INFO 4220: Homework 3

Due by: Thursday, March 14, 2019, 12 noon

General instructions:

1. Do make sure to follow the instructions on writing up (counter)examples (see handouts on CMS)—and do also make sure that you *substantiate your answers* with either a proof or a counterexample (depending on whether you are saying a logical statement is true or false). This **always** applies—to every problem, in all assignments, in this course!

2. The sage advice from the Hitchhiker's Guide continues to hold (and always will!):
(a) Don't panic. (b) This homework (like many other things) is Mostly Harmless.

3. Finish and turn in the homework, on CMS, *well in advance of the deadline.*

4. If (Not 3.), *i.e.*, if you fail to turn in your homework in time:

   (a) See the course homework policy (Lecture 1; CMS handout).

   (b) Realize that the policy does *not* say "email instructor asking for extension". In fact, it suggests that you should perhaps *not* do this, unless you have a Most Excellent Reason (these are *very* hard to come by, so you're statistically *very* unlikely to have one).

   (c) Let realization in 4(b) sink in.

   (d) Make peace with your situation (refering to 2(a) as often as necesssary).

   (e) Have a great rest-of-day. You might even want to whistle a jolly tune!

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1. *(Weak preferences: Rank-order preferences with* no *initial endowments.)* Let's start by rewinding all the way back to the serial dictatorship mechanism, which we saw is Pareto-efficient for the setting where agents have *strict* rank-order preferences over a set of items, and there are no initial endowments. Now suppose agents can have and report *weak* preferences in that setting: that is, an agent is now allowed to be indifferent between two (or more) objects that she finds acceptable; in other words, an agent may have ties amongst acceptable objects in her rank-ordered preference list.

   Consider the following extension of the serial dictatorship mechanism for weak preferences: If an agent has two (or more) objects tied at the top of her list (*i.e.*, that she likes equally) from amongst the remaining objects when her turn arrives, this mechanism randomly assigns her one of these objects, and then proceeds as usual.

   Assume preferences are complete—*i.e.*, all agents find all houses acceptable.

   (a) (*10pts.*) Is this mechanism Pareto-efficient when agents can have weak preferences? If yes, argue why it is Pareto-efficient; if not, provide a counterexample.

   Arguments in favor must be brief and to the point; your counterexample, if this is the route you take, must involve **exactly** 3 agents and 3 items.

   (b) (*5pts.*) Is this mechanism strategyproof? If yes, provide a brief argument why; if not, provide a counterexample.

2. *(Markets with mixed initial endowments.)* We saw two extremes of initial endowments in class—*no* agents have initial endowments, and *all* agents have initial endowments. In many real-life situations, we have some midway between these extremes. Consider, for instance, our friend Lisa from HW2. Lisa is now organizing a used clothing exchange-plus- giveaway, using donations she has already received in addition to what individuals may choose to bring in: Participants may enter the exchange either bringing an item of their own that they'd like to exchange for something nicer, if possible, or enter empty-handed, looking for a single donated item to take away. This is a situation where there are some agents who have no initial endowments (the participants who come in empty-handed), and some who do (the ones who bring in an item, looking to exchange it for something they like better, if possible—it would not do to send these participants away with an item they like less than the one they own already!).

   Throughout this problem, assume that (i) as usual, agents' preferences over items are strict and complete and (ii) that there are exactly the same total number of items as there are participants in the exchange.

   (a) *(15pts.)* Facing a situation which lies squarely between the two extremes (where no agents have initial endowments, and all agents have initial endowments), Lisa's first thought is to simply use the serial dictatorship mechanism, completely ignoring the initial endowments of agents who bring items to the exchange—after all, as we've seen, this is an excellent mechanism when agents do not have intial endowments. Is this mechanism (i) Pareto-efficient (ii) individually rational for the setting Lisa is actually dealing with? (Note that the serial dictatorship mechanism may use any (report-independent) priority ordering amongst participants.)

   (b) *(15pts.)* Lisa's second idea is to use the TTC algorithm, with the following adaptation. She 'endows' each participant who entered the exchange empty-handed with one of the donated items[1]; of course, each person who brought an item they own has an initial endowment already, which is that item. Lisa will then simply apply the TTC algorithm to this modified market where every agent now has an 'initial endowment', whether genuine or awarded by Lisa. Is this mechanism (i) Pareto-efficient (ii) individually rational for the setting Lisa is actually dealing with?

   (c) *(15 pts.)* Lisa's third idea is to combine the two excellent mechanisms she's learnt in the following way: she simply separates the market into two parts—agents who brought items, and the agents who did not (the items in the first submarket are the ones the agents brought along with them, while the items in the second submarket are the donated items). In other words, she splits the market into the part with agents who all have initial endowments (along with their items), and the part where none do (along with the remaining un-owned items). She assigns items in the submarket with initial endowments according to the TTC algorithm applied to that submarket, and items in the submarket with endowment-free agents and donated items according to the serial dictatorship mechanism.

   Is this mechanism (i) Pareto-efficient (ii) individually rational for the complete market that Lisa is actually dealing with?

3. *(Weak preferences: Markets with initial endowments.)* Now let's return to the scenario you've been looking at most recently in class—a market with initial endowments and *strict* rank-order preferences, where n agents, each of whom owns a house (her initial endownment), has a strict preference ordering over all the houses; you've just studied the concept of the core for such markets. In this problem, we will see what might happen when agents can have *weak* preferences, along with initial endowments.

   (a) *(10pts.)* Suppose we have a market with three agents with initial endowments (a, h_a), (b, h_b) and (c, h_c), with (strict) preferences as follows: a prefers houses in the order $h_b \succ h_c \succ h_a$, b prefers houses in the order $h_a \succ h_c \succ h_b$, and c prefers houses in the order $h_b \succ h_c \succ h_a$. What is the core matching in this market?

   (b) *(15pts.)* Now suppose b and c's preferences are as before, but a is *indifferent* between h_b and h_c: that is, a's preferences are now $\{h_b, h_c\} \succ h_a$. Is there a core matching[2] in this modified market?

[1] Which empty-handed entrant gets which donated item is determined according to some arbitrary permutation.

[2] Note that the *definition* of the core—the conditions a matching must satisfy to be in the core—does not depend on preferences being strict!

If your answer is yes, supply the matching and explain why; if it is no, explain why not.

(c) *(15pts.)* What if a and c's preferences are as in part (a), but now b is *indifferent* between h_a and h_c: that is, b's preferences are now $\{h_a, h_c\} \succ h_b$. Is there a core matching in the market with these preferences? Again, if your answer is yes, supply the matching and explain why; if it is no, explain why not.