Networks II: Market Design—Lecture 3
Matching Markets: One-sided Binary Preferences

ARPITA GHOSH

Dept. Of Information Science,
Cornell University

Logistics

- Clickers:
  - Registration site: atcsupport.cit.cornell.edu/pollsrvc/
  - Instructions to answer questions: https://teaching.cornell.edu/resource/using-iclicker-without-blackboard-or-canvas
  - Bring clickers to class starting next class
- FAQ: Technical writing—not an option offered with this course!
  - Reminder: Website
  - Reminder: CMS handouts

Recap from last lecture

- Matching markets without money: New issues
  - What *is* a 'good' allocation?
  - Need for new concepts, mechanisms, ...
- Matching markets with one-sided preferences: Agents only on one side
- The simplest setting: Binary preferences
  - Well-defined ideal allocations: Perfect matching
  - Existence of perfect matchings: Statement of Matching Theorem

Recap: What we're doing

- The setup:
  - Simplest form of preference: Binary (0-1)
  - A question: Can we assign students to rooms to 'make everyone happy'?
  - English to Math: Yes, if there is a *perfect matching* in corresponding bipartite graph
- Matching Theorem: A bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching *iff* it contains a constricted set.

Proving the Matching theorem: Where are we now?

- Recall key issue: Identifying a constricted set, knowing *only* that there is no perfect matching
- Structure of proof idea:
  - Start with a *maximum matching*
  - Try to enlarge it
  - Augmenting paths: A matching in a bipartite graph can be enlarged if there is an augmenting path
  - Show this produces a constricted set unless the matching was perfect
- Alternating Breadth-first Search: A way to look for augmenting paths so that a failure to find one produces a constricted set— unless the matching was perfect to start with

An aside

- Matching in bipartite graph can be enlarged if there is augmenting path: What about the converse?
- *It doesn't matter for our proof!* Recall:
  - No perfect matching: Maximum matching does not include all nodes
  - Start with a *maximum matching*
  - Try to enlarge maximum matching: Will fail
  - If matching is maximum, **any** method to enlarge matching must fail!
  - Failed attempt at enlarging returns a constricted set (unless original matching was perfect)
- Converses versus contrapositives

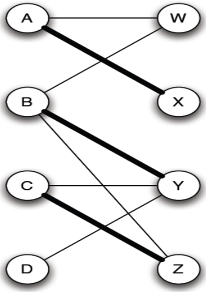
A procedure for finding augmenting paths

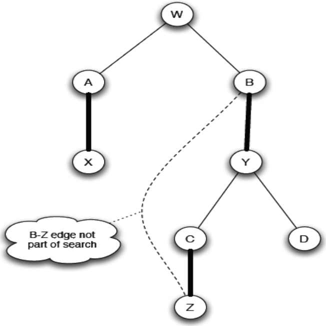
Alternating Breadth-First-Search (BFS)

- Start from an **unmatched** node on right
- Explore graph using *breadth-first search* (BFS)
- **But**: Only seek *alternating paths*
  - From RHS node: Use only non-matching edges to discover new nodes
  - From LHS node: Use only matching edges
- Note: BFS takes as input a *graph*, but Alternating BFS requires a graph *and* a matching!

An example of alternating BFS

- Alternating BFS on $\mathcal{M}$





Finding augmenting paths using Alternating BFS

- **If** alternating BFS produces layer with unmatched node from LHS
  - Move downward from unmatched node in layer 0 to unmatched LHS node
  - Edges on this path alternate between non-matching and matching
  - Augmenting path!
- Matching can be enlarged
- If not, say ABFS *'fails'*: No unmatched LHS nodes in any (odd) layer

Proving Hall's theorem: Where are we now?

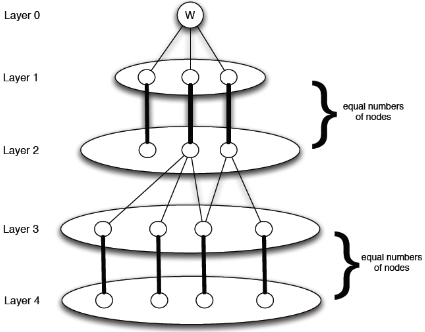
- Recall key issue: Identifying a constricted set, knowing *only* that there is no perfect matching
  - Start with a *maximum matching*
  - Try to enlarge it: Use Alternating BFS to find augmenting paths
  - (A matching in a bipartite graph can be enlarged if there is an augmenting path)
  - Show this produces a constricted set (unless the matching was perfect)

Producing a constricted set

- What does Alternating BFS have to do with constricted sets and perfect matchings?
  - To prove Hall's theorem: Show that there is a constricted set if there is no perfect matching
  - Start with a maximum matching that is not perfect
  - Failed ABFS: ABFS does not find an augmenting path
  - So we want to produce a constricted set from the output of such a failed Alternating BFS
- An aside: Will an ABFS fail or succeed if graph has, and we start with, a perfect matching?

