

Persistent Homology and TDA

Senior Thesis

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Standard Simplex - n -simplex

A n - simplex is denoted by $[v_0, v_1, \dots, v_n]$

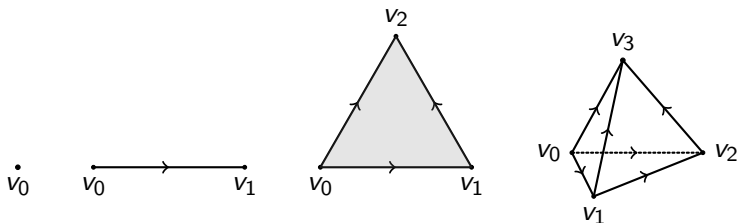


Figure 1: 0-simplex, 1-simplex, 2-simplex, 3-simplex

Δ – complex

Definition (Δ -complex)

A Δ – complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \rightarrow X$, with n depending on the index α , such that:

- 1 The restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$.
- 2 Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \rightarrow X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- 3 A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

Example - S^1

Consider $X = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. We are going to describe explicitly two maps $\sigma_0 : \Delta^0 \rightarrow S^1$, $\sigma_1 : \Delta^1 \rightarrow S^1$, which equip S^1 with the structure of a Δ -complex.

For the explicit description, keep in mind that $\Delta^0 = \{1\} \subseteq \mathbb{R}$ and that

$$\Delta^1 = \{(t_0, t_1) \mid t_0 + t_1 = 1\} = \{(t_0, 1 - t_0), t_0 \in [0, 1]\}$$

So in particular, any (continuous) map $\Delta^1 \rightarrow S^1$ is determined by and determines a (continuous) map $[0; 1] \rightarrow S^1$.

First, we define $\sigma_0 : \Delta^0 \rightarrow S^1$ by $\sigma_0(1) = (1, 0)$. Next, $\sigma_1 : \Delta^1 \rightarrow S^1$ is defined by $\sigma_1(t_0, t_1) = (\cos(2\pi t_0), \sin(2\pi t_0))$, which is clearly continuous. The map σ_1 is one-to-one on $\mathring{\Delta}^1$, and so is, trivially, σ_0 on $\mathring{\Delta}^0$. The images of the two maps cover the circle, with $\sigma_1(\mathring{\Delta}^1) = S^1 \setminus \{(1, 0)\}$ and $\sigma_0(\mathring{\Delta}^0) = \{(1, 0)\}$.

Finally, we check the compatibility property (2) of σ_1 and σ_0 .

The restrictions of σ_1 to the two faces of Δ^1 coincide with σ_0 :

$$\sigma_1|_{[e_1]} = \sigma_1(0, 1) = (\cos(0), \sin(0)) = (1, 0) = \sigma_0(1)$$

$$\sigma_1|_{[e_0]} = \sigma_1(1, 0) = (\cos(2\pi), \sin(2\pi)) = (1, 0) = \sigma_0(1)$$

Property (3) holds as well. There are numerous other Δ -complex structures on S^1 , and we shall discuss some of them later on.

Abstract Simplicial Complex

Definition (Abstract Simplicial Complex)

Given a finite set $\{1, 2, \dots, m\} (= [m])$ an *abstract simplicial complex* is a collection \mathcal{K} of subsets of $[m]$, such that:

- ① $\emptyset \in \mathcal{K}$
- ② $\{i\} \in \mathcal{K}$ (singletons are in \mathcal{K})
- ③ If $I \in \mathcal{K}$ and $J \subseteq I$, then $J \in \mathcal{K}$

The elements of $[m]$ are the *vertices*, and the elements of \mathcal{K} are the *simplices*, where $I \in \mathcal{K}$ is an $(|I| - 1)$ -simplex.

