

Persistent Homology and TDA

Senior Thesis

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December 13, 2020

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Standard Simplex - n -simplex

A n - simplex is denoted by $[v_0, v_1, \dots, v_n]$

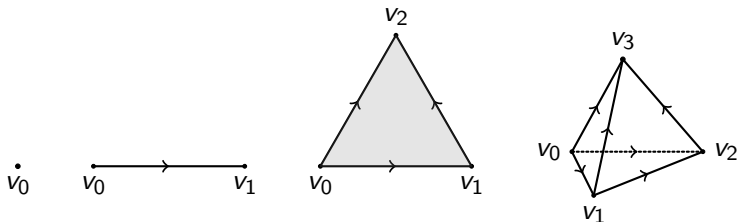


Figure 1: 0-simplex, 1-simplex, 2-simplex, 3-simplex

Δ – complex

Definition (Δ -complex)

A Δ – complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \rightarrow X$, with n depending on the index α , such that:

- 1 The restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$.
- 2 Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \rightarrow X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- 3 A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

Δ – complex – S^1

Consider $X = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. We are going to describe explicitly two maps $\sigma_0 : \Delta^0 \rightarrow S^1$, $\sigma_1 : \Delta^1 \rightarrow S^1$, which equip S^1 with the structure of a Δ -complex.

For the explicit description, keep in mind that $\Delta^0 = \{1\} \subseteq \mathbb{R}$ and that

$$\Delta^1 = \{(t_0, t_1) \mid t_0 + t_1 = 1\} = \{(t_0, 1 - t_0), t_0 \in [0, 1]\}$$

So in particular, any (continuous) map $\Delta^1 \rightarrow S^1$ is determined by and determines a (continuous) map $[0; 1] \rightarrow S^1$.

First, we define $\sigma_0 : \Delta^0 \rightarrow S^1$ by $\sigma_0(1) = (1, 0)$. Next, $\sigma_1 : \Delta^1 \rightarrow S^1$ is defined by $\sigma_1(t_0, t_1) = (\cos(2\pi t_0), \sin(2\pi t_0))$, which is clearly continuous. The map σ_1 is one-to-one on $\mathring{\Delta}^1$, and so is, trivially, σ_0 on $\mathring{\Delta}^0$. The images of the two maps cover the circle, with $\sigma_1(\mathring{\Delta}^1) = S^1 \setminus \{(1, 0)\}$ and $\sigma_0(\mathring{\Delta}^0) = \{(1, 0)\}$.

Finally, we check the compatibility property (2) of σ_1 and σ_0 .

The restrictions of σ_1 to the two faces of Δ^1 coincide with σ_0 :

$$\sigma_1|_{[e_1]} = \sigma_1(0, 1) = (\cos(0), \sin(0)) = (1, 0) = \sigma_0(1)$$

$$\sigma_1|_{[e_0]} = \sigma_1(1, 0) = (\cos(2\pi), \sin(2\pi)) = (1, 0) = \sigma_0(1)$$

Property (3) holds as well. There are numerous other Δ -complex structures on S^1 , and we shall discuss some of them later on.

Abstract Simplicial Complex

Definition (Abstract Simplicial Complex)

Given a finite set $\{1, 2, \dots, m\} (= [m])$ an *abstract simplicial complex* is a collection \mathcal{K} of subsets of $[m]$, such that:

- ① $\emptyset \in \mathcal{K}$
- ② $\{i\} \in \mathcal{K}$ (singletons are in \mathcal{K})
- ③ If $I \in \mathcal{K}$ and $J \subseteq I$, then $J \in \mathcal{K}$

The elements of $[m]$ are the *vertices*, and the elements of \mathcal{K} are the *simplices*, where $I \in \mathcal{K}$ is an $(|I| - 1)$ -simplex.

