **Connectivity**
  - **Cycles and cut sets:** Cycles in the original graph correspond to cut sets in the dual graph.
  - **Planarity**
  - **Satisfaction of the same Euler's formula as the original graph.**
  - **The original graph and its dual have the same number of edges**

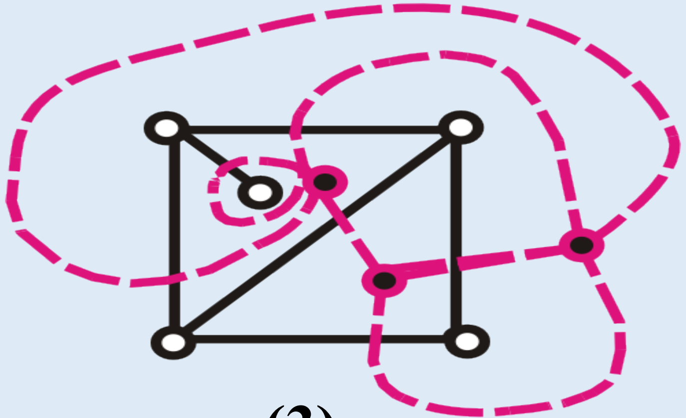
- **Example:** The black solid lines represent the original planar graph, and the red dashed lines represent its dual graph.



(1)



(2)



(3)

↳ Properties of the Dual Graph of a Planar Graph

- The dual graph ***G****** is a planar graph and a planar embedding.
- The dual graph ***G****** is ***connected***.
- If an edge ***e*** forms a ***cycle*** in ***G***, then the corresponding edge ***e****** in ***G****** is a ***cut-edge*** (bridge); if ***e*** is a bridge in ***G***, then the corresponding edge ***e****** in ***G****** forms a cycle.
- The dual graphs of isomorphic planar graphs are ***not necessarily isomorphic***.

  For example, in the previous illustration, planar graphs (1) and (3) are isomorphic, but their dual graphs are not isomorphic.

- **Theorem 6.18:** Let G^* be the dual graph of a connected planar graph G, n^*, m^*, r^* and n, m, r denote the number of vertices, edges, and faces of G^* and G, respectively.

(1) $n^* = r$

(2) $m^* = m$

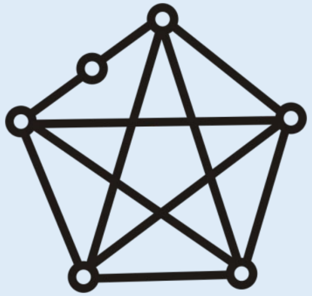
(3) $r^* = n$

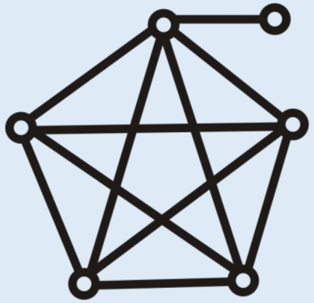
(4) If the vertex v_i^* of G^* lies in the face R_i of G, then $d(v_i^*) = \deg(R_i)$.

- **Example:** Draw all non-isomorphic simple connected non-planar graphs with 6 vertices and 11 edges.

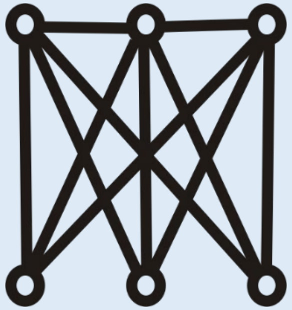
- **Solution:**

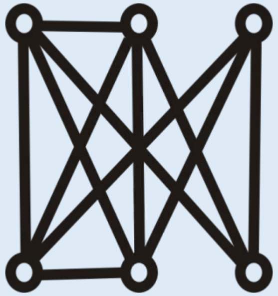
  (1) Add one vertex and one edge to K_5 (the complete graph with 5 vertices and 10 edges).





  (2) Add two edges to $K_{3,3}$ (the complete bipartite graph with 6 vertices and 9 edges).

