Mathematical definitions of the basic properties of the relation ***R*** on the set ***A***:

Let ***R*** be a relation on ***A***, then

(1) ***R*** is ***reflexive*** on ***A if and only if*** $I_A \subseteq R$

*R* at least contains all reflexive pairs (x,x)

(2) ***R*** is ***irreflexive*** on ***A if and only if*** $R \cap I_A = \varnothing$

R does not contain any (x,x)

(3) ***R*** is ***symmetric*** on ***A if and only if*** $R = R^{-1}$

(4) ***R*** is ***antisymmetric*** on ***A if and only if*** $R \cap R^{-1} \subseteq I_A$

R and its inverse relation contains only reflexive pairs ⟨x,x⟩

(5) ***R*** is ***transitive*** on ***A if and only if*** $R \circ R \subseteq R$

## ↳ Proof of Reflexivity of *R* on *A*

- To prove that $R$ is reflexive on $A$:
  - **Proof Pattern:**

    **For any $x$,**

    $x \in A \Rightarrow$ ……… …………….. ……. $\Rightarrow$ $<x, x> \in R$

    **Assume** easoning process Conclusion

e.g. **Example**: Prove that if $I_A \subseteq R$， then $R$ is reflexive on $A$.

**Proof**: **For any $x$,**

$$x \in A \Rightarrow <x, x> \in I_A \Rightarrow <x, x> \in R$$

Therefore, $R$ is reflexive on $A$.

- To prove that R is symmetric on A:
  - **Proof Pattern:**
    **For any <$x$, $y$>**

$$<x, y>\in R \Rightarrow \ldots\ldots \quad \ldots\ldots \ \ldots\ldots \quad \ldots\ldots \Rightarrow <y, x>\in R$$

**Assumption** Reasoning process Conclusion

e.g. **Example: Prove that if $R=R^{-1}$ , then $R$ is symmetric on $A$.**

**Proof: Let<$x$, $y$>**

$$<x, y>\in R \Rightarrow <y, x>\in R^{-1} \Rightarrow <y, x>\in R$$

Therefore, $R$ is symmetric on $A$.

- To prove that R is antisymmetric on A:
  - **Proof Pattern:**

    **For any $<x, y>$**

    $<x, y>\in R \wedge <y, x>\in R \Rightarrow$ …… …… …… ……. $\Rightarrow x=y$

    Assumption　　　Reasoning process　　　Conclusion

e.g. **Example: Prove that if $R\cap R^{-1}\subseteq I_A$, *then R is* antisymmetric on *A*.**

**Proof:** **For any $<x, y>$**

$$<x, y>\in R \wedge <y, x>\in R \Rightarrow <x, y>\in R \wedge <x, y>\in R^{-1}$$

$$\Rightarrow <x, y>\in R\cap R^{-1} \Rightarrow <x, y>\in I_A \Rightarrow x=y$$

Therefore, *R* is antisymmetric on *A*.

- To prove that R is transitive on *A*:
  - **Proof Pattern:**

**For any $<x, y>$, $<y, z>$**

$$<x, y>\in R \land <y, z>\in R \Rightarrow \ldots\ldots\ldots\ldots \Rightarrow <x, z>\in R$$

Assumption Reasoning process Conclusion

e.g. **Example 7**: Prove that if $R \circ R \subseteq R$ , **then** $R$ is transitive on $A$.

**Proof: Let** $<x, y>$, $<y, z>$

$$<x, y>\in R \land <y, z>\in R \Rightarrow <x, z>\in R\circ R \Rightarrow <x, z>\in R$$

Therefore, $R$ is transitive on $A$.

