Persistent Homology and TDA

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Contents

Chapter 1

Chain Complexes And Simplicial Homology

1.1 Introduction

The key method of algebraic topology is to assign various *algebraic structures* – groups, rings, modules – to topological spaces. This should be done in a *functorial way*. Roughly, functoriality means that maps of topological spaces (and compositions thereof) give rise to homomorphisms of the respective algebraic structures (and compositions thereof), and that the structures assigned to homeomorphic spaces are isomorphic. See (2.1) or [Hat02][Ch.2.3] for a more detailed discussion. In this way, we can think of these algebraic structures as *invariants* of the spaces under consideration. Questions about topological spaces are converted into questions about algebraic structures, which are typically "more rigid". This rigidity can be used to demonstrate, for example, that maps between certain spaces do not exist, or that certain spaces are not homeomorphic, etc.

Computing these algebraic invariants is a different matter altogether. There are notorious examples of invariants that are unknown or hard to compute even for simple enough spaces, such as spheres.

1.2 Δ-complexes

We begin now with a setup that allows for fairly easy calculations. We shall assign a collection of abelian groups to a topological space X *equipped with some additional structure*. This additional structure – called Δ*-complex structure* – is a way of "parametrizing" X by points, segments, triangles, tetrahedra (and their higher-dimensional analogues) and will be introduced in Definition 1.2.2. While the structure of a Δ-complex makes computations easy, it will be completely unclear whether the groups that we obtain are sensitive to this additional structure, or are in fact invariants of the space X itself. In other words, the functoriality of this construction will be completely unclear. This will be rectified in Chapter 2.

We start with the basic building blocks: simplices.

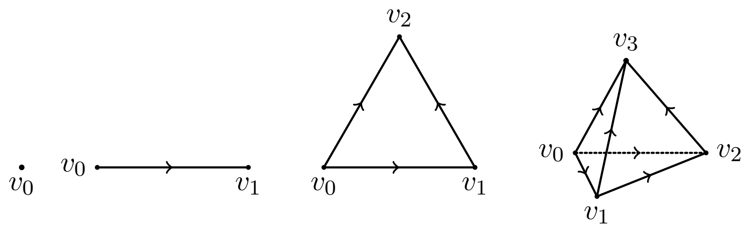
Definition 1.2.1 (Standard Simplex)**.** *The* standard n-simplex in $\mathbb{R}^{n+1}$ *is the convex hull* Δ^n *of the standard basis vectors* $\{e_0, \ldots, e_n\}$*, i.e.,*

$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \textit{ and } t_i \geq 0 \, \forall \, i\} \subseteq \mathbb{R}^{n+1}.$$

More generally, an n-simplex in $\mathbb{R}^{n+1}$ *is the convex hull* $[v_0, \ldots, v_n]$ *of any* $(n+1)$*-tuple of vectors* $v_0, \ldots, v_n \in \mathbb{R}^{n+1}$ *that do not lie in an* n*-dimensional hyperplane.*

Notice that with this definition, $\Delta^n = [e_0, \ldots, e_n]$.

Thus an n-simpex (plural simplices) is an n-dimensional analog of a triangle. A *0-simplex* is a point, a *1-simplex* is a line segment, a *2-simplex* is a triangle, *3-simplex* is a tetrahedron, as shown below.



The vectors v_i, determining $[v_0, ..., v_n]$ are the *vertices* of the simplex. In our calculations we are going to work with a chosen *ordering* of the vertices of the simplex. I.e., we are going to use "simplex" when we mean "a simplex together with an ordering of the vertices". This has a number of consequences. First, it determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts. Second, specifying an ordering of the vertices determines a canonical linear homeomorphism from Δ^n onto any n-simpex $[v_0, ..., v_n]$ preserving the order of vertices, namely, $\sum_i t_i \mathbf{e}_i \mapsto \sum_i t_i \mathbf{v}_i$. Once we fix an ordering of the vertices, we also obtain an orientation of the n-simplex, i.e., the sign of $\det(v_0, \ldots, v_n)$. Two orderings determine the same orientation when they differ by an even permutation.

By a *face* of a n-simplex we shall mean an $(n-1)$-simplex spanned by some n-tuple of vertices of the simplex. That is, the i-th face of $[v_0, \ldots, v_n]$ is $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$, where the hat indicates omission. Some sources refer to this face as an $(n-1)$-face, and talk, more generally about k-faces, for $0 \leq k \leq n-1$. These are k-simplices, obtained as the convex hull of a k-tuple of vertices. We always order the vertices of a face according to their order in the larger simplex.

Example 1.2.1. Consider the (second) face $[v_0, v_1, v_3]$ of the 3-simplex $[v_0, v_1, v_2, v_3]$ in $\mathbb{R}^4$. The canonical order-preserving map $\Delta^2 \to [v_0, v_1, v_3]$ is determined by $e_0 \mapsto v_0$, $e_1 \mapsto v_1$, $e_2 \mapsto v_3$. The canonical order-preserving map from Δ^1 to the edge $[v_1, v_3]$ is determined by $e_0 \mapsto v_1$, $e_1 \mapsto v_3$.

The *boundary* $\partial\Delta^n$ of the standard simplex is defined as the union of all the faces of Δ^n, and $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ denotes interior of Δ^n. We define analogously the boundary and interior of an arbitary simplex in $\mathbb{R}^n$. Notice that with this definition $\partial\Delta^0 = \varnothing$ and $\mathring{\Delta}^0 = \Delta^0$!

We now equip X with additional structure: "parametrization" of X by simplices of various dimensions that satisfies a number of compatibility conditions.

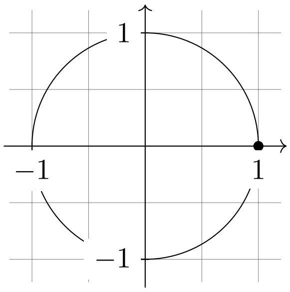
Definition 1.2.2 (Δ-complex)**.** *A $\Delta-$complex structure on a topological space X is a collection of maps $\{\sigma_\alpha : \Delta^n \to X\}_\alpha$, with $n = n(\alpha)$ depending on the index α, such that:*

1. *Each restriction $\sigma_\alpha|\mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\mathring{\Delta}^n$.*

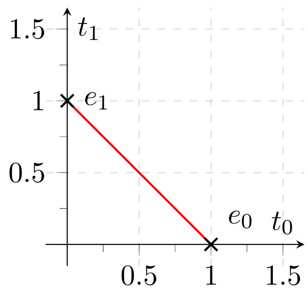
2. *Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here a face of Δ^n is identified with Δ^{n-1} via the canonical order-preserving linear homeomorphism.*

3. *A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α*

Example 1.2.2. Consider $X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$ and the following maps $\sigma_\alpha : \Delta^n \to X$.



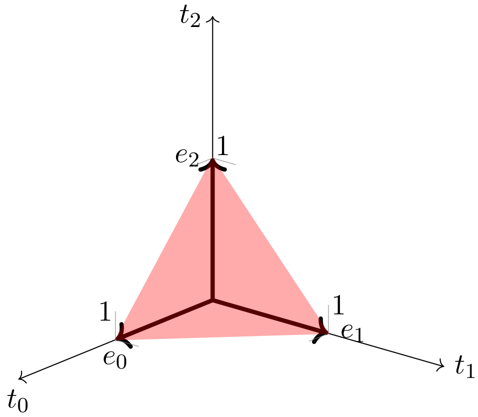
First, $\sigma_0 : \Delta^0 \to S^1$, with $\sigma_0(1) = (1, 0)$. Next, $\sigma_1 : \Delta^1 \to S^1$, defined by $\sigma_1(t_0, t_1) = (\cos(2\pi t_0), \sin(2\pi t_0))$
Recall (see the figure) that Δ^1 is the set of pairs (t_0, t_1) with $t_0 + t_1 = 1$, $t_0, t_1 \geq 0$, or equivalently, $\Delta^1 = \{(t_0, 1 - t_0)\ ,\ t_0 \in [0, 1]\} = [e_0, e_1]$



$\sigma|_{[e_0]} = \sigma_1(0) = (\cos(0), \sin(0)) = (1, 0)$
$\sigma|_{[e_1]} = \sigma_1(1) = (\cos(2\pi), \sin(2\pi)) = (1, 0)$

Example 1.2.3. Consider $\sigma_\alpha : \Delta^n \to X$ where $X = S^2 = \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$

For n = 2, $\alpha = 2$: $\sigma_2 : \Delta^2 \to S^2$
$\Delta^1 = \{(t_0, t_1, t_2) \,|\, t_0 + t_1 + t_2 = 1,\ t_i \geq 0\ for\, i = 1, 2, 3]\} = [e_0, e_1, e_2]$



The faces are $[e_1, e_2]$ or $(0, t_1, t_2)$, where $t_1 + t_2 = 1$,
$[e_0, e_2]$ or $(t_0, 0, t_2)$, where $t_0 + t_2 = 1$,
$[e_0, e_1]$ or $(t_0, t_1, 0)$, where $t_0 + t_1 = 1$.

$\sigma_2((t_0, t_1, t_2)) = \frac{(t_0, t_1, t_2)}{\sqrt{(t_0^2 + t_1^2 + t_2^2)}}$
$\sigma|_{[e_1, e_2]}(t_1, t_2) = \frac{(t_1, t_2)}{\sqrt{(t_1^2 + t_2^2)}}$, $\quad \sigma|_{[e_0, e_2]]}(t_0, t_2) = \frac{(t_0, t_2)}{\sqrt{(t_0^2 + t_2^2)}}$, $\quad \sigma|_{[e_0, e_1]}(t_1, t_2) = \frac{(t_0, t_1)}{\sqrt{(t_0^2 + t_1^2)}}$,

Here are some "non-examples": collections of maps $\sigma_\alpha : \Delta^n \to X$ that satisfy conditions (1) and (2), but not condition (3) from Definiton 1.2.2.

Example 1.2.4. Let $X = [0; 1]$ or more generally, any other non-discrete space, having infinitely many points. Then the collection of maps $\{\sigma_x : \Delta^0 \to X\}_{x \in X}$, $\sigma_x(1) = x$ satisfies (1) and (2), but not (3). We place one 0-simplex for each point of X, but points in $[0;1]$ are not open.

Another example is as follows. Let X be the unit square $X = [0;1] \times [0;1]$. Consider, for each $y \in [0;1]$, a map $\lambda_y : \Delta^0 \to X$, $\lambda_y(1) = (0, y)$, a map $\rho_y : \Delta^0 \to X$, $\rho_y(1) = (1, y)$ and a map $\sigma_y : \Delta^1 \to X$, $\sigma_y(t_0, t_1) = (t_0, y)$. In this way we place at height y one 1-simplex (as a horizontal segment), and two 0-simplices, as its left and right end. The collection of maps $\{\sigma_y, \lambda_y, \rho_y\}_{y\in[0;1]}$ satisfies (1) and (2), but not (3).

Thus condition (3) precludes inadequate choices of maps σ_α, for example, covering a manifold (with boundary) of dimension n by simplices of smaller dimension. More generally, condition (3) is important when one needs to deal with infinite collections of simplices, in particular, when dealing with infinite-dimensional (in appropriate sense) spaces X.

Usually simplicial homology is defined by simplicial complexes, which are the $\Delta-$*complexes* whose simpleces are uniquely determined by their vertices. In a simplicial complex any n-simplex has n+1 distinct vertices, and no other n-simplex has the same set of vertices.

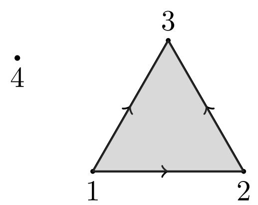
Another definition of simplicial complexes that will be important in indroducing Persistent Homology is a combinatorical description.