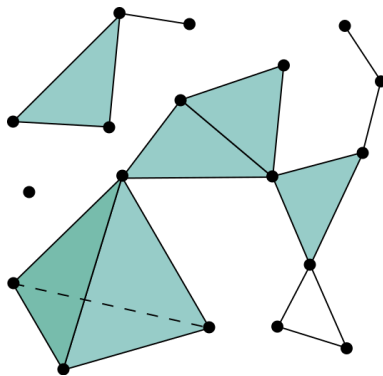
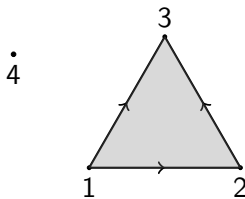


Figure 2: A simplicial 3-complex

Consider the following partially ordered set $V = \{1, 2, 3, 4\}$: The simplicial complex

$$\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$$



Given an abstract simplex \mathcal{K} , we can construct its *topological realization* as

$$|\mathcal{K}| = \bigcup_{\emptyset \neq I \in \mathcal{K}} (\text{Conv}(e_i), i \in I) \subseteq \mathbb{R}^m$$

where $\{e_i\}$ is in the standard basis $e_1, \dots, e_m \in \mathbb{R}^m$.

Definition (Chain complex)

Complex of abelian groups.

A chain complex is a sequence of homomorphisms of abelian groups:

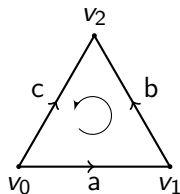
$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z} . The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\text{Im } \partial_{n+1} \subset \ker \partial_n$.

The boundary map $\partial_n : \Delta_n(\mathcal{X}) \rightarrow \Delta_{n-1}(\mathcal{X})$ for the would-be chain complex $\Delta_\bullet(X)$ is defined as

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \widehat{v_i}, \dots, v_n].$$

Boundary operator



$$\partial[v_0, v_1] = v_1 - v_0$$

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

Homology of a Chain Complex

Definition (Homology Group)

The n -th homology group of the chain complex is defined as the quotient group

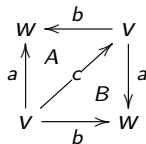
$$H_n = \frac{Z_n}{B_n} = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Elements of Z_n are called cycles and elements of B_n are called boundaries.

Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called homology classes. Two cycles representing the same homology class are said to be homologous. This means their difference is a boundary.

Computing Homology of \mathbb{RP}^2 in \mathbb{Z}

One way to calculate the homology groups of a projective plane \mathbb{RP}^2 is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z}

$$\left| \begin{array}{l} C_0 = \langle v, w \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geq 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

The n -th homology group is defined as $H_n = \ker \partial_n / \operatorname{Im} \partial_n$

The homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Let's compute H_1 :

$\ker \partial_1 = \langle a - b, c \rangle$ since

$$\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$$

The general element in C_1 : $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements $a-b$ and c

$\text{Im } \partial_2 = \langle -a + b + c, a - b + c \rangle$ since

$$\partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$$

$$H_1 = \frac{\ker \partial_1}{\text{Im } \partial_2} = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle}$$

The group $\langle a - b, c \rangle$ can be also generated by the elements

$m = a - b + c$, and c where $a - b = m - c$. So,

$$H_1 = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle} = \frac{\langle a-b+c, c \rangle}{\langle a-b+c, -a+b+c \rangle}$$

If we let $t = a - b + c$ then $-a + b + c = -t + 2c$ then the group $\langle t, -t + 2c \rangle$ can be also generated by the elements t and $2c$.

$$\text{In terms of } t \text{ and } c, H_1 = \frac{\langle t, c \rangle}{\langle t, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Maps of Chain Complexes

Let (C_\bullet, ∂) and (D_\bullet, δ) be two chain complexes. A map of chain complexes is a morphism f that is a sequence of homomorphisms $(f_n)_{n \in \mathbb{Z}}$:

$$\begin{array}{ccccccc}
 (C_\bullet, \partial) & C_\bullet & \dots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & \dots & C_\bullet \\
 & & & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & & \\
 (D_\bullet, \delta) & D_\bullet & \dots & \longrightarrow & D_n & \xrightarrow{\delta_n} & D_{n-1} & \xrightarrow{\delta_{n-1}} & D_{n-2} & \xrightarrow{\delta_{n-2}} & \dots & D_\bullet
 \end{array}$$

$$f_n : C_n \rightarrow D_n \quad \text{s.t.}, \quad f_{n-1} \circ \partial_n = \delta_n \circ f_n \quad \forall n \in \mathbb{Z}$$

$$\begin{array}{ccc}
 C_n & \xrightarrow{\partial_n} & C_{n-1} \\
 \downarrow f_n & & \downarrow f_{n-1} \\
 D_n & \xrightarrow{\delta_n} & D_{n-1}
 \end{array} \quad \text{commutes.}$$

