- **6.4.1 Bipartite Graphs**

  Necessary and sufficient conditions for a graph to be bipartite

  Matching, maximal matching, maximum matching, complete matching, perfect matching

- **6.4.2 Eulerian Graphs**

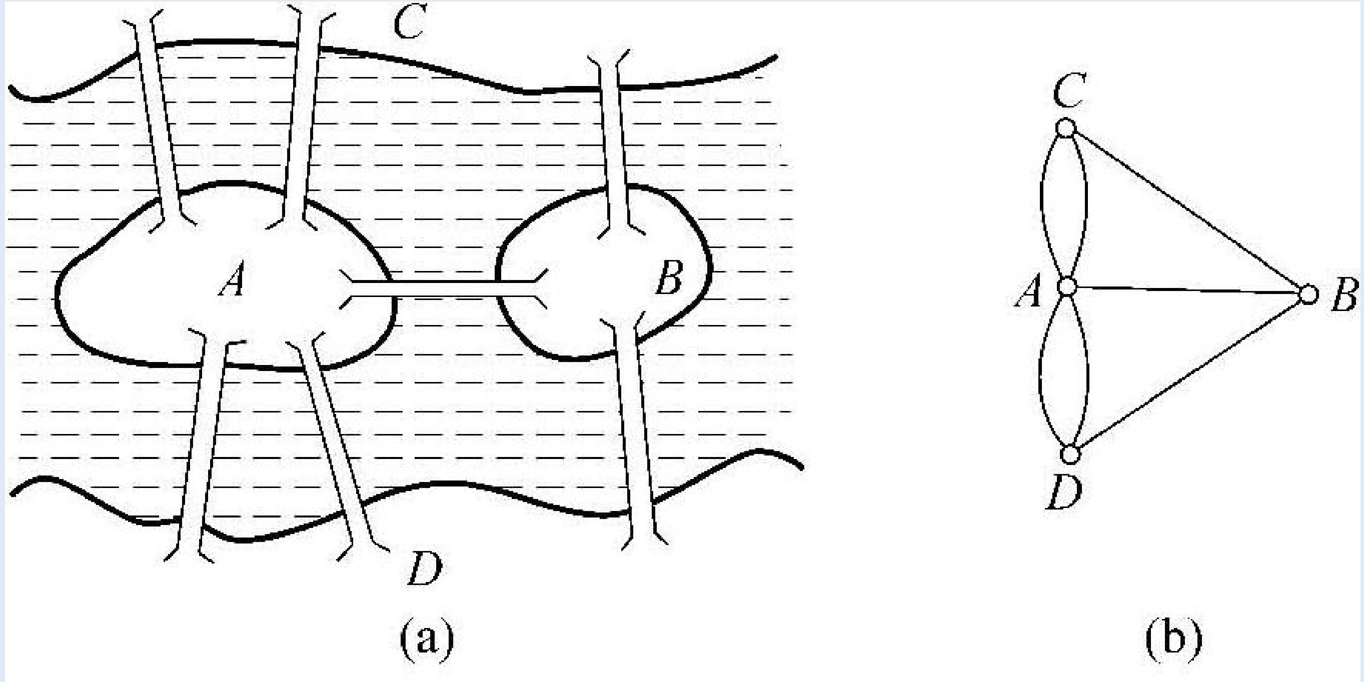
  Eulerian circuits (paths) and their necessary and sufficient conditions for existence

- **6.4.3 Hamiltonian Graphs**

  Hamiltonian circuits (paths) and the necessary and sufficient conditions for their existence

- **6.4.4 Planar Graphs**

Diagram of the Seven Bridges of Königsberg



↳ Eulerian Path (Circuit) and Eulerian Graph

- *Eulerian Path*: A path that passes through all vertices and **each edge exactly once**.
- *Eulerian Circuit*: A circuit that passes through all vertices and **each edge exactly once**.
- *Eulerian Graph*: A graph that contains an Eulerian circuit.
- **Notes:**
  - The above definitions apply to both **undirected** and **directed** graphs.
  - A **trivial graph** is considered an Eulerian graph.
  - An Eulerian path is a **simple path**, and an Eulerian circuit is a **simple circuit**.
  - **Loops (self-edges)** do **not affect** the Eulerian property of a graph.

