INFO 4220: Homework 5

Due by: Thursday May 3, 2019, 12 noon

General instructions:

1. The sage advice from the Hitchhiker's Guide continues to hold for this last homework (and always will, even after the course is over!):
   (a) Don't panic. (b) This homework (like many other things) is Mostly Harmless.
2. Finish and turn in the homework, on CMS, *well in advance of the deadline.*
3. If (Not 2.), *i.e.*, if you fail to turn in your homework in time:
   (a) See the course homework policy (Lecture 1; CMS handout).
   (b) Realize that the policy does *not* say "email instructor asking for extension". In fact, it suggests that you should perhaps *not* do this, unless you have a Most Excellent Reason (these are *very* hard to come by, so you're statistically *very* unlikely to have one).
   (c) Let realization in 3(b) sink in.
   (d) Make peace with your situation (refering to 1(a) as often as necessary).
   (e) Have a great rest-of-day. You might even want to whistle a jolly tune!

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1. A research study says that buyers on eBay are willing to pay a premium for buying from a seller with 2000 positive ratings and 1 negative rating, compared to a seller with 10 positive and no negative ratings. Let's try to understand why the total number of ratings being larger, even with a larger number of negative ratings, might make such a difference—in other words, you can think of this problem as a step towards quantifying the *value of information.* Suppose a buyer has no idea whether a seller is good (G) or bad (B), and models this by assuming that there is a 50% chance the seller is bad. Suppose a bad seller cheats buyers 30% of the time. A good seller never cheats buyers, but buyers might still have a bad experience with the seller 1% of the time due to circumstances outside

the seller's control—the package really was lost in the post, for example. (We'll assume that no such additional causes occur with the bad seller, so that you have a bad experience with the bad seller only if he is cheating you.)

(a) *(5pts.)* Suppose there are 10 ratings for a seller, and they are all positive. What is the probability that the seller being rated is actually bad?

(b) *(5pts.)* Now suppose there are 1000 positive ratings and 10 negative ratings. Now what is the probability that the seller being rated is actually bad? What if there are 1000 positive and 50 negative ratings?

(c) *(15pts.)* Now consider these two sellers, call them sellers A and B, where A has 10 positive and no negative ratings, and B has 1000 positive and 10 negative ratings. Suppose A and B are selling identical items that are worth v to the buyer. The buyer derives this value v if she has a good experience with the seller, and derives value 0 otherwise. How much more would the buyer be willing to pay to buy from seller B than A? (Continue with the assumption that the buyer believes sellers are equally likely to be good or bad.)

Solution. Let us use notation $+$ to denote a single positive rating (*i.e.*, a good experience for one buyer), and $-$ to denote a single negative rating (*i.e.*, a bad experience for one buyer) (so 10+ denotes ten positive ratings). The prior probabilities are $P(G) = 0.5, P(B) = 0.5$.
The problem tells us that $P(+ \mid G) = 0.99, P(- \mid G) = 0.01$, and $P(+ \mid B) = 0.70, P(- \mid B) = 0.30$.

(a) We want to calculate the posterior probabilities of a seller being bad, given that the prior probabilities were 1/2 each of being good or bad, and given the observations of these buyers. Since the ratings of buyers are assumed independent,

$$P(10+ \mid G) = P(+ \mid G)^{10} = 0.99^{10} = 0.9044$$
$$P(10+ \mid B) = P(+ \mid B)^{10} = 0.7^{10} = 0.0282.$$

Using Bayes rule,

$$\begin{aligned} P(B \mid 10+) &= \frac{P(10+ \mid B)P(B)}{P(10+ \mid B)P(B) + P(10+ \mid G)P(G)} \\ &= \frac{0.0282 \times 0.5}{0.0282 \times 0.5 + 0.9044 \times 0.5} \\ &= 0.0303. \end{aligned}$$

(b) First we start out by asking what is the probability of 1000 positive and 10 negative experiences if the seller is good (G), and if the seller is bad (B).

$$\begin{aligned}
P(1000+, 10- \mid G) &= \binom{1010}{10} \times P(+ \mid G)^{1000} \times P(- \mid G)^{10} \\
&= 0.99^{1000} \times 0.01^{10} \times \binom{1010}{10} \\
&= 0.1257
\end{aligned}$$

$$\begin{aligned}
P(1000+, 10- \mid B) &= \binom{1010}{10} \times P(+ \mid B)^{1000} \times P(- \mid B)^{10} \\
&= 0.7^{1000} \times 0.3^{10} \times \binom{1010}{10} \\
&= 2.154 \times 10^{-137}
\end{aligned}$$

