Persistent Homology and Topological Data Analysis

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Chapter 1

Chain Complexes And Simplicial Homology

1.1 Introduction

The key method of algebraic topology is to assign various *algebraic structures* – groups, rings, modules – to topological spaces. This should be done in a *functorial way*. Roughly, functoriality means that maps of topological spaces (and compositions thereof) give rise to homomorphisms of the respective algebraic structures (and compositions thereof), and that the structures assigned to homeomorphic spaces are isomorphic. See (2.1) or [Hat02][Ch.2.3] for a more detailed discussion. In this way, we can think of these algebraic structures as *invariants* of the spaces under consideration. Questions about topological spaces are converted into questions about algebraic structures, which are typically "more rigid". This rigidity can be used to demonstrate, for example, that maps between certain spaces do not exist, or that certain spaces are not homeomorphic, etc.

Computing these algebraic invariants is a different matter altogether. There are notorious examples of invariants that are unknown or hard to compute even for simple enough spaces, such as spheres.

1.2 Δ-complexes

We begin now with a setup that allows for fairly easy calculations. We shall assign a collection of abelian groups to a topological space X *equipped with some additional structure*. This additional structure – called Δ*-complex structure* – is a way of "parametrizing" X by points, segments, triangles, tetrahedra (and their higher-dimensional analogues) and will be introduced in Definition 1.2.2. While the structure of a Δ-complex makes computations easy, it will be completely unclear whether the groups that we obtain are sensitive to this additional structure, or are in fact invariants of the space X itself. In other words, the functoriality of this construction will be completely unclear. This will be rectified in Chapter 2.

We start with the basic building blocks: simplices.

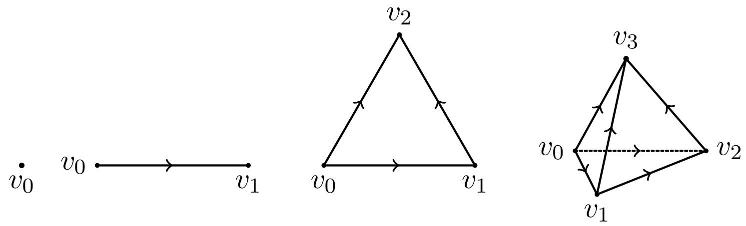
Definition 1.2.1 (Standard Simplex)**.** *The* standard n-simplex in $\mathbb{R}^{n+1}$ *is the convex hull* Δ^n *of the standard basis vectors* $\{e_0, \dots, e_n\}$*, i.e.,*

$$\Delta^n = \left\{ (t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1, \;\; t_i \geq 0 \,\forall\, i \right\} \subseteq \mathbb{R}^{n+1}.$$

More generally, an n-simplex in $\mathbb{R}^{n+1}$ *is the convex hull* $[v_0, \dots, v_n]$ *of any* $(n+1)$*-tuple of vectors* $v_0, \dots, v_n \in \mathbb{R}^{n+1}$ *that do not lie in an* n*-dimensional hyperplane.*

Notice that with this definition, $\Delta^n = [e_0, \dots, e_n]$.

Thus an n-simpex (plural simplices) is an n-dimensional analog of a triangle. A *0-simplex* is a point, a *1-simplex* is a line segment, a *2-simplex* is a triangle, *3-simplex* is a tetrahedron, as shown below.



The vectors v_i, determining $[v_0, ..., v_n]$ are the *vertices* of the simplex. In our calculations we are going to work with a chosen *ordering* of the vertices of the simplex. I.e., we are going to use "simplex" when we mean "a simplex together with an ordering of the vertices". This has a number of consequences. First, it determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts. Second, specifying an ordering of the vertices determines a canonical linear homeomorphism from Δ^n onto any n-simpex $[v_0, ..., v_n]$ preserving the order of vertices, namely, $\sum_i t_i \mathbf{e}_i \mapsto \sum_i t_i \mathbf{v}_i$. Once we fix an ordering of the vertices, we also obtain an orientation of the n-simplex, i.e., the sign of $\det(v_0, \dots, v_n)$. Two orderings determine the same orientation when they differ by an even permutation.

By a *face* of a n-simplex we shall mean an $(n-1)$-simplex spanned by some n-tuple of vertices of the simplex. That is, the i-th face of $[v_0, \dots, v_n]$ is $[v_0, \dots, \widehat{v_i}, \dots, v_n]$, where the hat indicates omission. Some sources refer to this face as an $(n-1)$-face, and introduce, more generally *k-faces*, for $0 \le k \le n-1$. These are k-simplices, obtained as the convex hull of a k-tuple of vertices. We always order the vertices of a face according to their order in the larger simplex.

Example 1.2.1. Consider the (second) face $[v_0, v_1, v_3]$ of the 3-simplex $[v_0, v_1, v_2, v_3]$ in $\mathbb{R}^4$. The canonical order-preserving map $\Delta^2 \to [v_0, v_1, v_3]$ is determined by $e_0 \mapsto v_0$, $e_1 \mapsto v_1$, $e_2 \mapsto v_3$. The canonical order-preserving map from Δ^1 to the edge $[v_1, v_3]$ is determined by $e_0 \mapsto v_1$, $e_1 \mapsto v_3$.

The *boundary* $\partial\Delta^n$ of the standard simplex is defined as the union of all the faces of Δ^n, and $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ denotes interior of Δ^n. We define analogously the boundary and interior of an arbitrary simplex in $\mathbb{R}^n$. Notice that with this definition $\partial\Delta^0 = \varnothing$ and $\mathring{\Delta}^0 = \Delta^0$!

We now equip X with additional structure: "parametrization" of X by simplices of various dimensions that satisfies a number of compatibility conditions.

Definition 1.2.2 (Δ-complex)**.** *A $\Delta-$complex structure on a topological space X is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}_\alpha$, with $n = n(\alpha)$ depending on the index α, such that:*

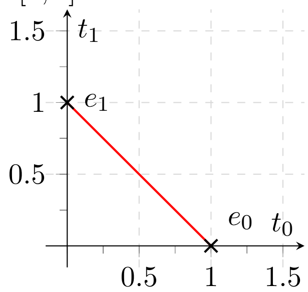
1. *Each restriction $\sigma_\alpha|\mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\mathring{\Delta}^n$.*
2. *Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here a face of Δ^n is identified with Δ^{n-1} via the canonical order-preserving linear homeomorphism.*
3. *A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α*

Example 1.2.2. Consider $X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$. We are going to describe explicitly two maps $\sigma_0 : \Delta^0 \to S^1$, $\sigma_1 : \Delta^1 \to S^1$, which equip S^1 with the structure of a Δ-complex.

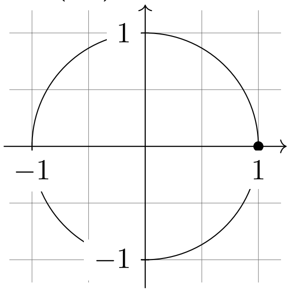
For the explicit description, keep in mind that $\Delta^0 = \{1\} \subseteq \mathbb{R}$ and that

$$\Delta^1 = \{(t_0, t_1) \,|t_0 + t_1 = 1\} = \{(t_0, 1 - t_0)\,,\, t_0 \in [0, 1]\}$$

So in particular, any (continuous) map $\Delta^1 \to S^1$ is determined by and determines a (continuous) map $[0;1] \to S^1$.



First, we define $\sigma_0 : \Delta^0 \to S^1$ by $\sigma_0(1) = (1,0)$. Next, $\sigma_1 : \Delta^1 \to S^1$ is defined by $\sigma_1(t_0,t_1) = (\cos(2\pi t_0), \sin(2\pi t_0))$, which is clearly continuous. The map σ_1 is one-to-one on $\mathring{\Delta}^1$, and so is, trivially, σ_0 on $\mathring{\Delta}^0$. The images of the two maps cover the circle, with $\sigma_1\left(\mathring{\Delta}^1\right) = S^1 \backslash \{(1,0)\}$ and $\sigma_0\left(\mathring{\Delta}^0\right) = \{(1,0)\}$.



Finally, we check the compatibility property (2) of σ_1 and σ_0. The restrictions of σ_1 to the two faces of Δ^1 coincide with σ_0:

$$\sigma_1|_{[e_1]} = \sigma_1(0,1) = (\cos(0), \sin(0)) = (1,0) = \sigma_0(1)$$
$$\sigma_1|_{[e_0]} = \sigma_1(1,0) = (\cos(2\pi), \sin(2\pi)) = (1,0) = \sigma_0(1)$$

Property (3) holds as well. There are numerous other Δ-complex structures on S^1, and we shall discuss some of them later on.

Example 1.2.3. Our next example is a Δ-complex structure on $X = S^2 = \{(x,y,z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1\}$.

Notice first that there is a homeomorphism between the L^1-sphere

$$\{(x_0,x_1,x_2) \mid |x_0| + |x_1| + |x_2| = 1\} \subseteq \mathbb{R}^3$$

and S^2, given by

$$(x_0,x_1,x_2) \longmapsto \left(\frac{x_0}{\sqrt{x_0^2+x_1^2+x_2^2}}, \frac{x_1}{\sqrt{x_0^2+x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_0^2+x_1^2+x_2^2}}, \right),$$

that is, by $\mathbf{x} \longmapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}$.

On the other hand, the L^1-sphere is the union of eight 2-simplices: one of them is the standard simplex Δ^2 in $\mathbb{R}^3$, while the others are obtained from Δ^2 by reflection with respect to a coordinate plane, i.e., by multiplying one of the variables by ± 1. Composing with the map between the L^1 and L^2 sphere, we obtain the eight continuous maps

$$\sigma_{s_0s_1s_2} : \Delta^2 = [e_0,e_1,e_2] \longrightarrow S^2,$$

$s_i \in \{\pm 1\}$, $i \in \{0,1,2\}$,

$$\sigma_{s_0s_1s_2} : \ (t_0,t_1,t_2) \longmapsto (s_0t_0, s_1t_1, s_2t_2) \longmapsto \frac{(s_0t_0, s_1t_1, s_2t_2)}{\sqrt{t_0^2+t_1^2+t_2^2}}.$$

Next, we have the three standard order-preserving homeomorphisms, identifying Δ^1 with the three faces (edges) of Δ^1, $[e_1, e_2]$, $[e_0, e_2]$, $[e_0, e_1]$. Each of these edges is contained in a coordinate plane in $\mathbb{R}^3$, and performing a reflection with respect to the coordinates planes orthogonal to the plane containing the edge gives rise to 12 maps from Δ^1 to the sphere.

Explicitly, these are the 12 continuous maps

$$\lambda_{0s_0s_1}, \lambda_{s_00s_1}, \lambda_{s_0s_10} : \Delta^1 \longrightarrow S^2,$$

where $s_i \in \{\pm 1\}$ and

$$\lambda_{0s_0s_1} : (t_0, t_1) \longmapsto (0, s_0t_0, s_1t_1) \longmapsto \frac{(0, s_0t_0, s_1t_1)}{\sqrt{t_0^2 + t_1^2}}$$

$$\lambda_{s_00s_1} : (t_0, t_1) \longmapsto (s_0t_0, 0, s_1t_1) \longmapsto \frac{(s_0t_0, 0, s_1t_1)}{\sqrt{t_0^2 + t_1^2}}$$

$$\lambda_{s_0s_10} : (t_0, t_1) \longmapsto (s_0t_0, s_1t_1, 0) \longmapsto \frac{(s_0t_0, s_1t_1, 0)}{\sqrt{t_0^2 + t_1^2}}.$$

Finally, we have six maps $\Delta^0 \to S^2$, obtained by composing the three inclusions of Δ^0 as a 0-face in Δ^2, c composed with reflections with respect to the coordinate planes. Explicitly, these are the 6 maps

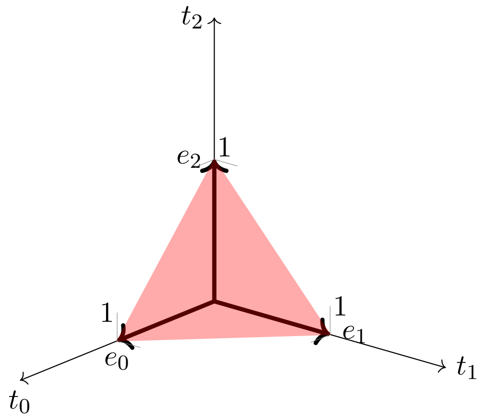
$$\mu_{s00}, \mu_{0s0}, \mu_{00s} : \Delta^1 \longrightarrow S^2,$$

where $s \in \{\pm 1\}$ and

$$\mu_{s00}(1) = (s, 0, 0),\ \mu_{0s0}(1) = (0, s, 0),\ \mu_{00s}(1) = (0, 0, s).$$

The images of the maps cover S^2, they are injective when restricted to the interiors of the simplices and are compatible. For example, $\sigma_{s_0s_1s_2}\big|_{[e_1,e_2]} = \lambda_{0s_1s_2}$, $\lambda_{s_0s_10}|_{[e_0]} = \mu_{s_000}$, etc. suppressing the canonical orientation-preserving homeomorphism between simplices. Moreover, property (3) is satisfied. We do not check it explicitly, but it follows from the fact that our maps are obtained as composition of inclusion of a face into a simplex, followed by a reflection and a homeomorphism between the L^1 and L^2 sphere.

Altogether, this Δ-complex structure on S^2 consists of the eight maps $\sigma_{s_0s_1s_2}$, the twelve maps $\lambda_{0s_0s_1}, \lambda_{s_00s_1}, \lambda_{s_0s_10}$ and the six maps $\mu_{s00}, \mu_{0s0}, \mu_{00s}$.



For completeness, we mention some "non-examples": collections of maps $\sigma_\alpha : \Delta^n \to X$ that satisfy conditions (1) and (2), but not condition (3) from Definition 1.2.2.

Example 1.2.4. Let $X = [0; 1]$ or more generally, any other non-discrete space, having infinitely many points. Then the collection of maps $\{\sigma_x : \Delta^0 \to X\}_{x \in X}$, $\sigma_x(1) = x$ satisfies (1) and (2), but not (3). We place one 0-simplex for each point of X, but points in $[0; 1]$ are not open.

Another example is as follows. Let X be the unit square $X = [0;1] \times [0;1]$. Consider, for each $y \in [0;1]$, a map $\lambda_y : \Delta^0 \to X$, $\lambda_y(1) = (0, y)$, a map $\rho_y : \Delta^0 \to X$, $\rho_y(1) = (1, y)$ and a map $\sigma_y : \Delta^1 \to X$, $\sigma_y(t_0, t_1) = (t_0, y)$. In this way we place at height y one 1-simplex (as a horizontal segment), and two 0-simplices, as its left and right end. The collection of maps $\{\sigma_y, \lambda_y, \rho_y\}_{y\in[0;1]}$ satisfies (1) and (2), but not (3).

Thus condition (3) precludes inadequate choices of maps σ_α, for example, covering a manifold (with boundary) of dimension n by simplices of smaller dimension. More generally, condition (3) is important when one needs to deal with infinite collections of simplices, in particular, when dealing with infinite-dimensional (in appropriate sense) spaces X.

Given a Δ-complex structure on X, we can recover the space just from the combinatorics of the maps $\left\{\sigma_\alpha : \Delta^{n(\alpha)} \to X\right\}_{\alpha\in A}$. More precisely, we can consider the disjoint union of simplices Δ^n_α, labelled by the respective maps σ_α. We then impose an equivalence relation $\sim$ on the disjoint union: the i-th face of Δ^n_α is identified with Δ^{n-1}_β via the canonical order-preserving isomorphism if $\sigma_\alpha|_{[e_0,\ldots,\widehat{e_i},\ldots e_n]}$ coincides with $\sigma_\beta : \Delta^{n-1} \to X$ (under the canonical isomorphism).

The properties (1), (2), (3) from Definition 1.2.2 guarantee that the quotient space is indeed homeomorphic to X:

$$X \simeq \coprod_{\alpha\in A} \Delta^n_\alpha \Big/ \sim .$$

Turning this viewpoint backwards, we can consider X as being built up, inductively, by simplices of dimension 0, 1, 2, ..., by imposing an equivalence relation, encoded combinatorially.

Example 1.2.5. Consider the Δ-complex structure on S^1, discussed in Example 1.2.2. It gives an identification of S^1 as a quotient of $\Delta^0 \sqcup \Delta^1$, i.e.,

$$S^1 \simeq \Delta^0_0 \coprod \Delta^1_1 \big/ \sim,$$

where the equivalence relation identifies the point Δ^0_0 with each of the endpoints of the 1-simplex Δ^1_1.

Similarly, the Δ-complex structure on S^2 from Example 1.2.3 identifies S^2 with a quotient of

$$\underbrace{\Delta^0 \sqcup \ldots \sqcup \Delta^0}_{6} \coprod \underbrace{\Delta^1 \sqcup \ldots \sqcup \Delta^1}_{12} \coprod \underbrace{\Delta^2 \sqcup \ldots \sqcup \Delta^2}_{8}.$$

The equivalence relation is encoded in the maps, and can be made explicit by labelling each simplex with the respective map, e.g., $\Delta^0_{\mu_{s00}}$, $\Delta^0_{\mu_{0s0}}$, $\Delta^0_{\mu_{00s}}$, $\Delta^1_{\lambda_{0s_0s_1}}, \ldots, \Delta^2_{\sigma_{s_0s_1s_2}}$.

One often constructs topological spaces as quotients, i.e., we write $X = Y/\sim$, where Y is a possibly simpler space. This is the case, for instance, for $X = S^1 \times S^1$, $X = \mathbb{RP}^2$, the Klein bottle, etc. In all of these examples $Y = [0;1] \times [0;1]$. One can try to build a Δ-complex structure on X by constructing first a Δ-complex structure on Y. We are going to see such examples in Section 1.4.

A Remark on Simplicial Complexes

The Δ-complex structure on S^1 from Example 1.2.2 involves one 1-simplex and one 0-simplex, and the gluing identifies the two vertices of Δ^1. We could, however, choose different Δ-structures, e.g., one involving two (or more) 1-simplices – see Section 1.4. In that case each 1-simplex in X will have two distinct vertices – i.e., the maps $\sigma_\alpha : \Delta^1 \to S^1$ do not identify the vertices of Δ^1. (Equivalently, the equivalence relation $\sim$ does not identify distinct vertices.)

More generally, we say that a Δ-complex (structure on a topological space) X is a *simplicial complex* (structure on X) if all simplices are uniquely determined by their vertices. That is, for all $n \in \mathbb{N}$ it is the case that each n-simplex in X has $n+1$ distinct vertices and there are no other n-simplices with this set of vertices. Now, in such a situation, every face of an n-simplex in X is an $(n-1)$-simplex in X, and so on.

But such a structure is a completely combinatorial object! It is specified by a finite set – the 0-complexes of our Δ-structure, together with finite subsets (corresponding to simplices in X), having the property that all of their subsets are also simplices.

Here is the formal combinatorial description of a simplicial complex, given as the definition of an *abstract simplicial complex.*

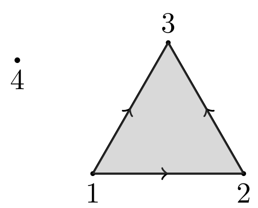
Definition 1.2.3 (Abstract Simplicial Complex)**.** *Given a finite set* $\{1, 2, \cdots, m\}(= [m])$ *an* abstract simplicial complex *is a collection* $\mathcal{K}$ *of subsets of* $[m]$*, such that:*

1. $\emptyset \in \mathcal{K}$
2. $\{i\} \in \mathcal{K}$ *(singletons are in* $\mathcal{K}$*)*
3. *If* $I \in \mathcal{K}$ *and* $J \subseteq I$*, then* $\mathcal{J} \in \mathcal{K}$

The elements of $[m]$ *are the* vertices*, and the elements of* $\mathcal{K}$ *are the* simplices*, where* $I \in \mathcal{K}$ *is an* $(|I|-1)$*-simplex.*

Sometimes, the condition $\varnothing \in \mathcal{K}$ is omitted. Abstract simplices are subsets of the power set of the given set of vertices, which are partially ordered by inclusion.

Example 1.2.6. Consider the following partially ordered set $V = \{1, 2, 3, 4\}$: The simplicial complex $\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$



Given an abstract simplex $\mathcal{K}$, we can construct its *topological realisation* as

$$|\mathcal{K}| = \bigcup_{\varnothing \neq I \in \mathcal{K}} (\mathrm{Conv}(e_i), i \in I) \subseteq \mathbb{R}^m$$

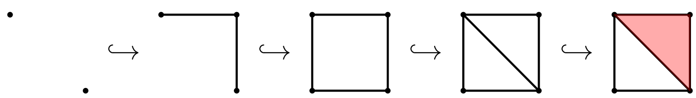
where $\{e_i\}$ is in the standard basis $e_1, \cdots e_m \in \mathbb{R}^m$.

