INFO 4220: Homework 2

Due by: Thursday, Feb 21 2019, 12 noon

General instructions:

1. Do make sure to follow the instructions on writing up (counter)examples[1]—and do also make sure that you *substantiate your answers* with either a proof or a counterexample (depending on whether you are saying a logical statement is true or false). This applies to every problem in this course!
2. The sage advice from the Hitchhiker's Guide continues to hold (and always will!):
   (a) Don't panic. (b) This homework (like many other things) is Mostly Harmless.
3. Finish and turn in the homework, on CMS, *well in advance of the deadline.*
4. If (Not 2.), *i.e.*, if you fail to turn in your homework in time:
   (a) See the course homework policy (Lecture 1; CMS handout).
   (b) Realize that the policy does *not* say "email instructor asking for extension". In fact, it suggests that you should perhaps *not* do this, unless you have a Most Excellent Reason (these are *very* hard to come by, so you're statistically *very* unlikely to have one).
   (c) Let realization in 3(b) sink in.
   (d) Make peace with your situation (refering to 1(a) as often as necesssary).
   (e) Have a great rest-of-day. You might even want to whistle a jolly tune!

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1. *(Binary preferences and Pareto-efficiency.)* Let's rewind all the way back to *binary preferences* (Lecture 2, 3), where each agent specifies the set of items that she finds acceptable (in other words, each agent supplies a yes/no preference for each item). We saw that such a market can be completely captured by an appropriate bipartite graph—the graph completely captures the agents' preferences, and a feasible allocation in the market is identical to a matching in this graph.

   The definition of Pareto-efficiency is not restricted to strict rank-order preferences, but is more generally applicable: A matching M is Pareto-efficient simply if it is not Pareto-dominated by any other matching with respect to the given preferences. That is, M is Pareto-efficient if there is no other matching M' such that every agent is at least as happy with their assignment in M', and at least one agent is strictly happier with their assignment in M', in comparison with their assignments in M.

   Reminder: All answers *must* be accompanied by proofs or counterexamples, as appropriate; for this question, and all others!

   (a) (*7pts.*) Suppose a matching M_1 is larger in size than a matching M_2 in some bipartite graph G. Can you conclude that M_1 Pareto-dominates M_2 (w.r.t the preferences represented by G)?

   (b) (*8pts.*) Suppose a matching M in a market represented by bipartite graph G is such that there is an augmenting path w.r.t matching M in G. Can M ever (*i.e.*, for any market G and matching M) be Pareto-efficient?

[1] (See the Homework guidelines handout on CMS.)

(c) (*15pts*) Recall that a maximum matching in a bipartite graph is a matching with the largest possible number of edges amongst all matchings in the graph. Is every maximum matching in a graph Pareto-efficient with respect to the preferences represented by the underlying graph (*i.e.*, for all possible maximum matchings, for all possible bipartite graphs of all possible sizes)?

2. *(Mechanisms for rank-order preferences.)* (*25pts.*) The serial dictatorship mechanism in a one-sided matching market fixes a priority order over *agents*, and assigns items to agents according to this priority order; we showed in class that this mechanism is strategyproof and Pareto-efficient when preferences are strict.

   Lisa is in charge of organizing a community giveaway of used winter clothing, and doesn't quite like the idea of a (serial) dictatorship to allocate the donations she has received. She wants to use a mechanism that allocates items to "those who want it most". Of course, there are many different ways to turn that concept into an algorithm—here is the mechanism she comes up with. Lisa decides that she will choose a priority order[2] g over *items* instead of individuals (agents), and process *items* according to the priority order, as follows: At each step i, she will assign the item $g(i)$ that has priority i to the agent a still remaining in the market who ranks it at the highest position in his (original) preference list. If multiple agents rank item $g(i)$ equally high, she will break ties according to a fixed, pre-announced, order on the agents. She then assigns item $g(i)$ to this agent a, and removes both the agent and the item from the market. (Assume agents all have *strict, complete* rank-order preferences, so that the mechanism at every step is well-defined.)

   So for example, if Lisa is now processing item h_3, and agent a_5 ranks it 2nd on his list and all other agents still remaining in the market rank this house 3rd or lower, then agent a_5 will be assigned item h_3. Also, note that agents do not update items' ranks after an item is removed from the market in each round (so for example, if agent a ranked an item x fourth and item y fifth in his list, y still remains ranked as a's fifth choice if item x is assigned to some other agent before a gets assigned an item).

   Is the mechanism created by Lisa strategyproof?

3. Consider the house allocation model (*i.e.*, rank-order preferences), where the input is a tuple $(A, H, \succ)$. Recall that a preference profile $\succ = \{\succ_1, \ldots, \succ_n\}$ is the n-tuple of preference rankings, one for each agent; the notation $\succ_i$ denotes the preference ranking of agent i.

   Suppose that the size of the market, $n = |A| = |H| = 3$. For uniformity and ease of grading, let's call the three agents a, b, c and the three houses x, y, z.

   Throughout this question, we will restrict ourselves, as usual, to strict and complete preferences.

   (a) (*10pts.*) With 3 houses and 3 agents, can you construct a *preference profile* $\succ$ such that there is a unique Pareto-efficient matching with respect to $\succ$?

   (b) (*10 pts.*) Again, with 3 houses and 3 agents, can you construct a preference profile for which *every* (perfect, *i.e.*, matching all three agents)[3] matching is Pareto-efficient? You must argue *why every* matching is Pareto-efficient w.r.t the preference profile you give in your answer, for full credit.

   (c) (*25pts.*) Now, returning to the issue of uniqueness as in (a): describe[4] the set of *all* preference profiles in our tiny three-agent market with the property in (a), *i.e.*, the set of all $\succ$ such that there is a unique Pareto-efficient matching w.r.t. $\succ$.

   Observe that to provide a complete answer to this question, you need to justify why the answer is *exactly* the set you describe: that is, you must provide an explanation as to why the preference profiles $\succ$ you claim have a unique Pareto-efficient matching do so, **and** why all other preference profiles must have strictly more than one Pareto-efficient matching. (In other words, of all possible (strict, complete) preference profiles in this size-three market, (i) which profiles have this property and why do they have this property, and (ii) how can you prove that all the other profiles do not have this property?)

[2] Note that this priority order is arbitrary: Lisa could choose any order (*i.e.*, any amongst all possible permutations on the set of items). However, the order is announced publicly to agents before they report their preferences.

[3] Confusion alert: We are in the *rank-order preferences* model now, not binary preferences! The qualifier perfect is meant to indicate that you needn't worry about matchings which are trivially not Pareto-efficient, or be concerned about matchings that might leave some agent unmatched.

[4] To pre-emptively answer a likely question: Yes, you can simply enumerate all the profiles you think have this property, but to receive full credit you will need to come up with the (correct!) principle behind the set of profiles you've enumerated, *i.e.*, a description.