Now we're just going to apply Bayes rule again, exactly as we did in the previous case.

$$\begin{aligned}
P(B \mid 1000+, 10-) &= \frac{P(1000 + 10- \mid B)P(B)}{P(1000 + 10- \mid B)P(B) + P(1000 + 10- \mid G)P(G)} \\
&= \frac{2.154 \times 10^{-137} \times 0.5}{2.154 \times 10^{-137} \times 0.5 + 0.1257 \times 0.5} \\
&= 1.714 \times 10^{-136}.
\end{aligned}$$

Notice that even though there are a full 10 negative ratings this time, the probability of the buyer actually being bad is very much smaller than when there were no negative ratings but only 10 positive ratings! This might help understand why having a larger number of reviews is useful.

With 1000 positive and 50 negative experiences, we have

$$\begin{aligned}
P(1000+, 50- \mid G) &= P(+ \mid G)^{1000} \times P(- \mid G)^{50} \times \binom{1050}{50} \\
&= 0.99^{1000} \times 0.01^{50} \times \binom{1050}{50} \\
&= 4.975 \times 10^{-19}, \\
P(1000+, 50- \mid B) &= P(+ \mid B)^{1000} \times P(- \mid B)^{50} \times \binom{1050}{50} \\
&= 0.7^{1000} \times 0.3^{50} \times \binom{1050}{50} \\
&= 1.037 \times 10^{-95}, \\
P(B \mid 1000+, 50-) &= \frac{P(1000+50- \mid B)P(B)}{P(1000+50- \mid B)P(B) + P(1000+50- \mid G)P(G)} \\
&= \frac{1.037 \times 10^{-95} \times 0.5}{1.037 \times 10^{-95} \times 0.5 + 4.975 \times 10^{-19} \times 0.5} \\
&= 2.084 \times 10^{-77}.
\end{aligned}$$

Notice that even with 50 negative experiences, the seller is overwhelmingly more likely to be good than when there were 10 positive and no negative ratings!

(c) Now we can put these probabilities together to reason about the price a buyer would be willing to pay each of these sellers, by calculating the expected value from the transaction to the buyer.

Amount willing to pay A:

$$v \times \Big(P(+|G) \times P(G \mid 10+) + P(+|B) \times P(B \mid 10+)\Big) +$$
$$0 \times \Big(P(-|G) \times P(G \mid 10+) + P(-|B) \times P(B \mid 10+)\Big)$$
$$= v \times \Big(0.99 \times (1 - 0.0303) + 0.7 \times 0.0303\Big) = 0.9812v.$$

Amount willing to pay B:

$$v \times \Big(P(+|G) \times P(G \mid 1000 + 10-) + P(+|B) \times P(B \mid 1000 + 10-)\Big) +$$
$$0 \times \Big(P(-|G) \times P(G \mid 1000 + 10-) + P(-|B) \times P(B \mid 1000 + 10-)\Big)$$
$$= v \times \Big(0.99 \times (1 - 1.714 \times 10^{-136}) + 0.7 \times 1.714 \times 10^{-136}\Big) \approx 0.99v.$$

The buyer is willing to pay $0.99v - 0.9812v = 0.0088v$ more to seller B than to seller A, which is approximately a 0.9% premium.

2. *(When does the market fail?)* We said in class that whether information asymmetry leads to market failure depends on the distribution of qualities in the seller population, and the buyer and seller valuations for items—informally, markets with a greater proportion of low-quality items, and a

smaller gap between buyer and seller values, are more likely candidates for market failure. In this problem, we're going to explore how market failure depends on these parameters, for the same toy market that we've been using with just two types of items (which we'll continue calling cars), and address the following question: How does the existence of an efficient self-fulfilling expectations equilibrium (SFEE) (*i.e.*, an equilibrium with $h = g$) depend on the buyer and seller values, and the fraction of good cars in the *population*)?

Specifically, suppose there are good cars and bad cars on the market; and a fraction g of used cars are good; good cars are valued at s_H by sellers and b_H by buyers, and bad cars are valued at s_L by sellers and b_L by buyers. (As always, there is information asymmetry in the market, where sellers know the quality of the cars they sell, whereas buyers do not; also, we'll continue to assume that there are more buyers than sellers.)

