Persistent Homology and TDA

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Contents

Chapter 1

Chain Complexes And Simplicial Homology

1.1 Δ-complexes

Definition 1.1.1 (Standard Simplex - *n-simplex*). *The standard n-simplex is a subset of* $\mathbb{R}^{n+1}$ *given by*

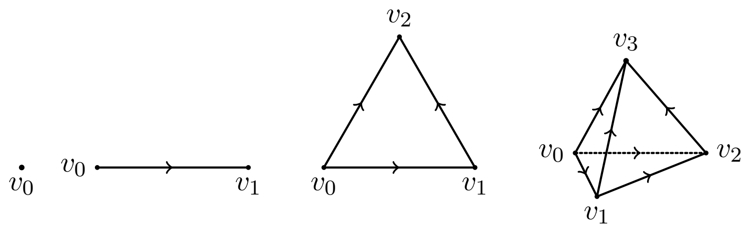
$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \text{ and } t_i \geq 0 \, \forall \, i\}$$

[Hat02, 103] whose vertices are unit vectors along the coordiante axis.

... Simplices in $\mathbb{R}^n$, ordering of the vertices and orientation ...

An *n-simpex* is an *n-dimensional* analog of a triangle. A *n-simpex* is denoted by $[v_0, ..., v_n]$, where v_i's are the vertices of the simplex. To compute homology is important to define the order of the vertices in a simplex. Ordering the vertices of a simplex $[v_0, ..., v_n]$ determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts. Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard *n-simpex* Δ^n onto any *n-simpex* $[v_0, ..., v_n]$ preserving the order of vertices $(t_0, t_1, ..., t_n) \mapsto \sum_i t_i v_i$ in $[v_0, ..., v_n]$ [Hat02, 103].

In $\mathbb{R}^n$ a *0-simplex* is a point, a *1-simplex* is a line segment, a *2-simplex* is a triangle, *3-simplex* is a tetrahedron, as shown below.



The boundary $\partial\Delta^n$ is defined as the union of all the faces of Δ^n, and $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ denotes interior of Δ^n.

Definition 1.1.2 (Δ-complex). *A* $\Delta-complex$ *structure on a space X is a collection of maps* $\sigma_\alpha : \Delta^n \to X$ *, with n depending on the index* α*, such that:*

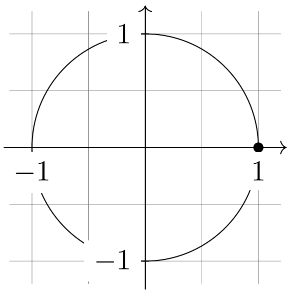
1. *The restriction* $\sigma_\alpha|\mathring{\Delta}^n$ *is injective, and each point of X is in the image of exactly one such restriction* $\sigma_\alpha|\mathring{\Delta}^n$*.*

2. *Each restriction of* σ_α *to a face of* Δ^n *is one of the maps* $\sigma_\beta : \Delta^{n-1} \to X$*. Here we are identifying the face of* Δ^n *with* Δ^{n-1} *by the canonical linear homeomorphism between them that preserves the ordering of the vertices.*

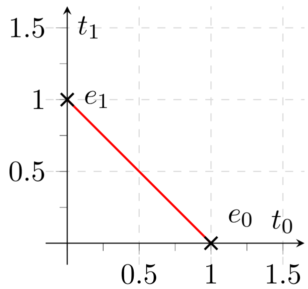
3. A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

.... Some explicit examples of Δ-complex structures on spaces. E.g., a closed interval $[0;1]$ $X = S^1$ with some *explicit* maps from Δ^1 (preferably several different ones) S^2 with some explicit maps. More examples on some quotient spaces, $S^1 \times S^1$, $\mathbb{RP}^2$, Klein bottle. ....

Example 1.1.1. Consider $\sigma_\alpha : \Delta^n \to X$ where $X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$



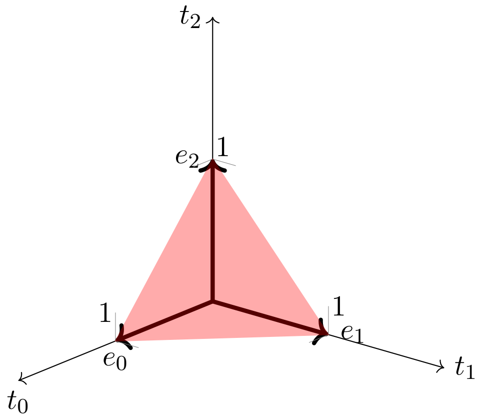
For n = 0, $\alpha = 0$: $\sigma_0 : \Delta^0 \to S^1$ where $\sigma_0(1) = (1, 0)$
For n = 1, $\alpha = 1$: In the figure below $t_0 + t_1 = 1$, $t_0, t_1 \geq 0$
$\Delta^1 = \{(t_0, 1 - t_0)\,,\, t_0 \in [0, 1]\} = [e_0, e_1]$



$\sigma_1 : \Delta^1 \to S^1$ where $t_0 \in [0, 1]$ where $\sigma_1(t_0) = (\cos(2\pi t_0), \sin(2\pi t_0))$
$\sigma|_{[e_0]} = \sigma_1(0) = (\cos(0), \sin(0)) = (1, 0)$
$\sigma|_{[e_1]} = \sigma_1(1) = (\cos(2\pi), \sin(2\pi)) = (1, 0)$

Example 1.1.2. Consider $\sigma_\alpha : \Delta^n \to X$ where $X = S^2 = \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$

For n = 2, $\alpha = 2$: $\sigma_2 : \Delta^2 \to S^2$
$\Delta^1 = \{(t_0, t_1, t_2) \,|\, t_0 + t_1 + t_2 = 1,\, t_i \geq 0 \; for\, i = 1, 2, 3]\} = [e_0, e_1, e_2]$



The faces are $[e_1, e_2]$ or $(0, t_1, t_2)$, where $t_1 + t_2 = 1$,
$[e_0, e_2]$ or $(t_0, 0, t_2)$, where $t_0 + t_2 = 1$,
$[e_0, e_1]$ or $(t_0, t_1, 0)$, where $t_0 + t_1 = 1$.

$\sigma_2((t_0, t_1, t_2)) = \frac{(t_0,t_1,t_2)}{\sqrt{(t_0^2+t_1^2+t_2^2)}}$
$\sigma|_{[e_1,e_2]}(t_1, t_2) = \frac{(t_1,t_2)}{\sqrt{(t_1^2+t_2^2)}}$, $\sigma|_{[e_0,e_2]]}(t_0, t_2) = \frac{(t_0,t_2)}{\sqrt{(t_0^2+t_2^2)}}$, $\sigma|_{[e_0,e_1]}(t_1, t_2) = \frac{(t_0,t_1)}{\sqrt{(t_0^2+t_1^2)}}$,

Usually simplicial homology is defined by simplicial complexes, which are the $\Delta-$*complexes* whose simpleces are uniquely determined by their vertices. In a simplicial complex any n-simplex has n+1 distinct vertices, and no other n-simplex has the same set of vertices.

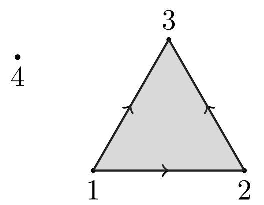
Another definition of simplicial complexes that will be important in indroducing Persistent Homology is a coombinatorical description.

Combinatorically we can define a simplicial complex as:

Definition 1.1.3 (Simplicial Complex). *Given a partially ordered set:* $V = \{1, 2, \cdots, m\}(= [m])$ *a simplicial complex is a collection* $\mathcal{K}$ *of subsets of* V*, such that:*

1. $\emptyset \in \mathcal{K}$
2. $\{i\} \in \mathcal{K}$ *(singleton)*
3. *If* $\mathcal{J} \subseteq I \in \mathcal{K} \Rightarrow \mathcal{J} \in \mathcal{K}$

Example 1.1.3. Consider the following partially ordereed set $V = \{1, 2, 3, 4\}$: The simplicial complex $\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$

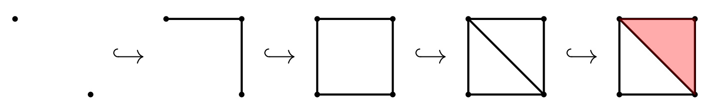


The topological realization of $\mathcal{K}$ is:
$|\mathcal{K}| = \bigcup_{I \in K \neq \emptyset} (Conv(e_i))$ where e_i is in the standard basis $e_1, \cdots e_n \in \mathbb{R}^n$

A subcomplex of $\mathcal{K}$ is a subset $L \subseteq \mathcal{K}$ that is also a simplicial complex. A *filtration* of complex $\mathcal{K}$ is a nested subsequence of complexes:

$$\emptyset = \mathcal{K}^0 \subseteq \mathcal{K}^1 \subseteq \dots \mathcal{K}^m = \mathcal{K}$$

For generality, we let $\mathcal{K}^i = \mathcal{K}^m$ for all $i \geq m$. $\mathcal{K}$ is called a filetered complex, and below there is a short example of a filtered complex:



1.2 Chain Complexes

Definition 1.2.1 (Chain complex). *Complex of abelian groups. Homology of a complex.*
A chain complex is a sequencee of homomorphisms of abelian groups:

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ *for each* n *in* $\mathbb{Z}$*. The equation* $\partial_n\partial_{n+1} = 0$ *is equivalent to the inclusion* $\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$.

.... As a remark: complex of R-modules, for a commutative ring R.

Chain complexes from a Δ-complex structure: definining the differential and checking the $\partial^2 = 0$ property. ....

The map ∂_n for a Δ-complex $\mathcal{X}$ is a boundary homomorphism $\partial_n : \Delta_n(\mathcal{X}) \to \Delta_{n-1}(\mathcal{X})$ where the action on a basis element of $\Delta_n(\mathcal{X})$ is:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[v_0, ..., \hat{v_i}, ..., v_n]$$

where the '^' symbol denotes the absence of that vertex.

