25th March, 2019

Outline of today's lecture.

(1) Prove HAM-CYCLE is NP-complete

(2) " U-HAM-CYCLE " " "

(3) Prove TSP is " " "
(we will also prove Metric-TSP is NP-complete).

HAM-CYCLE

**Defn**: A cycle C in a directed graph G that visits each node exactly once is called a

Hamiltonian cycle.

Input: Directed graph G.

Yes instance : If G contains a Hamiltonian cycle.

NO : o/w.

Thm 1 $\quad 3SAT \leq_p$ Ham-Cycle.

(Observe that Ham-cycle is in NP.)

Pf of Thm 1 $\quad \phi \xrightarrow{f} G$ directed graph

instance of 3SAT.

$I_{3SAT}$ : $\phi$ (3CNF)

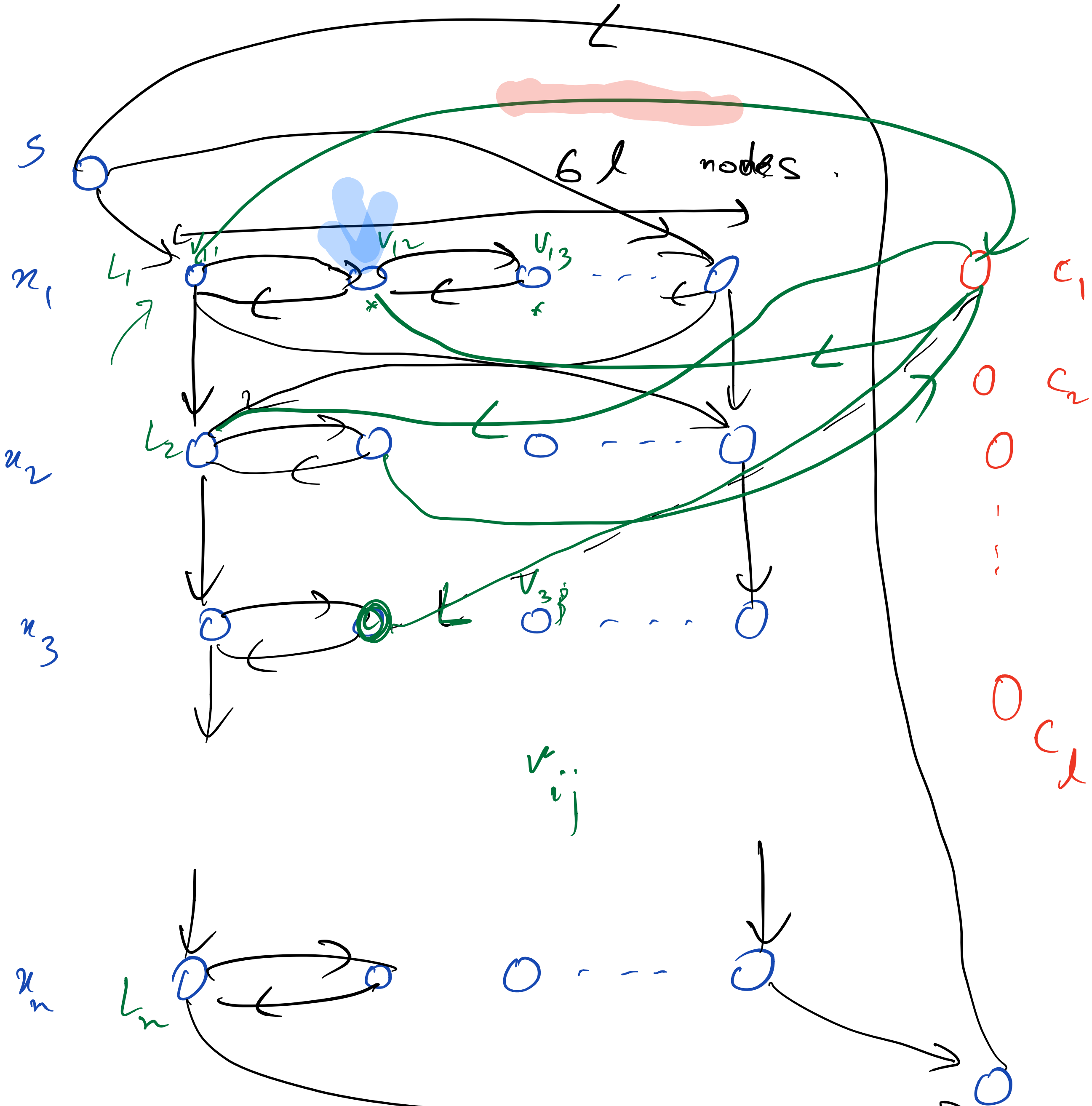
$\phi = C_1 \wedge C_2 \; --- \; \wedge C_\ell$

$C_i$ contains exactly 3 literals.