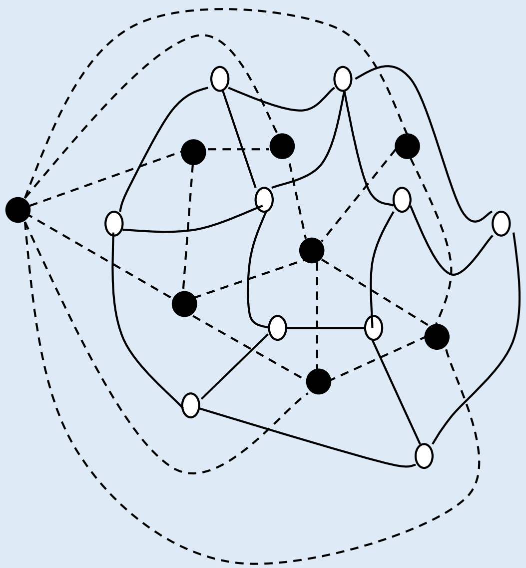


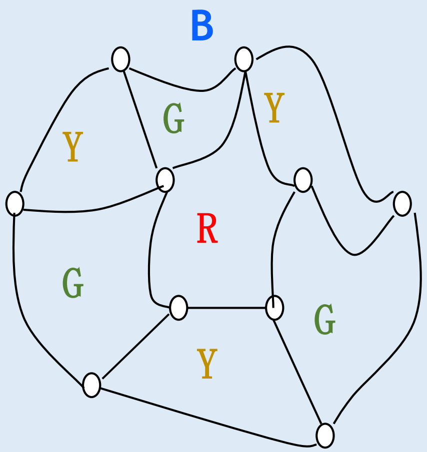


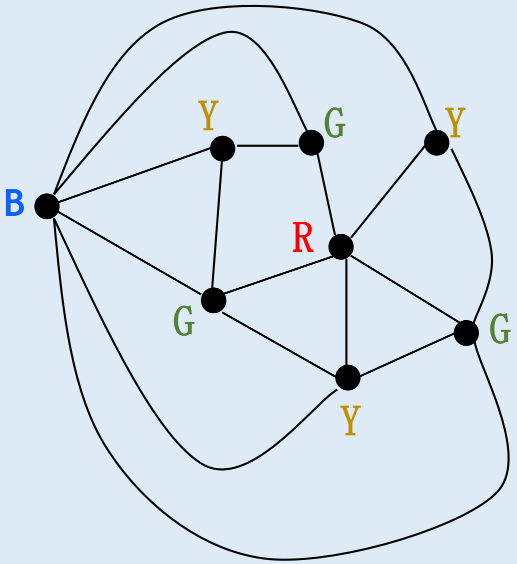
- The core objective of ***graph coloring problems*** is to avoid using the same color for adjacent or related elements (such as vertices, edges, or faces) under specific constraints, while using as few colors as possible.
- The *Four ColorTheorem* applies to the face coloring of planar graphs, whereas vertex coloring and edge coloring follow different rules and theoretical frameworks.
- *Four Color Theorem*: For any planar graph, it is possible to color all its faces using no more than four colors, in such a way that any two faces sharing a common boundary do not have the same color (i.e., ***every planar graph is 4-face-colorable***).
- The ***map coloring problem*** can be regarded as a specific instance of the face coloring problem for planar graph.

- **Map:** A planar embedding of a connected, ***bridgeless planar graph***, where each face represents a country. Two countries are said to be adjacent if they share a common boundary.
- **Map coloring (face coloring):** Assign a color to each country on the map such that ***adjacent countries receive different colors***.
- **Map coloring problem:** Color the map using ***as few colors as possible***.
- Map coloring can be ***transformed into the vertex coloring of a planar graph***. When G has no bridges, its dual graph G^* has no loops. Faces of G correspond to vertices of G^*, and two faces of G are adjacent if and only if the corresponding vertices in G^* are adjacent. Thus, ***face coloring of G is equivalent to vertex coloring of G^****.

- **Example:** Map Coloring and Vertex Coloring of Planar Graphs.