(a) First, let's warm up with a concrete example: suppose $s_H = 10$, $b_H = 15$, $s_L = 3$ and $b_L = 6$. For these values:

   i. *(5pts.)* Is $h = g$ a self-fulfilling expectations equilibrium (SFEE) if $g = 0.8$?

   ii. *(5pts.)* What if $g = 0.2$?

(b) Now let's use abstract values s_H, s_L, b_H and b_L for the sellers' and buyers' valuations of good and bad cars as defined above, and understand how the non-existence of an efficient equilibrium depends on the fraction of good cars in the population (*i.e.*, the distribution of qualities in the market), and on the buyer and seller valuations.

   i. *(5pts.) (Dependence on distribution of qualities.)* Explain why the *set* of values of g—the fraction of good cars in the population—for which there is an equilibrium with $h = g$ (*i.e.*, with the efficient outcome in which both good and bad cars are sold on the market) is an *interval*, of the form $[g^*, 1]$.

   ii. *(10pts.) (Dependence on items' valuations.)* This threshold g^* describes the minimum fraction of good cars that the market must contain for it to retain an efficient equilibrium $h = g$ with information asymmetry (the lower the threshold, the more bad sellers the market can tolerate without failure despite the information asymmetry). How does this threshold g^* depend on the values s_H, s_L, b_H, b_L?
   (To answer this question, it is enough to derive an expression for the value of g^* as a function of these parameters.)

Solution.

(a) i. First consider $g = 0.8$. If sellers of both good and bad cars put their cars on the market, *i.e.*, $h = g = 0.8$, buyers' expected value for a car sold on the market is:

$E(w) = h \times b_H + (1-h) \times b_L = 0.8 \times 15 + 0.2 \times 6 = 13.2 > s_H = 10$. Therefore, $h = g$ is an SFEE, and there is an equilibrium in which sellers of good cars are willing to put their cars on market. The price in this equilibrium, in general, will lie between 10 and 13.2; with the assumption that there are more buyers than sellers, the price will be be exactly equal to the buyers' expected value of 13.2.

ii. With $g = 0.2$, if sellers of both good and bad cars put their cars on the market, *i.e.*, $h = g = 0.2$, buyers' expected value for cars on the market is:
$0.2 \times 15 + 0.8 \times 6 = 7.8 < s_H = 10$. Therefore, sellers of good cars are not willing to put their cars on market: $h = g$ is not an SFEE when $g = 0.2$. (The only equilibrium when $g = 0.2$ is with $h = 0$, in which only bad cars are sold.)

(b) Now we'll just reproduce the same reasoning with the abstract values instead of numbers.

i. For the belief $h = g$ to be an SFEE, the buyer's expected value with $h = g$ must equal or exceed the sellers' valuation of a good car, *i.e.*: $g \times b_H + (1-g) \times b_L \geq s_H$. If there is an equilibrium with $h = g$ (*i.e.*, in which both good and bad cars are sold on the market) for some value of g, such an equilibrium in which both types of cars are sold must also exist for all $g' \geq g$, since the LHS, which is expected value of a car on the market to buyers (and therefore the price offered), is increasing in g (*i.e.*, the buyers' expected value at $h = g$ increases with the proportion of good cars on the market). So such a threshold g^* exists: this supports the informal statement that markets with more good sellers in the population are less suspectible to failure due to information asymmetry.

ii. At this threshold g^*, the expected value (with $h = g$) of a used car to the buyer, and therefore the price offered, is exactly *equal* to sellers' values for a good car (at any lower g, the sellers of good cars would not be willing to sell). This allows us to solve for g^*: $g^* \times b_H + (1-g^*) \times b_L = s_H$. Therefore,

$$g^* = \frac{s_H - b_L}{b_H - b_L}.$$

3. *(Repeated Prisoners' Dilemma: Discount factors and equilibria.)* Consider the infinitely repeated prisoners' dilemma, where the stage game has payoffs (r, r), (t, s), (s, t), and (p, p), and both agents have a discount factor of δ (all terminology is as defined in class; see Lecture 23 if necessary). To start with, let's consider a specific example with payoffs $(2, 2)$, $(3, 0)$, $(0, 3)$, and $(1, 1)$. Let's consider the grim-trigger strategy G that we discussed in class. We'll now reason about when (*i.e.*, for what values of the discount

factor δ) it is a Nash equilibrium for both agents to play according to the grim-trigger strategy in the infinitely repeated game.

