Classical Mechanics

Prof. Alexander Maloney

Physics 451: Classical Mechanics
Winter 2010

Editor: Zeyu Li

March 24, 2025

Contents

1 Lecture 1: Introduction, Degrees of Freedom, and Lagrangian Dynamics

1.1 Introduction

The objective of this course is to develop a deep understanding of classical dynamical systems using the framework of Lagrangian mechanics. This formulation elegantly generalizes Newtonian mechanics and provides powerful methods for dealing with complex, constrained systems.

Overview of Coordinate Systems

The description of a physical system can vary significantly with the choice of coordinate system. Below are some common coordinate representations:

Cartesian Coordinates. For a particle in three-dimensional space, the position, velocity, and acceleration are expressed as:

$$\begin{aligned} \mathbf{r} &= (x,\, y,\, z), \\ \mathbf{v} &= \dot{\mathbf{r}}, \\ \mathbf{a} &= \ddot{\mathbf{r}}. \end{aligned} \tag{1.1}$$

Polar Coordinates. For planar motion, using polar coordinates (r, θ):

$$\begin{aligned} \frac{d\hat{\mathbf{e}}_r}{dt} &= \dot{\theta}\,\hat{\mathbf{e}}_\theta, \\ \frac{d\hat{\mathbf{e}}_\theta}{dt} &= -\dot{\theta}\,\hat{\mathbf{e}}_r, \end{aligned} \tag{1.2}$$

so that the position and velocity become:

$$\begin{aligned} \mathbf{r} &= r\,\hat{\mathbf{e}}_r, \\ \mathbf{v} &= \dot{r}\,\hat{\mathbf{e}}_r + r\,\dot{\theta}\,\hat{\mathbf{e}}_\theta. \end{aligned} \tag{1.3}$$

Spherical Coordinates. For a particle in three-dimensional space described by spherical coordinates (r, θ, ϕ):

$$\begin{aligned} \mathbf{r} &= (r,\, \theta,\, \phi), \\ \mathbf{v} &= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\phi}\,\sin\theta\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{1.4}$$

1.2 Degrees of Freedom and Generalized Coordinates

In classical mechanics, the *degrees of freedom* (DOF) represent the number of independent parameters required to specify the configuration of a system.

Definition 1.1 (Degrees of Freedom)**. Mechanic definition:** The number of independent parameters needed to define a system's configuration.

Kinetic definition: For a rigid body, the DOF equals the number of independent movements (typically 3 translational and 3 rotational in three dimensions).

For a system of M particles in three-dimensional space with no constraints, the total degrees of freedom is $3M$. When N holonomic (i.e., integrable) constraints are present, the number of independent generalized coordinates reduces to:

$$\text{DOF} = 3M - N. \tag{1.5}$$

If there are additional k nonholonomic (non-integrable) constraints, these restrict the allowable velocities but do not reduce the configuration space's dimension. Thus, under the kinetic viewpoint, the effective degrees of freedom become:

$$\text{DOF}_{\text{kinetic}} = 3M - N - k. \tag{1.6}$$

Nevertheless, the number of generalized coordinates remains $3M - N$.

Examples of Constrained Systems

- **Two particles connected by a rod:**
  The rod fixes the distance between the particles, leading to:

$$\text{DOF} = 3 \times 2 - 1 = 5. \tag{1.7}$$

- **Four particles connected by rods with fixed angles:**
  With three constraints fixing the rod lengths and three additional constraints fixing the angles, the system behaves as a rigid body:

$$\text{DOF} = 3 \times 4 - 3\,(\text{length constraints}) - 3\,(\text{angular constraints}) = 6. \tag{1.8}$$

