EE2FH4
Electromagnetics I

Term II, January – April 2024

MATLAB Examples and Exercises (Set 4)

Prepared by: Dr. M. H. Bakr and C. He

Example: A surface charge of 5.0 μC/m^2 is located in the x-z plane in the region -2.0 m $\le x \le$ 2.0 m and -3.0 m $\le y \le$ 3.0 m. Find analytically the electric field at the point (0, 4.0, 0) m. Verify your answer using a MATLAB program that applies the principle of superposition.

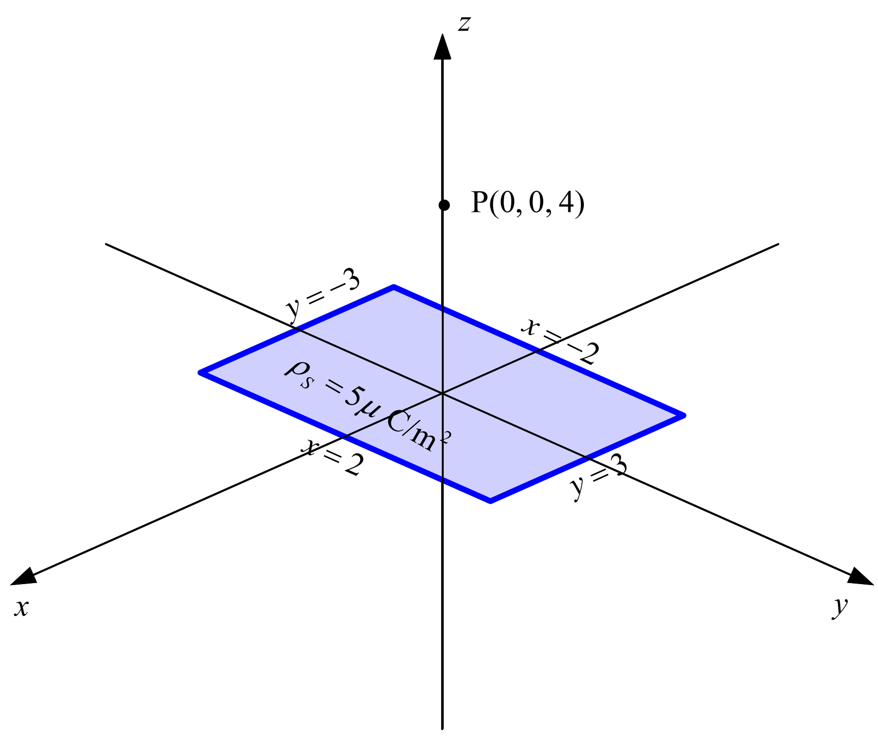

Figure 4.1 The surface charge of the example of Set 4.

Analytical solution:

As can be seen in Figure 4.2 , for any point A on the surface charge, we can find another point A' whose electric field at P has the same magnitude but the opposite sign of that of A in the direction parallel to the surface charge. Hence the electric field at P has only a *z* component .

$$dQ = \rho_s dS = \rho_s dxdy$$

$$dE = \frac{dQ}{4\pi\varepsilon |\mathbf{R}|^2}$$

$$dE_z = dE\cos\theta$$

$$\cos\theta = \frac{|\mathbf{R}_z|}{|\mathbf{R}|} = \frac{4}{\sqrt{(x-0)^2+(y-0)^2+(4-0)^2}} = \frac{4}{\sqrt{x^2+y^2+16}}$$

$$dE_z = \frac{\rho_S dxdz}{4\pi\varepsilon |\mathbf{R}|^2}\frac{4}{\sqrt{x^2+y^2+16}} = \frac{\rho_S}{\pi\varepsilon\left(\sqrt{x^2+z^2+16}\right)^3}dxdy$$

$$E_z = \iint_s dE_z$$

$$= \int_{y=-3}^{y=3}\int_{x=-2}^{x=2}\frac{\rho_S}{\pi\varepsilon\left(\sqrt{x^2+y^2+16}\right)^3}dxdy$$

