Persistent Homology and TDA

Senior Thesis

American University in Bulgaria
Departament of Mathematics and Science

Supervisor: Prof. Peter Dalakov

December 11, 2020

Outline

Standard Simplex - *n-simplex*

A $n - simplex$ is denoted by $[v_0, v_1, ..., v_n]$

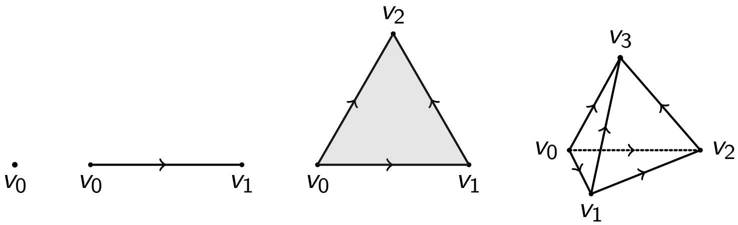


Figure 1: 0-simplex, 1-simpex, 2-simplex, 3-simplex

$\Delta - complex$

Definition (Δ-complex)

A $\Delta - complex$ structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha|\mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\mathring{\Delta}^n$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
3. A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

$\Delta - complex - S^1$

Consider $X = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$. We are going to describe explicitly two maps $\sigma_0 : \Delta^0 \to S^1$, $\sigma_1 : \Delta^1 \to S^1$, which equip S^1 with the structure of a Δ-complex.
For the explicit description, keep in mind that $\Delta^0 = \{1\} \subseteq \mathbb{R}$ and that

$$\Delta^1 = \{(t_0, t_1) \,|\, t_0 + t_1 = 1\} = \{(t_0, 1 - t_0)\,,\ t_0 \in [0, 1]\}$$

So in particular, any (continuous) map $\Delta^1 \to S^1$ is determined by and determines a (continuous) map $[0; 1] \to S^1$.

First, we define $\sigma_0 : \Delta^0 \to S^1$ by $\sigma_0(1) = (1, 0)$. Next, $\sigma_1 : \Delta^1 \to S^1$ is defined by $\sigma_1(t_0, t_1) = (\cos(2\pi t_0), \sin(2\pi t_0))$, which is clearly continuous. The map σ_1 is one-to-one on $\mathring{\Delta}^1$, and so is, trivially, σ_0 on $\mathring{\Delta}^0$. The images of the two maps cover the circle, with $\sigma_1\left(\mathring{\Delta}^1\right) = S^1 \backslash \{(1, 0)\}$ and $\sigma_0\left(\mathring{\Delta}^0\right) = \{(1, 0)\}$.

Finally, we check the compatibility property (2) of σ_1 and σ_0.
The restrictions of σ_1 to the two faces of Δ^1 coincide with σ_0:

$$\sigma_1|_{[e_1]} = \sigma_1(0, 1) = (\cos(0), \sin(0)) = (1, 0) = \sigma_0(1)$$

$$\sigma_1|_{[e_0]} = \sigma_1(1, 0) = (\cos(2\pi), \sin(2\pi)) = (1, 0) = \sigma_0(1)$$

Property (3) holds as well. There are numerous other Δ-complex structures on S^1, and we shall discuss some of them later on.

Abstract Simplicial Complex

Definition (Abstract Simplicial Complex)

Given a finite set $\{1, 2, \cdots, m\}(= [m])$ an *abstract simplicial complex* is a collection $\mathcal{K}$ of subsets of $[m]$, such that:

1. $\emptyset \in \mathcal{K}$
2. $\{i\} \in \mathcal{K}$ (singletons are in $\mathcal{K}$)
3. If $I \in \mathcal{K}$ and $J \subseteq I$, then $\mathcal{J} \in \mathcal{K}$

The elements of $[m]$ are the *vertices*, and the elements of $\mathcal{K}$ are the *simplices*, where $I \in \mathcal{K}$ is an $(|I| - 1)$-simplex.