- **Three particles connected by rods with one unfixed angle:**
  Removing one angular constraint results in:

$$\text{DOF} = 3 \times 3 - 2 = 7. \tag{1.9}$$

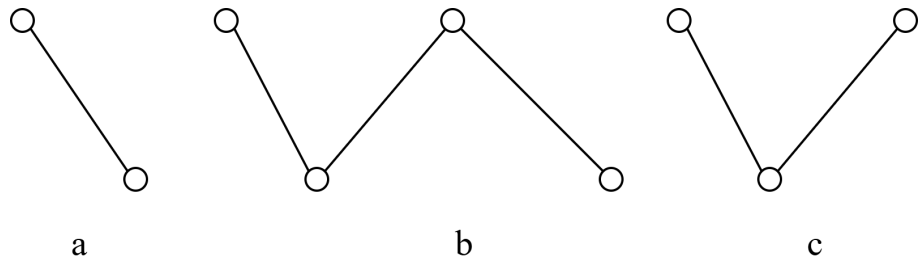

Figure 1.1: Examples of Constrained Systems

The Simple Pendulum

Consider the simple pendulum (see Figure 1.2). The motion of the pendulum is completely described by the angular displacement θ, thus it has one degree of freedom.

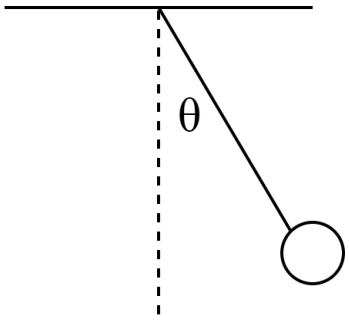

Figure 1.2: Simple Pendulum

Generalized Coordinates in Constrained Systems

For systems with constraints, we introduce generalized coordinates $\{q_i\}$ (with $i = 1, 2, \ldots, N$) such that the position of each particle α is expressed as:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, q_2, \ldots, q_N, t). \tag{1.10}$$

When constraints explicitly depend on time, they are termed *rheonomous*; otherwise, they are *scleronomous*. Moreover, if a constraint can be written as

$$f(q_1, q_2, \ldots, q_N, t) = 0,$$

(i.e., it does not involve velocities), it is a *holonomic* constraint; constraints involving velocities are classified as *nonholonomic*.

Nonholonomic systems frequently occur in practice. For example, consider a box moving on the surface of a sphere in figure 1.3: when the box loses contact with the surface, the effective degrees of freedom increase.

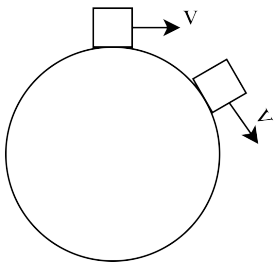

Figure 1.3: Example of a Nonholonomic System

1.3 Lagrangian Mechanics

Lagrangian mechanics reformulates the equations of motion for a system in terms of generalized coordinates. Consider a system described by generalized coordinates q_i $(i = 1, 2, \ldots, N)$; the positions of its constituent particles can be expressed as:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t). \tag{1.11}$$

The primary task is to determine the evolution of $q_i(t)$, governed by a set of N differential equations, known as the *Euler-Lagrange equations*.

From Newton to Lagrange

In the traditional Newtonian approach, one must:

1. Compute the forces $\mathbf{F}_\alpha$ acting on each particle.
2. Apply Newton's second law:
$$\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha, \tag{1.12}$$
yielding a set of second-order differential equations.
3. Express $\mathbf{r}_\alpha$ in terms of the generalized coordinates q_i.

This process can become cumbersome, particularly for systems with constraints.

Generalized Forces and Virtual Work

Consider an infinitesimal virtual displacement $\delta\mathbf{r}_\alpha$ of the particle positions, while time is held fixed. The corresponding virtual work done by the forces is given by:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta\mathbf{r}_\alpha. \tag{1.13}$$