Understanding the structure of a (failed) Alternating BFS

- Observation 1: Even-numbered layers (starting layer 0) contain RHS nodes; odd-numbered layers contain LHS nodes



- Observation 2: Odd layers have same number of nodes as *next* even layer— all connected by *matching* edges to distinct nodes in next layer

Understanding the structure of a (failed) Alternating BFS

- Observation 3: Every even-layer (*i.e.*, RHS) node in ABFS has all its neighbors present in some layer:
  - Each even-layer node (other than W) has its matched partner in previous layer
  - If any other neighbors not already explored in higher layer, will be added to the next layer (using non-matching edge)

- Aside: Is this also true of odd-layered nodes?

Extracting a constricted set

- Consider failed alternating BFS:
  - Failed ABFS always ends in even layer (Why?)
  - From Observation (2): Strictly more total nodes in even layers than in odd layers
  - From Observation (3): All even-layer nodes' neighbors are present somewhere in ABFS output
- Define S to be set of nodes in even layers
- S is a set of RHS nodes with strictly fewer neighbors than its size: S is a constricted set!

Extracting a constricted set

So what did we just learn?

- Let W be any unmatched node on the right-hand side in a bipartite matching. Then either there is an augmenting path beginning at W, or there is a constricted set containing W.
  - Alternating BFS starting at W either does not fail (produces unmatched LHS node in odd layer) or fails
  - Does not fail: Augmenting path from W to unmatched LHS node
  - Fails: Set of nodes in all even layers is constricted set

- Bipartite graph with no perfect matching
- Start with a maximum matching: Unmatched RHS node W
- No augmenting path (if yes, can enlarge matching)
- So alternating BFS from W produces constricted set!

Proof: Recap of ideas

- Start with maximum matching which is not perfect: There is an unmatched RHS node
- Maximum matching: Cannot be enlarged further
  - Alternating BFS does not produce augmenting path
- Structure of output of ABFS:
  - Starting layer 1 onwards, odd-even pairs have equal numbers of nodes
  - ABFS ends in an even layer

Proof summary: Contd.

- Consider set of nodes in even layers 0, 2, ...; call it S
  - Claim: S is a constricted set
  - All neighbors of nodes in S present in some odd layers in ABFS
  - Odd layers have exactly one less node than even layers including layer 0
- So: We started with a maximum matching that wasn't perfect, and produced a constricted set
  - Where did we use that the matching wasn't perfect?
  - Where did we use that the original matching was maximum?
  - But I haven't shown you how to find a maximum matching! Is the proof complete?

Recap: The Matching Theorem

- Theorem: A bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching *iff* it contains a constricted set.
- If: Easy
- Key issue for only if direction: Identifying a constricted set, knowing *only* that there is no perfect matching
  - Start with a *maximum matching*
  - Try to enlarge it : Augmenting paths, found using Alternating BFS
  - Cannot, since this was a maximum matching: ABFS fails to produce augmenting path
  - Show that output from failed ABFS produces a constricted set if maximum matching was not perfect

Proving the Matching Theorem: Why?

- What else does the proof buy us (beyond proving the result)?
  - Concept of augmenting paths
  - How to *find* a perfect matching (or a proof that none exists)
  - Maximum matchings
  - Improving matchings in other settings: More general preferences (key ideas used in School Choice!)

Finding a Perfect Matching

- Start with empty matching
- Start with any unmatched node on RHS
- Perform alternating BFS: Either find
    - Augmenting path: Enlarge matching
    - Constricted set: Proof of no perfect matching
    - (Why?)

What if?

- What if there is no perfect matching?
- Recall: ‘Make everyone happy’
- What if we can’t: No perfect matching?
- Binary preferences: Make maximum number happy

Maximum Matchings

- Will alternating BFS return a maximum matching?
- In general, no
- Seek augmenting path from *all*, rather than *any* unmatched node: Maximum matching
  - All unmatched RHS nodes in layer 0

What we did

- Asked how to decide if we can make all students happy
- Answer provided by Hall's theorem
- Proof tells us:
  - How to *construct* a perfect matching, if one exists
  - How to build a *maximum* matching, if one doesn't

Coming up next

- So far: One-sided markets with *binary* preferences
- Extending to richer preferences: Ranked preference lists
- What is a 'good' allocation?
- How do we find it?