Finding Minimum Cuts in Graphs

(6th May, '19)

Input: undirected graph $G = (V, E)$.

$|V| = n$, $|E| = m$.

Recall: A cut (A, B) of G is a partition of the vertices.

i.e $A \cup B = V$, $A \cap B = \emptyset$

$$\underbrace{E(A, B)}_{\text{cut edges}} = \{ (v, u) \in E : v \in A, u \in B \}$$

$|E(A, B)|$: cardinality of $E(A, B)$ is the size of the cut (A, B).

Task: Given G as input, find a cut of minimum size.

Algorithm using Max-flow:

Identify any arbitrary node $s \in V$, and this is the source node.

$\forall\ t \in V \setminus \{s\}$, define the flow network G_t as follows:

→ s is the source node

→ t is the sink node.

→ $\forall\ e = (u, v) \in E$, add directed nodes $u \to v$, $v \to u$ in G_t, with capacity 1.

for all $t \in V \setminus \{s\}$,

find min s-t cut

end for.

output the 'best solution' found.

Time complexity: $O\Big(\underbrace{\overset{\#\text{ of edges}}{m}\cdot\overset{\#\text{ max-capacity}}{n}}_{\text{F-F run time}}\cdot(n-1)\Big)$

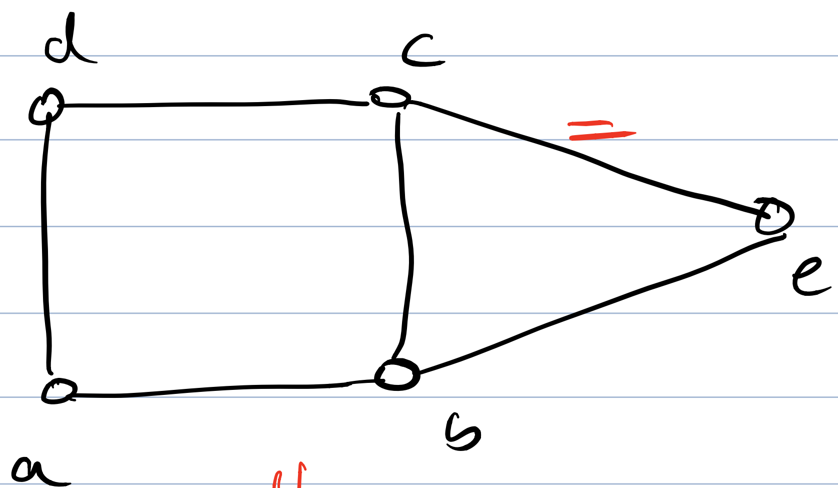
$= O(m\,n^2)$ algorithm.