Since the particle positions depend on the generalized coordinates, a variation in q_i induces:

$$\delta\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i. \tag{1.14}$$

Substituting into the virtual work expression,

$$\delta W = \sum_i \left(\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \delta q_i, \tag{1.15}$$

we define the *generalized force* Q_i as:

$$Q_i = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.16}$$

Kinetic Energy and its Variations

The kinetic energy of a system is

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha, \tag{1.17}$$

which can be expressed as a function of the generalized coordinates and their time derivatives, $T(q_i, \dot{q}_i, t)$. Since

$$\dot{\mathbf{r}}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t}, \tag{1.18}$$

it follows that

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.19}$$

Thus, the partial derivative of the kinetic energy with respect to q_i and $\dot{q}_i$ is:

$$\frac{\partial T}{\partial q_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \tag{1.20}$$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.21}$$

Taking the time derivative and using Newton's second law:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\right) \tag{1.22}$$

$$= \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i} \tag{1.23}$$

$$= Q_i + \frac{\partial T}{\partial q_i} \tag{1.24}$$

One obtains the important relation:

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}. \tag{1.25}$$

This result forms the foundation for deriving the Euler-Lagrange equations, which succinctly encapsulate the dynamics of the system.

1.4 The Principle of Virtual Work and d'Alembert's Principle

Principle of Virtual Work

Applied Forces. These are the external forces actively acting on the system. They can include gravity, friction (if it does work), external loads, etc. Applied forces are usually responsible for driving the motion or deformation of the system.

Constraint Forces. These forces arise from the constraints imposed on the system—such as rods, surfaces, or other geometric restrictions—that limit the motion of the system's parts. A key aspect of ideal constraints is that the constraint forces do no work during any *virtual displacement*.

The *principle of virtual work* states that for a system in equilibrium, the total virtual work done by the applied forces during any virtual displacement (consistent with the constraints) is zero:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = 0. \tag{1.26}$$

d'Alembert's Principle

d'Alembert's principle extends the principle of virtual work to dynamics. It introduces the concept of *inertial forces* (or *d'Alembert forces*) to reformulate Newton's second law into a form that is analogous to static equilibrium:

$$\sum_\alpha (\mathbf{F}_\alpha - m_\alpha \ddot{\mathbf{r}}_\alpha) \cdot \delta \mathbf{r}_\alpha = 0. \tag{1.27}$$

Here, the term $-m_\alpha \ddot{\mathbf{r}}_\alpha$ can be interpreted as an inertial force. By including these forces, the dynamic problem is converted into a statics-like problem where the total virtual work (including contributions from both applied and inertial forces) vanishes.

2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equations, and Hamiltonians

2.1 Conservative Forces and the Lagrangian

A **conservative force** is one for which the work done along any closed path is zero. Formally, for a particle labeled by α, this means:

$$\oint \mathbf{F}_\alpha \cdot d\mathbf{r}_\alpha = 0. \tag{2.1}$$

As a consequence, a conservative force can be expressed as the negative gradient of a potential energy function V:

$$\mathbf{F}_\alpha = -\nabla_\alpha V(\mathbf{r}_1, \ldots, \mathbf{r}_\alpha). \tag{2.2}$$

Since the particle positions $\mathbf{r}_\alpha$ depend on the generalized coordinates q_i, the potential energy may be written as

$$V(\mathbf{r}_\alpha) = V(q_i, t). \tag{2.3}$$

Using the chain rule, one finds:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -Q_i, \tag{2.4}$$

where Q_i is the generalized force associated with the coordinate q_i.

For conservative systems, the work done is path independent. Moreover, if the kinetic energy is expressed as $T(q_i, \dot{q}_i, t)$, then—as shown in Lecture 1—the generalized force takes the form:

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}. \tag{2.5}$$

2.1.1 The Lagrangian Function and the Euler-Lagrange Equations

Define the **Lagrangian** as:

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t). \tag{2.6}$$

Substituting the expression for F_i, the equation of motion becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0. \tag{2.7}$$

Because the potential energy V is independent of $\dot{q}_i$, the above expression can be neatly recast into the **Euler-Lagrange equations**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.8}$$

These N second-order differential equations (one for each degree of freedom) serve as the foundation for analyzing the dynamics of conservative systems.