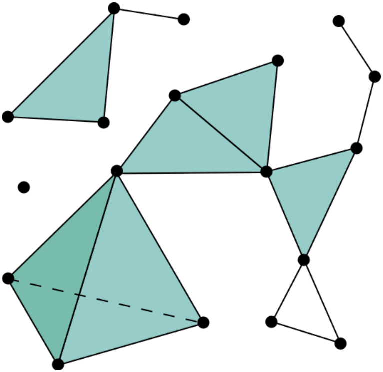
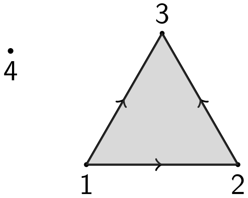


Figure 2: A simplicial 3-complex

Consider the following partially ordereed set $V = \{1, 2, 3, 4\}$: The simplicial complex
$\mathcal{K} = \{I = \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$



Given an abstract simplex $\mathcal{K}$, we can construct its *topological realization* as

$$|\mathcal{K}| = \bigcup_{\varnothing \neq I \in \mathcal{K}} (\mathrm{Conv}(e_i), i \in I) \subseteq \mathbb{R}^m$$

where $\{e_i\}$ is in the standard basis $e_1, \cdots e_m \in \mathbb{R}^m$.

Definition (Chain complex)

Complex of abelian groups.
A chain complex is a sequence of homomorphisms of abelian groups:

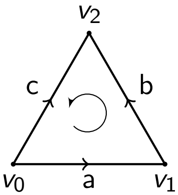
$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0=0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$. The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$.

The boundary map $\partial_n : \Delta_n(\mathcal{X}) \to \Delta_{n-1}(\mathcal{X})$ for the would-be chain complex $\Delta_\bullet(X)$ is defined as

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \dots, \widehat{v_i}, \dots, v_n].$$

Boundary operator



$$\partial[v_0, v_1] = v_1 - v_0$$

$$\partial[v_0, v_1, v_2] = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$$

Homology of a Chain Complex

Definition (Homology Group)

The n-th homology group of the chain complex is defined as the quotient group

$$H_n = \frac{Z_n}{B_n} = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Elements of Z_n are called cycles and elements of B_n are called boundaries.

Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called homology classes. Two cycles representing the same homology class are said to be homologous. This means their difference is a boundary.

Computing Homology of $\mathbb{RP}^2$ in $\mathbb{Z}$

One way to calculate the homology groups of a projective plain $\mathbb{RP}^2$ is by triangulating it into two 2-simplices A and B, upper triangle and lower one respectively.

$$\begin{tikzcd} w & v \arrow[l, "b"'] \arrow[d, "a"] \arrow[dl, phantom, "A" near end, "B" near start] \\ v \arrow[u, "a"] \arrow[ur, "c"] \arrow[r, "b"'] & w \end{tikzcd}$$

We can construct the following chain complex which is a sequence of homomorphisms of abelian groups:

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n in $\mathbb{Z}$

$$\left|\begin{array}{l} C_0 = \langle v, w \rangle \\ C_1 = \langle a, b, c \rangle \\ C_2 = \langle A, B \rangle \\ C_n = \{0\} \quad \forall n \geqslant 3 \end{array}\right. ,$$

$$0 \xrightarrow{\partial_3 = 0} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0 = 0} 0$$

The n-th homology group is defined as $H_n = \ker \partial_n / \operatorname{Im} \partial_n$
The homology groups of the projective plane are:

$$H_n^{\Delta}(\mathbb{RP}^2) \simeq \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

Let's compute H_1:
$\ker \partial_1 = \langle a-b, c\rangle$ since
$\partial_1(\alpha a+\beta b+\gamma c) = (\alpha+\beta)(w-v) = 0 \implies \alpha = -\beta$
The general element in C_1: $(\alpha a+\beta b+\gamma c) = \alpha(a-b)+\gamma c$, so the $\ker \partial_1$ can be generated by the elements a-b and c
$\operatorname{Im} \partial_2 = \langle -a+b+c,\ a-b+c\rangle$ since
$\partial_2(\alpha A+\beta B) = \alpha(-a+b+c)+\beta(a-b+c)$
$H_1 = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\langle a-b, c\rangle}{\langle -a+b+c, a-b+c\rangle}$
The group $\langle a-b, c\rangle$ can be also generated by the elements
$m = a-b+c, and\ c$ where $a-b = m-c$. So,
$H_1 = \frac{\langle a-b, c\rangle}{\langle -a+b+c, a-b+c\rangle} = \frac{\langle a-b+c, c\rangle}{\langle a-b+c, -a+b+c\rangle}$
If we let $t = a-b+c$ then $-a+b+c = -t+2c$ then the group $\langle t,\ -t+2c\rangle$ can be also generated by the elements $t\ and\ 2c$.
In terms of t and c, $H_1 = \frac{\langle t, c\rangle}{\langle t, 2c\rangle} = \frac{\langle c\rangle}{\langle 2c\rangle} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$