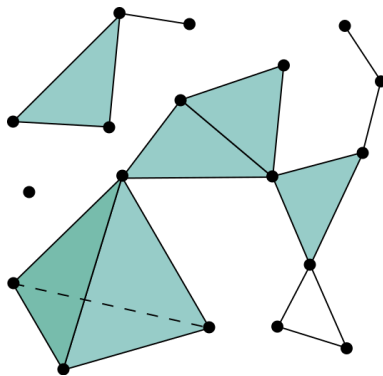
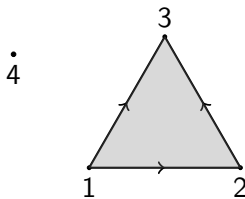


Figure 2: A simplicial 3-complex

Consider the following partially ordered set $V = \{1, 2, 3, 4\}$: The simplicial complex

$$\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$$



Given an abstract simplex \mathcal{K} , we can construct its *topological realization* as

$$|\mathcal{K}| = \bigcup_{\emptyset \neq I \in \mathcal{K}} (\text{Conv}(e_i), i \in I) \subseteq \mathbb{R}^m$$

where $\{e_i\}$ is in the standard basis $e_1, \dots, e_m \in \mathbb{R}^m$.

Definition (Chain complex)

Complex of abelian groups.

A chain complex is a sequence of homomorphisms of abelian groups:

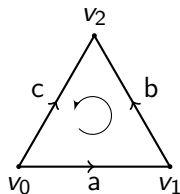
$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z} . The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\text{Im } \partial_{n+1} \subset \ker \partial_n$.

The boundary map $\partial_n : \Delta_n(\mathcal{X}) \rightarrow \Delta_{n-1}(\mathcal{X})$ for the would-be chain complex $\Delta_\bullet(X)$ is defined as

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|[\nu_0, \dots, \widehat{\nu_i}, \dots, \nu_n].$$

Boundary operator



$$\partial[v_0, v_1] = v_1 - v_0$$

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

Homology of a Chain Complex

Definition (Homology Group)

The n -th homology group of the chain complex is defined as the quotient group

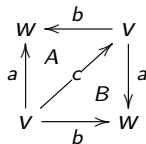
$$H_n = \frac{Z_n}{B_n} = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Elements of Z_n are called cycles and elements of B_n are called boundaries.

Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called homology classes. Two cycles representing the same homology class are said to be homologous. This means their difference is a boundary.

Computing Homology of \mathbb{RP}^2 in \mathbb{Z}

One way to calculate the homology groups of a projective plane \mathbb{RP}^2 is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.



We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3=0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in \mathbb{Z}

$$\left| \begin{array}{l} C_0 = \langle v, w \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geq 3 \end{array} \right. ,$$

$$0 \xrightarrow{\partial_3=0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0=0} 0$$

The n -th homology group is defined as $H_n = \ker \partial_n / \text{Im } \partial_n$

The homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Let's compute H_1 :

$\ker \partial_1 = \langle a - b, c \rangle$ since

$$\partial_1(\alpha a + \beta b + \gamma c) = (\alpha + \beta)(w - v) = 0 \implies \alpha = -\beta$$

The general element in C_1 : $(\alpha a + \beta b + \gamma c) = \alpha(a - b) + \gamma c$, so the $\ker \partial_1$ can be generated by the elements $a-b$ and c

$\text{Im } \partial_2 = \langle -a + b + c, a - b + c \rangle$ since

$$\partial_2(\alpha A + \beta B) = \alpha(-a + b + c) + \beta(a - b + c)$$

$$H_1 = \frac{\ker \partial_1}{\text{Im } \partial_2} = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle}$$

The group $\langle a - b, c \rangle$ can be also generated by the elements

$m = a - b + c$, and c where $a - b = m - c$. So,

$$H_1 = \frac{\langle a-b, c \rangle}{\langle -a+b+c, a-b+c \rangle} = \frac{\langle a-b+c, c \rangle}{\langle a-b+c, -a+b+c \rangle}$$

If we let $t = a - b + c$ then $-a + b + c = -t + 2c$ then the group $\langle t, -t + 2c \rangle$ can be also generated by the elements t and $2c$.

$$\text{In terms of } t \text{ and } c, H_1 = \frac{\langle t, c \rangle}{\langle t, 2c \rangle} = \frac{\langle c \rangle}{\langle 2c \rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$$