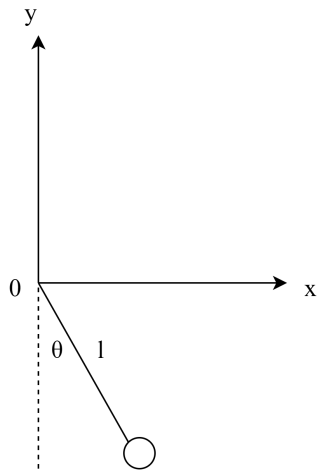

Figure 2.1: Schematic of a Simple Pendulum

2.1.2 A Worked Example: The Simple Pendulum

Consider a simple pendulum of mass m and length l (see Figure 2.1). The position of the pendulum bob in Cartesian coordinates is given by:

$$x = l \sin\theta, \tag{2.9}$$
$$y = -l \cos\theta. \tag{2.10}$$

Taking time derivatives yields:

$$\dot{x} = l \cos\theta \, \dot{\theta}, \tag{2.11}$$
$$\dot{y} = l \sin\theta \, \dot{\theta}. \tag{2.12}$$

Thus, the kinetic energy is:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\theta}^2. \tag{2.13}$$

Choosing the gravitational potential energy as

$$V = -mgl \cos\theta, \tag{2.14}$$

the Lagrangian becomes:

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta. \tag{2.15}$$

Applying the Euler-Lagrange equation for θ, we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin\theta = 0, \tag{2.16}$$

which simplifies to:

$$ml^2\ddot{\theta} + mgl \sin\theta = 0. \tag{2.17}$$

This is the familiar equation of motion for a simple pendulum.

In summary, given a system with M particles and N degrees of freedom, the following steps should be followed to determine the equations of motion using Lagrangian Mechanics:

1. Identify the generalized coordinates q_i that specify the system's configuration, and express the position vectors of the particles as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, where $\alpha = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$.
2. Calculate the kinetic energy $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha$ as a function of q_i and $\dot{q}_i$.
3. Compute the potential energy $V = V(\mathbf{r}_\alpha) = V(q_i, t)$.
4. Construct the Lagrangian $L = T - V$.
5. Apply the Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

2.1.3 Routh Equations &Nonholonomic Constraints

Supposing that there are k linear nonholonomic constraints of the form:

$$A_{rq}\dot{q}_r + B_r = 0, r = 1, 2, 3, \ldots, k, \tag{2.18}$$

we can use Routh equations to derive the equations of motion:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha + \lambda_r A_{r\alpha} \tag{2.19}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = \lambda A_{r\alpha}. \tag{2.20}$$

2.2 Hamiltonian Mechanics

While the Lagrangian formulation provides a powerful method for deriving the equations of motion, the Hamiltonian formulation offers an alternative perspective—often advantageous in areas such as canonical transformations, quantum mechanics, and statistical mechanics.

2.2.1 Definition of the Hamiltonian

For a system described by generalized coordinates q_i and velocities $\dot{q}_i$, the **Hamiltonian** is defined as the Legendre transform of the Lagrangian:

$$H(q_i, p_i, t) = \sum_i \dot{q}_i\, p_i - L(q_i, \dot{q}_i, t), \tag{2.21}$$

where the **generalized momentum** conjugate to q_i is given by:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.22}$$

2.2.2 Conservation of Energy and Time-Translation Symmetry

To understand the time evolution of H, consider its total time derivative:

$$\begin{aligned}\frac{dH}{dt} &= \sum_i \left(\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) - \frac{dL}{dt} \\ &= \sum_i \left(\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) - \left(\sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right)\right) - \frac{\partial L}{\partial t}\end{aligned} \tag{2.23}$$

Using the Euler-Lagrange equations, the bracketed term vanishes, leaving:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}. \tag{2.24}$$