Combinatorically we can define a simplicial complex as:

Definition 1.2.3 (Simplicial Complex). *Given a partially ordered set:* $V = \{1, 2, \cdots, m\}(= [m])$ *a simplicial complex is a collection* $\mathcal{K}$ *of subsets of* V*, such that:*

1. $\emptyset \in \mathcal{K}$
2. $\{i\} \in \mathcal{K}$ *(singleton)*
3. *If* $\mathcal{J} \subseteq I \in \mathcal{K} \Rightarrow \mathcal{J} \in \mathcal{K}$

Example 1.2.5. Consider the following partially ordereed set $V = \{1, 2, 3, 4\}$: The simplicial complex $\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$



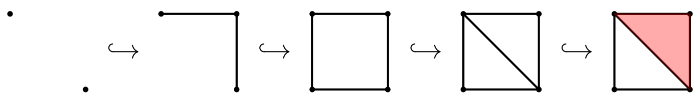
The topological realization of $\mathcal{K}$ is:

$|\mathcal{K}| = \underset{I\in K\neq\emptyset}{\cup}(Conv(e_i))$ where e_i is in the standard basis $e_1, \cdots e_n \in \mathbb{R}^n$

A subcomplex of $\mathcal{K}$ is a subset $L \subseteq \mathcal{K}$ that is also a simplicial complex. A *filtration* of complex $\mathcal{K}$ is a nested subsequence of complexes:

$$\emptyset = \mathcal{K}^0 \subseteq \mathcal{K}^1 \subseteq \ldots \mathcal{K}^m = \mathcal{K}$$

For generality, we let $\mathcal{K}^i = \mathcal{K}^m$ for all $i \geq m$. $\mathcal{K}$ is called a filetered complex, and below there is a short example of a filtered complex:



1.3 Chain Complexes

Definition 1.3.1 (Chain complex). *Complex of abelian groups. Homology of a complex. A chain complex is a sequencee of homomorphisms of abelian groups:*

$$... \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ *for each* n *in* $\mathbb{Z}$. *The equation* $\partial_n \partial_{n+1} = 0$ *is equivalent to the inclusion* $\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$.

The map ∂_n for a Δ-complex $\mathcal{X}$ is a boundary homomorphism $\partial_n : \Delta_n(\mathcal{X}) \to \Delta_{n-1}(\mathcal{X})$ where the action on a basis element of $\Delta_n(\mathcal{X})$ is:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, ..., \hat{v}_i, ..., v_n]$$

where the '^' symbol denotes the absence of that vertex.

Lemma 1.3.1. *The composition* $\partial^2 = 0$ *below is zero*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

Proof: For n = 3 :

$$\Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X)$$

Let us proof that $\partial_2 \partial_3 = 0$:
$\partial_3 \sigma = \sum_i (-1)^i \sigma | [v_0, ..., \hat{v}_i, v_3] = \sigma|[v_1, v_2, v_3] - \sigma|[v_0, v_2, v_3] + \sigma|[v_0, v_1, v_3] - \sigma|[v_0, v_1, v_2]$

$$\begin{aligned}
\partial_2 \partial_3(\sigma) &= \sigma|[v_2, v_3] - \sigma|[v_1, v_3] + \sigma|[v_1, v_2] \\
&= -\sigma|[v_2, v_3] + \sigma|[v_0, v_3] - \sigma|[v_0, v_2] \\
&= \sigma|[v_1, v_3] - \sigma|[v_0, v_3] + \sigma|[v_0, v_1] \\
&= -\sigma|[v_1, v_2] + \sigma|[v_0, v_2] - \sigma|[v_0, v_1] = 0
\end{aligned} \tag{1.1}$$

In case of n:

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n]) \\
&= \sum_j (-1)^j (\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n])|[v_0, ..., \hat{v}_j, ..., v_n] \\
&= \sum_{j<i} (-1)^i (-1)^j \sigma_| [v_0, ..., \hat{v}_j, ..., \hat{v}_i ..., v_n] + \sum_{j>i} (-1)^i (-1)^j \sigma_| [v_0, ..., \hat{v}_i, ..., \hat{v}_j ..., v_n] = 0
\end{aligned} \tag{1.2}$$

Remark: Chain complexes can be also defined over $R-modules$, where R is a commutative ring:

Definition 1.3.2. *(Chain complex of* $R-module$*) A Chain complex of R-modules is a sequence:*

$$(C_\bullet, d_\bullet) = (\ ... \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} ...\ )$$

where for each $n \in \mathbb{Z}$, C_n *is an* $R-module$ *and* $d_n \in Hom_R(C_n, C_{n-1})$ *satisfies* $d_n \circ d_{n+1} = 0$

In chain complexes of $R-modules$, n is the degree of the $R-module$ C_n. The $R-linear$ maps $d_n (n \in \mathbb{Z})$ are called differential maps.
Also , a complex $C_\bullet$ is called non-negative (resp. positive) if $C_n = 0$, for all $n \in \mathbb{Z}_{<0}$ (resp. $n \in \mathbb{Z}_{\leq 0}$)

Chain complexes together with morphisms of chain complexes (and composition given by degreewise composition of R-morphisms) form a category, which we will denote by $Ch({}_R Mod)$

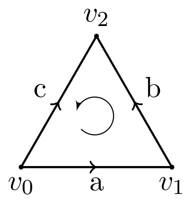
1.4 Homology Calculations: Examples

In the first example of Homology calculations, it is important to notice that the homology groups are calculated in two methods, with different Δ-structure on S^1. Even though, the circle is parametrized by different Δ-complex structures, the calculations below show that the Homology groups are the same.

1.4.1 S^1

Method I: Triangulation

To compute the homolgy group of the circle S^1 we can triangulate the circle in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2=0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1, v_2 \rangle \\ C_1 = \langle a, b, c \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2=0} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1, v_2 \rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v_1 - v_0) + \beta(v_2 - v_1) - \gamma(v_2 - v_0)$
$= (-\alpha + \gamma)v_0 + (\alpha - \beta)v_1 + (\beta - \gamma)v_2 = (\gamma - \alpha)v_0 + (\alpha - \beta)v_1 + (-(\gamma - \alpha) - (\alpha - \beta))v_2$

$$\operatorname{Im} \partial_1 = \left\{ \left(\begin{array}{c} (\gamma - \alpha) \\ (\alpha - \beta) \\ -(\gamma - \alpha) - (\alpha - \beta) \end{array} \right), \quad \alpha, \beta, \gamma \subseteq \mathbb{Z} \right\} \subseteq \mathbb{Z}^{\oplus 3}$$

Claim : There exist an isomorphism $\psi : \operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$\psi : \left(\begin{array}{c} (\gamma - \alpha) \\ (\alpha - \beta) \\ (\beta - \gamma) \end{array} \right) \mapsto \left(\begin{array}{c} (\gamma - \alpha) \\ (\alpha - \beta) \end{array} \right)$$

ψ is one-to-one since if $(\gamma - \alpha = 0 \;\&\; \alpha - \beta = 0) \Rightarrow \beta - \gamma = 0 \;\&\; \alpha = \beta = \gamma$

ψ is onto since given $\left(\begin{array}{c} m \\ n \end{array} \right) \in \mathbb{Z}^2$ there exist an element $\left(\begin{array}{c} m \\ n \\ -m-n \end{array} \right) \in \operatorname{Im} \partial_1$ such that

$\psi \left(\begin{array}{c} m \\ n \\ -m-n \end{array} \right) = \left(\begin{array}{c} m \\ n \end{array} \right)$, since ψ is one-to-one and onto, $\operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$

Claim: $\phi : \left(\mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z} \right) \simeq \mathbb{Z}$

First, let us take the map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3 \Big/ \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$\mathbb{Z}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (p+q+r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

where $p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$

So, $\varphi : \begin{pmatrix} p \\ q \\ r \end{pmatrix} \mapsto \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} = (p+q+r) \overline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Finally, $\phi : \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} \mapsto (p+q+r) \in \mathbb{Z}$, Clearly, ϕ is injective and surjective.

So, $H_0 \simeq \mathbb{Z}$

Second, let's compute H_1:

$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$

$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$

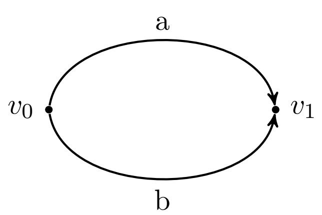
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Method II

To compute the homolgy group of the circle S^1 we can construct the circle, by two verteces and two edges, in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2 = 0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1 \rangle \\ C_1 = \langle a, b \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1 \rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b) = \alpha(v_1 - v_0) + \beta(v_1 - v_0) = (\alpha + \beta)(v_1 - v_0)$
$\operatorname{Im} \partial_1 = \langle v_1 - v_0 \rangle$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \frac{\langle v_1 - v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \langle v_1 \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \langle a - b \rangle$ since $\partial_1(\alpha a + \beta b) = (\alpha + \beta)(v_1 - v_0) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element $(a - b)$
$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle with a different $\Delta - complex$ on it are the same:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.4.2 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd}
v \arrow[r, "b"] \arrow[dr, phantom, "A", near start] & v \\
v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c" description] & v \arrow[u, "a"'] \arrow[ul, phantom, "B", near start]
\end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v \rangle$ since $\partial_0 = 0$

$\operatorname{Im}\partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v-v) + \beta(v-v) + \gamma(v-v) = 0$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im}\partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = C_1 = \langle a, b, c\rangle$ since $\partial_1 = 0$
$\operatorname{Im}\partial_2 = \langle a+b-c\rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(a+b-c) + \beta(a+b-c) = (\alpha+\beta)(a+b-c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im}\partial_2} = \frac{\langle a,b,c\rangle}{\langle a+b-c\rangle}$
The group $\langle a, b, c\rangle$ can be also generated by the elements $m = a+b-c, bandc$ where $a = m-b+c$.
So,
$H_1 = \frac{\langle a+b-c,b,c\rangle}{\langle a+b-c\rangle} = \langle b, c\rangle \simeq \mathbb{Z}\oplus\mathbb{Z}$

Last, let's compute H_2:
$\ker \partial_2 = \langle A - B\rangle$ since $\partial_2(\alpha A + \beta B) = (\alpha+\beta)(a+b-c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element A-B
$\operatorname{Im}\partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{\operatorname{Im}\partial_3} = \frac{\langle A-B\rangle}{\{0\}} = \langle A - B\rangle \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 2 \\ \mathbb{Z}\oplus\mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 3 \end{cases}$$

1.4.3 $\mathbb{RP}^2$

One way to calculate the homology groups of a projective plain $\mathbb{RP}^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd}
w & v \arrow[l, "b"'] \arrow[d, "a"] \\
v \arrow[u, "a"] \arrow[ur, "c"] \arrow[r, "b"'] & w
\end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v, w\rangle \\ C_1 = \langle a, b, c\rangle \\ C_2 = \langle A, B\rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \ker \partial_n / \operatorname{Im}\partial_n$

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v, w\rangle$ since $\partial_0 = 0$
$\operatorname{Im}\partial_1 = \langle w - v\rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(w-v) + \beta(w-v) + \gamma(v-v)$
$= (\alpha+\beta)(w-v)$

$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v,w \rangle}{\langle w-v \rangle} = \frac{\langle w-v,w \rangle}{\langle w-v \rangle} = \langle w \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \langle a-b, c \rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w-v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a + \beta b + \gamma c) = \alpha(a-b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements a-b and c
$\operatorname{Im} \partial_2 = \langle -a+b+c,\ a-b+c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(-a+b+c) + \beta(a-b+c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a-b,\, c \rangle}{\langle -a+b+c,\, a-b+c \rangle}$
The group $\langle a-b, c \rangle$ can be also generated by the elements $m = a-b+c, andc$ where $a-b = m-c$. So,
$H_1 = \frac{\langle a-b,\, c \rangle}{\langle -a+b+c,\, a-b+c \rangle} = \frac{\langle a-b+c,\, c \rangle}{\langle a-b+c,\, -a+b+c \rangle}$
If we let $t = a-b+c$ then $-a+b+c = -t+2c$ then the group $\langle t,\ -t+2c \rangle$ can be also generated by the elements t *and* $2c$.
In terms of t and c, $H_1 = \frac{\langle t,\, c \rangle}{\langle t,\, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Last, let's compute H_2:
$\ker \partial_2 = \{0\}$ since $\partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0$ only when $\alpha = \beta = 0$
$\operatorname{Im} \partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\{0\}}{\{0\}} = 0$

Finally, the homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.5 Maps of Complexes

In the previous sections, we considered boundary homomorphisms between abilean groups as part of a chain complex. In this section, we will draw our attention to maps between chain complexes.

