Classical Mechanics

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1 Lecture 1: Introduction, Degrees of Freedom, and Lagrangian Dynamics

1.1 Introduction

The primary goal of this course is to study the dynamics of classical systems, often referred to as "dynamical systems." We will analyze how these systems evolve over time. A simple example is a single particle moving in three-dimensional space, where its state is described by the dynamical variable, the position vector $\mathbf{r}$.

$$\mathbf{r} = (x_1, x_2, x_3) = \text{position}$$
$$\dot{\mathbf{r}} = \mathbf{v} = \text{velocity}$$
$$\ddot{\mathbf{r}} = \mathbf{a} = \text{acceleration}$$

Definition 1.1 (Dynamical Variables)**.** Dynamical variables are a set of continuous parameters that uniquely specify the state of a system at a given time.

For a system of M particles, each particle's position is a dynamical variable, denoted by $\mathbf{r}_\alpha(t)$, where $\alpha = 1, 2, ..., M$. This gives a total of $3M$ dynamical variables.

However, we are often interested in systems where these positions are constrained by relations, limiting their degrees of freedom.

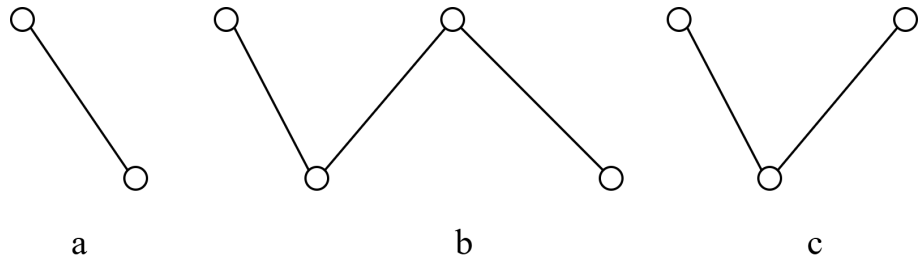

Figure 1.1: Examples of Constrained Systems

1.2 Degrees of Freedom

Definition 1.2 (Degrees of Freedom)**.** The number of independent variables required to completely specify the configuration of a system.

In general, for a 3D object composed of M moving parts, the number of degrees of freedom (DOF) is given by:

$$\text{DOF} = 3M - N \tag{1.1}$$

where N is the number of independent constraints imposed on the system.
Let's consider the examples in Figure 1.1:

- For case (a), two particles connected by a rod:
- $$\text{DOF} = 3 \times 2 - 1 = 5 \tag{1.2}$$

  Here, the constraint is that the distance between the two particles is fixed.

- For case (b), four particles connected by rods where all angles are fixed:

$$\text{DOF} = 3 \times 4 - 3 \text{ (lengths)} - 3 \text{ (angles)} = 3 \text{ (COM)} + 3 \text{ (orientations)} = 6 \tag{1.3}$$

This is equivalent to describing the motion of a rigid body in 3D space, which has 3 translational and 3 rotational degrees of freedom.

- For case (c), three particles connected by rods with one angle not fixed:

$$\text{DOF} = 3 \times 3 - 2 \text{ (lengths)} = 7 \tag{1.4}$$

It's important to note that dynamical variables are not restricted to Cartesian coordinates. We can use any set of independent variables that uniquely specify the system's state, such as:

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi) = \dots \tag{1.5}$$

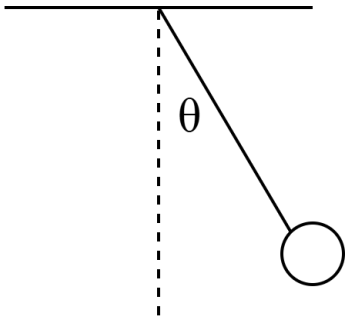

Figure 1.2: Pendulum Example

Consider the simple pendulum shown in Figure 1.2. The system has only 1 DOF. We could choose x, y, or θ as the dynamical variable.

We can introduce a set of generic degrees of freedom q_i, where $i = 1, 2, ..., N$, and N is the number of degrees of freedom. For a constrained system, the position of any part of the system can be written as a function of these generalized coordinates:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t), \quad \alpha = \text{index for particles} \tag{1.6}$$

This form allows for the possibility of an explicit time dependence. If the position vectors can be expressed in this way, the system is said to be **holonomic**. Otherwise, it is **nonholonomic**. Furthermore, if the relation between $\mathbf{r}_\alpha$ and q_i is time-independent (i.e., $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$), the system is **scleronomic**; otherwise, it is **rheonomic**.