Maps of Chain Complexes

Let $(C_\bullet, \partial)$ and $(D_\bullet, \delta)$ be two chain complexes. A map of chain complexes is a morphism f that is a sequence of homomorphisms $(f_n)_{n \in Z}$:

$$\begin{tikzcd}
(C_\bullet, \partial) & C_\bullet \; \dots \arrow[r] & C_n \arrow[r, "\partial n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial n-1"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r, "\partial n-2"] \arrow[d, "f_{n-2}"] & \dots \; C_\bullet \\
(D_\bullet, \delta) & D_\bullet \; \dots \arrow[r] & D_n \arrow[r, "\delta n"] & D_{n-1} \arrow[r, "\delta n-1"] & D_{n-2} \arrow[r, "\delta n-2"] & \dots \; D_\bullet
\end{tikzcd}$$

$$f_n : C_n \to D_n \quad s.t, \quad f_{n-1} \circ \partial n = \delta_n \circ f_n \; \forall n \in \mathbb{Z}$$

$$\begin{tikzcd}
C_n \arrow[r, "\partial n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "\delta n"] & D_{n-1}
\end{tikzcd} \quad \textit{commutes.}$$

Maps of Chain Complexes induce Maps on Homology

A homomorphism of chain complexes induces a homomorphism on the homology. The induced map can be defined as:

$$H_n(f) : H_n(C_\bullet) \to H_n(D_\bullet)$$
$$H_n(f) : [x] \mapsto [f_n(x)]$$

To prove the claim above it is enough to check that $H_n(f)$ is well-defined. We can prove well-defines by checking if cycles are send to cycles and boundaries to boundaries.

(1) Let us take a cycle $x \in C_n$, so that $x \in \ker(\partial_n)$, $\partial_n(x) = 0$

$$\begin{aligned} \delta_n \circ f_n(x) = f_{n-1} \circ \partial n(x) = f_{n-1}(0) = 0 &\Rightarrow f_n(x) \in \ker \delta_n, f_n(x) \textit{ is a cycle} \\ &\Rightarrow f_n(\ker \partial n) \subseteq \ker \delta_n \end{aligned}$$

So, cycles are send to cycles.

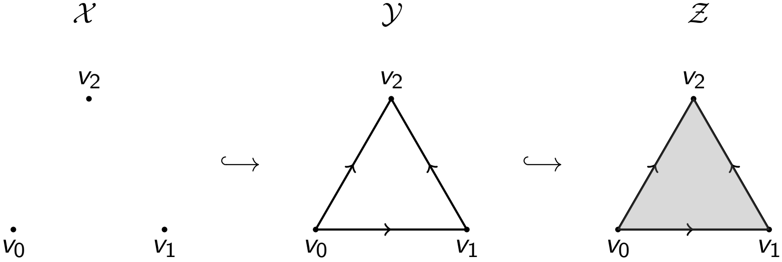
(2) Let us take a boundary $y \in C_n$, so that $y \in \operatorname{Im} \partial_{n+1} \Rightarrow \exists z \in C_{n+1}$ such that $\partial_{n+1}(z) = y$

$$
\begin{aligned}
& f_n(y) = f_n(\partial_{n+1}(z)) = \delta_{n+1}(f_{n+1}(z)) \\
& \Rightarrow f_n(y) \in \operatorname{Im} \partial_{n+1} f_n(y) \textit{ is a boundary} \\
& \Rightarrow f_n(Im\partial_{n+1}) \subseteq Im(\delta_{n+1})
\end{aligned}
$$