Maps of Chain Complexes induce Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$H_n(f) : H_n(C_\bullet) \rightarrow H_n(D_\bullet)$$

$$H_n(f) : [x] \mapsto [f_n(x)]$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-definedness by checking if cycles are sent to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$\begin{aligned} \delta_n \circ f_n(x) &= f_{n-1} \circ \partial_n(x) = f_{n-1}(0) = 0 \Rightarrow f_n(x) \in \ker \delta_n, f_n(x) \text{ is a cycle} \\ &\Rightarrow f_n(\ker \partial_n) \subseteq \ker \delta_n \end{aligned}$$

So, cycles are sent to cycles.

(2) Let us take a boundary $y \in C_n$, so that $y \in \text{Im } \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

$$\begin{aligned} f_n(y) &= f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z)) \\ &\Rightarrow f_n(y) \in \text{Im } \partial_{n+1} f_n(y) \text{ is a boundary} \\ &\Rightarrow f_n(\text{Im } \partial_{n+1}) \subseteq \text{Im}(\delta_{n+1}) \end{aligned}$$

So, boundaries are sent to boundaries.

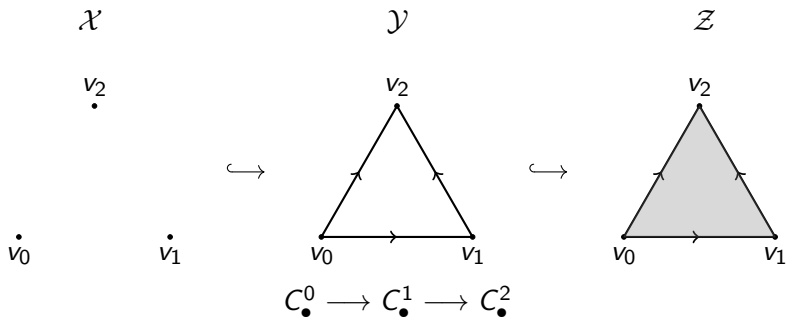
$$H_n(f) : H_n(C_\bullet) \rightarrow H_n(D_\bullet)$$

$$H_n(f) : \ker \partial_n / \text{Im}(\partial_{n+1}) \rightarrow \ker \delta_n / \text{Im}(\delta_{n+1})$$

$$[x] \mapsto [f_n(x)]$$

$$x + \text{Im } \partial_{n+1} \mapsto f_n(x) + f_n(\text{Im } \delta_{n+1}) = f_n(x) + \text{Im}(\delta_{n+1}) = [f_n(x)]$$

Filtered Complex



Maps of Complexes

$$C_{\bullet}^0 \longrightarrow C_{\bullet}^1 \longrightarrow C_{\bullet}^2$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 2 & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \\
 & \downarrow \partial_2^0=0 & & \downarrow \partial_2^1=0 & & \downarrow \partial_2^2=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & \\
 1 & 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
 & \downarrow \partial_1^0=0 & & \downarrow \partial_1^1=\partial_1^2 & & \downarrow \partial_1^2=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \\
 0 & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
 & \downarrow \partial_0^0=0 & & \downarrow \partial_0^1=0 & & \downarrow \partial_0^2=0 & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Maps of Complexes induce maps on Homology

$$H(C_\bullet^0) \longrightarrow H(C_\bullet^1) \longrightarrow H(C_\bullet^2)$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 2 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Consider a short exact sequence of chain complexes:

$$0 \rightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{\pi} C_{\bullet} \rightarrow 0$$

A_{\bullet} , B_{\bullet} , C_{\bullet} are chain complexes and i, π are maps between chain complexes where $\ker \pi = \operatorname{Im} i$, $\pi : \textit{surjective}$ and $i : \textit{injective}$ induces long exact sequence of homology:

$$\begin{aligned} H_{n+1}(C_{\bullet}) \xrightarrow{\partial_{n+1}} H_n(A_{\bullet}) \xrightarrow{i_*} H_n(B_{\bullet}) \xrightarrow{\pi_*} H_n(C_{\bullet}) \xrightarrow{\delta} H_{n-1}(A_{\bullet}) \rightarrow \\ H_{n-1}(B_{\bullet}) \rightarrow H_{n-1}(C_{\bullet}) \end{aligned}$$

The Equivalence of Simplicial and Singular Homology

singular n – *simplex* in a space X is a map $\sigma : \Delta^n \rightarrow X$. We considered a parameterization of a space by simplexes, where the maps $\sigma_\alpha : \Delta^n \rightarrow X$ had restrictions. If we only require that the σ map is continuous, then by definition that would be a singular n – *simplex* in a space X .