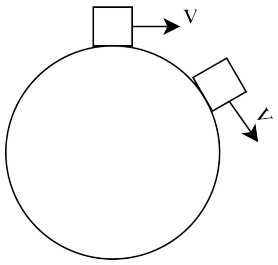

Figure 1.3: Nonholonomic System Example

Nonholonomic systems are common. In the example of the box on a surface in Figure 1.3, the DOF increases from 2 to 3 when the box leaves the surface.

1.3 Lagrangian Mechanics

Consider a system described by the generalized coordinates q_i, where $i = 1, 2, ..., N$ (number of DOF). The positions of the system's parts can be written as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$. The fundamental problem is to determine the time evolution of these generalized coordinates, $q_i(t)$. These will satisfy a set of N differential equations, known as the **equations of motion**.

Traditionally, Newton's laws of motion were used, which requires dealing with constraint forces:

1. Determine the force $\mathbf{F}_\alpha$ acting on each part of the system at position $\mathbf{r}_\alpha$.
2. Apply Newton's second law, which gives a system of second-order ordinary differential equations (ODEs):

$$\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha \tag{1.7}$$

3. Rewrite $\mathbf{r}_\alpha$ in terms of q_i. This can be challenging to actually implement!

Lagrangian mechanics provides an elegant way to avoid dealing with constraint forces directly. Consider an infinitesimal change in position, $\delta\mathbf{r}_\alpha$. The work done by the force is:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta\mathbf{r}_\alpha \tag{1.8}$$

A crucial question arises: how much work is done if we change the generalized coordinates from q_i to $q_i + \delta q_i$? Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, the variation in position can be written as:

$$\delta\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i \tag{1.9}$$

This assumes that we only consider variations in the generalized coordinates, keeping t constant. Thus:

$$\begin{aligned}\delta W &= \sum_\alpha \mathbf{F}_\alpha \cdot \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i\right) \\ &= \sum_i \left(\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right)\delta q_i\end{aligned}$$

We define the term in the parenthesis to be a **generalized force**:

$$F_i = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}$$

Here, F_i is the generalized force associated with the generalized coordinate q_i. It effectively represents the force component in the "allowed" direction defined by the variation in q_i.

Now let's consider the kinetic energy of a constrained system:

$$\begin{aligned}T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= T(q_i, \dot{q}_i, t)\end{aligned}$$

where,

$$\begin{aligned}\mathbf{r}_\alpha &= \mathbf{r}_\alpha(q_i, t) \\ \dot{\mathbf{r}}_\alpha &= \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t}\end{aligned}$$

Note that from the expression above, we have

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \tag{1.10}$$

We can compute the partial derivatives of the kinetic energy:

$$\begin{aligned}\frac{\partial T}{\partial q_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \\ \frac{\partial T}{\partial \dot{q}_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\end{aligned}$$

Now, let's compute the time derivative of $\frac{\partial T}{\partial \dot{q}_i}$:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) &= \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\right) \\ &= \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i} \\ &= F_i + \frac{\partial T}{\partial q_i}\end{aligned}$$

Therefore, we have the following important relation:

$$F_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \tag{1.11}$$

If we know the kinetic energy $T(q_i, \dot{q}_i, t)$, we can obtain the generalized force without directly calculating constraint forces! This provides a powerful generalization of $\mathbf{F} = m\mathbf{a}$ to a generic degree of freedom.

2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equations, and Hamiltonians

2.1 Lagrangian Mechanics and the Euler-Lagrange Equations

We now consider a simplification of the previous lecture: we assume that the forces $\mathbf{F}_\alpha$ acting on the system are conservative.

Definition 2.1 (Conservative Force)**.** A force $\mathbf{F}_\alpha$ is said to be **conservative** if the line integral of the force over any closed path is zero:

$$\oint \mathbf{F}_\alpha \cdot d\mathbf{r}_\alpha = 0 \tag{2.1}$$

This implies that the work done by the force in moving a particle between two points is independent of the path taken. A conservative force can be expressed as the negative gradient of a scalar potential function V:

$$\begin{aligned}\mathbf{F}_\alpha &= -\nabla_\alpha V(\mathbf{r}_1,\ \ldots,\ \mathbf{r}_\alpha) \\ &= -\frac{\partial}{\partial \mathbf{r}_\alpha} V(\mathbf{r}_1,\ \ldots,\ \mathbf{r}_\alpha)\end{aligned}$$

The work done in moving a particle from $\mathbf{r}_\alpha$ to $\mathbf{r}'_\alpha$ is given by $V(\mathbf{r}_\alpha) - V(\mathbf{r}'_\alpha)$. In this course, we will primarily consider systems with conservative forces.

Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, the potential energy $V(\mathbf{r}_\alpha)$ can also be written as a function of the generalized coordinates:

$$V(\mathbf{r}_\alpha) = V(q_i, t) \tag{2.2}$$

Using the chain rule, we obtain:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -F_i \tag{2.3}$$

Where F_i is the generalized force corresponding to the generalized coordinate q_i. From the previous lecture, we know that

$$F_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \tag{2.4}$$

Therefore, for a conservative force, we can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{2.5}$$

Since the potential energy V is not a function of $\dot{q}_i$, we have:

$$\frac{\partial V}{\partial \dot{q}_i} = 0 \tag{2.6}$$

We can now define the **Lagrangian** as $L = T - V$, where $L = L(q_i, \dot{q}_i, t)$. The equation of motion can then be rewritten as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.7}$$

These are the **Euler-Lagrange Equations**. This set of equations provides a powerful means to obtain the equations of motion for a mechanical system using the Lagrangian.

These are N differential equations, one for each degree of freedom. Typically, these are second-order ordinary differential equations (ODEs) for q_i.

In summary, given a system with M particles and N degrees of freedom, the following steps should be followed to determine the equations of motion using Lagrangian Mechanics:

1. Identify the generalized coordinates q_i that specify the system's configuration, and express the position vectors of the particles as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, where $\alpha = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$.

2. Calculate the kinetic energy $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha$ as a function of q_i and $\dot{q}_i$.
3. Compute the potential energy $V = V(\mathbf{r}_\alpha) = V(q_i, t)$.
4. Construct the Lagrangian $L = T - V$.
5. Apply the Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

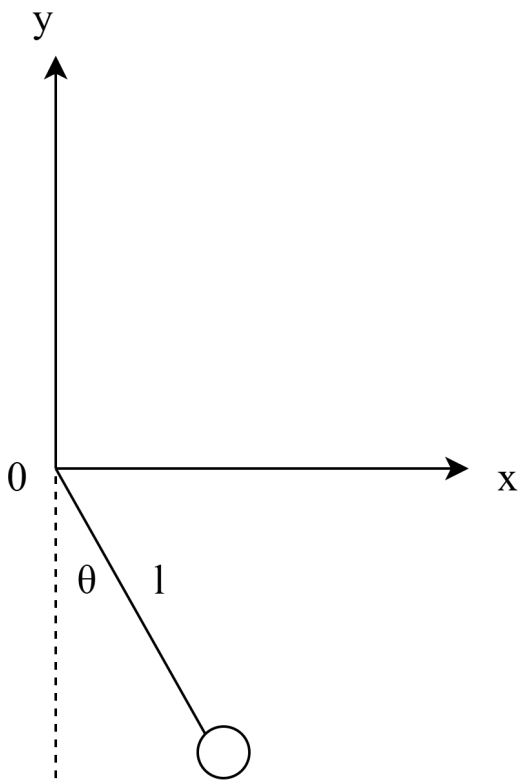

Figure 2.1: Simple Pendulum Example

As an example, let's consider the simple pendulum depicted in Figure 2.1. The position of the pendulum bob can be written as:

$$x = l \sin\theta$$
$$y = -l \cos\theta$$

The time derivatives of the coordinates are:

$$\dot{x} = l \cos\theta \cdot \dot{\theta}$$
$$\dot{y} = l \sin\theta \cdot \dot{\theta}$$

Thus, the kinetic energy is:

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2} m l^2 \dot{\theta}^2 \tag{2.8}$$

The potential energy is:

$$V = -mgy = -mgl \cos\theta \tag{2.9}$$

The Lagrangian for the pendulum is:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{2.10}$$

Applying the Euler-Lagrange equation we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) = ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{2.11}$$

Thus the equation of motion of the simple pendulum is:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{2.12}$$

This result is consistent with what is obtained using Newton's laws of motion.

