Persistent Homology and TDA

Kejsi Jonuzaj

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Contents

Chapter 1

Chain Complexes And Simplicial Homology

1.1 Δ-complexes

Definition 1.1.1 (Standard Simplex - *n-simplex*). *The standard n-simplex is a subset of* $\mathbb{R}^{n+1}$ *given by*

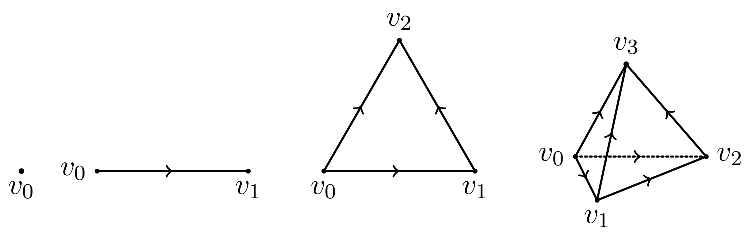
$$\Delta^n = \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \text{ and } t_i \geq 0 \,\forall\, i\}$$

[Hat02, 103] whose vertices are unit vectors along the coordiante axis.

... Simplices in $\mathbb{R}^n$, ordering of the vertices and orientation ...

An *n-simpex* is an *n-dimensional* analog of a triangle. A *n-simpex* is denoted by $[v_0, ..., v_n]$, where v_i's are the vertices of the simplex. To compute homology is important to define the order of the vertices in a simplex. Ordering the vertices of a simplex $[v_0, ..., v_n]$ determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts. Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard *n-simpex* Δ^n onto any *n-simpex* $[v_0, ..., v_n]$ preserving the order of vertices $(t_0, t_1, ..., t_n) \mapsto \sum_i t_i v_i$ in $[v_0, ..., v_n]$ [Hat02, 103].

In $\mathbb{R}^n$ a *0-simplex* is a point, a *1-simplex* is a line segment, a *2-simplex* is a triangle, *3-simplex* is a tetrahedron, as shown below.



The boundary $\partial\Delta^n$ is defined as the union of all the faces of Δ^n, and $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ denotes interior of Δ^n.

Definition 1.1.2 (Δ-complex). *A* $\Delta - complex$ *structure on a space X is a collection of maps* $\sigma_\alpha : \Delta^n \to X$ *, with n depending on the index* α*, such that:*

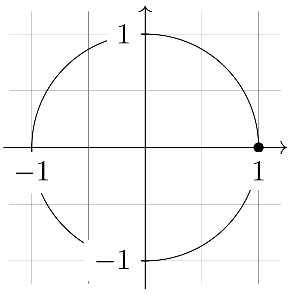
1. *The restriction* $\sigma_\alpha|\mathring{\Delta}^n$ *is injective, and each point of X is in the image of exactly one such restriction* $\sigma_\alpha|\mathring{\Delta}^n$*.*

2. *Each restriction of* σ_α *to a face of* Δ^n *is one of the maps* $\sigma_\beta : \Delta^{n-1} \to X$*. Here we are identifying the face of* Δ^n *with* Δ^{n-1} *by the canonical linear homeomorphism between them that preserves the ordering of the vertices.*

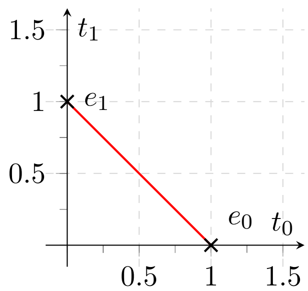
3. A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

.... Some explicit examples of Δ-complex structures on spaces. E.g., a closed interval $[0;1]$ $X = S^1$ with some *explicit* maps from Δ^1 (preferably several different ones) S^2 with some explicit maps. More examples on some quotient spaces, $S^1 \times S^1$, $\mathbb{RP}^2$, Klein bottle. ....

Example 1.1.1. Consider $\sigma_\alpha : \Delta^n \to X$ where $X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$



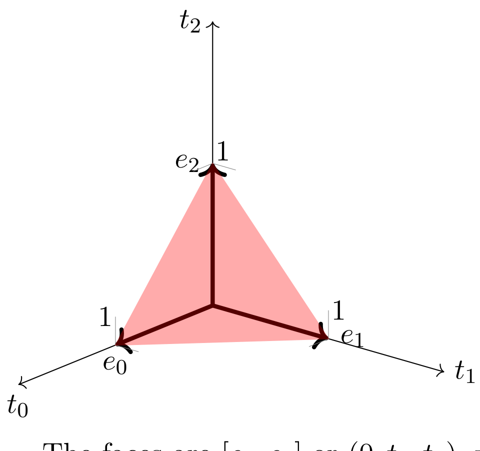
For n = 0, $\alpha = 0$: $\sigma_0 : \Delta^0 \to S^1$ where $\sigma_0(1) = (1, 0)$
For n = 1, $\alpha = 1$: In the figure below $t_0 + t_1 = 1$, $t_0, t_1 \geq 0$
$\Delta^1 = \{(t_0, 1 - t_0)\,,\, t_0 \in [0, 1]\} = [e_0, e_1]$



