22 Feb 2019

Announcements.

1. No lunch-with-the-instructors today. Let's reschedule.
2. Prelim 1 is graded, make-up prelim not yet.
   Will release P1 grades on Gradescope within 24 hrs.
   Make-up prelim grades: goal is by end of Feb. break.
3. Homework 4 (single question) is on CMS, due Friday night 11:59 pm.

Clicker question #1: I think my prelim grade was

(A) < 25 (B) 25-30 (C) 30-35 (D) 35-40 (E) 40-45

DIVIDE & CONQUER (Chapter 5)

E.g. mergesort, binary search, quicksort.

Recap of running time analysis for mergesort.

Mergesort sorts a list of n objects by:
- partitioning into lists of size $n/2$ ($\lfloor \frac{n}{2} \rfloor$ & $\lceil \frac{n}{2} \rceil$) $O(n)$
- recursively sorting each of them ?? $2 \cdot T(\frac{n}{2})$
- combining the results into a sorted list in $O(n)$ time.

Let $T(n)$ be the running time of Mergesort on size-n input.

$T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$ ← This is fine.

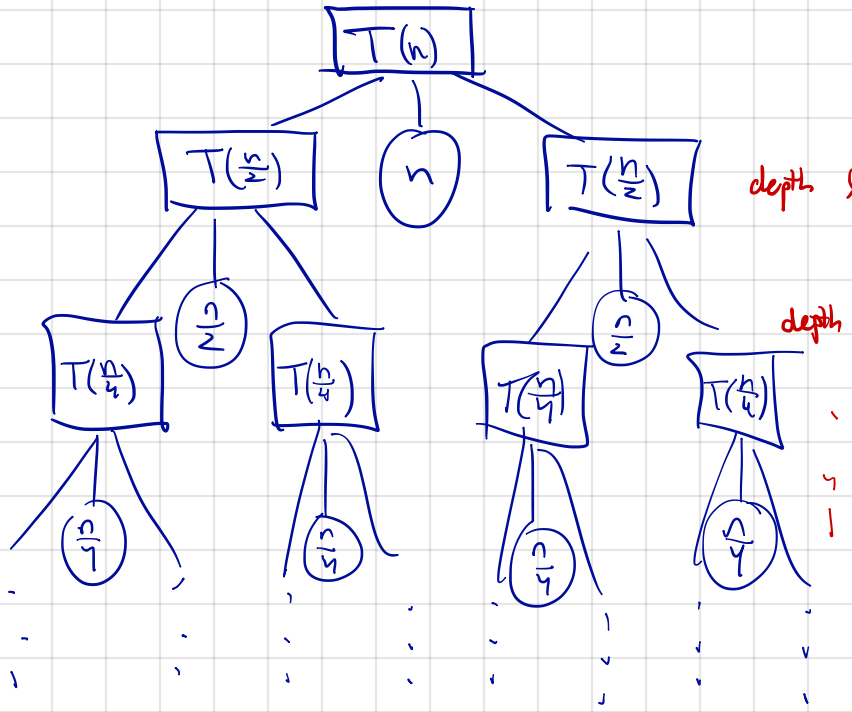
($T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + O(n)$.) ← Never mind ✓

$\forall n \geq 1$ $\quad T(n) = 2 \cdot T(\frac{n}{2}) + O(n) \qquad T(1) = O(1).$

$= 2 \cdot \left[2 \cdot T(\frac{n}{4}) + O(\frac{n}{2}) \right] + O(n)$

$= 2 \cdot \left[2 \cdot \left\{ 2 \cdot T(\frac{n}{8}) + O(\frac{n}{4}) \right\} + O(\frac{n}{2}) \right] + O(n)$

$= \ldots$ (iterate this substitution process)



depth 0

depth 1

depth $\log(n)$

Explanation: □ = sum of children

○ = leaf representing a number. (ignoring const factors.)

Lemma: The number represented by any □ is equal to the sum of the round-node leaves in its subtree.

Proof: Induction on depth of subtree.

At depth d we have 2^d round nodes labeled with $\frac{n}{2^d}$.

Sum of nodes in level d is n.

Sum of all round nodes in tree is $n \cdot (\#\text{ levels}) = n \cdot \log(n)$.

Multiplying numbers (and polynomials)

Current fastest algorithm: $O(n \cdot \log n \cdot 2^{\log^*(n)})$ (Fürer, 2007)

Plausibly fastest possible: $O(n \log n)$

Today's algorithm (Karatsuba): $O(n^{1.58\ldots})$

If $P_0(x) = a_0 x + b_0$

$P_1(x) = a_1 x + b_1$

then $P_0(x) \cdot P_1(x) = a_0 a_1 x^2 + (a_0 b_1 + a_1 b_0) x + b_0 b_1$

$= a_0 a_1 x^2 + [(a_0 + b_0)(a_1 + b_1) - a_0 a_1 - b_0 b_1] x + b_0 b_1$

To compute coefficients on 2nd line:

$c_0 = a_0 a_1$

$c_1 = (a_0 + b_0)(a_1 + b_1)$

$c_2 = b_0 b_1$

$P_0(x) P_1(x) = c_0 x^2 + (c_1 - c_0 - c_2) x + c_2$

(More subtraction, less multiplication)

Now say $P_0(x)$, $P_1(x)$ are polynomials of degree $\leq n-1$ where n is a power of 2.

