26th April, 2019.

Optimization Problems

Given a domain X, function $f : X \to \mathbb{R}$

$$\max_{x \in X} f(x) \quad \text{subject to}$$

$$\left\{ \text{constraints on } x \right\}$$

or

$$\min f(x) \quad \text{subject to}$$

$$\left\{ \text{constraints} \right\}.$$

In this class:

$X = \mathbb{R}^n$; $f : \mathbb{R}^n \to \mathbb{R}$ is a

linear function.

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \left\{ \begin{array}{c} \underline{\underline{\text{affine}}} \ \text{constraints} \\ \text{on } x \end{array} \right\}$$

Affine and Linear functions.

$f : \mathbb{R}^n \to \mathbb{R}$ is a linear function if $\forall\, \alpha, \beta \in \mathbb{R}$, $\forall\, x, y \in \mathbb{R}^n$,

(i) $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta f(y)$

Vector addition.

scalar addition.

(ii) $f(\vec{0}) = 0$.

Fact: If $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function, f must be of the form $f(x) = \sum_{i=1}^{n} c_i x_i$, c_i's are real numbers.

$f(x)$ is affine; $f(x) = \underbrace{\ell(x)}_{\text{linear function}} + \underbrace{b}_{\mathbb{R}}$.

<u>Reminder</u> / Notations) $\boxed{a \in \mathbb{R}^n,\ b \in \mathbb{R}^n \quad a \geq b \text{ if } a_i \geq b_i\ \forall i \in [n]}$

$a \in \mathbb{R}^n$ $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ column vector.

$a_i \in \mathbb{R}$.

For any $x \in \mathbb{R}^n$, x_i will denote its i^{th} coordinate.

<u>Matrices.</u> $M \in \mathbb{R}^{m \times n}$ be a $m \times n$ matrix on $\mathbb{R}$

M_{ij} will denote the $(i,j)^{th}$ entry.

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & \\ \vdots & & & \\ \vdots & & \cdots & M_{mn} \end{pmatrix}$$

$M_{m1} \quad - \quad - \quad \cdots$

M^T is a $n \times m$ matrix,

$$(M^T)_{ij} = M_{ji}$$

$M \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$,

$Mx = y \in \mathbb{R}^m$, where

$\forall i \in [n]$, $y_i = \sum_{j=1}^{n} M_{ij} x_j$

Linear Programming [with applications to approximating Vertex cover]

Standard set up:

Given: A is a $m \times n$ matrix on $\mathbb{R}$.

$b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, Task: $\min \quad c^T x \quad , x \in \mathbb{R}^n$

subject to

(i) $\quad x \geq 0 \qquad [x_i \geq 0 \ \forall i]$

(ii) $\quad Ax \geq b.$

Expanded form

$$\min \sum_{i=1}^{n} c_i x_i \qquad \text{Subject to}$$

$$x_j \geq 0 \qquad \forall j \in [n]$$

$$\sum_j A_{ij} x_j \geq b_i \qquad \forall i \in [m]$$

<u>Example:</u> $\quad A = \begin{bmatrix} -5 & 4 \\ -2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$c = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad , m = n = 2.$$