Classically, homology theories were introduced for simplicial complexes (while Δ-complexes were also called semi-simplicial complexes). Simplicial complexes have simple combinatorial description and are widely used in computer implementations of homology calculations. For the purposes of human understanding, however, Δ-complexes are much more convenient.

A subcomplex of $\mathcal{K}$ is a subset $L \subseteq \mathcal{K}$ that is also a simplicial complex. A *filtration* of complex $\mathcal{K}$ is a nested sub-sequence of complexes:

$$\emptyset = \mathcal{K}^0 \subseteq \mathcal{K}^1 \subseteq \ldots \mathcal{K}^m = \mathcal{K}$$

For generality, we let $\mathcal{K}^i = \mathcal{K}^m$ for all $i \geq m$. $\mathcal{K}$ is called a filtered complex, and below there is a short example of a filtered complex:



1.3 Chain Complexes and Homology

One of the key algebraic notions that we need is the notion of *chain complex.*

Definition 1.3.1 (Chain complex)**.** *Complex of abelian groups. Homology of a complex.* *A* chain complex $(C_\bullet, \partial_\bullet)$ *is a sequence of homomorphisms of abelian groups:*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each $n \in \mathbb{N}$. The equation $\partial_n\partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im}\partial_{n+1} \subset \ker\partial_n$*. The elements of* $\ker\partial_n$ *are called* n-cycles*, often denoted Z_n. The elements of* $\operatorname{Im}\partial_{n+1} \subseteq C_n$ *are called* n-boundaries*, often denoted B_n. The* n-th homology group *of the complex is the quotient $H_n(C_\bullet) = \ker\partial_n/\operatorname{Im}\partial_{n+1}$. The maps ∂_n are called* the differentials *or* the boundary maps *of the complex.*

One often considers a collection of groups C_n labelled by $n \in \mathbb{Z}$ – and in any case can extend from $n \in \mathbb{N}$ to $n \in \mathbb{Z}$ by setting $C_n = (0)$ for $n < 0$. The chain groups (n-chains) are often thought of as a single object, a *graded group* $C_\bullet = \bigoplus_n C_n$. We denoted by $C_\bullet[k]$ the graded group, whose grading is shifted by k, i.e., $(C_\bullet[k])_n = C_{n+k}$. The differentials ∂_n can be thought of as a map of graded groups $\partial : C_\bullet \to C_\bullet[-1]$, satisfying the condition $\partial^2 = 0$.

In algebraic topology, we construct various complexes of abelian groups, associated with a space X and study their homology groups.

We start with a space X, equipped with a Δ-complex structure. We associate with it a collection of groups $C_n := \Delta_n(X)$, $n \geq 0$ – the simplicial n-chains – as follows. The group $\Delta_n(X)$ is the free abelian group, generated by the (images of the) open simplices $e^n_\alpha := \sigma_\alpha(\mathring{\Delta}^n)$. Thus its elements are linear combinations with integer coefficients, $\sum_\alpha n_\alpha e^n_\alpha$. Equivalently, and more conveniently, we think of $\Delta_n(X)$ as the free group on the maps $\sigma_\alpha : \Delta^n \to X$.

The boundary map $\partial_n : \Delta_n(\mathcal{X}) \to \Delta_{n-1}(\mathcal{X})$ for the would-be chain complex $\Delta_\bullet(X)$ is defined as

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[v_0, ..., \widehat{v_i}, ..., v_n].$$

Lemma 1.3.1. *The composition $\partial^2 = 0$ below is zero*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)\ .$$

That is, $(\Delta_\bullet(X), \partial_\bullet)$ is a chain complex.

Proof: As a preliminary illustration, let us prove that $\partial_2\partial_3 = 0$:

$$\Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X)$$

We have, for $\sigma \in \Delta_3(X)$,

$$\partial_3\sigma = \sum_i (-1)^i \sigma|[v_0, ..., \widehat{v_i}, v_3] = \sigma|[v_1, v_2, v_3] - \sigma|[v_0, v_2, v_3] + \sigma|[v_0, v_1, v_3] - \sigma|[v_0, v_1, v_2]$$

$$\begin{aligned} \partial_2\partial_3(\sigma) &= \sigma|[v_2, v_3] - \sigma|[v_1, v_3] + \sigma|[v_1, v_2] \\ &= -\sigma|[v_2, v_3] + \sigma|[v_0, v_3] - \sigma|[v_0, v_2] \\ &= \sigma|[v_1, v_3] - \sigma|[v_0, v_3] + \sigma|[v_0, v_1] \\ &= -\sigma|[v_1, v_2] + \sigma|[v_0, v_2] - \sigma|[v_0, v_1] = 0 \end{aligned} \tag{1.1}$$

In case of arbitrary n the proof proceeds analogously.

$$
\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n]) \\
&= \sum_j (-1)^j (\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n])|[v_0, ..., \hat{v}_j, ..., v_n] \\
&= \sum_{j<i} (-1)^i(-1)^j \sigma_|[v_0, ..., \hat{v}_j, ..., \hat{v}_i ..., v_n] + \sum_{j>i} (-1)^i(-1)^j \sigma_|[v_0, ..., \hat{v}_i, ..., \hat{v}_j ..., v_n] = 0
\end{aligned} \tag{1.2}
$$

□

The homology groups of the complex $(\Delta_\bullet(X), \partial_\bullet)$ are called *the simplicial homology groups of* X and denoted by $H_n^\Delta(X)$, $n \geq 0$.

The superscript Δ in $H_n^\Delta(X)$ indicates that, properly speaking, these groups are defined for a Δ-complex X, i.e., for a topological space X with some additional structure. In fact, it is a very important result that these groups *do not* depend on the chosen Δ-complex structure. We are going to discuss this more in the next chapter. See also the comparison theorem for simplicial and singular homology in [Hat02][2.1]. In view of this, we may occasionally drop the superscript Δ and write $H_n(X)$ for the n-th simplicial homology of X.

Remark: Chain complexes can be also defined over $R-modules$, where R is a commutative ring:

Definition 1.3.2. *(Chain complex of R-module) A Chain complex of R-modules is a sequence:*

$$(C_\bullet, d_\bullet) = (\ ... \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} ...\ )$$

where for each $n \in \mathbb{Z}$*,* C_n *is an R-module and* $d_n \in Hom_R(C_n, C_{n-1})$ *satisfies* $d_n \circ d_{n+1} = 0$.

Chain complexes together with morphisms of chain complexes (and composition given by degree-wise composition of R-morphisms) form a category, which we will denote by $Ch({}_RMod)$

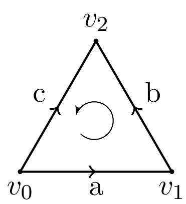
1.4 Homology Calculations: Examples

1.4.1 Homology of the circle S^1

We begin with computing the simplicial homology groups of the circle S^1. We will do so with respect to two different Δ-complex structures and will observe that, while the complexes $\Delta_n(S^1)$ will be different, their homology will be isomorphic.

Method I: Triangulation

To compute the homology group of the circle S^1 we can triangulate the circle in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2=0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1, v_2\rangle \\ C_1 = \langle a, b, c\rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2=0} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1, v_2\rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v_1 - v_0) + \beta(v_2 - v_1) - \gamma(v_2 - v_0)$
$= (-\alpha + \gamma)v_0 + (\alpha - \beta)v_1 + (\beta - \gamma)v_2 = (\gamma - \alpha)v_0 + (\alpha - \beta)v_1 + (-(\gamma - \alpha) - (\alpha - \beta))v_2$

$$\operatorname{Im} \partial_1 = \left\{ \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ -(\gamma - \alpha) - (\alpha - \beta) \end{pmatrix}, \quad \alpha, \beta, \gamma \subseteq \mathbb{Z} \right\} \subseteq \mathbb{Z}^{\oplus 3}$$

Claim : There exist an isomorphism $\psi : \operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$\psi : \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ (\beta - \gamma) \end{pmatrix} \mapsto \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \end{pmatrix}$$

ψ is one-to-one since if $(\gamma - \alpha = 0 \;\&\; \alpha - \beta = 0) \Rightarrow \beta - \gamma = 0 \;\&\; \alpha = \beta = \gamma$

ψ is onto since given $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$ there exist an element $\begin{pmatrix} m \\ n \\ -m-n \end{pmatrix} \in \operatorname{Im} \partial_1$ such that

$\psi \begin{pmatrix} m \\ n \\ -m-n \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$, since ψ is one-to-one and onto, $\operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$$

Claim: $\phi : \left(\mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z} \right) \simeq \mathbb{Z}$

First, let us take the map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3 \Big/ \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$$\mathbb{Z}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (p + q + r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$

So, $\varphi : \begin{pmatrix} p \\ q \\ r \end{pmatrix} \mapsto \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} = (p + q + r) \overline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Finally, $\phi : \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} \mapsto (p + q + r) \in \mathbb{Z}$, Clearly, ϕ is injective and surjective.

So, $H_0 \simeq \mathbb{Z}$

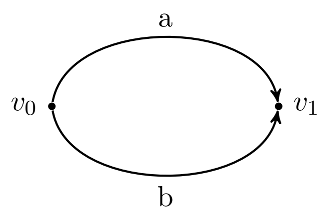
Second, let's compute H_1:
$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$
$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Method II

To compute the homology group of the circle S^1 we can construct the circle, by two vertices and two edges, in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2 = 0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1 \rangle \\ C_1 = \langle a, b \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1 \rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b) = \alpha(v_1 - v_0) + \beta(v_1 - v_0) = (\alpha + \beta)(v_1 - v_0)$
$\operatorname{Im} \partial_1 = \langle v_1 - v_0 \rangle$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \frac{\langle v_1 - v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \langle v_1 \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \langle a - b \rangle$ since $\partial_1(\alpha a + \beta b) = (\alpha + \beta)(v_1 - v_0) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element $(a - b)$
$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle with a different $\Delta - complex$ on it are the same:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0,1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.4.2 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} v \arrow[r, "b"] \arrow[dr, phantom, "A", near start] & v \\ v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c"] \arrow[ur, phantom, "B"', near end] & v \arrow[u, "a"'] \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v \rangle$ since $\partial_0 = 0$
$\operatorname{Im} \partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v - v) + \beta(v - v) + \gamma(v - v) = 0$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = C_1 = \langle a, b, c \rangle$ since $\partial_1 = 0$
$\operatorname{Im} \partial_2 = \langle a + b - c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(a + b - c) + \beta(a + b - c) = (\alpha + \beta)(a + b - c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a,b,c \rangle}{\langle a+b-c \rangle}$
The group $\langle a, b, c \rangle$ can be also generated by the elements $m = a+b-c, bandc$ where $a = m-b+c$. So,
$H_1 = \frac{\langle a+b-c,b,c \rangle}{\langle a+b-c \rangle} = \langle b, c \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$

Last, let's compute H_2:
$\ker \partial_2 = \langle A - B \rangle$ since $\partial_2(\alpha A + \beta B) = (\alpha + \beta)(a + b - c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element A-B
$\operatorname{Im} \partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\langle A-B \rangle}{\{0\}} = \langle A - B \rangle \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0,2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 3 \end{cases}$$

1.4.3 $\mathbb{RP}^2$

One way to calculate the homology groups of a projective plain $\mathbb{RP}^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} w \arrow[dr, phantom, "A" near start] & v \arrow[l, "b"'] \arrow[d, "a"] \\ v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c" description] & w \arrow[ul, phantom, "B" near start] \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v, w\rangle \\ C_1 = \langle a, b, c\rangle \\ C_2 = \langle A, B\rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus^2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus^3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus^2} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \ker\partial_n / \mathrm{Im}\,\partial_n$

First, let's compute H_0:
$\ker\partial_0 = C_0 = \langle v, w\rangle$ since $\partial_0 = 0$
$\mathrm{Im}\,\partial_1 = \langle w - v\rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(w - v) + \beta(w - v) + \gamma(v - v)$
$= (\alpha + \beta)(w - v)$
$H_0 = \frac{\ker\partial_0}{\mathrm{Im}\,\partial_1} = \frac{\langle v,w\rangle}{\langle w-v\rangle} = \frac{\langle w-v,w\rangle}{\langle w-v\rangle} = \langle w\rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker\partial_1 = \langle a - b, c\rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker\partial_1$ can be generated by the elements a-b and c
$\mathrm{Im}\,\partial_2 = \langle -a + b + c,\ a - b + c\rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$
$H_1 = \frac{\ker\partial_1}{\mathrm{Im}\,\partial_2} = \frac{\langle a-b,\, c\rangle}{\langle -a+b+c,\, a-b+c\rangle}$
The group $\langle a-b, c\rangle$ can be also generated by the elements $m = a-b+c, and c$ where $a-b = m-c$.
So,
$H_1 = \frac{\langle a-b,\, c\rangle}{\langle -a+b+c,\, a-b+c\rangle} = \frac{\langle a-b+c,\, c\rangle}{\langle a-b+c,\, -a+b+c\rangle}$
If we let $t = a-b+c$ then $-a+b+c = -t+2c$ then the group $\langle t,\ -t+2c\rangle$ can be also generated by the elements t *and* $2c$.
In terms of t and c, $H_1 = \frac{\langle t,\, c\rangle}{\langle t,\, 2c\rangle} = \frac{\langle c\rangle}{\langle 2c\rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Last, let's compute H_2:
$\ker\partial_2 = \{0\}$ since $\partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0$ only when $\alpha = \beta = 0$
$\mathrm{Im}\,\partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker\partial_2}{\mathrm{Im}\,\partial_3} = \frac{\{0\}}{\{0\}} = 0$

Finally, the homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.5 Maps of Complexes

In the previous sections, we considered boundary homomorphisms between abelian groups as part of a chain complex. In this section, we will draw our attention to maps between chain complexes.

Definition 1.5.1. *(Maps of Chain Complexes)*
Let $(C_\bullet, \partial)$ *and* $(D_\bullet, \delta)$ *be two chain complexes. A map of chain complexes is a morphism* f *that is a sequence of homomorphisms* $(f_n)_{n\in Z}$*:*

$$\begin{array}{ll} (C_\bullet, \partial) & C_\bullet \; \ldots \longrightarrow C_n \xrightarrow{\partial n} C_{n-1} \xrightarrow{\partial n-1} C_{n-2} \xrightarrow{\partial n-2} \ldots \; C_\bullet \\ & \qquad\qquad\quad \downarrow f_n \qquad\quad \downarrow f_{n-1} \qquad \downarrow f_{n-2} \\ (D_\bullet, \delta) & D_\bullet \; \ldots \longrightarrow D_n \xrightarrow{\delta n} D_{n-1} \xrightarrow{\delta n-1} D_{n-2} \xrightarrow{\delta n-2} \ldots \; D_\bullet \end{array}$$

$$f_n : C_n \to D_n \quad s.t, \quad f_{n-1} \circ \partial n = \delta_n \circ f_n \;\forall n \in \mathbb{Z}$$

$$\begin{array}{ccc} C_n & \xrightarrow{\partial n} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ D_n & \xrightarrow{\delta n} & D_{n-1} \end{array} \quad commutes.$$

1.5.1 Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$$
$$H_n(f) : [x] \mapsto [f_n(x)]$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-defines by checking if cycles are send to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$\delta_n \circ f_n(x) = f_{n-1} \circ \partial n(x) = f_{n-1}(0) = 0 \Rightarrow f_n(x) \in \ker \delta_n, f_n(x)\, is\, a\, cycle$$
$$\Rightarrow f_n(\ker \partial n) \subseteq \ker \delta_n$$

So, cycles are send to cycles.

(2) Let us take a boundary $y \in C_n$, so that $y \in \operatorname{Im} \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

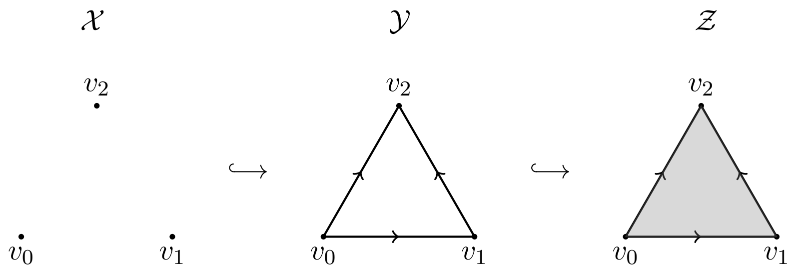
$$f_n(y) = f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z))$$
$$\Rightarrow f_n(y) \in \operatorname{Im} \partial_{n+1} f_n(y)\, is\, a\, boundary$$
$$\Rightarrow f_n(Im\partial_{n+1}) \subseteq Im(\delta_{n+1})$$

So, boundaries are send to boundaries.

$$
\begin{aligned}
&H_n(f): H_n(C_\bullet) \to H_n(D_\bullet)\\
&H_n(f): \ker \partial_n / \mathrm{Im}(\partial_{n+1}) \to \ker \delta_n / \mathrm{Im}(\delta_{n+1})\\
&\qquad [x] \mapsto [f_n(x)]\\
&\qquad x + \mathrm{Im}\, \partial_{n+1} \mapsto f_n(x) + f_n(\mathrm{Im}\, \delta_{n+1}) = f_n(x) + Im(\delta_{n+1}) = [f_n(x)]
\end{aligned}
$$

□

Let us consider an example between maps of complexes defined by the three spaces below.



$$C^0_\bullet \longrightarrow C^1_\bullet \longrightarrow C^2_\bullet$$

Maps between complexes:

$$C^0_\bullet \longrightarrow C^1_\bullet \longrightarrow C^2_\bullet$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d, "\partial_2^0=0"] & 0 \arrow[r] \arrow[d, "\partial_2^1=0"] & \mathbb{Z} \arrow[d, "\partial_2^2=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}"] \\
1 & 0 \arrow[r] \arrow[d, "\partial_1^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_1^1=\partial_1^2"] & \mathbb{Z}^3 \arrow[d, "\partial_1^2=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^1=0"] & \mathbb{Z}^3 \arrow[d, "\partial_0^2=0"] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

Induced maps on homology:

$$H(C^0_\bullet) \longrightarrow H(C^1_\bullet) \longrightarrow H(C^2_\bullet)$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Chapter 2

Singular Homology and Homotopy Invariance

In the previous chapter, we considered a parameterization of a space by simplexes, where the maps $\sigma_\alpha : \Delta^n \to X$ had restrictions defined in 1.2.2. If we only require that the σ map is continuous, then by definition that would be a singular $n-simplex$ in a space X. The lack of other restrictions on the map $\sigma : \Delta^n \to X$, convey that σ does not need to be a 'nice' embedding, in fact it can have singularities, where its image does not look like a simplex.

$C_n(X)$ is a free abelian group with generators the set of singular $n-simplexes$ in X: the continuous maps $\sigma : \Delta^n \to X$. The elements of $C_n(X)$ are singular $n-chains$ defined as $\sum_i(n_i\sigma_i)$ for $n_i \in \mathbb{Z}$. The boundary operator $\partial_n : C_n(x) \to C_{n-1}(X)$ is defined the same way as in simplicial $n-chains$, by the formula:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n]$$

By preserving the order of the vertices, the $\sigma|[v_0, ..., \hat{v_i}, ..., v_n]$ is identified with the map $\Delta^{n-1} \to X$. The proof of lemma 1.3.1, $\partial_2 = 0$, holds true also for singular simplexes. Therefore, the singular homology group is defined the same way: $H_n(x) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$. The elements in the ker are singular cycles, and the elements in the Im singular boundaries [Hat02].

Let us consider some explicit examples of $C_n(X)$:

- For a topological space X: $C_0(X)$ consists of all maps $\sigma : \Delta^0 \to X$, which means that $C_0(X)$ is a free group on the points of X.
- If X = $\mathbb{R}$: $C_1(X)$ consists of all maps $\sigma : \Delta^1 \to \mathbb{R}$, which means that $C_0(X)$ is a free group on continuous maps:

  $$[0, 1] \simeq \Delta^1 \to R$$

  In this case $C_1(X)$ can be considered as a vector space with vectors the maps: $[0, 1] \to \mathbb{R}$.

From the examples above, it is clear that the groups $C_n(X)$ can be so large to the point where the number of singular $n-simplexes$ in a space X is uncountable. It is not easy to see that even in singular homology where X is generated by a finite number of simplexes, $H_n(X)$ should be finite generated for all n, and that $H_n(X)$ should be 0 for $n > dim(X)$.