# ↳ Comparison Table of Properties of Relation *R*

| Relation Properties | Expression | Definition | Relation Matrix | Relation Diagram |
|---|---|---|---|---|
| Reflexivity | $I_A \subseteq R$ | $\forall x \in A, \exists \langle x,x \rangle \in R$ | Main diagonal elements are 1 | Every vertex has a loop |
| Irreflexivity | $R \cap I_A = \varnothing$ | $\forall x \in A, \exists \langle x,x \rangle \notin R$ | Main diagonal elements are 0 | No loops at any vertex |
| Symmetry | $R = R^{-1}$ | If $\langle x,y \rangle \in R$, then $\langle y,x \rangle \in R$ | The matrix is a symmetric matrix | If there is an edge between two vertices, it must be a directed edge (no undirected edge) |

# ↳ Comparison Table of Properties of Relation *R* (cont.)

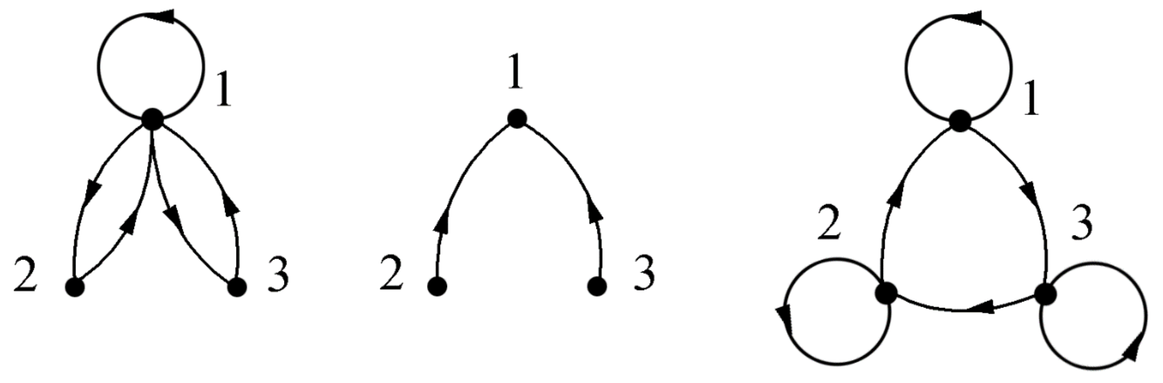


| Relation Properties | Expression | Definition | Relation Matrix | Relation Diagram |
|---|---|---|---|---|
| Antisymmetry | $R \cap R^{-1} \subseteq I_A$ | If Expression $<x,y> \in R$ and $x \neq y$, then $<y,x> \notin R$ | If $r_{ij}=1$, and $i \neq j$, Then $r_{ji}=0$ | If there is an edge between two points, it must be a directed edge (no bidirectional edges) |
| Transitivity | $R \circ R \subseteq R$ | If $<x,y> \in R$ and $<y,z> \in R$, then $<x,z> \in R$ | $M_{ij}^2=1 \Rightarrow M_{ij}=1$, $\forall i,j$ | If there is an edge from vertex $x_i$ to $x_j$, and an edge from $x_j$ to $x_k$, then there is also an edge from $x_i$ to $x_k$. |

## ↳ Determine the Relation Based on the Relation Diagram (e.g.)



e.g. **Example 8:** Determine the properties of the relationship in the figure below and explain the reasoning.




(a) (b) (c)

(a) Neither reflexive nor antireflexive; symmetric, not antisymmetric; not transitive.

(b) Antireflexive, not reflexive; antisymmetric, not symmetric; transitive.

(c) Reflexive, not antireflexive; antisymmetric, not symmetric; not transitive.