2.2 Hamiltonian Mechanics

While Lagrangian mechanics is conceptually valuable, we can take a further step by introducing the **Hamiltonian**. For a system with dynamical variables q_i and a Lagrangian $L(q_i, \dot{q}_i, t)$, we define the Hamiltonian H as:

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{2.13}$$

Let's compute the total time derivative of the Hamiltonian:

$$\begin{aligned}\frac{dH}{dt} &= \sum_i \left(\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) - \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) - \frac{\partial L}{\partial t} \\ &= \sum_i \dot{q}_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial t}\end{aligned}$$

Using the Euler-Lagrange equation, we have:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.14}$$

Thus, the equation simplifies to:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{2.15}$$

The total time derivative of the Hamiltonian is the negative of the partial time derivative of the Lagrangian. If L has no explicit time dependence, then:

$$\frac{dH}{dt} = 0 \tag{2.16}$$

This implies that H is a conserved quantity.

What is the **Hamiltonian** physically? If the constraints are time-independent, then $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$, and we have:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \tag{2.17}$$

We can then compute

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \tag{2.18}$$

$$\begin{aligned}\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} &= \sum_i \dot{q}_i \left(\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right) \\ &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i\right) \\ &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= 2T\end{aligned}$$

So, in this case:

$$\begin{aligned} H &= \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\ &= 2T - (T - V) \\ &= T + V \end{aligned}$$

Therefore, H represents the total energy of the system (when the constraints are time independent and the potential only depends on coordinates)!

From the above analysis, we can draw the following conclusions:

1. If the **Lagrangian** L does not depend explicitly on time, then the total energy of the system, represented by the Hamiltonian H, is conserved.
2. If the **Lagrangian** L does not depend explicitly on time, then the system possesses time-translation symmetry.

Thus, energy conservation is associated with time-translation symmetry.

Definition 2.2 (Noether's Theorem)**.** Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

This is an important theorem that relates symmetries and conservation laws. Let's look at another example to illustrate **Noether's Theorem**: Suppose $L(q_i, \dot{q}_i, t)$ is independent of q_i (though it could still depend on $\dot{q}_i$ and t). Then we have $L = L(\dot{q}_i, t)$, and

$$\frac{\partial L}{\partial q_i} = 0 \tag{2.19}$$

Then from the Euler-Lagrange equation we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{2.20}$$

So $\frac{\partial L}{\partial \dot{q}_i}$ is a conserved quantity. This is defined as the **momentum** conjugate to q_i.

Definition 2.3 (Generalized Momentum)**.** The **generalized momentum** p_i conjugate to q_i is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$. This momentum is conserved if the Lagrangian L does not depend on the corresponding coordinate q_i.

As a final example, consider a particle moving in a circle of radius R. The equations of motion can be written as:

$$\begin{aligned} x &= R\cos(\theta) \\ y &= R\sin(\theta) \end{aligned}$$

The time derivatives are:

$$\begin{aligned} \dot{x} &= -R\sin(\theta)\dot{\theta} \\ \dot{y} &= R\cos(\theta)\dot{\theta} \end{aligned}$$

The Lagrangian is thus:

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mR^2\dot{\theta}^2 \tag{2.21}$$

Note that L is independent of θ. The generalized momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{2.22}$$

This is recognized as the angular momentum.

In summary, a linear translational symmetry leads to conservation of linear momentum, a rotational symmetry leads to conservation of angular momentum, and time translation symmetry leads to conservation of energy.

3 Lecture 3: The Action Principle and the Calculus of Variations

We begin by considering **Fermat's principle**, which governs the path of light.

Definition 3.1 (Fermat's Principle)**.** Light travels along the path that minimizes the travel time between two points.

This can also be expressed by stating that light follows a path that minimizes the optical path length, given by

$$L = \int ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}. \tag{3.1}$$

This principle demonstrates that light minimizes a particular quantity along its trajectory. This naturally leads us to ask: does a similar principle apply to the trajectories of mechanical systems? The answer is yes.

Consider all possible paths $q_i(t)$ that a mechanical system could take through configuration space. For a given path, we define the **action** of the path, denoted by $S[q_i(t)]$, as:

$$S[q_i(t)] = \int_{t_{\text{initial}}}^{t_{\text{final}}} L(q_i, \dot{q}_i, t)\, dt \tag{3.2}$$

where L is the Lagrangian of the system. It turns out that the path that a mechanical system actually takes through configuration space is the one that **extremizes** the action. This is known as the **Principle of Least Action** or **Hamilton's Principle**. The action $S[q_i(t)]$ is a *functional*, which is a function of a function. We denote functionals using square brackets, e.g., $S[f]$, to distinguish them from functions like $f(x)$.

