Networks 2 hw2

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1. (Binary preferences and Pareto-efficiency.) Let's rewind all the way back to binary preferences (Lecture 2, 3), where each agent specifies the set of items that she finds acceptable (in other words, each agent supplies a yes/no preference for each item). We saw that such a market can be completely captured by an appropriate bipartite graph—the graph completely captures the agents' preferences, and a feasible allocation in the market is identical to a matching in this graph.

The definition of Pareto-efficiency is not restricted to strict rank-order preferences, but is more generally applicable: A matching M is Pareto-efficient simply if it is not Pareto-dominated by any other matching with respect to the given preferences. That is, M is Pareto-efficient if there is no other matching M' such that every agent is at least as happy with their assignment in M', and at least one agent is strictly happier with their assignment in M', in comparison with their assignments in M.

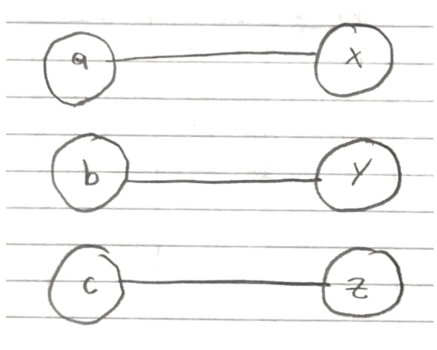
Reminder: All answers must be accompanied by proofs or counterexamples, as appropriate; for this question, and all others!

(a)(7pts.) Suppose a matching M_1 is larger in size than a matching M_2 in some bipartite graph G. Can you conclude that M_1 Pareto-dominates M_2 (w.r.t the preferences represented by G)?

Soln:

By counter example we can conclude that this statement is false.

Given a graph G in which there are 3 agents (a,b,c) and 3 items (x,y,z), we cans specify the acceptable items lists as a: {x} (a finds x acceptable), b: {y}, and c: {z}. This the graph described here corresponds to the image below.



For this counter example we can let M_1 = {a-x,b-y} and M_2 = {c-z}. Clearly the cardinality of M_1 is larger than the cardinality of M_2 (2>1), thus the first part of this question is satisfied. However, the claim that M_1 Pareto-dominates M_2 is false as in order for Pareto-domination to occur each element of must be at least as good in the dominating matching as it had been in the dominated matching. In this case c is worse off in M_1 as it goes from being matched in M_2 to being unmatched in M_1 . Thus, as M_1 does not Pareto-dominate M_2 and M_1 is larger in size than M_2 we can conclude that the claim is false.

(b) (8pts.) Suppose a matching M in a market represented by bipartite graph G is such that there is an augmenting path w.r.t matching M in G. Can M ever (i.e., for any market G and matching M) be Pareto-efficient?

Soln:

Claim: Suppose a matching M in a market represented by bipartite graph G is such that there is an augmenting path w.r.t matching M in G, then M cannot ever (i.e., for any market G and matching M) be Pareto-efficient.

Proof: For the sake of contradiction suppose that such a matching M is Pareto-efficient. By definition, this means that no other matching M' Pareto-dominates M. As an augmenting path exists, we know that this augmenting path includes all of the agents that had already been matched along the path as well as one additional match. We let M' be the set of all matchings in M not in the augmenting path U all matchings that are a part of the augmenting path. Then for each matched agent in M, they are matched in M' as they are either not a part of the augmenting path (in which case they are matched with the same as in M) or they were a part of the augmenting path and thus are matched again to a different viable pairing as the augmenting path ends in unmatched nodes, at least one agent goes from unmatched in M to matched in M'. Thus, as in M' all previously matched remain matched and at least one unmatched becomes better off getting matched. Thus, M' dominates M meaning M is not Pareto-efficient. This is a contradiction and the claim holds.

(c) (15pts) Recall that a maximum matching in a bipartite graph is a matching with the largest possible number of edges amongst all matchings in the graph. Is every maximum matching in a graph Pareto-efficient with respect to the preferences represented by the underlying graph (i.e., for all possible maximum matchings, for all possible bipartite graphs of all possible sizes)?