Lemma 1.2.1. *The composition $\partial^2 = 0$ below is zero*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

Proof: For n = 3 :

$$\Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X)$$

Let us proof that $\partial_2\partial_3 = 0$:
$\partial_3\sigma = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, v_3] = \sigma|[v_1, v_2, v_3] - \sigma|[v_0, v_2, v_3] + \sigma|[v_0, v_1, v_3] - \sigma|[v_0, v_1, v_2]$

$$\begin{aligned} \partial_2\partial_3(\sigma) &= \sigma|[v_2, v_3] - \sigma|[v_1, v_3] + \sigma|[v_1, v_2] \\ &= -\sigma|[v_2, v_3] + \sigma|[v_0, v_3] - \sigma|[v_0, v_2] \\ &= \sigma|[v_1, v_3] - \sigma|[v_0, v_3] + \sigma|[v_0, v_1] \\ &= -\sigma|[v_1, v_2] + \sigma|[v_0, v_2] - \sigma|[v_0, v_1] = 0 \end{aligned} \tag{1.1}$$

In case of n:

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n]) \\ &= \sum_j (-1)^j (\sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n])|[v_0, ..., \hat{v_j}, ..., v_n] \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma_{|}[v_0, ..., \hat{v_j}, ..., \hat{v_i} ..., v_n] + \sum_{j>i} (-1)^i (-1)^j \sigma_{|}[v_0, ..., \hat{v_i}, ..., \hat{v_j} ..., v_n] = 0 \end{aligned} \tag{1.2}$$

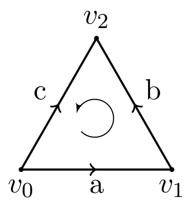
1.3 Homology Calculations: Examples

S^1 with several different Δ-complex structures. An interval $[0; 1]$. ... comment about diff delta structures resulting in same homology ...

1.3.1 S^1

Method I: Triangulation

To compute the homolgy group of the circle S^1 we can triangulate the circle in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2=0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1, v_2\rangle \\ C_1 = \langle a, b, c\rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2=0} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1, v_2\rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v_1 - v_0) + \beta(v_2 - v_1) - \gamma(v_2 - v_0)$
$= (-\alpha + \gamma)v_0 + (\alpha - \beta)v_1 + (\beta - \gamma)v_2 = (\gamma - \alpha)v_0 + (\alpha - \beta)v_1 + (-(\gamma - \alpha) - (\alpha - \beta))v_2$

$$\operatorname{Im} \partial_1 = \left\{ \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ -(\gamma - \alpha) - (\alpha - \beta) \end{pmatrix}, \quad \alpha, \beta, \gamma \subseteq \mathbb{Z} \right\} \subseteq \mathbb{Z}^{\oplus 3}$$

Claim : There exist an isomorphism $\psi : \operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$\psi : \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ (\beta - \gamma) \end{pmatrix} \mapsto \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \end{pmatrix}$$

ψ is one-to-one since if $(\gamma - \alpha = 0 \ \&\ \alpha - \beta = 0) \Rightarrow \beta - \gamma = 0 \ \&\ \alpha = \beta = \gamma$

ψ is onto since given $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$ there exist an element $\begin{pmatrix} m \\ n \\ -m-n \end{pmatrix} \in \operatorname{Im} \partial_1$ such that

$\psi \begin{pmatrix} m \\ n \\ -m-n \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$, since ψ is one-to-one and onto, $\operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$$

$$\text{Claim: } \phi : \left(\mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z} \right) \simeq \mathbb{Z}$$

First, let us take the map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3 \Big/ \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$$\mathbb{Z}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (p+q+r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where $p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$

$$\text{So, } \varphi : \begin{pmatrix} p \\ q \\ r \end{pmatrix} \mapsto \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} = \overline{(p+q+r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Finally, $\phi : \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} \mapsto (p+q+r) \in \mathbb{Z}$, Clearly, ϕ is injective and surjective.

So, $H_0 \simeq \mathbb{Z}$

Second, let's compute H_1:

$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$

$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$

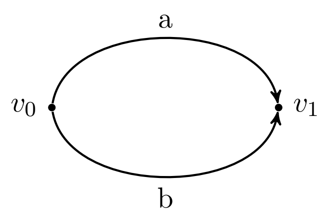
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Method II

To compute the homolgy group of the circle S^1 we can construct the circle, by two verteces and two edges, in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2 = 0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1 \rangle \\ C_1 = \langle a, b \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:

$\ker \partial_0 = C_0 = \langle v_0, v_1 \rangle$ since $\partial_0 = 0$

To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b) = \alpha(v_1 - v_0) + \beta(v_1 - v_0) = (\alpha + \beta)(v_1 - v_0)$

$\operatorname{Im} \partial_1 = \langle v_1 - v_0 \rangle$

$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \frac{\langle v_1 - v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \langle v_1 \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:

$\ker \partial_1 = \langle a - b \rangle$ since $\partial_1(\alpha a + \beta b) = (\alpha + \beta)(v_1 - v_0) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element $(a - b)$

$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle with a different $\Delta - \textit{complex}$ on it are the same:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.3.2 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} v \arrow[r, "b"] & v \\ v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c" description] \arrow[u, phantom, shift right=-4ex, "A"] & v \arrow[u, "a"'] \arrow[u, phantom, shift right=4ex, "B"] \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v \rangle$ since $\partial_0 = 0$
$\operatorname{Im} \partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v - v) + \beta(v - v) + \gamma(v - v) = 0$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = C_1 = \langle a, b, c \rangle$ since $\partial_1 = 0$
$\operatorname{Im} \partial_2 = \langle a + b - c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(a + b - c) + \beta(a + b - c) = (\alpha + \beta)(a + b - c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a,b,c \rangle}{\langle a+b-c \rangle}$
The group $\langle a, b, c \rangle$ can be also generated by the elements $m = a+b-c, bandc$ where $a = m-b+c$. So,
$H_1 = \frac{\langle a+b-c,b,c \rangle}{\langle a+b-c \rangle} = \langle b, c \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$

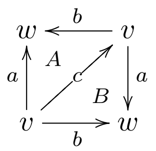
Last, let's compute H_2:
$\ker \partial_2 = \langle A - B \rangle$ since $\partial_2(\alpha A + \beta B) = (\alpha + \beta)(a + b - c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element A-B
$\operatorname{Im} \partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\langle A-B \rangle}{\{0\}} = \langle A - B \rangle \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 3 \end{cases}$$

1.3.3 $\mathbb{RP}^2$

One way to calculate the homology groups of a projective plain $\mathbb{RP}^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v, w \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \ker \partial_n / \operatorname{Im} \partial_n$

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v, w \rangle$ since $\partial_0 = 0$
$\operatorname{Im} \partial_1 = \langle w - v \rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(w - v) + \beta(w - v) + \gamma(v - v)$
$= (\alpha + \beta)(w - v)$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v,w \rangle}{\langle w-v \rangle} = \frac{\langle w-v,w \rangle}{\langle w-v \rangle} = \langle w \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \langle a - b, c \rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements a-b and c
$\operatorname{Im} \partial_2 = \langle -a + b + c,\ a - b + c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a-b,\, c \rangle}{\langle -a+b+c,\, a-b+c \rangle}$
The group $\langle a-b, c \rangle$ can be also generated by the elements $m = a-b+c, andc$ where $a-b = m-c$. So,
$H_1 = \frac{\langle a-b,\, c \rangle}{\langle -a+b+c,\, a-b+c \rangle} = \frac{\langle a-b+c,\, c \rangle}{\langle a-b+c,\, -a+b+c \rangle}$
If we let $t = a-b+c$ then $-a+b+c = -t+2c$ then the group $\langle t,\ -t+2c \rangle$ can be also generated by the elements $t \; and \; 2c$.
In terms of t and c, $H_1 = \frac{\langle t,\, c \rangle}{\langle t,\, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Last, let's compute H_2:
$\ker \partial_2 = \{0\}$ since $\partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0$ only when $\alpha = \beta = 0$

$\operatorname{Im}\partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = \frac{\{0\}}{\{0\}} = 0$

Finally, the homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.4 Maps of Complexes

In the previous sections, we considered boundary homomorphisms between abilean groups as part of a chain complex. In this section, we will draw our attention to maps between chain complexes.

Definition 1.4.1. *(Maps of Chain Complexes)*
Let $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ be two chain complexes. A map of chain complexes is a morphism f that is a sequence of homomorphisms $(f_n)_{n\in Z}$:

$$\begin{tikzcd}
(C_\bullet, \partial) & C_\bullet \ \dots \arrow[r] & C_n \arrow[r, "\partial n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial n-1"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r, "\partial n-2"] \arrow[d, "f_{n-2}"] & \dots \ C_\bullet \\
(D_\bullet, \delta) & D_\bullet \ \dots \arrow[r] & D_n \arrow[r, "\delta n"] & D_{n-1} \arrow[r, "\delta n-1"] & D_{n-2} \arrow[r, "\delta n-2"] & \dots \ D_\bullet
\end{tikzcd}$$

$$f_n : C_n \to D_n \quad s.t, \quad f_{n-1} \circ \partial n = \delta_n \circ f_n \ \forall n \in \mathbb{Z}$$

$$\begin{tikzcd}
C_n \arrow[r, "\partial n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "\delta n"] & D_{n-1}
\end{tikzcd} \quad \textit{commutes.}$$

1.4.1 Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$$
$$H_n(f) : [x] \mapsto [f_n(x)]$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-definess by checking if cycles are send to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$\delta_n \circ f_n(x) = f_{n-1} \circ \partial n(x) = f_{n-1}(0) = 0 \Rightarrow f_n(x) \in \ker\delta_n, f_n(x)\, is\, a\, cycle$$
$$\Rightarrow f_n(\ker\partial n) \subseteq \ker\delta_n$$

So, cycles are send to cycles.