Definition 1.5.1. *(Maps of Chain Complexes)*
Let $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ be two chain complexes. A map of chain complexes is a morphism f that is a sequence of homomorphisms $(f_n)_{n \in Z}$:

$$\begin{array}{lccccccccc} (C_\bullet, \partial) & C_\bullet & \dots \longrightarrow & C_n & \xrightarrow{\partial n} & C_{n-1} & \xrightarrow{\partial n-1} & C_{n-2} & \xrightarrow{\partial n-2} & \dots\ C_\bullet \\ & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & \\ (D_\bullet, \delta) & D_\bullet & \dots \longrightarrow & D_n & \xrightarrow{\delta n} & D_{n-1} & \xrightarrow{\delta n-1} & D_{n-2} & \xrightarrow{\delta n-2} & \dots\ D_\bullet \end{array}$$

$$f_n : C_n \to D_n \quad s.t, \quad f_{n-1} \circ \partial n = \delta_n \circ f_n \ \forall n \in \mathbb{Z}$$

$$\begin{array}{ccc} C_n & \xrightarrow{\partial n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{\delta n} & D_{n-1} \end{array} \quad \textit{commutes.}$$

1.5.1 Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$
\begin{aligned}
&H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet) \\
&H_n(f) : [x] \mapsto [f_n(x)]
\end{aligned}
$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-definess by checking if cycles are send to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$
\begin{aligned}
\delta_n \circ f_n(x) = f_{n-1} \circ \partial n(x) = f_{n-1}(0) = 0 &\Rightarrow f_n(x) \in \ker \delta_n, f_n(x)\, is\, a\, cycle \\
&\Rightarrow f_n(\ker \partial n) \subseteq \ker \delta_n
\end{aligned}
$$

So, cycles are send to cycles.

(2) Let us take a boundary $y \in C_n$, so that $y \in \operatorname{Im} \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

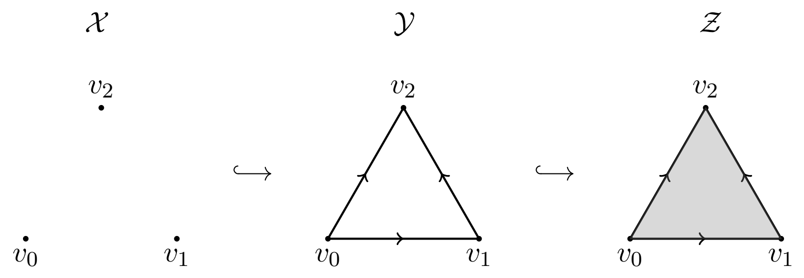
$$
\begin{aligned}
&f_n(y) = f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z)) \\
&\Rightarrow f_n(y) \in \operatorname{Im} \partial_{n+1} f_n(y)\, is\, a\, boundary \\
&\Rightarrow f_n(Im \partial_{n+1}) \subseteq Im(\delta_{n+1})
\end{aligned}
$$

So, boundaries are send to boundaries.

$$
\begin{aligned}
&H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet) \\
&H_n(f) : \ker \partial_n / \operatorname{Im}(\partial_{n+1}) \to \ker \delta_n / \operatorname{Im}(\delta_{n+1}) \\
&\qquad [x] \mapsto [f_n(x)] \\
&\qquad x + \operatorname{Im} \partial_{n+1} \mapsto f_n(x) + f_n(\operatorname{Im} \delta_{n+1}) = f_n(x) + Im(\delta_{n+1}) = [f_n(x)]
\end{aligned}
$$

□

Let us consider an example between maps of complexes defined by the three spaces below.



$$C^0_\bullet \longrightarrow C^1_\bullet \longrightarrow C^2_\bullet$$

Maps between complexes:

$$C_\bullet^0 \longrightarrow C_\bullet^1 \longrightarrow C_\bullet^2$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d, "\partial_2^0=0"] & 0 \arrow[r] \arrow[d, "\partial_2^1=0"] & \mathbb{Z} \arrow[d, "\partial_2^2=\begin{pmatrix}1\\1\\-1\end{pmatrix}"] \\
1 & 0 \arrow[r] \arrow[d, "\partial_1^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_1^1=\partial_1^2"] & \mathbb{Z}^3 \arrow[d, "\partial_1^2=\begin{pmatrix}-1&0&-1\\1&-1&0\\0&1&1\end{pmatrix}"] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^1=0"] & \mathbb{Z}^3 \arrow[d, "\partial_0^2=0"] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Induced maps on homology:

$$H(C_\bullet^0) \longrightarrow H(C_\bullet^1) \longrightarrow H(C_\bullet^2)$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Chapter 2

Singular Homology and Homotopy Invariance

In the previous chapter, we considered a parametrization of a space by simplexes, where the maps $\sigma_\alpha : \Delta^n \to X$ had restrictions defined in 1.2.2. If we only require that the σ map is continous, then by definition that would be a singular $n-simplex$ in a space X. The lack of other restrictions on the map $\sigma : \Delta^n \to X$, convey that σ does not need to be a 'nice' embedding, in fact it can have singularites, where its image does not look like a simplex.

$C_n(X)$ is a free abelian group with generators the set of singluar $n-simplexes$ in X: the continous maps $\sigma : \Delta^n \to X$. The elements of $C_n(X)$ are singular $n-chains$ defined as $\sum_i(n_i\sigma_i)$ for $n_i \in \mathbb{Z}$. The boundary operator $\partial_n : C_n(x) \to C_{n-1}(X)$ is defined the same way as in simplicial $n-chains$, by the formula:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n]$$

By preserving the order of the vertices, the $\sigma|[v_0, ..., \hat{v_i}, ..., v_n]$ is identified with the map $\Delta^{n-1} \to X$. The proof of lemma 1.3.1, $\partial_2 = 0$, holds true also for singular simplexes. Therefore, the singular homology group is defined the same way: $H_n(x) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$. The elements in the ker are singular cycles, and the elements in the Im singular boundaries.

Let us consider some explicit examples of $C_n(X)$:

- For a topological space X: $C_0(X)$ consists of all maps $\sigma : \Delta^0 \to X$, which means that $C_0(X)$ is a free group on the points of X.
- If X = $\mathbb{R}$: $C_1(X)$ consists of all maps $\sigma : \Delta^1 \to \mathbb{R}$, which means that $C_0(X)$ is a free group on continus maps:
$$[0, 1] \simeq \Delta^1 \to R$$
In this case $C_1(X)$ can be considered as a vector space with vectors the maps: $[0, 1] \to \mathbb{R}$.

From the examples above, it is clear that the groups $C_n(X)$ can be so large to the point where the number of singular $n-simplexes$ in a space X is uncountable. It is not easy to see that even in singular homology where X is generated by a finite number of simplexes, $H_n(X)$ should be finite generated for all n, and that $H_n(X)$ should be 0 for $n > dim(X)$.

At first glance, singular homology seems to be more general than simplicial homology, however if for an arbitrary space X , we define the singular complex $S(X)$ as a $\Delta - complex$ with one $n-simplex$ Δ^n_σ for each singular $n-simplex$ $\sigma : \Delta^n \to X$, then $H^\Delta_n(S(X))$ is the same as $H_n(X)$. In this case singular homology can be viewed as a special case of simplicial homology.

2.1 Homotopy invariance

A significant result that can be proven by singluar homology is that if two spaces X, Y are homeomorphic, the singular homology groups are isomorphic $H_n(X) \simeq H_n(Y)$.

More generally, A countinous map $f : X \to Y$ induces a chain map: $f_\# : C_n(X) \to C_n(Y)$.

$$f_\#(\sigma : \Delta^n \to X) = f \circ \sigma$$

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

The boundary operator of $f_\#$ is equal to $\partial_n(f \circ \sigma) = f \circ \partial_n(\sigma)$.

$H_n(f_\#) = f_* : H_n(X) \to H_n(Y)$

If we additionally require f to be a bijection and have a continous inverse f^{-1}, so f is a homoemorphism, then $f_\# : C_n(X) \simeq C_n(Y)$

$$\sigma \mapsto f \circ \sigma$$
$$\mu \circ f^{-1} \leftarrow\!\shortmid \mu$$

Then the induced singular homology map $f_* : H_n(X) \simeq H_n(Y)$ defines an isomorphism.

Moreover,

$$f_* \text{ preserves composition}, (f \circ g)_* = f_* \circ g_* \tag{2.1}$$
$$f_* \text{ preserves the identity}, id : X \to Y \text{ goes to } id_* : H_n(X) \to H_n(Y) \tag{2.2}$$

Category Theory Interpretation: If we consider Top to be the category of topological spaces where maps are countinous:

$$Hom_{Top}(X, Y) = \{f : X \to Y, f \text{ is continous}\},$$

and Ab the category of abelian groups were maps are group homomorphisms:

$$Hom_{Ab}(G, H) = \{\phi : G \to H, \phi \text{ is group homomorphism}\}$$

then for each $n \geq 0$:

$$H_n : Top \to Ab$$
$$X \rightsquigarrow H_n(X)$$
$$f : X \to Y \rightsquigarrow f_* : H_n(X) \to H_n(Y)$$

H_n is a functor and (2.1) (2.2) hold.