So, boundaries are send to boundaries.

$$
\begin{aligned}
H_n(f) &: H_n(C_\bullet) \to H_n(D_\bullet) \\
H_n(f) &: \ker \partial_n / \operatorname{Im}(\partial_{n+1}) \to \ker \delta_n / \operatorname{Im}(\delta_{n+1}) \\
& [x] \mapsto [f_n(x)] \\
& x + \operatorname{Im} \partial_{n+1} \mapsto f_n(x) + f_n(\operatorname{Im} \delta_{n+1}) = f_n(x) + Im(\delta_{n+1}) = [f_n(x)]
\end{aligned}
$$

Filtered Complex



$$C_\bullet^0 \longrightarrow C_\bullet^1 \longrightarrow C_\bullet^2$$

Maps of Complexes

$$
\begin{tikzcd}
C^0_\bullet \arrow[r] & C^1_\bullet \arrow[r] & C^2_\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d, "\partial_2^0=0"] & 0 \arrow[r] \arrow[d, "\partial_2^1=0"] & \mathbb{Z} \arrow[d, "\partial_2^2=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}"] \\
1 & 0 \arrow[r] \arrow[d, "\partial_1^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_1^1=\partial_1^2"] & \mathbb{Z}^3 \arrow[d, "\partial_1^2=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}"] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^0=0"] & \mathbb{Z}^3 \arrow[r] \arrow[d, "\partial_0^1=0"] & \mathbb{Z}^3 \arrow[d, "\partial_0^2=0"] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

Maps of Complexes induce maps on Homology

$$
\begin{tikzcd}
 & H(C^0_\bullet) \arrow[r] & H(C^1_\bullet) \arrow[r] & H(C^2_\bullet) \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

Exact Sequence

Consider a short exact sequence of chain complexes:

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$

$A_\bullet,\ B_\bullet,\ C_\bullet$ are chain complexes and i, π are maps between chain complexes where $\ker \pi = \operatorname{Im} i, \quad \pi$: *surjective and* i : *injective* induces long exact sequence of homology:

$$H_{n+1}(C_\bullet) \xrightarrow{\partial n+1} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(C_\bullet)$$

The Equivalence of Simplicial and Singular Homology

A singular $n-simplex$ in a space X is a map $\sigma : \Delta^n \to X$

$C_n(X)$ is a free abelian group with generators the set of singluar $n-simplexes$ in X: the continous maps $\sigma : \Delta^n \to X$.

Theorem

The homomorphisms $H_n^{\Delta}(X, A) \to H_n(X, A)$ are isomorphisms for all n and all Δ-complex pairs (X, A).

Mayer-Vietoris sequence

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a Mayer-Vietoris sequence, which is equally powerful but is sometimes more convenient to use. For a pair of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B , this exact sequence has the form

$$... \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to ... \to H_0(X)$$

The Mayer-Vietoris sequence is then the long exact sequence of homology groups associated to the short exact sequence of chain complexes formed by the short exact sequences

$$0 \to C_n(A \cap B) \xrightarrow{\Phi} C_n(A) \oplus C_n(B) \xrightarrow{\Psi} C_n(A + B) \to 0$$

The Nerve Theorem

Consider X is a topological space, and $X = \bigcup\limits_{\alpha} \mathcal{U}_\alpha$ where $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ are cover indexes.

Theorem (The Nerve Theorem)

If X is a paracompact space, and U is an open cover of X such that the intersection of any finite subfamily of U is either empty or contractible, then the realization of the nerve of U is homotopy equivalent to X.

If $\mathcal{X}$ is a metric space and $r \geq 0$:

Definition (Čech Complex)

Given a collection of points $\{x_\alpha\}$ in Euclidean space $\mathbb{E}^n$, the Čech Complex, $\mathcal{C}_\epsilon$, is the abstract simplicial complex whose k-simplices are determined by unordered (k+1)-uple of points $\{x_\alpha\}_0^k$ whose closed $\epsilon/2 - \mathit{ball}$ neighborhoods have a point of common intersection.

