Networks hw3

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1. (Weak preferences: Rank-order preferences with no initial endowments.) Let's start by rewinding all the way back to the serial dictatorship mechanism, which we saw is Pareto-efficient for the setting where agents have strict rank-order preferences over a set of items, and there are no initial endowments. Now suppose agents can have and report weak preferences in that setting: that is, an agent is now allowed to be indifferent between two (or more) objects that she finds acceptable; in other words, an agent may have ties amongst acceptable objects in her rank-ordered preference list.

Consider the following extension of the serial dictatorship mechanism for weak preferences: If an agent has two (or more) objects tied at the top of her list (i.e., that she likes equally) from amongst the remaining objects when her turn arrives, this mechanism randomly assigns her one of these objects, and then proceeds as usual.

Assume preferences are complete—i.e., all agents find all houses acceptable.

1. (a) (10pts.) Is this mechanism Pareto-efficient when agents can have weak preferences? If yes, argue why it is Pareto-efficient; if not, provide a counterexample.
   Arguments in favor must be brief and to the point; your counterexample, if this is the route you take, must involve exactly 3 agents and 3 items.

   This algorithm is not Pareto-efficient by the following counter example. 3 agents: a,b,c

   3 houses: x,y,z

   Preferences: a: x=y>z b: x>y>z c: z>x>y

   Using the extended serial dictatorship algorithm in this case with priority list a then b then c we get.

   a has first choice x tied with y, a could be randomly assigned to either x or y, we will say a is paired with x in this example. Next b will try to be paired with x but x is already paired so b must be paired with its second choice y. Finally, c is paired with its first choice z. However, this matching outcome is not Pareto-efficient as a could be paired with y and b could be paired with x and each agent would be just as well off while b would be better of receiving a higher preference choice. Thus, this matching would dominate the matching generated by the extended serial dictatorship algorithm.

   (b) (5pts.) Is this mechanism strategyproof? If yes, provide a brief argument why; if not, provide a counterexample.

   Claim: The extended serial dictatorship algorithm is strategyproof

Proof: This means that for all preference profiles $\succ_{-a}$ including the preferences of some agent a, a cannot improve the item it receives by changing it's preference profile. We can see this as the extended serial dictatorship algorithm will have the same priority order such that a is the ith agent in the priority. For all agents that come before a their choices are independent of a's preference profile and thus, when the algorithm gets to a the same set of remaining houses will be available in all cases. Given true preferences for a the extended serial dictatorship algorithm will select the highest ranked remaining house for a and assign it to a. Thus, as the remaining items is independent of a's preference profile and the algorithm assigns a's highest ranked item of those that are left, a cannot change it's profile in any way to improve the item given to it. Thus, the extended serial dictatorship algorithm is strategy-proof.

2. (Markets with mixed initial endowments.) We saw two extremes of initial endowments in class—no agents have initial endowments, and all agents have initial endowments. In many real-life situations, we have some midway between these extremes. Consider, for instance, our friend Lisa from HW2. Lisa is now organizing a used clothing exchange-plus- giveaway, using donations she has already received in addition to what individuals may choose to bring in: Participants may enter the exchange either bringing an item of their own that they'd like to exchange for something nicer, if possible, or enter empty-handed, looking for a single donated item to take away. This is a situation where there are some agents who have no initial endowments (the participants who come in empty-handed), and some who do (the ones who bring in an item, looking to exchange it for something they like better, if possible— it would not do to send these participants away with an item they like less than the one they own already!).

Throughout this problem, assume that (i) as usual, agents' preferences over items are strict and complete and (ii) that there are exactly the same total number of items as there are participants in the exchange.

1. (a) (15pts.) Facing a situation which lies squarely between the two extremes (where no agents have initial endowments, and all agents have initial endowments), Lisa's first thought is to simply use the serial dictatorship mechanism, completely ignoring the initial endowments of agents who bring items to the exchange—after all, as we've seen, this is an excellent mechanism when agents do not have intial endowments. Is this mechanism (i) Pareto-efficient (ii) individually rational for the setting Lisa is actually dealing with? (Note that the serial dictatorship mechanism may use any (report-independent) priority ordering amongst participants.)

   In this case the mechanism is Pareto-efficient but it is not Individually rational.

   (i)Claim: Serial Dictatorship with mixed initial endowments is Pareto-Efficient.

   Proof: This algorithm is specified to simply run the Serial Dictatorship mechanism regardless of initial endowments. Thus, this algorithm will take any initial endowments and un pair them. Then we are in the situation of Serial Dictatorship with strict complete preferences. This has been shown in class as having been Pareto-efficient and thus, we conclude that the claim holds.

   (ii)The mechanism is not Individually-rational as shown by the below counter example:

3 agents: a,b,c

3 houses: x,y,z

Preferences: a: x>y>z b: x>y>z c: z>x>y

Initial endowments: (a,x)

Priority ordering: c,b,a

The Serial Dictatorship mechanism will un pair the initial endowments and then run. Thus, c will pair with z, b will pair with x and a will be forced to pair with y. In this case a will be worse off than if a had just not participated in the market and thus, the mechanisms is not individually rational.

2. (b) (15pts.) Lisa's second idea is to use the TTC algorithm, with the following adaptation. She 'endows' each participant who entered the exchange empty-handed with one of the donated items[1]; of course, each person who brought an item they own has an initial endowment already, which is that item. Lisa will then simply apply the TTC algorithm to this modified market where every agent now has an 'initial endowment', whether genuine or awarded by Lisa. Is this mechanism (i) Pareto-efficient (ii) individually rational for the setting Lisa is actually dealing with?

(i) Claim: The adapted TTC mechanism described above is Pareto-efficient
Proof: Suppose for the sake of contradiction that the mechanism is not Pareto-Efficient. This means that the outputted matching must be dominated by some other matching. As the preferences are strict, there must be at least two agents who are better off in the dominant matching. These two or more agent which are better off within the dominant matching would in the adapted TTC mechanism have had to have at pointed to the house they are paired with in the dominant matching. In such a case, this agent would not move on to the next highest in their preference list unless another agent became paired with this house. In that case this agent that was just paired with the house would not be able to get better by trading within the remaining houses as this is their highest rated house remaining. Thus, a dominant matching could not include this pairing as another agent would have to become worse off which would make the dominant matching no longer dominant. This is a contradiction and thus the mechanism is Pareto-Efficient.

(ii) Claim: The adapted TTC mechanism is Individually Rational
Proof: Within the adapted TTC algorithm all agents who are unpaired before will automatically be better off after the mechanism as there are initially unpaired which is worse than when the mechanism finishes and then they will be paired. Thus, we must show that all agents who were previously endowed are not worse off. We can see this is true as the TTC mechanism initialize with each agent pointing to their favorite house and then removing cycles. In the case where an agent point to the house it was already endowed with they are removed and paired with this initial endowment. There is no way for an agent to end up paired with a lower house than their initial endowment as this would mean that they would have gone past the cycle in which they pointed to their initial endowment which is not possible. Thus, all initially endowed agents receive their initial endowments back or a higher ranked house. Thus, the adapted TTC mechanism is individually rational as no agent would have been better off not participating in the market at all.

3. (c) (15 pts.) Lisa's third idea is to combine the two excellent mechanisms she's learnt in the following way: she simply separates the market into two parts—agents who brought items, and the agents who did not (the items in the first submarket are the ones the agents brought along with them, while the items in the second submarket are the donated items). In other words, she splits the

market into the part with agents who all have initial endowments (along with their items), and the part where none do (along with the remaining un-owned items). She assigns items in the submarket with initial endowments according to the TTC algorithm applied to that submarket, and items in the submarket with endowment-free agents and donated items according to the serial dictatorship mechanism.

(i) By counter example we will show this mechanism is not Pareto-Efficient below:

2 agents: a,b 2 houses: y,z Initial endowments: (a,z) Preferences: a: y>z b: z>y

In the described mechanism there will be two submarkets, the Serial Dictatorship market which has agent b and house y and the TTC market which has pair (a,z). In the serial dictatorship market b will be paired with y and in the TTC market a will remain paired with z. Thus, the output will be (a,z) and (b,y). This matching is dominated by (a,y) and (b,z) as both agents would prefer this. Thus, the mechanism is not Pareto-Efficient.

(ii) Claim: The mechanism described above is Individually Rational.
Proof: We want to show that each agent is the same as they were previously or better off. In the Serial Dictatorship market all agents where previously unmatched. All will end up matched, thus all will be better off. In the case of the TTC market each agent comes in already paired. As was shown in part b all will get their initial endowed house or a better house. Thus as in both markets each individual is better off or the same, this mechanism is individually rational.

Is this mechanism (i) Pareto-efficient (ii) individually rational for the complete market that Lisa is actually dealing with?

3. (Weak preferences: Markets with initial endowments.) Now let's return to the scenario you've been looking at most recently in class—a market with initial endowments and strict rank-order preferences, where n agents, each of whom owns a house (her initial endownment), has a strict preference ordering over all the houses; you've just studied the concept of the core for such markets. In this problem, we will see what might happen when agents can have weak preferences, along with initial endowments.

1. (a) (10pts.) Suppose we have a market with three agents with initial endowments (a,h_a), (b,h_b) and (c,h_c), with (strict) preferences as follows: a prefers houses in the order $h_b \succ h_c \succ h_a$, b prefers housesintheorder$h_a \succ h_c \succ h_b$,andcprefershousesintheorder$h_b \succ h_c \succ h_a$. Whatisthe core matching in this market?

The core matching returned is (a,h_b), (b,h_a) and (c,h_c). This matching is core because no subset of members can trade within themselves to make at least one of them better off. As both a and b have their top choice and cannot improve, any possible subset would have to include agent c as c is the only agent which could get a better house. C would prefer house b but trading initial endowments with agent b would not work as agent b would end up worse off receiving its second choice instead of it's first. Thus, this matching is core as not subset can be formed.

(b) (15pts.) Now suppose b and c's preferences are as before, but a is indifferent between h_b and h_c: that is, a's preferences are now $\{h_b, h_c\} \succ h_a$. Is there a core matching[2] in this modified market?

There is a core matching in this case and it is (a,h_c), (b,h_a) and (c,h_b). In this case each agent receives their top choice. Thus, there cannot exist a subset in which the members would be better off trading within themselves as no agent can be better off than they already are in the matching and thus this subset cannot exist.

[1]Which empty-handed entrant gets which donated item is determined according to some arbitrary permutation.

[2]Note that the definition of the core—the conditions a matching must satisfy to be in the core—does not depend on preferences being strict!

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If your answer is yes, supply the matching and explain why; if it is no, explain why not.

(c) (15pts.) What if a and c's preferences are as in part (a), but now b is indifferent between h_a and h_c: that is, b's preferences are now $\{h_a,h_c\} \succ h_b$. Is there a core matching in the market with these preferences? Again, if your answer is yes, supply the matching and explain why; if it is no, explain why not.

The answer here is no, the reason that there is no core matching in this case is because in all cases there will be a subset of the matching which could be better off has they traded within themselves. With 3 agents and 3 houses there are a total of 6 matchings. Two of these matchings will have c paired with h_a in which case this is not a core matching as c would prefer to have traded within itself based on its initial endowment and thus these matchings are not core. In the remaining cases either a or b must be paired with h_a. In the case where h_a is paired with a then the subset of a and b would rather have traded within themselves as both could end up with their top choices and currently a does not have its top choice and thus these matching are not core. In the final case where h_a is paired with b, the subset of b and whichever a or c is paired with h_c would have preferred to have traded within themselves as either a or c could have ended up with h_b which they would have preferred to h_c which they are currently paired with and b would have their top choice, thus these matchings are not core. As all matchings are not core there does not exist a core matching for these preferences.