If we also consider the category, Homotopic Topology $HoTop$ of topological spaces where maps are continous up to homotopy, then we obtain the following commutative diagram:

$$\begin{tikzcd} Top \arrow[r, "H_n"] \arrow[d] & Ab \\ HoTop \arrow[ur, dotted] & \end{tikzcd}$$

$$Hom_{HoTop}(X, Y) = Hom_{Top}(X, Y)/\simeq$$

Remark: The continous maps $f, g : X \to Y$ are homotopic if

$$\exists\ H : X \times [0, 1] \to Y,$$
$$for\ x \in X\ and\ t \in [0, 1] : H(x, t) = H_t(x)$$
$$s.t\ H(x, 0) = f(x)\ and\ H(x, 1) = g(x)$$
$$i.e\ H|_{x \times \{0\}} = f\ and\ H|_{x \times \{1\}} = g$$

Let us consider some explicit homotopic maps:

- $f, g : \mathbb{R} \to \mathbb{R}$ where $f = id, f(x) = x \; \forall \; x \in X$, and $g = 0, \; g(x) = 0 \; \forall \; x \in X$
  $H : \mathbb{R} \times [0,1] \to \mathbb{R}$
  $H(x,t) = (1-t)x$, clearly $H(x,0) = x$ and $H(x,1) = 0$

- $f, g : S^1 \to \mathbb{R}^2$ where f *is an inclusion*, $f(x,y) = (x,y) \; \forall \; (x,y) \in S^1$, and $g = 0, \; g(x,y) = 0 \; \forall \; (x,y) \in S^1$
  $H : S^1 \times [0,1] \to \mathbb{R}^2$
  $H(x,y,t) = (1-t)\langle x,y \rangle$, clearly $H(x,y,0) = \langle x,y \rangle$ and $H(x,y,1) = 0$

If $f, g : X \to Y$ are homotopic, then the induced maps on homology $f_*, g_* : H_n(X) \to H_n(X)$, are the same $f_* = g_* \; \forall n$

Define $p_n : C_n^{sing}(X) \to C_{n+1}^{sing}(Y)$ s.t $f_\# - g_\# = \partial p + p \partial$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1}(X) \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \\
C_{n+1}(Y) \arrow[r, "\delta_{n+1}"] & C_n(Y) \arrow[r, "\delta_n"] & C_{n-1}(Y)
\end{tikzcd}$$

For $\sigma : \Delta^n \to X$ the map $p_n(\sigma) : \Delta^{n+1} \to Y$ should be a continous map.
$H : X \times [0,1] \to Y$, $H|_{x \times \{0\}} = f$ *and* $H|_{x \times \{1\}} = g$

$$\Delta^n \times [0,1] \xrightarrow{\sigma \times 1} X \times [0,1] \xrightarrow{H} Y$$

The idea is to write $\Delta^n \times [0,1]$ as union of Δ^{n+1}. Let us consider some explicit examples of the p maps:

- $p_0 : C_0(X) \to C_1(Y)$
  $p_0(\sigma) = H_0(\sigma \times 1)|_{[v_0 w_0]} : \Delta^1 \to Y$
  We can parametrize Δ^1, as $\Delta^1 = \{(t_0, t_1) | \; t_0 + t_1 = 1, \; t_0, t_1 \geq 1\}$
  $\Delta^1 = \{1\} \subseteq \mathbb{R}$, $\sigma(1) = q$

$$\begin{aligned}
\Delta^0 \times [0,1] &\simeq \Delta^1 \\
\{1\} \times \{t\} &\mapsto (1-t, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t), \; \textit{where } H(q,0) = f(q), \; H(q,1) = g(q)
\end{aligned}$$

- $p_1 : C_1(X) \to C_2(Y)$
  $p_1(\sigma) = \sum_{i=0}^{1} H_0(\sigma \times 1)|_{[v_0 \ldots w_1]} = H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}$

$$\begin{aligned}
\Delta^1 \times [0,1] &\simeq [0,1] \times [0,1] \\
((t_0, t_1), t) &\mapsto (t_0, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t), \; \textit{where } H(q,0) = f(q), \; H(q,1) = g(q)
\end{aligned}$$

$$\begin{tikzcd}
C_2(X) \arrow[r, "\partial_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_0(X) \arrow[r, "\partial_0"] \arrow[dl, "p_0"'] & 0 \\
C_2(Y) \arrow[r, "\delta_2"] & C_1(Y) \arrow[r, "\delta_1"] & C_0(Y) \arrow[r, "\delta_0"] & 0
\end{tikzcd}$$

From the diagram above: $\delta_2 p_1 + p_0\partial_1 : C_1(X) \to C_1(Y)$

$$\begin{aligned}
(p_0 \circ \partial_1)(\sigma) &= p_0(\sigma|_{[v_1]} - \sigma|_{[v_0]}) \\
&= H_0(\sigma \times 1)|_{[v_1 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma|_{[v_1]} \times 1)|_{[v_1 w_1]} - H_0(\sigma|_{[v_0]} \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma \times 1)|_{[v_1]\times[0,1]} - H_0(\sigma \times 1)|_{[v_0]\times[0,1]}
\end{aligned}$$

$$\begin{aligned}
(\delta_2 \circ p_1)(\sigma) &= \delta_2(H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}) \\
&= H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&\quad - H_0(\sigma \times 1)|_{[v_1 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]}
\end{aligned}$$

So , $(\delta_2 p_1 + p_0\partial_1)(\sigma) = H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]} = g \circ \sigma - f \circ \sigma = (g - f) \circ \sigma$

$$\Rightarrow \delta_2 p_1 + p_0 \partial_1 = g - f$$

Theorem 2.1.1. *If two chain maps $f_\bullet, g_\bullet : (C_\bullet, \partial) \to (D_\bullet, \delta)$ are chain-homotopic then they induce the same homomorphism on homology:*

More explicitly:

$$\begin{tikzcd}
\dots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, "p_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \dots \arrow[dl, "p_{n-2}"'] \\
\dots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \dots
\end{tikzcd}$$

$$\begin{aligned}
H_n(f) &: H_n(C_\bullet) \to H_n(D_\bullet) \\
H_n(f) &: [x] \mapsto [f_n(x)]
\end{aligned}$$

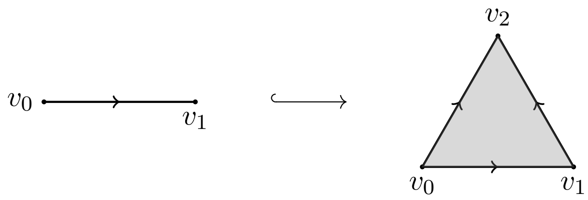
If $f_n - g_n = \delta_{n+1} p_n + p_{n-1}\partial_n \Rightarrow H_n(f) = H_n(g)$

Proof: Let us proof that the maps f_n, g_n induce the same homology.
For any $x \in \ker \partial_n \Rightarrow \partial_n x = 0$,
$(f_n - g_n)(x) = \delta_{n+1} p_n(x) + p_{n-1}\delta_n(x) = \delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$
$\Rightarrow (f_n - g_n)(x) \in \mathrm{Im}(\partial_{n+1})$

$$\left| \begin{aligned} H_n(f)([x]) &= [f_n(x)] \\ H_n(g)([x]) &= [g_n(x)] \end{aligned} \right. \Rightarrow [f_n(x)] - [g_n(x)] = [f_n(x) - g_n(x)] = [\delta_{n+1} p_n(x)] = [0]$$

since $\delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$.
So, $[f_n(x)] = [g_n(x)] \Rightarrow H_n(f) = H_n(g) \quad \square$

Example 2.1.1. Let X be a $1-simplex$ and Y a $2-simplex$:



$$C_\bullet(X,\partial): \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

$$D_\bullet(Y,\delta): \qquad \mathbb{Z} \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_0=0} 0$$

Let's introduce maps on the chain complexes:

$$f_n : C_n(X,\partial) \to C'_n(Y,\delta)$$
$$g_n : C_n(X,\partial) \to C'_n(Y,\delta)$$

$$\begin{tikzcd}
0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1=\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[dl, "p_1=id"'] \arrow[d, shift right, "f_1=\begin{bmatrix}0\\0\\1\end{bmatrix}"'] \arrow[d, shift left, "g_1=\begin{bmatrix}0\\1\\0\end{bmatrix}"] & \mathbb{Z}^{\oplus 2} \arrow[r, "\partial_0=0"] \arrow[dl] \arrow[d, shift right, "f_0=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}"'] \arrow[d, shift left, "g_0=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"] & 0 \\
\mathbb{Z} \arrow[r, "\delta_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_1=\begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_0=0"'] & 0
\end{tikzcd}$$

We can define $p_0 = \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$ and $p_1 = id_{\mathbb{Z}}$

For $n=0$:

$$f_0 - g_0 = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

$$\delta_1 p_0 + 0\circ\partial_0 = \begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

For the choice of the maps p_0, f_0, g_0: $f_0 \simeq g_0 \to$ homotopic equivalent

For $n=1$:

$$f_1 - g_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\delta_2 p_1 + p_0\circ\partial_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix} + \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

For the choice of the maps p_1, p_0, f_1, g_1: $f_1 \simeq g_1 \to$ homotopic equivalent
By theorem 2.1.1, $H_0(f) = H_0(g)$ and $H_1(f) = H_1(g)$.
More precicely, $H_0(f) = H_0(g) = id_{\mathbb{Z}}$ and $H_1(f) = H_1(g) = 0$, since $H_0(X) = H_0(Y) = Z$ and $H_n(X) = H_n(Y) = 0 \;\forall\; n > 0$

Lemma 2.1.1. *A chain complex $(C_\bullet, \partial)$ is contractable if id_C is homotopic equivalent to 0_C*

If $id_C \simeq 0_C$, then $H_n(C_\bullet) = 0 \ \forall n$
Examples:

- $C_\bullet: \ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$
  $H_0(C_\bullet) = \mathbb{Z} = H_1(C_\bullet)$ implies that $C_\bullet$ is not contractable

- $D_\bullet: \ \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$
  $H_0(D_\bullet) = \mathbb{Z}/2\mathbb{Z}$, $H_1(D_\bullet) = 0$ implies that $D_\bullet$ is not contractable

- $E_\bullet: \ \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$
  $H_0(E_\bullet) = 0 = H_1(E_\bullet) \Rightarrow$ need to check that $id_E \simeq 0_E$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] \arrow[dl, "p_1"'] \arrow[d, "id-0"] & \mathbb{Z} \arrow[r, "\partial_0=0"] \arrow[dl, "p_0"'] \arrow[d, "id-0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] & \mathbb{Z} \arrow[r, "\partial_0=0"] & 0
\end{tikzcd}$$

  We can assign $p_0 = id$, $p_1 = 0$.
  For a cycle $\sigma \in E_1 = \mathbb{Z}$:
  $(\partial_2 p_1 + p_0 \partial_1)(\sigma) = \partial_2 p_1(\sigma) + p_0 \partial_1(\sigma) = 0 + \sigma = \sigma = id - 0(\sigma) \Rightarrow id_E \simeq 0_E \Rightarrow (E_\bullet, \partial)$ is contractable

- $F_\bullet: \ \ldots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{\partial=2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \ldots$
  $\ker \partial = \operatorname{Im} \partial = (2) \Rightarrow H_n(F_\bullet) = 0 \ \forall \ n \Rightarrow$ need to check that $id_F \simeq 0_F$

$$\begin{tikzcd}
\ldots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[dl, "p_1"'] \arrow[d, "id-0"] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[dl, "p_0"'] \arrow[d, "id-0"] & \ldots \\
\ldots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \ldots
\end{tikzcd}$$

  In $\mathbb{Z}/4$ we have four classes $\overline{0}, \overline{1}, \overline{2}, \overline{4}$. The boudary operator $\partial = mult(2)$ maps the four classes only in two maps $\overline{0}, \overline{2}$. So, the ∂ cannot be surjective.
  For a cycle $\sigma \in \mathbb{Z}/4$:
  Since $((2)p_1 + p_0(2))(\sigma) \in (2)$, $((2)p_1 + p_0(2))(\sigma) \neq \sigma \Rightarrow (F_\bullet, \partial)$ is not contractable

Theorem 2.1.2. *Given topological spaces X, Y with maps $f : X \to Y$ and $g : Y \to X$*
If

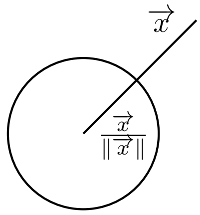
$$f \circ g \simeq id_Y$$
$$g \circ f \simeq id_X$$

where $\simeq$ denotes homotopic equivalence, then

$$H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y) \quad \forall \ n \in \mathbb{Z}$$

$H_n(X)$ is isomorphic to $H_n(Y)$, $\Rightarrow f_* = (g_*)^{-1}$where f_*, g_* are the induced maps on homology.

