11 Mar 2019

Bipartite Maximum Matching (§7.5)

Recap.

1. Ford-Fulkerson Max Flow Algorithm

Running time: $O(m\,C)$

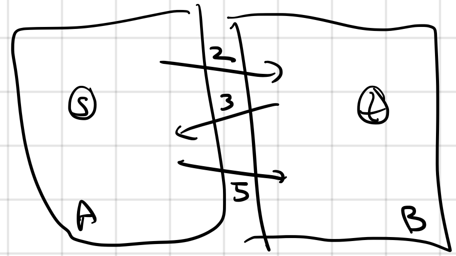
m ← # edges

C ← combined capacity of edges leaving s

Really C could be any upper bound on the min cut capacity.

2. Max-flow min-cut theorem.

$$\max\{v(f) \mid f \text{ an s-t flow}\} = \min\{c(A,B) \mid A,B \text{ an s-t cut}\}$$



$c(A,B) = 7$, not 10.

Given flow f and cut (A,B):

f is a max flow and (A,B) is a min cut

$\Longleftrightarrow$ $v(f) = c(A,B)$

$\Longleftrightarrow$ f saturates every edge from A to B and leaves empty every edge from B to A.

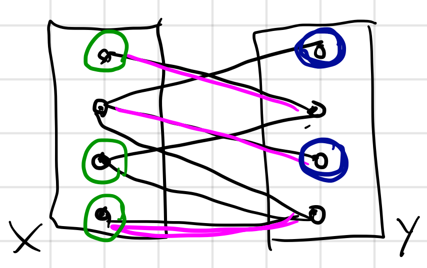
3. Flow integrality theorem

If G is a flow network with integer capacities, the set of maximum flows always includes an integer-valued flow.

The Bipartite Maximum Matching Problem

Given a bipartite graph $G = (V,E)$ where $V = X \cup Y$ and every edge has one endpoint in X, the other in Y.

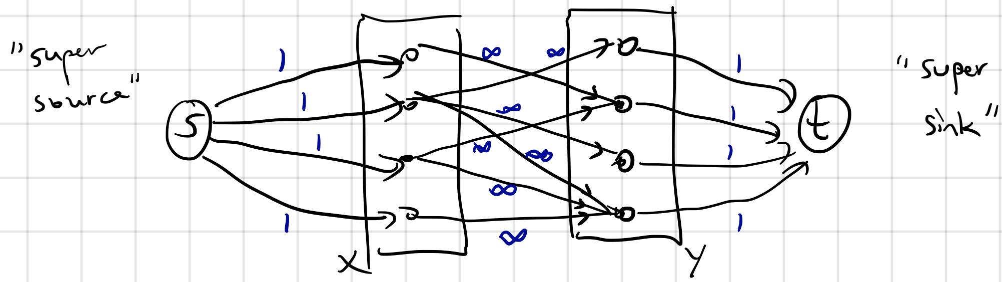
At most 2 of the 3 green nodes can be matched at the same time,

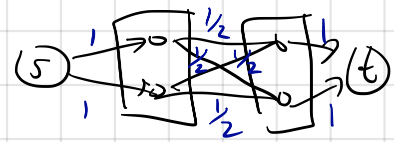


Problem: Find a matching with as many edges as possible.

At most 3 vertices in Y can be matched simultaneously, because the 2 blue nodes have only 1 neighbor

Designing an algorithm to solve bipartite max matching.





∞ is a symbol denoting $C+1$.

Create a flow network with vertex set:

- Source s
- sink t
- left nodes X
- right nodes Y

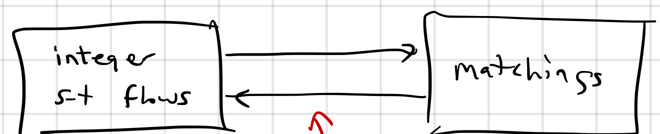
and edge set:

- (s, x_i) with capacity 1 $\forall\, x_i \in X$
- (y_j, t) " " " $\forall\, y_j \in Y$
- (x_i, y_j) " " ∞ $\forall\, (x_i, y_j) \in E$ ← edge set of bipartite graph

Use Ford-Fulkerson to compute an integer max flow in this network.

Let $M = \{ (x_i, y_j) \mid f(x_i, y_j) = 1 \}$.

Why is this correct?



bijection s.t. $v(f)$ equals size of corresponding matching.

Properties we must show:

① f is an int flow $\Rightarrow \{(x,y) \mid f(x,y)=1\}$ is a matching.

② M is a matching $\Rightarrow$ this thing is an integer flow.

$$f \longmapsto \{(x,y) \mid f(x,y)=1\}$$

$$\left\{\begin{array}{ll} f(s,x)=1 & \text{iff } x \in M \\ f(x,y)=1 & \text{iff } (x,y) \in M \\ f(y,t)=1 & \text{iff } y \in M \end{array}\right\} \longleftarrow M$$

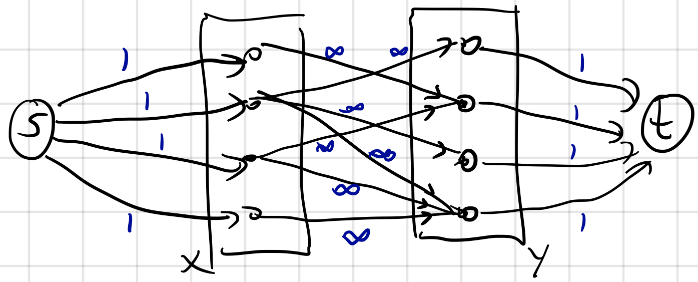
(2a) capacity (2b) conservation

③ $\underbrace{f \mapsto M \mapsto f}_{\text{identity}}$

④ $\underbrace{M \mapsto f \mapsto M}_{\text{identity}}$

⑤ If f maps to M then $v(f) = |M|$.

What does max-flow min-cut imply about bipartite matchings?



If (A,B) is a min s-t cut then $c(A,B) \leq C < \infty$.

Write

$$A = \{s\} \cup A_X \cup A_Y \qquad A_X := A \cap X$$
$$\qquad\qquad\qquad\qquad A_Y := A \cap Y.$$

$$B = \{t\} \cup B_X \cup B_Y$$

Edges from A_X to B_Y have capacity ∞ so a min-cut cannot have any such edges.

$\Longrightarrow$ every edge leaving A_X goes to A_Y.

2 types of edges crossing the cut: $\{s\}$ to B_X, A_Y to $\{t\}$.

$$c(A,B) = |B_X| + |A_Y|.$$