Maps of Chain Complexes

Let (C_\bullet, ∂) and (D_\bullet, δ) be two chain complexes. A map of chain complexes is a morphism f that is a sequence of homomorphisms $(f_n)_{n \in \mathbb{Z}}$:

$$\begin{array}{ccccccc}
 (C_\bullet, \partial) & C_\bullet & \dots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & \dots & C_\bullet \\
 & & & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} & & & \\
 (D_\bullet, \delta) & D_\bullet & \dots & \longrightarrow & D_n & \xrightarrow{\delta_n} & D_{n-1} & \xrightarrow{\delta_{n-1}} & D_{n-2} & \xrightarrow{\delta_{n-2}} & \dots & D_\bullet
 \end{array}$$

$$f_n : C_n \rightarrow D_n \quad \text{s.t.}, \quad f_{n-1} \circ \partial_n = \delta_n \circ f_n \quad \forall n \in \mathbb{Z}$$

$$\begin{array}{ccc}
 C_n & \xrightarrow{\partial_n} & C_{n-1} \\
 \downarrow f_n & & \downarrow f_{n-1} \\
 D_n & \xrightarrow{\delta_n} & D_{n-1}
 \end{array} \quad \text{commutes.}$$

Maps of Chain Complexes induce Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$H_n(f) : H_n(C_\bullet) \rightarrow H_n(D_\bullet)$$

$$H_n(f) : [x] \mapsto [f_n(x)]$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-definedness by checking if cycles are sent to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$\begin{aligned} \delta_n \circ f_n(x) &= f_{n-1} \circ \partial_n(x) = f_{n-1}(0) = 0 \Rightarrow f_n(x) \in \ker \delta_n, f_n(x) \text{ is a cycle} \\ &\Rightarrow f_n(\ker \partial_n) \subseteq \ker \delta_n \end{aligned}$$

So, cycles are sent to cycles.

(2) Let us take a boundary $y \in C_n$, so that $y \in \text{Im } \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

$$\begin{aligned} f_n(y) &= f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z)) \\ &\Rightarrow f_n(y) \in \text{Im } \partial_{n+1} f_n(y) \text{ is a boundary} \\ &\Rightarrow f_n(\text{Im } \partial_{n+1}) \subseteq \text{Im}(\delta_{n+1}) \end{aligned}$$

So, boundaries are sent to boundaries.

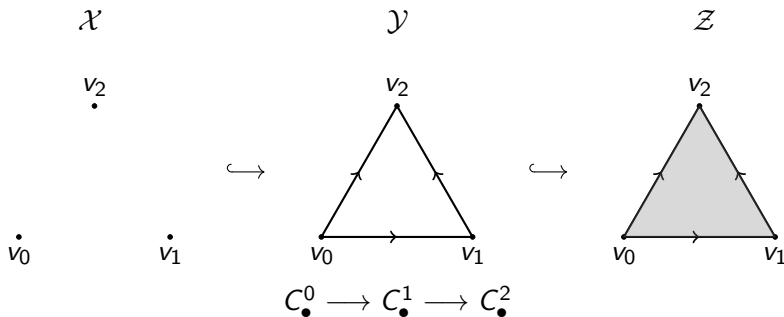
$$H_n(f) : H_n(C_\bullet) \rightarrow H_n(D_\bullet)$$

$$H_n(f) : \ker \partial_n / \text{Im}(\partial_{n+1}) \rightarrow \ker \delta_n / \text{Im}(\delta_{n+1})$$

$$[x] \mapsto [f_n(x)]$$

$$x + \text{Im } \partial_{n+1} \mapsto f_n(x) + f_n(\text{Im } \delta_{n+1}) = f_n(x) + \text{Im}(\delta_{n+1}) = [f_n(x)]$$

