Topological Data Analysis

Introducing Persistent Homology

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November 30, 2020

Outline

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   - *n − simplex*
   - *Δ − complex*
   - Chain Complex
   - Computing Homology
2. Maps of Complexes and Maps on Homology
3. Čech and Vietoris-Rips Complex
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4. Persistent Homology
   - Persistance
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Standard Simplex - *n-simplex*

A $n - simplex$ is denoted by $[v_0, v_1, ..., v_n]$

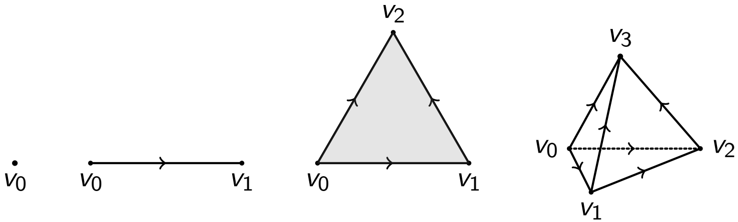


Figure 1: 0-simplex, 1-simpex, 2-simplex, 3-simplex

Δ − *complex*

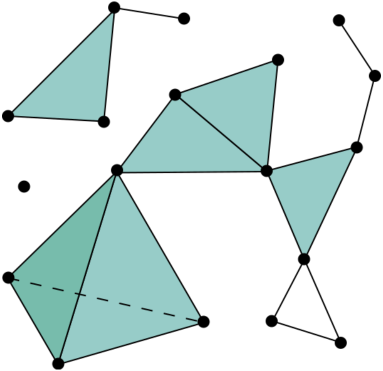


Figure 2: A simplicial 3-complex

Definition (Chain complex)

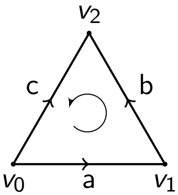
Complex of abelian groups.
A chain complex is a sequence of homomorphisms of abelian groups:

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$. The equation $\partial_n\partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im}\partial_{n+1} \subset \ker\partial_n$.

where the boundary homomorphisms ∂_n of $[v_0, v_1, \dots, v_n]$ is a defined as $\Sigma_i(-1)^i[v_0, \dots, \hat{v}_i, \dots, v_n]$ where the '^' symbol denotes the absence of that vertex.

Boundary operator



$$\partial[v_0, v_1] = v_1 - v_0$$

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

Homology of a Chain Complex

Definition (Homology Group)

The n-th homology group of the chain complex is defined as the quotient group

$$H_n = \frac{Z_n}{B_n} = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Elements of Z_n are called cycles and elements of B_n are called boundaries.

Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called homology classes. Two cycles representing the same homology class are said to be homologous. This means their difference is a boundary.

Computing Homology of S^1 in $\mathbb{Z}$

Space $\mathcal{X} = S^1$

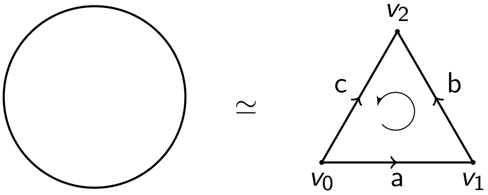


Figure 3: Triangulation of S^1

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2 = 0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1, v_2 \rangle \\ C_1 = \langle a, b, c \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2 = 0} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_0 = 0} 0$$

H_0 - # of connected components

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.
First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1, v_2 \rangle$ since $\partial_0 = 0$.
To calculate $\operatorname{Im} \partial_1$, let's compute:
$\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v_1 - v_0) + \beta(v_2 - v_1) - \gamma(v_2 - v_0) =$
$(\gamma - \alpha)v_0 + (\alpha - \beta)v_1 + (-(\gamma - \alpha) - (\alpha - \beta))v_2$

$$\operatorname{Im} \partial_1 = \left\{ \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ -(\gamma - \alpha) - (\alpha - \beta) \end{pmatrix}, \quad \alpha, \beta, \gamma \subseteq \mathbb{Z} \right\} \subseteq \mathbb{Z}^{\oplus^3}$$

There exist an isomorphism $\operatorname{Im} \partial_1 \simeq \mathbb{Z}^2$

$$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$$

H_1 - # of holes

Second, let's compute H_1:

$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$

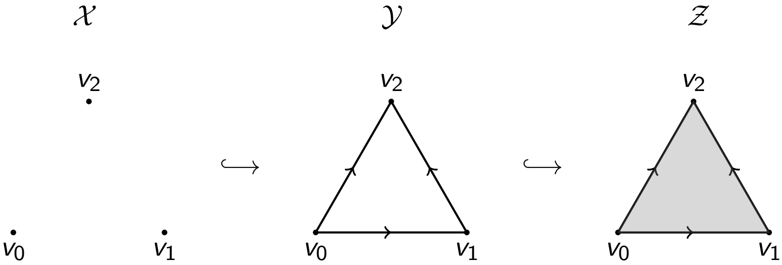
$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$

$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Filtered Complex



$$C^0_\bullet \longrightarrow C^1_\bullet \longrightarrow C^2_\bullet$$

Maps of Complexes

$$
\begin{tikzcd}
C^0_\bullet \arrow[r] & C^1_\bullet \arrow[r] & C^2_\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d, "\partial_2^0=0"] & 0 \arrow[r] \arrow[d, "\partial_2^1=0"] & \mathbb{Z} \arrow[d, "\partial_2^2=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}"] \\
1 & 0 \arrow[r] \arrow[d, "\partial_1^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_1^1=\partial_1^2"] & \mathbb{Z}^3 \arrow[d, "\partial_1^2=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^1=0"] & \mathbb{Z}^3 \arrow[d, "\partial_0^2=0"] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

Maps of Complexes induce maps on Homology

$$
\begin{tikzcd}
H(C^0_\bullet) \arrow[r] & H(C^1_\bullet) \arrow[r] & H(C^2_\bullet)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

If $\mathcal{X}$ is a metric space and $r \geq 0$:

Definition

The Čech Complex has vertix set $\mathcal{X}$ and simplex $[v_0, v_1, ..., v_n]$ when

$$\bigcap_{i=0}^{n} \mathcal{B}(v_i; r/2) \neq \varnothing$$

Definition

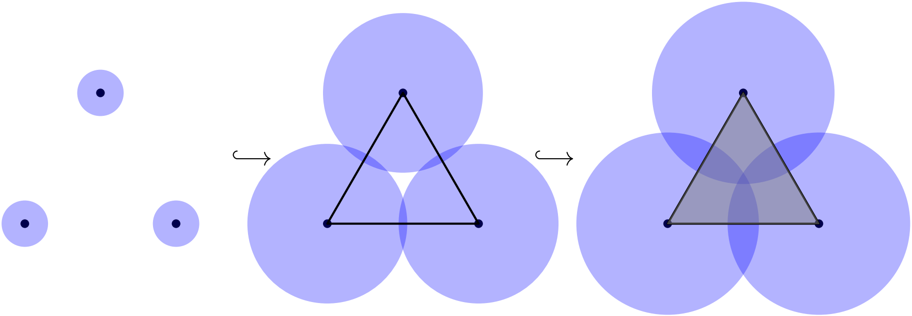
The Vietoris Rips Complex has vertix set $\mathcal{X}$ and simplex $[v_0, v_1, ..., v_n]$ when

$$d(v_i, v_j) \leq r \; \forall i, j$$

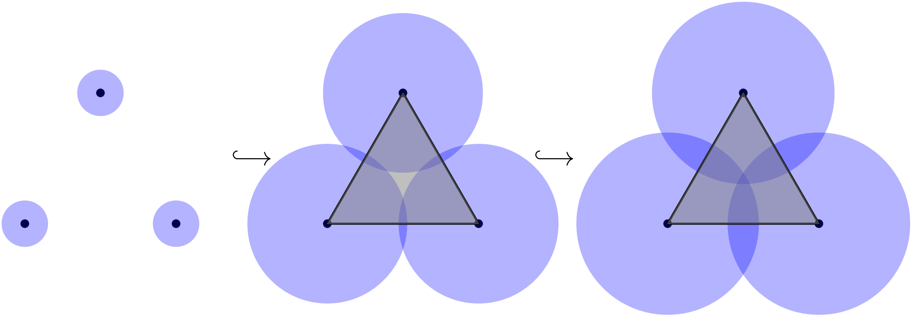
Relation of Čech and Vietoris-Rips Complex: For each $\epsilon > 0$, there is a chain inclusion maps

$$\mathcal{R} \hookrightarrow \mathcal{C}_{\epsilon\sqrt{2}} \hookrightarrow \mathcal{R}_{\epsilon\sqrt{2}}$$