At first glance, singular homology seems to be more general than simplicial homology, however if for an arbitrary space X , we define the singular complex $S(X)$ as a $\Delta - complex$ with one $n-simplex$ Δ^n_σ for each singular $n-simplex$ $\sigma : \Delta^n \to X$, then $H^\Delta_n(S(X))$ is the same as $H_n(X)$. In this case singular homology can be viewed as a special case of simplicial homology.

2.1 Homotopy Invariance

A significant result that can be proven by singular homology is that if two spaces X, Y are homeomorphic, the singular homology groups are isomorphic $H_n(X) \simeq H_n(Y)$.

More generally, A continuous map $f : X \to Y$ induces a chain map: $f_\# : C_n(X) \to C_n(Y)$.

$$f_\#(\sigma : \Delta^n \to X) = f \circ \sigma$$

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

The boundary operator of $f_\#$ is equal to $\partial_n(f \circ \sigma) = f \circ \partial_n(\sigma)$.
$H_n(f_\#) = f_* : H_n(X) \to H_n(Y)$

If we additionally require f to be a bijection and have a continuous inverse f^{-1}, so f is a homeomorphism, then $f_\# : C_n(X) \simeq C_n(Y)$

$$\sigma \mapsto f \circ \sigma$$
$$\mu \circ f^{-1} \leftarrow\!\shortmid \mu$$

Then the induced singular homology map $f_* : H_n(X) \simeq H_n(Y)$ defines an isomorphism.

Moreover,

$$f_* \ \textit{preserves composition}, (f \circ g)_* = f_* \circ g_* \tag{2.1}$$

$$f_* \ \textit{preserves the identity}, id : X \to Y \ \textit{goes to} \ id_* : H_n(X) \to H_n(Y) \tag{2.2}$$

Category Theory Interpretation: If we consider Top to be the category of topological spaces where maps are continuous:

$$Hom_{Top}(X, Y) = \{f : X \to Y, f \ \textit{is continous}\},$$

and Ab the category of abelian groups were maps are group homomorphisms:

$$Hom_{Ab}(G, H) = \{\phi : G \to H, \phi \ \textit{is group homomorphism}\}$$

then for each $n \geq 0$:

$$\begin{aligned} H_n : Top &\to Ab \\ X &\rightsquigarrow H_n(X) \\ f : X \to Y &\rightsquigarrow f_* : H_n(X) \to H_n(Y) \end{aligned}$$

H_n is a functor and (2.1) (2.2) hold.
If we also consider the category, Homotopic Topology $HoTop$ of topological spaces where maps are continuous up to homotopy, then we obtain the following commutative diagram:

$$\begin{tikzcd} Top \arrow[r, "H_n"] \arrow[d] & Ab \\ HoTop \arrow[ur, dotted] & \end{tikzcd}$$

$$Hom_{HoTop}(X, Y) = Hom_{Top}(X, Y) / \simeq$$

Remark: The continuous maps $f, g : X \to Y$ are homotopic if

$$\begin{aligned} &\exists \ H : X \times [0, 1] \to Y, \\ &for \ x \in X \ and \ t \in [0, 1] : H(x, t) = H_t(x) \\ &s.t \ H(x, 0) = f(x) \ and \ H(x, 1) = g(x) \\ &i.e \ H|_{x \times \{0\}} = f \ and \ H|_{x \times \{1\}} = g \end{aligned}$$

Let us consider some explicit homotopic maps:

- $f, g : \mathbb{R} \to \mathbb{R}$ where $f = id, f(x) = x \ \forall \ x \in X$, and $g = 0, \ g(x) = 0 \ \forall \ x \in X$
  $H : \mathbb{R} \times [0,1] \to \mathbb{R}$
  $H(x,t) = (1-t)x$, clearly $H(x,0) = x$ and $H(x,1) = 0$

- $f, g : S^1 \to \mathbb{R}^2$ where f *is an inclusion*, $f(x,y) = (x,y) \ \forall \ (x,y) \in S^1$, and $g = 0, \ g(x,y) = 0 \ \forall \ (x,y) \in S^1$
  $H : S^1 \times [0,1] \to \mathbb{R}^2$
  $H(x,y,t) = (1-t)\langle x,y \rangle$, clearly $H(x,y,0) = \langle x,y \rangle$ and $H(x,y,1) = 0$

If $f, g : X \to Y$ are homotopic, then the induced maps on homology $f_*, g_* : H_n(X) \to H_n(X)$, are the same $f_* = g_* \ \forall n$

Define $p_n : C_n^{sing}(X) \to C_{n+1}^{sing}(Y)$ s.t $f_\# - g_\# = \partial p + p \partial$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1}(X) \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \\
C_{n+1}(Y) \arrow[r, "\delta_{n+1}"] & C_n(Y) \arrow[r, "\delta_n"] & C_{n-1}(Y)
\end{tikzcd}$$

For $\sigma : \Delta^n \to X$ the map $p_n(\sigma) : \Delta^{n+1} \to Y$ should be a continuous map.
$H : X \times [0,1] \to Y$, $H|_{x \times \{0\}} = f \ and \ H|_{x \times \{1\}} = g$

$$\Delta^n \times [0,1] \xrightarrow{\sigma \times 1} X \times [0,1] \xrightarrow{H} Y$$

The idea is to write $\Delta^n \times [0,1]$ as union of Δ^{n+1}. Let us consider some explicit examples of the p maps:

- $p_0 : C_0(X) \to C_1(Y)$
  $p_0(\sigma) = H_0(\sigma \times 1)|_{[v_0 w_0]} : \Delta^1 \to Y$
  We can parameterise Δ^1, as $\Delta^1 = \{(t_0, t_1) | \ t_0 + t_1 = 1, \ t_0, t_1 \geq 1\}$
  $\Delta^1 = \{1\} \subseteq \mathbb{R}$, $\sigma(1) = q$

$$\begin{aligned}
\Delta^0 \times [0,1] &\simeq \Delta^1 \\
\{1\} \times \{t\} &\mapsto (1-t, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t), \ where \ H(q,0) = f(q), \ H(q,1) = g(q)
\end{aligned}$$

- $p_1 : C_1(X) \to C_2(Y)$
  $p_1(\sigma) = \sum_{i=0}^{1} H_0(\sigma \times 1)|_{[v_0 \ldots w_1]} = H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}$

$$\begin{aligned}
\Delta^1 \times [0,1] &\simeq [0,1] \times [0,1] \\
((t_0, t_1), t) &\mapsto (t_0, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t), \ where \ H(q,0) = f(q), \ H(q,1) = g(q)
\end{aligned}$$

$$\begin{tikzcd}
C_2(X) \arrow[r, "\partial_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_0(X) \arrow[r, "\partial_0"] \arrow[dl, "p_0"'] & 0 \\
C_2(Y) \arrow[r, "\delta_2"] & C_1(Y) \arrow[r, "\delta_1"] & C_0(Y) \arrow[r, "\delta_0"] & 0
\end{tikzcd}$$

From the diagram above: $\delta_2 p_1 + p_0 \partial_1 : C_1(X) \to C_1(Y)$

$$\begin{aligned}
(p_0 \circ \partial_1)(\sigma) &= p_0(\sigma|_{[v_1]} - \sigma|_{[v_0]}) \\
&= H_0(\sigma \times 1)|_{[v_1 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma|_{[v_1]} \times 1)|_{[v_1 w_1]} - H_0(\sigma|_{[v_0]} \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma \times 1)|_{[v_1] \times [0,1]} - H_0(\sigma \times 1)|_{[v_0] \times [0,1]}
\end{aligned}$$

$$\begin{aligned}
(\delta_2 \circ p_1)(\sigma) &= \delta_2(H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}) \\
&= H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&- H_0(\sigma \times 1)|_{[v_1 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]}
\end{aligned}$$

So , $(\delta_2 p_1 + p_0 \partial_1)(\sigma) = H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]} = g \circ \sigma - f \circ \sigma = (g - f) \circ \sigma$

$$\Rightarrow \delta_2 p_1 + p_0 \partial_1 = g - f$$

Theorem 2.1.1. *If two chain maps $f_\bullet, g_\bullet : (C_\bullet, \partial) \to (D_\bullet, \delta)$ are chain-homotopic then they induce the same homomorphism on homology:*

More explicitly:

$$\begin{tikzcd}
\dots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, "p_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \dots \arrow[dl, "p_{n-2}"'] \\
\dots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \dots
\end{tikzcd}$$

$$\begin{aligned}
&H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet) \\
&H_n(f) : [x] \mapsto [f_n(x)]
\end{aligned}$$

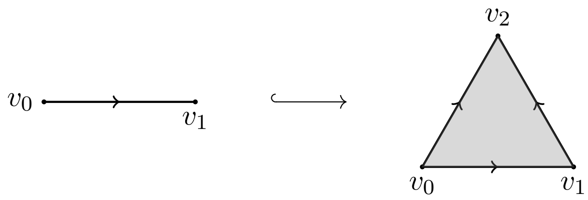
If $f_n - g_n = \delta_{n+1} p_n + p_{n-1} \partial_n \Rightarrow H_n(f) = H_n(g)$

Proof: Let us proof that the maps f_n, g_n induce the same homology.
For any $x \in \ker \partial_n \Rightarrow \partial_n x = 0$,
$(f_n - g_n)(x) = \delta_{n+1} p_n(x) + p_{n-1} \delta_n(x) = \delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$
$\Rightarrow (f_n - g_n)(x) \in \mathrm{Im}(\partial_{n+1})$

$$\left| \begin{array}{l} H_n(f)([x]) = [f_n(x)] \\ H_n(g)([x]) = [g_n(x)] \end{array} \right. \Rightarrow [f_n(x)] - [g_n(x)] = [f_n(x) - g_n(x)] = [\delta_{n+1} p_n(x)] = [0]$$

since $\delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$.
So, $[f_n(x)] = [g_n(x)] \Rightarrow H_n(f) = H_n(g)$ $\square$

Example 2.1.1. Let X be a $1-simplex$ and Y a $2-simplex$:



$$C_\bullet(X,\partial): \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

$$D_\bullet(Y,\delta): \qquad \mathbb{Z} \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_0=0} 0$$

Let's introduce maps on the chain complexes:

$$f_n : C_n(X,\partial) \to C'_n(Y,\delta)$$
$$g_n : C_n(X,\partial) \to C'_n(Y,\delta)$$

$$\begin{tikzcd}
0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1=\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[dl, "p_1=id"'] \arrow[d, shift right, "f_1=\begin{bmatrix}0\\0\\1\end{bmatrix}"'] \arrow[d, shift left, "g_1=\begin{bmatrix}0\\1\\0\end{bmatrix}"] & \mathbb{Z}^{\oplus 2} \arrow[r, "\partial_0=0"] \arrow[dl] \arrow[d, shift right, "f_0=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}"'] \arrow[d, shift left, "g_0=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"] & 0 \\
\mathbb{Z} \arrow[r, "\delta_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_1=\begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_0=0"'] & 0
\end{tikzcd}$$

We can define $p_0 = \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$ and $p_1 = id_{\mathbb{Z}}$

For $n=0$:

$$f_0 - g_0 = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

$$\delta_1 p_0 + 0\circ\partial_0 = \begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

For the choice of the maps p_0, f_0, g_0: $f_0 \simeq g_0 \to$ homotopic equivalent

For $n=1$:

$$f_1 - g_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\delta_2 p_1 + p_0\circ\partial_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix} + \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

For the choice of the maps p_1, p_0, f_1, g_1: $f_1 \simeq g_1 \to$ homotopic equivalent
By theorem 2.1.1, $H_0(f) = H_0(g)$ and $H_1(f) = H_1(g)$.
More precisely, $H_0(f) = H_0(g) = id_{\mathbb{Z}}$ and $H_1(f) = H_1(g) = 0$, since $H_0(X) = H_0(Y) = Z$ and $H_n(X) = H_n(Y) = 0 \; \forall \; n > 0$

Lemma 2.1.1. *A chain complex $(C_\bullet, \partial)$ is contractable if id_C is homotopic equivalent to 0_C*

If $id_C \simeq 0_C$, then $H_n(C_\bullet) = 0 \; \forall n$
Examples:

- $C_\bullet: \;\; \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$
  $H_0(C_\bullet) = \mathbb{Z} = H_1(C_\bullet)$ implies that $C_\bullet$ is not contractable

- $D_\bullet: \;\; \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$
  $H_0(D_\bullet) = \mathbb{Z}/2\mathbb{Z}$, $H_1(D_\bullet) = 0$ implies that $D_\bullet$ is not contractable

- $E_\bullet: \;\; \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$
  $H_0(E_\bullet) = 0 = H_1(E_\bullet) \Rightarrow$ need to check that $id_E \simeq 0_E$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] \arrow[dl, "p_1"'] \arrow[d, "id-0"] & \mathbb{Z} \arrow[r, "\partial_0=0"] \arrow[dl, "p_0"'] \arrow[d, "id-0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] & \mathbb{Z} \arrow[r, "\partial_0=0"] & 0
\end{tikzcd}$$

  We can assign $p_0 = id$, $p_1 = 0$.
  For a cycle $\sigma \in E_1 = \mathbb{Z}$:
  $(\partial_2 p_1 + p_0 \partial_1)(\sigma) = \partial_2 p_1(\sigma) + p_0 \partial_1(\sigma) = 0 + \sigma = \sigma = id - 0(\sigma) \Rightarrow id_E \simeq 0_E \Rightarrow (E_\bullet, \partial)$ is contractable

- $F_\bullet: \;\; \ldots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{\partial=2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \ldots$
  $\ker \partial = \operatorname{Im} \partial = (2) \Rightarrow H_n(F_\bullet) = 0 \; \forall \; n \Rightarrow$ need to check that $id_F \simeq 0_F$

$$\begin{tikzcd}
\ldots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[dl, "p_1"'] \arrow[d, "id-0"] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[dl, "p_0"'] \arrow[d, "id-0"] & \ldots \\
\ldots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \ldots
\end{tikzcd}$$

  In $\mathbb{Z}/4$ we have four classes $\overline{0}, \overline{1}, \overline{2}, \overline{4}$. The boundary operator $\partial = mult(2)$ maps the four classes only in two maps $\overline{0}, \overline{2}$. So, the ∂ cannot be surjective.
  For a cycle $\sigma \in \mathbb{Z}/4$:
  Since $((2)p_1 + p_0(2))(\sigma) \in (2)$, $((2)p_1 + p_0(2))(\sigma) \neq \sigma \Rightarrow (F_\bullet, \partial)$ is not contractable

Theorem 2.1.2. *Given topological spaces X, Y with maps $f : X \to Y$ and $g : Y \to X$*
If

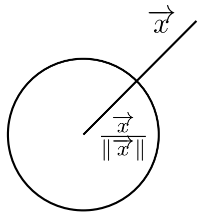
$$f \circ g \simeq id_Y$$
$$g \circ f \simeq id_X$$

where $\simeq$ denotes homotopic equivalence, then

$$\begin{tikzcd}
H_n(X) \arrow[r, bend left, "f_*"] & H_n(Y) \arrow[l, bend left, "g_*"]
\end{tikzcd} \quad \forall \; n \in \mathbb{Z}$$

$H_n(X)$ is isomorphic to $H_n(Y)$, $\Rightarrow f_* = (g_*)^{-1}$where f_*, g_* are the induced maps on homology.

Example 2.1.2. Let X be the n-dimensional sphere and Y the (n+1)-dimensional real coordinate space without the origin, $X = S^n$ and $Y = \mathbb{R}^{n+1}/\{0\}$



$$f : S^n \hookrightarrow \mathbb{R}^{n+1}/\{0\} \textit{ is the usual inclusion}$$
$$g : \mathbb{R}^{n+1}/\{0\} \to S^n \ s.t \ g(\overrightarrow{x}) = \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$$

Clearly, $g \circ f \simeq id_S^n$ while $f \circ g : \overrightarrow{x} \mapsto \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|} \neq id_{\mathbb{R}^{n+1}/\{0\}}$
Let's prove that $f \circ g$ is homotopic equivalent to $id_{\mathbb{R}^{n+1}/\{0\}}$: We can construct a function

$$\begin{aligned} F : [0,1] \times \mathbb{R}^{n+1}/\{0\} &\longrightarrow \mathbb{R}^{n+1}/\{0\} \\ F(t, \overrightarrow{x}) &= (t(f \circ g) + (1-t)1_{\mathbb{R}^{n+1}/\{0\}})\overrightarrow{x} \\ &= t(\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) + (1-t)\overrightarrow{x} \end{aligned}$$

Clearly, $F(0, \overrightarrow{x}) = \overrightarrow{x} = id_{\mathbb{R}^{n+1}/\{0\}}$ and $F(1, \overrightarrow{x}) = (\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) = f \circ g$.
So, $f \circ g \simeq id_{\mathbb{R}^{n+1}/\{0\}}$, and by theorem 2.1.2 $\Rightarrow f_* = (g_*)^{-1}$.

Example 2.1.3. Let X be a $1-simplex$ and Y a $0-simplex$:

$$C_\bullet(X, \partial) : \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

$$C'_\bullet(Y, \partial') : \qquad 0 \xrightarrow{0} 0 \xrightarrow{\partial'_1 = 0} \mathbb{Z} \xrightarrow{\partial'_0 = 0} 0$$

Let's introduce maps on the chain complexes:

$$\begin{aligned} f_n &: C_n(X, \partial) \to C'_n(Y, \partial') \\ g_n &: C'_n(Y, \partial') \to C_n(X, \partial) \end{aligned}$$

$$\begin{tikzcd} 0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, "0"] & \mathbb{Z}^{\oplus 2} \arrow[r, "\partial_0 = 0"] \arrow[dl, "p_0"'] \arrow[d, shift left, "f_0 = {[1\ 1]}"] & 0 \\ 0 \arrow[r, "0"] & 0 \arrow[r, "\partial'_1 = 0"] & \mathbb{Z} \arrow[r, "\partial'_0 = 0"] \arrow[u, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} = g_0"] & 0 \end{tikzcd}$$

For $n = 0$:

$$f_0 \circ g_0 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = id_{\mathbb{Z}}$$
$$g_0 \circ f_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq id_{\mathbb{Z}^{\oplus 2}}$$

Let's prove that $g_0 \circ f_0$ is homotopic equivalent to $id_{\mathbb{Z}^{\oplus 2}}$:
For an arbitrary element $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^{\oplus^2}$:
$(g_0 \circ f_0 - id_{\mathbb{Z}^{\oplus 2}}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ and $(\partial_1' p_0 + 0 \circ \partial_0) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Longrightarrow g_0 \circ f_0 \simeq id_{\mathbb{Z}^{\oplus 2}}$
For $n = 1$:

$$\begin{aligned} f_1 \circ g_1 &\simeq id_0 = 0 \\ g_1 \circ f_1 &\simeq id_{\mathbb{Z}} \end{aligned}$$

So, by theorem 2.1.2 $\Rightarrow f_* = (g_*)^{-1}$.

2.2 Exact Sequences

Definition 2.2.1. *A sequence of homomorphisms:*

$$\dots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots$$

is exact if $\ker \alpha_n = \operatorname{Im} \alpha_{n+1} \ \forall \ n$.

$\operatorname{Im} \alpha_{n+1} \subseteq \ker \alpha_n$ is equivalent to $\alpha_n \alpha_{n+1} = 0$ since $(A_\bullet, \alpha)$ is a chain complex.
& $\ker \alpha_n \subset \operatorname{Im} \alpha_{n+1} \Rightarrow H_n$ *is trivial* : $H_n = 0 \ \forall \ n$

Examples of short exact sequences:

1. $0 \to A \xrightarrow{\alpha} B$ *is exact* $\iff \ker \alpha = 0, \alpha$ *is injective*
2. $A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B, \alpha$ *is surjective*
3. $0 \to A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B$ *and* $\ker \alpha = \{0\}, \alpha$ *is an isomorphism*
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is exact* $\iff$
   (a) $\ker \alpha = \operatorname{Im}(0 \to A) = 0 \ \Rightarrow \alpha$ *is injective*
   (b) $\operatorname{Im} \beta = C \Rightarrow \beta$ *is surjective*
   (c) $\ker \beta = \operatorname{Im} \alpha$

   So, β induces an isomorphism $C \simeq \frac{B}{\operatorname{Im} \alpha}$. C can written as $C \simeq B/A$ if α is an inclusion of A as a subgroup of B.