(a) *(5pts.)* Suppose player P1 is playing the grim-trigger strategy G. If player P2 also plays G, then since the game starts in state (C,C) at time $n = 0$, applying the strategy G will keep both players in the same states (C,C) for all steps $n \geq 0$. What is the total payoff to player P2 playing G in the infinitely repeated Prisoners' Dilemma game, when the discount factor is δ?

(b) *(7pts.)* Now suppose P1 plays according to the strategy G, but P2 deviates at some step $N \geq 0$ by selecting D. **Given** that P1 (continues to) play the grim-trigger *strategy* G,
(i) What is player P2's best response in steps $N+1$ onwards (*i.e.*, in steps $N+1, N+2, \ldots$): that is, at what steps $n \geq N+1$ should she choose to play C, and when should she choose D (having deviated at step N)?
(ii) Does your answer to this question depend on the value of δ?

(c) *(8pts.)* What is the total payoff to P2 if she decides to play according to the grim-trigger strategy for $n < N$, and then play D for $n \geq N$, while P1 continues to play the grim-trigger strategy G throughout?

(d) *(10pts.)* From parts (a), (b) and (c), can you now reason for what values of δ is it a Nash equilibrium for both agents to play according to the grim-trigger strategy G? (You should both derive the condition on δ, as well as explain how you arrived at the conclusion that the grim-trigger strategy is an equilibrium for these values of δ.)

Solution.
$r = 2$, $t = 3$, $s = 0$, $p = 1$, $0 < \delta < 1$.

(a) The total payoff is

$$r + r\delta + r\delta^2 + r\delta^3 + \cdots = \sum_{n=0}^{\infty} r\delta^n = r\sum_{n=0}^{\infty} \delta^n = \frac{r}{1-\delta} = \frac{2}{1-\delta}.$$

(b) (i) Because P2 played D on step N, according to the grim-trigger strategy, P1 will play D for all steps starting from step $N+1$ onwards. In any step in which P1 plays D, the best response for P2 is to play D, since that yields a positive payoff $(1 \cdot \delta^n)$ compared to the payoff of 0 from playing C. So P2 should always play D from step $N+1$ onwards.

(ii) Note that this argument about playing D in any given step, and therefore every step $N+1$ onwards, does not dependent on the value of δ. So the best response of P2 playing D for step $N+1$ onwards does not depend on δ.

(c) Let's think about the total payoff to P2 as the sum of the payoffs from steps 0 to $N-1$, at step N, and step $N+1$ onwards .

From step 0 to step $N-1$, both players play C. Payoff for P2: $\sum_{n=0}^{N-1} r\delta^n$.

At step N, P1 plays C and P2 plays D. Payoff for P2: $t\delta^N$.

From step $N+1$ onwards, both players play D. Payoff for P2: $\sum_{n=N+1}^{\infty} p\delta^n$.

Thus the total payoff for P2 is:

$$\begin{aligned}
\sum_{n=0}^{N-1} r\delta^n + t\delta^N + \sum_{n=N+1}^{\infty} p\delta^n &= \frac{r(1-\delta^N)}{1-\delta} + \frac{t\delta^N(1-\delta)}{1-\delta} + \frac{p\delta^{N+1}}{1-\delta} \\
&= \frac{r - r\delta^N + t\delta^N - t\delta^{N+1} + p\delta^{N+1}}{1-\delta} \\
&= \frac{r + (t-r)\delta^N - (t-p)\delta^{N+1}}{1-\delta} \\
&= \frac{2 + (3-2)\delta^N - (3-1)\delta^{N+1}}{1-\delta} \\
&= \frac{2 + \delta^N - 2\delta^{N+1}}{1-\delta}.
\end{aligned}$$

(Note: If you denoted the first step as step 1 instead of step 0, the answer to this question would be

$$\sum_{n=0}^{N-2} r\delta^n + t\delta^{N-1} + \sum_{n=N}^{\infty} p\delta^n = \frac{r + (t-r)\delta^{N-1} - (t-p)\delta^N}{1-\delta} = \frac{2 + \delta^{N-1} - 2\delta^N}{1-\delta},$$

which doesn't change the answer to future questions. Both results have been given full credit.)