Claim: Every maximum matching in a graph is pareto-efficient with respect to the preferences represented by the underlying graph.

Proof: For the sake of contradiction suppose that there exists some maximum matching M who is not Pareto-efficient. Thus, there exists some other matching for this graph M' which pareto dominates M. The conditions for Pareto domination require at least one agent is better off in M' than M and all remaining agents are at least as good in M' as in M. Thus in order for M' to improve for at least one agent of M, an agent must go from unmatched in M to matched in M'. However, as M is a maximum matching, M' can contain at most the same number of matchings and thus matched agents as M. However, as M' must contain at least one matched agent who was unmatched in M, and all other agents who were previously matched in M must remain matched as to not violate the second condition of Pareto dominance. M' thus contains all matched agents of M with at least one additional agent, thus $|M'| \geq |M|+1$. This is a contradiction as M is a max matching and thus M' cannot be a larger matching.

2. (Mechanisms for rank-order preferences.) (25pts.) The serial dictatorship mechanism in a one-sided matching market fixes a priority order over agents, and assigns items to agents according to this priority order; we showed in class that this mechanism is strategyproof and Pareto-efficient when preferences are strict.

Lisa is in charge of organizing a community giveaway of used winter clothing, and doesn't quite like the idea of a (serial) dictatorship to allocate the donations she has received. She wants to use a mechanism that allocates items to "those who want it most". Of course, there are many different ways to turn that concept into an algorithm—here is the mechanism she comes up with. Lisa decides that she will choose a priority order[2] g over items instead of individuals (agents), and process items according to the priority order, as follows: At each step i, she will assign the item g(i) that has priority i to the agent a still remaining in the market who ranks it at the highest position in his (original) preference list. If multiple agents rank item g(i) equally high, she will break ties according to a fixed, pre-announced, order on the agents. She then assigns item g(i) to this agent a, and removes both the agent and the item from the market. (Assume agents all have strict, complete rank-order preferences, so that the mechanism at every step is well-defined.)

So for example, if Lisa is now processing item h_3, and agent a_5 ranks it 2nd on his list and all other agents still remaining in the market rank this house 3rd or lower, then agent a_5 will be assigned item h_3. Also, note that agents do not update items' ranks after an item is removed from the market in each round (so for example, if agent a ranked an item x fourth and item y fifth in his list, y still remains ranked as a's fifth choice if item x is assigned to some other agent before a gets assigned an item).

Is the mechanism created by Lisa strategyproof?

The mechanism created by Lisa is not strategyproof as we will show by counter example. Consider a situation in which there are three people (a,b,c) and three items (x,y,z). The true preferences of each person are a: x >y>z, b: z>y>x, and c: x>z>y. Lisa at the beginning could announce any order for the items to be assigned in, for this example we will let the order she announces be x, z, y and she also announces an arbitrary order of tiebreaking priorities: a, c, b. Now in the case that each person reports their true preferences to the algorithm, x will be assigned to a, then z to b, and y to c. As c gets its third choice, it can create a strategy to try and improve what item it gets. For example, if a and b report their true preferences while c reports c: z>x>y, then the algorithm will output x->a, next both b and c have the same ranking of z thus the tie breaker gives z to c, and finally y is assigned to b. Thus, when c uses a strategy, it can improve the item it gets in this case from its third choice item to its second choice item. Thus, as a strategy can be used to produce such a result, the algorithm designed by Lisa is not strategyproof.

3. Consider the house allocation model (i.e., rank-order preferences), where the input is a tuple $(A, H, \succ)$. Recall that a preference profile $\succ = \{\succ_1, \ldots, \succ_n\}$ is the n-tuple of preference rankings, one for each agent; the notation $\succ_i$ denotes the preference ranking of agent i.