Let us consider a short exact sequence of chain complexes:

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$

$A_\bullet$, $B_\bullet$, $C_\bullet$ are chain complexes and i, π are maps between chain complexes where $\ker \pi = \operatorname{Im} i$, $\quad \pi$: *surjective and* i : *injective*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : \arrow[r] & A_n \arrow[r, "\partial"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "i_{n-2}"] & {} \\
B_\bullet : \arrow[r] & B_n \arrow[r, "\partial"] \arrow[d, "\pi_n"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "\pi_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "\pi_{n-2}"] & {} \\
C_\bullet : \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The induced sequence on homology:

$$H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \; \forall \; n \tag{2.3}$$

$\pi \circ i = 0 \Rightarrow \pi_* \circ i_* = 0, \quad H_n(\pi \circ i) = H_n(\pi) \circ H_n(i)$
(2.3) <u>need not</u> be a short exact sequence. However, we can create a long exact sequence of homology:

$$H_{n+1}(C_\bullet) \xrightarrow{\partial n+1} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet)$$

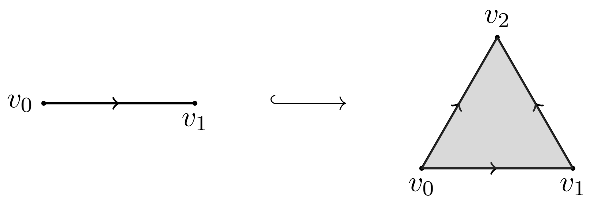
$$0 \to \operatorname{Im} \delta_{n+1} \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \to \ker \delta_n \to 0$$

where the δ map is defined as:

$$\delta : H_n(C) \to H_{n-1}(A)$$
$$[c] \mapsto [a]$$

For a element $b \in B_n$ there exists $c = \pi_n(b)$ since π is onto.
If we apply the boundary map $\partial : B_n \to B_{n-1}$ on b, then $\partial b \in B_{n-1}$, $\pi_n(\partial b) = \partial(\pi_n(b)) = 0$
We can take an element $a \in A_{n-1}$ such that $i(a) = \partial(b)$
$\partial(\partial b) = \partial(i(a)) = i(\partial a) \Rightarrow \partial a = 0 \; \textit{since i is injective} \; \Rightarrow \partial b = (0) \in A_{n-1}$

Example 2.2.1. Let consider X to be a $1 - simplex$ and Y a $2 - simplex$



A short exact sequence of chain complexes for X, Y:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial=\begin{bmatrix} -1 \\ 1 \end{bmatrix}"] \arrow[d, "i_1=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}"'] & \mathbb{Z}^{\oplus 2} \arrow[r] \arrow[d, "i_0=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}"] & 0 \\
B_\bullet : & 0 \arrow[r] & \mathbb{Z}^{\oplus 3} \arrow[r, "\partial=\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}"] \arrow[d, "\pi_1"] & \mathbb{Z}^{\oplus 3} \arrow[r] \arrow[d, "\pi_0"] & 0 \\
C_\bullet : & 0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The short exact sequence of complexes induces a long exact sequence on homology:

$$0 \to H_1(A) \to H_1(B) \to H_1(C) \xrightarrow{\delta} H_0(A) \to H_0(B) \to H_0(C) \to 0$$

Explicitly, the long exact sequence is:

$$0 \to 0 \to \mathbb{Z} \to H_1(C) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\alpha} H_0(C) \to 0$$

$$\ker(1) = \{0\} = \operatorname{Im}\delta$$
$$\ker(\alpha) = \operatorname{Im}(1) = \mathbb{Z}$$

$$C_1 = \mathbb{Z}^{\oplus^3} / \operatorname{Im} i_1 \simeq \mathbb{Z}^{\oplus^2}$$
$$C_0 = \mathbb{Z}^{\oplus^3} / \operatorname{Im} i_0 \simeq \mathbb{Z}$$

The boundary operator between C_1, C_0:

$$\partial : C_1 \to C_0$$
$$\partial : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a+b)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} mod(\operatorname{Im} i_1) \xmapsto{\partial=\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} -b \\ -a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \end{bmatrix}$$

$$H_1(C) = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \right\} \simeq \mathbb{Z}$$
$$H_0(C) = C_0 / C_0 = 0$$

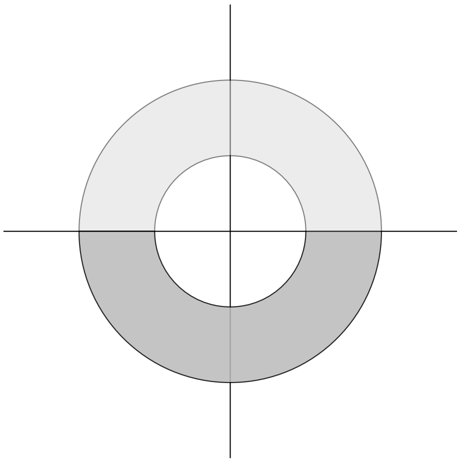
Let us take an example when $Y \subseteq X$ subspace, the short exact sequence of chain complexes is:

$$0 \to C_n(Y) \to C_n(X) \to C_n(X)/C_n(Y) \to 0$$

We get a long exact sequence on homology:

$$H_n(Y) \to H_n(X) \to H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y) \to H_{n-1}(X) \to \cdots$$

Example 2.2.2. Consider X to be the annulus, and the shaded area $Y \subseteq X$



$$\begin{aligned}
&X = \{(x,y)|1 \leq x^2 + y^2 \leq 2\} \\
&Y = \{(x,y)|1 \leq x^2 + y^2 \leq 2, y \leq 0\} \\
&\delta \ : \ H_1(X,Y) \to H_0(Y) \\
&\delta([\sigma]) = [\partial\sigma] = [\sigma(1) - \sigma(0)]
\end{aligned}$$

In "good cases" $H_n(X,Y) = H_n(X/Y)$

For two topological spaces $Y \subseteq X$ "pairs of spaces".
We can construct the following chain complex:
$0 \to C_n(Y) \xrightarrow{i} C_n(X) \xrightarrow{\pi} C_n(X)/C_n(Y) \to 0, \quad C_n(X,Y)$ *are relative chains*
Elements of C_n: $\sigma : \Delta^n \to Y \subseteq X$

$$\begin{gathered}
\partial : C_n(X,Y) \to C_{n-1}(X,Y) \\
\delta \ (\text{mod } C_n(Y)) \mapsto \partial\delta \ (\text{mod } C_{n-1}(Y)) \\
H_n(X,Y) \equiv H_n(C_\bullet(X,Y)) = Z_n(X,Y)/B_n(X,Y), \quad \textit{cycle/boundary}
\end{gathered}$$

Definition 2.2.2 (Retraction)**.** *Consider $Y \subseteq X$, a retraction of X onto Y is a map $r : X \to Y$ such that, $r(X) = Y$ and $r^2 = r$.*

i.e. $r(y) = Y$ if $y \in Y$
$i : Y \to X \quad r \circ i = id_Y \quad i \circ r \neq id_x$ but $r_* \circ i_* = id$ in homology

Definition 2.2.3 (Deformation retract). *Consider $Y \subseteq X$: subspace*
Y is a deformation retract of X if there is a <u>*homotopy*</u> *between id_X and a retraction $r : X \to Y$*

$$(F_t) \quad F_t : X \to X \quad \begin{array}{l} F_0 : id_X \\ F_1 : X \to Y \quad F_1|_Y = id_Y \\ F_1(X) = Y \end{array} \Bigg| \; F_0 \simeq F_1 \textit{ homotopic}, \; id_X \simeq r$$

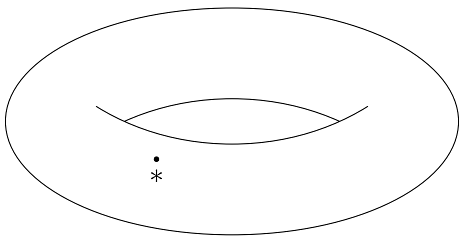
(X, Y) is a "good pair" if

- $Y \subseteq X$ - closed
- There is open $V \subseteq X$, *such that* V is a deformation retracts on Y.

Example 2.2.3.

$(\mathbb{R}^{n+1}, S^n)$ is a good pair
$S^n \subseteq \mathbb{R}^{n+1}$ - closed
$S^n \subseteq (\mathbb{R}^{n+1}/\{0\})$ and is a deformation retracted of it

Example 2.2.4. Consider X to be a torus, and Y a point on its surface:



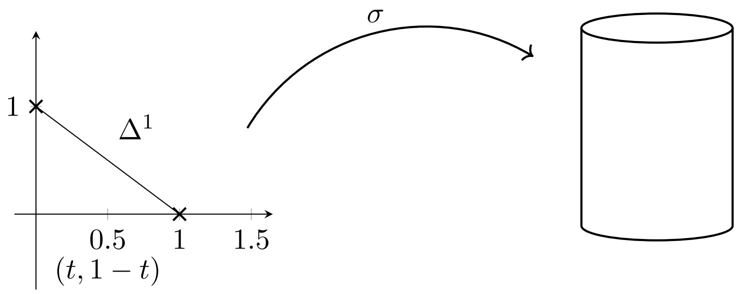
$$Y = \underset{pt}{\{*\}} \hookrightarrow X$$
$$0 \to C_n(Y) \to C_n(X) \to C_n(X/*) \to 0$$
$$H_n(Y) = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$$
$$\cdots \to H_n(Y) \to H_n(X) \to H_n(X, *) \to H_{n-1}(Y) \to \cdots$$
$$\cdots \to H_0(Y) \to H_0(X) \to H_0(X, *) \to 0$$
$$n > 0 : \quad H_n(X, \star) = H_n(X)$$
$$n = 0 : \quad 0 \to \mathbb{Z} \to H_0(X) \to H_0(X, *) \to 0$$
$$H_0(X, \star) = H_0(x)/\mathbb{Z}, \quad i.e. \; \begin{array}{l} H_0(X) = \mathbb{Z}^d \\ H_0(X) = \mathbb{Z}^{d-1} \end{array}$$

Remark: Sometimes one introduces "reduced homology"

$$\cdots \to C_n(X) \to C_{n-1} \to \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0, \quad \tilde{H}_n(X) \text{ - reduced homology}$$
$$\sum n_i \sigma_i \mapsto \sum_i n_i \in C_0(X)$$
$$\tilde{H}_n(X) = \begin{cases} H_n(X), n > 0 \\ H_0^{sing}(X) = H_0(X) \oplus \mathbb{Z} \end{cases}$$
$$H_n(X, *) \to \tilde{H}_n(X)$$

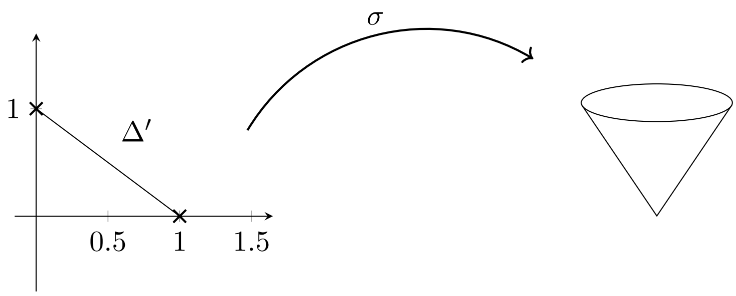
Let us consider some examples of Reduced Homology:

Example 2.2.5. Consider $X = S^1 \times [0,1]$, and $Y = S^1 \times \{0\}$:



$$\sigma : \Delta^1 \mapsto S^1 \times [0,1]$$
$$\sigma : t \mapsto (\cos 2\pi t, \sin 2\pi t, 1)$$

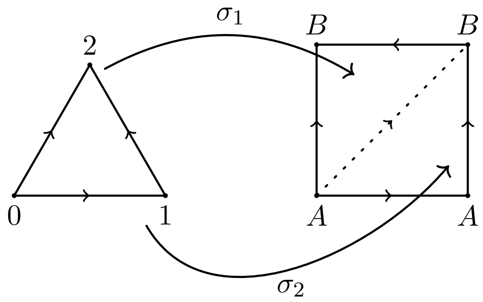
The boundary operator on σ: $\partial\sigma = 0 \rightsquigarrow [\sigma] \in H_1(X)$
The space X is homotopic equivalent to S^1 so, $H_1(X) \simeq H_1(S^1) \simeq \mathbb{Z}$, considered in 2.1.
Let us take the quotient map X/Y. The space Y will contract into a point creating a cone.



In $H_1(X,Y)$ the class $[\sigma] \in H_1(X)$ goes to 0 and the long exact sequence on homology is:

$$H_1(Y) \to H_1(X) \to H_1(X,Y) \to H_0(X,Y) \to H_0(Y) \to H_0(X) \to 0$$

where $H_1(Y), H_1(X), H_0(Y), H_0(X) \simeq \mathbb{Z}$.



$$\partial\sigma_1 = \sigma_1|_{[12]} - \sigma_1|_{[02]} + \sigma_1|_{[01]}$$
$$\partial\sigma_2 = \sigma_2|_{[12]} - \sigma_2|_{[02]} + \sigma_2|_{[01]}$$
$$\partial(\sigma_1 + \sigma_2) = \sigma_1|_{[12]} + \sigma_2|_{[01]}$$

We can choose $\sigma = \sigma_1|_{[12]} = \partial(\sigma_1 + \sigma_2) - \sigma_2|_{[01]}$, $\sigma_2|_{[01]} \in C_1(Y)$
$\sigma \in B_1(X,Y) \Rightarrow [\sigma] = 0 \in H_1(X,Y)$

Chapter 3

Persistent Homology

Persistent homology (PH) is a method used in topological data analysis (TDA) to study qualitative features of data that persist across multiple scales. Due to its construction, persistent homology computations are robust to perturbation of the data [OPT+17]. Also, PH is used to extract relevant features of the data, and separate them from noise.

An important result in computing persistent homology is that the persistent homology of a filtered d-dimensional simplicial complex is simply the standard homology of a particular graded module over a polynomial ring [ZC05].
Before introducing Persistent homology and understanding its calculation, we need to provide some preliminary concept from algebra.

We begin by reviewing graded modules and rings and then stating the structure of finitely generated modules over principal ideal domains. Then, referring to the notions introduced in Chapter 1, we provide some comments of the reduction algorithm used for computing simplicial homology. We conclude this chapter by describing persistent homology.

3.1 Background

3.1.1 Graded Rings and Modules

A graded ring is a ring $\langle R, +, \cdot \rangle$ equipped with a direct sum decomposition of Abelian groups $R \cong \bigoplus_{i \in \mathbb{Z}} R_i$, so multiplication is defined by bilinear pairings $R_n \otimes R_m \to R_{n+m}$.

Elements in a single R_i are called homogeneous and have degree i, $\deg e = i$ for all $e \in R_i$ [Art11].

Example 3.1.1. $R = A[t]$, where (A - commutative ring)

$$R_0 = A, \quad R_1 = \{at, a \in A\}, \quad \dots \, , R_i = \{at^i, a \in A\}$$

Example 3.1.2. $R = \mathbb{R}[x, y, z]$

$$R_i = \{cx^{d_1} y^{d_2} z^{d_3} | \sum_{k=1}^{3} d_k = i\}$$

For example $R_1 \simeq \mathbb{R}^3$ as a vector space $\{ax + by + cz\}$

A graded module M over a graded ring R is a module equipped with a direct sum decomposition $M \cong \bigoplus_{i \in \mathbb{Z}} M_i$, so that the action of R on M is defined is bilinear pairings $R_n \otimes M_m \to M_{n+m}$.

A graded ring (module) is non-negatively graded if $R_i = 0$ ($M_i = 0$) for all $i < 0$.

Note: R is a PID if it's a domain (no zero divisors) & all its ideal are principal

For example:

$$R = \mathbb{Z},\ I = (n),\ n \in \mathbb{Z}$$
$$R = k[t],\ k\ -\ field$$

The structure theorem describes finitely generated modules and graded modules over PIDs.

Theorem 3.1.1 (Structure Theorem)**.** *If D is a PID, then every finitely generated D-module is isomorphic to a direct sum of cyclic D-module. That is, it decomposes uniquely into the form*

$$D^{\beta} \oplus (\bigoplus_i D/d_i D)), \tag{3.1}$$

for $d_i \in D$, $\beta \in \mathbb{Z}$, such that $d_i | d_{i+1}$. Similarly, every graded module M over a graded PID D decomposes uniquely into the form

$$(\bigoplus_i \Sigma^{\alpha_i} D) \oplus (\bigoplus_i \Sigma^{\gamma_j} D/d_j D)), \tag{3.2}$$

where $d_j \in D$ are homogeneous elements so that $d_j | d_{j+1}$, $\alpha_i, \gamma_j in \mathbb{Z}$, and Σ^{α} denotes an α-shift upward in grading.

The free portion on the left is a vector includes generators that may generate an infinite number of elements. Decomposition (3.1) has a vector space of dimension β. The torsional portion on the right includes generators that may generate a finite number of elements. These torsional elements are also homogeneous. Intuitively then, the theorem describes finitely generated modules and graded modules as structures that look like vector spaces but also have some dimensions that are "finite" in size.

Example 3.1.3. Let us take $D = k[t]$ – graded ring(e.g. $\mathbb{R}[t]$) then:

$$D = \underset{\underset{M_0}{\parallel}}{k} \oplus \underset{\underset{M_1}{\parallel}}{kt} \oplus \underset{\underset{M_2}{\parallel}}{kt^2} \oplus \cdots$$

is also a graded module over itself.

$$M = \sum^{\alpha} D = t^{\alpha} k[t] \subseteq k[t] \text{ is an ideal of } D \Rightarrow D \text{ - module}$$
$$M = \underset{\underset{k}{\wr|}}{M_{\alpha}} \oplus \underset{\underset{kt}{\wr|}}{M_{\alpha+1}} \oplus \cdots$$
$$(\sum^{\alpha} D)_i = D_{\alpha+i}$$

3.1.2 Reduction

The reduction algorithm is the standard method used in computing homology. For simplicity, we describe the method for integer coefficients. However, the method applies also to modules over arbitrary PIDs.

Given C_k, we can use as standard basis the oriented k-simplices:

$$\partial_k : \underset{\{e_i\}}{C_k} \to \underset{\{\hat{e_i}\}}{C_{k-1}}$$

$(C_k, C_{k-1}$: free abelian groups or free R-module)

M_k : standard matrix representation of ∂_k

$$e \cdot u : \left(e_1, \cdots, e_{m_k}\right) \begin{pmatrix} u_1 \\ \vdots \\ u_{m_k} \end{pmatrix} \in C_k$$

$$\partial_k(eu) = \hat{e}(M_u)$$

The null-space of M_k corresponds to Z_k and its range-space to $B_{k?1}$. The reduction algorithm derives alternate bases for the chain groups, relative to which the matrix for ∂_k is diagonal. The algorithm utilises the following elementary column operations on M_k:

- exchange column i, and column j,
- multiply column i by -1
- replace column i by (column i) + q(column j), where $q \in \mathbb{Z}$ and $i \neq j$

The algorithm also uses elementary row operation that are similarly defined. The idea of the reduction algorithm is to systematically modify the bases of C_k and $C_{k?1}$ using elementary operations so that it reduce M_k to its Smith normal form:

Row-operation

$$R_i \mapsto R_i + qR_j \text{ on } M$$

$$M \mapsto i \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & q \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\text{A}} M$$

$$R_2 \mapsto R_2 + 2R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} M$$

$$\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e_1 & e_2 & 2e_2 + e_3 \end{pmatrix}$$

Column operations:

$$M \mapsto MB, B_2 \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & q & & 1 \end{pmatrix}$$

Interpretation of row/column operations on a matrix of a map in terms of changing bases.

$$\partial : \xi = \underline{e}u \mapsto \hat{e}Mu \text{ where } u = [\xi]\underline{e}$$

Suppose we perform a column-operation $M \mapsto MB$. This is supposed to change the matrix of a map - keeping the map unchanged

$$\xi = \underset{\underline{e}B(B^{-1}u)}{\underset{\|}{\underline{e}u}} \mapsto \hat{e}Mu = \hat{e}MB(B^{-1}u)$$

If we set $\underline{e}' = \underline{e}B$

$\partial : \xi = \underline{e}'v \mapsto \underline{\hat{e}}MBv$

$v = [\xi]_{\underline{e}'}$

Similarly, suppose we perform a row-operation $M \mapsto AM$. Then

$$\begin{aligned} \xi = \underline{e}u \mapsto \underline{\hat{e}}Mu &= \underline{\hat{e}}A^{-1}AMu \\ &= \underline{\hat{e}}'AMu \end{aligned}$$

That is, if $M = [\partial]_{\underline{e}\underline{\hat{e}}}$, then

$$\begin{aligned} &AMB = [\partial]_{\underline{e}B,\underline{\hat{e}}A^{-1}} \\ &A = I + qE_{ij} : R_i \mapsto R_i + qR_j \text{ (via } M \mapsto AM) \\ &A^{-1} = I - qE_{ij} \\ &\underline{\hat{e}}A^{-1} = (\hat{e}_1, \cdots, \hat{e}_{m_{k-1}}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \cdots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} i = \\ &= (\hat{e}_1, \cdots, \hat{e}_j - q\hat{e}_i, \cdots), \text{i.e. } \hat{e}_j \mapsto \hat{e}_j - q\hat{e}_i \end{aligned}$$

$$\begin{aligned} &B = I + qE_{ji} : C_i \mapsto C_i + qC_j (\text{via } M \mapsto MB) \\ &\underline{e}B = (e_1, \cdots, e_{m_k}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \cdots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} j = (e_1, \cdots, e_i + qe_j, \cdots) \end{aligned}$$

Let $V \simeq \mathbb{R}^n$(be a vector space)
$B = \{e_1, \cdots, e_n\}$

$$V = \underline{e}X = (e_1, \cdots, e_a)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i \quad (e_i \in V, x_i \in \mathbb{R})$$

$B^1 = \{f_1, \cdots, f_n\}$

$$v = \underline{e}X = \underbrace{\underline{e}T^{-1}}_{\underline{\mathrm{f}}}(\underbrace{TX}_{\mathrm{Y}}) = (f_1, \cdots, f_n)Y$$

$(f_1, \cdots, f_n) = (l_1, \cdots, l_n)T^{-1}$
$Y = TX$

Smith normal form (for PID):

$$\exists A, B:$$

$$\partial_k \equiv AMB = \left(\begin{array}{c|c} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_l \end{pmatrix} & \overbrace{\quad}^{} 0 \\ \hline 0 & 0 \end{array}\right)^{b_i | b_{ix1}}$$

$$rank Z_k = m_k - e_k$$

$$rank\ H_k = m_k - e_k - e_{k+1}$$

3.2 The Persistence Module

In this section we will combine the homology of all the complexes in the filtration into a single algebraic structure. We then establish a correspondence that reveals a simple description over fields. Most significantly, we illustrate that the persistent homology of a filtered complex is simply the standard homology of a particular graded module over a polynomial ring.