In standard calculus, we minimize functions of a finite number of variables. Here, we are tasked with minimizing a *functional*, which depends on an entire curve. This requires a new set of tools, which fall under the domain of the **Calculus of Variations**.

To understand this, let's consider a general problem. Suppose we have a function $F(y(x), y'(x), x)$, and we want to minimize the functional:

$$I[y(x)] = \int_{x_0}^{x_1} F(y(x), y'(x), x)\, dx \tag{3.3}$$

where $y(x)$ is defined on the interval $x_0 \leq x \leq x_1$ and $y'(x) = \frac{dy}{dx}$. The goal is to find the function $y(x)$ that extremizes $I[y(x)]$.

We proceed as follows:

- Fix the values of $y(x_0)$ and $y(x_1)$.
- Consider a curve $y(x)$ connecting these endpoints.
- Introduce a small perturbation to this curve: $y(x) \to y(x) + \delta y(x)$.
- Compute the variation δI due to this perturbation, and require that the term linear in δy vanishes.
- Since we have fixed the endpoints, we require $\delta y(x_0) = \delta y(x_1) = 0$.

The change in the functional due to the perturbation is:

$$\delta I = I[y(x) + \delta y(x)] - I[y(x)] = \int_{x_0}^{x_1} \left(F(y + \delta y, y' + \delta y', x) - F(y, y', x)\right) dx \tag{3.4}$$

Using a first-order Taylor expansion of F, we have:

$$\delta I \approx \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) \tag{3.5}$$

Note that $\delta y' = \frac{d}{dx}(\delta y)$. Integrating the second term by parts (with $u = \frac{\partial F}{\partial y'}$ and $dv = \frac{d}{dx}(\delta y)\, dx$), and using the fact that $\delta y(x_0) = \delta y(x_1) = 0$, we get:

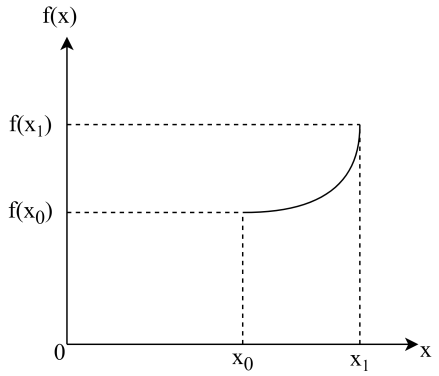

Figure 3.1: Calculus of Variations: Varying a Curve

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} \delta y - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y \right) \tag{3.6}$$

Combining the terms, we obtain:

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \delta y \tag{3.7}$$

For δI to be zero for an arbitrary perturbation $\delta y(x)$, the term inside the parentheses must vanish, giving us the **Euler-Lagrange Equation**:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \tag{3.8}$$

The quantity

$$\frac{\delta I}{\delta y(x)} = \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \tag{3.9}$$

is called the **functional derivative** of $I[y(x)]$ with respect to $y(x)$.

The above result can be easily generalized to a functional that depends on N functions $y_i(x)$, rather than just one:

$$I[y_i(x)] = \int dx F(y_i, \frac{dy_i}{dx}, x), \quad i = 1, 2, \ldots, N \tag{3.10}$$

Following the same procedure as before, we obtain the condition for extremizing I:

$$\frac{\delta I}{\delta y_i} = \frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y_i'} \right) = 0 \tag{3.11}$$

This Euler-Lagrange equation must hold for all i.

In the context of mechanics, we have that the action is defined as $S[q_i(t)] = \int L \, dt$, where $L\left(q_i, \dot{q}_i, t\right)$ is the Lagrangian. Thus, the extremization of the action implies the following Euler-Lagrange equations:

$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{3.12}$$

These are the equations of motion for the system.

Let us apply the calculus of variations to an elementary problem.

Example 3.1. Prove that the shortest distance between two points is a line.