$$= \frac{\rho_S}{\pi\varepsilon}\int_{y=-3}^{y=3}\int_{x=-2}^{x=2}\frac{1}{\left(\sqrt{x^2+y^2+16}\right)^3}dxdy$$

$$= \frac{\rho_S}{\pi\varepsilon}\int_{y=-3}^{y=3}\int_{x=-2}^{x=2}\frac{1}{\left(\sqrt{x^2+a^2}\right)^3}dxdy \quad (\text{let } a^2 = y^2+16)$$

$$= \frac{\rho_S}{\pi\varepsilon}\int_{y=-3}^{y=3}\left.\frac{x}{a^2\sqrt{a^2+x^2}}\right|_{x=-2}^{x=2}dy$$

$$= \frac{\rho_S}{\pi\varepsilon}\int_{y=-3}^{y=3}\frac{4}{\left(y^2+16\right)\sqrt{y^2+20}}dy$$

let $y = \sqrt{20}\tan\alpha$

then $dy = \sqrt{20}\sec^2\alpha d\alpha$, $\sqrt{y^2+20} = \sqrt{20}\sec\alpha$,

and $y^2+16 = 20\tan^2\alpha + 16$

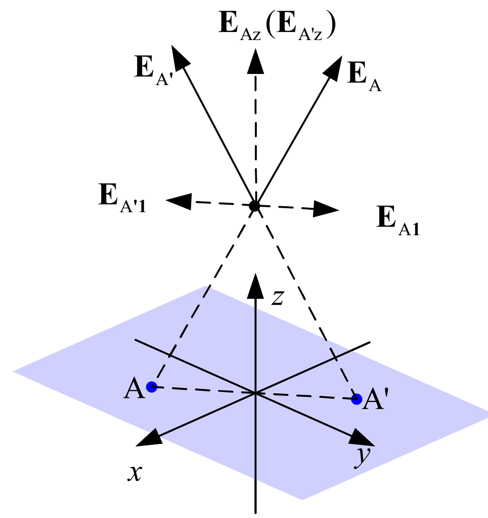

Figure 4.2 The field components.

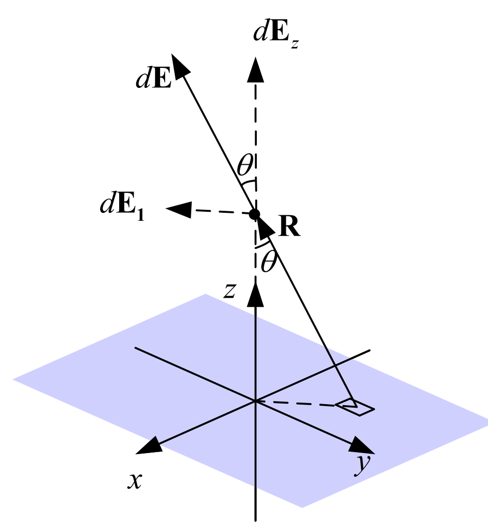

Figure 4.3 Field decomposition.

therefore

$$E_z = \frac{4\rho_S}{\pi\varepsilon}\int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\sqrt{20}\sec^2\alpha d\alpha}{\left(\sqrt{20}\sec\alpha\right)\left(20\tan^2\alpha+16\right)}$$

$$= \frac{4\rho_S}{\pi\varepsilon}\int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\sec\alpha d\alpha}{20\tan^2\alpha+16}$$

$$= \frac{4\rho_S}{\pi\varepsilon}\int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\frac{1}{\cos\alpha}d\alpha}{20\frac{\sin^2\alpha}{\cos^2\alpha}+16}$$

$$= \frac{4\rho_S}{\pi\varepsilon}\int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\cos\alpha d\alpha}{20\sin^2\alpha+16\cos^2\alpha}$$

$$= \frac{4\rho_S}{\pi\varepsilon}\int_{\alpha=\alpha_1}^{\alpha=\alpha_2} \frac{\cos\alpha d\alpha}{4\sin^2\alpha+16} = \frac{\rho_S}{\pi\varepsilon}\int_{y=-3}^{y=3} \frac{\cos\alpha d\alpha}{\sin^2\alpha+4}$$

