22 Mar 2019

Recap.....

Def. P is the set of decision problems solvable by poly-time alg.

NP is the set of decision problems that have a poly-time verifier: an algorithm V with two inputs x & y, called "problem instance" & "witness", s.t. V runs in poly-time in worst case and $\forall x$ $A(x) = \text{yes} \iff \exists y$ s.t. $|y| \le \text{poly}(|x|)$ and $V(x,y) = \text{yes}$.

Observe the asymmetry: verifier for "yes" but not "no".

E.g. for Indep Set, the verifier $V(x,y)$ accepts input

x = input instance of Indep set, i.e. $x = (G, k)$

y = subset of $V(G)$.

Verifier tests that $|y| \ge k$ and $O(k)$

$\forall$ edge $e = (u,v)$, $\{u,v\} \not\subseteq y$. $O(m)$

Def. A decision problem B is NP-hard if every problem $A \in NP$ satisfies $A \le_p B$.

B is NP-Complete if it is NP-hard and $B \in NP$.

"Easy enough that solutions to B can be easily verified. Hard enough that every ~~other~~ NP problem reduces to it."

Theorem. (Cook-Levin) 3SAT is NP-Complete.

Corollary. For a decision problem $B \in NP$ the following are equivalent.

1. B is NP-Complete
2. $3SAT \leq_p B$
3. $\exists$ an NP-complete problem A such that $A \leq_p B$.

Proof. (1) $\Rightarrow$ (2). $3SAT \in NP$, B is NP-Complete, $\therefore$ $3SAT \leq_p B$.

(2) $\Rightarrow$ (3). By Cook-Levin.

(3) $\Rightarrow$ (1). By transitivity of $\leq_p$.

Assuming A is NP-complete and $A \leq_p B$, if A' is any other problem in NP,

$$A' \underset{\substack{\uparrow \\ A \text{ is NP-C}}}{\leq_p} A \underset{\substack{\uparrow \\ \text{By assumption}}}{\leq_p} B \qquad \therefore \qquad A' \underset{\substack{\uparrow \\ \text{transitivity.}}}{\leq_p} B$$

This shows B is NP-hard. Since $B \in NP$ by assumption, B is NP-Complete.

Corollary. Any 2 NP-complete problems can be reduced to ~~one~~ another.

E.g. IND SET $\leq_p$ 3SAT. See if you can find a reduction!

Recipe for showing a problem B is NP-Complete.

1. $B \in NP$. Describe a verifier. Confirm V runs in poly time.
2. Choose a problem, A, that you know to be NP-Complete. Describe a reduction from A to B, i.e. a function $f: I_A \to I_B$.
3. Confirm reduction runs in poly time.
4. $\forall x \in I_A$ $\quad A(x) = \text{'yes'} \Longrightarrow B(f(x)) = \text{'yes'}$
5. $\forall x \in I_A$ $\quad B(f(x)) = \text{'yes'} \Longrightarrow A(x) = \text{'yes'}$

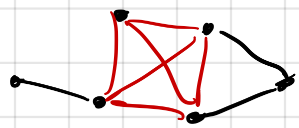
Correctness of the reduction

Don't "reduce in the wrong direction."

To show B is NP complete, reduce **from** some other problem A.

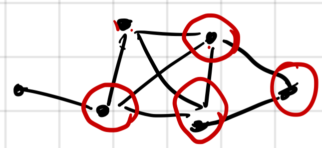
Some other NP-Complete Problems

- CLIQUE. Given (G,k) G undirected graph, $k \in \mathbb{N}$. Does G contain a set S with $|S| \geq k$ such that every 2 elements of S are joined by an edge.



(A graph with a clique of size 4.)

- VERTEX COVER. Given (G,k) as before. Does G contain a set S with $|S| = k$ such that every edge has at least one endpoint is S.



(A vertex cover of size 4.)

IND SET $\leq_p$ CLIQUE. The reduction is $f(G,k) = (\overline{G}, k)$

$\overline{G}$ has same vertex set as G, but the complementary edge set: $(u,v) \in E(\overline{G}) \Leftrightarrow (u,v) \notin E(G)$.

S is independent in $G \Leftrightarrow S$ is a clique in $\overline{G}$.

IND SET $\leq_p$ VTX COVER. The reduction is $f(G,k) = (G, |V(G)| - k)$.

$$\begin{array}{ccc} S \text{ is independent in } G & \Longleftrightarrow & \overline{S} \text{ is a vertex cover of } G \\ \Updownarrow & & \Updownarrow \\ \forall (u,v) \in E,\ u \notin S \text{ or } v \notin S & \Longleftrightarrow & \forall (u,v) \in E,\ u \in \overline{S} \text{ or } v \in \overline{S} \end{array}$$

- SET COVER. Given sets $S_1, S_2, \ldots, S_n \subset [m]$ and $k \in \mathbb{N}$

  Are there k sets in the given collection whose union is $[m]$?

Reduce from?

(A) 3SAT
(B) IS
(C) CLIQUE
(D) VERTEX COVER

"Cover" occurs in both problems
If satisfied for k, also satisfied $\forall k' > k$.

<u>Reduction:</u> Number edges of G as $e_1, e_2, \ldots, e_m$.
$\forall$ vertex v create set S_v
defined as $\{ i \mid e_i \text{ has } v \text{ as an endpoint} \}$.