## ↳ Relation Between Operations and Properties



| | Reflexivity | Irreflexivity | Symmetry | Antisymmetry | Transitivity |
|---|---|---|---|---|---|
| $R_1^{-1}$ | √ | √ | √ | √ | √ |
| $R_1 \cap R_2$ | √ | √ | √ | √ | √ |
| $R_1 \cup R_2$ | √ | √ | √ | × | × |
| $R_1 - R_2$ | × | √ | √ | √ | × |
| $R_1 \circ R_2$ | √ | × | × | × | × |

- 4.3.1 Definition and Determination of Relation Properties
  - Reflexivity and Irreflexivity
  - Symmetry and Antisymmetry
  - Transitivity
- 4.3.2 Closure of Relations
  - Definition of Closure
  - Closure Calculation
  - Warshall's Algorithm

- **Definition 4.17: $r(R)$, $s(R)$ and $t(R)$**

Let $R$ be a relation on a non-empty set $A$. The **reflexive (symmetric or transitive) closure** of $R$ is a relation $R'$ *on* $A$, *such that* $R'$ satisfies the following conditions:

- $R'$ is reflexive (symmetric or transitive).
- $R \subseteq R'$
- For any reflexive (symmetric or transitive) relation $R''$ **on A that contains $R$, we have $R' \subseteq R''$ .**

The reflexive closure of $R$ is usually denoted by $r(R)$, the symmetric closure by $s(R)$, and the transitive closure by $t(R)$.

## ↳ Construction of the Transitive Closure of Relation *R*

- For a relation $R$ on a non-empty set $A$, the reflexive closure $r(R)$, symmetric closure $s(R)$, and transitive closure $t(R)$ can be constructed.
- The ***reflexive closure*** $R'$ of $R$ is a relation obtained by adding all necessary pairs to ensure reflexivity, and it is the smallest superset. It can be defined as: $R'=R\cup\{(a,a)\mid a\in A\}$
- The ***symmetric closure*** $R'$ of $R$ is a relation obtained by adding all necessary pairs to ensure symmetry, and it is the smallest superset. It can be defined as: $R'=R\cup\{(b,a)\mid(a,b)\in R\}$

## ↳ Construction of the Transitive Closure of Relation $R$



- The ***transitive closure*** $\boldsymbol{R'}$ of $\boldsymbol{R}$ is a relation obtained by adding all necessary pairs to ensure transitivity, and it is the smallest superset. It can be defined as:

For each pair of elements a,c∈A, if there exist one or more elements $b_1,b_2,\ldots,b_n$ such that $(a,b_1),(b_1,b_2),\ldots,(b_{n-1},b_n),(b_n,c)$ are all in $R$ , then (a,c) should be in $R'$.

## ↳ Closure Theorem of Relations

- **Theorem 4.7:** Closure Theorem of Relations.

  Let $R$ be a relation on $A$, then we have:

  (1) $r(R)=R \cup R^0$

  (2) $s(R)=R \cup R^{-1}$

  (3) $t(R)=R \cup R^2 \cup R^3 \cup \ldots$

**Explanation:**

- For a finite set $A$ (**where** $|A|=n$) , the union in (3) will have at most $R^n$.
- *If* $R$ is reflexive, then $r(R)=R$; **If** $R$ is symmetric, then $s(R)=R$; **If** $R$ is transitive, then $t(R)=R$.

- **Proof of Theorem 4.7** (Proving (1)).
  - Proof of **(1)** $r(R)=R\cup R^0$ , It is sufficient to show that $R\cup R^0$ satisfies the closure definition.
  - Proof that $R\cup R^0$ is a reflexive relation

    Since $R\cup R^0$ contains $R$ , **and by** $I_A\subseteq R\cup R^0$, we can conclude that $R\cup R^0$ is reflexive on $A$.
  - Proof that $R\cup R^0$ is the smallest reflexive relation containing $R$ .

    We need to show that no reflexive relation smaller than $R\cup R^0$ exists that contains $R$.

    Assume $R'$ is a reflexive relation that contains $R$ *and* and is smaller than $R\cup R^0$ $I_A\subseteq R'$, $R\subseteq R'$, Therefore, we have $R\cup R^0=I_A\cup R\subseteq R'$ , which contradicts the assumption that $R'$ is smaller than $R\cup R^0$.

