Discrete Mathematics 2025 Spring

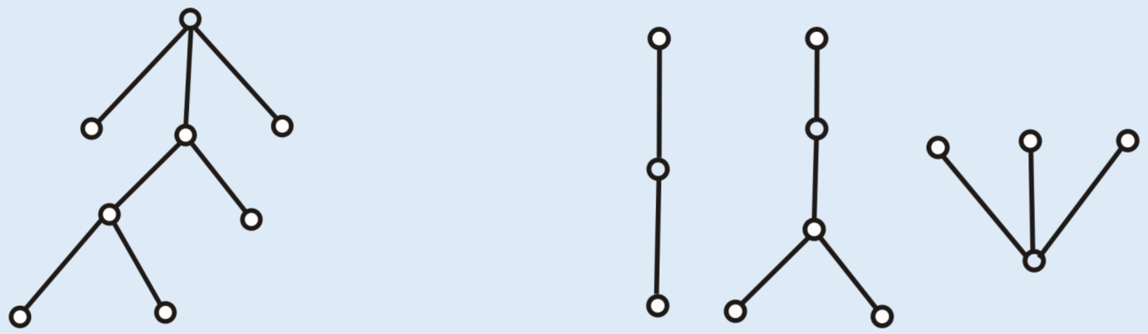
魏可佶 kejiwei@tongji.edu.cn

- 7.1 Undirected Trees
- 7.2 Rooted Trees and Their Applications

- 7.1.1 Definition and Properties of Undirected Trees
- 7.1.2 Spanning Trees

↳ Undirected Trees, Trivial Trees, and Forests

- *Undirected Tree*: A connected undirected graph with ***no cycles***.
- *Trivial Tree*: A trivial graph (a graph with ***only one vertex*** and no edges).
- *Forest*: A disconnected undirected graph in which ***each connected component is a tree***.
- *Leaf*: A vertex in a tree with ***degree 1***.
- *Branching Point*: A vertex in a tree with ***degree ≥ 2***.
- **Example:**



(a) (b)

- **Theorem 7.1:** Let G=<V,E> be an undirected graph of order n with m edges. The following statements are equivalent:

(1) G is connected and contains *no cycles* (i.e., G is a tree).

(2) There exists a *unique path* between any two vertices in G.

(3) G is connected and $m=n-1$.

(4) G contains no cycles and $m=n-1$.

(5) G contains no cycles, but adding an edge between any two non-adjacent vertices creates exactly one simple cycle.

(6) G is connected, and every edge in G is a bridge.

- **Proof: (1)⇒(2)**

  ①By the definition of connectivity, there is a path between any two vertices.

  ②Now suppose there are two different paths between some pair of vertices. Then these two paths together form a cycle, which contradicts the definition of a tree.

- **Proof: (2)⇒(3)**

  ①Clearly, the graph is connected. We now prove that $m=n-1$ by induction on n.

  ②**Base case:** When $n=1$, obviously $m=0$, so the conclusion holds.

- **Proof:** (2)$\Rightarrow$(3)
  - ③ **Inductive step:** Assume the conclusion holds for all graphs with up to $n \leq k$ ($k \geq 1$), consider $n=k+1$.
  - ④Select any edge $e=(u,v)$, Since the path between u and v is unique, ***e must be the only path connecting them.***
  - ⑤Removing e disconnects the graph into two connected components.
  - ⑥Let these components have n_1 and n_2 vertices, and m_1 and m_2 edges, respectively, where $n_1 \leq k$, $n_2 \leq k$.
- **Therefore,** $m_1=n_1-1$, $m_2=n_2-1$, $m= m_1+m_2+1= n-1$.

- **Proof: (3)⇒(4)**

① Suppose G contains a cycle. Then, by removing any edge from the cycle, the resulting graph remains connected.

② Repeat this process until no cycles remain. The resulting graph is a tree with n vertices and m-r edges, $r>0$.

③ From (1)⇒(2)⇒(3), we know $m-r = n-1$, which contradicts the assumption that $m=n-1$.

- **Proof: (4)⇒(1)** We only need to prove that G is connected.

① Assume G is disconnected and has p connected components, where $p(p>1)$.

② Let the k-th component have n_k vertices and m_k edges, **by (1)⇒(2)⇒(3),** $m_k= n_k-1$, **get** $m= n-p$, which contradicts the assumption that $m=n-1$.

- **Proof: (1)⇒(5)**

① From **(1)⇒(2)**, there exists a unique path between any two non-adjacent vertices.

② Therefore, adding a new edge between them will create exactly one simple cycle.

- **Proof: (5)⇒(6)**

① First, there must be a path between any two non-adjacent vertices. Otherwise, adding a new edge between them would not create a cycle. Hence, ***G*** is connected.

② Second, if removing any edge leaves ***G*** still connected, then that edge must lie on a cycle—contradicting the assumption that ***G*** has no cycles.

- **Proof: (6)⇒(1)**

① If *G* contains a cycle, then removing any edge on that cycle would leave *G* connected—contradicting the assumption that every edge is a bridge.

②Thus, *G* has no cycles and is connected, so *G* is a tree.

- **Theorem 7.2:**

  An undirected tree of order n (with $n>1$) has at least two leaves.

- **Proof：**

  ① Let the tree have n vertices $n(n>1)$ k leaves (vertices of degree 1), and m edges.

  ② By the andshaking Lemma, we have: $2m \geq k + 2(n-k)$.

  ③ According to **Theorem 7.1** (the number of edges in a tree is always $m=n-1$), we substitute:

  $2(n-1) \geq k + 2(n-k)$

  $k \geq 2$.