Suppose that the size of the market, $n = |A| = |H| = 3$. For uniformity and ease of grading, let's call the three agents a, b, c and the three houses x, y, z.

Throughout this question, we will restrict ourselves, as usual, to strict and complete preferences.

1. (a) (10pts.) With 3 houses and 3 agents, can you construct a preference profile $\succ$ such that there is

   a unique Pareto-efficient matching with respect to $\succ$?

   An example preference profile for each agent such that a unique Pareto-efficient matching exist with respect to this profile is {(a:x>y>z), (b:y>z>x), (c:z>x>y)}. The unique Pareto-efficient matching in this case is {a-x,b-y,c-z}. All three agents have their first choice in this case, thus for all other matchings that are not this one at least two agents will not have their first choice and thus this matching will Pareto-dominate it meaning this is the only Pareto-efficient matching.

2. (b) (10 pts.) Again, with 3 houses and 3 agents, can you construct a preference profile for which every (perfect, i.e., matching all three agents)[3] matching is Pareto-efficient? You must argue why every matching is Pareto-efficient w.r.t the preference profile you give in your answer, for full credit.

An example preference profile for each agent such every perfect matching is a Pareto-efficient matching with respect to this profile is {(a:x>y>z), (b: x>y>z), (c: x>y>z)}. In every matching within these preference profiles the person who receives x will have their first choice, the person receiving y will have their second choice, and the person receiving z will have their third choice. In all cases no matching dominates another matching as in all cases at least one of the three people must be worse off in the other matching. As in all matchings there is a 1^{st}, 2^{nd}, and 3^{rd} choice in order for one person to move up in terms of their choice another person must move down. As no perfect matching can be dominated by another perfect matching every perfect matching is Pareto-efficient.

3. (c) (25pts.) Now, returning to the issue of uniqueness as in (a): describe[4] the set of all preference profiles in our tiny three-agent market with the property in (a), i.e., the set of all $\succ$ such that there is a unique Pareto-efficient matching w.r.t. $\succ$.
   Observe that to provide a complete answer to this question, you need to justify why the answer is exactly the set you describe: that is, you must provide an explanation as to why the preference profiles $\succ$ you claim have a unique Pareto-efficient matching do so, and why all other preference profiles must have strictly more than one Pareto-efficient matching. (In other words, of all possible (strict, complete) preference profiles in this size-three market, (i) which profiles have this property and why do they have this property, and (ii) how can you prove that all the other profiles do not have this property?)

   (i) The only preference profiles in which there is a unique Pareto-efficient matching are profiles in which the each agent has a different first choice. These profiles will have a unique Pareto-efficient matching for the same reasons as listed in (a). Regardless, of the other preferences in their preference profiles, All three agents have their first choice in this case, thus for all other matchings that are not this one at least two agents will not have their first choice and thus this matching will Pareto-dominate it meaning this is the only Pareto-efficient matching.
   Suppose that another Pareto efficient match exists. This means that no other matching dominates it. As this matching is different from the matching of each agent with their first choice, thus an agent must have their second choice. If an agent has their second choice comparing to the matching in which each agent has their top choice this person who has their second choice will be improving as they will now have their first choice. As all others are also

paired with their first choice they will at the very least be just as well off. Thus the matching in which all agents are paired with their first choice satisfies both conditions of Pareto-dominance over this proposed second Pareto-efficient matching and thus in fact a second pareto-efficient matching does not exist.

(ii) All other preference profiles by definition include at least two of the three agent shaving the same first choice. In such a case, on Pareto-efficient matching is one of these two agents with their first choice and the other does not get their first choice. In such a case this matching cannot be dominated as The agent who did not receive their first choice cannot improve their item without another agent being worse off as the other agent has their first choice. In such a case if we allow the matching to swap the items received by these agents who have the same first choice, we see that this matching will also be Pareto-efficient for the same reasons. Thus, all preference profiles where each agent doesn't have a different 1^{st} choice will result in no unique Pareto-efficient matching being present.