Taking into consideration the construction of a filtered simplicial complex introduced in section 1.2, we can construct a filtered chain complex:

$$\begin{array}{ccccccccc} (0) & \subseteq & C_\bullet^1 & \subseteq & C_\bullet^2 & \subseteq & \cdots & \subseteq & C_\bullet^m \\ \| & & & & & & & & \| \\ C_\bullet^0 & & & & & & & & C_\bullet \end{array}$$

Definition 3.2.1. *(Persistent Homology Group). Given a filtered complex, the i-th complex K^i has associated boundary operators ∂_k^i, matrices M_k^i , and groups C_k^i, Z_k^i, B_k^i, and H_k^i for all $i, k \geq 0$ The p-persistent k-th homology group of K^i is*

$$H_k^{i,p} = Z_k^i \Big/ (B_k^{i+p} \cap Z_k^i)$$

For $p = 0$, this is the usual homology formula:

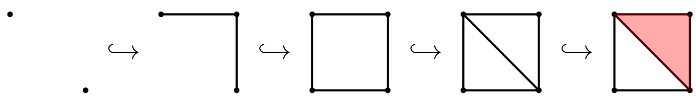
$$H_k(C_\bullet^i) = Z_k^i / (B_k^i \cap Z_k^i) = Z_k^i / B_k^i$$

Definition 3.2.2. *(Persistence Complex)*

A persistence complex $\mathcal{C}$ is a family of chain complexes $\{C_*^i\}_{i\geq 0}$ over R, together with a chain map's $f_i : C_*^i \to C_*^{i+1}$ so that we have the following diagram:

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \dots$$

Example 3.2.1. Let us consider the following filtered simplicial complex, and the filtered chain complex:



$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
0 \arrow[r, hook, "f^0"] \arrow[d] & \mathbb{Z}^2 \arrow[r, hook, "f^2"] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook, "f^3"] \arrow[d] & \mathbb{Z}^5 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
\mathbb{Z}^2 \arrow[r, hook, "f^0"] \arrow[d] & \mathbb{Z}^3 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
0 \arrow[r, hook, "f^0"] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
C_\bullet^0 \arrow[r, hook, "f^0"] & C_\bullet^1 \arrow[r, hook, "f^1"] & C_\bullet^2 \arrow[r, hook, "f^2"] & C_\bullet^3 \arrow[r, hook, "f^3"] & C_\bullet^4
\end{tikzcd}$$

Definition 3.2.3. *(Persistence Module). A persistence module $\mathcal{M}$ is a family of $R-$modules, M^i, together with homomorphism $\varphi^i : M^i \to M^{i+1}\}$*

Suppose we have a persistence module $\mathcal{M} = \{M^i, \varphi^i : M^i \to M^{i+1}\}$ over a ring R, We can equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by

$$\alpha(M) = \underset{i\geq 0}{\oplus} M_i$$

, where the $R - module$ structure is the sum of the structures on the individual components, and where the action of t is given by:

$$t \cdot (m^0, m^1, \cdots) = (0, \varphi^0(m^0), \varphi^1(m^1), \cdots)$$

$$\left(\begin{array}{c|c|c|c}
0 & & & \\
\hline
\varphi^0 & 0 & & \\
\hline
 & \varphi^1 & 0 & \\
\hline
 & & \varphi^2 &
\end{array}\right)$$

t simply shift elements of the module up in gradation.

Example 3.2.2.

$$
\begin{aligned}
&\varphi : \mathbb{R}^n \to \mathbb{R} \\
&\varphi : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (A_1, \cdots, A_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A_1 x_1 + \cdots + A_n x_n \\
&(\varphi_1, \cdots, \varphi_{n-1}) \mathbb{R}^n \to \mathbb{R}^{n-1} \\
&\underline{x} \mapsto \begin{pmatrix} A_{11} & \cdots & A_{m-1} \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&\text{e.g.} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

Theorem 3.2.1. *(Correspondence) The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated non-negatively graded modules over $R[t]$.*

The proof is the Artin-Rees theory in commutative algebra [Eis95]

The Correspondence theorem gives us a simple decomposition when the ground ring is a field F. In this case the graded ring $F[t]$ is a PID and its only graded ideals are homogeneous of form (t_n), so the structure of the $F[t]-module$ is described by sum (3.2) in structure theorem 3.1.1:

$$
(\bigoplus_i \Sigma^{\alpha_i} F[t]) \oplus (\bigoplus_j \Sigma^{\gamma_j} F[t]/(t^{n^j})). \tag{3.3}
$$

By using the result from the correspondence theorem, we can build a singe structure that contains all the simplicial complexes in the filtration. The computation start with the direct sum of complexes, and develops into a larger space graded according to the filtration order. The filtration order represents the time when each simplex enters the filtration. The main idea is that the filtration ordering is encoded in the coefficient polynomial ring. For examples on how this encoding is calculated, Chapter 4

3.3 Čech and Vietoris-Rips Complex

In a ever-increasing world of data, the need for data analysis techniques and method is very high. Topological Data Analysis is a relatively new sphere in Data Analysis, that is bringing new insights in the study of data.

The principal themes is previous surveys from Carlsson, de Silva, Harer, Zomorodian and other, are:

1. It is useful to replace a set of data points with a family of simplicial complexes, indexed by a proximity parameter.

2. It is beneficial to use persistent homology to calculate these topological complexes.

3. It is helpful to encode the persistent homology of a data set in the form of a parameterised version of a Betti number: a barcode

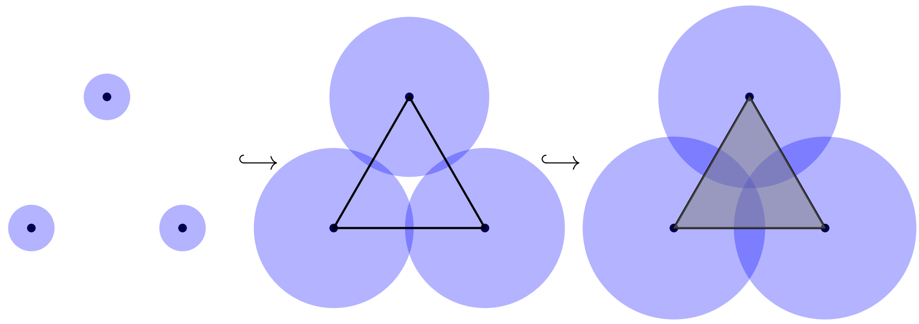
Data is collected from different number of sources, and comes in a variety of form and shapes. Often data is represented as a unordered sequence of points in a Euclidean $n-dimentional$ space $\mathbb{E}^n$. The overall shape of data might provide important information about the underlying phenomena that the data represents. A **point cloud data** is the type of dataset where global feature are present and important. It is exactly this type of data that is tackled by persistent homology. [Ghr07]

Naturally, the question of how to represent a collection of points $\{x_\alpha\}$, rises. A straightforward approach would be to use the point cloud as vertices of a combinatorial graph whose edges are determined by proximity, for example within a specific distance ϵ. A graph of this type can capture the connection of the point cloud, but it will fail to detect any higher order features beyond clustering. A good way to tackle this issue, it to complete the graph to a simplicial complex. The choice of how to fill in the higher dimensional simplices of the proximity graph will result in different global representation.

Two more prominent ways on how to fill in the higher dimensional simplices are: Čech and Vietoris-Rips Complex:

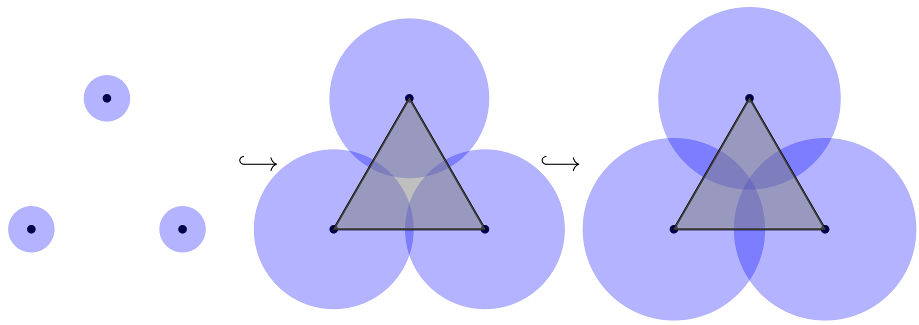
Definition 3.3.1. *(Čech Complex) Given a collection of points $\{x_\alpha\}$ in Euclidean space $\mathbb{E}^n$, the Čech Complex, $\mathcal{C}_\epsilon$, is the abstract simplicial complex whose k-simplices are determined by unordered (k+1)-uple of points $\{x_\alpha\}_0^k$ whose closed $\epsilon/2-ball$ neighbourhoods have a point of common intersection.*

For example:



Definition 3.3.2. *(Vietoris-Rips Complex) Given a collection of points $\{x_\alpha\}$ in Euclidean space $\mathbb{E}^n$, the Rips Complex, $\mathcal{R}_\epsilon$, is the abstract simplicial complex whose k-simplices correspond to unordered (k+1)-uple of points $\{x_\alpha\}_0^k$ which are pairwise within distance ϵ.*

For example:



A useful tool that is relevant to building our argument is Mayer-Vietoris sequences which is the analog for homology of van Kampen's theorem for the fundamental group.

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a Mayer-Vietoris sequence, which is equally powerful but is sometimes more convenient to use. For a pair of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B , this exact sequence has the form

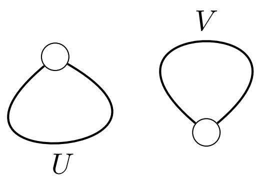
$$... \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to ... \to H_0(X) \to 0$$

The Mayer-Vietoris sequence is then the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

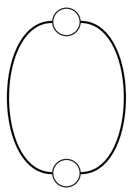
$$0 \to C_n(A \cap B) \xrightarrow{\Phi} C_n(A) \oplus C_n(B) \xrightarrow{\Psi} C_n(A + B) \to 0$$

If we think of X, not only as the union of two interiors but as a topological space, where $X = \underset{\alpha}{\cup}\ \mathcal{U}_\alpha$ where $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ are cover indexes, $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n\}$, given a cover $\mathcal{U}$ (not necessarily open) of a space X, the nerve of $\mathcal{U}$ is a simplicial complex $\mathcal{N}(\mathcal{U})$ whose n-simplices consist of sets of $n + 1$ elements of $\mathcal{U}$ with non-empty intersections. Vertices of $\mathcal{N}(\mathcal{U}) = \{1, 2, \cdots, m\}$ are $(k - 1)$ - faces if $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_k} \neq \emptyset$ for $\{i_1, \cdots, i_k\} \in \mathcal{N}(\mathcal{U})$

Example 3.3.1. Let's break the circle S^1 into the following spaces U, V:



where $\mathcal{U} = S' \backslash \{0, 1\}$ *and* $\mathcal{V} = S' \backslash \{1, 0\}$
$S^1 = \mathcal{U} \cup V$
The cover $\mathcal{U} = \{U, V\}$

$\underset{U_1}{\mathcal{U}} \cap \underset{U_2}{V} =$ 

$$\{\{1, 2\}, \{1\}, \{2\}, \emptyset\} = \mathcal{N}(\mathcal{U})$$

The Čech complex $\mathcal{C}_\epsilon$ is the nerve of the collection of closed discs of radius $\epsilon/2$ around the points of the cloud, thought of as a cover of its union.

Theorem 3.3.1. *(The Nerve Theorem)*
If X is a paracompact space, and U is an open cover of X such that the intersection of any finite subfamily of U is either empty or contractible, then the realisation of the nerve of U is homotopy equivalent to X.[Hat02]

The nerve theorem shows that we can recover X by the homotopic equivalence of the $\mathcal{N}(\mathcal{U})$. Actually, the nerve theorem also shows that the Čech complex is a better topological model for the point cloud data. However, computing persistent homology using the Čech complex turns out to be difficult and computationally expensive, because the program should either have precise distances between vertices or load in memory and the entire complex. This is why we also define the Rips complex that is often referred to as a 'flag' complex. The Vietoris-Rips complex is completely determined by its 1-skeleton, so there is not need to load the entire complex in memory. To compute homology, it is enough to store its 1-skeleton, and reconstruct higher dimensional simplexes whenever it is needed.

It is not easy to show why we can use the Rips complex to compute homology without linking it to the Čech complex and take advantage of its nice properties, stated in the nerve theorem above.

Using the triangle inequality, if closed balls of radius $\epsilon/2$ of a set of n points, intersect then the pairwise distance between the points is $\leq \epsilon$. Therefore, there is a embedding $\mathcal{R}_{\epsilon/2} \hookrightarrow \mathcal{C}_\epsilon$. This result is easily seen in the examples of three points given above.

Furthermore, if n + 1 points are within distance $\epsilon/2$ of one another, then each of them is contained in the intersection of the closed balls of radius $\epsilon/2$. Thus we also have an embedding $\mathcal{R}_{\epsilon/2} \hookrightarrow \mathcal{C}_\epsilon$.

Combining the two results above, we can construct the following chain of embeddings:

$$\mathcal{C}_{\epsilon/2} \hookrightarrow \mathcal{R}_{\epsilon/2} \hookrightarrow \mathcal{C}_\epsilon \hookrightarrow \mathcal{R}_\epsilon$$

This embedding shows that the Rips complex is bounded on both sides by a Čech complex, and every topological feature that persists from ϵ to ϵ', will also persists in a Rips complex if $\frac{\epsilon}{\epsilon'} \geq 2$.

In fact this lower bound can be further improved to $\sqrt{2}$, which can be proved using the result from [dSG07].

In conclusion, the Čech complex is a good model for the "shape" of the point cloud data, and because of the chain of embeddings above, the Rips complex not only facilities computation and its persistent homology precisely models that of the Čech complex.

We will see in section 4.1, how the Vietoris Rips complex is used in computing barcodes.

Chapter 4

Computing Persistent Homology

So far we have build the following setup to compute persistence homology on persistence chain complex:

$$\begin{aligned}
&Persistence\ complex: C^0_\bullet \hookrightarrow C^1_\bullet \hookrightarrow C^2_\bullet \hookrightarrow \cdots \\
&Persistent\ Homology\ Group: \ H^{i,p}_k = Z^i_k / B^{i+p}_k \cap Z^i_k \\
&where\ Z^i_k = \ker(\partial^i_k : C^i_k \to C^i_{k-1}) = \operatorname{Im}((H^i_k = H_k(C^i_\bullet) \to (H_k(C^{i+p}_\bullet))
\end{aligned}$$

For a complete picture, we can encapsulate the process of computing and visualising persistent homology, by the following "persistent homology pipeline" stated in [RB19]:

$$\{\text{finite metric space}\} \longrightarrow \{\text{filtered simplicial complex}\} \longrightarrow \{\text{barcodes / persistence diagrams}\},$$

where a barcode is defined as a multiset of non-empty intervals of the form either $[x, y) \subset \mathbb{R}$ *or* $[x, \infty]$

The following theorem, which is a key classification result in [ZC05], shows that all the information in a filtered system of vector spaces can be encoded into barcodes.

Theorem 4.0.1. *Let $\mathbb{F}$ be a field. There is a bijection between the set of finite barcodes and the set of isomorphism classes of filtered $\mathbb{F}$-vectors spaces of finite type*

The connection between the filtered simplicial complexes and filtered $\mathbb{F}$-vectors spaces, will be clarified in the computations below.

In order to understand the computations, it is important to pay attention to the following remarks.

Remarks on Computing Persistent Homology

- Any ideal $\omega \subseteq R$ in a commutative ring in as R - module in a natural way: $\forall a \in \omega, \forall r \in R, r \cdot a \in \omega$ and $\omega \subseteq R$ is an ab$\dot{\text{g}}$roup(R is a module over itself & ω is a submodule) Hence, for any

$$\begin{array}{ccc}
 & & n \in \mathbb{N} \\
 & (t^n) & \subseteq k[t] \\
 & \| & \\
\{ \ t^n & P(t), P \in k[t]\} & \\
\| & & \\
t^n k[t] & &
\end{array}$$

is a free $k[t]$ - module.

- $R = k[t]$ has a natural structure of <u>graded ring</u>. The deg i - elements are the (non-zero) elements of the line $R_i := kt^i \subseteq k[t]$ (in general: $R = \oplus R_i, R_iR_j \subseteq R_{i+j}$). The ideal $(t^n) \subseteq k[t]$ is then a graded $k[t]$ - module (in general, this means $M = \underset{i}{\oplus} M_j, R = \underset{i}{\oplus} R_i, R_i \cdot M_j \subseteq M_{i+j}$) $(t^n) = t^n, k[t] = \underset{i \geq 0}{\oplus} \underbrace{kt^{n+i}}_{(t^n)}$
  i.e.
  degree - i elements of (t^n) are the monomials of degree $(n+i)$

- While $(t^n) \subseteq k[t]$ is an ideal, so is $k[t]$ - module and $k[t]$ - submodule; and a graded $k[t]$ - module, it is <u>not</u> a graded $k[t]$ - submodule!

- The grading of (t^n) is <u>not</u> the grading that is induced by the <u>ambient</u> $k[t]$: it is <u>shifted up</u> by n: t^{n+i}, as an alternative of $(t^n) = t^n k[t]$, has degree i, not degree $(n+i)$.

- In general, for a graded module M over a graded ring R we can define the <u>twist of M by n</u>, $M[n]$, also denoted by $\sum^n M$, is defined by $(\sum^n M)_i = M_{n+i}$.
  i.e., by redefining/shifting the grading up by n. We see that we can identify $\sum^n k[t]$ with $(t^n) = t^n k[t]$ as <u>graded modules</u> $(\sum^n k[t])_i = k[t]_{n+i} = kt^{n+i} \xrightarrow{id} kt^n \cdot t^i$

- A map of free $k[t]$ - modules $k[t] \to k[t]$ is multiplication by some polynomial $p(t)$ i.e. of the kind $q(t) \mapsto p(t)q(t)$ ($p(t)$ is the image of $1 \in k[k]$). More generally, a map of free $k[t]$ - module, $k[t]^{\oplus^m} \to k[t]^{\oplus^r}$ is given by some $r \times m$ matrix with $k[t]$ - entries.

- A map (morphism) of <u>graded R-modules</u> $\varphi : M \to N$ is a map of R - modules (i.e.: R - linear, i.e.: $\varphi(rm) = r\varphi(m), \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$) which <u>preserves the degrees</u>. That is $\varphi(M_i) \subseteq N_i, \forall i$.
  In particular, a map of graded modules $\varphi : \underset{\underset{(t^n)}{\wr|}}{\sum^n k[t]} \to \underset{\underset{(t^p)}{\wr|}}{\sum^p k[t]}$ must send $(t^n)_i = kt^{n+i}$ to $(t^p)_i = kt^{p+i}$.