$\sigma_1 : \Delta^1 \to S^1$ where $t_0 \in [0, 1]$ where $\sigma_1(t_0) = (\cos(2\pi t_0), \sin(2\pi t_0))$
$\sigma|_{[e_0]} = \sigma_1(0) = (\cos(0), \sin(0)) = (1, 0)$
$\sigma|_{[e_1]} = \sigma_1(1) = (\cos(2\pi), \sin(2\pi)) = (1, 0)$

Example 1.1.2. Consider $\sigma_\alpha : \Delta^n \to X$ where $X = S^2 = \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 + z^2 = 1\}$

For n = 2, $\alpha = 2$: $\sigma_2 : \Delta^2 \to S^2$
$\Delta^1 = \{(t_0, t_1, t_2) \,|\, t_0 + t_1 + t_2 = 1,\, t_i \geq 0 \; for \, i = 1, 2, 3]\} = [e_0, e_1, e_2]$



The faces are $[e_1, e_2]$ or $(0, t_1, t_2)$, where $t_1 + t_2 = 1$,
$[e_0, e_2]$ or $(t_0, 0, t_2)$, where $t_0 + t_2 = 1$,
$[e_0, e_1]$ or $(t_0, t_1, 0)$, where $t_0 + t_1 = 1$.

$\sigma_2((t_0,t_1,t_2)) = \frac{(t_0,t_1,t_2)}{\sqrt{(t_0^2+t_1^2+t_2^2)}}$
$\sigma|_{[e_1,e_2]}(t_1,t_2) = \frac{(t_1,t_2)}{\sqrt{(t_1^2+t_2^2)}}$, $\sigma|_{[e_0,e_2]]}(t_0,t_2) = \frac{(t_0,t_2)}{\sqrt{(t_0^2+t_2^2)}}$, $\sigma|_{[e_0,e_1]}(t_1,t_2) = \frac{(t_0,t_1)}{\sqrt{(t_0^2+t_1^2)}}$,

1.2 Chain Complexes

Definition 1.2.1 (Chain complex)**.** *Complex of abelian groups. Homology of a complex. A chain complex is a sequencee of homomorphisms of abelian groups:*

$$... \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ *for each* n *in* $\mathbb{Z}$*. The equation* $\partial_n\partial_{n+1} = 0$ *is equivalent to the inclusion* $\operatorname{Im}\partial_{n+1} \subset \ker \partial_n$.

.... As a remark: complex of R-modules, for a commutative ring R.

Chain complexes from a Δ-complex structure: defininig the differential and checking the $\partial^2 = 0$ property. ....

The map ∂_n for a Δ-complex $\mathcal{X}$ is a boundary homomorphism $\partial_n : \Delta_n(\mathcal{X}) \to \Delta_{n-1}(\mathcal{X})$ where the action on a basis element of $\Delta_n(\mathcal{X})$ is:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[v_0, ..., \hat{v}_i, ..., v_n]$$

where the '^' symbol denotes the absence of that vertex.

Lemma 1.2.1. *The composition* $\partial^2 = 0$ *below is zero*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

Proof: For n = 3 :

$$\Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X)$$

Let us proof that $\partial_2\partial_3 = 0$:
$\partial_3\sigma = \sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, v_3] = \sigma|[v_1, v_2, v_3] - \sigma|[v_0, v_2, v_3] + \sigma|[v_0, v_1, v_3] - \sigma|[v_0, v_1, v_2]$

$$\begin{aligned} \partial_2\partial_3(\sigma) &= \sigma|[v_2, v_3] - \sigma|[v_1, v_3] + \sigma|[v_1, v_2] \\ &= -\sigma|[v_2, v_3] + \sigma|[v_0, v_3] - \sigma|[v_0, v_2] \\ &= \sigma|[v_1, v_3] - \sigma|[v_0, v_3] + \sigma|[v_0, v_1] \\ &= -\sigma|[v_1, v_2] + \sigma|[v_0, v_2] - \sigma|[v_0, v_1] = 0 \end{aligned} \tag{1.1}$$

In case of n:

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n]) \\ &= \sum_j (-1)^j (\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n])|[v_0, ..., \hat{v}_j, ..., v_n] \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma_|[v_0, ..., \hat{v}_j, ..., \hat{v}_i ..., v_n] + \sum_{j>i} (-1)^i (-1)^j \sigma_|[v_0, ..., \hat{v}_i, ..., \hat{v}_j ..., v_n] = 0 \end{aligned} \tag{1.2}$$

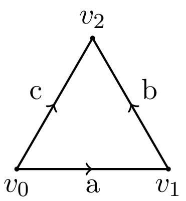
1.3 Homology Calculations: Examples

S^1 with several different Δ-complex structures. An interval $[0;1]$.