- **Proof of (3) $t(R)=R\cup R^2\cup R^3\cup\ldots$**
  - Consider arbitrary pairs **$<x,y>$ and $<y,z>$**

    $$<x,y>\in R\cup R^2\cup R^3\cup\ldots. \wedge <y,z>\in R\cup R^2\cup R^3\cup\ldots.$$

    $$\Rightarrow <x,z>\in R\cup R^2\cup R^3\cup\ldots.$$

    Therefore, by the transitivity of $R\cup R^2\cup R^3\cup\ldots$. **We have**

    $$t(R)\subseteq R\cup R^2\cup R^3\cup\ldots$$

  - Next, we prove by induction that $R^n\subseteq t(R)$.

    *For* $n$=1, the statement is obviously true. Assume it holds for $n$=$k$.

    *For any* $<x,y>$, **we have**

    $$<x,y>\in R^{k+1}\Rightarrow <x,y>\in R^k\circ R \Rightarrow \exists t\ (<x,t>\in R^k \wedge <t,y>\in R)$$

    $$\Rightarrow \exists t\ (<x,t>\in t(R)\wedge <t,y>\in t(R)) \Rightarrow <x,y>\in t(R) \quad (t(R) \text{ transitive})$$

    Thus, $R\cup R^2\cup R^3\cup\ldots\subseteq t(R)$

- Let the relation matrices of $R$, $r(R)$, $s(R)$, $t(R)$ be $M$, $M_r$, $M_s$ and $M_t$ , respectively. Then, we have:

$$M_r = M + E$$

$$M_s = M + M'$$

$$M_t = M + M^2 + M^3 + \ldots$$

- where $E$ is the identity matrix of the same order as $M$, and $M'$ is the transpose of $M$.

**Note:** In the above equations, the matrix elements are added using logical addition.

## ↳ Closure Operations on Relation Graphs

- Let the relation graphs of $R$, $r(R)$, $s(R)$, $t(R)$ be denoted by $G$, $G_r$, $G_s$, $G_t$ , respectively.

  **Then**, the vertex sets of $G_r$, $G_s$, $G_t$ are the same as the vertex set of $G$.
- In addition to the edges of $G$ , new edges are added in the following ways:
  - For each vertex in $G$, if there is no cycle, add a cycle. The resulting graph is $G_r$.
  - For each directed edge $x_i \rightarrow x_j$, **(with $i \neq j$)**, add a reverse edge $x_i \rightarrow x_i$ . The resulting graph is $G_s$.
  - For each vertex $x_i$ **in** $G$, examine all paths starting from $x_i$ , If there is no edge from $x_i$ to any node $x_j$ in the path, add the corresponding edge. After checking all vertices, the resulting graph is $G_t$ .

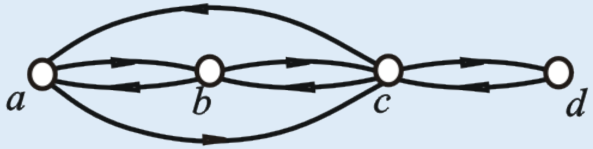
## ↳ Closure Operations on Relation Graphs (e.g.)



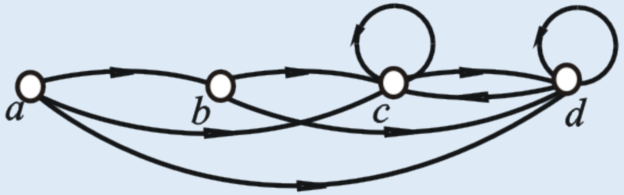
e.g. **Example:** Let $A=\{a,b,c,d\}$, $R=\{\langle a,b\rangle,\langle a,c\rangle,\langle b,c\rangle,\langle c,d\rangle,\langle d,c\rangle\}$, $R$ and $r(R)$, $s(R)$, $t(R)$ the relation graph is shown.