Thus, if the Lagrangian is *time-independent* (i.e., $\partial L/\partial t = 0$), then:

$$\frac{dH}{dt} = 0, \tag{2.25}$$

implying that the Hamiltonian is conserved. For many systems, especially when constraints are time-independent and V depends only on coordinates, the Hamiltonian corresponds to the total energy:

$$H = T + V. \tag{2.26}$$

2.2.3 Another Example: Particle Moving in a Circle

Consider a particle of mass m moving along a circle of fixed radius R. Using the angular coordinate θ, the position is parameterized as:

$$x = R\cos\theta, \tag{2.27}$$
$$y = R\sin\theta. \tag{2.28}$$

The kinetic energy is:

$$T = \frac{1}{2}mR^2\dot{\theta}^2. \tag{2.29}$$

Since the motion is confined to a circle, there is no potential energy (or it is constant, and can be set to zero):

$$V = 0. \tag{2.30}$$

Thus, the Lagrangian is:

$$L = \frac{1}{2}mR^2\dot{\theta}^2. \tag{2.31}$$

The generalized momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \tag{2.32}$$

which is recognized as the *angular momentum*. The Hamiltonian then is:

$$H = \dot{\theta}\, p_\theta - L = mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\dot{\theta}^2 = \frac{1}{2}mR^2\dot{\theta}^2 = T. \tag{2.33}$$

Since L is independent of θ, the angular momentum p_θ is conserved. This result reflects the underlying rotational symmetry of the system.

2.3 Noether's Theorem and Symmetries

An elegant feature of Lagrangian mechanics is its ability to relate continuous symmetries to conservation laws through **Noether's Theorem**. In essence:

> *Every continuous symmetry of the action of a physical system has a corresponding conservation law.*

For example:

- **Time-translation symmetry** (i.e., $\partial L/\partial t = 0$) implies conservation of energy.
- **Spatial translation symmetry** leads to conservation of linear momentum.
- **Rotational symmetry** results in conservation of angular momentum.

Furthermore, if a coordinate q_i does not appear explicitly in the Lagrangian (so that $\partial L/\partial q_i = 0$), the corresponding generalized momentum $p_i = \partial L/\partial \dot{q}_i$ is conserved.

3 Lecture 3: The Action Principle and the Calculus of Variations

We begin by considering **Fermat's principle**, which governs the path of light.

Definition 3.1 (Fermat's Principle)**.** Light travels along the path that minimizes the travel time between two points.

This can also be expressed by stating that light follows a path that minimizes the optical path length, given by

$$L = \int ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}. \tag{3.1}$$

This principle demonstrates that light minimizes a particular quantity along its trajectory. This naturally leads us to ask: does a similar principle apply to the trajectories of mechanical systems? The answer is yes.

Consider all possible paths $q_i(t)$ that a mechanical system could take through configuration space. For a given path, we define the **action** of the path, denoted by $S[q_i(t)]$, as:

$$S[q_i(t)] = \int_{t_{\text{initial}}}^{t_{\text{final}}} L(q_i, \dot{q}_i, t)\, dt \tag{3.2}$$

where L is the Lagrangian of the system. It turns out that the path that a mechanical system actually takes through configuration space is the one that **extremizes** the action. This is known as the **Principle of Least Action** or **Hamilton's Principle**. The action $S[q_i(t)]$ is a *functional*, which is a function of a function. We denote functionals using square brackets, e.g., $S[f]$, to distinguish them from functions like $f(x)$.

In standard calculus, we minimize functions of a finite number of variables. Here, we are tasked with minimizing a *functional*, which depends on an entire curve. This requires a new set of tools, which fall under the domain of the **Calculus of Variations**.

To understand this, let's consider a general problem. Suppose we have a function $F(y(x), y'(x), x)$, and we want to minimize the functional:

$$I[y(x)] = \int_{x_0}^{x_1} F(y(x), y'(x), x)\, dx \tag{3.3}$$

where $y(x)$ is defined on the interval $x_0 \leq x \leq x_1$ and $y'(x) = \frac{dy}{dx}$. The goal is to find the function $y(x)$ that extremizes $I[y(x)]$.