[ Variables: $x_1, ---, x_n$

literals: $x_1, --, x_n, \bar{x}_1, ---, \bar{x}_n$ ]

Goal: $\phi$ is satisfiable iff $G$ has a Hamiltonian cycle.




$C_1 = x_1 \vee \underline{\underline{\bar{x}_2}} \vee x_3$

If variable $x_i$ appears in clause $C_j$:

Choose nodes from layer $L_i$.

choose $6(j-1)+1^{th}$ and $(6(j-1)+2)^{th}$ nodes from $L_i$.

If literal $x_i$ appears: $v_{i,6(j-1)+1} \to c_j$

$c_j \to v_{i,6(j-1)+2}$

If literal $\bar{x}_i$ appears:

$v_{i,6j+2} \to c_j$

$c_j \to v_{i,6j+1}$

<u>Claim:</u> $\phi$ is satisfiable iff $G$ has a Hamiltonian cycle.

<u>Pf:</u> ($\Rightarrow$) Suppose $\phi$ has a satisfying assignment $A$.

If $x_i \to T$ then

traverse left to right in layer $i$.

$x_i \to F$

trav - " - layer $i$

right to left.

For each $C_j$, there is some literal $z_i$ (which is $x_i$ or $\overline{x_i}$)

which makes it true.

Visit the node $C_j$ when traversing $x_i$.

($\Leftarrow$) Suppose $G$ has a

Hamiltonian cycle.

"Claim" the layers are traversed in order (ie $L_i, L_{i+1}, \ldots$)

and further each $L_i$ is

traversed 'left to right' or

'right to left'.

If we use $v_{11} \to C_1$

then we must use $c_1 \to v_{12}$.
(this is general result for all other nodes)

A: $x_i \to T$ if left to right travers.
$x_i \to F$ if right to left trav.

$\square$

## U-HAM-CYCLE

Input: undirected graph G.

Yes instance iff G has a simple cycle visiting all nodes of G.

(each node is visited exactly once)

$$\text{HAM-CYCLE} \le_P \text{U-HAM-CYCLE}$$

$$G \xrightarrow{f} H$$

directed graph

undirected graph.

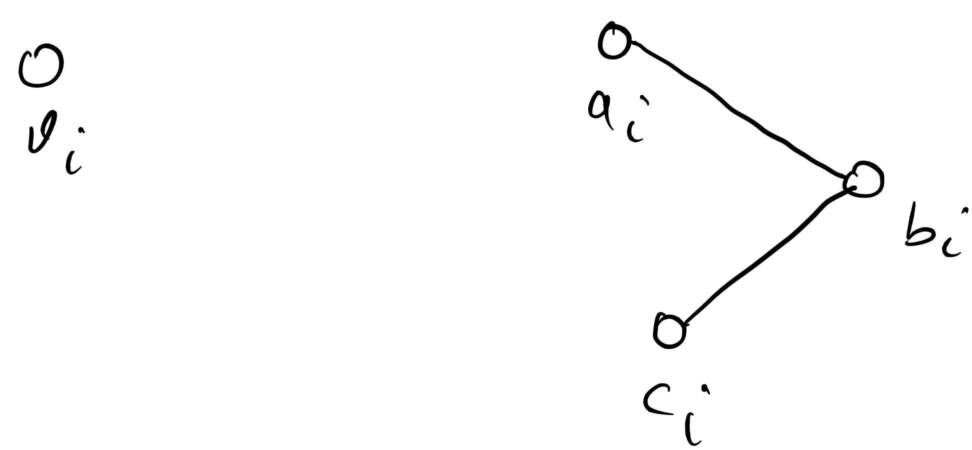
G has a Hamiltonian cycle iff

H " " " .

