CS 4820, Spring 2019	**Homework 9, Problem 1**
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(1) *(10 points)* Let $\mathcal{L} \subseteq \{0,1\}^*$ be a language, and let $\overline{\mathcal{L}}$ be its complement, i.e. the set of all finite binary strings that don't belong to $\mathcal{L}$. Prove that $\mathcal{L}$ is decidable if and only if both $\mathcal{L}$ and $\overline{\mathcal{L}}$ are recognizable.

Claim: $\mathcal{L}$ is decidable iff both $\mathcal{L}$ and $\overline{\mathcal{L}}$ are recognizable.

Sub claim: $\mathcal{L}$ is decidable if both $\mathcal{L}$ and $\overline{\mathcal{L}}$ are recognizable.

Proof: Suppose $\mathcal{L}$ is decidable. By definition, this means there exists some function M which can either reject or accept on every possible input x as being a member of $\mathcal{L}$. M cannot possibly fail to terminate in finite time by the definition of decidable. Using this same function M, it is clear that for all inputs which are elements of $\mathcal{L}$, M will accept x. Thus, M recognizes $\mathcal{L}$. Using a slightly modified M which switches the outputs (if M accepts x then x is not in the language and if M rejects x then x is a member of the language), then this modified M recognizes $\overline{\mathcal{L}}$. As both $\overline{\mathcal{L}}$ and $\mathcal{L}$ are recognizable the subclaim holds.

Sub claim: if both $\mathcal{L}$ and $\overline{\mathcal{L}}$ are recognizable, then $\mathcal{L}$ is decidable.

Proof: Suppose that both $\mathcal{L}$ and $\overline{\mathcal{L}}$ are recognizable. By definition of recognizable, we know there must exist some M which recognizes $\mathcal{L}$ and some M' which recognizes $\overline{\mathcal{L}}$. Thus given some input x which is an element of $\{0,1\}^*$ we can run both M and M' on input x. If M accepts x then it is a member of $\mathcal{L}$ and if x is accepted by M' then it is a member of $\overline{\mathcal{L}}$ and thus not a member of $\mathcal{L}$. As both $\mathcal{L}$ and $\overline{\mathcal{L}}$ recognized by M and M' respectively, we know that M and M' must accept all members of their respective languages in finite time. Running M and M' together will always see at least one terminate in finite time as all inputs which are an element of $\{0,1\}^*$ are either an element of $\mathcal{L}$ or $\overline{\mathcal{L}}$. Thus, on all possible inputs we can either accept or reject all inputs in finite time based on the outputs of M and M' running together as our function. As we can either accept or reject every input as either and element of $\mathcal{L}$ or not an element of $\mathcal{L}$ in finite time, $\mathcal{L}$ is decidable.

Proof: As both sub claims hold, we can conclude that the original claim holds.