1. In markets with one-sided rank-order preferences, we know that there is always some agent who receives her top-ranked house in any Pareto-efficient matching. (Do you know this? Yes, you do!: This follows from the result we stated in class (Abdilkadiroglu and Sonmez'99) saying that every Pareto-efficient matching can be obtained as the output of some serial dictatorship mechanism (i.e., with some partic- ular priority ordering of the agents), which implies that the top agent in this priority order obtains her favorite item).

Now we're going to investigate similar ideas in markets with two-sided rank-order preferences, i.e., when both sides of the market have preferences over the agents on the other side. Assume that there are an equal number of agents on each side of the market, and that all agents have strict, complete preferences.

1. (a) (5 pts.) Consider the marriage model. Does every stable matching contain a pair (m,w) such that m and w are at the top of each other's priority lists, for all possible markets (i.e., all possible number of agents and preference rankings of agents)?

No, every stable matching does not contain a pair (m,w) such that m and w are at the top of each other's priority lists by the following counter example:

Men: a and b Woman y and z

Prefs:

a: y>z

b:z>y

y:b>a

z:a>b

matching a-y and b-z

This matching is stable in that it contains no unstable pairs as both a and b received their top choices and thus they cannot prefer any matching to their current matching. However, neither woman y or woman z received their top choice and thus not all stable matching contain a pair (m,w) where m is w's top choice and w is m's top choice.

2. (b) (10 pts.) Suppose there is a man m and a woman w who are at the top of each other's priority lists. Does every stable matching in this market contain the pair (m, w)?

Claim: if there is a man m and a woman w who are at the top of each other's priority lists, then every stable matching in this market must contain the pair (m,w).

Proof: Suppose for the sake of contradiction that there exists a stable matching in such a market that does not contain (m,w). As (m,w) is not a pair in the market then the two pairs (m,w') and (m',w) must be in the market where m' and w' are some other man and woman in the market. As m is w's top choice and w is m's top choice, m prefers w>w' and w prefers m>m'. This defines and unstable pair and thus the matching is not stable. As this is a contradiction with the assumption of the matching being stable, we can conclude that if (m,w) must be a pair in every stable matching in a market where m is w's top choice and w is m's top choice.

3. (c) (15pts.) Now suppose the market has n men and n women, where n ≥ 2, and there is a man m and a woman w who are at the bottom of each other's priority lists[1]. Can such a pair (m,w) ever belong to a stable matching?

Yes, such a pair can belong to a stable matching as shown in the example below.

Men: a and b Woman y and z

Prefs:

a: y>z

b:y>z

y:a>b

z:a>b

matching a-y and b-z

In this matching no unstable pairs can exist as both and y cannot prefer another match to their current match meaning they cannot be part of an unstable pair. As the only potentially unstable pairs are a-z and b-y, as each of these pairs include either a or y which both already have their top choice, clearly the matching is stable. Man b and woman z both received their bottom choice in this stable matching and thus the claim is shown.

[1](Note that if a man is on a woman's priority list, this implies that he is acceptable to her, and vice versa).

2. (Existence of stable matchings: Examining non-pairwise grouping.) (35 pts.) In class, we used the Gale- Shapley algorithm to prove the existence of stable matchings in two-sided matching markets where each agent has rank-order preferences over agents on the other side, and each agent wants to be matched to at most one other agent. The deferred acceptance algorithm can be deceptive in its simplicity (and familiarity, for some), leading to an unfortunate tendency to not gape at the fact that stable matchings even exist. In this problem, we're going to investigate the marvelousness of the very existence of stable matchings.

Stable matchings do exist in some many-to-one matching markets, for example where (e.g. colleges') preferences over groups (e.g., of students) are derived from their preference rankings over individuals. There are other markets, however, such as firms employing workers, where a firm might have preferences over sets of workers that cannot be derived from a rank-ordered list over individual workers (for reasons such as complementarity of skillsets).