Filtered Complex



Maps of Complexes

$$C_{\bullet}^0 \longrightarrow C_{\bullet}^1 \longrightarrow C_{\bullet}^2$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 2 & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \\
 & \downarrow \partial_2^0=0 & & \downarrow \partial_2^1=0 & & \downarrow \partial_2^2=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} & \\
 1 & 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
 & \downarrow \partial_1^0=0 & & \downarrow \partial_1^1=\partial_1^2 & & \downarrow \partial_1^2=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \\
 0 & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^3 & \\
 & \downarrow \partial_0^0=0 & & \downarrow \partial_0^1=0 & & \downarrow \partial_0^2=0 & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Maps of Complexes induce maps on Homology

$$H(C_\bullet^0) \longrightarrow H(C_\bullet^1) \longrightarrow H(C_\bullet^2)$$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 2 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Exact Sequence

Consider a short exact sequence of chain complexes:

$$0 \rightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{\pi} C_{\bullet} \rightarrow 0$$

A_{\bullet} , B_{\bullet} , C_{\bullet} are chain complexes and i, π are maps between chain complexes where $\ker \pi = \operatorname{Im} i$, $\pi : \text{surjective}$ and $i : \text{injective}$ induces long exact sequence of homology:

$$H_{n+1}(C_{\bullet}) \xrightarrow{\partial_{n+1}} H_n(A_{\bullet}) \xrightarrow{i_*} H_n(B_{\bullet}) \xrightarrow{\pi_*} H_n(C_{\bullet}) \xrightarrow{\delta} H_{n-1}(A_{\bullet}) \rightarrow H_{n-1}(B_{\bullet}) \rightarrow H_{n-1}(C_{\bullet})$$

The Equivalence of Simplicial and Singular Homology

A singular n – *simplex* in a space X is a map $\sigma : \Delta^n \rightarrow X$

$C_n(X)$ is a free abelian group with generators the set of singular n – *simplexes* in X : the continuous maps $\sigma : \Delta^n \rightarrow X$.

Theorem

The homomorphisms $H_n^\Delta(X, A) \rightarrow H_n(X, A)$ are isomorphisms for all n and all Δ -complex pairs (X, A) .

Mayer-Vietoris sequence

In addition to the long exact sequence of homology groups for a pair (X, A) , there is another sort of long exact sequence, known as a Mayer-Vietoris sequence, which is equally powerful but is sometimes more convenient to use. For a pair of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B , this exact sequence has the form

$$\dots \rightarrow H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \rightarrow \dots \rightarrow H_0(X)$$

The Mayer-Vietoris sequence is then the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

$$0 \rightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \rightarrow 0$$

The Nerve Theorem

Consider X is a topological space, and $X = \bigcup_{\alpha} \mathcal{U}_{\alpha}$ where $\mathcal{U} = \{\mathcal{U}_{\alpha}\}_{\alpha \in A}$ are cover indexes.

Theorem (The Nerve Theorem)

If X is a paracompact space, and U is an open cover of X such that the intersection of any finite subfamily of U is either empty or contractible, then the realization of the nerve of U is homotopy equivalent to X .

If \mathcal{X} is a metric space and $r \geq 0$:

Definition (Čech Complex)

Given a collection of points $\{x_\alpha\}$ in Euclidean space \mathbb{E}^n , the Čech Complex, \mathcal{C}_ϵ , is the abstract simplicial complex whose k -simplices are determined by unordered $(k+1)$ -uple of points $\{x_\alpha\}_0^k$ whose closed $\epsilon/2$ - ball neighborhoods have a point of common intersection.

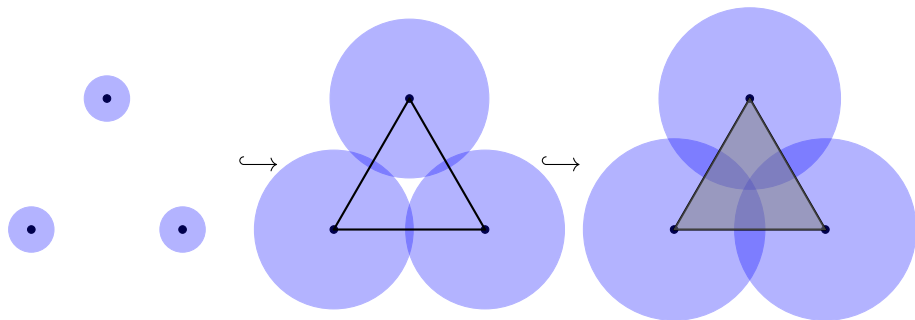
Definition (Vietoris-Rips Complex)

Given a collection of points $\{x_\alpha\}$ in Euclidean space \mathbb{E}^n , the Rips Complex, \mathcal{R}_ϵ , is the abstract simplicial complex whose k -simplices correspond to unordered $(k+1)$ -uple of points $\{x_\alpha\}_0^k$ which are pairwise within distance ϵ .