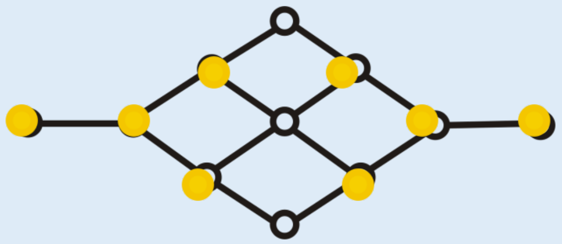
- **Theorem 6.8:**

  (1) An undirected graph ***G*** has an ***Eulerian circuit*** if and only if ***G*** is connected and has no vertices of odd degree.

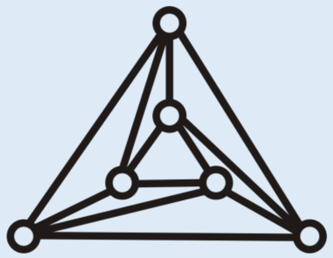
- (2) An undirected graph ***G*** has an ***Eulerian path*** but not an Eulerian circuit if and only if ***G*** is connected and has exactly two vertices of odd degree, with all other vertices having even degree. These two odd-degree vertices are the endpoints of every Eulerian path.

6.4.2 Eulerian Graphs

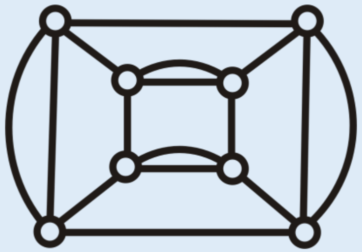
↳ Eulerian Graph Theorem (for undirected graphs)(e.g.)



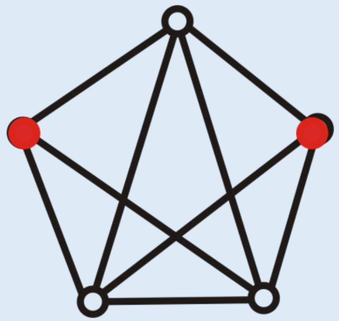
No Eulerian Path



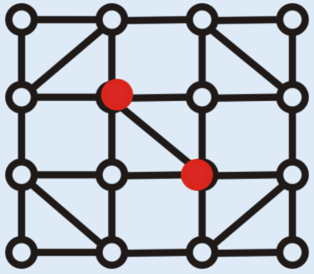
Eulerian Graph



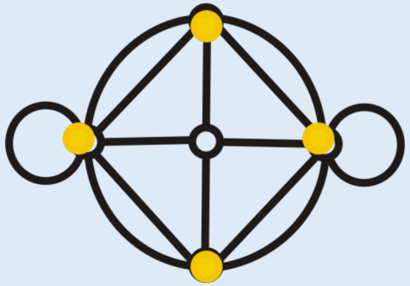
Eulerian Graph



Eulerian Path, not Eulerian Graph



Eulerian Path, not Eulerian Graph



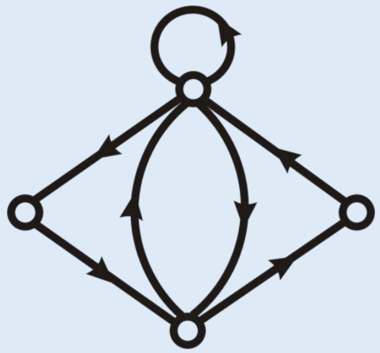
No Eulerian Path

- **Theorem 6.9:**

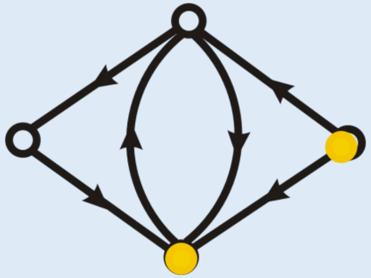
(1) A directed graph ***D*** has an ***Eulerian circuit*** if and only if ***D*** is connected and the **in-degree equals the out-degree** for every vertex.