let $u=\sin\alpha$ then $du=\cos\alpha d\alpha$ therefore

$$E_y = \frac{\rho_S}{\pi\varepsilon}\int_{u=u_1}^{u=u_2} \frac{du}{u^2+4}$$

$$= \frac{\rho_S}{\pi\varepsilon}\times\frac{1}{2}\arctan\frac{u}{2}\Big|_{y=-3}^{y=3}.$$

as we can see from Figure 4.4, $y=\sqrt{20}\tan\alpha$ and $u=\sin\alpha$. The relationship between u and z is given by

$$u = \frac{y}{\sqrt{y^2+20}}$$

therefore

$$E_y = \frac{\rho_S}{\pi\varepsilon}\times\frac{1}{2}\arctan\frac{u}{2}\Big|_{=-3/\sqrt{29}}^{u=3/\sqrt{29}}$$

$$= \frac{5\times10^{-6}}{\pi\times\frac{1}{36\pi}\times10^{-9}}\times\frac{1}{2}\times 2\arctan\frac{\frac{3}{\sqrt{29}}}{2} = 4.8898\times10^4$$

$\mathbf{E} = E_y\mathbf{a}_y = 4.8898\times10^4\mathbf{a}_y$ V/m

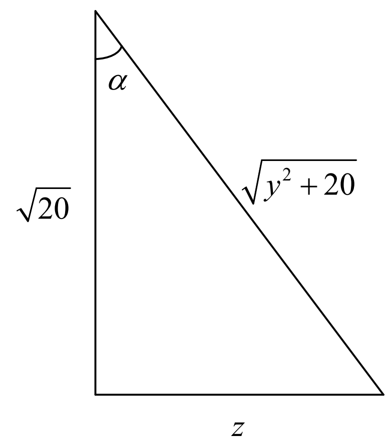

Figure 4.4 $\boldsymbol{u} = \sin\alpha = \frac{\boldsymbol{y}}{\sqrt{\boldsymbol{y}^2+20}}$.

MATLAB Solution:

To write a MATLAB code to solve this problem, we equally divide the surface into many cells each with a length Δy and a width Δx. Each cell has a charge of $\Delta Q = \rho_s \Delta x \Delta y$. When Δx and Δy are very small, the electric field generated by this cell is very close to that generated by a point charge with a charge ΔQ located at the center of the cell. Hence the electric field generated by the surface charge at point P is given by

$$\mathbf{E} \approx \sum_{j=1}^{m}\sum_{i=1}^{n} \Delta\mathbf{E}_{j,i} = \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{\rho_s \Delta x \Delta y}{4\pi\varepsilon |\mathbf{R}_{j,i}|^3}\mathbf{R}_{j,i}$$

where $\mathbf{R}_{j,i}$ is the vector pointing from the center of a cell to the observation point, as shown in Figure 4.5. The location of the center of a cell is given by $x = -2 + \frac{\Delta x}{2} + \Delta x(i-1)$, $y = -3 + \frac{\Delta y}{2} + \Delta y(j-1)$ and $z = 0$. The MATLAB code is given in the next page.