We proceed as follows:

- Fix the values of $y(x_0)$ and $y(x_1)$.
- Consider a curve $y(x)$ connecting these endpoints.
- Introduce a small perturbation to this curve: $y(x) \to y(x) + \delta y(x)$.
- Compute the variation δI due to this perturbation, and require that the term linear in δy vanishes.
- Since we have fixed the endpoints, we require $\delta y(x_0) = \delta y(x_1) = 0$.

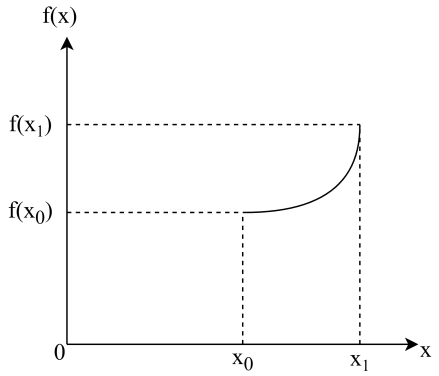

Figure 3.1: Calculus of Variations: Varying a Curve

The change in the functional due to the perturbation is:

$$\delta I = I[y(x) + \delta y(x)] - I[y(x)] = \int_{x_0}^{x_1} (F(y + \delta y, y' + \delta y', x) - F(y, y', x))\, dx \quad (3.4)$$

Using a first-order Taylor expansion of F, we have:

$$\delta I \approx \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) \quad (3.5)$$

Note that $\delta y' = \frac{d}{dx}(\delta y)$. Integrating the second term by parts (with $u = \frac{\partial F}{\partial y'}$ and $dv = \frac{d}{dx}(\delta y)\, dx$), and using the fact that $\delta y(x_0) = \delta y(x_1) = 0$, we get:

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} \delta y - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y \right) \quad (3.6)$$

Combining the terms, we obtain:

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \delta y \quad (3.7)$$

For δI to be zero for an arbitrary perturbation $\delta y(x)$, the term inside the parentheses must vanish, giving us the **Euler-Lagrange Equation**:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad (3.8)$$

The quantity

$$\frac{\delta I}{\delta y(x)} = \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \quad (3.9)$$

is called the **functional derivative** of $I[y(x)]$ with respect to $y(x)$.

The above result can be easily generalized to a functional that depends on N functions $y_i(x)$, rather than just one:

$$I[y_i(x)] = \int dx F(y_i, \frac{dy_i}{dx}, x), \quad i = 1, 2, \ldots, N \quad (3.10)$$

Following the same procedure as before, we obtain the condition for extremizing I:

$$\frac{\delta I}{\delta y_i} = \frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \tag{3.11}$$

This Euler-Lagrange equation must hold for all i.

In the context of mechanics, we have that the action is defined as $S[q_i(t)] = \int L\, dt$, where $L(q_i, \dot{q}_i, t)$ is the Lagrangian. Thus, the extremization of the action implies the following Euler-Lagrange equations:

$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{3.12}$$

These are the equations of motion for the system.

Let us apply the calculus of variations to an elementary problem.

Example 3.1. Prove that the shortest distance between two points is a line.

Proof. The length of the path can be written as:

$$L = \int ds \sqrt{(\dot{x}^2 + \dot{y}^2)} \tag{3.13}$$

Here $\dot{x}$ represents $\frac{dx}{ds}$. The Euler-Lagrange equations give us:

$$\frac{\delta L}{\delta x(s)} = -\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.14}$$

This can be simplified to:

$$\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.15}$$

Similarly, for y we get

$$\frac{d}{ds}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.16}$$

From equation (3.15) we can get:

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} \tag{3.17}$$

$$\frac{d}{ds}(\log \dot{x}) = \frac{d}{ds}(\log \dot{y}) \tag{3.18}$$

which means:

$$\frac{d}{ds} \log \frac{\dot{x}}{\dot{y}} = 0 \tag{3.19}$$

i.e.,

$$\frac{dx}{dy} = \text{constant} \tag{3.20}$$

Thus the shortest path between two points is a line. ■

We have found that

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{const} \tag{3.21}$$

Likewise,

$$p_y = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{const} \tag{3.22}$$

So we can get:

$$\frac{p_x}{p_y} = \frac{\dot{x}}{\dot{y}} = \text{const} \tag{3.23}$$

This shows that $\frac{dx}{dy}$ is constant, which implies a straight line. Also, the quantities p_x and p_y are constants of motion.

