Persistent Homology and TDA

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Contents

Chapter 1

Chain Complexes And Simplicial Homology

1.1 Δ-complexes

Definition 1.1.1 (Standard Simplex)**.** *...*

Simplices in $\mathbb{R}^n$, ordering of the vertices and orientation

Definition 1.1.2 (Δ-complex)**.** *...*

Some explicit examples of Δ-complex structures on spaces. E.g., a closed interval $[0;1]$ $X = S^1$ with some *explicit* maps from Δ^1 (preferably several different ones) S^2 with some explicit maps. More examples on some quotient spaces, $S^1 \times S^1$, $\mathbb{RP}^2$, Klein bottle.

1.2 Chain Complexes

Definition 1.2.1 (Chain complex)**.** *Complex of abelian groups. Homology of a complex.*

As a remark: complex of R-modules, for a commutative ring R.

Chain complexes from a Δ-complex structure: definining the differential and checking the $\partial^2 = 0$ property.

1.3 Homology Calculations: Examples

S^1 with several different Δ-complex structures. An interval $[0;1]$.

1.3.1 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd}
v \arrow[r, "b"] \arrow[dr, phantom, "A", near start] & v \\
v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c"] \arrow[ur, phantom, "B"', near end] & v \arrow[u, "a"']
\end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{Im \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = <v>$ since $\partial_0 = 0$
$Im \partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v - v) + \beta(v - v) + \gamma(v - v) = 0$
$H_0 = \frac{\ker \partial_0}{Im \partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = C_1 = <a, b, c>$ since $\partial_1 = 0$
$Im \partial_2 = <a + b - c>$ since $\partial_2(\alpha A + \beta B) = \alpha(a + b - c) + \beta(a + b - c) = (\alpha + \beta)(a + b - c)$
$H_1 = \frac{\ker \partial_1}{Im \partial_2} = \frac{<a,b,c>}{<a+b-c>}$
The group $<a, b, c>$ can be also generated by the elements $m = a + b - c, b \; and \; c$ where $a = m - b + c$. So,
$H_1 = \frac{<a+b-c,b,c>}{<a+b-c>} = < b, c > \simeq \mathbb{Z} \oplus \mathbb{Z}$

Last, let's compute H_2:
$\ker \partial_2 = <A - B>$ since $\partial_2(\alpha A + \beta B) = (\alpha + \beta)(a + b - c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element A-B
$Im \partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{Im \partial_3} = \frac{<A-B>}{\{0\}} = <A - B> \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 3 \end{cases}$$

1.3.2 $\mathbb{RP}^2$

One way to calculate the homology groups of a projective plain $\mathbb{R}P^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} w & v \arrow[l, "b"'] \arrow[d, "a"] \\ v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c"] \arrow[u, phantom, "A", near end, shift right=4] & w \arrow[u, phantom, "B", near start, shift left=4] \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = <v, w> \\ C_1 = <a, b, c> \\ C_2 = <A, B> \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

The n-th homology group is defined as $H_n = \ker \partial_n / \mathrm{Im}\partial_n$

First, let's compute H_0:
$\ker \partial_0 = C_0 = <v, w>$ since $\partial_0 = 0$
$Im\partial_1 = <w - v>$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(w - v) + \beta(w - v) + \gamma(v - v)$
$= (\alpha + \beta)(w - v)$
$H_0 = \frac{\ker \partial_0}{Im\partial_1} = \frac{<v,w>}{<w-v>} = \frac{<w-v,w>}{<w-v>} = <w> \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = <a - b, c>$ since $\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements a-b and c
$Im\partial_2 = <-a + b + c,\ a - b + c>$ since $\partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$
$H_1 = \frac{\ker \partial_1}{Im\partial_2} = \frac{<a-b,\, c>}{<-a+b+c,\, a-b+c>}$
The group $< a - b, c >$ can be also generated by the elements $m = a - b + c, and\ c$ where $a - b = m - c$. So,
$H_1 = \frac{<a-b,\, c>}{<-a+b+c,\, a-b+c>} = \frac{<a-b+c,\, c>}{<a-b+c,\, -a+b+c>}$
If we let $t = a - b + c$ then $-a + b + c = -t + 2c$ then the group $<t,\ -t + 2c>$ can be also generated by the elements $t\ and\ 2c$.
In terms of t and c, $H_1 = \frac{<t,\, c>}{<t,\, 2c>} = \frac{<c>}{<2c>} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Last, let's compute H_2:
$\ker \partial_2 = \{0\}$ since $\partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0$ only when $\alpha = \beta = 0$
$Im\partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{Im\partial_3} = \frac{\{0\}}{\{0\}} = 0$

Finally, the homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{R}P^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$