Networks II: Market Design—Lecture 9
Markets with Initial Endowments: The Core

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Logistics

- Speed feedback: What could slowing down mean?
  - More time for clicker questions
  - More time to discuss answers to clicker questions
  - More time on definitions—they're a handful!
  - Just speak slower (and/or pause more)!
  - No changes I can think of in-class: Please don't change speed (zzzz) and/or Offline time necessary

- Markets with initial endowments: $((a_k, h_k)_{k\in\{1,\ldots,n\}}, \succ)$
  - 'Housing market': Agent a_k *owns* house h_k
  - Each agent a has strict preferences $\succ_a$ over houses

- What properties should a good matching have?
  - Pareto efficiency (PE): Matching M is Pareto efficient if
    - There is no matching $\hat{M}$ such that $\hat{M}(a) \succeq_a M(a)$ for *all* agents a, and $\hat{M}(b) \succ_b M(b)$ for *some* agent b
  - *'Individual rationality'* (IR): No agent is better off not participating in the market
    - $M(a_k) \succeq_{a_k} h_k$
  - Core: No *group* of agents can do better by not participating in the market and trading their initial endowments amongst themselves

Checking in: An example

- Agents a,b,c; Houses x,y,z
  - Initial endowments: a-x, b-y, c-z
  - Preferences: a: $y \succ z \succ x$; b: $y \succ x \succ z$; c: $x \succ y \succ z$

- Pareto-efficient: PE; Individually rational: IR

- Matching $M_1 =$ (a-y, b-x, c-z) is:
  - [A] PE [B] IR [C] Both PE and IR [D] Neither
  - PE does not imply IR!

- Matching $M_2 =$ (a-x, b-y, c-z) is:
  - [A] PE [B] IR [C] PE and IR [D] Neither
  - IR does not imply PE!

Checking in: Understanding individual rationality

- Let M_0 be 'initial matching', *i.e.*, $M_0(a_k) = h_k$. Any individually rational matching M Pareto-dominates M_0.
  - A True
  - B False

- M_0 is itself individually rational, and does not Pareto-dominate itself!
  - Reminder: For matching $\hat{M}$ to Pareto-dominate matching M,
    - $\hat{M}(a) \succeq_a M(a)$ for all agents a, **and**
    - $\hat{M}(b) \succ_b M(b)$ **for at least one agent** b**!**
  - So: No matching can Pareto-dominate itself

- Core: No *group* of agents can do better by not participating in the market and trading their initial endowments amongst themselves
- Matching M is in the **core** if there is **no** coalition of agents $C \subseteq A$, and a matching $\hat{M}$, such that
  - For any $a \in C$, $\hat{M}(a)$ is the *initial house* of some $a' \in C$
  - $\hat{M}(a) \succeq_a M(a)$ for all $a \in C$
  - $\hat{M}(b) \succ_b M(b)$ for some $b \in C$

Understanding the core: Double negatives

- Observe common structure in definitions of Pareto-efficiency, core: A matching is 'good' if it is **not** 'not-good'
  - Matching M is in the core if there is **no** coalition of agents $C \subseteq A$, and a matching $\hat{M}$, such that
    1. For any $a \in C$, $\hat{M}(a)$ is the *initial house* of some $a' \in C$
    2. $\hat{M}(a) \succeq_a M(a)$ for all $a \in C$
    3. $\hat{M}(b) \succ_b M(b)$ for some $b \in C$
  - A matching is not in the core if there is some coalition $C \subseteq A$, and a matching $\hat{M}$, such that [1]-[3] hold
  - A matching is in the core if it is not not-in-the-core

Understanding the core: Double negatives

- Pareto-efficiency: A matching M is Pareto-efficient if it is **not** not Pareto-efficient (*i.e.*, if it is not Pareto-dominated)

- Pareto-efficiency for mechanisms: A mechanism φ is Pareto-efficient if it not not-Pareto-efficient, *i.e.*, does not return a not-Pareto-efficient matching on any input $(A, H, \succ)$

- Strategyproofness: A mechanism φ is strategyproof if it is **not** not-strategyproof (recall definition)

(A matching M is in the **core** if there is no coalition of agents $B \subseteq A$, and a matching $\hat{M}$, such that

- For any $a \in B$, $\hat{M}(a)$ is the *initial house* of some $b \in B$, and
- $\hat{M}(a) \succeq_a M(a)$ for all $a \in B$ and $\hat{M}(b) \succ_b M(b)$ for some $b \in B$)