4 Lecture 4: Lagrange Multipliers, Near Equilibrium Dynamics & Oscillators

4.1 Lagrange Multipliers: Constrained Optimization

Our standard Lagrangian approach requires expressing the positions of all particles in terms of generalized coordinates, $r_\alpha(q_i, t)$. This can be complex making it difficult to obtain an explicit form for the Lagrangian,$L(q_i, \dot{q}_i, t)$. A natural question arises: can we formulate mechanics directly in terms of the particle positions, $\mathbf{r}_\alpha$, and impose constraints directly at the level of the equations of motion, rather than at the level of the action? This is where Lagrange multipliers become a valuable tool.

4.1.1 Review of Lagrange Multipliers in Calculus

Let's first recall how Lagrange multipliers work in standard calculus. Suppose we want to minimize a function $F(x, y)$. Without constraints, this involves simply finding where the partial derivatives vanish:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \tag{4.1}$$

However, given by $g(x, y) = 0$, if we want to minimize $F(x, y)$ subject to a constraint, we introduce a Lagrange multiplier λ. We then minimize the modified function: $F(x, y) + \lambda\, g(x, y)$. This leads to the following set of equations:

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \tag{4.2}$$

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \tag{4.3}$$

$$g(x, y) = 0 \tag{4.4}$$

Notice that we now have three equations for three unknowns: x, y, and λ. The first two equations define the stationary points of F subject to the constraint, and the third equation ensures that the constraint is satisfied. This method provides a systematic way to solve constrained optimization problems.

4.1.2 Lagrange Multipliers in the Calculus of Variations

The same principle can be extended to the calculus of variations. Let's say we have M functions $y_i(x), i = 1, 2, \cdots, M$, and we want to minimize an action integral:

$$I[y_i(x)] = \int dx\ F(y_i, \dot{y}_i, x) \tag{4.5}$$

Furthermore, suppose we have $M - N$ constraints of the form $G_k[y_i] = 0, k = 1, 2, \cdots, M - N$. The traditional approach would be to express all M functions, y_i, in terms of N independent degrees of freedom, however, we can avoid this by using Lagrange multipliers. To incorporate the constraints, we modify the action integral as follows:

$$\tilde{I}[y_i, \lambda_k] = \int dx\ \left(F(y_i, \dot{y}_i, x) + \sum_k \lambda_k(x)\, G_k[y_i] \right) \tag{4.6}$$

Here, $\lambda_k(x)$ are now functions of x, the independent variable of the integral. We treat $\tilde{I}$ as the functional we want to minimize, and obtain its Euler-Lagrange equations:

$$\frac{\delta \tilde{I}}{\delta y_i} = \frac{\delta I}{\delta y_i} + \sum_k \lambda_k \frac{\delta G_k}{\delta y_i} = 0 \tag{4.7}$$

$$G_k[y_i] = 0 \tag{4.8}$$

This gives us a total of $M + (M - N)$ equations. We have M equations from the functional derivative with respect to each y_i, plus $M - N$ equations from the constraints. These equations involve the M unknowns y_i and the $M - N$ Lagrange multiplier functions, $\lambda_k(x)$. These can, in principle, be solved to obtain the solution to the constrained variational problem.

Example 4.1. Compute the equations of motion in again, but this time using Lagrange multipliers.