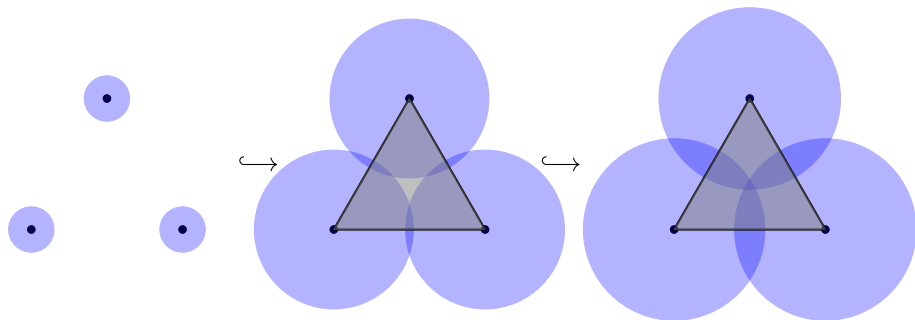
Relation of Čech and Vietoris-Rips Complex: For each $\epsilon > 0$, there is a chain inclusion maps

$$\mathcal{R} \hookrightarrow \mathcal{C}_{\epsilon\sqrt{2}} \hookrightarrow \mathcal{R}_{\epsilon\sqrt{2}}$$

Čech Complex Example



Vietoris-Rips Complex Example



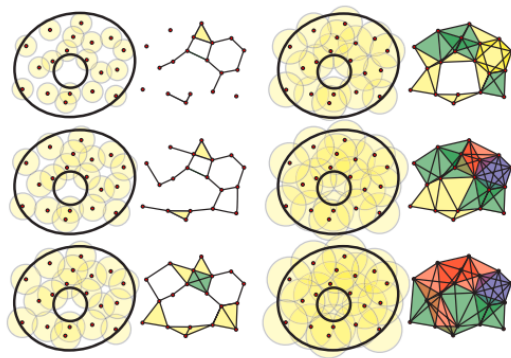


Figure 3: A sequence of Rips Complex from a point cloud data set that represent an annulus

Definition

Given a filtered complex, the i -th complex K^i has associated boundary operators ∂_k^i , matrices M_k^i , and groups C_k^i , Z_k^i , B_k^i , and H_k^i for all $i, k \geq 0$. The p -persistent k -th homology group of K^i is

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

Example: For $p = 0$, this is the usual homology formula:

$$H_k(C_\bullet^i) = Z_k^i / (B_k^i \cap Z_k^i) = Z_k^i / B_k^i$$

Definition (Persistence Complex)

A persistence complex \mathcal{C} is a family of chain complexes $\{C_*^i\}_{i \geq 0}$ over R , together with a chain map's $f_i : C_*^i \rightarrow C_*^{i+1}$ so that we have the following diagram:

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \dots$$

Definition (Persistence Module)

A persistence module \mathcal{M} is a family of R – *modules*, M^i , together with homomorphism $\varphi^i : M^i \rightarrow M^{i+1}$

Suppose we have a persistence module $\mathcal{M} = \{M^i, \varphi^i : M^i \rightarrow M^{i+1}\}$ over a ring R . We can equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by

$$\alpha(M) = \bigoplus_{i \geq 0} M_i$$

, where the R -module structure is the sum of the structures on the individual components, and where the action of t is given by:

$$t \cdot (m^0, m^1, \dots) = (0, \varphi^0(m^0), \varphi^1(m^1), \dots)$$

$$\begin{pmatrix} 0 & & & \\ \varphi^0 & 0 & & \\ & \varphi^1 & 0 & \\ & & \varphi^2 & \end{pmatrix}$$

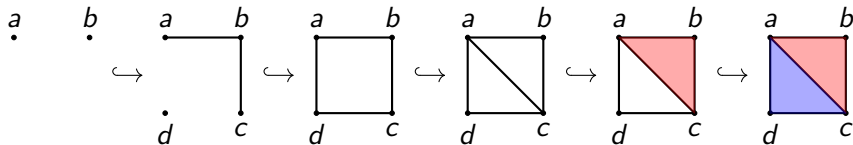
t simply shifts elements of the module up in gradation.