$R$


$r(R)$




$s(R)$




$t(R)$

## ↳ Warshall's Algorithm for Transitive Closure



- **Algorithm Idea:** Consider a sequence of matrices $M_0$, $M_1$, …, $M_n$ of size $n$+1, where the element in the i-th row and j-th column of matrix $M_k$ is denoted as $M_k[i,j]$. For k=0,1,…,n, $M_k[i,j]$=1 if and only if there exists a path from $x_i$ to $x_j$ in the relation graph of $R$ , and this path passes through only the vertices in $\{x_1, x_2, …, x_k\}$ except for the endpoints. It is easy to prove that M0M_0M0 is the relation matrix of $R$ , and $M_n$ corresponds to the transitive closure of $R$ .
- **Warshall Algorithm:** Starting from $M_0$, calculate $M_1$, $M_2$, …, until $M_n$ .

  **From $M_k[i, j]$ *to compute* $M_{k+1}[i, j]$: $i, j \in V$.**

  **The vertex set $V_1=\{1,2, …, k\}$, $V_2=\{k+2, …, n\}$, $V=V_1 \cup \{k+1\} \cup V_2$,**

  **$M_{k+1}[i,j]=1 \Leftrightarrow$ There exists a path *i to j* .**

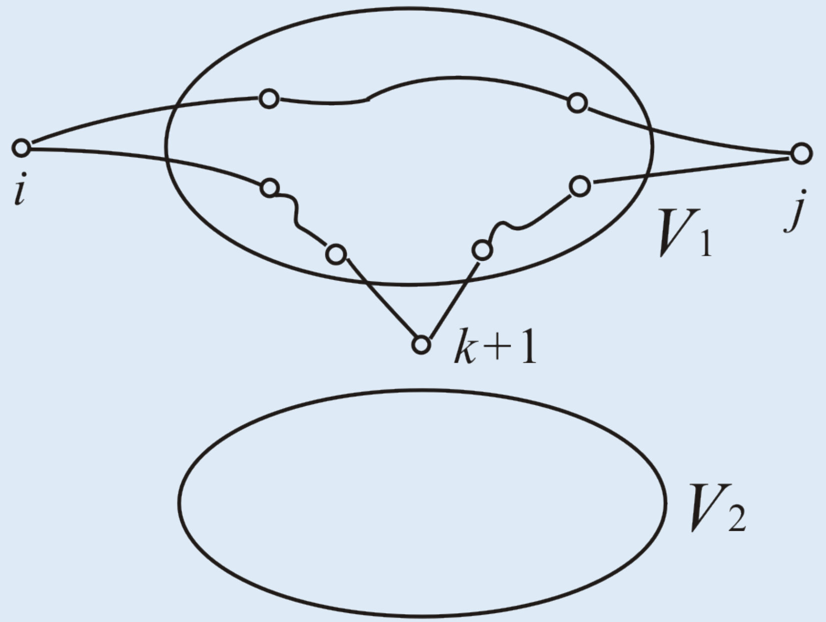
  that only passes through the points in $V_1 \cup \{k+1\}$.

- These paths are divided into two categories:
  - **Category 1:** Paths that only pass through the points in $V_1$
  - **Category 2:** Paths that pass through point $k+1$

  For **Category 1** paths: $M_k[i,j]=1$

  For **Category 2** paths:

  $$M_k[i,k+1]=1 \wedge M_k[k+1,j]=1$$



- **Algorithm 4.1: Warshall** Algorithm

Input：$M$ （relation matrix of $R$ ）

Output：$M_t$ （relation matrix of $t(R)$）

1. $M_t \leftarrow M$
2. for $k \leftarrow 1$ to $n$ do
3.     for $i \leftarrow 1$ to $n$ do
4.         for $j \leftarrow 1$ to $n$ do
5. $M_t[i, j] \leftarrow M_t[i, j] + M_t[i, k] \cdot M_t[k, j]$

Time Complexity: $T(n)=O(n^3)$

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**Objective：**

**Key Concepts ：**