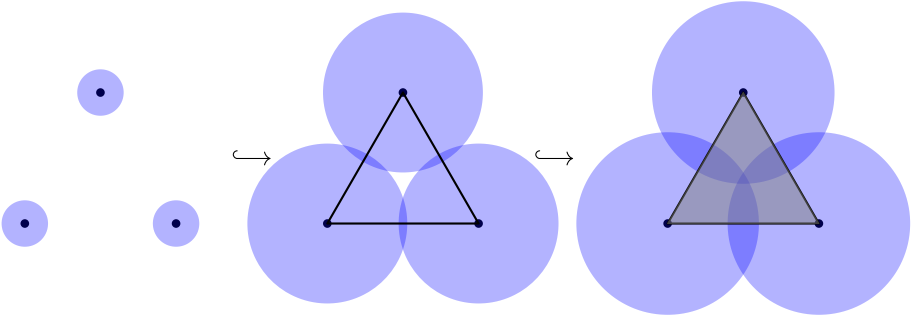
Definition (Vietoris-Rips Complex)

Given a collection of points $\{x_\alpha\}$ in Euclidean space $\mathbb{E}^n$, the Rips Complex, $\mathcal{R}_\epsilon$, is the abstract simplicial complex whose k-simplices correspond to unordered (k+1)-uple of points $\{x_\alpha\}_0^k$ which are pairwise within distance ϵ.

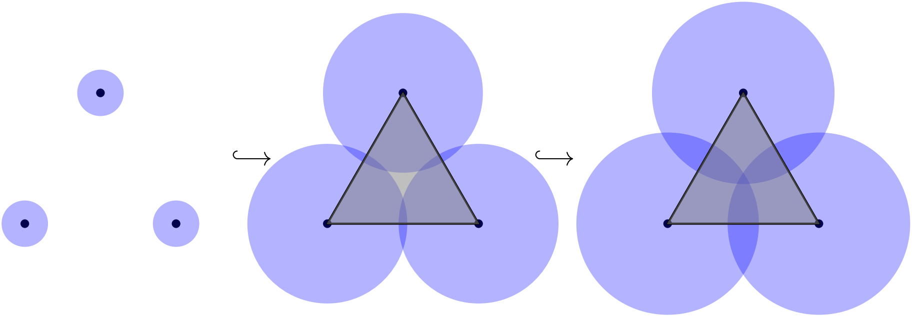
Relation of Čech and Vietoris-Rips Complex: For each $\epsilon > 0$, there is a chain inclusion maps

$$\mathcal{R} \hookrightarrow \mathcal{C}_{\epsilon\sqrt{2}} \hookrightarrow \mathcal{R}_{\epsilon\sqrt{2}}$$

Čech Complex Example



Vietoris-Rips Complex Example



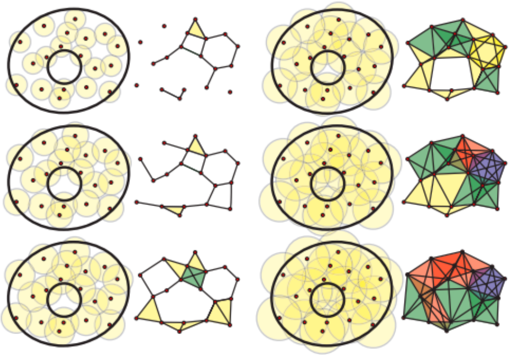


Figure 3: A sequence of Rips Complex from a point clound data set that represent an annulus

Definition

Given a filtered complex, the i-th complex K^i has associated boundary operators ∂^{i_k}, matrices M_k^i , and groups C_k^i, Z_k^i, B_k^i, and H_k^i for all $i, k \geq 0$ The p-persistent k-th homology group of K^i is

$$H_k^{i,p} = Z_k^i \Big/ (B_k^{i+p} \cap Z_k^i)$$

Example: p = i = k = 1: $H_1^{1,1} = Z_1^1 / (B_2^1 \cap Z_1^1) \, \mathbb{Z} / (\{0\} \cap \mathbb{Z}) = \mathbb{Z}$