(d) For the grim-trigger strategy G to be a Nash equilibrium, a player needs to have a higher payoff when she sticks to playing strategy G (which is her payoff in (a)) than when she deviates (as calculated in (c)). Thus, for every step N, starting at step 0 onwards,

$$\begin{aligned}
\frac{2}{1-\delta} &> \frac{2 + \delta^N - 2\delta^{N+1}}{1-\delta} \\
2 &> 2 + \delta^N - 2\delta^{N+1} && \text{(assuming } 1-\delta > 0\text{)} \\
2\delta^{N+1} &> \delta^N \\
\delta &> \frac{1}{2} && \text{(assuming } \delta > 0\text{).}
\end{aligned}$$

4. *(20pts.) (Repeated Prisoners' Dilemma: Payoffs and discount factors.)* In the previous question, we saw how to reason about whether the grim-trigger strategy is a Nash equilibrium, and how the discount factor plays

a role in determining whether or not G is an equilibrium set of strategies in the infinitely repeated prisoners' dilemma.

In this problem, we're going to try to understand how the payoffs r, s, t and p in the stage game affect how patient players need to be to sustain cooperation via the grim-trigger strategy—*i.e.*, how the minimum value of the discount factor δ at which both players playing according to G is an equilibrium depends on the payoffs in the game.

Hint: For this problem, think about how to derive an expression for δ^* *in terms of the payoffs* in the stage game (r, s, t, p), such that the grim-trigger strategy is an equilibrium for all $\delta \geq \delta^*$. You can use that expression to answer the questions below; this will save you the effort of doing the same calculations with different numbers again and again for each part.

(a) Suppose the reward r from cooperating goes up to 2.5 from 2, whereas s, t, p remain unchanged (*i.e.*, s, t, p are the same as in Q2). Following the same arguments as in Q2, for what values of δ is the grim-trigger strategy now a Nash equilibrium?

(b) Now let's change the values of the temptation t from 3 to 5, while $r = 2$, $s = 0$ and $p = 1$ remain as before. What is the new minimum value of δ required for G to be an equilibrium set of strategies?

(c) Let's change p next—say $p = \frac{1}{2}$ while $r = 2$, $s = 0$ and $t = 3$ remain as before. Now what is the new minimum value of δ required for G to be an equilibrium?

(d) Finally, supposing that r, t, p remain the same as in Q2, how does a change in s, say from 0 to -1, affect the minimum discount factor δ beyond which G is an equilibrium?

Solution.
Let's use the more general expression of the payoffs when grim-trigger strategy G was used (2a) and deviated (2c) as calculated in Q3.

For the grim-trigger strategy G to be a Nash equilibrium, a player needs to have a higher payoff when she sticks to G than when she deviates. Thus,

$$\begin{aligned}
\frac{r}{1-\delta} &> \frac{r + (t-r)\delta^N - (t-p)\delta^{N+1}}{1-\delta} \\
r &> r + (t-r)\delta^N - (t-p)\delta^{N+1} && \text{(assuming } 1-\delta > 0\text{)} \\
(t-p)\delta^{N+1} &> (t-r)\delta^N \\
(t-p)\delta &> (t-r) && \text{(assuming } \delta > 0\text{)} \\
\delta &> \frac{(t-r)}{(t-p)} && \text{(assuming } t-p > 0\text{).}
\end{aligned}$$

(Note that t must be bigger than p in a prisoner's dilemma game, so the condition $t - p > 0$ is indeed satisfied.)

(a) $r = 2.5$, $t = 3$, $s = 0$, $p = 1$

$$\delta > \frac{(t-r)}{(t-p)} = \frac{(3-2.5)}{(3-1)} = \frac{1}{4}$$

(b) $r = 2$, $t = 5$, $s = 0$, $p = 1$

$$\delta > \frac{(t-r)}{(t-p)} = \frac{(5-2)}{(5-1)} = \frac{3}{4}$$

(c) $r = 2$, $t = 3$, $s = 0$, $p = 1/2$

$$\delta > \frac{(t-r)}{(t-p)} = \frac{(3-2)}{(3-1/2)} = \frac{2}{5}$$

(d) Note that s does not figure in the payoff comparison at all— so the value of s does not influence the threshold value of δ beyond which grim-trigger is a Nash equilibrium!