Classical Mechanics (McGill University)

Alexander Maloney, Zeyu Li

January 13, 2025

1 Lecture 1: Introduction, Degrees of Freedom & Lagrangian Dynamics

1.1 Introduction

Our goal is to study the dynamics in classical systems (”dynamical systems”). For example, consider a particle moving in 3D, a dynamical system with a dynamical variable $\mathbf{r}$.

$$\begin{aligned}\mathbf{r} &= (x_1, x_2, x_3) = \text{position}\\ \dot{\mathbf{r}} &= \mathbf{v}\\ \ddot{\mathbf{r}} &= \mathbf{a}\end{aligned}$$

Definition 1.1 (Dynamical Variables)**.** A set of continuous parameters which uniquely specify the state of the system.

For example, consider the motion of a system, which is uniquely specified by $\mathbf{r}(t)$: M particles with $3M$ variables $\mathbf{r}_\alpha(t)$, $\alpha = 1, 2, ..., M$.

However, we will be interested in systems where these positions are constrained, i.e., $\mathbf{r}_\alpha$ obey some relations.

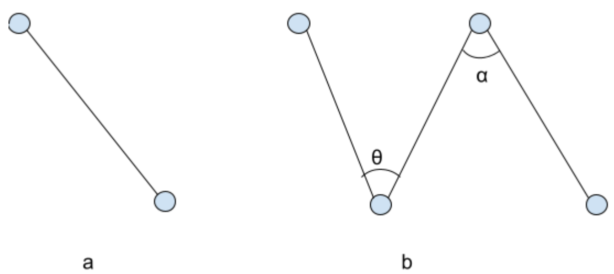

Figure 1: Rigid Body

1.2 Degrees of Freedom

Definition 1.2 (Degrees of Freedom)**.** Number of variables required to uniquely specify the system.

For example, if we have a 3D object which consists of M moving parts, then we have:

$$\#\text{ degrees of freedom} = 3M - N$$

where N is the number of constraints in this system.

Let’s take a look at the Figure 1. For a,

$$\#\text{ degrees of freedom} = 3 \times 2 - 1 = 5\text{ DOF}$$

For b (all angles are fixed),

$$\#\text{ degrees of freedom} = 3 \times 4 - 3\text{ lengths} - 3\text{ angles} = 3\text{ COM} + 3\text{ orientations} = 6\text{ DOF}$$

For c (the angle is not fixed),

$$\#\text{ degrees of freedom} = 3 \times 3 - 2\text{ lengths} = 7\text{ DOF}$$

What needs to be noticed is that dynamic variables don’t have to be the usual Cartesian coordinates.

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi)...$$

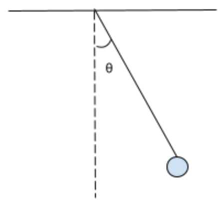

Figure 2: Pendulum Example

Consider the pendulum example in Figure 2. There is only 1 DOF, so you can choose x, y, or θ to depict the motion of the pendulum.

Let's introduce the concept of Generic Degrees of Freedom $q_i, i = 1, 2, ..., N$, where N is the number of degrees of freedom. In this way, for a constrained system, the position of any part of the system will be a function of q_i.

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t),\ \alpha = \#\text{ parts}$$

Here we allow any part of the system to have explicit dependence on time. If we can write $\mathbf{r}_\alpha(q_i, t)$ for a system, then the system (or sometimes we say the constraints of the system) is **holonomic**. Otherwise, the system is **nonholonomic**. For these systems, if the relations are time independent, then the system is **scleronomic**. Otherwise, the system is **rheonomic**.

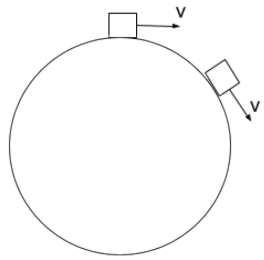

Figure 3: Rigid Body

Nonholonomic systems are common in the real world. Consider the example in Figure 3, where DOF changes from 2 to 3 if the box flies free.

1.3 Lagrangian Mechanics

Consider a dynamical system $q_i, i = 1, 2, .., \#$ DOF. For a typical mechanical system, the positions of the various parts can be written as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, and the basic problem for this system is to determine the $q_i(t)$. $q_i(t)$ satisfy a system of N differential equations known as **Equations of Motions**.

In the past, we typically used the old way of Newton's Law, which requires constraint forces:

1. Determine the force F_α on a part of the system r_α
2. Use the 2^{nd} order ordinary differential equations (ODEs) for r_α:

$$\mathbf{F}_\alpha = m\ddot{\mathbf{r}}_\alpha$$

3. Rewrite $\mathbf{r}_\alpha$ in terms of q_i, and we can get 2^{nd} order ODEs for $\mathbf{r}_\alpha$, which is easier to said than done!