- As the shifts $\sum^n k[t]$ are still free modules, any $k[t]$ - module homomorphism $\varphi : (t^n) \to (t^n)$ is determined by some polynomial $p(t) \in k[t]$, i.e. by the image of a generator: $\varphi(t^n) = p(t)t^p$ as $\varphi(t^n q(t)) = q(t)\varphi(t^n)$. <u>However</u> if φ is a <u>graded</u> module homomorphism, we must have that $\varphi(t^n) = \underbrace{(mt^p)}_{p(t)} t^p$, ie, that $p(t)$ be homogeneous. Similarly, for direct sums $\underset{i}{\oplus}(t^{ni}) \to \underset{i}{\oplus}(t^{pj})$.

- <u>In particular:</u> if $n \geq p$, any map of modules $\varphi : k[t] \to k[t], 1 \mapsto p(t)$ determines a map of <u>graded</u> modules

$$
\begin{aligned}
&(t^n) = \sum^n k[t] \to (t^p) = \sum^p k[t] \\
&t^n \mapsto (p(t)t^{n-p})t^p
\end{aligned}
$$

  E.g, the identity map on $k[t]$ induces $t^n \mapsto t^{n-p} \cdot t^p$ (The "matrix element" of the identity on $k[t]$ is t^{n-p}).

Example 4.0.1. Consider X to be a space consisting of two points and Y, of an edge. Then we can construct the following persistence complex:

$$C_\bullet^0 \xhookrightarrow{\quad f_0 \quad} C_\bullet^1$$

$$\begin{tikzcd}
1 & (0) \arrow[r, hook] \arrow[d, "\partial_1=0"'] & \mathbb{Z} \arrow[d, "\partial_1=\begin{bmatrix}-1\\1\end{bmatrix}"] \\
0 & \mathbb{Z}^2 \arrow[r, hook] \arrow[d, "\partial_0=0"'] & \mathbb{Z}^2 \arrow[d, "\partial_0=\begin{bmatrix}0 & 0\end{bmatrix}"] \\
 & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

$\underline{p = 0}$

$$\begin{aligned}
H_k^{i,0} &= Z_k^i / B_k^i \cap Z_k^i \\
H_1^{0,0} &= (0) \quad H_1^{1,0} = (0) \\
H_0^{0,0} &= \mathbb{Z}^2 \quad H_0^{1,0} \simeq \mathbb{Z}
\end{aligned}$$

$\underline{p = 1}$

$$\begin{aligned}
H_k^{i,1} &= Z_k^i / B_k^{i+1} \cap Z_k^i \\
H_1^{0,1} &= Z_0^1 / B_1^1 \cap Z_1^0 = (0) \quad H_1^{1,1} = (0) \\
H_0^{0,1} &= Z_0^0 / B_0^1 \cap Z_0^0 = \mathbb{Z}^2 / \operatorname{Im} \begin{bmatrix}-1\\1\end{bmatrix} \simeq \mathbb{Z}
\end{aligned}$$

The final complex is $0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \to 0$. The matrix of ∂_1 is

$$M_1 = \begin{bmatrix}-1\\1\end{bmatrix}.$$

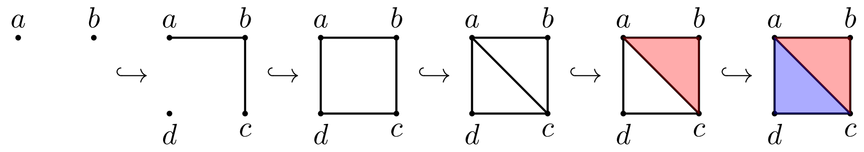
This gives rise to the complexes of $\mathbb{Z}[t] - modules$.

$$0 \to \mathbb{Z}[t] \xrightarrow{\begin{bmatrix}-1\\1\end{bmatrix}} \mathbb{Z}t^{\oplus 2} \to 0$$

and of graded $\mathbb{Z}[t]$ -modules.

$$0 \to \mathbb{Z}[t] \xrightarrow{\begin{bmatrix} -t \\ t \end{bmatrix}} \underset{\mathbb{Z}t^{\oplus 2}}{\underset{\parallel}{(1)^{\oplus 2}}} \to 0$$

Example 4.0.2. Let us consider the following filtered complex:



$$\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
 & C^0_\bullet \arrow[r] & C^1_\bullet \arrow[r] & C^2_\bullet \arrow[r] & C^3_\bullet \arrow[r] & C^4_\bullet \arrow[r] & C^5_\bullet \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[r] \arrow[d, "\partial_1^3"] & \mathbb{Z}^5 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[d] \\
0 & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

Some explicit computations are:
Here the matrix of ∂_1^3 is

$$M_1 = \left(\begin{array}{c|ccccc}
 & ab & bc & cd & ad & ac \\
\hline
d & 0 & 0 & 1 & 1 & 0 \\
c & 0 & 1 & -1 & 0 & 1 \\
b & 1 & -1 & 0 & 0 & 0 \\
a & -1 & 0 & 0 & -1 & -1
\end{array}\right)$$

If we introduce the structure of a graded module over $k[t]$ (on $R[t]$) where degrees of a simplex signify the appearance in the filtration:

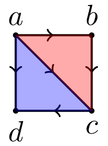
- $a, b \longleftrightarrow$ deg 0
- $c, d, ab, bc \longleftrightarrow$ deg 1
- $ab, dc \longleftrightarrow$ deg2

- $ac \longleftrightarrow$ deg 3
- $abc \longleftrightarrow$ deg 4
- $adc \longleftrightarrow$ deg 5

If we work in the graded PID $(\mathbb{Z}/2\mathbb{Z})[t]$

$$M_1 = \left(\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right)$$

* Everything mentioned in Remarks works similarly for $\mathbb{Z}[t]$ or $A[t]$, A - common ring. In particular, consider the last simplicial complex in the filtration:



The chain complex is $0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^5 \xrightarrow{\partial_1} \mathbb{Z}^4 \to 0$. The matrices M_2 and M_1 of ∂_2 and ∂_1 without the indicated bases are:

$$M_2 = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \begin{pmatrix} abc & acd \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Then ∂_1 (ie M_1) induces a map of $\mathbb{Z}[t]$ - modules $\mathbb{Z}[t]^{\oplus 5} \xrightarrow{M_1} \mathbb{Z}[t]^{\oplus 4}$:

$$\underline{V} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{pmatrix} \mapsto M_1 \underline{V}.$$

Using the grading (filtration) on the simplicial complex we get a chain complex of graded $\mathbb{Z}[t]$ - modules
$0 \to (t^4) \oplus (t^5) \xrightarrow{\partial_2^1} (t)^{\oplus 2} \oplus (t^2)^{\oplus 2} \oplus (t^3) \xrightarrow{\partial_1^1} (t)^{\oplus 2} \oplus (1)^{\oplus 2} \to 0$
by the procedure defined on [ZC05]. The respective matrices are (without same bases)

$$M_2' = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \begin{pmatrix} abc & acd \\ t^3 & 0 \\ t^3 & 0 \\ 0 & t^3 \\ 0 & -t^3 \\ -t^3 & t^3 \end{pmatrix}, M_1' = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & t & t & 0 \\ 0 & 1 & -t & 0 & t^2 \\ t & -t & 0 & 0 & 0 \\ -t & 0 & 0 & -t^2 & -t^2 \end{pmatrix}$$

Note: In Carlsson-Zomordian there is no difference in notation between M_i and M_i'. Also, for the algorithm they order the elements such that the degree decreases down to rows. If we work in $\mathbb{R}[t]$, the matrices will be the same. In $\mathbb{Z}/2\mathbb{Z}[t]$ there won't be difference between ± 1.

4.1 Computing Vietoris-Rips persistence barcodes using Ripser

In order to explicitly calculate persistence homology groups of point cloud data, I have created a Python scrip, that randomly generates points in a topological space, with and without noise, and then by using Ripser.py package, it calculates and plots Vietoris-Rips persistence barcodes. The project can be found in persitent homology repository. The library used is actually a wrapper around the original software written in C++. Before introducing details and results of my implementation, we will consider details in optimisation, methodology, computation and implementation of the Ripser software.

Ripser is a software design and implemented to calculate Vietoris-Rips persistence barcodes. The algorithm uses an implicit representation of the co-boundary operator and of the filtration order. Ripser has been optimized by avoiding any explicit construction and storage of the filtration co-boundary matrix, resulting in a significant improvement in time and memory usage.

The predominant approach to persistence computation consist of two steps: the generation of a filtration boundary matrix, and the computation of persistence barcodes using matrix reduction. Often, the construction of the filtration boundary matrix becomes the bottleneck for the computation of Vietoris-Rips barcodes.

That is why, Ripser takes a different approach by avoiding construction and storage of the whole filtration boundary matrix. The algorithm discards parts of the matrix and recomputes them when necessary. To reduce the memory usage, Ripser does not use an explicit matrix data structure, but uses a arithmetic operator, that recomputes the co-boundary map of a simplex whenever needed. The filtration is specified using another algorithmic operator for comparing simplices with respect to their appearance in the filtration order.

The computation of persistent homology implemented in Ripser is based on matrix reduction and uses four key optimisations in order to achieve an efficient implementation:

1. Clearing birth columns: Avoid computation of unnecessary cycles, by using the spectral structure of a boundary matrix D, $D^2 = 0$
2. Cohomology : Using Cohomology to compute persistence barcodes, since it is faster
3. Implicit representation of boundary and reduced boundary matrices: Decouple the description of the filtration and of the boundary operator, representing the boundary matrix only algorithmically instead of explicitly, and to avoid the storage of the entire unreduced and reduced boundary matrices, retaining only the much smaller reduction matrix encoding the column operations
4. Apparent and emergent pairs : The construction of the co-boundary matrix columns can be shortcut when a certain easily identified type of persistence pair, called an emergent co-face pair, is encountered

To construct the algorithm, Ripser Software makes use of simplicial complexes and filtaration introduced in chapter 1, with focus on the Vietoris Ripser complex, defined in 3.3.2. Despite the Vietris -Rips filtration, a re-indexing and refinement of the filtration is used. In order to compute persistent homology, one needs to apply one further step of re-indexing, refining the essential filtration to an essential simplex-wise one.

To ease the computation of the Vietoris-Rips filtration, Ripser makes use of sub-level sets of functions and persistent homology. The Ripser package considers only simplicial homology with coefficients in prime field $\mathbb{F}_p$. The homology computation is made possible by using simplex-wise refinement. A simplex-wise filtration gives rise to a filtration boundary matrix, which is the matrix of the boundary operator of the chain complex $C_*(K)$ with respect to the ordered basis given by the oriented simplices in filtration order.

During computation simplices are indexed in a combinatorial number system. Also, Ripser defines a refinement of the Vietoris-Rips filtration to an essential simplex-wise filtration, as required for the computation of persistent homology. The simplexes are ordered by increasing diameter, then by increasing dimension and then by decreasing lexicographic vertex order. The result is called the *lexicographically refined Vietoris-Rips filtration*.

4.1.1 Computation and Implementation

In the centre of the algorithm for computing persistent homology, there is the matrix reduction method. The matrix reduction method is similar to the one introduced in the previous section. However, since the matrix reduction process can be expensive in time and memory usage, a technique for clearing columns is used. Another interesting approach is the choice to opt for computing persistence barcodes using co-homology instead of homology of Vietoris-Rips filtration. This approach is backed up by de Silva at all [dSMVJ11], and it further optimises the clearing of columns. Furthermore, as already mentioned before, the matrix reduction is done implicitly. Apart from the computation techniques mentioned above, there are several more optimisations that are presented in detail in [Bau19]

In regards to the implementation, some important data structures of the C++ implementation of Ripser algorithm are:

- Input: The input for Ripser is a finite metric space (X, d), encoded in a comma (or whitespace, or other nonnumerical character) separated list as either a distance matrix (full, lower, or upper triangular part), or as a list of points in some Euclidean space
- Vertices and Simplices: Vertices are identified with natural numbers $0, ..., n?1$, where n is the cardinality of the input space. Simplices are indexed by natural numbers according to the combinatorial number system.
- Coefficients: Ripser supports the computation of persistent homology with coefficients in a prime field $\mathbb{F}_p$, for any prime number $p < 2^{16}$
- Column and matrix data structures: The basic data type for entries in a $(diameter_entry_t)$ boundary or coefficient matrix is a tuple consisting of a simplex index $(index_t)$, a floating point value $(value_t)$ caching the diameter of the simplex with that index, and a coefficient $(coeff_t)$ if coefficients are enabled.

4.1.2 Details and Results

The Python project is designed and implemented as a command line tool, that would allow the user to enter the name of a common space, the number of points, and the presence or absence of noise.

The application can be supplied with 3 different options - “space”, “points”, and ”no_noise“. The options “space” (–space or -s) stands for the different types of spaces that can be used. By default a sphere in $\mathbb{R}^3$ shall be applied. Other options include a cylinder in $\mathbb{R}^3$ (cylinder_3), a torus in $\mathbb{R}^3$ (torus_3) and $\mathbb{R}^4$ (torus_4), klein bottle in $\mathbb{R}^4$ (klein_bottle_4), projective plane in $\mathbb{R}^4$ (pro_plane_4) and $\mathbb{R}^6$ (pro_plane_6). After that, we have a “points” (–points or -p) parameter, indicating the number of points to be generated, where by default 500 shall be outputted. Last, the “no_noise” (-n or –no-noise) parameter is whether or not to add noise to the generated points - it being enabled by default.

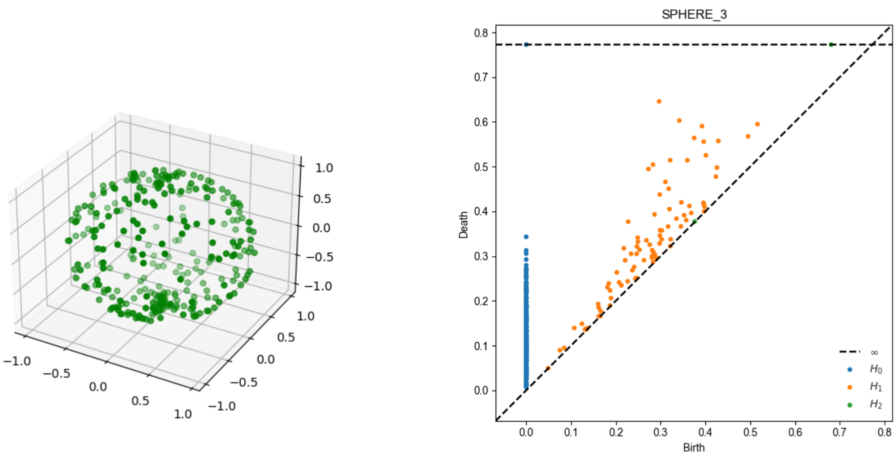
The application uniformly generates the number of points inputted by the user into a square 1×1. Then the resulting points are saved in a n-dimensional array and used as a domain for the functions that are parameterizing surfaces that are included in the application. The points in the square are mapped on the surface of a common space. When the parameter ”$-n$” is passed through the command line to the application, noise is introduced in the mapping of the points to the appropriate surface. The noise can generated randomly as uniform distribution or a Gaussian distribution, where the user can choose the σ and standard deviation, or the bounds for the uniform segment as an argument parameter.

If the space is embedded in $\mathbb{R}^3$ the points generated on the surface will be plotted in a 3D graph. Regardless the generated points will be outputted on the console. Then the application will pop out a figure of the calculation of Vietoris-Rips persistence diagram up to the homology specified by the user again as a argument parameter. Each persistent diagram is a pair (birth time, death time). After that, another figure will pop up showing separately the computations of the Vietoris-Rips persistence barcodes in the respective homology. The last graph plots life time of each point instead of birth and death.

Note: It is often useful to use persistence diagrams that present point plotted in $\mathbb{R}^2$ in a birth-death ratio, where there is an entire diagonal consisting of size zero bars. However, there is a direct translation of a persistent diagram to a barcode one: a point in a birth-death ratio shows the start and end point(ϵ distance) of a barcode.

Results

Sphere in $\mathbb{R}^3$.
Command: *python generator.py –space sphere_3 -points 500 -n*



(a) Plot. (b) Birth-death persistent diagram.

Figure 4.1: Plot and Death birth graphs.

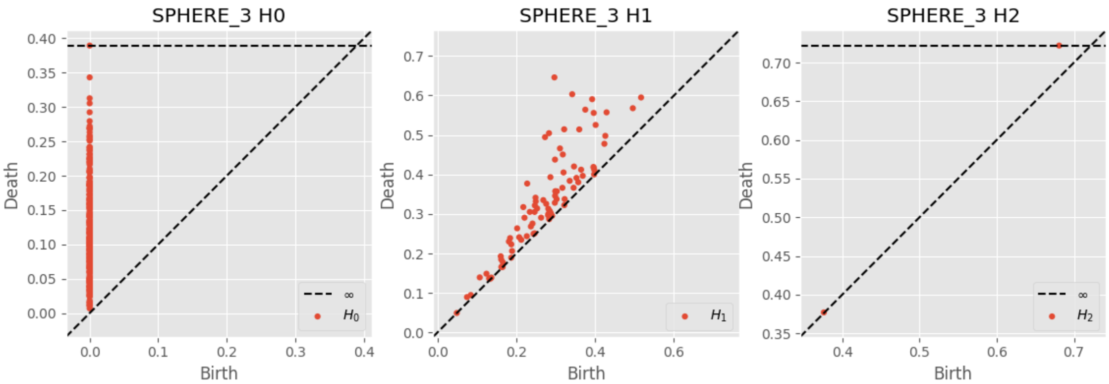


Figure 4.2: Birth-death persistent diagram separate homology.

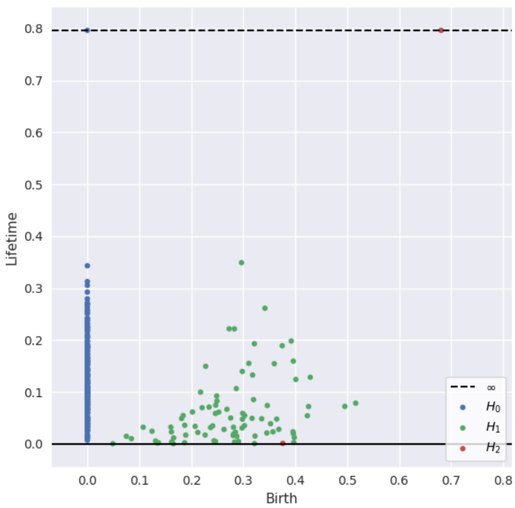
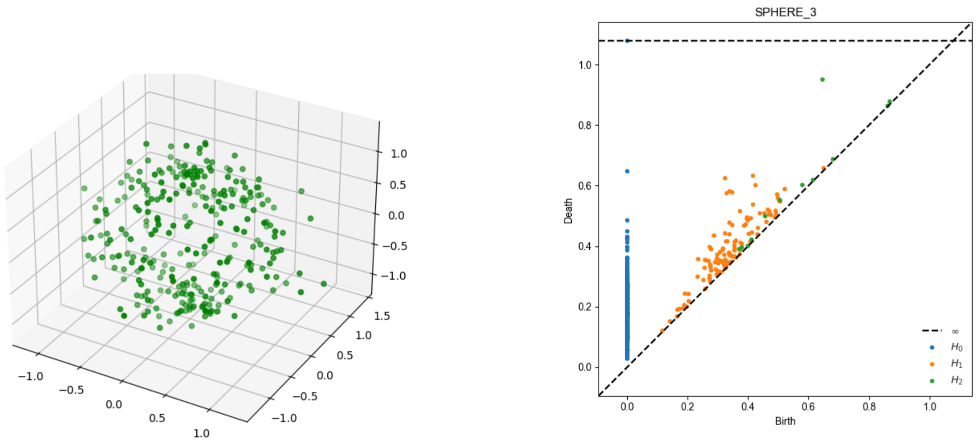


Figure 4.3: Lifetime persistent diagram.

Sphere in $\mathbb{R}^3$ with noise.
Command: *python generator.py –space sphere_3 -points 500*



(a) Plot. (b) Birth-death persistent diagram.

Figure 4.4: Plot and Death birth graphs.

