Networks II: Market Design—Lecture 14 Matching Markets with Non-transferable Utilities: Two-sided preferences

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Logistics

- Project groups, titles, one-line description:
  - (Due Thursday March 21, 12 noon on CMS)
  - Please submit as **groups**
  - **One** submission per group, not one submission per student!

Recap: Last time

- Two-sided preferences: What are 'good matchings'?
- Conceptual similarity between one and two-sided preferences in markets without money:
  - No single cardinal measure for matching's goodness
  - Cannot compare goodness of all matchings
  - Good matchings: 'Not-bad' matchings
- Marriage model (Gale-Shapley'62): n men and n women, with *strict* rank-order preferences over opposite side
  - Assume for now that preferences are also complete

Good matchings in two-sided markets

- **Blocking pair**: (m, w) blocks matching μ if m and w both prefer each other to partners in μ
- **Stable** matching: Matching with no blocking pairs
  - Matchings where no agent is unhappy *and* can do something about it

Stable allocations with two-sided preferences

- Immediate questions:
  - How does stability relate to Pareto-efficiency, core in two-sided markets?
  - Does a stable matching always *exist*? Can we find one if it does exist?
  - What are the properties of stable matchings?
  - Stability is theoretically appealing, but does it matter in real life?

Pareto-efficiency and stability

Pareto-efficiency in two-sided markets:

- Matching μ' Pareto-dominates matching μ if
  - All agents weakly prefer partners in μ' to partners in μ
  - At least one agent strictly prefers partner in μ'
- Matching μ is PE if there is no matching μ' that Pareto-dominates it

An example

- 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3)
- Preferences:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_1	m_1	m_1	m_1
w_2	w_3	w_2	m_2	m_3	m_2
w_3	w_2	w_3	m_3	m_2	m_3

- Is the matching $m_1 - w_1, m_2 - w_2, m_3 - w_3$ Pareto-efficient?
  A Yes
  B No
- What about matching $m_1 - w_1, m_2 - w_3, m_3 - w_2$?
  A Yes
  B No

The core in a marriage market

- Allocation belongs to core if *no* subset of agents can improve upon it without coming to market
- Matching μ' *dominates* matching μ if there is a subset of agents $A \subseteq M \cup W$ that collectively *strictly prefer* μ' to μ:
  - For any agent $a \in A$, $\mu'(a) \in A$
  - $\mu'(a) \succ_a \mu(a)$
  - How does this definition differ from Pareto-dominance?
- Core: Set of *undominated* matchings

An example

- 3 men (m_1, m_2, m_3) and 3 women (w_1, w_2, w_3)
- Preferences:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_3	m_2	m_3	m_1
w_2	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_1	m_1	m_2	m_2

- Is the matching $m_1 - w_1, m_2 - w_2, m_3 - w_3$ in the core?
  A Yes
  B No
- Is it Pareto-efficient?
- Is matching $m_1 - w_3, m_2 - w_1, m_3 - w_2$ in the core?

Pareto-efficiency and stability

- Pareto-efficiency in two-sided markets: Matching μ' Pareto-dominates matching μ if
  - All agents weakly prefer partners in μ' to partners in μ
  - At least one agent strictly prefers partner in μ'
- Matching μ is PE if no matching μ' that Pareto-dominates it
- Is every Pareto-efficient matching stable?
  - A Yes
  - B No
- No: Simple example with two agents on each side
  - Both men prefer w_1 to w_2, both women prefer m_2 to m_1
  - Matching $(m_1 - w_1, m_2 - w_2)$ is Pareto-efficient, but not stable!

Pareto-efficiency and stability

- What about the converse: Are stable matchings PE?
  (Recall preferences are strict)
  - Suppose μ is not PE: μ is Pareto-dominated by a matching μ'
  - Consider an agent, say, m^*, who strictly prefers μ':
    $\mu'(m^*) \succ_{m^*} \mu(m^*)$
  - Let $w^* = \mu'(m^*)$: $m^* \succ_{w^*} \mu(w^*)$
    (Why? Strict preferences, definition of Pareto-dominance)
  - But then (m^*, w^*) blocks μ: μ is not stable!

Theorem

Suppose preferences are strict. Then, every stable matching is Pareto-efficient!

- Not PE $\Rightarrow$ Not stable: So Stable $\Rightarrow$ PE (Contrapositive)

Core and stability

- Core requires "more" than stability in matching markets:
  - Stability: No *pair* (m, w) can improve
  - Core: No *group* of agents can improve
- How do the core and stability relate?
  - Is every core matching stable?
  - Is every stable matching in the core?

Q1: Is every core matching stable?

- Consider any matching μ: Suppose μ is not stable
  - Blocking pair (m, w) with $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$
  - μ dominated by any matching μ' containing (m, w) via coalition $A = (m, w)$
  - So μ cannot be in the core either
  - Contrapositive: Every matching μ in core is stable

Q2: Is every stable matching in the core?

- Consider any matching μ: Suppose μ is not in the core
  - μ is dominated by a matching μ' via some coalition A
  - Consider any $m \in A$, and let $w = \mu'(m) \in A$
  - By definition of domination, $w \succ_m \mu(m)$, *and* $m \succ_w \mu(w)$
  - But then (m, w) blocks μ: μ is not stable!
  - Contrapositive: Every stable matching μ is in core

Core and stability

- Our two questions:
  - Is every core matching stable? Yes
  - Is every stable matching in the core? Yes!