1.3.1 S^1

Method I: Triangulation

To compute the homolgy group of the circle S^1 we can triangulate the circle in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2=0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v_0, v_1, v_2\rangle \\ C_1 = \langle a, b, c\rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_2=0} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1, v_2\rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im}\partial_1$, let's compute $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v_1 - v_0) + \beta(v_2 - v_1) - \gamma(v_2 - v_0)$
$= (-\alpha + \gamma)v_0 + (\alpha - \beta)v_1 + (\beta - \gamma)v_2 = (\gamma - \alpha)v_0 + (\alpha - \beta)v_1 + (-(\gamma - \alpha) - (\alpha - \beta))v_2$

$$\operatorname{Im}\partial_1 = \left\{ \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ -(\gamma - \alpha) - (\alpha - \beta) \end{pmatrix}, \quad \alpha, \beta, \gamma \subseteq \mathbb{Z} \right\} \subseteq \mathbb{Z}^{\oplus 3}$$

Claim : There exist an isomorphism $\psi : \operatorname{Im}\partial_1 \simeq \mathbb{Z}^2$

$$\psi : \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \\ (\beta - \gamma) \end{pmatrix} \mapsto \begin{pmatrix} (\gamma - \alpha) \\ (\alpha - \beta) \end{pmatrix}$$

ψ is one-to-one since if $(\gamma - \alpha = 0 \;\&\; \alpha - \beta = 0) \Rightarrow \beta - \gamma = 0 \;\&\; \alpha = \beta = \gamma$

ψ is onto since given $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^2$ there exist an element $\begin{pmatrix} m \\ n \\ -m - n \end{pmatrix} \in \operatorname{Im}\partial_1$ such that

$\psi \begin{pmatrix} m \\ n \\ -m - n \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$, since ψ is one-to-one and onto, $\operatorname{Im}\partial_1 \simeq \mathbb{Z}^2$

$$H_0 = \frac{\ker \partial_0}{\operatorname{Im}\partial_1} = \mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$$

Claim: $\phi : \left(\mathbb{Z}^3 \Big/ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} \oplus \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z} \right) \simeq \mathbb{Z}$

First, let us take the map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3 \Big/ \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$\mathbb{Z}^3 \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (p+q+r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

where $p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathbb{Z}$

So, $\varphi : \begin{pmatrix} p \\ q \\ r \end{pmatrix} \mapsto \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} = (p+q+r) \overline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Finally, $\phi : \overline{\begin{pmatrix} p \\ q \\ r \end{pmatrix}} \mapsto (p+q+r) \in \mathbb{Z}$, Clearly, ϕ is injective and surjective.

So, $H_0 \simeq \mathbb{Z}$

Second, let's compute H_1:

$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$

$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$

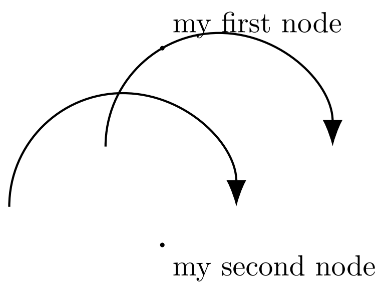
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Method II

To compute the homolgy group of the circle S^1 we can construct the circle, by two verteces and two edges, in the following way:



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_2 = 0} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left|\begin{array}{l} C_0 = \langle v_0, v_1 \rangle \\ C_1 = \langle a, b \rangle \\ C_n = \{0\} \quad \forall n \geqslant 2 \end{array}\right. ,$$

$$0 \xrightarrow{\partial_2 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v_0, v_1 \rangle$ since $\partial_0 = 0$
To calculate $\operatorname{Im} \partial_1$, let's compute $\partial_1(\alpha a + \beta b) = \alpha(v_1 - v_0) + \beta(v_1 - v_0) = (\alpha + \beta)(v_1 - v_0)$
$\operatorname{Im} \partial_1 = \langle v_1 - v_0 \rangle$
$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \frac{\langle v_1 - v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \langle v_1 \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \left\{ \begin{pmatrix} m \\ m \\ m \end{pmatrix}, m \in \mathbb{Z} = \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathbb{Z} \simeq \mathbb{Z}$
$\operatorname{Im} \partial_2 = \{0\}$ since $C_2 = \{0\}$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\ker \partial_1}{\{0\}} = \ker \partial_1 \simeq \mathbb{Z}$

Finally, the homology groups of the circle are:

$$H_n^{\Delta}(S^1) \simeq \left\{ \begin{array}{ll} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{array} \right.$$

1.3.2 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd}
v \arrow[r, "b"] \arrow[dr, phantom, "A", near start] & v \\
v \arrow[u, "a"] \arrow[ur, "c"] \arrow[r, "b"'] & v \arrow[u, "a"'] \arrow[ul, phantom, "B", near start]
\end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left|\begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array}\right. ,$$

$$0 \xrightarrow{\partial_3 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$.