Theorem (Correspondence)

The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated non-negatively graded modules over $R[t]$.

The Correspondence theorem gives us a simple decomposition when the ground ring is a field F . In this case the graded ring $F[t]$ is a PID and its only graded ideals are homogeneous of form (t^n) , so the structure of the $F[t]$ – *module* is described in structure theorem:

$$(\bigoplus_i \Sigma^{\alpha_i} F[t]) \oplus (\bigoplus_j \Sigma^{\gamma_j} F[t]/(t^{n_j})). \quad (1)$$

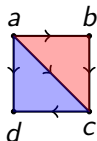


0 1 2 3 4 5

$$C_{\bullet}^0 \longrightarrow C_{\bullet}^1 \longrightarrow C_{\bullet}^2 \longrightarrow C_{\bullet}^3 \longrightarrow C_{\bullet}^4 \longrightarrow C_{\bullet}^5$$

2	0	\longrightarrow	0	\longrightarrow	0	\longrightarrow	0	\longrightarrow	\mathbb{Z}	\longrightarrow	\mathbb{Z}^2
	\downarrow		\downarrow		\downarrow		\downarrow		\downarrow		\downarrow
1	0	\longrightarrow	\mathbb{Z}^2	\longrightarrow	\mathbb{Z}^4	\longrightarrow	\mathbb{Z}^5	\longrightarrow	\mathbb{Z}^5	\longrightarrow	\mathbb{Z}^5
	\downarrow		\downarrow		\downarrow		$\downarrow \partial_1^3$		\downarrow		\downarrow
0	\mathbb{Z}^2	\longrightarrow	\mathbb{Z}^4	\longrightarrow	\mathbb{Z}^4	\longrightarrow	\mathbb{Z}^4	\longrightarrow	\mathbb{Z}^4	\longrightarrow	\mathbb{Z}^4
	\downarrow		\downarrow		\downarrow		\downarrow		\downarrow		\downarrow
	0		0		0		0		0		0

In particular, consider the last simplicial complex in the filtration:



The chain complex is $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^5 \xrightarrow{\partial_1} \mathbb{Z}^4 \rightarrow 0$. The matrices M_2 and M_1 of ∂_2 and ∂_1 without the indicated bases are:

$$M_2 = \begin{matrix} & \begin{pmatrix} abc & acd \end{pmatrix} \\ \begin{matrix} ab \\ bc \\ cd \\ ad \\ ac \end{matrix} & \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \end{matrix}, M_1 = \begin{matrix} & \begin{pmatrix} ab & bc & cd & ad & ac \end{pmatrix} \\ \begin{matrix} d \\ c \\ b \\ a \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix} \end{matrix}$$

Then ∂_1 (ie M_1) induces a map of $\mathbb{Z}[t]$ - modules $\mathbb{Z}[t]^{\oplus 5} \xrightarrow{M_1} \mathbb{Z}[t]^{\oplus 4}$:

$$\underline{V} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{pmatrix} \mapsto M_1 \underline{V}.$$