(2) Let us take a boundary $y \in C_n$, so that $y \in \operatorname{Im} \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

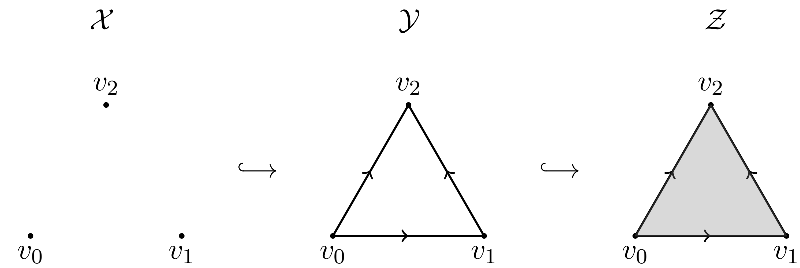
$$\begin{aligned} &f_n(y) = f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z)) \\ &\Rightarrow f_n(y) \in \operatorname{Im} \partial_{n+1} f_n(y)\, is\, a\, boundary \\ &\Rightarrow f_n(Im\partial_{n+1}) \subseteq Im(\delta_{n+1}) \end{aligned}$$

So, boundaries are send to boundaries.

$$\begin{aligned} &H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet) \\ &H_n(f) : \ker \partial_n / \operatorname{Im}(\partial_{n+1}) \to \ker \delta_n / \operatorname{Im}(\delta_{n+1}) \\ &\qquad [x] \mapsto [f_n(x)] \\ &\qquad x + \operatorname{Im} \partial_{n+1} \mapsto f_n(x) + f_n(\operatorname{Im} \delta_{n+1}) = f_n(x) + Im(\delta_{n+1}) = [f_n(x)] \end{aligned}$$

□

Let us consider an example between maps of complexes defined by the three spaces below.



$$C_\bullet^0 \longrightarrow C_\bullet^1 \longrightarrow C_\bullet^2$$

Maps between complexes:

$$C_\bullet^0 \longrightarrow C_\bullet^1 \longrightarrow C_\bullet^2$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d, "\partial_2^0=0"] & 0 \arrow[r] \arrow[d, "\partial_2^1=0"] & \mathbb{Z} \arrow[d, "\partial_2^2=\begin{pmatrix}1\\1\\-1\end{pmatrix}"] \\
1 & 0 \arrow[r] \arrow[d, "\partial_1^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_1^1=\partial_1^2"] & \mathbb{Z}^3 \arrow[d, "\partial_1^2=\begin{pmatrix}-1&0&-1\\1&-1&0\\0&1&1\end{pmatrix}"] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^1=0"] & \mathbb{Z}^3 \arrow[d, "\partial_0^2=0"] \\
 & 0 & 0 & 0
\end{tikzcd}$$

Induced maps on homology:

$$H(C^0_\bullet) \longrightarrow H(C^1_\bullet) \longrightarrow H(C^2_\bullet)$$

$$\begin{array}{cccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
2 & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
1 & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0
\end{array}$$

Chapter 2

Singular Homology and Homotopy Invariance

In the previous chapter, we considered a parametrization of a space by simplexes, where the maps $\sigma_\alpha : \Delta^n \to X$ had restrictions defined in 1.1.2. If we only require that the σ map is continous, then by definition that would be a singular $n - simplex$ in a space X. The lack of other restrictions on the map $\sigma : \Delta^n \to X$, convey that σ does not need to be a 'nice' embedding, in fact it can have singularites, where its image does not look like a simplex.

$C_n(X)$ is a free abelien group with generators the set of singluar $n - simplexes$ in X: the continous maps $\sigma : \Delta^n \to X$. The elements of $C_n(X)$ are singular $n - chains$ defined as $\sum_i (n_i \sigma_i)$ for $n_i \in \mathbb{Z}$. The boundary operator $\partial_n : C_n(x) \to C_{n-1}(X)$ is defined the same way as in simplicial $n - chains$, by the formula:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n]$$

By preserving the order of the vertices, the $\sigma|[v_0, ..., \hat{v_i}, ..., v_n]$ is identified with the map $\Delta^{n-1} \to X$. The proof of lemma 1.2.1, $\partial_2 = 0$, holds true also for singular simplexes. Therefore, the singular homology group is defined the same way: $H_n(x) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$. The elements in the ker are singular cycles, and the elements in the Im singular boundaries.

Let us consider some explicit examples of $C_n(X)$:

- For a topological space X: $C_0(X)$ consists of all maps $\sigma : \Delta^0 \to X$, which means that $C_0(X)$ is a free group on the points of X.
- If X = $\mathbb{R}$: $C_1(X)$ consists of all maps $\sigma : \Delta^1 \to \mathbb{R}$, which means that $C_0(X)$ is a free group on continus maps:

$$[0, 1] \simeq \Delta^1 \to R$$

  In this case $C_1(X)$ can be considered as a vector space with vectors the maps: $[0, 1] \to \mathbb{R}$.

From the examples above, it is clear that the groups $C_n(X)$ can be so large to the point where the number of singular $n - simplexes$ in a space X is uncountable. It is not easy to see that even in singular homology where X is generated by a finite number of simplexes, $H_n(X)$ should be finite generated for all n, and that $H_n(X)$ should be 0 for $n > dim(X)$.

At first glance, singular homology seems to be more general than simplicial homology, however if for an arbitrary space X , we define the singular complex $S(X)$ as a $\Delta - complex$ with one $n - simplex$ Δ^n_σ for each singular $n - simplex$ $\sigma : \Delta^n \to X$, then $H^\Delta_n(S(X))$ is the same as $H_n(X)$. In this case singular homology can be viewed as a special case of simplicial homology.

By the definition of singular homology groups $H_n(X)$, it follows that if two spaces X, Y are homeomorphic, the singular homology groups are isomorphic $H_n(X) \simeq H_n(Y)$.

More generally,
A countinous map $f : X \to Y$ induces a chain map: $f_{\#} : C_n(X) \to C_n(Y)$.

$$f_{\#}(\sigma : \Delta^n \to X) = f \circ \sigma$$

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

The boundary operator of $f_{\#}$ is equal to $\partial_n(f \circ \sigma) = f \circ \partial_n(\sigma)$.
$H_n(f_{\#}) = f_* : H_n(X) \to H_n(Y)$

If we additionally require f to be a bijection and have a continous inverse f^{-1}, so f is a homoemorphism, then $f_{\#} : C_n(X) \simeq C_n(Y)$

$$\sigma \mapsto f \circ \sigma$$
$$\mu \circ f^{-1} \mapsfrom \mu$$

Then the induced singular homology map $f_* : H_n(X) \simeq H_n(Y)$ defines an isomorphism.
Moreover,

$$f_* \textit{ preserves composition}, (f \circ g)_* = f_* \circ g_* \tag{2.1}$$
$$f_* \textit{ preserves the identity}, id : X \to Y \textit{ goes to } id_* : H_n(X) \to H_n(Y) \tag{2.2}$$

Category Theory Interpretation: If we consider Top to be the category of topological spaces where maps are countinous:

$$Hom_{Top}(X, Y) = \{f : X \to Y, f \textit{ is continous}\},$$

and Ab the category of abelian groups were maps are group homomorphisms:

$$Hom_{Ab}(G, H) = \{\phi : G \to H, \phi \textit{ is group homomorphism}\}$$

then for each $n \geq 0$:

$$H_n : Top \to Ab$$
$$X \rightsquigarrow H_n(X)$$
$$f : X \to Y \rightsquigarrow f_* : H_n(X) \to H_n(Y)$$

H_n is a functor and (2.1) (2.2) hold.
If we also consider the category, Homotopic Topology $HoTop$ of topological spaces where maps are continous up to homotopy, then we obtain the following commutative diagram:

$$\begin{tikzcd} Top \arrow[r, "H_n"] \arrow[d] & Ab \\ HoTop \arrow[ur, dotted] & \end{tikzcd}$$

$$Hom_{HoTop}(X, Y) = Hom_{Top}(X, Y) /\simeq$$

Remark: The continous maps $f, g : X \to Y$ are homotopic if

$$\exists\ H : X \times [0, 1] \to Y,$$
$$for\ x \in X\ and\ t \in [0, 1] : H(x, t) = H_t(x)$$
$$s.t\ H(x, 0) = f(x)\ and\ H(x, 1) = g(x)$$
$$i.e\ H|_{x \times \{0\}} = f\ and\ H|_{x \times \{1\}} = g$$

Let us consider some explicit homotopic maps:

- $f, g : \mathbb{R} \to \mathbb{R}$ where $f = id, f(x) = x \;\forall\; x \in X$, and $g = 0,\; g(x) = 0 \;\forall\; x \in X$
  $H : \mathbb{R} \times [0,1] \to \mathbb{R}$
  $H(x,t) = (1-t)x$, clearly $H(x,0) = x$ and $H(x,1) = 0$
- $f, g : S^1 \to \mathbb{R}^2$ where f *is an inclusion*, $f(x,y) = (x,y) \;\forall\; (x,y) \in S^1$, and $g = 0,\; g(x,y) = 0 \;\forall\; (x,y) \in S^1$
  $H : S^1 \times [0,1] \to \mathbb{R}^2$
  $H(x,y,t) = (1-t)\langle x,y\rangle$, clearly $H(x,y,0) = \langle x,y\rangle$ and $H(x,y,1) = 0$

If $f, g : X \to Y$ are homotopic, then the induced maps on homology $f_*, g_* : H_n(X) \to H_n(X)$, are the same $f_* = g_* \;\forall n$

Define $p_n : C_n^{sing}(X) \to C_{n+1}^{sing}(Y)$ s.t $f_\# - g_\# = \partial p + p\partial$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1}(X) \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \\
C_{n+1}(Y) \arrow[r, "\delta_{n+1}"] & C_n(Y) \arrow[r, "\delta_n"] & C_{n-1}(Y)
\end{tikzcd}$$

For $\sigma : \Delta^n \to X$ the map $p_n(\sigma) : \Delta^{n+1} \to Y$ should be a continous map.
$H : X \times [0,1] \to Y$, $H|_{x\times\{0\}} = f$ *and* $H|_{x\times\{1\}} = g$

$$\Delta^n \times [0,1] \xrightarrow{\sigma\times 1} X \times [0,1] \xrightarrow{H} Y$$