Example 2.1.2. Let X be the n-dimentional sphere and Y the (n+1)-dimentional real cordinate space without the origin, $X = S^n$ and $Y = \mathbb{R}^{n+1}/\{0\}$



$$f : S^n \hookrightarrow \mathbb{R}^{n+1}/\{0\} \textit{ is the usual inclusion}$$

$$g : \mathbb{R}^{n+1}/\{0\} \to S^n \; s.t \; g(\overrightarrow{x}) = \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$$

Clearly, $g \circ f \simeq id_S^n$ while $f \circ g : \overrightarrow{x} \mapsto \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|} \neq id_{\mathbb{R}^{n+1}/\{0\}}$
Let's prove that $f \circ g$ is homotopic equivalent to $id_{\mathbb{R}^{n+1}/\{0\}}$: We can construct a function

$$\begin{aligned} F : [0,1] \times \mathbb{R}^{n+1}/\{0\} &\longrightarrow \mathbb{R}^{n+1}/\{0\} \\ F(t, \overrightarrow{x}) &= (t(f \circ g) + (1-t)1_{\mathbb{R}^{n+1}/\{0\}})\overrightarrow{x} \\ &= t(\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) + (1-t)\overrightarrow{x} \end{aligned}$$

Clearly, $F(0, \overrightarrow{x}) = \overrightarrow{x} = id_{\mathbb{R}^{n+1}/\{0\}}$ and $F(1, \overrightarrow{x}) = (\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) = f \circ g$.
So, $f \circ g \simeq id_{\mathbb{R}^{n+1}/\{0\}}$, and by theorem 2.1.2 $\Rightarrow f_* = (g_*)^{-1}$.

Example 2.1.3. Let X be a $1-simplex$ and Y a $0-simplex$:

$$C_\bullet(X, \partial) : \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus^2} \xrightarrow{\partial_0 = 0} 0$$

$$C'_\bullet(Y, \partial') : \qquad 0 \xrightarrow{0} 0 \xrightarrow{\partial'_1 = 0} \mathbb{Z} \xrightarrow{\partial'_0 = 0} 0$$

Let's introduce maps on the chain complexes:

$$f_n : C_n(X, \partial) \to C'_n(Y, \partial')$$
$$g_n : C'_n(Y, \partial') \to C_n(X, \partial)$$

$$\begin{tikzcd} 0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, "0"] & \mathbb{Z}^{\oplus 2} \arrow[r, "\partial_0 = 0"] \arrow[dl, "p_0"'] \arrow[d, shift left, "f_0 = {[1\ 1]}"] & 0 \\ 0 \arrow[r, "0"] & 0 \arrow[r, "\partial'_1 = 0"] & \mathbb{Z} \arrow[r, "\partial'_0 = 0"] \arrow[u, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} = g_0"] & 0 \end{tikzcd}$$

For $n = 0$:

$$f_0 \circ g_0 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = id_{\mathbb{Z}}$$

$$g_0 \circ f_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq id_{\mathbb{Z}^{\oplus 2}}$$

Let's prove that $g_0 \circ f_0$ is homotopic equivalent to $id_{\mathbb{Z}^{\oplus 2}}$:
For an arbitrary element $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^{\oplus^2}$:
$(g_0 \circ f_0 - id_{\mathbb{Z}^{\oplus 2}}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ and $(\partial'_1 p_0 + 0 \circ \partial_0) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Longrightarrow g_0 \circ f_0 \simeq id_{\mathbb{Z}^{\oplus 2}}$
To be continued and rechecked

2.2 Exact Sequences

Definition 2.2.1. *A sequence of homomorphisms:*

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$$

is <u>*exact*</u> *if* $\ker \alpha_n = \operatorname{Im} \alpha_{n+1}\ \forall\ n$.

$\operatorname{Im} \alpha_{n+1} \subseteq \ker \alpha_n$ is equivalent to $\alpha_n \alpha_{n+1} = 0$ since $(A_\bullet, \alpha)$ is a chain complex.
& $\ker \alpha_n \subset \operatorname{Im} \alpha_{n+1} \Rightarrow H_n$ *is trivial* : $H_n = 0\ \forall\ n$

Examples of short exact sequences:

1. $0 \to A \xrightarrow{\alpha} B$ *is exact* $\iff \ker \alpha = 0, \alpha$ *is injective*
2. $A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B, \alpha$ *is surjective*
3. $0 \to A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B$ *and* $\ker \alpha = \{0\}, \alpha$ *is an isomorphism*
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is exact* $\iff$
   (a) $\ker \alpha = \operatorname{Im}(0 \to A) = 0\ \Rightarrow \alpha$ *is injective*
   (b) $\operatorname{Im} \beta = C \Rightarrow \beta$ *is surjective*
   (c) $\ker \beta = \operatorname{Im} \alpha$

   So, β induces an isomorphism $C \simeq \frac{B}{\operatorname{Im} \alpha}$. C can written as $C \simeq B/A$ if α is an inclusion of A as a subgroup of B.

Let us consider a short exact sequence of chain complexes:

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$

$A_\bullet,\ B_\bullet,\ C_\bullet$ are chain complexes and i, π are maps between chain complexes where
$\ker \pi = \operatorname{Im} i, \quad \pi$: *surjective and* i : *injective*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : \arrow[r] & A_n \arrow[r, "\partial"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "i_{n-2}"] & {} \\
B_\bullet : \arrow[r] & B_n \arrow[r, "\partial"] \arrow[d, "\pi_n"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "\pi_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "\pi_{n-2}"] & {} \\
C_\bullet : \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The induced sequence on homology:

$$H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet)\ \forall\ n \tag{2.3}$$

$\pi \circ i = 0 \Rightarrow \pi_* \circ i_* = 0, \quad H_n(\pi \circ i) = H_n(\pi) \circ H_n(i)$
(2.3) <u>need not</u> be a short exact sequence. However, we can create a long exact sequence of

homology:

$$H_{n+1}(C_\bullet) \xrightarrow{\partial n+1} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet)$$

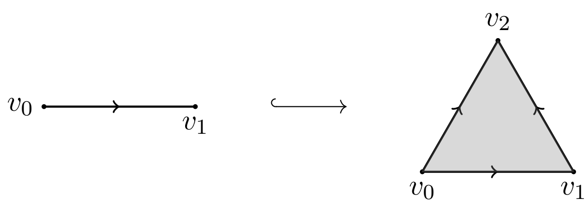
$$0 \to \operatorname{Im} \delta_{n+1} \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \to \ker \delta_n \to 0$$

where the δ map is defined as:

$$\delta : H_n(C) \to H_{n-1}(A)$$
$$[c] \mapsto [a]$$

For a element $b \in B_n$ there exists $c = \pi_n(b)$ since π is onto.
If we apply the boudary map $\partial : B_n \to B_{n-1}$ on b, then $\partial b \in B_{n-1}$, $\pi_n(\partial b) = \partial(\pi_n(b)) = 0$
We can take an element $a \in A_{n-1}$ such that $i(a) = \partial(b)$
$\partial(\partial b) = \partial(i(a)) = i(\partial a) \Rightarrow \partial a = 0$ *since i is injective* $\Rightarrow \partial b = (0) \in A_{n-1}$

Example 2.2.1. Let consider X to be a $1-$*simplex* and Y a $2-$*simplex*



A short exact sequence of chain complexes for X, Y:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial=\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[d, "i_1=\begin{bmatrix}0\\0\\1\end{bmatrix}"'] & \mathbb{Z}^{\oplus 2} \arrow[r] \arrow[d, "i_0=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}"] & 0 \\
B_\bullet : & 0 \arrow[r] & \mathbb{Z}^{\oplus 3} \arrow[r, "\partial=\begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}"] \arrow[d, "\pi_1"] & \mathbb{Z}^{\oplus 3} \arrow[r] \arrow[d, "\pi_0"] & 0 \\
C_\bullet : & 0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The short exact sequence of complexes induces a long exact sequence on homology:

$$0 \to H_1(A) \to H_1(B) \to H_1(C) \xrightarrow{\delta} H_0(A) \to H_0(B) \to H_0(C) \to 0$$

Explicitly, the long exact sequence is:

$$0 \to 0 \to \mathbb{Z} \to H_1(C) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\alpha} H_0(C) \to 0$$

$$\ker(1) = \{0\} = \operatorname{Im}\delta$$
$$\ker(\alpha) = \operatorname{Im}(1) = \mathbb{Z}$$

$$C_1 = \mathbb{Z}^{\oplus^3} / \operatorname{Im} i_1 \simeq \mathbb{Z}^{\oplus^2}$$
$$C_0 = \mathbb{Z}^{\oplus^3} / \operatorname{Im} i_0 \simeq \mathbb{Z}$$

The boundary operator between C_1, C_0:

$$\partial : C_1 \to C_0$$
$$\partial : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a+b)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} mod(\operatorname{Im} i_1) \xmapsto{\partial = \left[\begin{smallmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right]} \begin{bmatrix} -b \\ -a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \end{bmatrix}$$

$$H_1(C) = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \right\} \simeq \mathbb{Z}$$
$$H_0(C) = C_0 / C_0 = 0$$

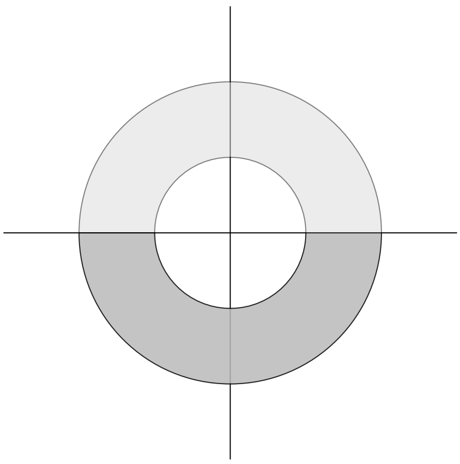
Let us take an example when $Y \subseteq X$ subspace, the short exact sequence of chain complexes is:

$$0 \to C_n(Y) \to C_n(X) \to C_n(X)/C_n(Y) \to 0$$

We get a long exact sequence on homology:

$$H_n(Y) \to H_n(X) \to H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y) \to H_{n-1}(X) \to \cdots$$

Example 2.2.2. Consider X to be the annulus, and the shaded area $Y \subseteq X$



$$
\begin{aligned}
&X = \{(x,y)|1 \leq x^2 + y^2 \leq 2\} \\
&Y = \{(x,y)|1 \leq x^2 + y^2 \leq 2, y \leq 0\} \\
&\delta \ : \ H_1(X,Y) \to H_0(Y) \\
&\delta([\sigma]) = [\partial\sigma] = [\sigma(1) - \sigma(0)]
\end{aligned}
$$

In "good cases" $H_n(X,Y) = H_n(X/Y)$

For two topological spaces $Y \subseteq X$ "pairs of spaces".
We can construct the following chain complex:
$0 \to C_n(Y) \xrightarrow{i} C_n(X) \xrightarrow{\pi} C_n(X)/C_n(Y) \to 0, \quad C_n(X,Y)$ *are relative chains*
Elements of C_n: $\sigma : \Delta^n \to Y \subseteq X$

$$
\begin{gathered}
\partial : C_n(X,Y) \to C_{n-1}(X,Y) \\
\delta \ (\mathrm{mod}\ C_n(Y)) \mapsto \partial\delta \ (\mathrm{mod}\ C_{n-1}(Y)) \\
H_n(X,Y) \equiv H_n(C_\bullet(X,Y)) = Z_n(X,Y)/B_n(X,Y), \quad \textit{cycle/boundary}
\end{gathered}
$$

Definition 2.2.2 (Retraction). *Consider $Y \subseteq X$, a retraction of X onto Y is a map $r : X \to Y$ such that, $r(X) = Y$ and $r^2 = r$.*

i.e. $r(y) = Y$ if $y \in Y$
$i : Y \to X \quad r \circ i = id_Y \quad i \circ r \neq id_x$ but $r_* \circ i_* = id$ in homology

Definition 2.2.3 (Deformation retract). *Consider $Y \subseteq X$: subspace*
Y is a deformation retract of X if there is a <u>homotopy</u> between id_X and a retraction $r : X \to Y$

$$
(F_t) \quad F_t : X \to X \quad \left.\begin{array}{l} F_0 : id_X \\ F_1 : X \to Y \quad F_1|_Y = id_Y \\ F_1(X) = Y \end{array}\right| F_0 \simeq F_1 \ \textit{homotopic}, \ id_X \simeq r
$$

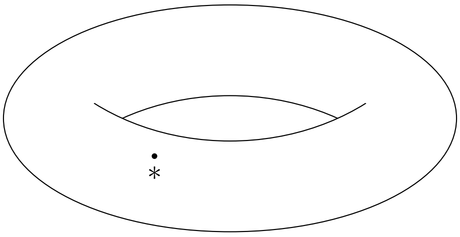
(X,Y) is a "good pair" if

- $Y \subseteq X$ - closed
- There is open $V \subseteq X,$ *such that* V is a deformation retracts on Y.