Theorem

Suppose preferences are strict. Then, the core of the marriage market equals the set of stable matchings.

The core in a marriage market: Off to brain gym!

Second time we're meeting the core: Do the two definitions relate?

- One-sided preferences with initial endowments:
  - A matching μ is in the core if there is no coalition of agents C and matching μ' such that
    - For any agent $a \in C$, $\mu'(a)$ is initial endowment of some agent in C
    - For all $a \in C$: $\mu'(a) \succeq_a \mu(a)$
    - For at least one $a^* \in C$: $\mu'(^*a) \succ_{a^*} \mu(a^*)$
- Two-sided marriage market:
  - Matching μ' *dominates* matching μ if there is a subset of agents $A \subseteq M \cup W$ that **collectively** *strictly prefer* μ' to μ:
    - For any agent $a \in A$, $\mu'(a) \in A$
    - $\mu'(a) \succ_a \mu(a)$

The core in a marriage market: Off to brain gym!

- Consider two alternative definitions for the core: Matching μ' *dominates* matching μ if there is a subset of agents A s.t. (i) For any agent $a \in A$, $\mu'(a) \in A$, and
  - Option D1: (ii) $\mu'(a) \succ_a \mu(a)$ for all $a \in A$
  - Option D2: (ii) $\mu'(a) \succeq_a \mu(a)$ for all $a \in A$, and $\mu'(a) \succ_a \mu(a)$ for at least one $a \in A$
- How do these definitions relate to each other?
  - What does 'relate' mean: I.e., how to formalize this question?
  - Say definition (property) D is at least as "strong" as D' if every instance satisfying D also satisfies D'
  - Say two definitions (properties) are
    - Equivalent if each is at least as strong as the other
    - Unrelated if neither is stronger than other

The core in a marriage market: Off to brain gym!

- Two alternative definitions: Matching μ' *dominates* matching μ if there is a subset of agents A s.t.
  (i) For any agent $a \in A$, $\mu'(a) \in A$, and
  - Option D1: (ii) $\mu'(a) \succ_a \mu(a)$ for all $a \in A$
  - Option D2: (ii) $\mu'(a) \succeq_a \mu(a)$ for all $a \in A$, and $\mu'(a) \succ_a \mu(a)$ for at least one $a \in A$

- How do these definitions relate to each other?
  - A D1 is at least as strong as D2 (If μ' "D1-dominates" μ, it also "D2-dominates" μ (for all μ', μ))
  - B D2 is at least as strong as D1 (If μ' "D2-dominates" μ, it also "D1-dominates" μ (for all μ', μ)
  - C D1 and D2 are equivalent (Both [A] and [B] are true)
  - D D1 and D2 are unrelated (Neither [A] nor [B] is true)

The core in a marriage market: Off to brain gym!

- Recall: μ' dominates μ if there is a subset of agents A s.t.
  - (i) For any agent $a \in A$, $\mu'(a) \in A$, and
  - Option D1: (ii) $\mu'(a) \succ_a \mu(a)$ for all $a \in A$
  - Option D2: (ii) $\mu'(a) \succeq_a \mu(a)$ for all $a \in A$, and $\mu'(a) \succ_a \mu(a)$ for at least one $a \in A$)
- Easy: D1 is at least as strong as D2 (If μ' "D1-dominates" μ, it also "D2-dominates" μ)
- D2 is also at least as strong as D1! (If μ' "D2-dominates" μ, it also "D1-dominates" μ): Why?
  - Consider set A; agent a with $\mu'(a) \succ_a \mu(a)$ (D2: such a exists)
  - If $\mu'(a) \succ_a \mu(a)$, $\mu'(a) \neq \mu(a)$ (Why? Preferences are strict)
  - $b = \mu'(a) \in A$ (Why? By definition D2)
  - So b must also strictly prefer $a = \mu'(b)$ to $\mu(b) \neq a$!
  - So: (i) Agents who strictly prefer μ' come in pairs, and (ii) there is at least one such pair in A
  - Consider subset $A' \subseteq A$ of all such pairs: A' shows μ' D1-dominates μ!

The core in a marriage market

- Similar exercise: Two notions of dominance in two-sided marriage markets
  - D1: Matching μ' Pareto-dominates matching μ if
    - All agents weakly prefer partners in μ' to partners in μ
    - At least one agent strictly prefers partner in μ'
  - D2: Matching μ' *dominates* matching μ if there is a subset of agents that collectively *strictly prefer* μ' to μ:
    - For any agent $a \in A$, $\mu'(a) \in A$
    - $\mu'(a) \succ_a \mu(a)$
- How do these notions relate to each other?
  - If matching μ' Pareto-dominates μ, it also dominates μ
  - But if μ' dominates μ, it need not Pareto-dominate μ!
  - So: Core $\Rightarrow$ PE, but PE $\not\Rightarrow$ Core

Immediate questions

- How does stability relate to Pareto-efficiency, core in two-sided markets?
  - Stable matchings are Pareto-efficient
  - But not all Pareto-efficient matchings are stable
  - Every stable matching is in the core, and every core matching is stable
- Does a stable matching always *exist*? If yes, how can we find one?
- What are the properties of stable matchings?
- Stability is a pretty abstraction, but does it matter in reality?