The idea is to write $\Delta^n \times [0,1]$ as union of Δ^{n+1}. Let us consider some explicit examples of the p maps:

- $p_0 : C_0(X) \to C_1(Y)$
  $p_0(\sigma) = H_0(\sigma \times 1)|_{[v_0 w_0]} : \Delta^1 \to Y$
  We can parametrize Δ^1, as $\Delta^1 = \{(t_0,t_1)|\; t_0 + t_1 = 1,\; t_0, t_1 \geq 1\}$
  $\Delta^1 = \{1\} \subseteq \mathbb{R}$, $\sigma(1) = q$

$$\begin{aligned}
\Delta^0 \times [0,1] &\simeq \Delta^1 \\
\{1\} \times \{t\} &\mapsto (1-t, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t),\; where\; H(q,0) = f(q),\; H(q,1) = g(q)
\end{aligned}$$

- $p_1 : C_1(X) \to C_2(Y)$
  $p_1(\sigma) = \sum_{i=0}^{1} H_0(\sigma \times 1)|_{[v_0 \dots w_1]} = H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}$

$$\begin{aligned}
\Delta^1 \times [0,1] &\simeq [0,1] \times [0,1] \\
((t_0,t_1),t) &\mapsto (t_0, t) \\
\sigma \times 1 : (1,t) &\mapsto (\sigma(1), t) \\
H_0(\sigma \times 1) : (1,t) &\mapsto H(q,t),\; where\; H(q,0) = f(q),\; H(q,1) = g(q)
\end{aligned}$$

$$
\begin{tikzcd}
C_2(X) \arrow[r, "\partial_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_0(X) \arrow[r, "\partial_0"] \arrow[dl, "p_0"'] & 0 \\
C_2(Y) \arrow[r, "\delta_2"] & C_1(Y) \arrow[r, "\delta_1"] & C_0(Y) \arrow[r, "\delta_0"] & 0
\end{tikzcd}
$$

From the diagram above: $\delta_2 p_1 + p_0\partial_1 : C_1(X) \to C_1(Y)$

$$
\begin{aligned}
(p_0 \circ \partial_1)(\sigma) &= p_0(\sigma|_{[v_1]} - \sigma|_{[v_0]}) \\
&= H_0(\sigma \times 1)|_{[v_1 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma|_{[v_1]} \times 1)|_{[v_1 w_1]} - H_0(\sigma|_{[v_0]} \times 1)|_{[v_0 w_0]} \\
&= H_0(\sigma \times 1)|_{[v_1]\times[0,1]} - H_0(\sigma \times 1)|_{[v_0]\times[0,1]}
\end{aligned}
$$

$$
\begin{aligned}
(\delta_2 \circ p_1)(\sigma) &= \delta_2(H_0(\sigma \times 1)|_{[v_0 w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1 w_1]}) \\
&= H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_0]} \\
&\quad - H_0(\sigma \times 1)|_{[v_1 w_1]} + H_0(\sigma \times 1)|_{[v_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]}
\end{aligned}
$$

So , $(\delta_2 p_1 + p_0\partial_1)(\sigma) = H_0(\sigma \times 1)|_{[w_0 w_1]} - H_0(\sigma \times 1)|_{[v_0 v_1]} = g \circ \sigma - f \circ \sigma = (g - f) \circ \sigma$

$$\Rightarrow \delta_2 p_1 + p_0\partial_1 = g - f$$

Theorem 2.0.1. *If two chain maps* $f_\bullet, g_\bullet : (C_\bullet, \partial) \to (D_\bullet, \delta)$ *are chain-homotopic then they induce the same homomorphism on homology:*

More explicitly:

$$
\begin{tikzcd}
\dots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, "p_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "p_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "p_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \dots \arrow[dl, "p_{n-2}"'] \\
\dots \arrow[r, "\delta_{n+2}"] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & D_{n-1} \arrow[r, "\delta_{n-1}"] & \dots
\end{tikzcd}
$$

$$
\begin{aligned}
H_n(f) &: H_n(C_\bullet) \to H_n(D_\bullet) \\
H_n(f) &: [x] \mapsto [f_n(x)]
\end{aligned}
$$

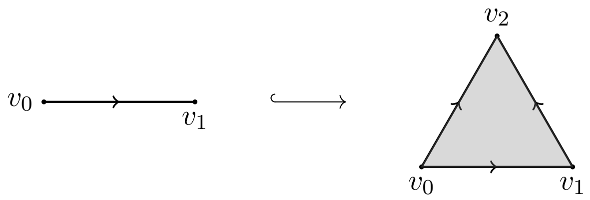
If $f_n - g_n = \delta_{n+1}p_n + p_{n-1}\partial_n \Rightarrow H_n(f) = H_n(g)$

Proof: Let us proof that the maps f_n, g_n induce the same homology.
For any $x \in \ker \partial_n \Rightarrow \partial_n x = 0$,
$(f_n - g_n)(x) = \delta_{n+1}p_n(x) + p_{n-1}\delta_n(x) = \delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$
$\Rightarrow (f_n - g_n)(x) \in \mathrm{Im}(\partial_{n+1})$

$$
\left|\begin{aligned} H_n(f)([x]) &= [f_n(x)] \\ H_n(g)([x]) &= [g_n(x)] \end{aligned}\right. \Rightarrow [f_n(x)] - [g_n(x)] = [f_n(x) - g_n(x)] = [\delta_{n+1}p_n(x)] = [0]
$$

since $\delta_{n+1}(p_n(x)) \in \mathrm{Im}(\partial_{n+1})$.
So, $[f_n(x)] = [g_n(x)] \Rightarrow H_n(f) = H_n(g) \quad \square$

Example 2.0.1. Let X be a $1-simplex$ and Y a $2-simplex$:



$$C_\bullet(X,\partial): \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

$$D_\bullet(Y,\delta): \qquad \mathbb{Z} \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_0=0} 0$$

Let's introduce maps on the chain complexes:

$$f_n : C_n(X,\partial) \to C'_n(Y,\delta)$$
$$g_n : C_n(X,\partial) \to C'_n(Y,\delta)$$

$$\begin{tikzcd}
0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1=\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[dl, "p_1=id"'] \arrow[d, shift right, "f_1=\begin{bmatrix}0\\0\\1\end{bmatrix}"'] \arrow[d, shift left, "g_1=\begin{bmatrix}0\\1\\0\end{bmatrix}"] & \mathbb{Z}^{\oplus 2} \arrow[r, "\partial_0=0"] \arrow[dl] \arrow[d, shift right, "f_0=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}"'] \arrow[d, shift left, "g_0=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"] & 0 \\
\mathbb{Z} \arrow[r, "\delta_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_1=\begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}"'] & \mathbb{Z}^{\oplus 3} \arrow[r, "\delta_0=0"'] & 0
\end{tikzcd}$$

We can define $p_0 = \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}$ and $p_1 = id_{\mathbb{Z}}$

For $n=0$:

$$f_0 - g_0 = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

$$\delta_1 p_0 + 0\circ\partial_0 = \begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\\0&1\end{bmatrix}$$

For the choice of the maps p_0, f_0, g_0: $f_0 \simeq g_0 \to$ homotopic equivalent

For $n=1$:

$$f_1 - g_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\delta_2 p_1 + p_0\circ\partial_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1\end{bmatrix} + \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

For the choice of the maps p_1, p_0, f_1, g_1: $f_1 \simeq g_1 \to$ homotopic equivalent

By theorem 2.0.1, $H_0(f) = H_0(g)$ and $H_1(f) = H_1(g)$.

More precicely, $H_0(f) = H_0(g) = id_{\mathbb{Z}}$ and $H_1(f) = H_1(g) = 0$, since $H_0(X) = H_0(Y) = Z$ and $H_n(X) = H_n(Y) = 0 \ \forall\ n > 0$

Lemma 2.0.1. *A chain complex $(C_\bullet, \partial)$ is contractable if id_C is homotopic equivalent to 0_C*

If $id_C \simeq 0_C$, then $H_n(C_\bullet) = 0 \; \forall n$
Examples:

- $C_\bullet : \;\; \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$
  $H_0(C_\bullet) = \mathbb{Z} = H_1(C_\bullet)$ implies that $C_\bullet$ is not contractable

- $D_\bullet : \;\; \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$
  $H_0(D_\bullet) = \mathbb{Z}/2\mathbb{Z}$, $H_1(D_\bullet) = 0$ implies that $D_\bullet$ is not contractable

- $E_\bullet : \;\; \mathbb{Z} \xrightarrow{1} \mathbb{Z} \longrightarrow 0$
  $H_0(E_\bullet) = 0 = H_1(E_\bullet) \Rightarrow$ need to check that $id_E \simeq 0_E$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] \arrow[d, "id-0"] \arrow[dl, "p_1"'] & \mathbb{Z} \arrow[r, "\partial_0=0"] \arrow[d, "id-0"] \arrow[dl, "p_0"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1=1"] & \mathbb{Z} \arrow[r, "\partial_0=0"] & 0
\end{tikzcd}$$

  We can assign $p_0 = id$, $p_1 = 0$.
  For a cycle $\sigma \in E_1 = \mathbb{Z}$:
  $(\partial_2 p_1 + p_0 \partial_1)(\sigma) = \partial_2 p_1(\sigma) + p_0 \partial_1(\sigma) = 0 + \sigma = \sigma = id - 0(\sigma) \Rightarrow id_E \simeq 0_E \Rightarrow (E_\bullet, \partial)$ is contractable

- $F_\bullet : \;\; \dots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{\partial=2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \dots$
  $\ker \partial = \operatorname{Im} \partial = (2) \Rightarrow H_n(F_\bullet) = 0 \; \forall \; n \Rightarrow$ need to check that $id_F \simeq 0_F$

$$\begin{tikzcd}
\dots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[d, "id-0"] \arrow[dl, "p_1"'] & \mathbb{Z}/4 \arrow[r, "2"] \arrow[d, "id-0"] \arrow[dl, "p_0"'] & \dots \\
\dots \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r, "2"] & \dots
\end{tikzcd}$$