Example 2.2.3.

$$
\begin{aligned}
&(\mathbb{R}^{n+1}, S^n) \text{ is a good pair} \\
&S^n \subseteq \mathbb{R}^{n+1} \text{ - closed} \\
&S^n \subseteq (\mathbb{R}^{n+1}/\{0\}) \text{ and is a deformation retracted of it}
\end{aligned}
$$

Example 2.2.4. Consider X to be a torus, and Y a point on its surface:



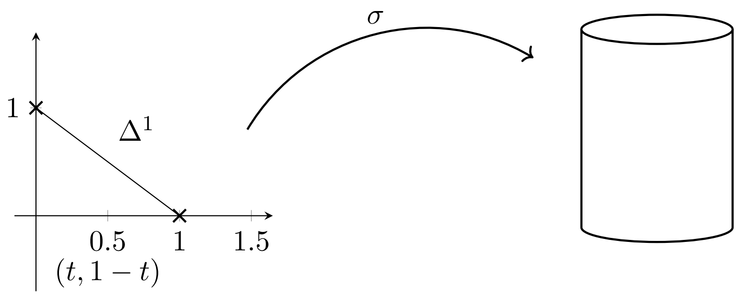
$$Y = \underset{pt}{\{*\}} \hookrightarrow X$$
$$0 \to C_n(Y) \to C_n(X) \to C_n(X/*) \to 0$$
$$H_n(Y) = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases}$$
$$\cdots \to H_n(Y) \to H_n(X) \to H_n(X, *) \to H_{n-1}(Y) \to \cdots$$
$$\cdots \to H_0(Y) \to H_0(X) \to H_0(X, *) \to 0$$
$$n > 0: \quad H_n(X, \star) = H_n(X)$$
$$n = 0: \quad 0 \to \mathbb{Z} \to H_0(X) \to H_0(X, *) \to 0$$
$$H_0(X, \star) = H_0(x)/\mathbb{Z}, \quad i.e. \begin{array}{l} H_0(X) = \mathbb{Z}^d \\ H_0(X) = \mathbb{Z}^{d-1} \end{array}$$

Remark: Sometimes one introduces “reduced homolgy”

$$\cdots \to C_n(X) \to C_{n-1} \to \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0, \quad \tilde{H}_n(X) \text{ - reduced homology}$$
$$\sum n_i \sigma_i \mapsto \sum_i n_i \in C_0(X)$$
$$\tilde{H}_n(X) = \begin{cases} H_n(X), n > 0 \\ H_0^{sing}(X) = H_0(X) \oplus \mathbb{Z} \end{cases}$$
$$H_n(X, *) \to \tilde{H}_n(X)$$

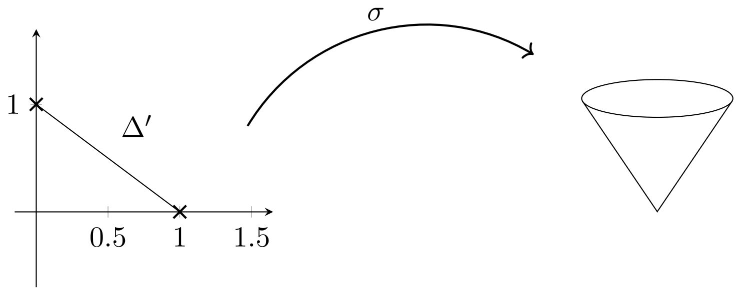
Let us consider some examples of Reduced Homology:

Example 2.2.5. Consider $X = S^1 \times [0, 1]$, and $Y = S^1 \times \{0\}$:



$$\sigma : \Delta^1 \mapsto S^1 \times [0, 1]$$
$$\sigma : t \mapsto (\cos 2\pi t, \sin 2\pi t, 1)$$

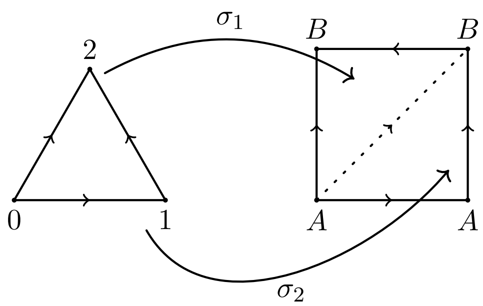
The boundary operator on σ: $\partial\sigma = 0 \rightsquigarrow [\sigma] \in H_1(X)$
The space X is homotopic equivalent to S^1 so, $H_1(X) \simeq H_1(S^1) \simeq \mathbb{Z}$, considered in 2.1.
Let us take the quotient map X/Y. The space Y will contract into a point creating a cone.



In $H_1(X, Y)$ the class $[\sigma] \in H_1(X)$ goes to 0 and the long exact sequence on homolgy is:

$$H_1(Y) \to H_1(X) \to H_1(X, Y) \to H_0(X, Y) \to H_0(Y) \to H_0(X) \to 0$$

where $H_1(Y), H_1(X), H_0(Y), H_0(X) \simeq \mathbb{Z}$.



$$\partial\sigma_1 = \sigma_1|_{[12]} - \sigma_1|_{[02]} + \sigma_1|_{[01]}$$
$$\partial\sigma_2 = \sigma_2|_{[12]} - \sigma_2|_{[02]} + \sigma_2|_{[01]}$$
$$\partial(\sigma_1 + \sigma_2) = \sigma_1|_{[12]} + \sigma_2|_{[01]}$$

We can choose $\sigma = \sigma_1|_{[12]} = \partial(\sigma_1 + \sigma_2) - \sigma_2|_{[01]}$, $\sigma_2|_{[01]} \in C_1(Y)$
$\sigma \in B_1(X, Y) \Rightarrow [\sigma] = 0 \in H_1(X, Y)$

Chapter 3

Persistent Homology

Persistent homology (PH) is a method used in topological data analysis (TDA) to study qualitative features of data that persist across multiple scales. Due to its construction, persistent homology computations are robust to pertubation of the data. Also, PH is used to extract relevant features of the data, and separate them from noise.

An important result in computing persistent homology is that the persistent homology of a filtered d-dimensional simplicial complex is simply the standard homology of a particular graded module over a polynomial ring [ZC05].
Before intruducing Persistent homology and understanding its calculation, we need to provide some priliminary concept from albegra.
We begin by reviewing graded modules and rings and then stating the structure of finitely generated modules over principal ideal domains. Then, refering to the notions intruduced in Chapter 1, we provide some comments of the reduction algorithm used for computing simplicial homology. We conclude this chapter by describing persistent homology.

3.1 Background

3.1.1 Graded Rings and Modules

A graded ring is a ring $\langle R, +, \cdot \rangle$ equipped with a direct sum decomposition of Abelian groups $R \cong \bigoplus_{i \in \mathbb{Z}} R_i$, so multiplication is defined by bilinear pairings $R_n \otimes R_m \to R_{n+m}$.

Elements in a single R_i are called homogeneous and have degree i, $\deg e = i$ for all $e \in R_i$.

Example 3.1.1. $R = A[t]$, where (A - commutative ring)

$$R_0 = A, \quad R_1 = \{at, a \in A\}, \quad \dots, R_i = \{at^i, a \in A\}$$

Example 3.1.2. $R = \mathbb{R}[x, y, z]$

$$R_i = \{cx^{d_1}y^{d_2}z^{d_3} | \sum_{k=1}^{3} d_k = i\}$$

For example $R_1 \simeq \mathbb{R}^3$ as a vector space $\{ax + by + cz\}$

A graded module M over a graded ring R is a module equipped with a direct sum decomposition $M \cong \bigoplus_{i \in \mathbb{Z}} M_i$, so that the action of R on M is defined is bilinear pairings $R_n \otimes M_m \to M_{n+m}$.

A graded ring (module) is non-negatively graded if $R_i = 0$ ($M_i = 0$) for all $i < 0$.

Note: R is a PID if it's a domain (no zero divisors) & all its ideal are principal

For example:

$$R = \mathbb{Z},\ I = (n),\ n \in \mathbb{Z}$$
$$R = k[t],\ k\ -\ field$$

The structure theorem describes finitely generated modules and graded modules over PIDs.

Theorem 3.1.1 (Structure Theorem)**.** *If D is a PID, then every finitely generated D-module is isnomorphic to a direct sum of cyclic D-module. That is, it decomposes uniquely into the form*

$$D^{\beta} \oplus (\bigoplus_i D/d_i D)), \tag{3.1}$$

for $d_i \in D$, $\beta \in \mathbb{Z}$, such that $d_i | d_{i+1}$. Similarly, every graded module M over a graded PID D decomposes uniquely into the form

$$(\bigoplus_i \Sigma^{\alpha_i} D) \oplus (\bigoplus_i \Sigma^{\gamma_j} D/d_j D)), \tag{3.2}$$

where $d_j \in D$ are homogeneous elements so that $d_j | d_{j+1}$, $\alpha_i, \gamma_j in \mathbb{Z}$, and Σ^{α} denotes an α-shift upward in grading.

The free portion on the left is a vector includes generators that may generate an infinite number of elements. Decomposition (3.1) has a vector space of dimension β. The torsional portion on the right includes generators that may generate a finite number of elements. These torsional elements are also homogeneous. Intuitively then, the theorem describes finitely generated modules and graded modules as structures that look like vector spaces but also have some dimensions that are "finite" in size.

Example 3.1.3. Let us take $D = k[t]$ – graded ring(e.g. $\mathbb{R}[t]$) then:

$$D = \underset{\substack{\parallel \\ M_0}}{k} \oplus \underset{\substack{\parallel \\ M_1}}{kt} \oplus \underset{\substack{\parallel \\ M_2}}{kt^2} \oplus \cdots$$

is also a graded module over itself.

$$M = \sum^{\alpha} D = t^{\alpha} k[t] \subseteq k[t] \text{ is an ideal of } D \Rightarrow D \text{ - module}$$
$$M = \underset{\substack{\wr| \\ k}}{M_{\alpha}} \oplus \underset{\substack{\wr| \\ kt}}{M_{\alpha+1}} \oplus \cdots$$
$$(\sum^{\alpha} D)_i = D_{\alpha+i}$$

3.1.2 Reduction

The reduction algorithm is the standard method used in computing homology. For simplicity, we describe the method for integer coefficents. However, the method applies also to modules over arbitrary PIDs.