For each node $v_i$ in G,

construct 3 nodes in H.

$[a_i, b_i, c_i]$



$a_i$: all incoming nodes in G.

$c_i$: all outgoing " in G.

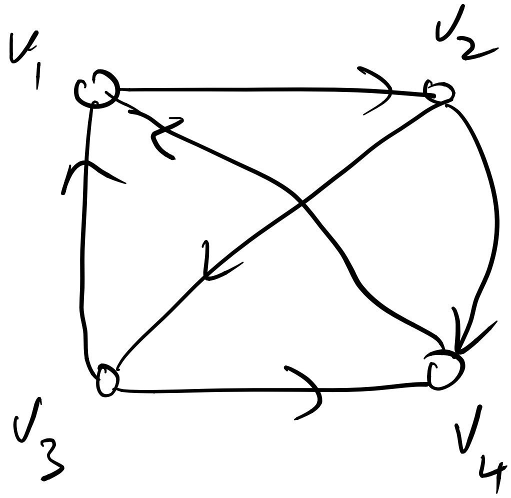
## Edges in $H$:

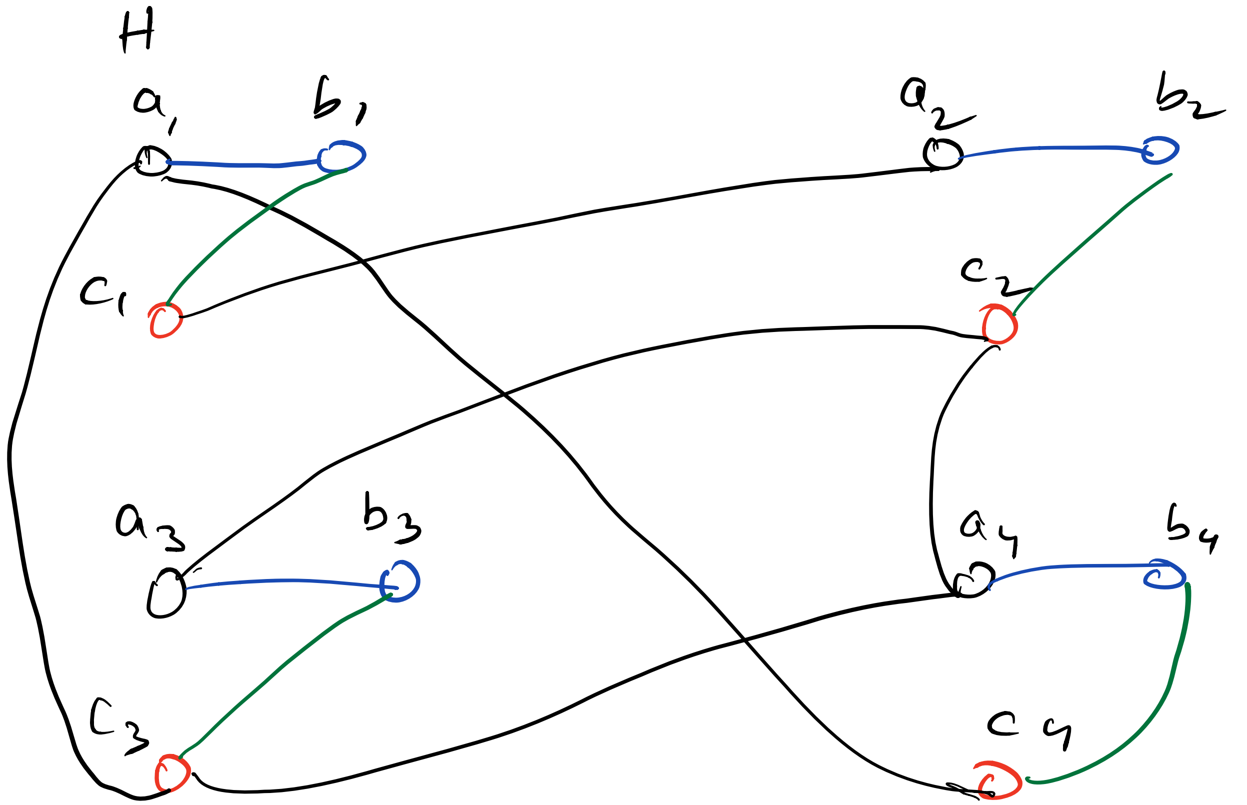
$\forall i$, $(a_i, b_i)$ and $(b_i, c_i)$.