First, let's compute H_0:
$\ker \partial_0 = C_0 = \langle v \rangle$ since $\partial_0 = 0$

$\operatorname{Im}\partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v-v) + \beta(v-v) + \gamma(v-v) = 0$
$H_0 = \frac{\ker\partial_0}{\operatorname{Im}\partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker\partial_1 = C_1 = \langle a,b,c\rangle$ since $\partial_1 = 0$
$\operatorname{Im}\partial_2 = \langle a+b-c\rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(a+b-c) + \beta(a+b-c) = (\alpha+\beta)(a+b-c)$
$H_1 = \frac{\ker\partial_1}{\operatorname{Im}\partial_2} = \frac{\langle a,b,c\rangle}{\langle a+b-c\rangle}$
The group $\langle a,b,c\rangle$ can be also generated by the elements $m = a+b-c, bandc$ where $a = m-b+c$.
So,
$H_1 = \frac{\langle a+b-c,b,c\rangle}{\langle a+b-c\rangle} = \langle b,c\rangle \simeq \mathbb{Z}\oplus\mathbb{Z}$

Last, let's compute H_2:
$\ker\partial_2 = \langle A-B\rangle$ since $\partial_2(\alpha A + \beta B) = (\alpha+\beta)(a+b-c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element A-B
$\operatorname{Im}\partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker\partial_2}{\operatorname{Im}\partial_3} = \frac{\langle A-B\rangle}{\{0\}} = \langle A-B\rangle \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^{\Delta}(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0,2 \\ \mathbb{Z}\oplus\mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 3 \end{cases}$$

1.3.3 $\mathbb{RP}^2$

One way to calculate the homology groups of a projective plain $\mathbb{RP}^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} w & v \arrow[l, "b"'] \arrow[d, "a"] \\ v \arrow[u, "a"] \arrow[r, "b"'] \arrow[ur, "c"] \arrow[ur, phantom, "A", shift left=3ex] \arrow[ur, phantom, "B"', shift right=3ex] & w \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n\partial_{n+1} = 0$ for each n in $\mathbb{Z}$ and

$$\left| \begin{array}{l} C_0 = \langle v,w\rangle \\ C_1 = \langle a,b,c\rangle \\ C_2 = \langle A,B\rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

The n-th homology group is defined as $H_n = \ker\partial_n / \operatorname{Im}\partial_n$

First, let's compute H_0:
$\ker\partial_0 = C_0 = \langle v,w\rangle$ since $\partial_0 = 0$
$\operatorname{Im}\partial_1 = \langle w-v\rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(w-v) + \beta(w-v) + \gamma(v-v)$
$= (\alpha+\beta)(w-v)$

$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\langle v,w \rangle}{\langle w-v \rangle} = \frac{\langle w-v,w \rangle}{\langle w-v \rangle} = \langle w \rangle \simeq \mathbb{Z}$

Second, let's compute H_1:
$\ker \partial_1 = \langle a-b, c \rangle$ since $\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w-v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a + \beta b + \gamma c) = \alpha(a-b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements a-b and c
$\operatorname{Im} \partial_2 = \langle -a+b+c,\, a-b+c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(-a+b+c) + \beta(a-b+c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle}$
The group $\langle a-b, c \rangle$ can be also generated by the elements $m = a-b+c, andc$ where $a-b = m-c$. So,
$H_1 = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle} = \frac{\langle a-b+c, c \rangle}{\langle a-b+c, -a+b+c \rangle}$
If we let $t = a-b+c$ then $-a+b+c = -t+2c$ then the group $\langle t, -t+2c \rangle$ can be also generated by the elements $t \, and \, 2c$.
In terms of t and c, $H_1 = \frac{\langle t, c \rangle}{\langle t, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Last, let's compute H_2:
$\ker \partial_2 = \{0\}$ since $\partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0$ only when $\alpha = \beta = 0$
$\operatorname{Im} \partial_3 = \{0\}$ since $C_3 = \{0\}$
$H_2 = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\{0\}}{\{0\}} = 0$

Finally, the homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

1.4 Maps of Complexes

1.4.1 Maps on Homology

Bibliography

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.