Now we need to come up with a way to eliminate the need to use constraint forces: **Lagrangian Mechanics**!

If we change $\mathbf{r}_\alpha$ to $\mathbf{r}_\alpha + \delta\mathbf{r}_\alpha$, then the work done is:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \delta\mathbf{r}_\alpha$$

This raises a question: how much work is done if we change q_i to $q_i + \delta q_i$? Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, we can get (here we only consider one degree of freedom):

$$\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i$$

$$\begin{aligned}
\delta W &= \sum_\alpha \mathbf{F}_\alpha \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i\right) \\
&= \sum_i \left(\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right)\delta q_i \\
\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} &= F_i
\end{aligned}$$

Here we call F_i a **generalized force** associated with the variable q_i, and F_i is the force in the "allowed directions".

Now let's discuss the kinetic energy of a constrained system:

$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\
&= T(q_i, \dot{q}_i, t)
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_\alpha &= \mathbf{r}_\alpha(q_i, t) \\
\dot{\mathbf{r}}_\alpha &= \sum_i \frac{\partial \mathbf{r}_\alpha}{q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{t}
\end{aligned}$$

Since:

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{q_i}$$

We can get:

$$\begin{aligned}
\frac{\partial T}{\partial q_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \\
\frac{\partial T}{\partial \dot{q}_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) &= \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\right) \\
&= \sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i} \\
&= \mathbf{F}_i + \frac{\partial T}{\partial q_i}
\end{aligned}$$

So we can get:

$$\mathbf{F}_i = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i}$$

If we know $T(q_i, \dot{q}_i, t)$, we can write down the generalized force without computing a constraint! We can get a generalization of $\mathbf{F} = m\mathbf{a}$ to a generic degree of freedom!

2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equation & Hamiltonians

2.1 Lagrangian Mechanics & Euler-Lagrange Equation

Consider the following simplification: consider the case where the force $\mathbf{F}_\alpha$ is conservative. A force $\mathbf{F}_\alpha$ is **conservative** if:

$$\oint \mathbf{F}_\alpha \, d\mathbf{r}_\alpha = 0$$

i.e., the work done to change the state of the system is independent of the path through the space of $\mathbf{r}_\alpha$. For a conservative force,

$$\begin{aligned}\mathbf{F}_\alpha &= \nabla_\alpha V(\mathbf{r}_1, \ ..., \ \mathbf{r}_\alpha) \\ &= -\frac{\partial}{\partial \mathbf{r}_\alpha} V(\mathbf{r}_1, \ ..., \ \mathbf{r}_\alpha)\end{aligned}$$

And the work done to change the state of the system from $\mathbf{r}_\alpha$ to $\mathbf{r}'_\alpha$ is $V(\mathbf{r}'_\alpha) - V(\mathbf{r}_\alpha)$. In this class, we will mostly consider conservative forces.

Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, we can write $V(\mathbf{r}_\alpha)$ as:

$$V(\mathbf{r}_\alpha) = V(q_i, t)$$

From **Chain Rule**, we can get:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\mathbf{F}_i$$

Then for a conservative force, we can get:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Since V is not a function of q_i, we can get:

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

So we can rewrite the EOM above by defining **Lagrangian** $L = T - V$, $L = L(q_i, \dot{q}_i, t)$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

This EOM is called **Euler-Lagrange Equation**. We can see that for a general dynamic system, if we can compute $L = T - V$, then we can find the equations of motion!

This is a set of N differential equations, one for each DOF, where N is the total number of DOF. Typically, these are 2^{nd} order ODEs for q_i.

To summarize, given a system of M parts and N degrees of freedom, it is advised to follow the following steps:

1. Identify some dynamic variables q_i, and write down $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, where $\alpha = 1, 2, ..., M$, $i = 1, 2, ..., N$.
2. Compute $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha$ as a function of q_i.
3. Compute $V = V(\mathbf{r}_\alpha) = V(q_i, t)$.
4. Let $L = T - V$.
5. We can get equations of motion: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$.