Proof. The length of the path can be written as:

$$L = \int ds \sqrt{(\dot{x}^2 + \dot{y}^2)} \tag{3.13}$$

Here $\dot{x}$ represents $\frac{dx}{ds}$. The Euler-Lagrange equations give us:

$$\frac{\delta L}{\delta x\,(s)} = -\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0 \tag{3.14}$$

This can be simplified to:

$$\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0 \tag{3.15}$$

Similarly, for y we get

$$\frac{d}{ds}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0 \tag{3.16}$$

From equation (3.15) we can get:

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} \tag{3.17}$$

$$\frac{d}{ds}\left(\log \dot{x}\right) = \frac{d}{ds}\left(\log \dot{y}\right) \tag{3.18}$$

which means:

$$\frac{d}{ds}\log\frac{\dot{x}}{\dot{y}} = 0 \tag{3.19}$$

i.e.,

$$\frac{dx}{dy} = \text{constant} \tag{3.20}$$

Thus the shortest path between two points is a line. ■

We have found that

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \text{const} \tag{3.21}$$

Likewise,

$$p_y = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \text{const} \tag{3.22}$$

So we can get:

$$\frac{p_x}{p_y} = \frac{\dot{x}}{\dot{y}} = \text{const} \tag{3.23}$$

This shows that $\frac{dx}{dy}$ is constant, which implies a straight line. Also, the quantities p_x and p_y are constants of motion.

4 Lecture 4: Lagrange Multipliers, Near Equilibrium Dynamics & Oscillators

4.1 Lagrange Multipliers: Constrained Optimization

Our standard Lagrangian approach requires expressing the positions of all particles in terms of generalized coordinates, $r_\alpha(q_i, t)$. This can be complex, making it difficult to obtain an explicit form for the Lagrangian, $L(q_i, \dot{q}_i, t)$. A natural question arises: can we formulate mechanics directly in terms of the particle positions, $\mathbf{r}_\alpha$, and impose constraints directly at the level of the equations of motion, rather than at the level of the action? This is where Lagrange multipliers become a valuable tool.

4.1.1 Review of Lagrange Multipliers in Calculus

Let's first recall how Lagrange multipliers work in standard calculus. Suppose we want to minimize a function $F(x, y)$. Without constraints, this involves simply finding where the partial derivatives vanish:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \tag{4.1}$$

However, if we want to minimize $F(x, y)$ subject to a constraint given by $g(x, y) = 0$, we introduce a Lagrange multiplier, λ. We then minimize the modified function: $F(x, y) + \lambda\, g(x, y)$. This leads to the following set of equations:

$$\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 \\ g(x, y) &= 0 \end{aligned}$$

Notice that we now have three equations for three unknowns: x, y, and λ. The first two equations define the stationary points of F subject to the constraint, and the third equation ensures that the constraint is satisfied. This method provides a systematic way to solve constrained optimization problems.

4.1.2 Lagrange Multipliers in the Calculus of Variations

The same principle can be extended to the calculus of variations. Let's say we have M functions $y_i(x), i = 1, 2, \cdots, M$, and we want to minimize an action integral:

$$I[y_i(x)] = \int dx\; F(y_i, \dot{y}_i, x) \tag{4.2}$$

Furthermore, suppose we have $M - N$ constraints of the form $G_k[y_i] = 0, k = 1, 2, \cdots, M - N$. The traditional approach would be to express all M functions, y_i, in terms of N independent degrees of freedom, however, we can avoid this by using Lagrange multipliers. To incorporate the constraints, we modify the action integral as follows:

$$\tilde{I}[y_i, \lambda_k] = \int dx\; \left(F(y_i, \dot{y}_i, x) + \sum_k \lambda_k(x)\, G_k[y_i] \right) \tag{4.3}$$

Here, $\lambda_k(x)$ are now functions of x, the independent variable of the integral. We treat $\tilde{I}$ as the functional we want to minimize, and obtain its Euler-Lagrange equations:

$$\begin{aligned} \frac{\delta \tilde{I}}{\delta y_i} = \frac{\delta I}{\delta y_i} + \sum_k \lambda_k \frac{\delta G_k}{\delta y_i} &= 0 \\ G_k[y_i] &= 0 \end{aligned}$$

This gives us a total of $M + (M - N)$ equations. We have M equations from the functional derivative with respect to each y_i, plus $M - N$ equations from the constraints. These equations involve the M unknowns y_i and the $M - N$ Lagrange multiplier functions, $\lambda_k(x)$. These can, in principle, be solved to obtain the solution to the constrained variational problem.