  In $\mathbb{Z}/4$ we have four classes $\overline{0}, \overline{1}, \overline{2}, \overline{4}$. The boudary operator $\partial = mult(2)$ maps the four classes only in two maps $\overline{0}, \overline{2}$. So, the ∂ cannot be surjective.
  For a cycle $\sigma \in \mathbb{Z}/4$:
  Since $((2)p_1 + p_0(2))(\sigma) \in (2)$, $((2)p_1 + p_0(2))(\sigma) \neq \sigma \Rightarrow (F_\bullet, \partial)$ is not contractable

Theorem 2.0.2. *Given topological spaces X, Y with maps $f : X \to Y$ and $g : Y \to X$*
If

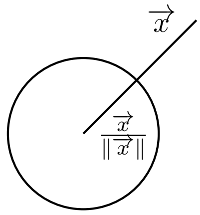
$$f \circ g \simeq id_Y$$
$$g \circ f \simeq id_X$$

where $\simeq$ denotes homotopic equivalence, then

$$\begin{tikzcd}
H_n(X) \arrow[r, bend left, "f_*"] & H_n(Y) \arrow[l, bend left, "g_*"]
\end{tikzcd} \quad \forall \; n \in \mathbb{Z}$$

$H_n(X)$ is isomorphic to $H_n(Y)$, $\Rightarrow f_* = (g_*)^{-1}$where f_*, g_* are the induced maps on homology.

Example 2.0.2. Let X be the n-dimentional sphere and Y the (n+1)-dimentional real cordinate space without the origin, $X = S^n$ and $Y = \mathbb{R}^{n+1}/\{0\}$



$$f : S^n \hookrightarrow \mathbb{R}^{n+1}/\{0\} \textit{ is the usual inclusion}$$

$$g : \mathbb{R}^{n+1}/\{0\} \to S^n \ s.t \ g(\overrightarrow{x}) = \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$$

Clearly, $g \circ f \simeq id_S^n$ while $f \circ g : \overrightarrow{x} \mapsto \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|} \neq id_{\mathbb{R}^{n+1}/\{0\}}$
Let's prove that $f \circ g$ is homotopic equivalent to $id_{\mathbb{R}^{n+1}/\{0\}}$: We can construct a function

$$\begin{aligned} F : [0,1] \times \mathbb{R}^{n+1}/\{0\} &\longrightarrow \mathbb{R}^{n+1}/\{0\} \\ F(t, \overrightarrow{x}) &= (t(f \circ g) + (1-t)1_{\mathbb{R}^{n+1}/\{0\}})\overrightarrow{x} \\ &= t(\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) + (1-t)\overrightarrow{x} \end{aligned}$$

Clearly, $F(0, \overrightarrow{x}) = \overrightarrow{x} = id_{\mathbb{R}^{n+1}/\{0\}}$ and $F(1, \overrightarrow{x}) = (\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}) = f \circ g$.
So, $f \circ g \simeq id_{\mathbb{R}^{n+1}/\{0\}}$, and by theorem 2.0.2 $\Rightarrow f_* = (g_*)^{-1}$.

Example 2.0.3. Let X be a $1-simplex$ and Y a $0-simplex$:

$$C_\bullet(X, \partial) : \qquad 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus^2} \xrightarrow{\partial_0 = 0} 0$$

$$C'_\bullet(Y, \partial') : \qquad 0 \xrightarrow{0} 0 \xrightarrow{\partial'_1 = 0} \mathbb{Z} \xrightarrow{\partial'_0 = 0} 0$$

Let's introduce maps on the chain complexes:

$$\begin{aligned} f_n &: C_n(X, \partial) \to C'_n(Y, \partial') \\ g_n &: C'_n(Y, \partial') \to C_n(X, \partial) \end{aligned}$$

$$\begin{tikzcd}
0 \arrow[r, "\partial_2"] & \mathbb{Z} \arrow[r, "\partial_1"] \arrow[dl, "p_1"'] \arrow[d, "0"] & \mathbb{Z}^{\oplus^2} \arrow[r, "\partial_0=0"] \arrow[dl, "p_0"'] \arrow[d, shift left, "f_0={[1\ 1]}"] & 0 \\
0 \arrow[r, "0"] & 0 \arrow[r, "\partial'_1=0"] & \mathbb{Z} \arrow[r, "\partial'_0=0"] \arrow[u, shift left, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]=g_0"] & 0
\end{tikzcd}$$

For $n = 0$:

$$f_0 \circ g_0 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = id_{\mathbb{Z}}$$

$$g_0 \circ f_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq id_{\mathbb{Z}^{\oplus 2}}$$

Let's prove that $g_0 \circ f_0$ is homotopic equivalent to $id_{\mathbb{Z}^{\oplus 2}}$:
For an arbitrary element $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^{\oplus^2}$:
$(g_0 \circ f_0 - id_{\mathbb{Z}^{\oplus 2}}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$ and $(\partial'_1 p_0 + 0 \circ \partial_0) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Longrightarrow g_0 \circ f_0 \simeq id_{\mathbb{Z}^{\oplus 2}}$
To be continued and rechecked

2.0.1 Exact Sequences

Definition 2.0.1. *A sequence of homomorphisms:*

$$\dots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \dots$$

is <u>*exact*</u> *if* $\ker \alpha_n = \operatorname{Im} \alpha_{n+1}$ $\forall$ n.

$\operatorname{Im} \alpha_{n+1} \subseteq \ker \alpha_n$ is equivalent to $\alpha_n \alpha_{n+1} = 0$ since $(A_\bullet, \alpha)$ is a chain complex.
& $\ker \alpha_n \subset \operatorname{Im} \alpha_{n+1} \Rightarrow H_n$ *is trivial* : $H_n = 0 \;\forall\; n$

Examples of short exact sequences:

1. $0 \to A \xrightarrow{\alpha} B$ *is exact* $\iff \ker \alpha = 0, \alpha$ *is injective*
2. $A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B, \alpha$ *is surjective*
3. $0 \to A \xrightarrow{\alpha} B \to 0$ *is exact* $\iff \operatorname{Im} \alpha = B$ *and* $\ker \alpha = \{0\}, \alpha$ *is an isomorphism*
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is exact* $\iff$
   (a) $\ker \alpha = \operatorname{Im}(0 \to A) = 0 \;\Rightarrow \alpha$ *is injective*
   (b) $\operatorname{Im} \beta = C \Rightarrow \beta$ *is surjective*
   (c) $\ker \beta = \operatorname{Im} \alpha$

   So, β induces an isomorphism $C \simeq \frac{B}{\operatorname{Im} \alpha}$. C can written as $C \simeq B/A$ if α is an inclusion of A as a subgroup of B.

Let us consider a short exact sequence of chain complexes:

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$

$A_\bullet$, $B_\bullet$, $C_\bullet$ are chain complexes and i, π are maps between chain complexes where $\ker \pi = \operatorname{Im} i$, $\quad \pi$: *surjective and* i : *injective*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : & {} \arrow[r] & A_n \arrow[r, "\partial"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "i_{n-2}"] & {} \\
B_\bullet : & {} \arrow[r] & B_n \arrow[r, "\partial"] \arrow[d, "\pi_n"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "\pi_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "\pi_{n-2}"] & {} \\
C_\bullet : & {} \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & {} \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

The induced sequence on homology:

$$H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \;\forall\; n \tag{2.3}$$

$\pi \circ i = 0 \Rightarrow \pi_* \circ i_* = 0, \quad H_n(\pi \circ i) = H_n(\pi) \circ H_n(i)$
(2.3) <u>need not</u> be a short exact sequence. However, we can create a long exact sequence of

homology:

$$H_{n+1}(C_\bullet) \xrightarrow{\partial n+1} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet)$$

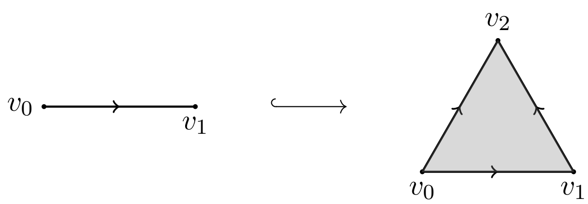
$$0 \to \operatorname{Im}\delta_{n+1} \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{\pi_*} H_n(C) \to \ker\delta_n \to 0$$

where the δ map is defined as:

$$\delta : H_n(C) \to H_{n-1}(A)$$
$$[c] \mapsto [a]$$

For a element $b \in B_n$ there exists $c = \pi_n(b)$ since π is onto.
If we apply the boudary map $\partial : B_n \to B_{n-1}$ on b, then $\partial b \in B_{n-1}$, $\pi_n(\partial b) = \partial(\pi_n(b)) = 0$
We can take an element $a \in A_{n-1}$ such that $i(a) = \partial(b)$
$\partial(\partial b) = \partial(i(a)) = i(\partial a) \Rightarrow \partial a = 0$ *since i is injective* $\Rightarrow \partial b = (0) \in A_{n-1}$

Example 2.0.4. Let consider X to be a $1 - simplex$ and Y a $2 - simplex$



A short exact sequence of chain complexes for X, Y:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
A_\bullet : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial=\begin{bmatrix}-1\\1\end{bmatrix}"] \arrow[d, "i_1=\begin{bmatrix}0\\0\\1\end{bmatrix}"'] & \mathbb{Z}^{\oplus 2} \arrow[r] \arrow[d, "i_0=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}"] & 0 \\
B_\bullet : & 0 \arrow[r] & \mathbb{Z}^{\oplus 3} \arrow[r, "\partial=\begin{bmatrix}0&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}"] \arrow[d, "\pi_1"] & \mathbb{Z}^{\oplus 3} \arrow[r] \arrow[d, "\pi_0"] & 0 \\
C_\bullet : & 0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The short exact sequence of complexes induces a long exact sequence on homology:

$$0 \to H_1(A) \to H_1(B) \to H_1(C) \xrightarrow{\delta} H_0(A) \to H_0(B) \to H_0(C) \to 0$$