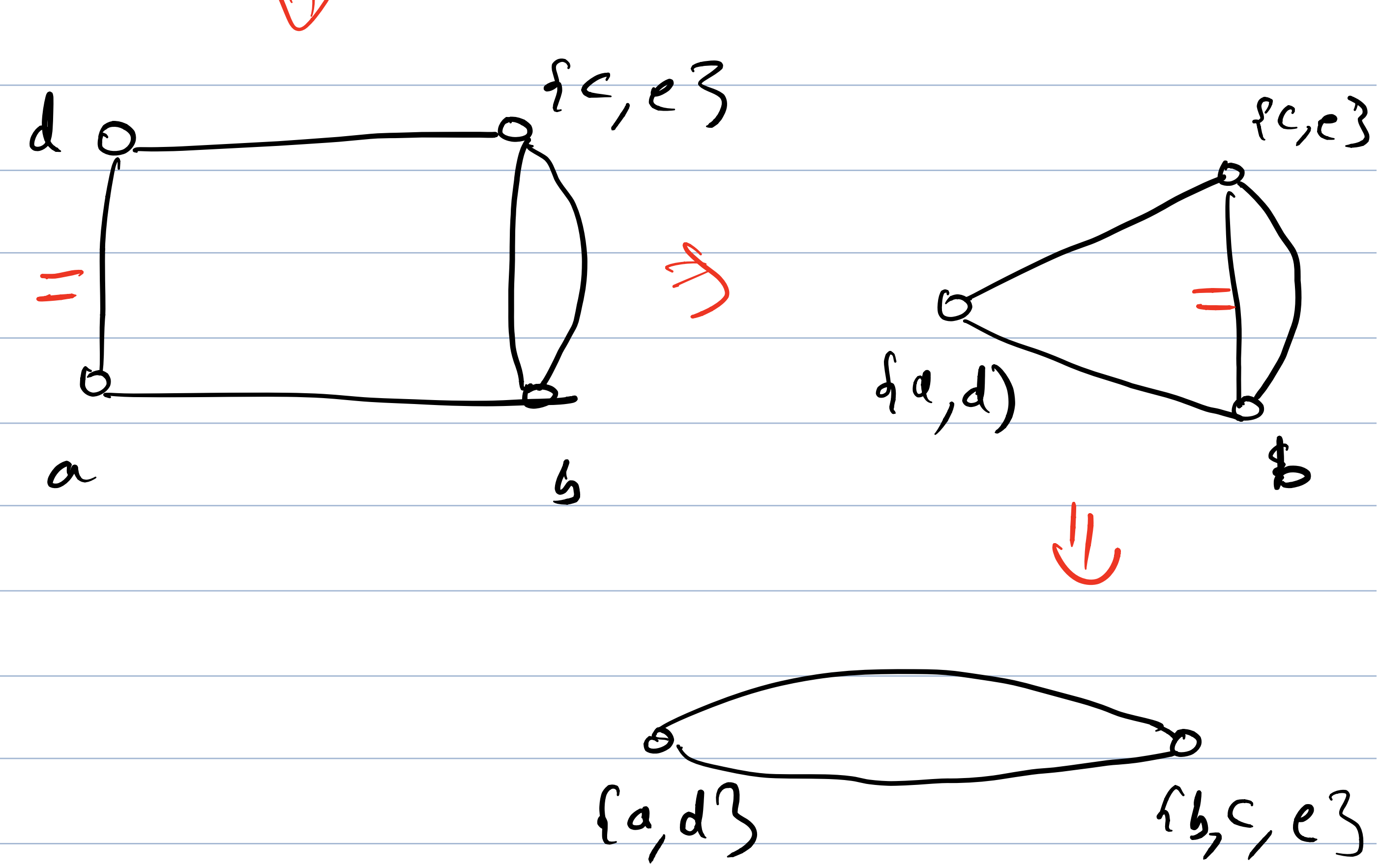
A randomized algorithm for min-cut

/ Karger's Min-Cut Algorithm.

'Simple and elegant'.

'Notion of "contractions" of edges.





[Defn: (Multigraph) Graph with multiple edges allowed between any two vertices.]

More formally, after a sequence of contractions on a multigraph G to get G', each node of G' is labelled with a disjoint subset of V; further the labels form a partition of V.

$$G' \xrightarrow[\text{edge } e]{\text{contract}} G'', \quad e = (S_1, S_2)$$

$S_1, S_2 \subseteq V$, disjoint

"Combine" the nodes labelled S_1, S_2 in G' to a single node labelled $S_1 \cup S_2$;

any edge incident on S_1 or S_2 in G' is incident to $S_1 \cup S_2$ in G''. (no self loops).

Karger's Algorithm

Alg 1 :

(1) $G' = G$

// if G' has 1 node, return "undefined".

(2) while G' has **more** than 2 nodes,

pick a random edge e in G', and update G' by contracting e.

(3) Output (A, B), where A, B are labels of the two nodes in G'.

From now on, assume $n \geq 2$.

Claim 1 (A, B) is a cut of G.

Pf. Follows from def^n of contractions.

$\square$.

Claim 2: Alg 1 outputs a min-cut with prob $\geq \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$.

Pf: (below claims hold for multigraph)

Fact 1: $\sum_{v \in V} \deg(v) = 2|E| = 2m$

Corollary 1: $\exists\ v \in V$ s.t

$$\deg(v) \leq \frac{2m}{n}.$$

Pf: (Averaging argument)

$$\frac{1}{|V|} \sum_{v \in V} \deg(v) = \frac{2m}{n}$$

$$\mathbb{E}_{v \sim \text{random vertex in } V} \left[\deg(v) \right] = \frac{2m}{n} .$$

$\square$.

<u>Corollary 2:</u> Size of a min cut in $G \le \frac{2m}{n}$.

<u>Pf.</u> $\exists\, v \in V$, $\deg(v) \le \frac{2m}{n}$.

$\Rightarrow$ size of $(\{v\}, V \setminus \{v\})$ is $\le \frac{2m}{n}$

$\Rightarrow$ min-cut $\le \frac{2m}{n}$ $\square$.

Let (A_{min}, B_{min}) be a min-cut of G.

<u>Claim</u>: Alg 1 outputs (A_{min}, B_{min}) iff no edges in $E(A_{min}, B_{min})$ is contracted during execution of Alg 1.

Pf. Follows directly. (easy to check both directions) □.

<u>Obs.</u> If G' is obtained by a series of contractions from G, any cut of G' is also a cut of G.

→ Prob. that $e \in E(A_{min}, B_{min})$ is <u>not selected</u> in the first iteration of Alg 1. $\geq 1 - \frac{2}{n}$.

→ Prob " $e \in E(A_{min}, B_{min})$ is <u>not selected</u> in iteration 2 $\geq 1 - \frac{2}{n-1}$

⋮

→ Prob. " $e \in E(A_{min}, B_{min})$ is not selected in iteration $n-2$ $\geq 1 - \frac{2}{3}$.

Prob. Alg 1 succeeds

$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots$

$$\left(\quad n \right)\left(\quad n-1 \right)$$

$$\left(1 - \frac{2}{3}\right)$$

$$= \frac{n-\cancel{2}}{n} \cdot \frac{n-\cancel{3}}{n-1} \quad \frac{n-4}{\cancel{n-2}} \quad \frac{n-5}{\cancel{n-3}}$$

$$\cancel{-}\;-\;-\;\cancel{-} \quad \frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{3}}$$

$$= \frac{2}{n \cdot (n-1)} = \frac{1}{\binom{n}{2}} .$$

$\blacksquare$.

Alg 2: set $l = 100 \cdot \binom{n}{2}$.

Repeat Alg 1 l times,

and output 'best solution'.

Prob Alg 2 **does not** succeed

$\equiv$ Alg 1 fails l times.

$$\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{l}$$

$$\leq e^{-\frac{l}{\binom{n}{2}}} \leq e^{-100}.$$

[uses the fact that

$1 + x \leq e^{x}$ for all $x \in \mathbb{R}$]

Alg 2 succeed with proba ≥ 0.999.

Running time : $O(n^4)$ time.

Can be improved to :

$O(m n^2 \log n)$

using Union-Find data-structure