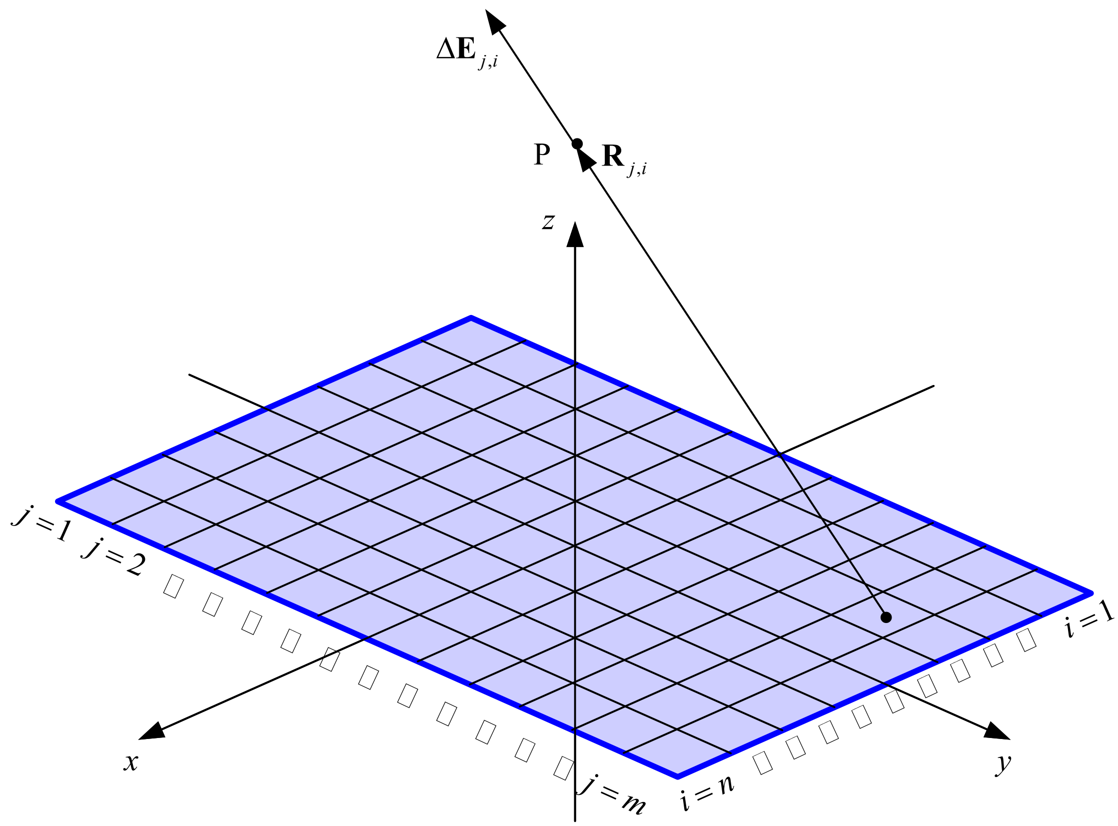

Figure 4.5 The utilized discretization in the MATLAB code.

MATLAB code:

```
clc; %clear the command line
clear; %remove all previous variables

Epsilono=8.854e-12; %use permittivity of air
D=5e-6; %the surface charge density
P=[0 0 4]; %the position of the observation point
E=zeros(1,3); % initialize E=(0 ,0, 0)

Number_of_x_Steps=100;%initialize discretization in the x direction
Number_of_y_Steps=100;%initialize discretization in %the z direction


x_lower=-2; %the lower boundary of x
x_upper=2; %the upper boundary of x
y_lower=-3; %the lower boundary of y
y_upper=-2; %the upper boundary of y
dx=(x_upper- x_lower)/Number_of_x_Steps; %the x increment or the width of a grid
dy=(y_upper- y_lower)/Number_of_y_Steps; %The y increment or the length of a grid
ds=dx*dy; %the area of a single grid
dQ=D*ds; % the charge on a single grid

for j=1: Number_of_y_Steps
    for i=1: Number_of_x_Steps
        x= x_lower +dx/2+(i-1)*dx; %the x component of the center of a grid
        y= y_lower +dy/2+(j-1)*dy; %the y component of the center of a grid
        R=P-[x y 0];%  vector R is the vector seen from the center of the grid to the
observation point
        RMag=norm(R); % magnitude of vector R
        E=E+(dQ/(4*Epsilono*pi* RMag ^3))*R; % get contribution to the E field
    end
end
```

Running result:

Command Window

```
>> E

E =

  1.0e+004 *

   -0.0000    0.0000    4.8833

>>
```

Comparing the MATLAB answer and the analytical answer we see that there is a slight difference. This difference is a result of the finite discretization of the surface *S*.

Exercise: Given the surface charge density, ρ_s=2.0 μC/m^2, existing in the region ρ<1.0 m, z = 0, and zero elsewhere, find **E** at P (ρ = 0, z = 1.0) and write a MATLAB program to verify your answer.
.