(2) A directed graph D has an ***Eulerian path*** but not an Eulerian circuit if and only if ***D*** is connected and there is **one vertex** whose in-degree exceeds its out-degree by 1, and **one vertex** whose out-degree exceeds its in-degree by 1, with all other vertices having equal in-degree and out-degree.

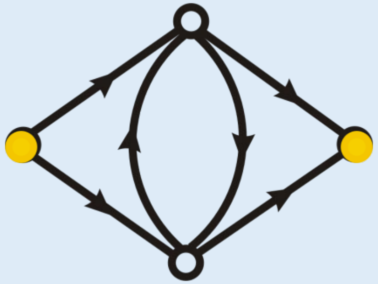
↳ Eulerian Graph Theorem (for directed graphs)(e.g.)



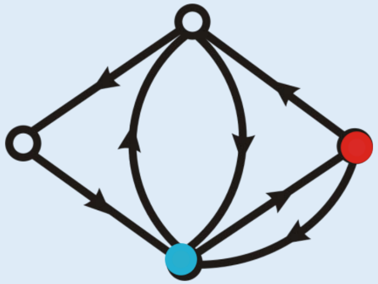
Eulerian Graph



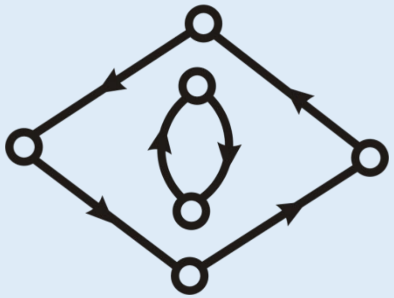
No Eulerian Path



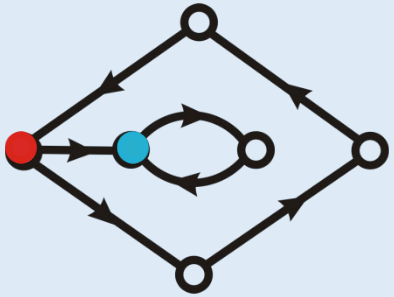
No Eulerian Path



Eulerian Path, not Circuit



No Eulerian Path



Eulerian Path, not Circuit

6.4 Several Special Types of Graphs

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- **6.4.1 Bipartite Graphs**

  Necessary and sufficient conditions for a graph to be bipartite

  Matching, maximal matching, maximum matching, complete matching, perfect matching

- **6.4.2 Eulerian Graphs**

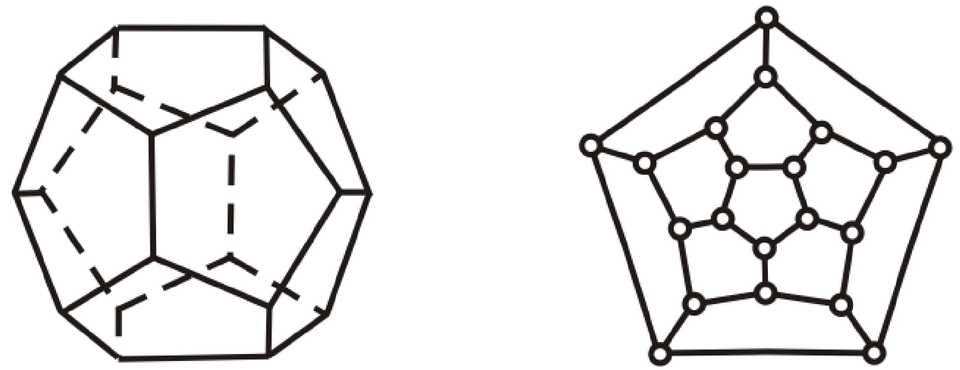
  Eulerian circuits (paths) and their necessary and sufficient conditions for existence

