

Persistent Homology and TDA

Kejsi Jonuzai

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Chapter 1

Chain Complexes And Simplicial Homology

1.1 Δ -complexes

Definition 1.1.1 (Standard Simplex). ...

Simplices in \mathbb{R}^n , ordering of the vertices and orientation

Definition 1.1.2 (Δ -complex). ...

Some explicit examples of Δ -complex structures on spaces. E.g., a closed interval $[0; 1]$ $X = S^1$ with some *explicit* maps from Δ^1 (preferably several different ones) S^2 with some explicit maps. More examples on some quotient spaces, $S^1 \times S^1$, \mathbb{RP}^2 , Klein bottle.

1.2 Chain Complexes

Definition 1.2.1 (Chain complex). *Complex of abelian groups. Homology of a complex.*

As a remark: complex of R -modules, for a commutative ring R .

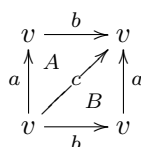
Chain complexes from a Δ -complex structure: defining the differential and checking the $\partial^2 = 0$ property.

1.3 Homology Calculations: Examples

S^1 with several different Δ -complex structures. An interval $[0; 1]$.

1.3.1 Torus

One way to calculate the homology groups of a torus T is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z} and

$$\left| \begin{array}{l} C_0 = \langle v \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geq 3 \end{array} \right. ,$$

The n -th homology group is defined as $H_n = \frac{\ker \partial_n}{\text{Im} \partial_{n+1}}$.

First, let's compute H_0 :

$\ker \partial_0 = C_0 = \langle v \rangle$ since $\partial_0 = 0$

$\text{Im} \partial_1 = \{0\}$ since $\partial_1(\alpha a + \beta b + \gamma c) = \alpha(v - v) + \beta(v - v) + \gamma(v - v) = 0$

$H_0 = \frac{\ker \partial_0}{\text{Im} \partial_1} = C_0 \simeq \mathbb{Z}$

Second, let's compute H_1 :

$\ker \partial_1 = C_1 = \langle a, b, c \rangle$ since $\partial_1 = 0$

$\text{Im} \partial_2 = \langle a + b - c \rangle$ since $\partial_2(\alpha A + \beta B) = \alpha(a + b - c) + \beta(a + b - c) = (\alpha + \beta)(a + b - c)$

$H_1 = \frac{\ker \partial_1}{\text{Im} \partial_2} = \frac{\langle a, b, c \rangle}{\langle a + b - c \rangle}$

The group $\langle a, b, c \rangle$ can be also generated by the elements $m = a + b - c, b$ and c where $a = m - b + c$. So,

$H_1 = \frac{\langle a + b - c, b, c \rangle}{\langle a + b - c \rangle} = \langle b, c \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$

Last, let's compute H_2 :

$\ker \partial_2 = \langle A - B \rangle$ since $\partial_2(\alpha A + \beta B) = (\alpha + \beta)(a + b - c) = 0 \implies \alpha = -\beta$ so the kernel is generated by the element $A - B$

$\text{Im} \partial_3 = \{0\}$ since $C_3 = \{0\}$

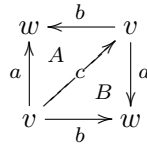
$H_2 = \frac{\ker \partial_2}{\text{Im} \partial_3} = \frac{\langle A - B \rangle}{\{0\}} = \langle A - B \rangle \simeq \mathbb{Z}$

Finally, the homology groups of the torus are:

$$H_n^\Delta(T) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$$

1.3.2 \mathbb{RP}^2

One way to calculate the homology groups of a projective plain \mathbb{RP}^2 is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z} and

$$\left| \begin{array}{l} C_0 = \langle v, w \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geq 3 \end{array} \right. ,$$

The n-th homology group is defined as $H_n = \ker \partial_n / \text{Im} \partial_n$

First, let's compute H_0 :

$$\ker \partial_0 = C_0 = \langle v, w \rangle \text{ since } \partial_0 = 0$$

$$\text{Im} \partial_1 = \langle w - v \rangle \text{ since } \partial_1(\alpha a + \beta b + \gamma c) = \alpha(w - v) + \beta(w - v) + \gamma(v - v)$$

$$= (\alpha + \beta)(w - v)$$

$$H_0 = \frac{\ker \partial_0}{\text{Im} \partial_1} = \frac{\langle v, w \rangle}{\langle w - v \rangle} = \frac{\langle w - v, w \rangle}{\langle w - v \rangle} = \langle w \rangle \simeq \mathbb{Z}$$

Second, let's compute H_1 :

$$\ker \partial_1 = \langle a - b, c \rangle \text{ since } \partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$$

The general element in C_1 : $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements $a - b$ and c

$$\text{Im} \partial_2 = \langle -a + b + c, a - b + c \rangle \text{ since } \partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$$

$$H_1 = \frac{\ker \partial_1}{\text{Im} \partial_2} = \frac{\langle a - b, c \rangle}{\langle -a + b + c, a - b + c \rangle}$$

The group $\langle a - b, c \rangle$ can be also generated by the elements $m = a - b + c$, and c where $a - b = m - c$. So,

$$H_1 = \frac{\langle a - b, c \rangle}{\langle -a + b + c, a - b + c \rangle} = \frac{\langle a - b + c, c \rangle}{\langle a - b + c, -a + b + c \rangle}$$

If we let $t = a - b + c$ then $-a + b + c = -t + 2c$ then the group $\langle t, -t + 2c \rangle$ can be also generated by the elements t and $2c$.

$$\text{In terms of } t \text{ and } c, H_1 = \frac{\langle t, c \rangle}{\langle t, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Last, let's compute H_2 :

$$\ker \partial_2 = \{0\} \text{ since } \partial_2(\alpha A + \beta B) = (-\alpha + \beta)a + (\alpha - \beta)b + (\alpha + \beta)c = 0 \text{ only when } \alpha = \beta = 0$$

$$\text{Im} \partial_3 = \{0\} \text{ since } C_3 = \{0\}$$

$$H_2 = \frac{\ker \partial_2}{\text{Im} \partial_3} = \frac{\{0\}}{\{0\}} = 0$$

Finally, the homology groups of the projective plane are:

$$H_n^\Delta(\mathbb{R}P^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$