The lack of other restrictions on the map $\sigma : \Delta^n \rightarrow X$, convey that σ does not need to be a ‘nice’ embedding, in fact it can have singularities, where its image does not look like a simplex.

$C_n(X)$ is a free abelian group with generators the set of singular n – *simplexes* in X : the continuous maps $\sigma : \Delta^n \rightarrow X$.

Theorem

The homomorphisms $H_n^\Delta(X, A) \rightarrow H_n(X, A)$ are isomorphisms for all n and all Δ -complex pairs (X, A) .

In addition to the long exact sequence of homology groups for a pair (X, A) , there is another sort of long exact sequence, known as a Mayer-Vietoris sequence, which is equally powerful but is sometimes more convenient to use. For a pair of subspaces $A, B \subseteq X$ such that X is the union of the interiors of A and B , this exact sequence has the form

If \mathcal{X} is a metric space and $r \geq 0$:

Definition

The Čech Complex has vertex set \mathcal{X} and simplex $[v_0, v_1, \dots, v_n]$ when

$$\bigcap_{i=0}^n \mathcal{B}(v_i; r/2) \neq \emptyset$$

Definition

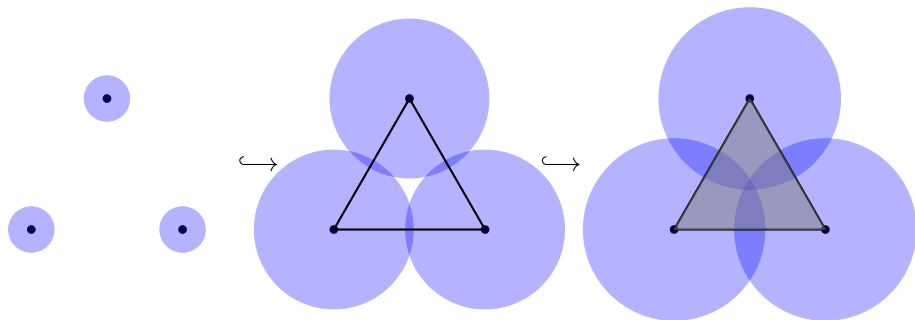
The Vietoris Rips Complex has vertex set \mathcal{X} and simplex $[v_0, v_1, \dots, v_n]$ when

$$d(v_i, v_j) \leq r \quad \forall i, j$$

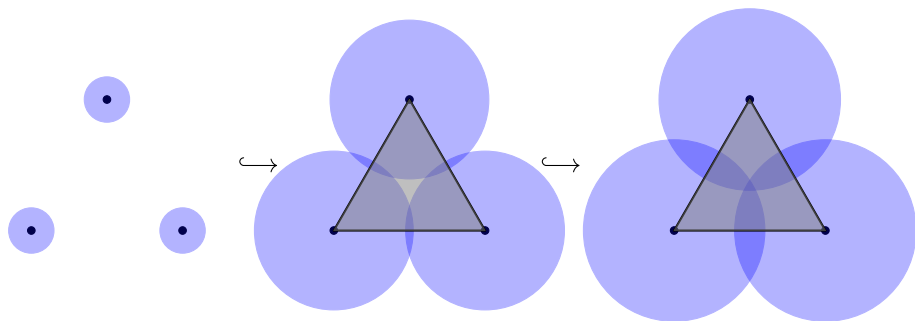
Relation of Čech and Vietoris-Rips Complex: For each $\epsilon > 0$, there is a chain inclusion maps

$$\mathcal{R} \hookrightarrow \mathcal{C}_{\epsilon\sqrt{2}} \hookrightarrow \mathcal{R}_{\epsilon\sqrt{2}}$$