- **6.4.3 Hamiltonian Graphs**

  Hamiltonian circuits (paths) and the necessary and sufficient conditions for their existence

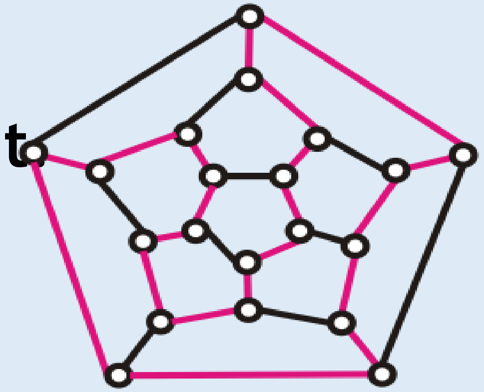
- **6.4.4 Planar Graphs**

W.Hamilton, 1859



↳ Hamiltonian Path (Circuit) and Hamiltonian Graph

- ***Hamiltonian Path***: A **path** that visits every vertex in the graph **exactly once.**
- ***Hamiltonian Circuit***: A **circuit** that visits every vertex in the graph **exactly once.**
- ***Hamiltonian Graph***: A graph that **contains a Hamiltonian circuit.**
- **Notes:**
  - A Hamiltonian path is a ***elementary*** **path.**
  - A Hamiltonian circuit is a ***elementary*** **circuit.**
  - A graph having a Hamiltonian path does **not necessarily** have a Hamiltonian circuit.
  - **Loops** and **parallel edges** do **not affect** the Hamiltonian property of a graph.



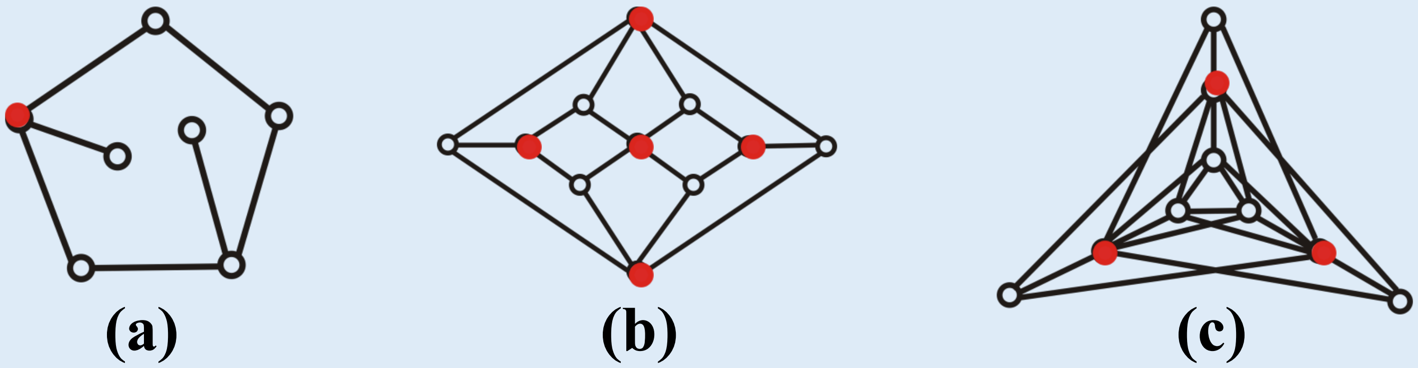
- **Theorem 6.10:** If an undirected graph $G=<V,E>$ is a Hamiltonian graph, then for any non-empty proper subset $V_1 \subset V$, The number of connected components in $G-V_1$ satisfies: $p(G-V_1) \leq |V_1|$.
- **Proof:**

  Let C be a Hamiltonian circuit in G. Then:p(C−V1)≤|V1|, **since** $p(C-V_1) \leq |V_1|$. And because $C \subseteq G$, Hence, $p(G-V_1) \leq p(C-V_1) \leq |V_1|$.