Consider a many-to-one matching market with 2 firms F_1 and F_2, and 3 workers w_1, w_2, w_3. Each worker can join only one firm, and the workers' preferences over firms are w_1: $F_2 \succ F_1$, w_2: $F_2 \succ F_1$, w_3: $F_1 \succ F_2$. Firms can potentially hire more than one worker, and have preferences over sets of work- ers. The firms' preferences are as follows: (i) F_1: $\{w_1, w_3\} \succ \{w_1, w_2\} \succ \{w_2, w_3\} \succ \{w_1\} \succ \{w_2\}$, and (ii) F_2: $\{w_1, w_3\} \succ \{w_2, w_3\} \succ \{w_1, w_2\} \succ \{w_3\} \succ \{w_1\} \succ \{w_2\}$. (As usual, we drop any unacceptable worker sets from the preference ordering, i.e., any set of workers not listed in a firm's preference list is unacceptable to that firm.)

A matching is stable if it is individually rational, and if there is no pair (F,w) that is not currently matched and that would prefer to be matched to each other.

(a) Enumerate the set of all (valid) matchings in this market specified above.
{M1={(F_1, $\{w_1, w_3\}$),(F_2,$\{w_2\}$)}, M2={(F_1, $\{w_2, w_3\}$),(F_2,$\{w_1\}$)}, M3={(F_1, $\{w_1, w_2\}$),(F_2,$\{w_3\}$)}, M4={(F_2, $\{w_1, w_3\}$),(F_1,$\{w_2\}$)}, M1={(F_2, $\{w_2, w_3\}$),(F_1,$\{w_1\}$)}}

(b) Are any of them stable? (If you say a matching is not stable, you must point out a blocking pair demonstrating this.)
Not stable: M1,M2,M3, M4 M5
All matching are unstable with blocking pairs for each listed as follows, M1:(F_2,$\{w_1\}$), M1:(F_2,$\{w_2\}$), M1:(F_2,$\{w_1\}$), M1:(F_1,$\{w_1\}$), and M1:(F_1,$\{w_1\}$).

3. (Comparing matchings.) The man-proposing deferred acceptance algorithm produces the stable match- ing μ^*_M which is preferred by all men to all other stable matchings, and similarly women prefer the matching μ^*_W produced by the woman-proposing deferred acceptance algorithm. It is quite surprising that the two matchings μ^*_M and μ^*_W could be compared with all other stable matchings, i.e., that all men agree about which matching they prefer (and similarly for women).
   (a) (15pts.) First, let's explore whether other kinds of 'good' matchings are comparable as well. Are an arbitrary pair of Pareto-efficient matchings comparable? That is, is it possible to compare (as defined in class) every pair of matchings that are each Pareto-efficient with respect to the agent's preferences, for all possible (strict) preferences and any number of agents?

No, it is not possible by the following market and matching example:

Men: a and b Woman y and z

Prefs:

a: y>z

b:y>z

y:a>b

z:a>b

m1: a-y and b-z and m2: a-z and b-y

Prefer:

a and y prefers m1

b and z prefers m2

As neither matching can dominate the other as at least one agent prefers the other matching in both cases, both matchings are pareto-efficient. These matchings are not comparable as a prefers m1 and b prefers m2 thus, they cannot be compared according to men's preferences. These matchings are not comparable as y prefers m1 and z prefers m2 thus, they cannot be compared according to woman's preferences. Thus, the matchings are pareto-efficient but they are not comparable.

(b) (20pts.) When there is a unique stable matching, the man-proposing and woman-proposing algorithm must, by neccessity, return the same outcome. Is the converse also true—that is, if the man-proposing and woman-proposing DA algorithms return the same outcome on an instance, is there only one stable matching for that instance? (Hint: Can you use what you've learnt about comparing matchings to answer this question concisely?)

Claim: if the man-proposing and woman-proposing DA algorithms return the same outcome on an instance then there is only one stable matching for that instance.
Proof: As was shown in class the man-proposing DA algorithm will return a stable matching which is Man-optimal which means that of the stable matchings, each man will weakly prefer this matching to all other stable matchings. Similarly, As was shown in class the woman-proposing DA algorithm will return a stable matching which is Woman-optimal which means that of the stable matchings, each woman will weakly prefer this matching to all other stable matchings. As these algorithms produce the same matching each agent (man and woman) weakly prefer this matching to all other stable matchings. This weak preference of this matching to all other stable matching means that this matching will dominate all other stable matchings. In order for a matching to be different and stable it cannot be Pareto-dominated by any other matching. However, as the matching produced is optimal for both all men and women among stable matchings, there cannot be any other stable matchings as there would be at least one pair which would be blocking as it would rather be paired as in the matching produced by the algorithm.