Given C_k, we can use as standard basis the oriented k-simplices:

$$\partial_k : \underset{\{e_i\}}{C_k} \to \underset{\{\hat{e_i}\}}{C_{k-1}}$$

$(C_k, C_{k-1}$: free abelian groups or free R-module)

M_k : standard matrix representation of ∂_k

$$e \cdot u : \begin{pmatrix} e_1, \cdots, e_{m_k} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{m_k} \end{pmatrix} \in C_k$$

$$\partial_k(eu) = \hat{e}(M_u)$$

The null-space of M_k corresponds to Z_k and its range-space to $B_{k?1}$. The reduction algorithm derives alternate bases for the chain groups, relative to which the matrix for ∂_k is diagonal. The algorithm utilizes the following elementary column operations on M_k:

- exchange column i, and column j,
- multiply column i by -1
- replace column i by (column i) + q(column j), where $q \in \mathbb{Z}$ and $i \neq j$

The algorithm also uses elementatry row operation that are similary defined. The idea of the reduction algorithm is to systematically modify the bases of C_k and $C_{k?1}$ using elementary operations so that it reduce M_k to its Smith normal form:

Row-operation

$$R_i \mapsto R_i + qR_j \text{ on } M$$

$$M \mapsto i \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & q \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\text{A}} M$$

$$R_2 \mapsto R_2 + 2R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} M$$

$$\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e_1 & e_2 & 2e_2 + e_3 \end{pmatrix}$$

Column operations:

$$M \mapsto MB, B_2 \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & q & & 1 \end{pmatrix}$$

Interpretation of row/column operations on a matrix of a map in terms of changing bases.

$$\partial : \xi = \underline{e}u \mapsto \hat{e}Mu \textit{ where } u = [\xi]\underline{e}$$

Suppose we perform a column-operation $M \mapsto MB$. This is supposed to change the matrix of a map - keeping the map unchanged

$$\xi = \underset{\underline{e}B(B^{-1}u)}{\underset{\|}{\underline{e}u}} \mapsto \hat{e}Mu = \hat{e}MB(B^{-1}u)$$

If we set $\underline{e}' = \underline{e}B$

$\partial : \xi = \underline{e}'v \mapsto \underline{\hat{e}}MBv$

$v = [\xi]_{\underline{e}'}$

Similarly, suppose we perform a row-operation $M \mapsto AM$. Then

$$\begin{aligned}\xi = \underline{e}u \mapsto \underline{\hat{e}}Mu &= \underline{\hat{e}}A^{-1}AMu \\ &= \underline{\hat{e}}'AMu\end{aligned}$$

That is, if $M = [\partial]_{\underline{e}\underline{\hat{e}}}$, then

$$AMB = [\partial]_{\underline{e}B,\underline{\hat{e}}A^{-1}}$$

$$A = I + qE_{ij} : R_i \mapsto R_i + qR_j \text{ (via } M \mapsto AM)$$

$$A^{-1} = I - qE_{ij}$$

$$\underline{\hat{e}}A^{-1} = (\hat{e}_1, \cdots, \hat{e}_{m_{k-1}}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \cdots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} i =$$

$$= (\hat{e}_1, \cdots, \hat{e}_j - q\hat{e}_i, \cdots), \text{i.e. } \hat{e}_j \mapsto \hat{e}_j - q\hat{e}_i$$

$$B = I + qE_{ji} : C_i \mapsto C_i + qC_j (\text{via } M \mapsto MB)$$

$$\underline{e}B = (e_1, \cdots, e_{m_k}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \cdots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} j = (e_1, \cdots, e_i + qe_j, \cdots)$$

$$\begin{aligned}
&\text{Let } V \simeq \mathbb{R}^n (\text{be a vector space}) \\
&B = \{e_1, \cdots, e_n\} \\
&V = \underline{e}X = (e_1, \cdots, e_a) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i \quad (e_i \in V, x_i \in \mathbb{R}) \\
&B^1 = \{f_1, \cdots, f_n\} \\
&v = \underline{e}X = \underbrace{\underline{e}T^{-1}}_{\underline{\text{f}}}(\underbrace{TX}_{\text{Y}}) = (f_1, \cdots, f_n)Y \\
&(f_1, \cdots, f_n) = (l_1, \cdots, l_n)T^{-1} \\
&Y = TX
\end{aligned}$$

Smith normal form (for PID):

$$\begin{aligned}
&\exists A, B : \\
&\partial_k \equiv AMB = \left(\begin{array}{c|c} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_l \end{pmatrix} & \overbrace{}^{} 0 \\ \hline 0 & 0 \end{array} \right)^{b_i | b_{ix1}} \\
&rank Z_k = m_k - e_k \\
&rank\ H_k = m_k - e_k - e_{k+1}
\end{aligned}$$

3.2 The Persistence Module

In this section we will combine the homology of all the complexes in the filtration into a single algebraic structure. We then establish a correspondence that reveals a simple description over fields. Most significantly, we illustrate that the persistent homology of a filtered complex is simply the standard homology of a particular graded module over a polynomial ring.

Taking into consideration the construction of a filtered simplicial complex intruduced in section 1.2, we can construct a filtered chain complex:

$$\begin{array}{ccccccccc} (0) & \subseteq & C_\bullet^1 & \subseteq & C_\bullet^2 & \subseteq & \cdots & \subseteq & C_\bullet^m \\ \| & & & & & & & & \| \\ C_\bullet^0 & & & & & & & & C_\bullet \end{array}$$

Definition 3.2.1. *(Persistent Homology Group). Given a filtered complex, the i-th complex K^i has associated boundary operators ∂_k^i, matrices M_k^i , and groups C_k^i, Z_k^i, B_k^i, and H_k^i for all $i, k \geq 0$ The p-persistent k-th homology group of K^i is*

$$H_k^{i,p} = Z_k^i \Big/ (B_k^{i+p} \cap Z_k^i)$$

For $p = 0$, this is the usual homology formula:

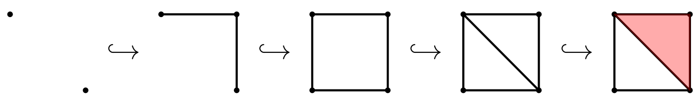
$$H_k(C_\bullet^i) = Z_k^i / (B_k^i \cap Z_k^i) = Z_k^i / B_k^i$$

Definition 3.2.2. *(Persistence Complex)*

A persistence complex $\mathcal{C}$ is a family of chain complexes $\{C_*^i\}_{i\geq 0}$ over R, together with a chain map's $f_i : C_*^i \to C_*^{i+1}$ so that we have the following diagram:

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \dots$$

Example 3.2.1. Let us consider the following filtered simplicial complex, and the filtered chain complex:



$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
0 \arrow[r, hook, "f^0"] \arrow[d] & \mathbb{Z}^2 \arrow[r, hook, "f^2"] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook, "f^3"] \arrow[d] & \mathbb{Z}^5 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
\mathbb{Z}^2 \arrow[r, hook, "f^0"] \arrow[d] & \mathbb{Z}^3 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
0 \arrow[r, hook, "f^0"] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

$$C_\bullet^0 \xhookrightarrow{f^0} C_\bullet^1 \xhookrightarrow{f^1} C_\bullet^2 \xhookrightarrow{f^2} C_\bullet^3 \xhookrightarrow{f^3} C_\bullet^4$$

Definition 3.2.3. *(Persistence Module). A persistance module $\mathcal{M}$ is a family of $R-modules$, M^i, together with homoemorphism $\varphi^i : M^i \to M^{i+1}\}$*

Suppose we have a persistance module $\mathcal{M} = \{M^i, \varphi^i : M^i \to M^{i+1}\}$ over a ring R, We can equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by

$$\alpha(M) = \bigoplus_{i\geq 0} M_i$$

, where the $R-module$ structure is the sum of the structures on the individual componets, and where the action of t is given by:

$$t \cdot (m^0, m^1, \cdots) = (0, \varphi^0(m^0), \varphi^1(m^1), \cdots)$$

$$\left(\begin{array}{c|c|c|c}
0 & & & \\ \hline
\varphi^0 & 0 & & \\ \hline
 & \varphi^1 & 0 & \\ \hline
 & & \varphi^2 &
\end{array}\right)$$

t simply shift elements of the module up in gradation.

Example 3.2.2.

$$\varphi : \mathbb{R}^n \to \mathbb{R}$$
$$\varphi : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (A_1, \cdots, A_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A_1 x_1 + \cdots + A_n x_n$$
$$(\varphi_1, \cdots, \varphi_{n-1})\mathbb{R}^n \to \mathbb{R}^{n-1}$$
$$\underline{x} \mapsto \begin{pmatrix} A_{11} & \cdots & A_{m-1} \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$
$$\text{e.g.} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

Theorem 3.2.1. *(Correspondence) The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated non-negatively graded modules over $R[t]$.*

The Correspondence theorem gives us a simple decomposition when the ground ring is a field F. In this case the graded ring $F[t]$ is a PID and its only graded ideals are homogeneous of form (t_n), so the structure of the $F[t] - module$ is described by sum (3.2) in structure theorem 3.1.1:

$$(\bigoplus_i \Sigma^{\alpha_i} F[t]) \oplus (\bigoplus_j \Sigma^{\gamma_j} F[t]/(t^{n^j})). \tag{3.3}$$

Chapter 4

Computing Persistent Homology

So far we have build the following setup to compute persistance homology on persistance chain complex:

$$\begin{aligned}
&Persistence\ complex : C_\bullet^0 \hookrightarrow C_\bullet^1 \hookrightarrow C_\bullet^2 \hookrightarrow \cdots \\
&Persistent\ Homology\ Group : \ H_k^{i,p} = Z_k^i / B_k^{i+p} \cap Z_k^i \\
&where\ Z_k^i = \ker(\partial_k^i : C_k^i \to C_{k-1}^i) = \operatorname{Im}((H_k^i = H_k(C_\bullet^i) \to (H_k(C_\bullet^{i+p}))
\end{aligned}$$

Example 4.0.1. Consider X to be a space consisting of two points and Y, of an edge. Then we can construct the following persitence complex:

$$\begin{tikzcd}
C_\bullet^0 \arrow[r, hook, "f_0"] & C_\bullet^1
\end{tikzcd}$$

$$\begin{tikzcd}
1 & (0) \arrow[r, hook] \arrow[d, "\partial_1 = 0"'] & \mathbb{Z} \arrow[d, "\partial_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}"] \\
0 & \mathbb{Z}^2 \arrow[r, hook] \arrow[d, "\partial_0 = 0"'] & \mathbb{Z}^2 \arrow[d, "\partial_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}"] \\
 & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

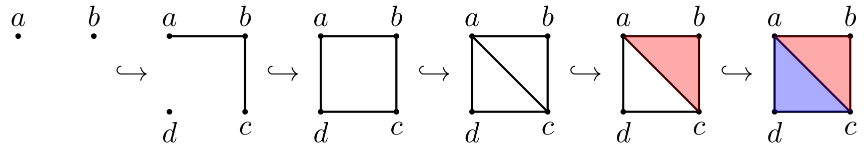
$\underline{p = 0}$

$$\begin{aligned}
&H_k^{i,0} = Z_k^i / B_k^i \cap Z_k^i \\
&H_1^{0,0} = (0) \quad H_1^{1,0} = (0) \\
&H_0^{0,0} = \mathbb{Z}^2 \quad H_0^{1,0} \simeq \mathbb{Z}
\end{aligned}$$

$\underline{p = 1}$

$$
\begin{aligned}
H_k^{i,1} &= Z_k^i / B_k^{i+1} \cap Z_k^i \\
H_1^{0,1} &= Z_0^1 / B_1^1 \cap Z_1^0 = (0) \quad H_1^{1,1} = (0) \\
H_0^{0,1} &= Z_0^0 / B_0^1 \cap Z_0^0 = \mathbb{Z}^2 / \operatorname{Im} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \simeq \mathbb{Z}
\end{aligned}
$$

