27 Mar 2019 Subset Sum. (NP Complete problems in arithmetic)

Remark about Euclidean TSP.

Input: list of points in $\mathbb{R}^2$ with int coords $(x_1, y_1), \ldots, (x_n, y_n)$
integer k
Question: $\exists$? a TSP tour that visits all points and with combined (Euclidean) length $\leq k$

This problem is NP-hard. NOT KNOWN TO BE NP-COMPLETE! We don't know a verifier.

Subset Sum Problem:

Input: list of integers $w_1, \ldots, w_n$ (in binary)
target sum W (in binary)
Question: $\exists$? $S \subseteq [n]$ s.t. $\sum_{i \in S} w_i = W$.

Remark: When we worked on dyn prog we saw an algorithm that solves this in time $O(nW)$. ← exponential in input size, which is $O(n \log W)$

Boolean table: $T[i,j] = \begin{cases} \text{TRUE} & \text{if a subset of } \{w_1, \ldots, w_i\} \text{ sums up to } j \\ \text{FALSE} & \text{otherwise.} \end{cases}$

$i = 0, \ldots, n$; $j = 0 \ldots W$.

$T[0,j] = \begin{cases} \text{TRUE} & \text{if } j = 0 \\ \text{FALSE} & \text{if } j > 0 \end{cases}$

$(i > 0)$

$T[i,j] = \begin{cases} T[i-1,j] & \text{if } w_i > j \\ T[i-1,j] \vee T[i-1, j-w_i] & \text{if } w_i \leq j \end{cases}$

3SAT $\leq_p$ SUBSET SUM:

Variables: $x_1, \ldots, x_n$

Clauses: $C_1, \ldots, C_\ell$

(1) Convert each variable to a pair of numbers.

$x_1 \longmapsto \quad w_1 = 1, \quad w_2 = 2$

$x_2 \longmapsto \quad w_3 = 4, \quad w_4 = 8$

$x_3 \longmapsto \quad w_5 = 16, \quad w_6 = 32$

$\vdots$

$x_n \longmapsto \quad w_{2n-1} = 4^{2n-2}, \quad w_{2n} = 2^{2n-1}$

This is not good enough: nothing to prevent us from choosing both w_1 & w_2 which corresponds to $x_1 = T$ & $x_1 = F$.

(2)

$x_1 \longmapsto$

1 0 1	F
1 1 0	T
0 0 1	
0 1 0	
1 1 1	W

$x_2 \longmapsto$

101	000	F
110	000	T
001	000	
010	000	
111	111	W