Čech Complex Example



Vietoris-Rips Complex Example



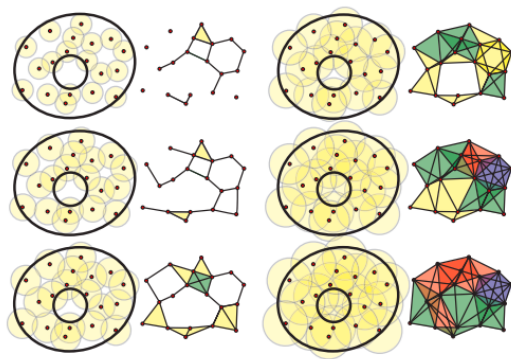


Figure 3: A sequence of Rips Complex from a point cloud data set that represent an annulus

Definition

Given a filtered complex, the i -th complex K^i has associated boundary operators ∂^i_k , matrices M^i_k , and groups C^i_k , Z^i_k , B^i_k , and H^i_k for all $i, k \geq 0$. The p -persistent k -th homology group of K^i is

$$H^{i,p}_k = Z^i_k / (B^{i+p}_k \cap Z^i_k)$$

Example: $p = i = k = 1$: $H^{1,1}_1 = Z^1_1 / (B^2_1 \cap Z^1_1) \mathbb{Z} / (\{0\} \cap \mathbb{Z}) = \mathbb{Z}$

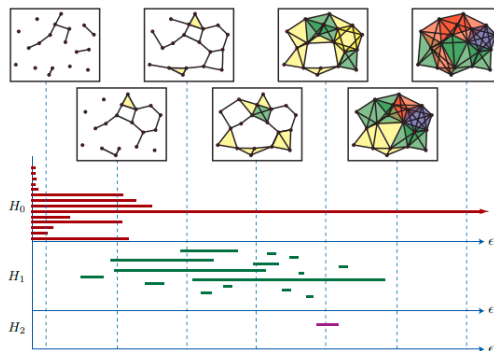


Figure 4: An example of barcode representations of the homology of the sampling of points in an annulus

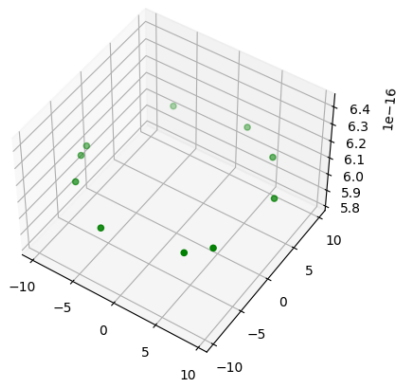


Figure 5: 10 points on a cycle

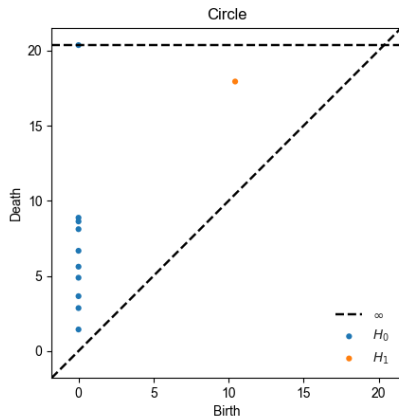


Figure 6: Persistent diagram of homology of circle (10 points)

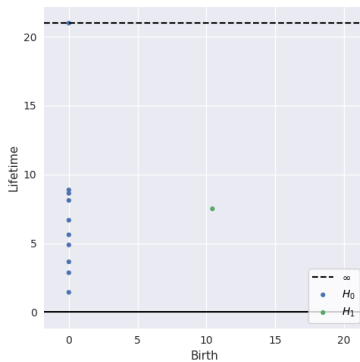


Figure 7: Lifetime diagram of homology of circle (10 points)

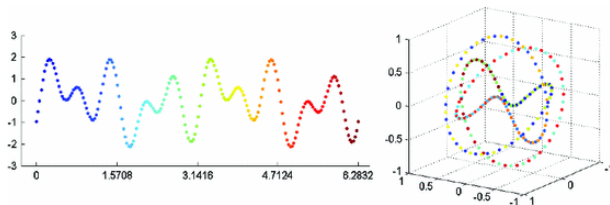


Figure 8: Periodicity in Timeseries