Networks II: Market Design—Lecture 11
Markets with Initial Endowments: The Core

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Recap: Existence of core matching

Theorem (Shapley and Scarf 1974)

Assume agents have strict preferences. There exists a core matching for any housing market.

Proof by construction: *Top Trading Cycles (TTC) algorithm*

- $p_k = (a_k, h_k)$: Pair of agent k and her initial endowment h_k
- P: (Running) set of pairs p_k
- Directed graph $G(P)$: Directed edge from $p_i = (a_i, h_i) \in P$ to $p_j = (a_j, h_j) \in P$ if item h_j is agent a_i's top preference amongst items in P
- Each agent points to (pair corresponding to) her most-preferred item in P

- Initialize P with all agent-item pairs. Repeat until P is empty:
  - Create graph $G(P)$
  - There exists at least one cycle in $G(P)$ and no cycles intersect
  - Assign each agent in a cycle the item she is pointing to
  - Update P to remove (pairs p_k corresponding to) all cycles in $G(P)$ from P
- True ([A]) or False ([B])?
  1. There is exactly one cycle in $G(P)$ at every step of algorithm
  2. At any step: Each agent in P belongs to exactly one cycle in $G(P)$
  3. Suppose that at step l, a_1's second choice is h_2 and a_2's top choice is h_1. If a_1 does not belong to a cycle in step l, then
     1. a_2 does not belong to a cycle either
     2. a_1 and a_2 will both be removed in step $l+1$

Proving TTC returns a core matching

- Let M_T be matching returned by TTC
- *If* M_T is not in the core: There exists coalition $C^* \subseteq A$ that 'prefers' a different matching $\hat{M}$
  - $C_{strict} \subseteq C^*$: Agents who *strictly* prefer allocation under $\hat{M}$ to M
  - a_0: *An* agent who is matched **earliest** in TTC in C_{strict}
  - Item $\hat{M}(a_0)$ is owned by an agent $a_1 \in C^*$
  - a_1 obtains an item belonging to $a_2 \in C^* \cap C_l'$ both in $\hat{M}$ and M_T
  - ... and so on until $a^* \in C^* \cap C_l'$ obtains $\hat{M}(a_0)$ in both $\hat{M}$ and M_T
- A contradiction: Two different agents a_0 and $a^* \neq a_0$ are allocated same item in $\hat{M}$, so $\hat{M}$ is not a matching

TTC returns a core matching

- Can $\hat{M}(a_0)$ be a_0's own initial endowment? [A] Yes [B] No
- What if C_{strict} equals C^*?
- Why can't a_1 be in the same step l as a_0?

Understanding matchings under initial endowments

- Allocation in markets *without* initial endowments
  - Agent set $A = \{a_1, \ldots, a_n\}$, item set $H = \{h_1, \ldots, h_n\}$
  - Allocation represented as matching M in *bipartite* graph
  - M is set of edges: $(a_i, h_j) \in M$ if agent a_i is assigned item h_j in matching M
- Allocation in markets *with* initial endowments
  - Agent-item pairs $\{(a_1, h_1), \ldots, (a_n, h_n)\}$: Item h_i is agent a_i's initial endowment
  - Allocation can still be represented as matching M in bipartite graph
  - **But** can also be represented via *directed, non-bipartite* graph

Understanding matchings under initial endowments

- Given **any matching** M in initial-endowments market, can construct directed graph G_M:
  - Vertices: Agent-item pairs $P = \{(a_i, h_i)\}$
  - Create directed edge from i to j if a_i is assigned item h_j in M: $M(a_i) = h_j$
  - G_M: An alternative *directed, non-bipartite*, graph representation of matching M
- Fact: G_M can be decomposed into non-intersecting cycles for **any** such matching M
  - Each cycle consists of agents who trade their initial endowments amongst themselves in M
  - '**Such**' matching: All agents matched (complete preferences)

Understanding matchings under initial endowments

- Recall: Matching M is in the **core** if there is no coalition of agents $B \subseteq A$, and a matching $\hat{M}$, such that
  - For any $a \in B$, $\hat{M}(a)$ is the *initial house* of some $b \in B$, and
  - $\hat{M}(a) \succeq_a M(a)$ for all $a \in B$
  - $\hat{M}(b) \succ_b M(b)$ for some $b \in B$
- Revisit core in terms of $G_M, G_{\hat{M}}$
- Revisiting TTC algorithm proof:
  - TTC algorithm produces matching M: Cycles in G_M are 'top trading cycles', where agents get 'current-best' items
  - Consider hypothetical matching $\hat{M}$ that is better for some cycle of agents in $G_{\hat{M}}$: Find contradiction