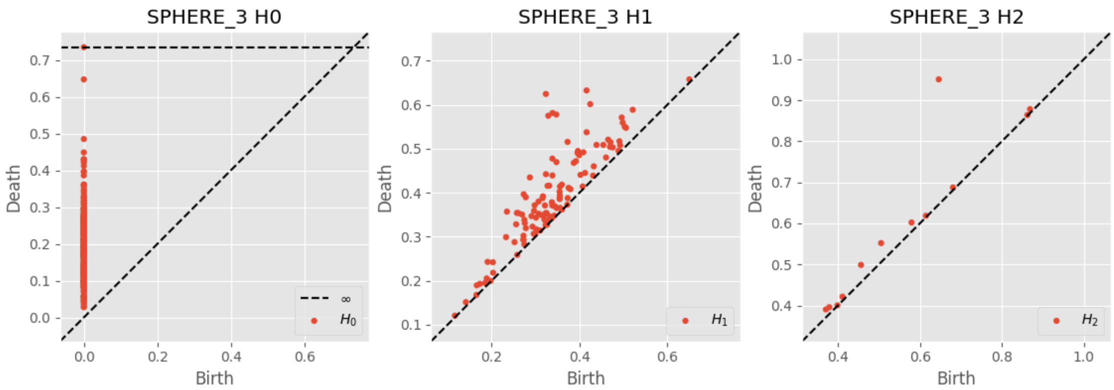


Figure 4.5: Birth-death persistent diagram separate homology.

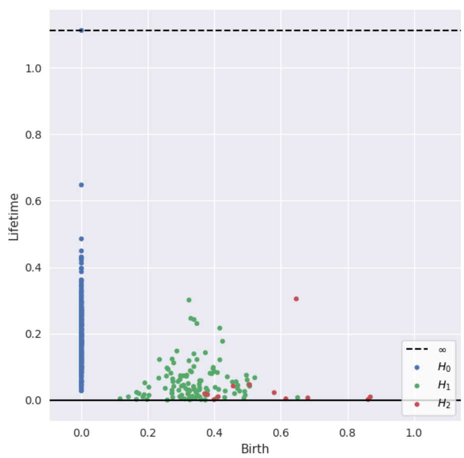
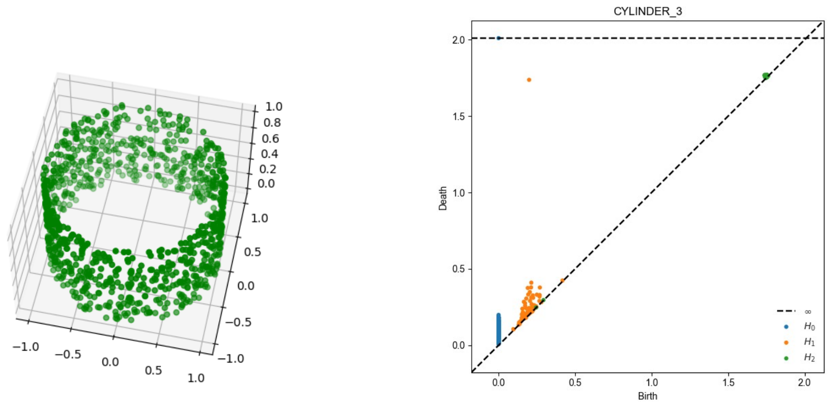


Figure 4.6: Lifetime persistent diagram.

Cylinder in $\mathbb{R}^3$.

Command: *python generator.py –space cylinder_3 -points 500 -n*



(a) Plot. (b) Birth-death persistent diagram.

Figure 4.7: Plot and Death birth graphs.

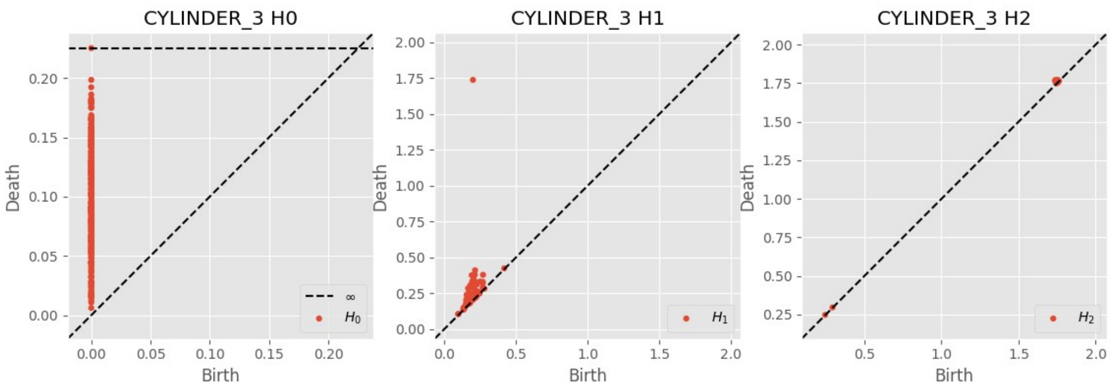


Figure 4.8: Birth-death persistent diagram separate homology.

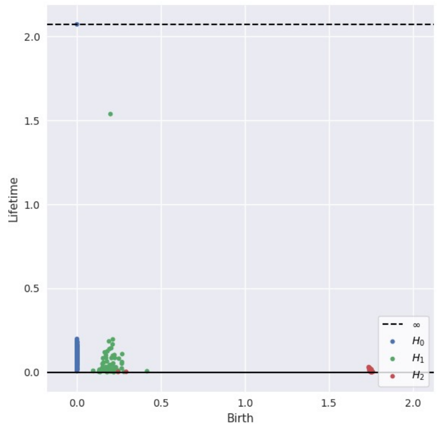
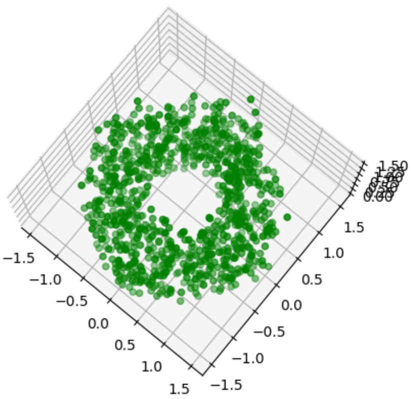
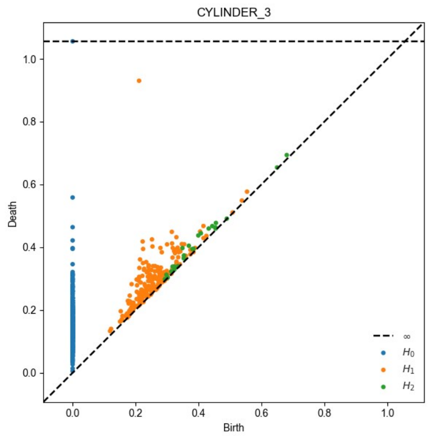


Figure 4.9: Lifetime persistent diagram.

Cylinder in $\mathbb{R}^3$ with noise.
Command: *python generator.py –space cylinder_3 -points 500*



(a) Plot.



(b) Birth-death persistent diagram.

Figure 4.10: Plot and Death birth graphs.

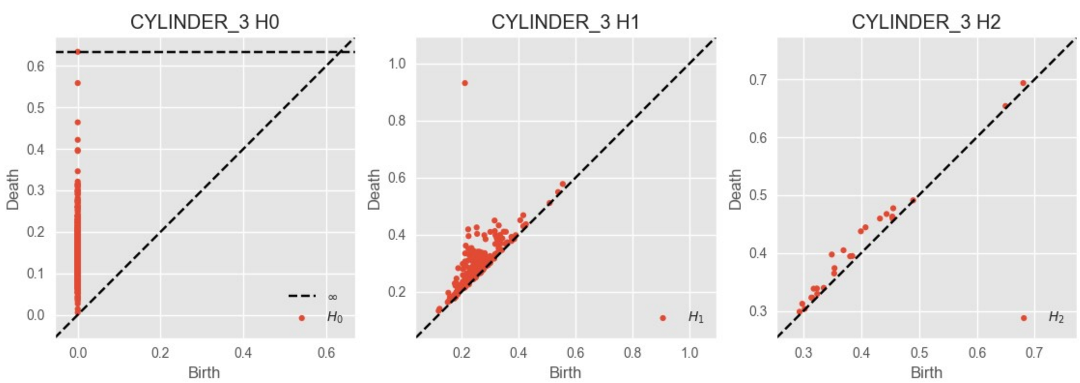


Figure 4.11: Birth-death persistent diagram separate homology.

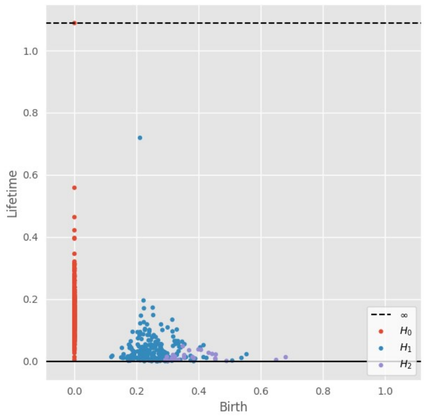


Figure 4.12: Lifetime persistent diagram.

Projective plane in $\mathbb{R}^4$.
Command: *python generator.py –space pro_plane_4 –points 300 -n –mean 1 -d 0.2 –up_to_homology 2*

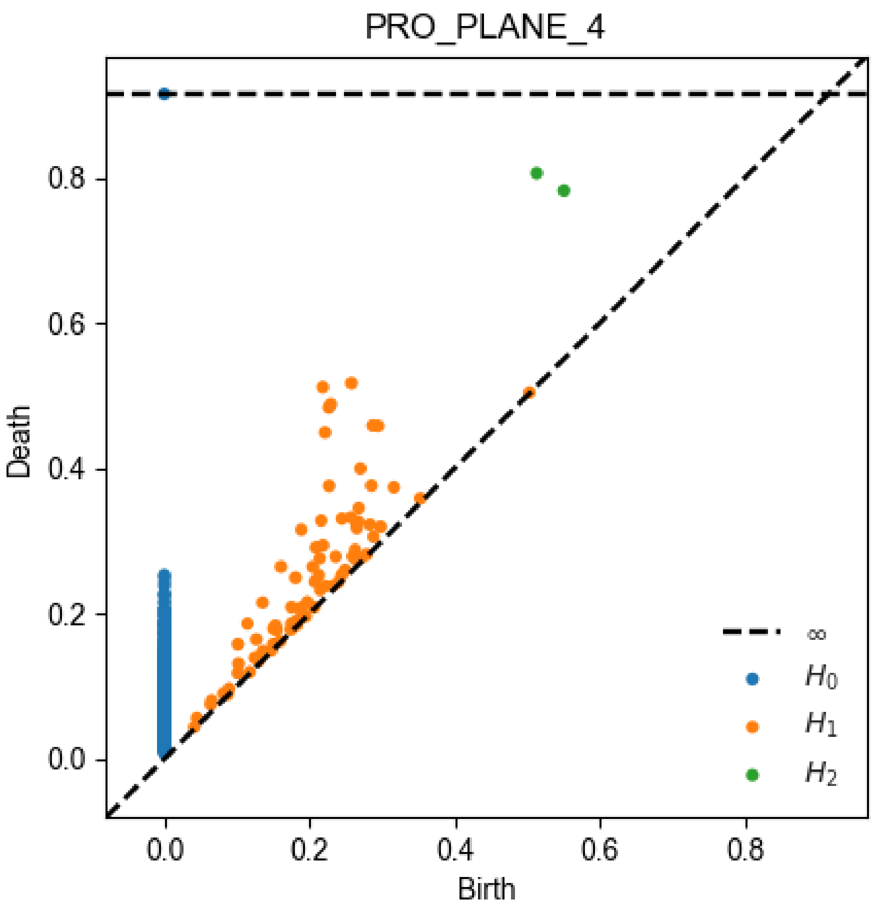


Figure 4.13: Plot and Death birth graphs.

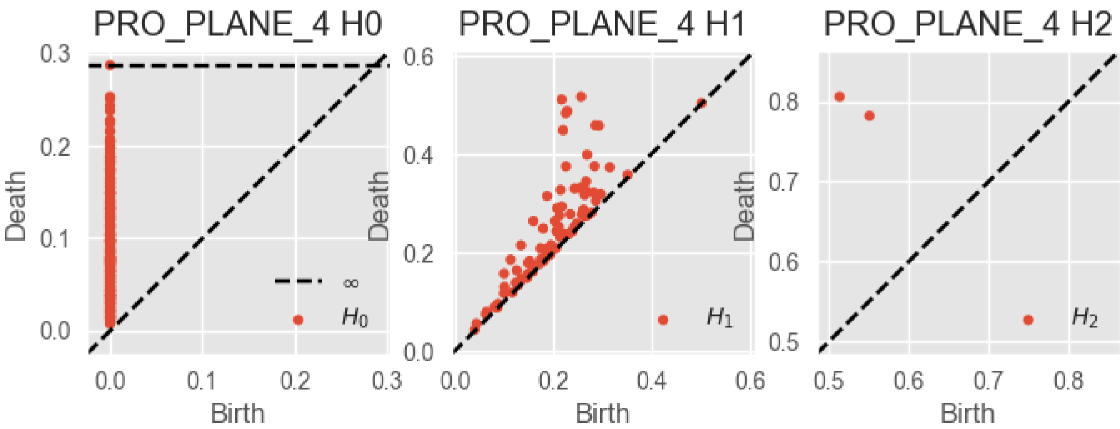


Figure 4.14: Birth-death persistent diagram separate homology.

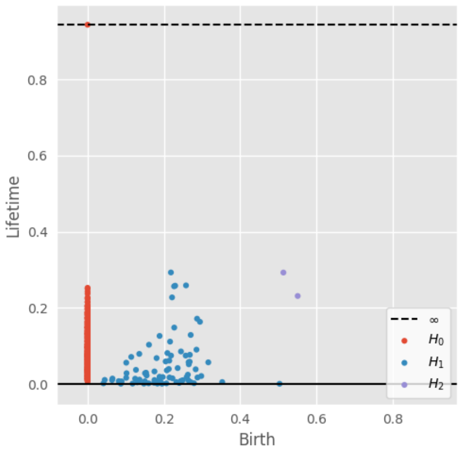


Figure 4.15: Lifetime persistent diagram.

Projective plane in $\mathbb{R}^4$ with noise.
Command: *python generator.py –space pro_plane_4 –points 300 –mean 1 -d 0.2 –up_to_homology 2*

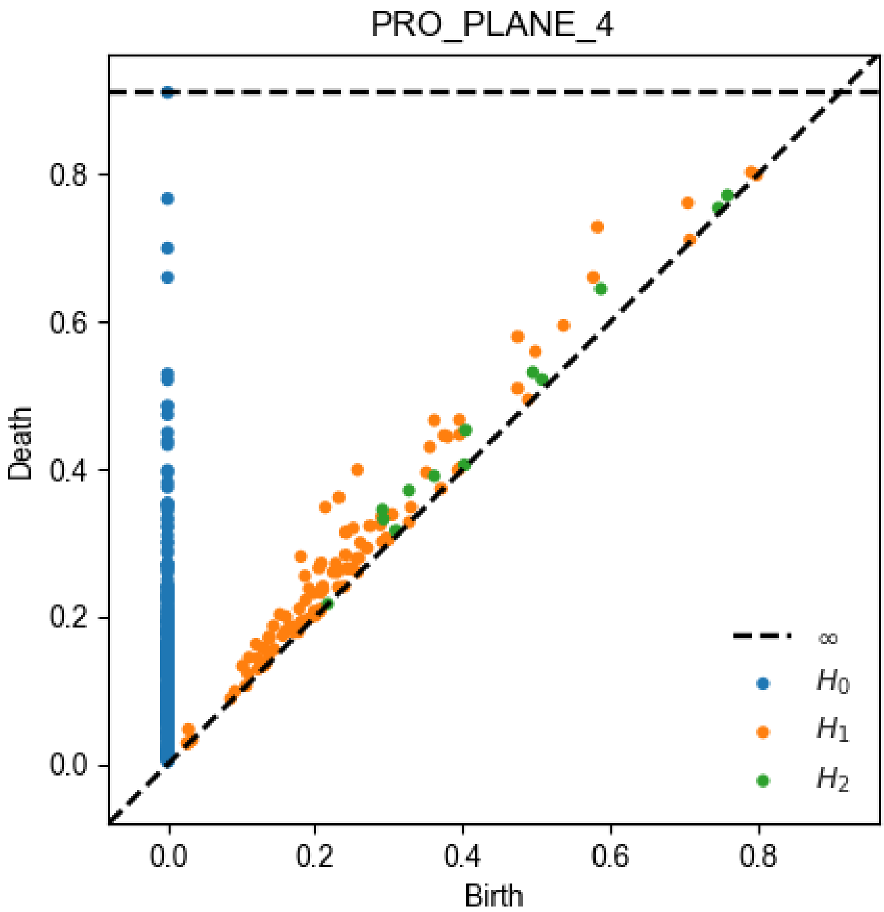


Figure 4.16: Plot and Death birth graphs.

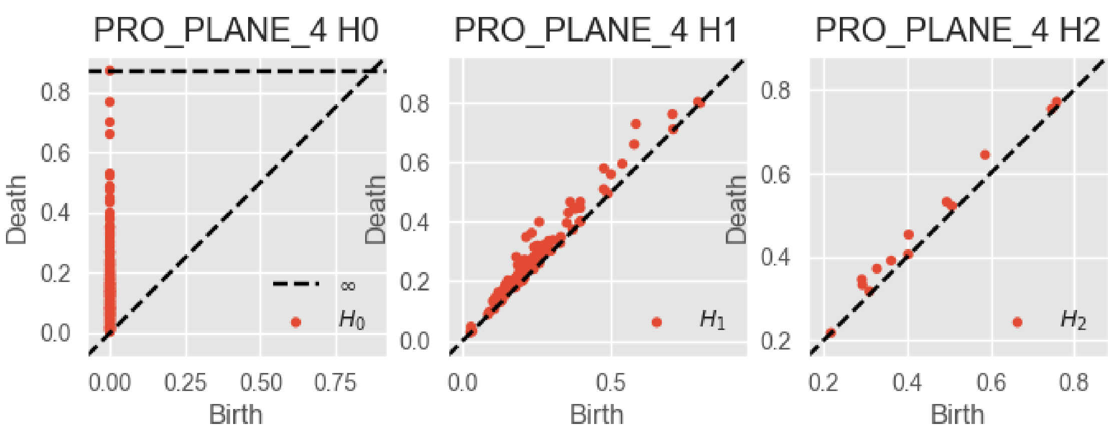


Figure 4.17: Birth-death persistent diagram separate homology.

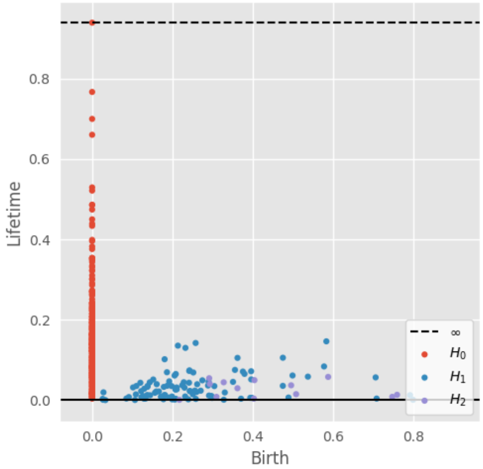


Figure 4.18: Lifetime persistent diagram.

Torus in $\mathbb{R}^4$.
Command: *python generator.py –space torus_4 –points 200 -n –mean 1 -d 0.2 –up_to_homology 3*

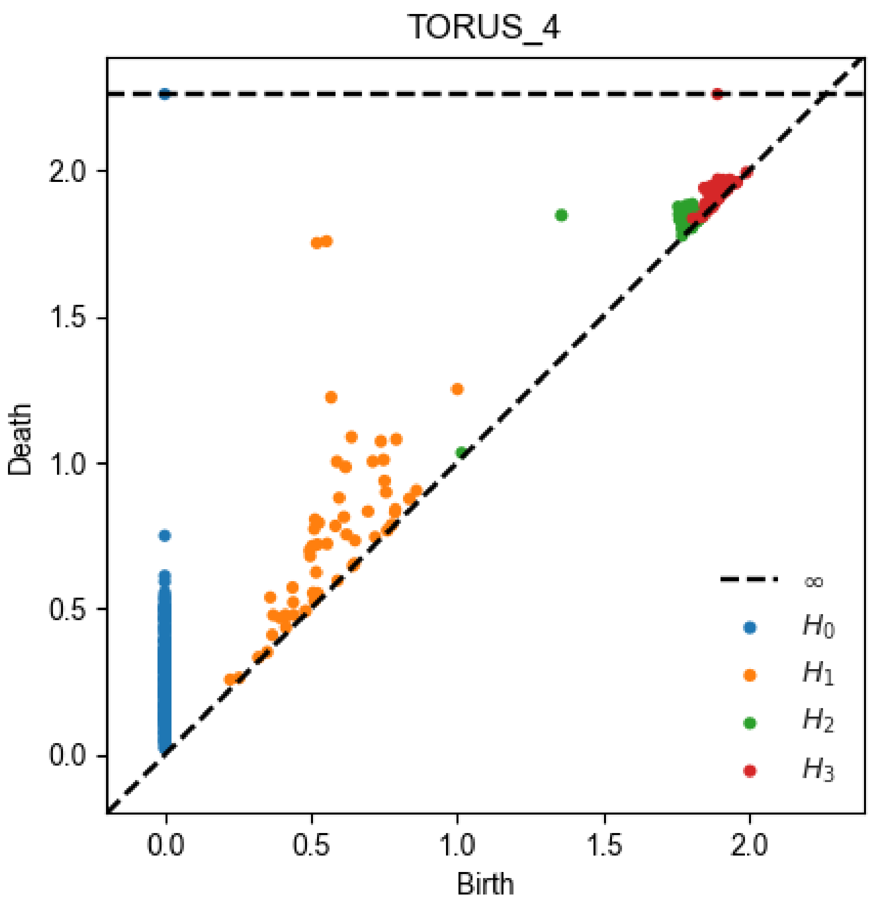


Figure 4.19: Plot and Death birth graphs.