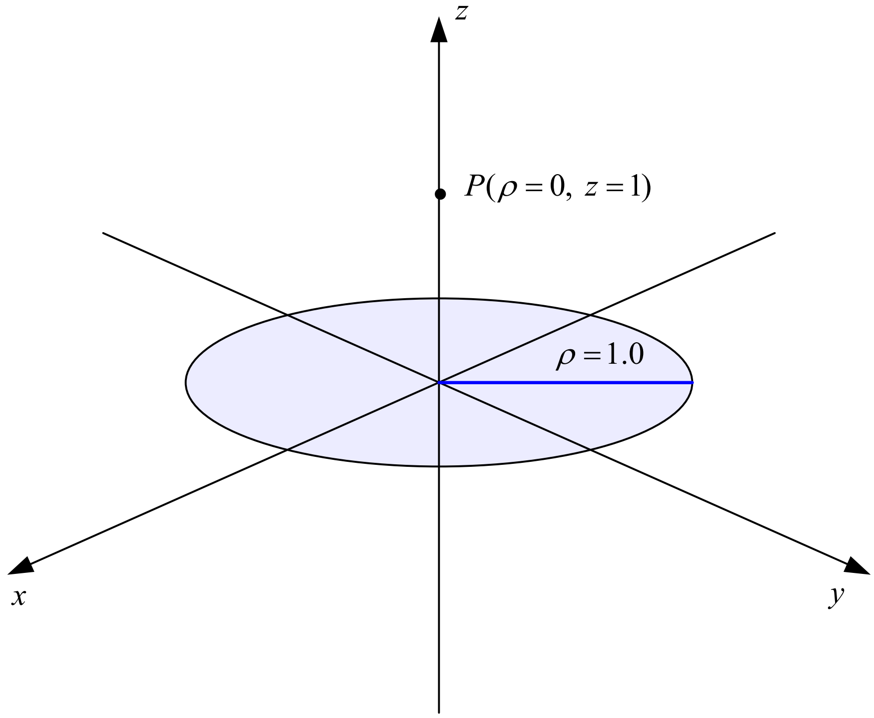

Figure 4.6 The problem of the exercise of Set 4.

Exercise: Given the surface charge density, ρ_s=2.0 μC/m^2, existing in the region ρ<1.0 m, z = 0, and zero elsewhere, find **E** at P (ρ = 0, z = 1.0) and write a MATLAB program to verify your answer.

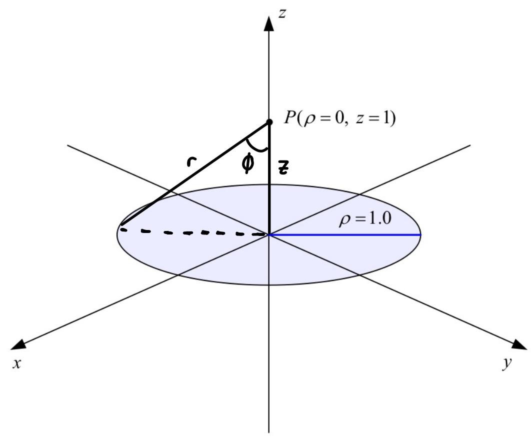

Figure 4.6 The problem of the exercise of Set 4.

analgtical:

$$d\underline{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{r} \cdot \cos\phi$$

$$= \int_0^1\int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \cdot \frac{\boxed{dQ}}{\boxed{r}} \cdot \boxed{\cos\phi} = \int_0^1\int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \cdot \frac{\boxed{\rho_s \rho d\rho d\phi}}{\boxed{\rho^2 + z^2}} \cdot \boxed{\frac{z}{\sqrt{\rho^2+z^2}}}$$

$$= \frac{\rho_s z}{4\pi\varepsilon_0} \int_0^1\int_0^{2\pi} \frac{\rho d\rho d\phi}{(\rho^2+z^2)^{2/3}}$$

$$= \frac{\rho_s z}{4\pi\varepsilon_0} \left[2\pi \cdot \left(\frac{1}{2} \cdot \frac{-2}{\sqrt{\rho^2+1}} \right)_0^1 \right]$$

$$\boxed{= 3.309 \times 10^4 \text{ V/m } \underline{a}_z}$$

MATLAB OUTPUT: