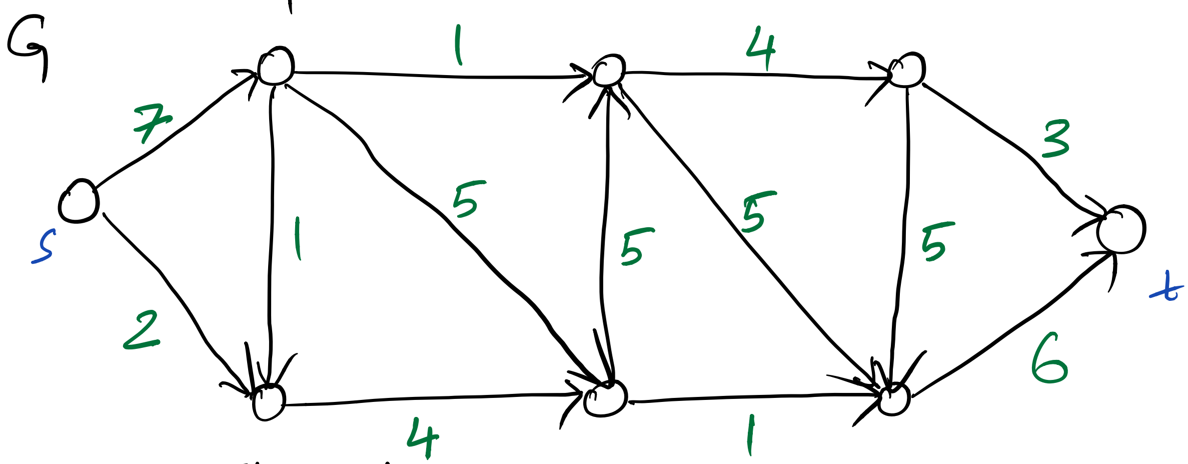
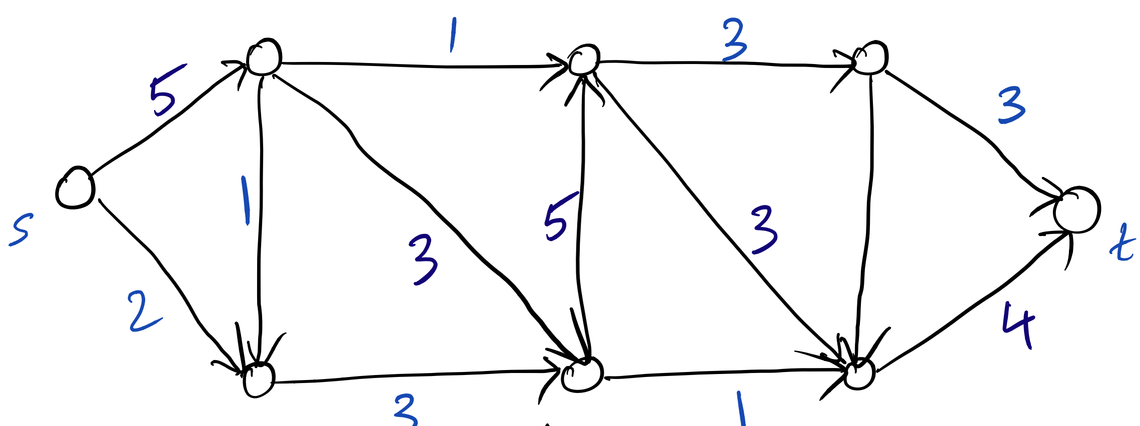
An example : 3/8/2019 .

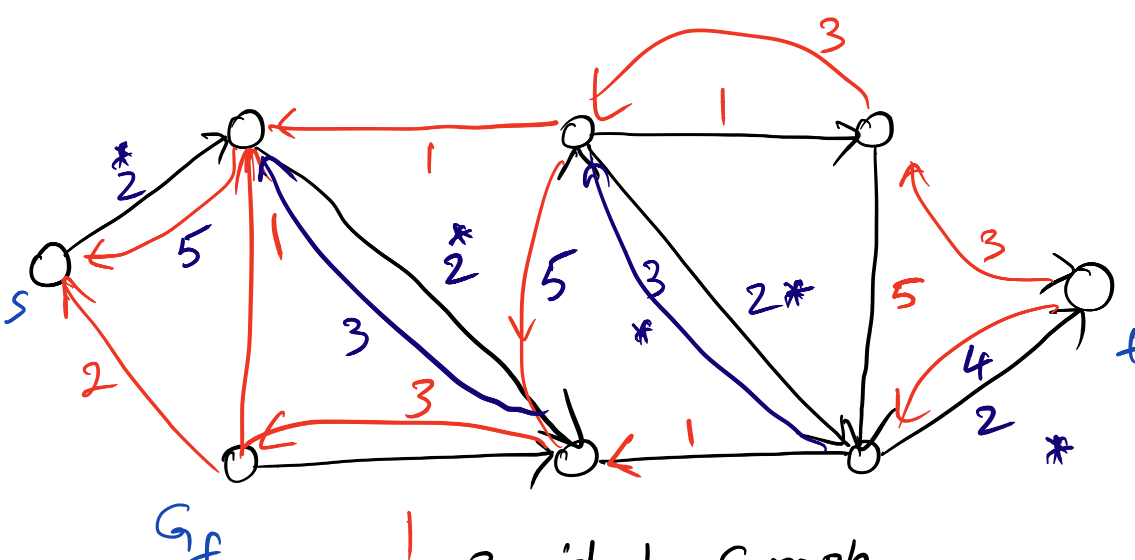
G



Flow network



Flow



G_f Residual Graph .

$$\text{Max flow} = \sum_{e=(s,u)} c(e) = 7.$$

Max Flow Min Cut Theorem / Proof of correctness of F-F algorithm.