- Agents a,b,c; Houses x,y,z
  - Initial endowments: a-z, b-y, c-x
  - Preferences: a: y $\succ$ z $\succ$ x; b: x $\succ$ y $\succ$ z; c: z $\succ$ y $\succ$ x
- Matching $M_2 =$ (a-z, b-x, c-y): (i) Is M_2 in the core? (ii) Does coalition $\{a, c\}$ demonstrate M_2 is not in the core?
  - No, and no!:
    - Consider coalition $C = A$, matching $\hat{M} =$ (a-y, b-x, c-z): No agent is unhappier, and agents a, c strictly happier
    - c cannot improve by deviating out of market only with a: Coalition {a,b,c} is required!

Desiderata: One-sided markets with initial endowments

What properties should a good matching have?

- **Pareto efficiency:** Matching M is Pareto efficient if
  - There is no matching $\hat{M}$ such that $\hat{M}(a) \succeq_a M(a)$ for *all* agents a, and $\hat{M}(b) \succ_b M(b)$ for *some* agent b

- **Individual rationality:** No agent is better off not participating in the market
  - $M(a_k) \succeq_{a_k} h_k$

- **Core:** Matching M is in the core if there is **no** coalition of agents $C \subseteq A$, and a matching $\hat{M}$, such that
  - For any $a \in C$, $\hat{M}(a)$ is the *initial house* of some $a' \in C$, and
  - $\hat{M}(a) \succeq_a M(a)$ for all $a \in C$ and $\hat{M}(b) \succ_b M(b)$ for some $b \in C$

Properties of matchings in the core

(A matching M is in the **core** if there is no coalition of agents $B \subseteq A$, and a matching $\hat{M}$, such that

- For any $a \in B$, $\hat{M}(a)$ is the *initial house* of some $b \in B$, and
- $\hat{M}(a) \succeq_a M(a)$ for all $a \in B$ and $\hat{M}(b) \succ_b M(b)$ for some $b \in B$)

- Is every core matching individually rational (IR)?

  A Yes
  B No

  – Yes: Consider a one-person "coalition" $B = \{a\}$

- Is every core matching Pareto efficient (PE)?

  A Yes
  B No

  – Yes: Consider the coalition $B = A$

- Is every individually rational (IR) matching in the core?
  - A Yes
  - B No
  - – Market with two agents and houses
    - Initial endowments a-x, b-y
    - Preferences a: $y \succ x$, b: $x \succ y$
    - 'Initial' matching a-x, b-y is not PE: Every core matching is PE!

- Is every Pareto-efficient matching in the core?
  - A Yes
  - B No
  - – Market with two agents and houses
    - Initial endowments a-x, b-y
    - Preferences a: $y \succ x$, b: $y \succ x$
    - Matching a-y, b-x is not IR: Every core matching is IR!

Just checking: The core

- The 'initial matching' $M_0(a_k) = h_k$ is always
  - A Individually rational
  - B Belongs to the core
  - C Neither
- Every core matching M Pareto-dominates the 'initial' matching $M_0(a_k) = h_k$
  - A True
  - B False

The core so far

- Every core matching is individually rational (IR) and Pareto-efficient (PE)
  - Consider one-person "coalition" $B = \{a\}$ and grand coalition $B = A$ respectively
- Is every IR matching in the core?
  - No: Simple 2-agent, 2-item example where swapping improves allocation
- Is every PE matching in the core?
  - No, again: A Pareto-efficient matching need not be individually rational!
- What if a matching is both PE and IR—is such a matching always in the core?

- Agents a, b, c; Houses x, y, z
  - Preferences $\succ$: a: z $\succ$ y $\succ$ x; b: z $\succ$ x $\succ$ y; c: y $\succ$ x $\succ$ z
  - Initial endowments: a owns x, b owns y, c owns z

  - Is matching $M_1 =$ (a-z, b-x, c-y) (i) Pareto-efficient? (ii) Individually rational?
  - Is matching $M_2 =$ (a-x, b-z, c-y) (i) Pareto-efficient? (ii) Individually rational?

  - Which of these matchings is in the core?
    - A Only M_1
    - B Only M_2
    - C Neither M_1 nor M_2
    - D Both M_1 and M_2

- A matching may be both individually rational *and* Pareto-efficient, but still not belong to the core!