Explicitly, the long exact sequence is:

$$0 \to 0 \to \mathbb{Z} \to H_1(C) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\alpha} H_0(C) \to 0$$

$$\ker(1) = \{0\} = \operatorname{Im}\delta$$
$$\ker(\alpha) = \operatorname{Im}(1) = \mathbb{Z}$$

$$C_1 = \mathbb{Z}^{\oplus^3}/\operatorname{Im} i_1 \simeq \mathbb{Z}^{\oplus^2}$$
$$C_0 = \mathbb{Z}^{\oplus^3}/\operatorname{Im} i_0 \simeq \mathbb{Z}$$

The boundary operator between C_1, C_0:

$$\partial : C_1 \to C_0$$
$$\partial : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a+b)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} mod(\operatorname{Im} i_1) \xmapsto{\partial = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} -b \\ -a \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a+b \end{bmatrix}$$

$$H_1(C) = \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \right\} \simeq \mathbb{Z}$$
$$H_0(C) = C_0/C_0 = 0$$

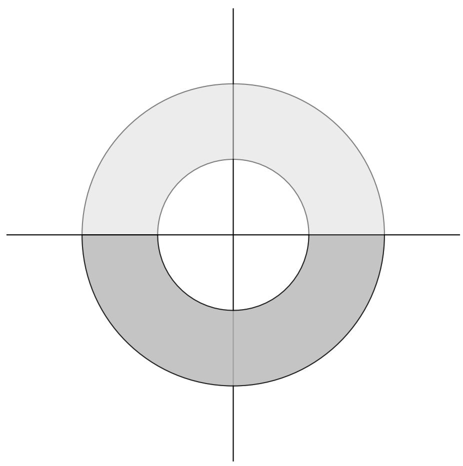
Let us take an example when $Y \subseteq X$ subspace, the short exact sequence of chain complexes is:

$$0 \to C_n(Y) \to C_n(X) \to C_n(X)/C_n(Y) \to 0$$

We get a long exact sequence on homology:

$$H_n(Y) \to H_n(X) \to H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y) \to H_{n-1}(X) \to \cdots$$

Example 2.0.5. Consider X to be the annulus, and the shaded area $Y \subseteq X$



$$
\begin{aligned}
&X = \{(x,y) | 1 \leq x^2 + y^2 \leq 2\} \\
&Y = \{(x,y) | 1 \leq x^2 + y^2 \leq 2, y \leq 0\} \\
&\delta \ : \ H_1(X,Y) \to H_0(Y) \\
&\delta([\sigma]) = [\partial \sigma] = [\sigma(1) - \sigma(0)]
\end{aligned}
$$

In "good cases" $H_n(X,Y) = H_n(X/Y)$

For two topological spaces $Y \subseteq X$ “pairs of spaces”.
We can construct the following chain complex:
$0 \to C_n(Y) \xrightarrow{i} C_n(X) \xrightarrow{\pi} C_n(X)/C_n(Y) \to 0, \quad C_n(X,Y)$ *are relative chains*
Elements of C_n: $\sigma : \Delta^n \to Y \subseteq X$

$$
\begin{gathered}
\partial : C_n(X,Y) \to C_{n-1}(X,Y) \\
\delta \pmod{C_n(Y)} \mapsto \partial\delta \pmod{C_{n-1}(Y)} \\
H_n(X,Y) \equiv H_n(C_\bullet(X,Y)) = Z_n(X,Y)/B_n(X,Y), \quad \textit{cycle/boundary}
\end{gathered}
$$

Definition 2.0.2 (Retraction)**.** *Consider $Y \subseteq X$, a retraction of X onto Y is a map $r : X \to Y$ such that, $r(X) = Y$ and $r^2 = r$.*

i.e. $r(y) = Y$ if $y \in Y$
$i : Y \to X \quad r \circ i = id_Y \quad i \circ r \neq id_x$ but $r_* \circ i_* = id$ in homology

Definition 2.0.3 (Deformation retract)**.** *Consider $Y \subseteq X$: subspace*
Y is a deformation retract of X if there is a homotopy between id_X and a retraction $r : X \to Y$

$$
(F_t) \quad F_t : X \to X \quad \left.\begin{array}{l} F_0 : id_X \\ F_1 : X \to Y \quad F_1|_Y = id_Y \\ F_1(X) = Y \end{array}\right| F_0 \simeq F_1 \ \textit{homotopic}, \ id_X \simeq r
$$

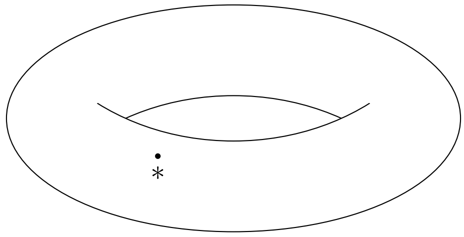
(X,Y) is a "good pair" if

- $Y \subseteq X$ - closed
- There is open $V \subseteq X$, *such that* V is a deformation retracts on Y.

Example 2.0.6.

$$
\begin{aligned}
&(\mathbb{R}^{n+1}, S^n) \text{ is a good pair} \\
&S^n \subseteq \mathbb{R}^{n+1} \text{ - closed} \\
&S^n \subseteq (\mathbb{R}^{n+1}/\{0\}) \text{ and is a deformation retracted of it}
\end{aligned}
$$

Example 2.0.7. Consider X to be a torus, and Y a point on its surface:



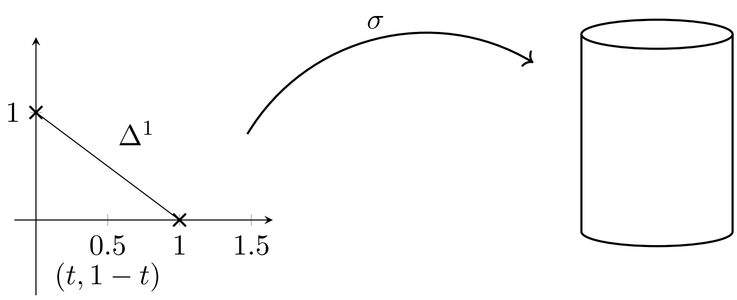
$$
\begin{aligned}
&Y = \underset{pt}{\{*\}} \hookrightarrow X \\
&0 \to C_n(Y) \to C_n(X) \to C_n(X/*) \to 0 \\
&H_n(Y) = \begin{cases} 0 & n \neq 0 \\ \mathbb{Z} & n = 0 \end{cases} \\
&\cdots \to H_n(Y) \to H_n(X) \to H_n(X,*) \to H_{n-1}(Y) \to \cdots \\
&\cdots \to H_0(Y) \to H_0(X) \to H_0(X,*) \to 0 \\
&n > 0: \quad H_n(X,\star) = H_n(X) \\
&n = 0: \quad 0 \to \mathbb{Z} \to H_0(X) \to H_0(X,*) \to 0 \\
&H_0(X,\star) = H_0(x)/\mathbb{Z}, \quad i.e. \begin{array}{l} H_0(X) = \mathbb{Z}^d \\ H_0(X) = \mathbb{Z}^{d-1} \end{array}
\end{aligned}
$$

Remark: Sometimes one introduces “reduced homolgy”

$$
\begin{aligned}
&\cdots \to C_n(X) \to C_{n-1} \to \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0, \quad \tilde{H}_n(X) \text{ - reduced homology} \\
&\sum n_i\sigma_i \mapsto \sum_i n_i \in C_0(X) \\
&\tilde{H}_n(X) = \begin{cases} H_n(X), n > 0 \\ H_0^{sing}(X) = H_0(X) \oplus \mathbb{Z} \end{cases} \\
&H_n(X,*) \to \tilde{H}_n(X)
\end{aligned}
$$

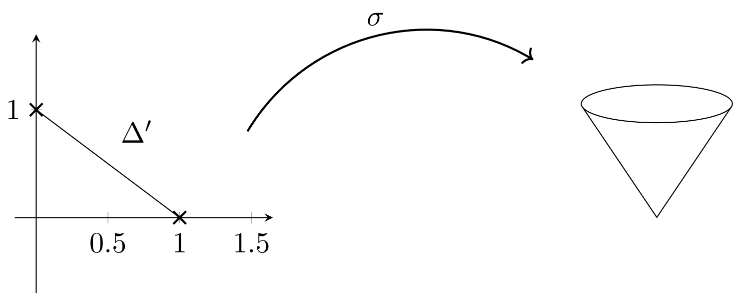
Let us consider some examples of Reduced Homology:

Example 2.0.8. Consider $X = S^1 \times [0,1]$, and $Y = S^1 \times \{0\}$:



$$
\begin{aligned}
&\sigma : \Delta^1 \mapsto S^1 \times [0,1] \\
&\sigma : t \mapsto (\cos 2\pi t, \sin 2\pi t, 1)
\end{aligned}
$$

The boundary operator on σ: $\partial\sigma = 0 \rightsquigarrow [\sigma] \in H_1(X)$
The space X is homotopic equivalent to S^1 so, $H_1(X) \simeq H_1(S^1) \simeq \mathbb{Z}$, considered in 2.
Let us take the quotient map X/Y. The space Y will contract into a point creating a cone.



In $H_1(X,Y)$ the class $[\sigma] \in H_1(X)$ goes to 0 and the long exact sequence on homolgy is:

$$H_1(Y) \to H_1(X) \to H_1(X,Y) \to H_0(X,Y) \to H_0(Y) \to H_0(X) \to 0$$

where $H_1(Y), H_1(X), H_0(Y), H_0(X) \simeq \mathbb{Z}$.