Using the grading (filtration) on the simplicial complex we get a chain complex of graded $\mathbb{Z}[t]$ - modules

$$0 \rightarrow (t^4) \oplus (t^5) \xrightarrow{\partial_2^1} (t)^{\oplus 2} \oplus (t^2)^{\oplus 2} \oplus (t^3) \xrightarrow{\partial_1^1} (t)^{\oplus 2} \oplus (1)^{\oplus 2} \rightarrow 0$$

by the procedure defined on. The respective matrices are (without same bases)

$$M'_2 = \begin{matrix} ab \\ bc \\ cd \\ ad \\ ac \end{matrix} \begin{pmatrix} abc & acd \\ t^3 & 0 \\ t^3 & 0 \\ 0 & t^3 \\ 0 & -t^3 \\ -t^3 & t^3 \end{pmatrix}, M'_1 = \begin{matrix} d \\ c \\ b \\ a \end{matrix} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & t & t & 0 \\ 0 & 1 & -t & 0 & t^2 \\ t & -t & 0 & 0 & 0 \\ -t & 0 & 0 & -t^2 & -t^2 \end{pmatrix}$$

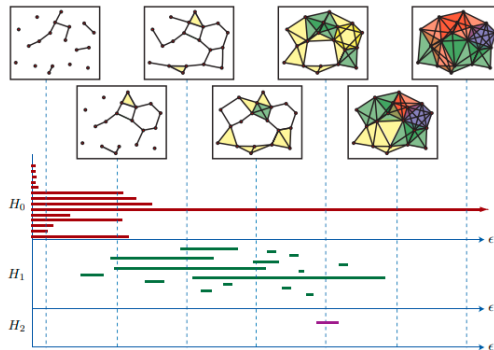


Figure 4: An example of barcode representations of the homology of the sampling of points in an annulus

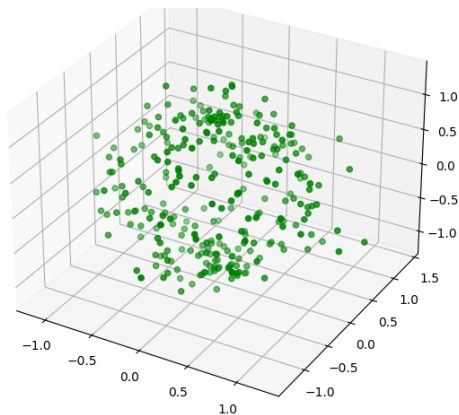


Figure 5: Points on a sphere with noise

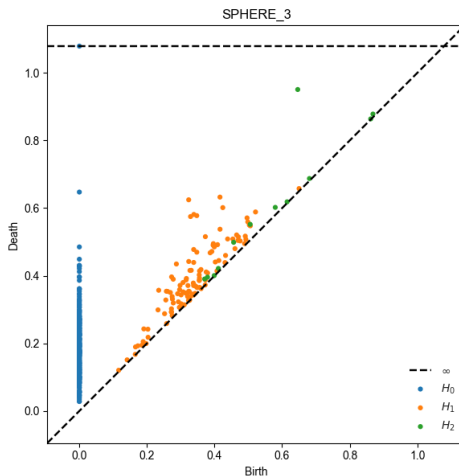


Figure 6: Persistent diagram of homology of sphere (300 points)

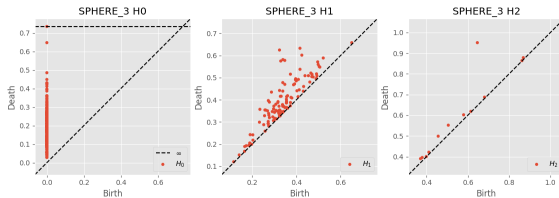


Figure 7: Persistent diagram of homology of sphere, separate (300 points)

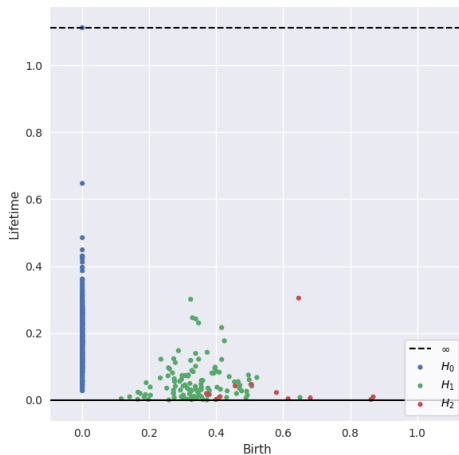


Figure 8: Lifetime persistence

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