$$\min_{x \in \mathbb{R}^2} \quad c^T x \qquad \text{subject to}$$

$$Ax \geq b$$

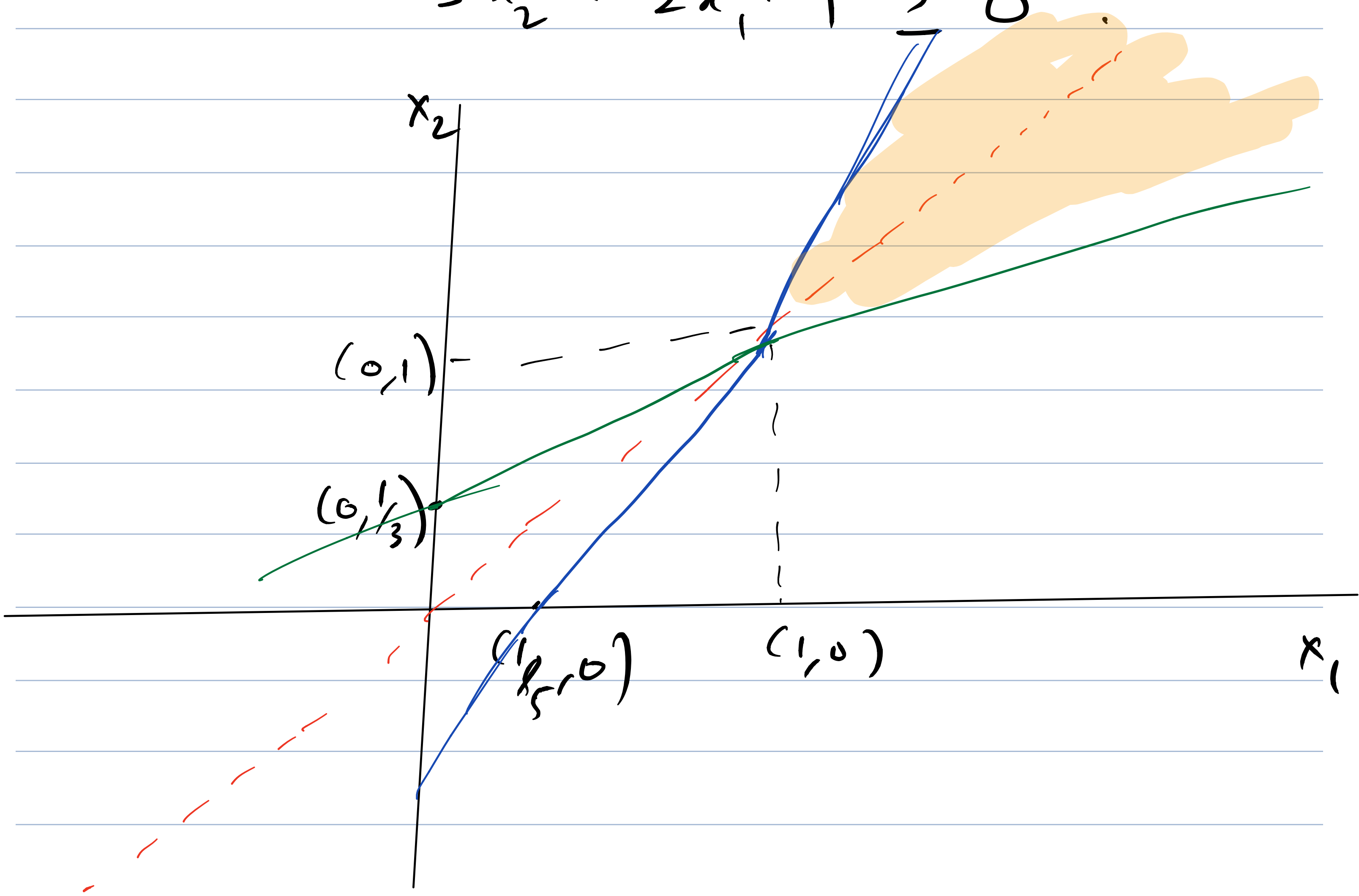
$$x \geq 0$$

i.e $\quad \min_{x \in \mathbb{R}^2} \quad x_2 - x_1 \quad ,$

$$x_1 \geq 0 , \qquad x_2 \geq 0$$

$$4x_2 - 5x_1 + 1 \geq 0$$

$$3x_2 - 2x_1 - 1 \geq 0$$



① We know how to solve linear programs (LPs) in polynomial time.

② LP's are widely applied
- Flow networks are a special case.

Integer linear Programming (ILP)

min $c^T x$ subject to

(i) $Ax \geq b$

(ii) $x \geq 0$

(iii) $x \in \mathbb{Z}^n$. new addition constraint.

Thm ILP is NP-hard.

Pf: VC $\leq_p$ ILP
(vertex-cover)

[we will not prove this in class, but enough hints will be given to get a proof].

2-Approximation algorithm for Vertex Cover.

* Weighted Vertex cover [WVC]

given: $G = (V, E)$; $w \in \mathbb{R}^n$, $w \geq 0$. [weights on the vertices]

Task: Find a vertex cover $S \subseteq [n]$ of G which minimizes $\sum_{i \in S} w_i$.

if $w = \vec{1}$, then WVC is VC.