FIGURE here:

$$\begin{aligned}
\partial\sigma_1 &= \sigma_1|_{[12]} - \sigma_1|_{[02]} + \sigma_1|_{[01]} \\
\partial\sigma_2 &= \sigma_2|_{[12]} - \sigma_2|_{[02]} + \sigma_2|_{[01]} \\
\partial(\sigma_1 + \sigma_2) &= \sigma_1|_{[12]} + \sigma_2|_{[01]}
\end{aligned}$$

We can choose $\sigma = \sigma_1|_{[12]} = \partial(\sigma_1 + \sigma_2) - \sigma_2|_{[01]}$, $\sigma_2|_{[01]} \in C_1(Y)$
$\sigma \in B_1(X,Y) \Rightarrow [\sigma] = 0 \in H_1(X,Y)$

Chapter 3

Persistent Homology

Graded Rings and Modules.

A graded ring is a ring $\langle R, +, \cdot \rangle$ equipped with a direct sum decomposition of Abelian groups $R \cong \bigoplus_{i \in \mathbb{Z}} R_i$, so multiplication is defined by bilinear pairings $R_n \otimes R_m \to R_{n+m}$.

Elements in a single R_i are called homogeneous and have degree i, $\deg e = i$ for all $e \in R_i$.

Example 3.0.1. $R = A[t]$, where (A - commutative ring)

$$R_0 = A, \quad R_1 = \{at, a \in A\}, \quad \dots, R_i = \{at^i, a \in A\}$$

Example 3.0.2. $R = \mathbb{R}[x, y, z]$

$$R_i = \{cx^{d_1} y^{d_2} z^{d_3} | \sum_{k=1}^{3} d_k = i\}$$

For example $R_1 \simeq \mathbb{R}^3$ as a vector space $\{ax + by + cz\}$

A graded module M over a graded ring R is a module equipped with a direct sum decomposition $M \cong \bigoplus_{i \in \mathbb{Z}} M_i$, so that the action of R on M is defined is bilinear pairings $R_n \otimes M_m \to M_{n+m}$.

A graded ring (module) is non-negatively graded if $R_i = 0$ ($M_i = 0$) for all $i < 0$.

Reminder: R is a PID if it's a domain (no zero divisors) & $\forall$ ideal is principal

For example:

$$R = \mathbb{Z},\ I = (n),\ n \in \mathbb{Z}$$
$$R = k[t],\ k\ -\ field$$

Theorem 3.0.1 (Structure Theorem)**.** *If D is a PID, then every finitely generated D-module is isnomorphic to a direct sum of cyclic D-module. That is, it decomposes uniquely into the form*

$$D^\beta \oplus (\bigoplus_i D/d_i D)), \tag{3.1}$$

for $d_i \in D$, $\beta \in \mathbb{Z}$, such that $d_i | d_{i+1}$. Similarly, every graded module M over a graded PID D decomposes uniquely into the form

$$(\bigoplus_i \Sigma^{\alpha_i} D) \oplus (\bigoplus_i \Sigma^{\gamma_j} D/d_j D)), \tag{3.2}$$

where $d_j \in D$ are homogeneous elements so that $d_j | d_{j+1}$, $\alpha_i, \gamma_j in \mathbb{Z}$, and Σ^α denotes an α-shift upward in grading.

The free portion on the left is a vector includes generators that may generate an infinite number of elements. Decomposition (3.1) has a vector space of dimension β. The torsional portion on the right includes generators that may generate a finite number of elements. These torsional elements are also homogeneous. Intuitively then, the theorem describes finitely generated modules and graded modules as structures that look like vector spaces but also have some dimensions that are "finite" in size.

Example 3.0.3. Let us take $D = k[t]$ – graded ring(e.g. $\mathbb{R}[t]$) then:

$$D = \underset{\substack{\| \\ M_0}}{k} \oplus \underset{\substack{\| \\ M_1}}{kt} \oplus \underset{\substack{\| \\ M_2}}{kt^2} \oplus \cdots$$

is also a graded module over itself.

$$M = \sum^{\alpha} D = t^{\alpha} k[t] \subseteq k[t] \text{ is an ideal of } D \Rightarrow D \text{ - module}$$
$$M = \underset{\substack{\wr| \\ k}}{M_{\alpha}} \oplus \underset{\substack{\wr| \\ kt}}{M_{\alpha+1}} \oplus \cdots$$
$$(\sum^{\alpha} D)_i = D_{\alpha+i}$$

Reduction:

$$\partial_k : \underset{\{e_i\}}{C_k} \to \underset{\{\hat{e_i}\}}{C_{k-1}}$$
$$(C_k, C_{k-1} : \text{free abelian groups or free R-module})$$
$$\text{Get } M_k : \text{ matrix of } \partial_K$$
$$e \cdot u : (e_1, \cdots, e_{m_k}) \begin{pmatrix} u_1 \\ \vdots \\ u_{m_k} \end{pmatrix} \in C_k$$
$$\partial_k(eu) = \hat{e}(M_u)$$

Row-operation

$$R_i \mapsto R_i + qR_j \text{ on } M$$
$$M \mapsto i \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & q \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\text{A}} M$$
$$R_2 \mapsto R_2 + 2R_3$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} M$$
$$\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e_1 & e_2 & 2e_2 + e_3 \end{pmatrix}$$

Column operations:

$$M \mapsto MB, B_2 \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \vdots & \ddots & \\ & & q & & 1 \end{pmatrix}$$

Interpretation of row/column operations on a matrix of a map in terms of changing bases.

$$\partial : \xi = \underline{e}u \mapsto \hat{e}Mu \textit{ where } u = [\xi]\underline{e}$$

Suppose we perform a column-operation $M \mapsto MB$. This is supposed to change the matrix of a map - keeping the map unchanged

$$\xi = \underset{\underline{e}B(B^{-1}u)}{\overset{\underline{e}u}{\parallel}} \mapsto \hat{e}Mu = \hat{e}MB(B^{-1}u)$$

If we set $\underline{e}' = \underline{e}B$

$\partial : \xi = \underline{e}'v \mapsto \underline{\hat{e}}MBv$

$v = [\xi]_{\underline{e}'}$

Similarly, suppose we perform a row-operation $M \mapsto AM$. Then

$$\begin{aligned} \xi = \underline{e}u \mapsto \underline{\hat{e}}Mu &= \underline{\hat{e}}A^{-1}AMu \\ &= \underline{\hat{e}}'AMu \end{aligned}$$

That is, if $M = [\partial]_{\underline{e}\underline{\hat{e}}}$, then

$AMB = [\partial]_{\underline{e}B,\underline{\hat{e}}A^{-1}}$

$A = I + qE_{ij} : R_i \mapsto R_i + qR_j$ (via $M \mapsto AM$)

$A^{-1} = I - qE_{ij}$

$$\underline{\hat{e}}A^{-1} = (\hat{e}_1, \cdots, \hat{e}_{m_{k-1}}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \dots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} i =$$

$$= (\hat{e}_1, \cdots, \hat{e}_j - q\hat{e}_i, \cdots), \text{i.e. } \hat{e}_j \mapsto \hat{e}_j - q\hat{e}_i$$

$B = I + qE_{ji} : C_i \mapsto C_i + qC_j$(via $M \mapsto MB$)

$$\underline{e}B = (e_1, \cdots, e_{m_k}) \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \dots & q \\ & & & 1 & \\ & & & & 1 \end{pmatrix} j = (e_1, \cdots, e_i + qe_j, \cdots)$$

Let $V \simeq \mathbb{R}^n$(be a vector space)
$B = \{e_1, \cdots, e_n\}$

$$V = \underline{e}X = (e_1, \cdots, e_a)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i \quad (e_i \in V, x_i \in \mathbb{R})$$

$B^1 = \{f_1, \cdots, f_n\}$

$$v = \underline{e}X = \underbrace{\underline{e}T^{-1}}_{\underline{\mathrm{f}}}(\underbrace{TX}_{\mathrm{Y}}) = (f_1, \cdots, f_n)Y$$

$(f_1, \cdots, f_n) = (l_1, \cdots, l_n)T^{-1}$
$Y = TX$

Smith normal form (for PID):

$\exists A, B:$

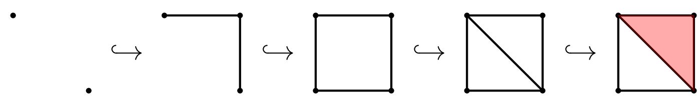
$$\partial_k \equiv AMB = \left(\begin{array}{c|c} \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_l \end{pmatrix} & \overbrace{0} \\ \hline 0 & 0 \end{array}\right)^{b_i | b_{ix1}}$$

$rank Z_k = m_k - e_k$
$rank\ Free(H_k) = m_k - l_k - l_{k+1}$

<u>On persistence:</u>
We have a "filtered chain complex"

$$\begin{array}{ccccccc} (0) & \subseteq C^1_\bullet \subseteq C^2_\bullet \subseteq \cdots \subseteq & C^m_\bullet \\ \| & & \| \\ C^0_\bullet & & C_\bullet \end{array}$$

It may come from a fitted simplicial complex:



$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
0 \arrow[r, hook, "f^1"] \arrow[d] & \mathbb{Z}^2 \arrow[r, hook, "f_2"] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^5 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
\mathbb{Z}^2 \arrow[r, hook, "f^1"] \arrow[d] & \mathbb{Z}^3 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[r, hook] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
0 \arrow[r, hook, "f^1"] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
C^1_\bullet \arrow[r, hook, "f^1"] & C^2_\bullet \arrow[r, hook, "f^2"] & C^3_\bullet \arrow[r, hook, "f^3"] & C^4_\bullet \arrow[r, hook, "f^4"] & C^5_\bullet
\end{tikzcd}$$

Definition 3.0.1. *(Persistant Homology Group)*

Given a filtered complex, the i-th complex K^i has associated boundary operators ∂_k^i, matrices M_k^i , and groups C_k^i, Z_k^i, B_k^i, and H_k^i for all $i, k \geq 0$ The p-persistent k-th homology group of K^i is

$$H_k^{i,p} = Z_k^i \Big/ (B_k^{i+p} \cap Z_k^i)$$

$$\text{If } p = 0 : H_k(C_\bullet^i) = Z_k^i/(\underbrace{B_k^i \cap Z_k^i}_{\equiv B_k^i})$$

"Persistence cx"