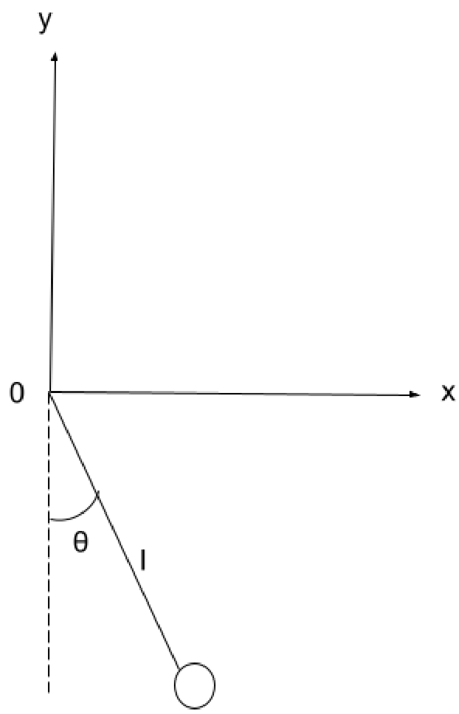

Figure 4: Pendulum Example

For example, let's take a look at the pendulum in Figure 4.

$$
\begin{aligned}
x &= l \sin \theta \\
y &= -l \cos \theta
\end{aligned}
$$

So we can get:

$$
\begin{aligned}
\dot{x} &= l \cos \theta \cdot \dot{\theta} \\
\dot{y} &= l \sin \theta \cdot \dot{\theta}
\end{aligned}
$$

So the kinetic energy can expressed as:

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) = \frac{1}{2} m l^2 \dot{\theta}^2$$

The potential energy is:

$$V = -mgy = -mgl \cos \theta$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$

From **Euler-Lagrange Equation** we can get:

$$m l^2 \ddot{\theta} + mgl \sin \theta = 0$$

This is exactly the same as we get using Newton's Law!

2.2 Hamiltonian Mechanics

The Lagrangian Mechanics is conceptually useful. But what if we step further? We have dynamic variables q_i, $L(q, q_i, t)$, so we define the **Hamiltonian** as $H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$.

$$\frac{dH}{dt} = \sum_i \left(\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) - \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) - \frac{\partial q_i}{\partial t}$$

We know:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

So we can get the following equation:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

The total time derivative of H is the explicit time derivative of L! If L has no explicit time dependence, then we can get:

$$\frac{dH}{dt} = 0$$

i.e., H is conserved.

So what is the **Hamiltonian**? If the constraints are time independent,

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$$

Then we can get

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}_\alpha} \cdot \dot{\mathbf{r}_\alpha}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}$$

$$\begin{aligned}
\sum_i \dot{q}_i \frac{\partial L}{\partial q_i} &= \sum_i \dot{q}_i \left(\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \\
&= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \left(\sum_i \frac{\partial r_\alpha}{\partial q_i} \dot{q}_i \right) \\
&= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\
&= 2T
\end{aligned}$$

So we can get:

$$\begin{aligned}
H = \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - L \\
&= 2T - (T - V) \\
&= T + V
\end{aligned}$$

So H is the total energy of the system (in many cases)!

We can get the following conclusions from the analysis above:

1. If the **Lagrangian** L is time independent, then the total energy of the system is conserved.
2. If the **Lagrangian** L is time independent, the system has time translation symmetry.

So conserved energy is equivalent to time translation symmetry.

Definition 2.1 (Noether's theorem)**.** Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

Let's look at another example of **Noether's theorem**: Imagine $L(q_i, \dot{q}_i, t)$ is independent of q_i (although it could depend on q_i). In this way, we can represent $L = L(\dot{q}_i, t)$, so we can get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

So $\frac{\partial L}{\partial \dot{q}_i}$ is conserved. This is the **momentum**!

Definition 2.2 (Momentum)**.** The **momentum** p_i conjugated to q_i is defined as $\frac{\partial L}{\partial \dot{q}_i}$. This momentum is conserved if the Lagrangian L is independent of the coordinate q_i.

3 Lecture 3: Action Principle & Calculus of Variations

3.1 Action Principle

For example, let's take a look at the free particle moving in a circle.

We have the following equations of motion:

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

$$\dot{x} = -R\sin(\theta)\dot{\theta}$$
$$\dot{y} = R\cos(\theta)\dot{\theta}$$

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mR^2\dot{\theta}^2$$

So L is independent of θ.

$$p_\theta = mR^2\dot{\theta} = \text{angular momentum}$$

Definition 3.1 (Fermat's Principle)**.** Light travels along the shortest path between two points.

Precisely the length of a curve is given by $L = \int ds$. It will be minimized for the trajectory of light.

Does every mechanical system have a principle like this? Yes, it is called the **Action Principle**.

This is called the **Least Action Principle**, or **Hamilton's Principle**.

The path that the system takes is the one that "minimizes" $S[q_i(t)]$.