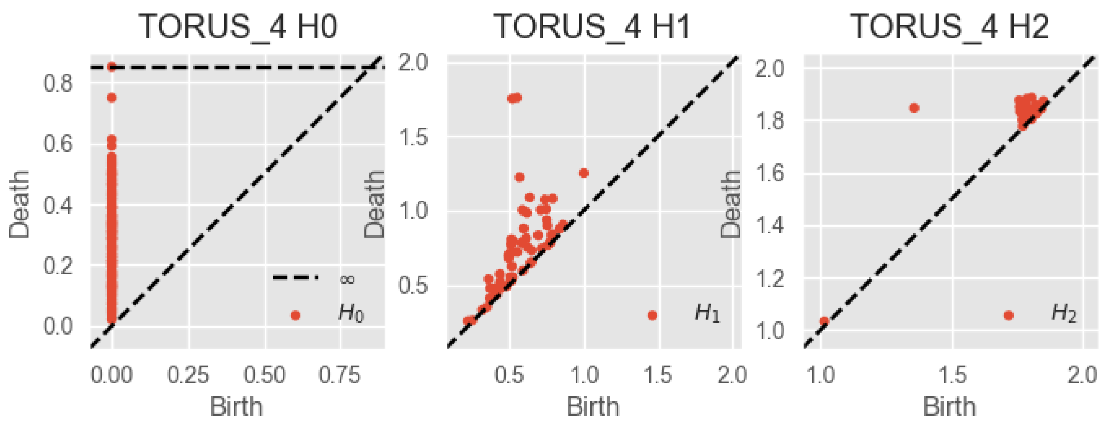


Figure 4.20: Birth-death persistent diagram separate homology.

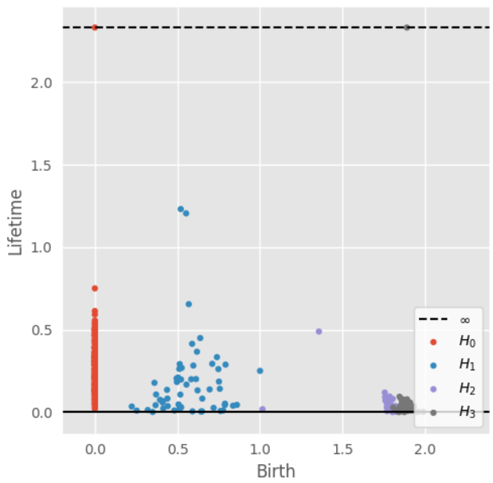


Figure 4.21: Lifetime persistent diagram.

Torus in $\mathbb{R}^4$ with noise.
Command: *python generator.py –space pro_plane_4 –points 300 –mean 1 -d 0.2 –up_to_homology 2*

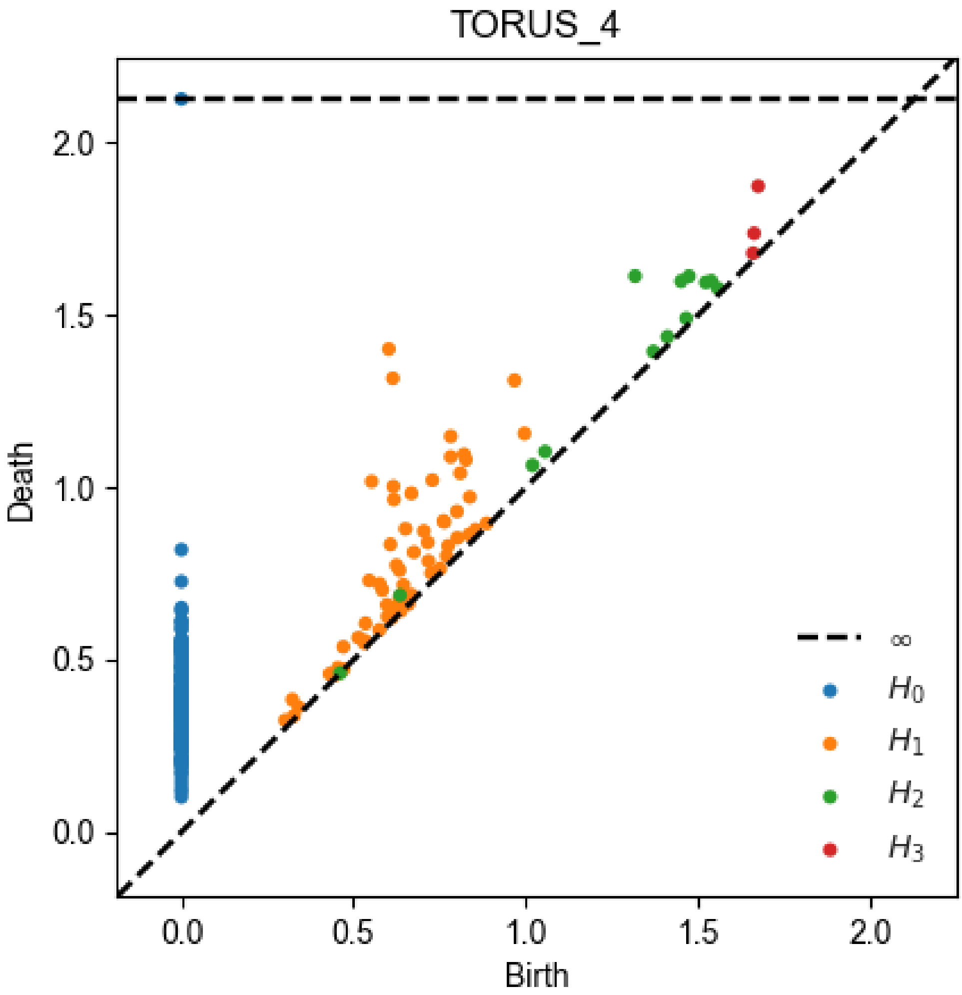


Figure 4.22: Plot and Death birth graphs.

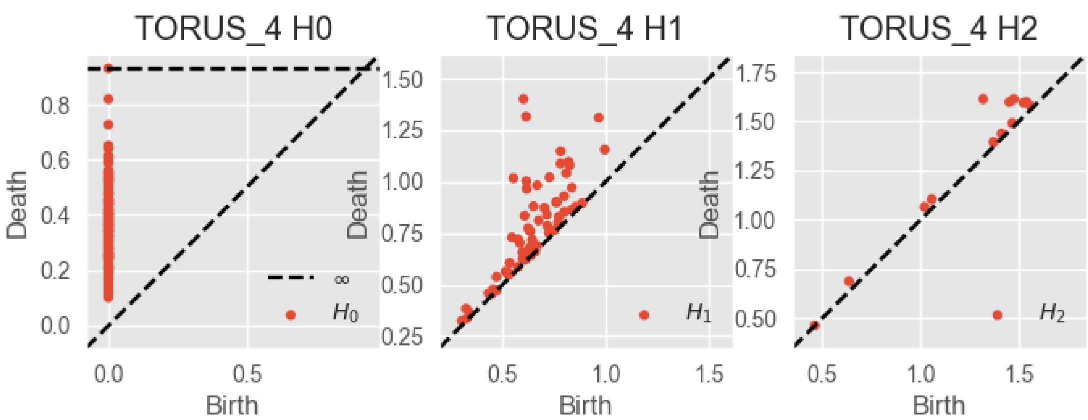


Figure 4.23: Birth-death persistent diagram separate homology.

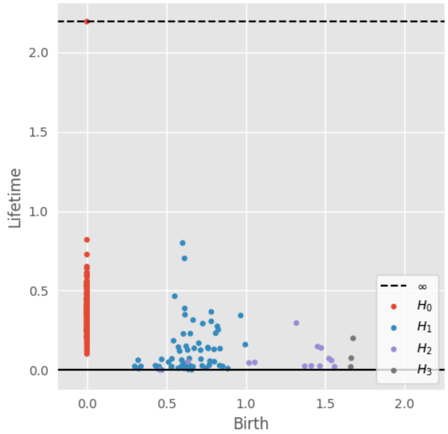


Figure 4.24: Lifetime persistent diagram.

Klein Bottle in $\mathbb{R}^4$.

Command: *python generator.py –space klein_bottle_4 –points 200 -n –mean 1 -d 0.2 –up_to_homology 3*

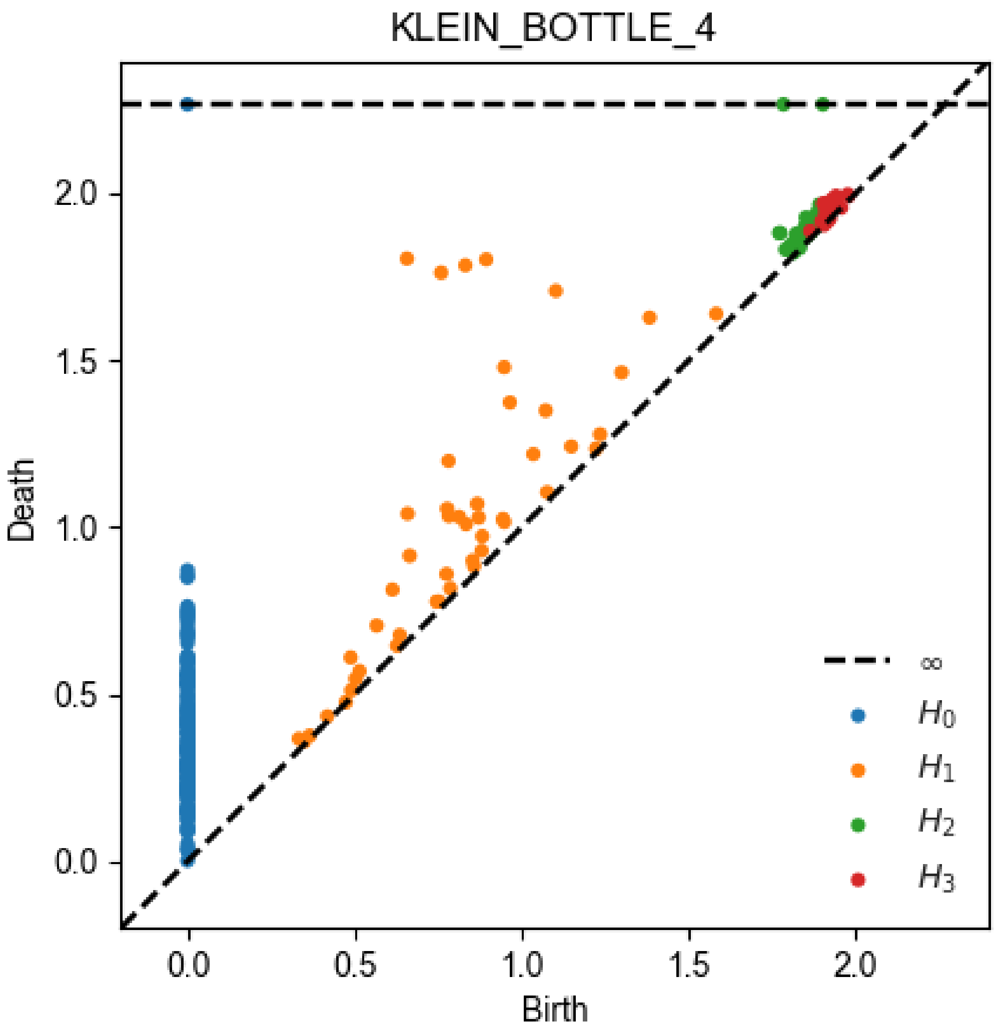


Figure 4.25: Plot and Death birth graphs.

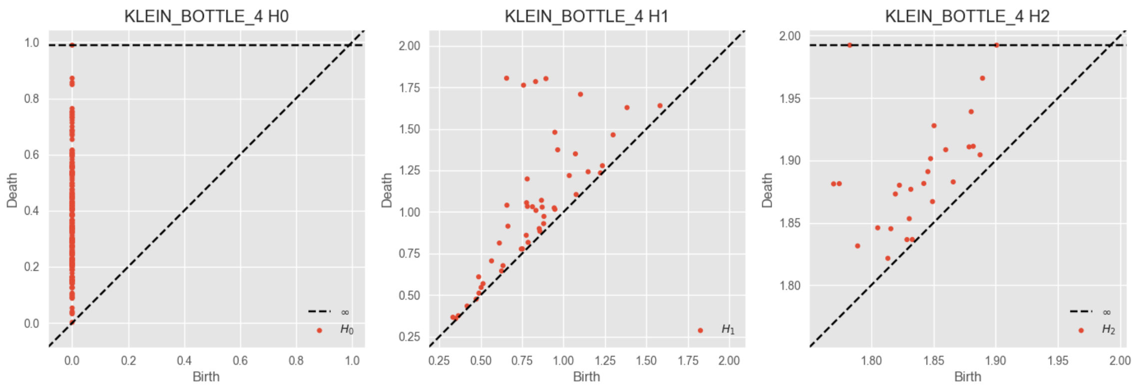


Figure 4.26: Birth-death persistent diagram separate homology.

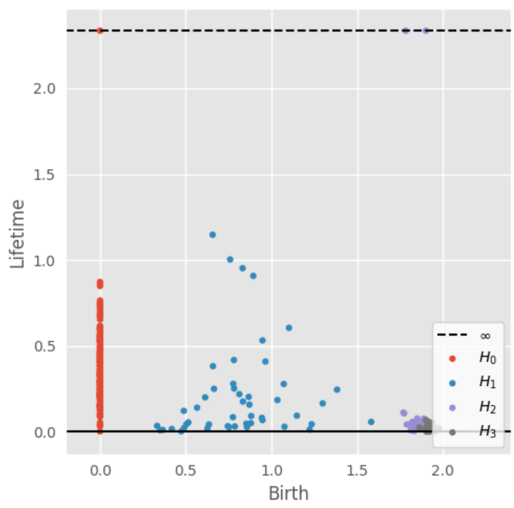


Figure 4.27: Lifetime persistent diagram.

Klein Bottle in $\mathbb{R}^4$ with noise.
Command: *python generator.py –space klein_bottle_4 –points 200 –mean 1 -d 0.2 –up_to_homology 3*

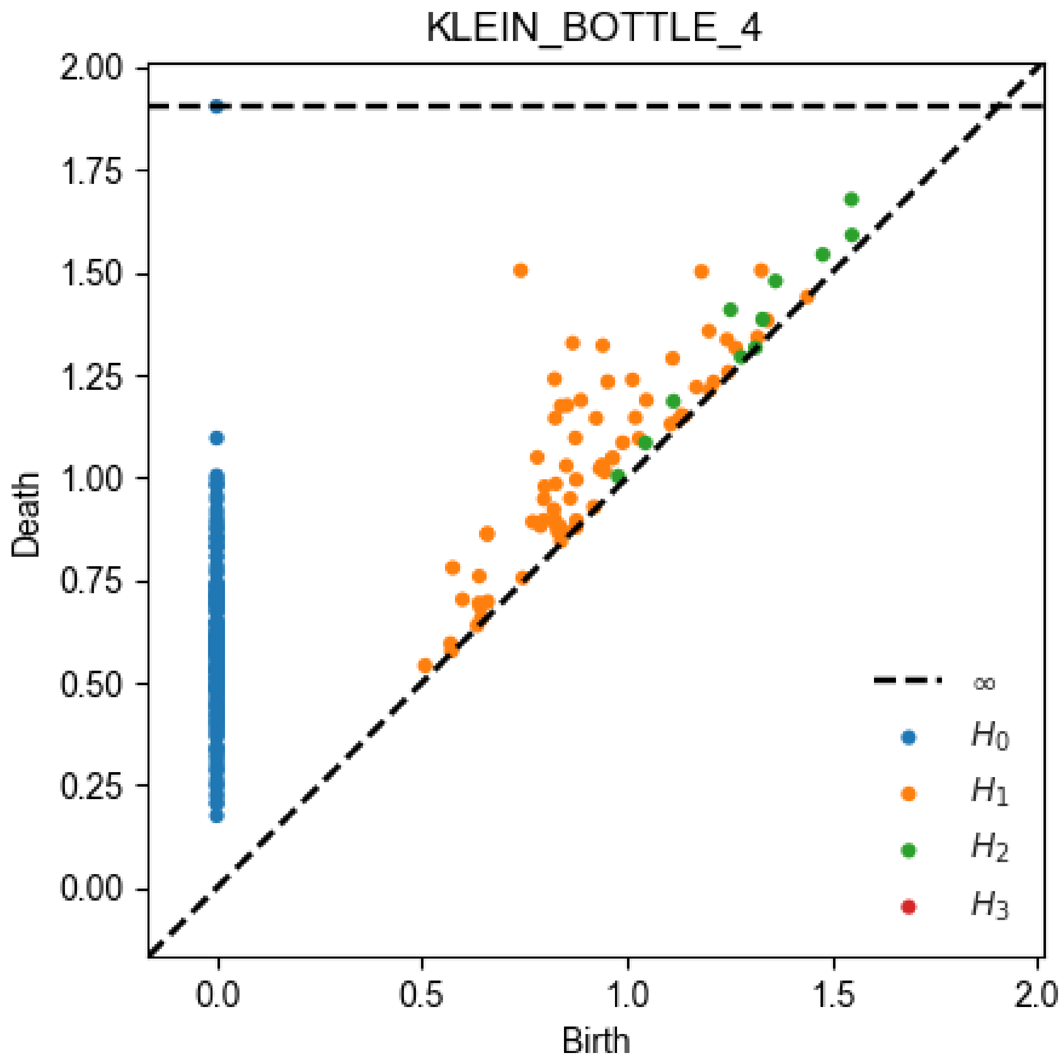


Figure 4.28: Plot and Death birth graphs.

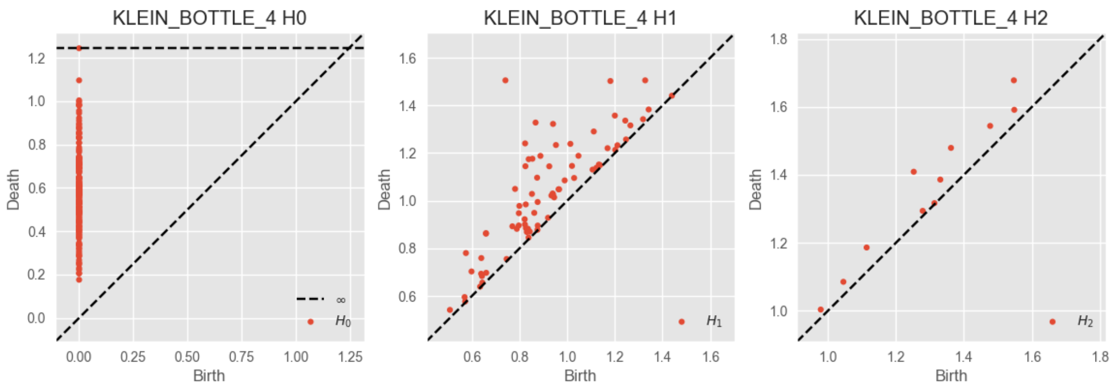


Figure 4.29: Birth-death persistent diagram separate homology.

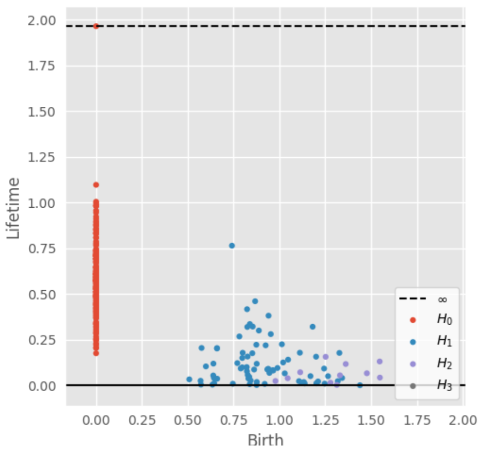


Figure 4.30: Lifetime persistent diagram.

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