- **Corollary:**

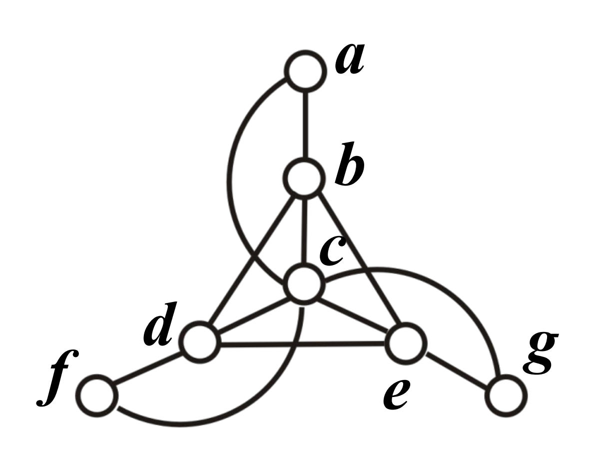
  A graph with a **cut vertex** is not a Hamiltonian graph.

- **Example:** Prove that each of the following graphs is not a Hamiltonian graph.



There exists a Hamiltonian path in (c).

- **Example:** Prove that the graph on the right is not a Hamiltonian graph.



- **Proof:**
  - Assume there exists a Hamiltonian circuit. ***a*,*f*,*g*** are the node of degree 2, edge **(*a*,*c*)**, **(*f*,*c*)** and **(*g*,*c*)** must all be included in the Hamiltonian circuit. As a result, vertex ***c*** would appear three times, which is a **contradiction**.
  - Moreover, the graph satisfies the condition of **Theorem 6.10**, which shows that the condition is **necessary but not sufficient**. The graph has a Hamiltonian path.

- **Theorem 6.11：**

  Let ***G*** be a simple undirected graph of order ***n*** (***n***≥3).

  - If the sum of the degrees of any two non-adjacent vertices is at least ***n*−1**, then ***G*** contains a ***Hamiltonian path***.
  - If the sum is at least ***n***, then ***G*** contains a ***Hamiltonian circuit***, i.e., ***G*** is a ***Hamiltonian graph***.

- **Corollary：**
  - Let ***G*** be a simple undirected graph of order ***n*** (***n*≥3**), **If** $\delta(G) \geq n/2$**,** then ***G*** is a Hamiltonian graph.
  - When ***n*≥3**, the complete graph K_n is Hamiltonian; when ***r*=*s*≥2**, the complete bipartite graph $K_{r,s}$ is Hamiltonian.

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- **Theorem 6.12:**
  Let ***D*** be a **directed graph** of order n (n≥2).
  If the underlying undirected graph (obtained by ignoring the directions of all edges) contains a subgraph K_n , then ***D*** contains a ***Hamiltonian path***.

- **Example:** There are 7 people:
  - A speaks English.
  - B speaks English and Chinese.
  - C speaks English, Italian, and Russian.
  - D speaks Japanese and Chinese.
  - E speaks German and Italian.
  - F speaks French, Japanese, and Russian.
  - G speaks French and German

Can they be seated around a round table so that **each person can communicate with both neighbors?**

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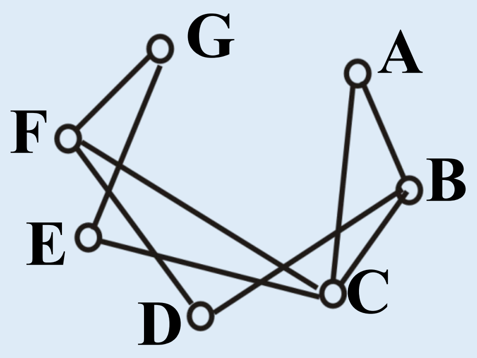
↳ Hamiltonian Circuits: Roundtable Seating for Communication

- **Solve:**

(1) Construct an undirected graph where each person is a vertex, and there is an edge between two people **if and only if** they speak a common language.