For each edge $(v_i, v_j)$ in $G$,
edge $(c_i, a_j)$ in $H$.

## Example.

$G$.






**Claim:** G has a Hamiltonian cycle, iff H has a Hamiltonian cycle.

**Proof sketch:** ($\Rightarrow$) Suppose G has a Hamiltonian cycle $C = (v_{i_1}, v_{i_2}, \ldots, v_{i_n}, v_{i_1})$.

Then, $H$ has the Hamiltonian cycle:

$a_{i_1}\ b_{i_1}\ c_{i_1}\ a_{i_2}\ b_{i_2}\ c_{i_2}\ a_{i_3}\ b_{i_3}\ c_{i_3}$

$---- \ a_{i_m}\ b_{i_m}\ c_{i_m}\ a_{i_1}\ b_{i_1}\ c_{i_1}$.

($\Leftarrow$) suppose $H$ has a Hamiltonian cycle. Note each $b_i$ must have either $a_i$ or $c_i$ as the preceding node and the next node in the cycle.

Thus, the cycle in $H$ is either:

$a_{i_1}\ b_{i_1}\ c_{i_1}\ a_{i_2}\ b_{i_2}\ c_{i_2} \ ---- \ a_{i_m}\ b_{i_n}\ c_{i_n}$
$a_{i_1}\ b_{i_1}\ c_{i_1}$

or

$c_{i_1}\ b_{i_1}\ a_{i_1}\ c_{i_2}\ b_{i_2}\ a_{i_2}\ \cdots\ c_{i_n}\ b_{i_n}\ a_{i_n}$

$c_{i_1}\ b_{i_1}\ a_{i_1}$.

In either case it is easy to see that a cycle exists in $G$.

Traveling Salesman Problem (TSP)

Input : $n$ points $v_1, v_2, \ldots, v_n$,

a function $d(v_i, v_j) \to \mathbb{R}^{\geq 0}$,

integer $K$.

Output : Yes if there is a way to visit all cities exactly once, and return to start city with cost $\leq K$.

Tour: $v_{i_1} \to v_{i_2} \to v_{i_3} \to \cdots \to v_{i_n} \to v_{i_1}$.

Cost of the tour:

$$\left( \sum_{l=1}^{n-1} d(v_{i_l}, v_{i_{l+1}}) \right) + d(v_{i_n}, v_{i_1})$$

Witness: $v_{i_1} \to v_{i_2} \cdots \to v_{i_{n+1}}$

TSP $\in$ NP.

Def^n: Metric (1) $\forall x$, $d(x,x) = 0$

(2) $\forall x, y$, $d(x,y) = d(y,x)$

(Triangle ineq.) (3) $\forall x,y,z$ $d(x,z) \le d(x,y) + d(y,z)$

Thm Metric-TSP is NP-complete.

Pf: (i) Metric-TSP $\in$ NP.

(ii) U-HAM-CYCLE $\le_p$ Metric-TSP

Input: Undirected graph $G$.

Yes: If $\exists$ Ham cycle.

$$G \xrightarrow{f} v_1, \ldots, v_n,$$

metric $d$,

integer $K$.

For vertex $u_i$ in $G \to$ Point $v_i$.

If $(u_i, u_j) \in E(G)$,

$$d(v_i, v_j) = 1.$$

If $(u_i, u_j) \notin E(G)$,

$$d(v_i, v_j) = 2.$$

$$K = n.$$

If $C$ is a Ham. cycle in $G$,
let $C = (u_{i_1}, u_{i_2}, \ldots, u_{i_n}, u_{i_1})$.

The tour $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_n} \to v_{i_1}$

has cost exactly $n$.

Now suppose there is a tour of cost at most $n$.

Let the tour be: $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_n} \to v_{i_1}$.

Since each $d(v_i, v_j) \geq 1$, the cost of the tour must be exactly $n$.

Thus, each $d(v_{i_j}, v_{i_{j+1}}) = 1 \ \forall j$.

$\Rightarrow (u_{i_1}, u_{i_2}, \ldots, u_{i_n}, u_{i_1})$ must be a cycle in $G$.

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