Classical Mechanics (McGill University)

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1 Lecture 1: Introduction, Degrees of Freedom & Lagrangian Dynamics

1.1 Introduction

Our goal is to study the dynamics in classical systems ("dynamical systems"). For example, consider a particle moving in 3D, a dynamical system with a dynamical variable $\mathbf{r}$.

$$\mathbf{r} = (x_1, x_2, x_3) = \text{position}$$
$$\dot{\mathbf{r}} = \mathbf{v}$$
$$\ddot{\mathbf{r}} = \mathbf{a}$$

Definition 1.1 (Dynamical Variables)**.** A set of continuous parameters which uniquely specify the state of the system.

For example, consider the motion of a system, which is uniquely specified by $\mathbf{r}(t)$: M particles with $3M$ variables $\mathbf{r}_\alpha(t)$, $\alpha = 1, 2, ..., M$.

However, we will be interested in systems where these positions are constrained, i.e., $\mathbf{r}_\alpha$ obey some relations.

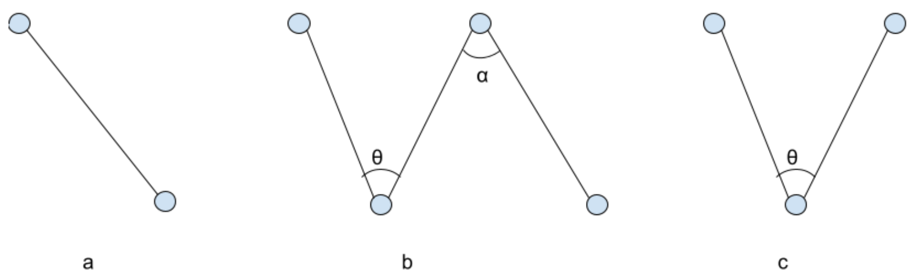

Figure 1: Rigid Body

1.2 Degrees of Freedom

Definition 1.2 (Degrees of Freedom)**.** Number of variables required to uniquely specify the system.

For example, if we have a 3D object which consists of M moving parts, then we have:

$$\# \text{ degrees of freedom} = 3M - N$$

where N is the number of constraints in this system.

Let's take a look at the Figure 1. For a,

$$\# \text{ degrees of freedom} = 3 \times 2 - 1 = 5 \text{ DOF}$$

For b (all angles are fixed),

$$\# \text{ degrees of freedom} = 3 \times 4 - 3 \text{ lengths} - 3 \text{ angles} = 3 \text{ COM} + 3 \text{ orientations} = 6 \text{ DOF}$$

For c (the angle is not fixed),

$$\# \text{ degrees of freedom} = 3 \times 3 - 2 \text{ lengths} = 7 \text{ DOF}$$

What needs to be noticed is that dynamic variables don't have to be the usual Cartesian coordinates.

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi)...$$

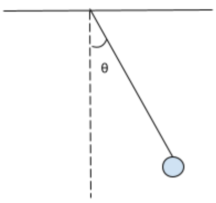

Figure 2: Pendulum Example

Consider the pendulum example in Figure 2. There is only 1 DOF, so you can choose x, y, or θ to depict the motion of the pendulum.

Let's introduce the concept of Generic Degrees of Freedom $q_i, i = 1, 2, ..., N$, where N is the number of degrees of freedom. In this way, for a constrained system, the position of any part of the system will be a function of q_i.

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t),\ \alpha = \#\text{ parts}$$

Here we allow any part of the system to have explicit dependence on time. If we can write $\mathbf{r}_\alpha(q_i, t)$ for a system, then the system (or sometimes we say the constraints of the system) is **holonomic**. Otherwise, the system is **nonholonomic**. For these systems, if the relations are time independent, then the system is **scleronomic**. Otherwise, the system is **rheonomic**.

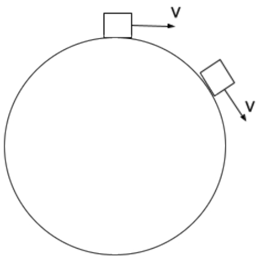

Figure 3: Rigid Body

Nonholonomic systems are common in the real world. Consider the example in Figure 3, where DOF changes from 2 to 3 if the box flies free.

1.3 Lagrangian Mechanics

Consider a dynamical system $q_i, i = 1, 2, .., \#$ DOF. For a typical mechanical system, the positions of the various parts can be written as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, and the basic problem for this system is to determine the $q_i(t)$. $q_i(t)$ satisfy a system of N differential equations known as **Equations of Motions**.

In the past, we typically used the old way of Newton's Law, which requires constraint forces:

1. Determine the force F_α on a part of the system r_α
2. Use the 2$^{\text{nd}}$ order ordinary differential equations (ODEs) for r_α:

$$\mathbf{F}_\alpha = m\ddot{\mathbf{r}}_\alpha$$

3. Rewrite $\mathbf{r}_\alpha$ in terms of q_i, and we can get 2$^{\text{nd}}$ order ODEs for $\mathbf{r}_\alpha$, which is easier to said than done!

Now we need to come up with a way to eliminate the need to use constraint forces: **Lagrangian Mechanics**!

If we change $\mathbf{r}_\alpha$ to $\mathbf{r}_\alpha + \delta\mathbf{r}_\alpha$, then the work done is:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \delta \mathbf{r}_\alpha$$

This raises a question: how much work is done if we change q_i to $q_i + \delta q_i$? Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, we can get (here we only consider one degree of freedom):

$$\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \delta q_i$$

$$\begin{aligned}\delta W &= \sum_\alpha \mathbf{F}_\alpha \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \delta q_i \right) \\ &= \sum_i \left(\sum_\alpha \frac{F_\alpha \partial \mathbf{r}_\alpha}{\partial q_i} \right) \delta q_i \\ \sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} &= F_i\end{aligned}$$

Here we call F_i a **generalized force** associated with the variable q_i, and F_i is the force in the "allowed directions".

Now let's discuss the kinetic energy of a constrained system:

$$\begin{aligned}T &= \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= T(q_i, \dot{q}_i, t)\end{aligned}$$

$$\begin{aligned}\mathbf{r}_\alpha &= \mathbf{r}_\alpha(q_i, t) \\ \dot{\mathbf{r}}_\alpha &= \sum_i \frac{\partial \mathbf{r}_\alpha}{q_i} \dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{t}\end{aligned}$$

Since:

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{q_i}$$

We can get:

$$\begin{aligned}\frac{\partial T}{\partial q_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \\ \frac{\partial T}{\partial \dot{q}_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) &= \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \right) \\ &= \sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i} \\ &= \mathbf{F}_i + \frac{\partial T}{\partial q_i}\end{aligned}$$

So we can get:

$$F_i = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i}$$

If we know $T(q_i, \dot{q}_i, t)$, we can write down the generalized force without computing a constraint! We can get a generalization of $\mathbf{F} = m\mathbf{a}$ to a generic degree of freedom!