Def^n: (s-t cut of G (V,E)) A, B is an s-t cut of G if A, B partitions V and $s \in A$, $t \in B$.

Notation: $E(A,B) = \{e=(u,v) : u \in A, v \in B\}$

Capacity of an s-t cut:

Let (A,B) be an s-t cut of G.

$$c(A,B) = \sum_{e \in E(A,B)} c(e).$$

G.

Def^n: f^{out}, f^{in} of subsets of vertices.

For any $A \subseteq V$,

$$f^{out}(A) = \sum_{e \in E(A,\bar{A})} f(e)$$

$$f^{in}(A) = \sum_{e \in E(\bar{A},A)} f(e).$$

$\bar{A} = V \setminus A$.

Lemma: Let f be an s-t flow on G,

and (A, B) is any s-t cut.

Then,
$$v(f) = f^{out}(A) - f^{in}(A)$$

Proof: Recall, $v(f) = \sum_{e=(s,u)} f(e)$ — ①

For any $v \in V \setminus \{s, t\}$,

$$f^{in}(v) = f^{out}(v)$$ [Conservation of flow]

$$f^{out}(A) - f^{in}(A)$$

$$= \sum_{e \in E(A, \bar{A})} f(e) - \sum_{e \in E(\bar{A}, A)} f(e).$$

— ①

$$v(f) = \underbrace{\sum_{v \in A} \left(f^{in}(v) - f^{out}(v) \right)},$$

since ~~for~~ $v \in A \setminus \{s\}$,

we know $f^{in}(v) = f^{out}(v)$

So the only surviving term is $f^{out}(s)$.

For any e with both end points in A (or $\bar{A}$), it appears once as '+' and once '−'ve. □

<u>**Lemma:**</u> For any s-t cut (A, B), any flow f,

$$v(f) = f^{in}(B) - f^{out}(B)$$

<u>Pf</u>. Argue as above. □.

any s-t cut

Corrolary: For any flow f, a (A,B),

$$v(f) \leq c(A,B).$$

Proof:

$$v(f) \leq f^{out}(A) = \sum_{e \in E(A,B)} f(e)$$

$$\leq \sum_{e \in E(A,B)} c(e) = c(A,B)$$

$\square$.

Optimality of F-F

Thm If f is a flow, s.t there is no s-t path in G_f, then $\exists$ an (s,t)-cut A,B s.t

$$v(f) = c(A,B).$$

Cor. The flow f in the above theorem must be a max-flow. Similarly, (A,B) must be a min-cut (i.e a cut with minimal capacity).

Pf: Define, $A = \{ v : \exists$ is a path from s to v in $G_f \}$.

$B = V \setminus A$.

(i) (A, B) is an s-t cut of G.

Pf: There is no s-t path in G_f,

$s \in A$; $t \in B$.
A, B is a partition of V.

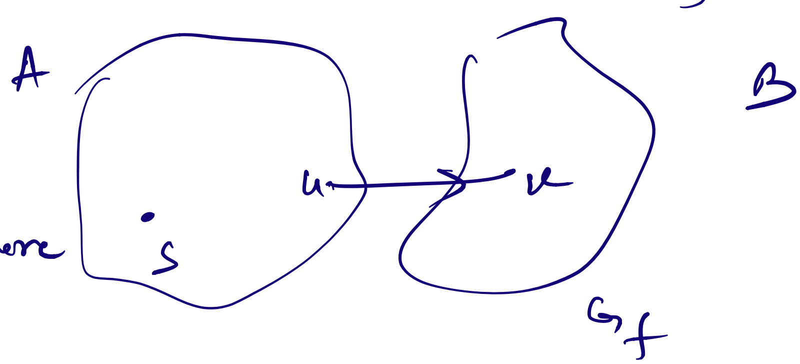
(ii) For any $e \in E(A, B)$,

$$f(e) = c(e)$$

Pf: If not, say $f(e) < c(e)$

Let $e = (u, v)$.

In G_f, there $c_f(e) > 0$.



$\Rightarrow \exists$ a path from s to v

which contradicts that
$v \in B$, (by definition of B)

(iii) $e \in E(B, A) \Rightarrow f(e) = 0$.

$$\Rightarrow v(f) = f^{out}(A) - f^{in}(A)$$

$$= C(A, B)$$

<u>Cor.</u> There is an $O(C \cdot m)$ algorithm to find a min-cut of G.

<u>Pf:</u> Do BFS on G_f,

where f is a max-flow to
construct $A = \{v \in V : \exists$ a path in G_f from s to $v\}$.

<u>Integrality Thm:</u> If all capacities of a flow network G are integers, then there is a max flow f which is integral.
($\forall e \in E$, $f(e)$ is an integer).