Definition (Persistence Complex)

A persistence complex $\mathcal{C}$ is a family of chain complexes $\{C_*^i\}_{i\geq 0}$ over R, together with a chain map's $f_i : C_*^i \to C_*^{i+1}$ so that we have the following diagram:

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \dots$$

Definition (Persistence Module)

A persistence module $\mathcal{M}$ is a family of $R-modules$, M^i, together with homomorphism $\varphi^i : M^i \to M^{i+1}\}$

Suppose we have a persistence module $\mathcal{M} = \{M^i, \varphi^i : M^i \to M^{i+1}\}$ over a ring R, We can equip $R[t]$ with the standard grading and define a graded module over $R[t]$ by

$$\alpha(M) = \bigoplus_{i \geq 0} M_i$$

, where the $R - module$ structure is the sum of the structures on the individual components, and where the action of t is given by:

$$t \cdot (m^0, m^1, \cdots) = (0, \varphi^0(m^0), \varphi^1(m^1), \cdots)$$

$$\left(\begin{array}{c|c|c|c} 0 & & & \\ \hline \varphi^0 & 0 & & \\ \hline & \varphi^1 & 0 & \\ \hline & & \varphi^2 & \end{array}\right)$$

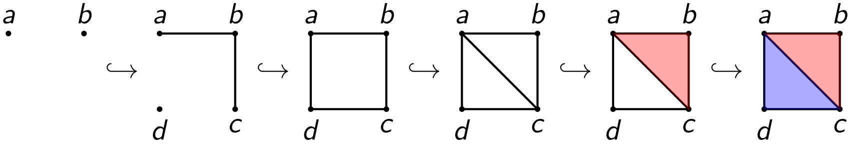
t simply shift elements of the module up in gradation.

Theorem (Correspondence)

The correspondence α defines an equivalence of categories between the category of persistence modules of finite type over R and the category of finitely generated non-negatively graded modules over $R[t]$.

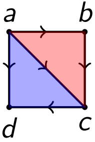
The Correspondence theorem gives us a simple decomposition when the ground ring is a field F. In this case the graded ring $F[t]$ is a PID and its only graded ideals are homogeneous of form (t_n), so the structure of the $F[t] - module$ is described in structure theorem:

$$(\bigoplus_i \Sigma^{\alpha_i} F[t]) \oplus (\bigoplus_j \Sigma^{\gamma_j} F[t]/(t^{n^j})). \tag{1}$$



$$
\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
 & C^0_\bullet \arrow[r] & C^1_\bullet \arrow[r] & C^2_\bullet \arrow[r] & C^3_\bullet \arrow[r] & C^4_\bullet \arrow[r] & C^5_\bullet \\
2 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[r] \arrow[d, "\partial_1^3"] & \mathbb{Z}^5 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[d] \\
0 & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[d] \\
 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

In particular, consider the last simplicial complex in the filtration:



The chain complex is $0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^5 \xrightarrow{\partial_1} \mathbb{Z}^4 \to 0$. The matrices M_2 and M_1 of ∂_2 and ∂_1 without the indicated bases are:

$$M_2 = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \left(\begin{array}{cc} abc & acd \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{array} \right), M_1 = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \left(\begin{array}{ccccc} ab & bc & cd & ad & ac \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{array} \right)$$

Then ∂_1 (ie M_1) induces a map of $\mathbb{Z}[t]$ - modules $\mathbb{Z}[t]^{\oplus 5} \xrightarrow{M_1} \mathbb{Z}[t]^{\oplus 4}$:

$$\underline{V} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \\ p_5(t) \end{pmatrix} \mapsto M_1 \underline{V}_\bullet$$

Using the grading (filtration) on the simplicial complex we get a chain complex of graded $\mathbb{Z}[t]$ - modules