- Let M_T be matching returned by TTC: *If* M_T is not in core, there exists coalition C^* that 'prefers' a different matching $\hat{M}$
    - $a_0 \in C^*$: First agent in 'TTC order' to *strictly prefer* $\hat{M}$ to M_T
    - Item $\hat{M}(a_0)$ is owned by an agent $a_1 \in C^*$: Agent a_1 must have been removed by TTC in a strictly earlier step, say in cycle C_l
    - a_1 obtains item belonging to $a_2 \in C^* \cap C_l$ both in $\hat{M}$ and M_T
    - $\ldots$, and so on until $a^* \in C^* \cap C_l$ obtains $\hat{M}(a)$ both at $\hat{M}$ and M_T
- A contradiction: Two different agents a_0 and agent $a^* \neq a_0$ are allocated same item in $\hat{M}$, so $\hat{M}$ is not a matching

Understanding the set of core matchings: An example

- Housing market with 4 agents, 4 items
  - Initial endowments (a_i, h_i) as usual
  - Preferences are:

$$
\begin{aligned}
a_1 &: h_2 \succ h_3 \succ h_1 \succ h_4 \\
a_2 &: h_3 \succ h_1 \succ h_2 \succ h_4 \\
a_3 &: h_4 \succ h_3 \succ h_2 \succ h_1 \\
a_4 &: h_3 \succ h_1 \succ h_2 \succ h_4
\end{aligned}
$$

- Is $M_1 = (a_1 - h_2, a_2 - h_1, a_3 - h_4, a_4 - h_3)$ [A] Yes [B] No
  (i) in the core? (ii) Pareto-efficient?
- Is matching $M_2 = (a_1 - h_2, a_2 - h_3, a_3 - h_4, a_4 - h_1)$ in the core?
- Can you locate any other matching in the core?

Theorem (Roth and Postlewaite 1977)

The matching produced by Gale's TTC algorithm is the **unique** *core matching.*

- Let M_T be matching returned by TTC; $\hat{M} \neq M_T$ be hypothetical other matching in core
- a_0: Any of **first** agents with $\hat{M}(a) \neq M(a)$, according to order of being matched in TTC
- C_l: Set of agents in same TTC cycle as agent a_0
- C_l is a coalition demonstrating $\hat{M}$ is not a core matching:
  - $M_T(a)$ is an item owned by some other agent in C_l for all $a \in C_l$
  - $M_T(a) \succeq_a \hat{M}(a)$ for all $a \in C_l$ (Why?)
    - $\hat{M}(a') = M(a')$ for all agents a' matched *before* C_l in TTC
  - $\hat{M}(a_0) \neq M_T(a_0)$ and $a_0 \in C_l$: $M_T(a_0) \succ_{a_0} \hat{M}(a_0)$

Properties of TTC algorithm

- TTC algorithm: Produces a matching in the core **for** reported preference profile $\succ$
- But what about with respect to *true* preferences?
- Notational aside: TTC algorithm also called the ‘core mechanism’
  - Recall: Unique core matching
  - Core mechanism: Return matching in the core of market $((a_k, h_k), \succ)$

Theorem (Roth’82)

The TTC mechanism is strategyproof.

- So TTC mechanism produces a core matching wrt true preferences!

- Consider the following argument:
  - TTC induces an ordering amongst agents (the TTC order)
  - When an agent's turn comes, he/she gets her top choice amongst remaining items
  - Therefore, no agent has an incentive to lie
  - Therefore, TTC mechanism is strategyproof
- Is this argument correct? [A] Yes [B] No

Characterization of the core mechanism

- Recall question from markets without initial endowments: Why use serial dictatorship?
  - Characterization result: Any mechanism with certain desirable properties is a serial dictatorship
- Why use TTC mechanism in markets with initial endowments?

Theorem (Ma'94)

The core mechanism is the only mechanism that is individually rational, Pareto-efficient, and strategyproof for a 'housing market'.