Now we treat x & y as independent variables, and they subject to $x^2+y^2-l^2 = 0$.

$$G(x, y) = x^2 + y^2 - l^2 \tag{4.9}$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{4.10}$$

$$L + \lambda G = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda \left(x^2 + y^2 - l^2\right) \tag{4.11}$$

So by taking the derivative of $L + \lambda G$ with respect to x, we can get:

$$m\ddot{x} - 2\lambda x = 0 \tag{4.12}$$

$$m\ddot{y} + mg - 2\lambda y = 0 \tag{4.13}$$

$$x^2 + y^2 = l^2 \tag{4.14}$$

From 4.12 and 4.13 we get:

$$m\frac{\ddot{x}}{x} = m\left(\frac{\ddot{x}}{x} + \frac{g}{y}\right) \tag{4.15}$$

$$\ddot{x}y - \ddot{y}x - gx = 0 \tag{4.16}$$

we get the time derivatives of the coordinates are:

$$\dot{x} = l\cos\theta \cdot \dot{\theta} \tag{4.17}$$

$$\dot{y} = l\sin\theta \cdot \dot{\theta} \tag{4.18}$$

So the second time derivatives of the coordinates are:

$$\ddot{x} = l\left(\cos\theta\ \ddot{\theta} - \sin\theta\ \dot{\theta}^2\right) \tag{4.19}$$

$$\ddot{y} = l\left(\sin\theta\ \ddot{\theta} + \cos\theta\ \dot{\theta}^2\right) \tag{4.20}$$

Take 4.19 and 4.20 into 4.16, we can get:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{4.21}$$

This is the same as what we got in.

4.2 Oscillators

So far we have written down equations of motion. Now we will solve them for a simple oscillator. Consider a general 1D system, where the degree of freedom is q and the langrangian is $L(q, \dot{q})$. An equilibrium solution is one where $q = \text{const} = q_0$, and q is independent of time.

The equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{4.22}$$

is a 2^{nd} order differential equation. Solutions depend on initial values of q, $\dot{q}$ at the same time t.

For an equilibrium solution, $q = q_0$, $\dot{q} = 0$ will remain at rest only if $\ddot{q} = 0$.

$$L = T - V \to 0 = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} \tag{4.23}$$

This vanishes when $\frac{\partial V}{\partial q}\big|_{q_0} = 0$.

There are two types of equilibrium solutions, namely stable equilibrium and unstable equilibrium. Consider a small perterbation around an equilibrium point $q = q_0 = 0$, let's expand around the equilibrium point:

$$L = A + Bq + C\dot{q} + Dq^2 + E\dot{q}^2 + Fq\dot{q} + \cdots \tag{4.24}$$

The EOM will remain unchanged if you add total time derivative to L:

$$L \to L + \frac{d}{dt}J \tag{4.25}$$

$$S \to S + \sum_{t_0}^{t_1} dt\ \frac{d}{dt}J = S + J(t_1) - J(t_0) \tag{4.26}$$

I can ignore A, C and E, so we can get:

$$\frac{\partial V}{\partial q} = 0 \to B = 0 \tag{4.27}$$

$$L = F\dot{q} + Dq^2 = 2F\left(\frac{1}{2}\dot{q}^2 + \frac{D}{2F}q^2\right) \tag{4.28}$$

This is equivalent to

$$L = \frac{1}{2}\left(\dot{q}^2 + \frac{D}{F}q^2\right) \tag{4.29}$$

Let the frequency ω be:

$$\omega^2 = -\frac{D}{F} \tag{4.30}$$

So the equations of motion are:

$$\ddot{q} - \frac{D}{F}q = 0 \tag{4.31}$$

which is,

$$\ddot{q} + \omega^2 q = 0 \tag{4.32}$$

If ω is real ($\frac{D}{F} > 0$), then the solutions to the equations of motion are:

$$q = A\sin\omega t + B\cos\omega t \tag{4.33}$$

The solution oscillates around $q = 0$ with frequency ω. In this case the equilibrium is stable.

If the ω is imaginary ($\frac{D}{F} < 0$), then we can get:

$$q = Ae^{i\omega t} + Be^{-i\omega t} \tag{4.34}$$

generically (unless we choose initial condition with $A = 0$) the perterbation grows in time. The equilibrium is then unstable.