(2) ACEGFDBA is a *Hamiltonian circuit*; they can be seated in this order around the table.

6.4 Several Special Types of Graphs

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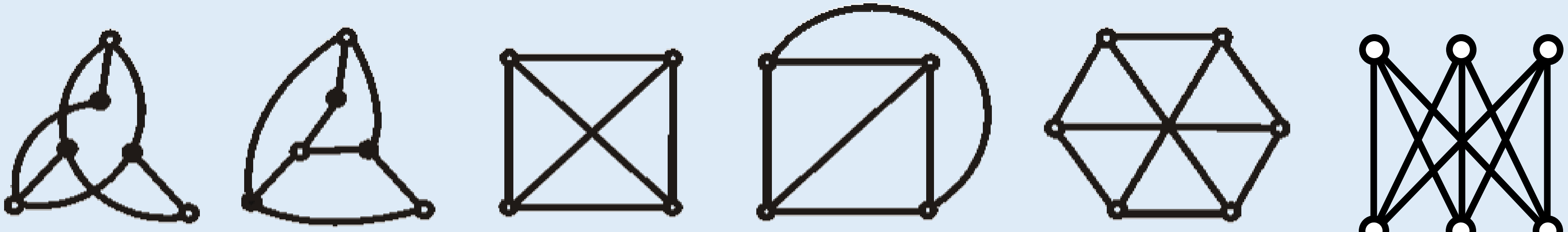
- **6.4.4 Planar Graphs**

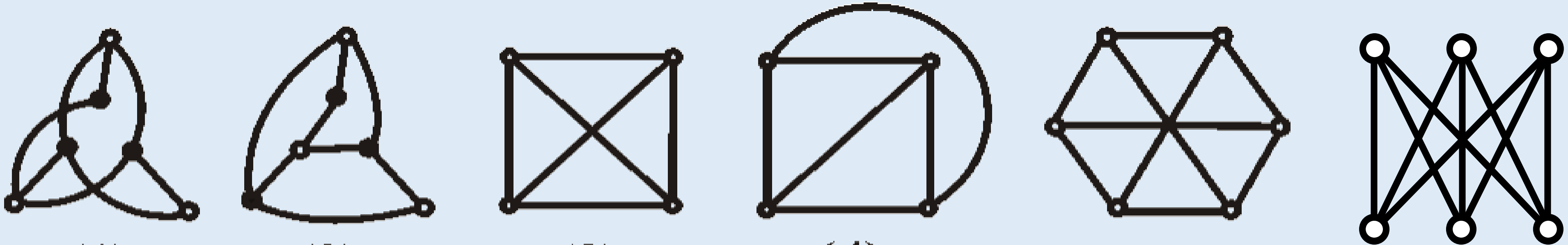
- **Definition 6.12:** A graph ***G*** is called a ***planar graph*** if it can be drawn in the plane such that its edges do not intersect except at the vertices. The drawing of the graph with no edge intersections is called a ***planar embedding*** of ***G***. A graph that does not have a planar embedding is called a ***non-planar graph***.
- **Example**: Determine whether the following graph is a planar graph.



(1) (2) (3) (4) (5) (6)

↳ Planar Graphs and Planar Embeddings(e.g.)

- **Example**: Determine whether the following graph is a planar graph.