$P_0(x) = A_0(x) \cdot x^{n/2} + B_0(x)$

$P_1(x) = A_1(x) \cdot x^{n/2} + B_1(x)$

A_0, A_1, B_0, B_1 are polyn's of degree $\leq \frac{n}{2} - 1$.

$P_0(x) \cdot P_1(x) = A_0(x) A_1(x) x^n +$

$[(A_0(x) + B_0(x)) \cdot (A_1(x) + B_1(x)) - A_0(x) A_1(x) - B_0(x) B_1(x)] x^{n/2}$

$+ B_0(x) B_1(x)$.

KARATSUBA (P_0, P_1): If degree (P_0), degree $(P_1) < n$ and $n = 1$: just multiply them.

Else

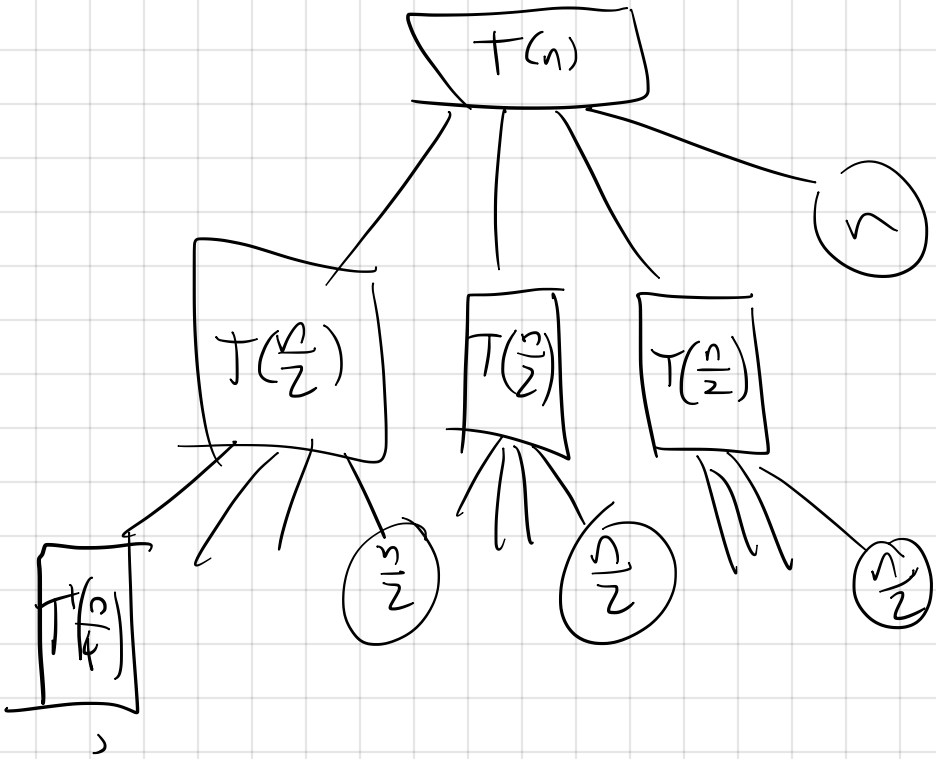
$C_0(x) = A_0(x) A_1(x)$

$C_1(x) = (A_0 + B_0)(A_1 + B_1)$ } 3 lines called Recursively

$C_2(x) = B_0 \cdot B_1$

Output $C_0(x) \cdot x^n + [C_1(x) - C_0(x) - C_2(x)] \cdot x^{n/2} + C_2(x)$.

$O(n)$



depth $\log(n)$

Level d has 3^d nodes labeled with $\frac{n}{2^d}$.

Running time: $\sum_{d=0}^{\log_2(n)} 3^d \cdot \frac{n}{2^d}$

$= n \cdot \sum_{d=0}^{\log_2(n)} \left(\frac{3}{2}\right)^d$

$= O\left(n \cdot \left(\frac{3}{2}\right)^{\log_2(n)}\right)$.

Sum(geom series) = O(largest term)

$= O\left(n \cdot 2^{\log_2(3/2) \cdot \log_2(n)}\right)$

$= O\left(n \cdot 2^{\log_2(n) \cdot \log_2(3/2)}\right) = O\left(n \cdot n^{\log_2(3/2)}\right) = O(n^{1.58\ldots})$

$\log_2(3/2) \approx 0.58\ldots$