$\{C_\bullet^i\} :\geq 0 \quad f : C_\bullet^i \to C_\bullet^{i+1}$ for chain maps

"Persistence mod"

$$\begin{aligned}
&\{M^i, \varphi^i : M^i \to M^{i+1}\} = M \\
&\alpha(M) = \bigoplus_{i \geq 0} M_i \\
&t \cdot (m^0, m^1, \cdots) = (0, \varphi^0(m^0), \varphi^1(m^1), \cdots) \\
&\left(\begin{array}{c|c|c|c}
0 & & & \\ \hline
\varphi^0 & 0 & & \\ \hline
 & \varphi^1 & 0 & \\ \hline
 & & \varphi^2 &
\end{array}\right)
\end{aligned}$$

Theorem 3.0.2. *(Correspondence) The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated non-negatively graded modules over $R[t]$.*

$$\varphi : \mathbb{R}^n \to \mathbb{R}$$

$$\varphi : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (A_1, \cdots, A_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A_1 x_1 + \cdots + A_n x_n$$

$$(\varphi_1, \cdots, \varphi_{n-1})\mathbb{R}^n \to \mathbb{R}^{n-1}$$

$$\underline{x} \mapsto \begin{pmatrix} A_{11} & \cdots & A_{m-1} \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\text{e.g.} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

setup: chain complex with a filtration

$$\begin{aligned}
C_\bullet^0 \hookrightarrow C_\bullet^1 &\hookrightarrow C_\bullet^2 \hookrightarrow \cdots \\
H_k^{i-p} &= Z_k^i / B_k^{i+p} \cap Z_k^i \\
Z_k^i &= \ker(\partial_k^i : C_k^i \to C_{k-1}^i) \\
= \mathrm{Im}((H_k^i &= H_k | C_\bullet^i) \to (H_k | C_\bullet^{i+p}))
\end{aligned}$$

Ex:

$$C^0_\bullet \qquad\qquad\qquad C^1_\bullet$$

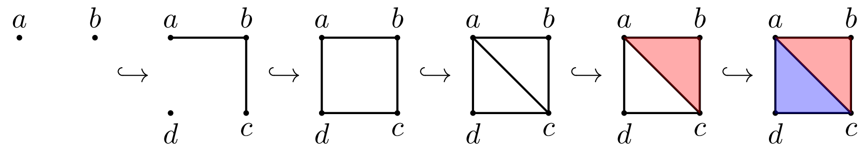
$$\begin{tikzcd}
1 & (0) \arrow[r, hook] \arrow[d, "\partial_1=0"'] & \mathbb{Z} \arrow[d, "\partial_1=\begin{bmatrix}-1\\1\end{bmatrix}"] \\
0 & \mathbb{Z}^2 \arrow[r, hook] \arrow[d, "\partial_0=0"'] & \mathbb{Z}^2 \arrow[d, "\partial_0=\begin{bmatrix}0 & 0\end{bmatrix}"] \\
 & 0 \arrow[r, hook] & 0
\end{tikzcd}$$

<u>$p=0$</u>

$$\begin{aligned}
&H_k^{i,0} = Z_k^i / B_k^i \cap Z_k^i \\
&H_1^{0,0} = (0) \quad H_1^{1,0} = (0) \\
&H_0^{0,0} = \mathbb{Z}^2 \quad H_0^{1,0} \simeq \mathbb{Z}
\end{aligned}$$

<u>$p=1$</u>

$$\begin{aligned}
&H_k^{i,1} = Z_k^i / B_k^{i+1} \cap Z_k^i \\
&H_1^{0,1} = Z_0^1 / B_1^1 \cap Z_1^0 = (0) \quad H_1^{1,1} = (0) \\
&H_0^{0,1} = Z_0^0 / B_0^1 \cap Z_0^0 = \mathbb{Z}^2 / \operatorname{Im}\begin{bmatrix}-1\\1\end{bmatrix} \simeq \mathbb{Z}
\end{aligned}$$



$$\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
 & C^0_\bullet \arrow[r] & C^1_\bullet \arrow[r] & C^2_\bullet \arrow[r] & C^3_\bullet \arrow[r] & C^4_\bullet \arrow[r] & C^5_\bullet \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[r] \arrow[d, "\partial_1^3"] & \mathbb{Z}^5 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[d] \\
0 & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

<u>Exc</u> Compute explicitly some $H_k^{i,p}$
e.g. $H_1^{3,2} = Z_1^3 = B_1^5 \cap Z_1^3$
Here the matrix of ∂_1^3 is

$$\begin{array}{cc}
 & \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\
\begin{array}{c} d \\ c \\ b \\ a \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix}
\end{array} = M_1$$

Introduce the structure of a graded module over $k[t]$ (on $R[t]$)
Degrees $\iff$ appearance in the translation

- $a, b\}$ deg 0
- $\left.\begin{array}{c} c, d \\ ab, bc \end{array}\right\}$ deg 1
- ab, dc deg2
- ac deg 3
- abc deg 4
- adc deg 5

If we work in $(\mathbb{Z}/2\mathbb{Z})[t]$

$$M_1 = \begin{array}{cc}
 & \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\
\begin{array}{c} d \\ c \\ b \\ a \end{array} & \begin{pmatrix} 0 & 0 & t & t & 0 \\ 0 & 1 & t & 0 & t^2 \\ t & t & 0 & 0 & 0 \\ t & 0 & 0 & t^2 & t^3 \end{pmatrix}
\end{array}$$

Remarks:
* Any ideal $\omega \subseteq R$ in a commutative ring in as R - module in a natural way: $\forall a \in \omega, \forall r \in R, r \cdot a \in \omega$ and $\omega \subseteq R$ is an abgroup(R is a module over itself & ω is a submodule)
* Hence for any

$$\begin{array}{cc} & n \in \mathbb{N} \\ (t^n) & \subseteq k[t] \\ \| & \\ \{ \underset{\substack{\| \\ t^n k[t]}}{t^n} \ P(t), P \in k[t]\} & \end{array}$$

is a $k[t]$ - module. This is a free $k[t]$ - module.
* $R = k[t]$ has a natural structure of graded ring. The deg i - elements are the (non-zero) elements of the line $R_i := kt^i \subseteq k[t]$ (in general: $R = \oplus R_i, R_iR_j \subseteq R_{i+j}$). The ideal $(t^n) \subseteq k[t]$ is then a graded $k[t]$ - module (in general, this means $M = \underset{i}{\oplus} M_j, R = \underset{i}{\oplus} R_i, R_i \cdot M_j \subseteq M_{i+j}$)
$(t^n) = t^n, k[t] = \underset{i \geq 0}{\oplus} \underbrace{kt^{n+i}}_{(t^n)}$
i.e.
degree - i elements of (t^n) are the monorvials of degree $(n+i)$
* While $(t^n) \subseteq k[t]$ is an ideal, so $k[t]$ - module - and so $k[t]$ - submodule; and a graded $k[t]$ - module, it is not a graded $k[t]$ - submodule!
The grading of (t^n) is not the grading that is induced by the ambient $k[t]$: it is shifted up by n: t^{n+i}, as an alternative of $(t^n) = t^n k[t]$, has degree i, not degree $(n+i)$.
* In general, for a graded module M over a graded ring R we can define the twist of M by n, $M[n]$, also denoted by $\sum^n M$, is defined by $(\sum^n M)_i = M_{n+i}$.
i.e., by redefining/shifting the grading up by n. We see that we can identify $\sum^n k[t]$ with $(t^n) = t^n k[t]$ as graded modules $(\sum^n k[t])_i = k[t]_{n+i} = kt^{n+i} \xrightarrow{id} kt^n \cdot t^i$
* A map of free $k[t]$ - modules $k[t] \to k[t]$ is multiplication by some polynomial $p(t)$ ie of the kind $p(t) \mapsto p(t)q(t)$ ($p(t)$ is the image of $1 \in k[k]$). More generally, a map of free $k[t]$ - module, $k[t]^{\oplus m} \to k[t]^{\oplus r}$ is given by some $r \times m$ matrix with $k[t]$ - entries.
* A map (morphism) if graded R-modules $\varphi : M \to N$ is a map of R - modules (ie: R - linear, ie: $\varphi(rm) = r\varphi(m), \varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$) which preserves the degrees. That is $\varphi(M_i) \subseteq N_i, \forall i$.
* In particular, a map of graded modules $\varphi : \underset{\substack{\wr\| \\ (t^n)}}{\sum^n k[t]} \to \underset{\substack{\wr\| \\ (t^p)}}{\sum^p k[t]}$ must send $(t^n)_i = kt^{n+i}$ to $(t^p)_i = kt^{p+i}$.
* As the shifts $\sum^n k[t]$ are still free modules, any $k[t]$ - module homomorphism $\varphi : (t^n) \to (t^n)$ is determined by some polynomial $p(t) \in k[t]$, ie by the image of a generator: $\varphi(t^n) = p(t)t^p$ as $\varphi(t^n q(t)) = q(t)\varphi(t^n)$. However if φ is a graded module homomorphism, we must have that $\varphi(t^n) = \underbrace{(mt^p)}_{p(t)} t^p$, ie, that $p(t)$ be homogeneous. Similarly, for direct sums $\underset{i}{\oplus}(t^{ni}) \to \underset{i}{\oplus}(t^{pj})$.
* In particular: if $n \geq p$, any map of modules $\varphi : k[t] \to k[t], 1 \mapsto p(t)$ determines a map of graded modules

$$\begin{aligned} &(t^n) = \sum^n k[t] \to (t^p) = \sum^p k[t] \\ &t^n \mapsto (p(t)t^{n-p})t^p \end{aligned}$$

E.g, the identity map on $k[t]$ induces $t^n \mapsto t^{n-p} \cdot t^p$ (The "matrix element" of the identity on

$k[t]$ is t^{n-p}). Ditto for direct sums.
* Everything so far works similarly for $\mathbb{Z}[t]$ or $A[t]$, A - common ring. In particular, look again at the simplicial complex [ZC05]

Bibliography

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