(1) (2) (3) (4) (5) (6)

- **Solution:**
  - The graphs **(1)** to **(4)** are ***planar graphs***. **(2)** is a ***planar embedding*** of **(1)**, and **(4)** is a ***planar embedding*** of **(3)**.
  - **(5)** is the **complete graph** K_5, which is a typical ***non-planar grap***h.
  - **(6)** is the **complete bipartite graph** $K_{3,3}$, which is a typical ***non-planar graph***.

- Let G be a planar embedding.
  - ***Faces*** **of G**: Each region into which the plane is divided by the edges of G.
  - ***Infinite face*** **(outer face)**: The face with infinite area, denoted by R_0 .
  - ***Finite faces*** **(inner faces)**: Faces with finite areas, denoted by R_1, R_2,…, R_k.
  - ***Boundary*** **of face R_i** : The set of loops formed by the edges that enclose R_i.
  - ***Degree*** **of face R_i** : The length of the boundary of R_i, denoted by $\deg(R_i)$.
- **Note**: The **boundary** of a face may consist of simple **loops**, elementary **cycles**, or even more complex loops, and in some cases, it may be the union of disconnected loops.

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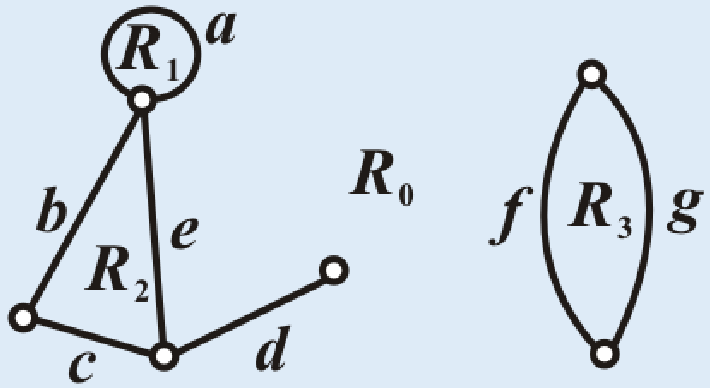
- **Example:** The diagram on the right has 4 faces.

R_1 Boundary: a

R_2 Boundary: bce

R_3 Boundary: fg

R_0 Boundary: $abcdde$, fg



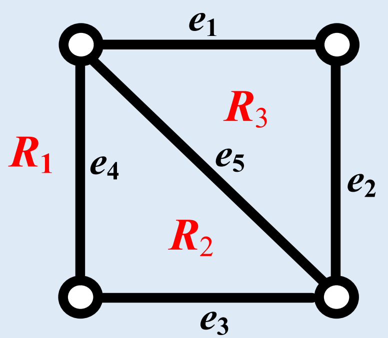
deg(R_1)= 1

deg(R_2)= 3

deg(R_3)= 2

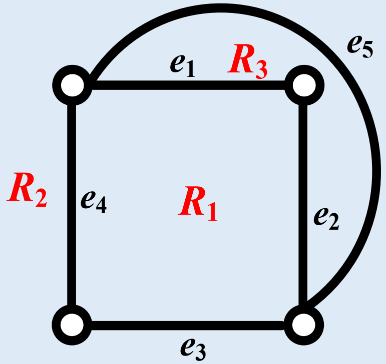
deg(R_0)= 8

■ **Example:** The two diagrams on the right are planar embeddings of the same planar graph.



(1)

R_1 is the **outer face** in **(1)** and the inner face in **(2)**.
R_2 is the **inner face** in **(1)** and the outer face in **(2)**.