Čech Complex Example



Vietoris-Rips Complex Example



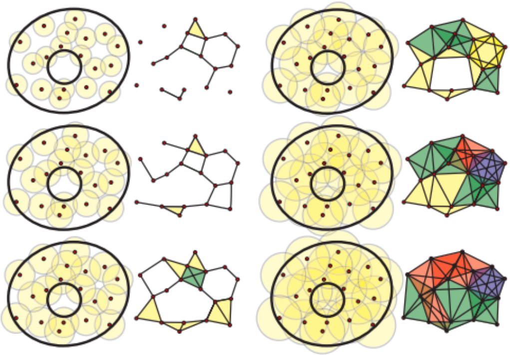


Figure 4: A sequence of Rips Complex from a point clound data set that represent an annulus

Definition

Given a filtered complex, the i-th complex K^i has associated boundary operators ∂_k^i, matrices M_k^i , and groups C_k^i, Z_k^i, B_k^i, and H_k^i for all $i, k \geq 0$
The p-persistent k-th homology group of K^i is

$$H_k^{i,p} = Z_k^i \Big/ (B_k^{i+p} \cap Z_k^i)$$

Example: p = i = k = 1: $H_1^{1,1} = Z_1^1 \,/ (B_2^1 \cap Z_1^1)\, \mathbb{Z} \,/ (\{0\} \cap \mathbb{Z}) = \mathbb{Z}$

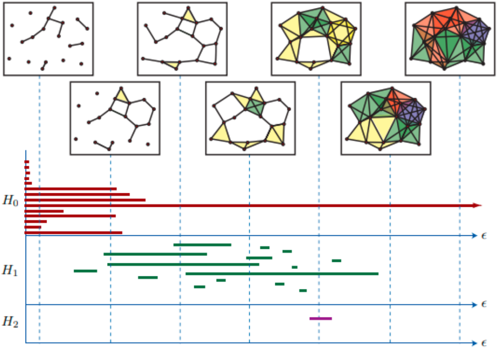


Figure 5: An example of barcode representations of the homology of the sampling of points in an annulus

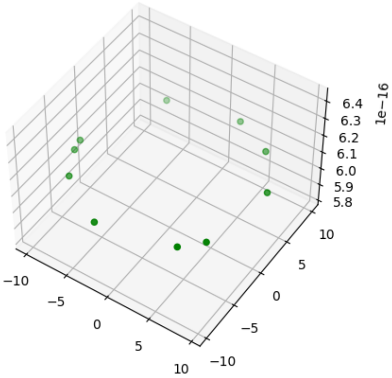


Figure 6: 10 points on a cicle

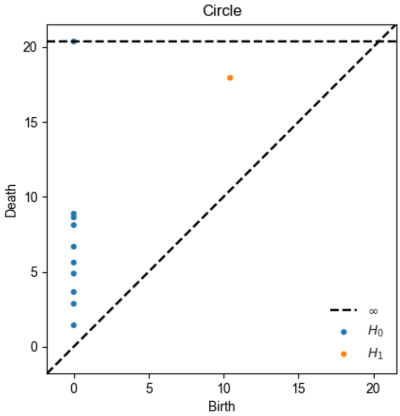


Figure 7: Persistent diagram of homology of circle (10 points)

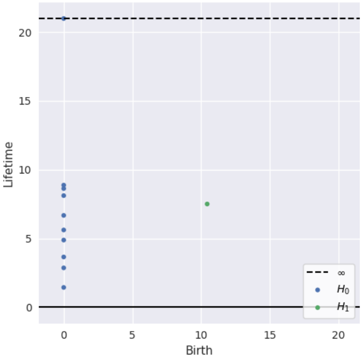


Figure 8: Lifetime diagram of homology of circle (10 points)

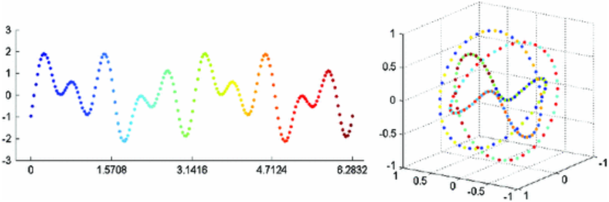


Figure 9: Peridicity in Timeseries

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