The following ILP captures WVC:

$$\min_{x \in \mathbb{R}^n} w^T x ,$$

(i) $x_i + x_j \geq 1 \quad \forall (i,j) \in E$

(ii) $\forall i \in [n] \quad x_i \in \{0,1\}$.

$\equiv \forall i \in [n], \quad 0 \leq x_i \leq 1,$

(iii) $x \in \mathbb{Z}^n$.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ \vdots \\ m \end{array} \overset{\text{vertices}}{\begin{array}{c} 1 \ 2 \ \cdots i \cdots \quad j \quad m \\ \left(\begin{array}{cccccccccc} & & & & & & & & & \\ & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & & & & \end{array} \right) \end{array}}$$

edges

Let $\bar{x}$ be the solution of ILP.

<u>Claim:</u> Let $S = \{ i : \bar{x}_i = 1 \}$.

S is the required solution of the WVC instance.

Pf: Direct from construction $\square$.

An approximation algorithm for WVC.

<u>Natural idea</u>: drop the 'integer constraint' on x'.

<u>LP formulation</u>.

$$\min_{x \in \mathbb{R}^n} w^T x,$$

(i) $x_i + x_j \geq 1 \quad \forall (i,j) \in E$

(ii) $\forall i \in [n], \quad 0 \leq x_i \leq 1,$

→ Solve the above LP.

→ Let $\bar{x} \in \mathbb{R}^n$ denote the solution of the LP.

→ Roundoff each $\bar{x}_i$ to nearest integer

[if $\bar{x}_i \in [0, 1/2)$, set $x_i^* = 0$

else set $x_i^* = 1$].

$\underline{\underline{x^*}}$ is the 'integer' solution.

<u>Claim1:</u> $S = \{ i : x_i^* = 1 \}$ is a VC of G.

<u>Claim 2:</u> $w^T x^* \leq 2 w^T \bar{x}$.

<u>Proof of Claim 1:</u> For each edge $(i,j) \in E$,

$$\bar{x}_i + \bar{x}_j \geq 1 .$$

Since $0 \leq \bar{x}_i , \bar{x}_j \leq 1$,

either $\bar{x}_i \geq 1/2$ or $\bar{x}_j \geq 1/2$.

$\downarrow$ $\downarrow$

$x_i^* = 1$ or $x_j^* = 1$.

$$\Rightarrow \quad x_i^* + x_j^* \geq 1 \quad \forall (i,j) \in E.$$

$\square$.

<u>Proof of Claim 2:</u>

x_{opt} be an optimal solution of ILP.

(i) $w^T \bar{x} \leq w^T x_{opt}$. ✓.

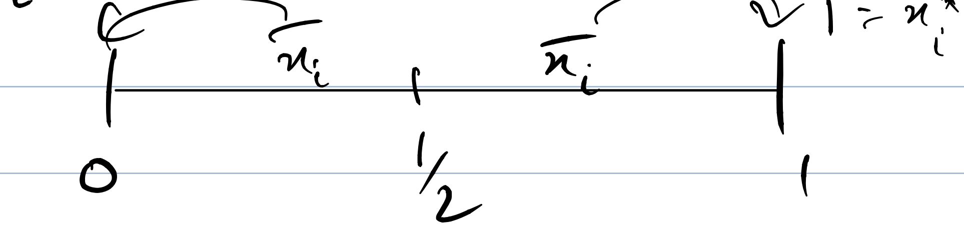
Since LP is optimizing over larger domain than ILP.

$$\underbrace{w^T x^* \leq 2\, w^T \bar{x}}_{\text{need to prove this!}} \underset{\text{(using (i))}}{\leq} 2\, w^T x_{opt}.$$

For each i,

$$x_i^* \leq 2\, \bar{x}_i. \quad ✓.$$

$x_i^* = 0$



$\Rightarrow$ Since $w \geq 0$,

$$w^T x^* = \sum w_i x_i^* \leq 2 \sum w_i \bar{x}_i$$
$$= 2\, w^T \bar{x}.$$

$\Box$.

Approximation algorithm for Knapsack.

Knapsack : items $1, \ldots, n$

(integers) weight $w_1, \ldots, w_n$.

(integers) values $v_1, \ldots, v_n$.

$$\underset{S \subseteq [n]}{\text{maximize}} \quad \sum_{i \in S} v_i \qquad \text{subject to}$$