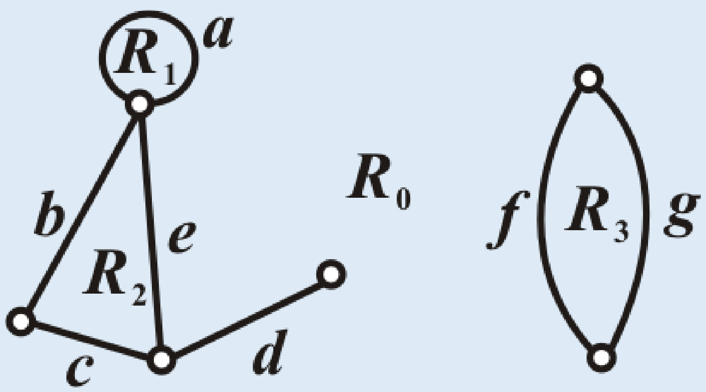
(2)

■ **Explanation:**

(1) A planar graph can have **multiple different forms of planar embeddings**, all of which are isomorphic.

(2) Any face of a planar graph can be considered the outer face through a **transformation** (such as geodesic projection).

- **Theorem 6.13**: The *sum of the degrees* of all faces in a planar graph is equal to **twice the number of edges.**
- **Proof**: An edge either serves as a common boundary for two faces or appears twice in the boundary of a single face. When calculating the sum of the degrees of all faces, each edge is counted exactly twice.
- **For example:** In the diagram below, the sum of the degrees of the faces is equal to: $\sum_{i=0}^{3} \boldsymbol{deg(R_i)} = \mathbf{8+1+3+2} = \mathbf{2}|\{\boldsymbol{a, b, c, d, e, f, g}\}|$



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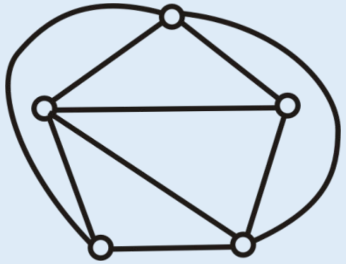
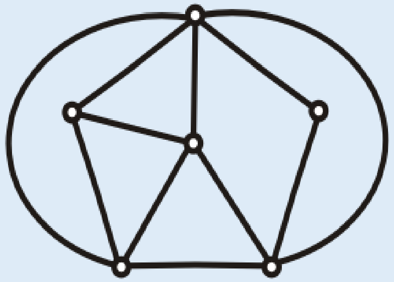
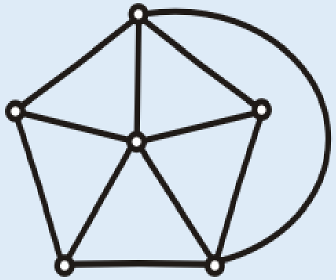
- **Definition 6.13:** If G is a simple planar graph, and the graph obtained by adding a new edge between any two non-adjacent vertices is non-planar, then G is called a ***maximal planar graph***.
- **Example:**
  - K_1, K_2, K_3, K_4 are all maximal planar graphs.
  - **(1)** is K_5 with one edge removed, which is a maximal planar graph. **(2)** and **(3)** are not.

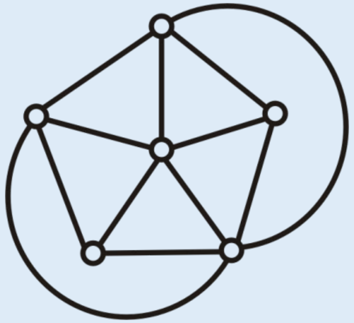

(1)


(2)

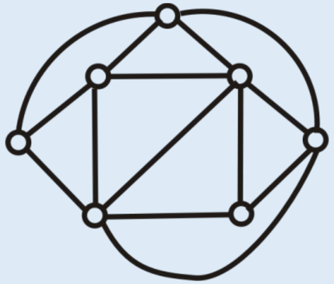

(3)

↳Properties of Maximal Planar Graphs: Connected and triangular.

- A maximal planar graph is ***connected***.
- Let ***G*** be a simple graph of order ***n*** (***n***≥3). A necessary and sufficient condition for ***G*** to be a maximal planar graph is that the ***degree of each face in G is 3***. （Triangulation）
- **Example:**



Maximal planar graph



The degree of the outer face is 4.
It is a non-maximal planar graph.