INFO 4220: Homework 4

Due by: Thursday March 28, 2019, 12 noon

General instructions:

1. Do make sure to follow the instructions on writing up (counter)examples (CMS guidelines)—and continue to be sure that you substantiate your answers with either a proof or a counterexample (depending on whether you are saying a logical statement is true or false)!
2. The sage advice from the Hitchhiker's Guide continues to hold (and always will!):
   (a) Don't panic. (b) This homework (like many other things) is Mostly Harmless.
3. Finish and turn in the homework, on CMS, *well in advance of the deadline.*
4. If (Not 3.), *i.e.*, if you fail to turn in your homework in time:
   (a) See the course homework policy (Lecture 1; CMS handout).
   (b) Realize that the policy does *not* say "email instructor asking for extension". In fact, it suggests that you should perhaps *not* do this, unless you have a Most Excellent Reason (these are *very* hard to come by, so you're statistically *very* unlikely to have one).
   (c) Let realization in 4(b) sink in.
   (d) Make peace with your situation (refering to 2(a) as often as necesssary).
   (e) Have a great rest-of-day. You might even want to whistle a jolly tune!

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1. In markets with *one-sided* rank-order preferences, we know that there is always some agent who receives her top-ranked house in *any* Pareto-efficient matching. (Do you know this? Yes, you do!: This follows from the result we stated in class (Abdilkadiroglu and Sonmez'99) saying that every Pareto-efficient matching can be obtained as the output of some serial dictatorship mechanism (*i.e.*, with some particular priority ordering of the agents), which implies that the top agent in this priority order obtains her favorite item).

   Now we're going to investigate similar ideas in markets with *two-sided* rank-order preferences, *i.e.*, when both sides of the market have preferences over the agents on the other side. Assume that there are an equal number of agents on each side of the market, and that all agents have strict, complete preferences.

   (a) (*5 pts.*) Consider the marriage model. Does every stable matching contain a pair (m, w) such that m and w are at the top of each other's priority lists, for all possible markets (*i.e.*, all possible number of agents and preference rankings of agents)?
   (b) (*10 pts.*) Suppose there is a man m and a woman w who are at the top of each other's priority lists. Does every stable matching in this market contain the pair (m, w)?
   (c) (*15pts.*) Now suppose the market has n men and n women, where $n \geq 2$, and there is a man m and a woman w who are at the *bottom* of each other's priority lists[1]. Can such a pair (m, w) ever belong to a stable matching?

[1](Note that if a man is on a woman's priority list, this implies that he is acceptable to her, and vice versa).

2. (Existence of stable matchings: Examining non-pairwise grouping.) *(35 pts.)* In class, we used the Gale-Shapley algorithm to prove the existence of stable matchings in two-sided matching markets where each agent has rank-order preferences over agents on the other side, and each agent wants to be matched to at most one other agent. The deferred acceptance algorithm can be deceptive in its simplicity (and familiarity, for some), leading to an unfortunate tendency to not gape at the fact that stable matchings even exist. In this problem, we're going to investigate the marvelousness of the very existence of stable matchings.

   Stable matchings do exist in some many-to-one matching markets, for example where (e.g. colleges') preferences over groups (e.g., of students) are derived from their preference rankings over individuals. There are other markets, however, such as firms employing workers, where a firm might have preferences over *sets* of workers that cannot be derived from a rank-ordered list over individual workers (for reasons such as complementarity of skillsets).

   Consider a many-to-one matching market with 2 firms F_1 and F_2, and 3 workers w_1, w_2, w_3. Each worker can join only one firm, and the workers' preferences over firms are w_1: $F_2 \succ F_1$, w_2: $F_2 \succ F_1$, w_3: $F_1 \succ F_2$. Firms can potentially hire more than one worker, and have preferences over *sets* of workers. The firms' preferences are as follows: (i) F_1: $\{w_1, w_3\} \succ \{w_1, w_2\} \succ \{w_2, w_3\} \succ \{w_1\} \succ \{w_2\}$, and (ii) F_2: $\{w_1, w_3\} \succ \{w_2, w_3\} \succ \{w_1, w_2\} \succ \{w_3\} \succ \{w_1\} \succ \{w_2\}$. (As usual, we drop any unacceptable worker sets from the preference ordering, *i.e.*, any set of workers not listed in a firm's preference list is unacceptable to that firm.)

   A matching is stable if it is individually rational, and if there is no pair (F, w) that is not currently matched and that would prefer to be matched to each other.

   (a) Enumerate the set of all (valid) matchings in this market specified above.

   (b) Are any of them stable? (If you say a matching is not stable, you must point out a blocking pair demonstrating this.)

3. (Comparing matchings.) The man-proposing deferred acceptance algorithm produces the stable matching μ_M^* which is preferred by all men to all other stable matchings, and similarly women prefer the matching μ_W^* produced by the woman-proposing deferred acceptance algorithm. It is quite surprising that the two matchings μ_M^* and μ_W^* could be *compared* with all other stable matchings, *i.e.*, that all men agree about which matching they prefer (and similarly for women).

   (a) *(15pts.)* First, let's explore whether other kinds of 'good' matchings are comparable as well. Are an arbitrary pair of *Pareto-efficient* matchings comparable? That is, is it possible to compare (as defined in class) *every* pair of matchings that are each Pareto-efficient with respect to the agent's preferences, for all possible (strict) preferences and any number of agents?

   (b) *(20pts.)* When there is a unique stable matching, the man-proposing and woman-proposing algorithm must, by neccessity, return the same outcome. Is the converse also true—that is, if the man-proposing and woman-proposing DA algorithms return the same outcome on an instance, is there only one stable matching for that instance? (Hint: Can you use what you've learnt about comparing matchings to answer this question concisely?)