$0 \to (t^4) \oplus (t^5) \xrightarrow{\partial_2^1} (t)^{\oplus 2} \oplus (t^2)^{\oplus 2} \oplus (t^3) \xrightarrow{\partial_1^1} (t)^{\oplus 2} \oplus (1)^{\oplus 2} \to 0$
by the procedure defined on. The respective matrices are (without same bases)

$$M_2' = \begin{array}{c} \\ ab \\ bc \\ cd \\ ad \\ ac \end{array} \begin{pmatrix} abc & acd \\ t^3 & 0 \\ t^3 & 0 \\ 0 & t^3 \\ 0 & -t^3 \\ -t^3 & t^3 \end{pmatrix}, M_1' = \begin{array}{c} \\ d \\ c \\ b \\ a \end{array} \begin{pmatrix} ab & bc & cd & ad & ac \\ 0 & 0 & t & t & 0 \\ 0 & 1 & -t & 0 & t^2 \\ t & -t & 0 & 0 & 0 \\ -t & 0 & 0 & -t^2 & -t^2 \end{pmatrix}$$

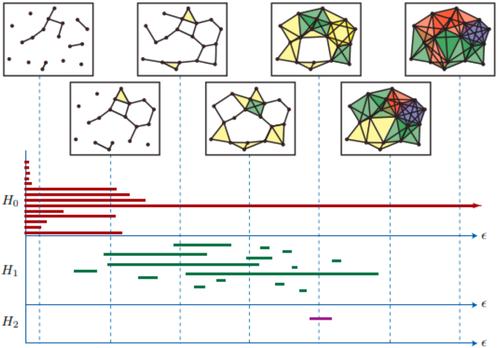


Figure 4: An example of barcode representations of the homology of the sampling of points in an annulus

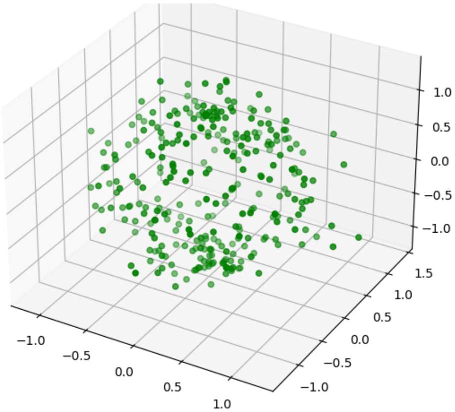


Figure 5: Points on a sphere with noise

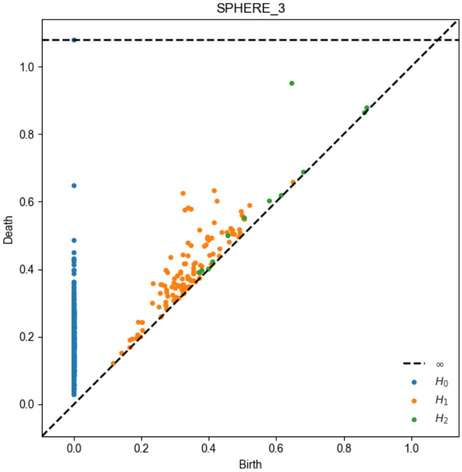


Figure 6: Persistent diagram of homology of sphere (300 points)

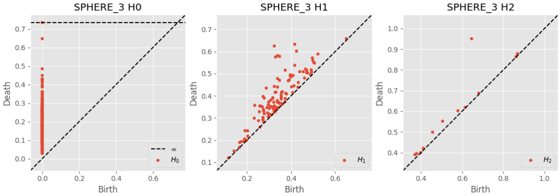


Figure 7: Persistent diagram of homology of sphere, separate (300 points)

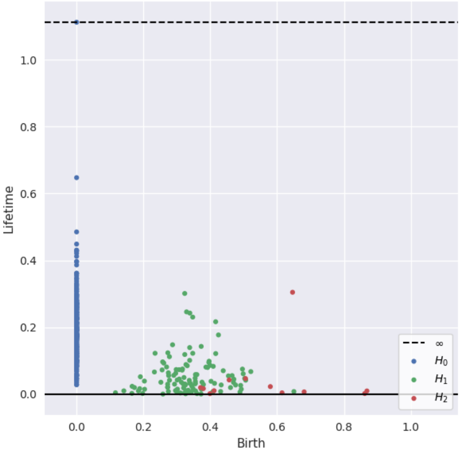


Figure 8: Lifetime persistence

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