Example 4.0.2. Let us consider the following filtered complex:



$$
\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
 & C_\bullet^0 \arrow[r] & C_\bullet^1 \arrow[r] & C_\bullet^2 \arrow[r] & C_\bullet^3 \arrow[r] & C_\bullet^4 \arrow[r] & C_\bullet^5 \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[r] \arrow[d, "\partial_1^3"] & \mathbb{Z}^5 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[d] \\
0 & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Some explicit computations are:
Here the matrix of ∂_1^3 is

$$
\left(\begin{array}{c|ccccc}
 & ab & bc & cd & ad & ac \\
\hline
d & 0 & 0 & 1 & 1 & 0 \\
c & 0 & 1 & -1 & 0 & 1 \\
b & 1 & -1 & 0 & 0 & 0 \\
a & -1 & 0 & 0 & -1 & -1
\end{array}\right) = M_1
$$

Introduce the structure of a graded module over $k[t]$ (on $R[t]$)
Degrees $\iff$ appearance in the translation

- $a, b\}$ deg 0

- $\left.\begin{array}{c} c, d \\ ab, bc \end{array}\right\}$ deg 1
- ab, dc deg2
- ac deg 3
- abc deg 4
- adc deg 5

If we work in $(\mathbb{Z}/2\mathbb{Z})[t]$

$$M_1 = \left(\begin{array}{c|ccccc} & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right)$$

Remarks:
* Any ideal $\omega \subseteq R$ in a commutative ring in as R - module in a natural way: $\forall a \in \omega, \forall r \in R, r \cdot a \in \omega$ and $\omega \subseteq R$ is an abgroup(R is a module over itself & ω is a submodule)
* Hence for any

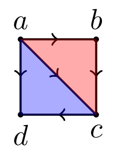
$$\begin{array}{cc} & n \in \mathbb{N} \\ (t^n) & \subseteq k[t] \\ \| & \\ \{ \underset{\overset{\|}{t^n k[t]}}{t^n} \; P(t), P \in k[t]\} & \end{array}$$

is a $k[t]$ - module. This is a free $k[t]$ - module.
* $R = k[t]$ has a natural structure of graded ring. The deg i - elements are the (non-zero) elements of the line $R_i := kt^i \subseteq k[t]$ (in general: $R = \oplus R_i, R_i R_j \subseteq R_{i+j}$). The ideal $(t^n) \subseteq k[t]$ is then a graded $k[t]$ - module (in general, this means $M = \underset{i}{\oplus} M_j, R = \underset{i}{\oplus} R_i, R_i \cdot M_j \subseteq M_{i+j}$)
$(t^n) = t^n, k[t] = \underset{i \geq 0}{\oplus} \underbrace{kt^{n+i}}_{(t^n)}$
i.e.
degree - i elements of (t^n) are the monorvials of degree $(n + i)$
* While $(t^n) \subseteq k[t]$ is an ideal, so $k[t]$ - module - and so $k[t]$ - submodule; and a graded $k[t]$ - module, it is not a graded $k[t]$ - submodule!
The grading of (t^n) is not the grading that is induced by the ambient $k[t]$: it is shifted up by n: t^{n+i}, as an alternative of $(t^n) = t^n k[t]$, has degree i, not degree $(n + i)$.
* In general, for a graded module M over a graded ring R we can define the twist of M by n, $M[n]$, also denoted by $\sum^n M$, is defined by $(\sum^n M)_i = M_{n+i}$.
i.e., by redefining/shifting the grading up by n. We see that we can identify $\sum^n k[t]$ with $(t^n) = t^n k[t]$ as graded modules $(\sum^n k[t])_i = k[t]_{n+i} = kt^{n+i} \xrightarrow{id} kt^n \cdot t^i$
* A map of free $k[t]$ - modules $k[t] \to k[t]$ is multiplication by some polynomial $p(t)$ ie of the kind $p(t) \mapsto p(t)q(t)$ ($p(t)$ is the image of $1 \in k[k]$). More generally, a map of free $k[t]$ - module, $k[t]^{\oplus m} \to k[t]^{\oplus r}$ is given by some $r \times m$ matrix with $k[t]$ - entries.
* A map (morphism) if graded R-modules $\varphi : M \to N$ is a map of R - modules (ie: R - linear, ie: $\varphi(rm) = r\varphi(m), \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$) which preserves the degrees. That is $\varphi(M_i) \subseteq N_i, \forall i$.

* In particular, a map of graded modules $\varphi : \underset{(t^n)}{\underset{\wr|}{\sum^n k[t]}} \to \underset{(t^p)}{\underset{\wr|}{\sum^p k[t]}}$ must send $(t^n)_i = kt^{n+i}$ to $(t^p)_i = kt^{p+i}$.
* As the shifts $\sum^n k[t]$ are still free modules, any $k[t]$ - module homomorphism $\varphi : (t^n) \to (t^n)$ is determined by some polynomial $p(t) \in k[t]$, ie by the image of a generator: $\varphi(t^n) = p(t)t^p$ as $\varphi(t^n q(t)) = q(t)\varphi(t^n)$. <u>However</u> if φ is a <u>graded</u> module homomorphism, we must have that $\varphi(t^n) = \underbrace{(mt^p)}_{p(t)} t^p$, ie, that $p(t)$ be homogeneous. Similarly, for direct sums $\bigoplus_i (t^{ni}) \to \bigoplus_i (t^{pj})$.
* <u>In particular:</u> if $n \geq p$, any map of modules $\varphi : k[t] \to k[t], 1 \mapsto p(t)$ determines a map of <u>graded</u> modules

$$\begin{aligned} &(t^n) = \sum^n k[t] \to (t^p) = \sum^p k[t] \\ &t^n \mapsto (p(t)t^{n-p})t^p \end{aligned}$$

E.g, the identity map on $k[t]$ induces $t^n \mapsto t^{n-p} \cdot t^p$ (The "matrix element" of the identity on $k[t]$ is t^{n-p}). Ditto for direct sums.
* Everything so far works similarly for $\mathbb{Z}[t]$ or $A[t]$, A - common ring. In particular, look again at the simplicial complex



The chain complex is $0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^5 \xrightarrow{\partial_1} \mathbb{Z}^4 \to 0$. The matrices M_2 and M_1 of ∂_2 and ∂_1 without the indicated bases are:

$$M_2 = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \begin{pmatrix} abc & acd \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Then ∂_1 (ie M_1) induces a map of $\mathbb{Z}[t]$ - modules $\mathbb{Z}[t]^{\oplus 5} \xrightarrow{M_1} \mathbb{Z}[t]^{\oplus 4}$:

$$\underline{V} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{pmatrix} \mapsto M_1 \underline{V}_{\bullet}$$

Using the grading (filtration) on the simplicial *cx* we get a chain complex of graded $\mathbb{Z}[t]$ - modules
$0 \to (t^4) \oplus (t^5) \xrightarrow{\partial_2^1} (t)^{\oplus 2} \oplus (t^2)^{\oplus 2} \oplus (t^3) \xrightarrow{\partial_1^1} (t)^{\oplus 2} \oplus (1)^{\oplus 2} \to 0$
by the procedure on pg. The respective matrices are (without same bases)

$$M_2' = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \begin{pmatrix} abc & acd \\ t^3 & 0 \\ t^3 & 0 \\ 0 & t^3 \\ 0 & -t^3 \\ -t^3 & t^3 \end{pmatrix}, M_1' = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & t & t & 0 \\ 0 & 1 & -t & 0 & t^2 \\ t & -t & 0 & 0 & 0 \\ -t & 0 & 0 & -t^2 & -t^2 \end{pmatrix}$$

Note: In Carlsson-Zomordian there is no difference in notation between M_i and M_i'. Also, for the algorithm they order the elements such that the degree decreases down to rows. If we work in $\mathbb{R}[t]$, the matrices will be the same. In $\mathbb{Z}/2\mathbb{Z}[t]$ there won't be difference between ± 1.
The example from p.1

The final cx is $0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \to 0$. The matrix of ∂_1 is

$$M_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

. This gives rise to the complexes of $\mathbb{Z}[t] - modules$.

$$0 \to \mathbb{Z}[t] \xrightarrow{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \mathbb{Z}t^{\oplus 2} \to 0$$

and of graded $\mathbb{Z}[t]$ -modules.

$$0 \to \mathbb{Z}[t] \xrightarrow{\begin{bmatrix} -t \\ t \end{bmatrix}} \underset{\mathbb{Z}t^{\oplus 2}}{\overset{(1)^{\oplus 2}}{\|}} \to 0$$

4.1 Computing Vietoris-Rips persistence barcodes using Ripser

Ripser is a software design and implemented to calculate Vietoris-Rips persistance barcodes. The algorithm uses an implicit representation of the coboundary operator and of the filtration order. Ripser has been optimized by avoiding any explicit construction and storage of the filtration coboundary matrix, resulting in a significant imporvement in time and memory usage.

The predominant approach to persistence computation consist of two steps: the generation of a filtration boundary matrix, and the computation of persistence barcodes using matrix reduction. Often, the construction of the filtration boundary matrix becomes the bottleneck for the computation of Vietoris-Rips barcodes.

That is why, Ripser takes a different approach by avioding construction and storage of the whole filtration boundary matrix. The algorithm discards parts of the matrix and recomputes them when necessary. To reduce the memory usage, Ripser does not use an explicit matrix data structure, but uses a arithmetic operator, that recomputes the coboundary map of a simplex whenever needed. The filtration is specified using another algorithmic operator for comparing simplices with respect to their appearance in the filtration order.

The computation of persistent homology implemented in Ripser is based on matrix reduction and uses four key optimizations in order to achieve an efficient implementation:

1. Clearing birth columns: Avoid computation of unnecessary cycles, by using the spectrial structure of a boundary matris D, $D^2 = 0$

2. Cohomology : Using Cohomology to compute persistence barcodes, since it is faster

3. Implicit representation of boundary and reduced boundary matrices: Decouple the description of the filtration and of the boundary operator, representing the boundary matrix

only algorithmically instead of explicitly, and to avoid the storage of the entire unreduced and reduced boundary matrices, retaining only the much smaller reduction matrix encoding the column operations

4. Apparent and emergent pairs : The construction of the coboundary matrix columns can be shortcut when a certain easily identified type of persistence pair, called an emergent coface pair, is encountered

To construct the algorithm, Ripser Software makes use of simplicial complexes and filtaration intruduced in chapter 1, with focus on the Vietoris Ripser complex, introduced in section [kejsi ref]. Despite the Vietris -Rips filtration, a reindexing and refinement of the filtration is used. In order to compute persistent homology, one needs to apply one further step of reindexing, refining the essential filtration to an essential simplexwise one.

To ease the computation of the Vietoris-Rips filtration, Ripser makes use of sublevel sets of fucntions and persistent homology. The Ripser package considers only simplicial homology with coefficents in prime field $\mathbb{F}_p$. The homology computation is made possible by using simplexwise refinement. A simplexwise filtration gives rise to a filtration boundary matrix, which is the matrix of the boundary operator of the chain complex $C_*(K)$ with respect to the ordered basis given by the oriented simplices in filtration order.

During computation simplices are indexed in a combinatorial number system. Also, Ripser defines a refinement of the Vietoris-Rips filtration to an essential simplexwise filtration, as required for the computation of persistent homology. The simplexes are ordered by increasing diameter, then by increasing dimension and then by decreasing lexicographic vertex order. The result is called the *lexicographically refined Vietoris-Rips filtration.*

4.1.1 Computation and Implementation

In the center of the algorithm for computing persistent homology, there is the matrix reduction method. The matrix reduction method is similar to the one intruduced in the previous section. However, since the matrix reduation process can be expensive in time and memory usage, a technique for clearing columns is used. Another interesting approach is the choice to opt for computing persistence barcodes using cohomology instead of homology of Vietoris-Rips filtration. This approach is backed up by de Silva et all [dSMVJ11], and it further optimizes the clearing of columns. Furthermore, as already mentioned before, the matrix reduction is done implicitly. Apart from the computation techniqes mentioned above, there are several more optimizations that are presented in detail in [Bau19]

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