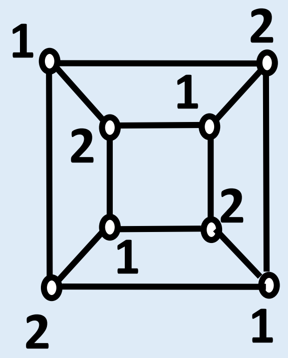


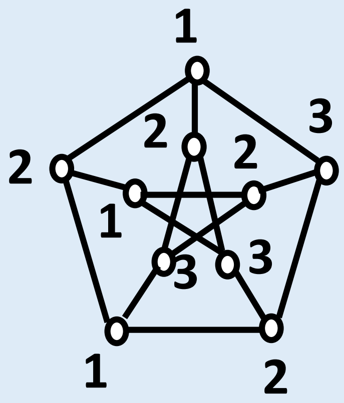


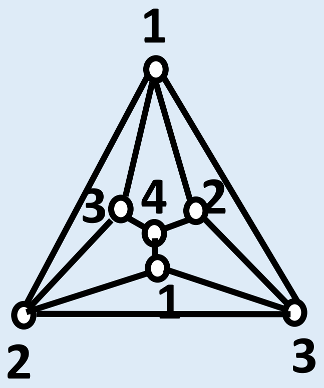


↳ Graph coloring (e.g.)

- **Example:** Provide a coloring using as few colors as possible.







- **Example：** A program has six variables x_i for i=1,2,…,6, where the following pairs of variables need to be used simultaneously: x_1 with x_4, x_1 with x_5 , x_2 with x_5 , x_2 with x_6 , x_3 with x_4, x_3 with x_6, x_4 with x_5 , and x_5 with x_6. Assign each variable to a register. Variables that need to be used simultaneously cannot be assigned to the same register. **Question:** What is the minimum number of registers needed? How should the variables be assigned?

- **Solution:**

  ① The problem is ***transformed into a vertex coloring problem*** of a graph: each variable x_i is treated as a vertex, and the "simultaneous usage" relationship between variables indicates the presence of an edge between the corresponding vertices.

■ **Solution:**

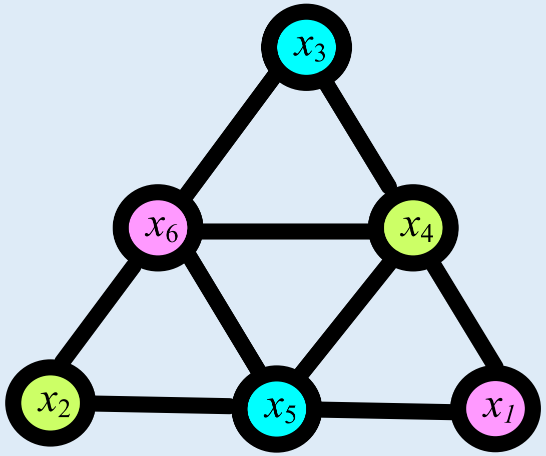
① The problem is ***transformed into a vertex coloring problem*** of a graph: each variable $\boldsymbol{x_i}$ is treated as a vertex, and the "simultaneous usage" relationship between variables indicates the presence of an edge between the corresponding vertices.

② ***Construct the graph*** by defining the vertex set and the edge set.

③ Build the graph and apply the principle of ***vertex coloring*** to determine the **chromatic number** (the minimum number of colors needed), ensuring that adjacent vertices are assigned different colors.

④ Based on the chromatic number, determine the ***minimum number of registers required*** and the corresponding assignment scheme.

- **Result:**



The register allocation scheme using three registers is as follows:

Register 0: Assigned to variables $\mathbf{x_4}$ and $\mathbf{x_2}$.

Register 1: Assigned to variables $\mathbf{x_5}$ and $\mathbf{x_3}$.

Register 2: Assigned to variables $\mathbf{x_6}$ and $\mathbf{x_1}$.

- **Four Color Conjecture (1850s)**

  → **Five Color Theorem (Heawood, 1890)**

  → **Four Color Theorem (Appel and Haken, 1976)**
- **Theorem (Four Color Theorem):** ***Every planar graph is 4-colorable***.
- The Four Color Theorem ***guarantees the existence of a four-coloring scheme for any planar graph***, but finding a specific coloring usually relies on concrete algorithms and techniques.
- Common ***coloring algorithms*** include greedy algorithms, backtracking algorithms, and heuristic search methods such as simulated annealing and genetic algorithms.

Objective：

Key Concepts：