

# Classical Mechanics

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# 1 Lecture 1: Introduction, Degrees of Freedom, and Lagrangian Dynamics

## 1.1 Introduction

The primary goal of this course is to study the dynamics of classical systems, often referred to as "dynamical systems." We will analyze how these systems evolve over time. A simple example is a single particle moving in three-dimensional space, where its state is described by the dynamical variable, the position vector  $\mathbf{r}$ .

$$\mathbf{r} = (x_1, x_2, x_3) = \text{position}$$

$$\dot{\mathbf{r}} = \mathbf{v} = \text{velocity}$$

$$\ddot{\mathbf{r}} = \mathbf{a} = \text{acceleration}$$

**Definition 1.1** (Dynamical Variables). Dynamical variables are a set of continuous parameters that uniquely specify the state of a system at a given time.

For a system of  $M$  particles, each particle's position is a dynamical variable, denoted by  $\mathbf{r}_\alpha(t)$ , where  $\alpha = 1, 2, \dots, M$ . This gives a total of  $3M$  dynamical variables.

However, we are often interested in systems where these positions are constrained by relations, limiting their degrees of freedom.

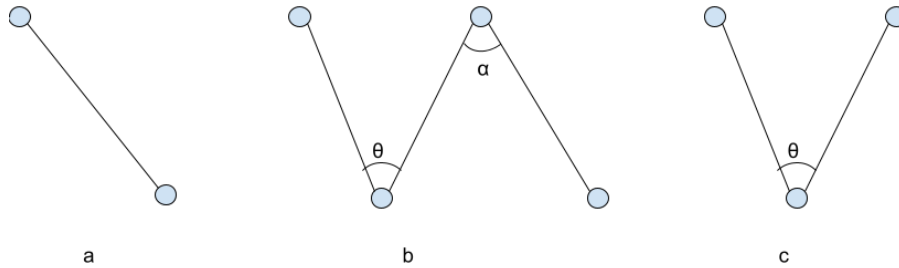


Figure 1: Examples of Constrained Systems

## 1.2 Degrees of Freedom

**Definition 1.2** (Degrees of Freedom). The number of independent variables required to completely specify the configuration of a system.

In general, for a 3D object composed of  $M$  moving parts, the number of degrees of freedom (DOF) is given by:

$$\text{DOF} = 3M - N$$

where  $N$  is the number of independent constraints imposed on the system.

Let's consider the examples in Figure 1:

- For case (a), two particles connected by a rod:

$$\text{DOF} = 3 \times 2 - 1 = 5$$

Here, the constraint is that the distance between the two particles is fixed.

- For case (b), four particles connected by rods where all angles are fixed:

$$\text{DOF} = 3 \times 4 - 3 \text{ (lengths)} - 3 \text{ (angles)} = 3 \text{ (COM)} + 3 \text{ (orientations)} = 6$$

This is equivalent to describing the motion of a rigid body in 3D space, which has 3 translational and 3 rotational degrees of freedom.

- For case (c), three particles connected by rods with one angle not fixed:

$$\text{DOF} = 3 \times 3 - 2 (\text{lengths}) = 7$$

It's important to note that dynamical variables are not restricted to Cartesian coordinates. We can use any set of independent variables that uniquely specify the system's state, such as:

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi) = \dots$$

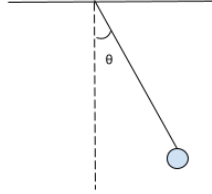


Figure 2: Pendulum Example

Consider the simple pendulum shown in Figure 2. The system has only 1 DOF. We could choose  $x$ ,  $y$ , or  $\theta$  as the dynamical variable.

We can introduce a set of generic degrees of freedom  $q_i$ , where  $i = 1, 2, \dots, N$ , and  $N$  is the number of degrees of freedom. For a constrained system, the position of any part of the system can be written as a function of these generalized coordinates:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t), \quad \alpha = \text{index for particles}$$

This form allows for the possibility of an explicit time dependence. If the position vectors can be expressed in this way, the system is said to be **holonomic**. Otherwise, it is **nonholonomic**. Furthermore, if the relation between  $\mathbf{r}_\alpha$  and  $q_i$  is time-independent (i.e.,  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$ ), the system is **scleronomic**; otherwise, it is **rheonomic**.

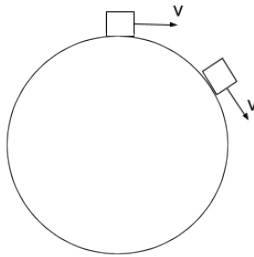


Figure 3: Nonholonomic System Example

Nonholonomic systems are common. In the example of the box on a surface in Figure 3, the DOF increases from 2 to 3 when the box leaves the surface.

### 1.3 Lagrangian Mechanics

Consider a system described by the generalized coordinates  $q_i$ , where  $i = 1, 2, \dots, N$  (number of DOF). The positions of the system's parts can be written as  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$ . The fundamental problem is to determine the time evolution of these generalized coordinates,  $q_i(t)$ . These will satisfy a set of  $N$  differential equations, known as the **equations of motion**.

Traditionally, Newton's laws of motion were used, which requires dealing with constraint forces:

1. Determine the force  $\mathbf{F}_\alpha$  acting on each part of the system at position  $\mathbf{r}_\alpha$ .
2. Apply Newton's second law, which gives a system of second-order ordinary differential equations (ODEs):

$$\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha$$

3. Rewrite  $\mathbf{r}_\alpha$  in terms of  $q_i$ . This can be challenging to actually implement!

Lagrangian mechanics provides an elegant way to avoid dealing with constraint forces directly. Consider an infinitesimal change in position,  $\delta \mathbf{r}_\alpha$ . The work done by the force is:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha$$

A crucial question arises: how much work is done if we change the generalized coordinates from  $q_i$  to  $q_i + \delta q_i$ ? Since  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$ , the variation in position can be written as:

$$\delta \mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \delta q_i$$

This assumes that we only consider variations in the generalized coordinates, keeping  $t$  constant. Thus:

$$\begin{aligned} \delta W &= \sum_\alpha \mathbf{F}_\alpha \cdot \left( \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \delta q_i \right) \\ &= \sum_i \left( \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \delta q_i \end{aligned}$$

We define the term in the parenthesis to be a **generalized force**:

$$F_i = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}$$

Here,  $F_i$  is the generalized force associated with the generalized coordinate  $q_i$ . It effectively represents the force component in the "allowed" direction defined by the variation in  $q_i$ .

Now let's consider the kinetic energy of a constrained system:

$$\begin{aligned} T &= \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= T(q_i, \dot{q}_i, t) \end{aligned}$$

where,

$$\begin{aligned} \mathbf{r}_\alpha &= \mathbf{r}_\alpha(q_i, t) \\ \dot{\mathbf{r}}_\alpha &= \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t} \end{aligned}$$

Note that from the expression above, we have

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{\partial q_i}$$

We can compute the partial derivatives of the kinetic energy:

$$\begin{aligned} \frac{\partial T}{\partial q_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \\ \frac{\partial T}{\partial \dot{q}_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \end{aligned}$$

Now, let's compute the time derivative of  $\frac{\partial T}{\partial \dot{q}_i}$ :

$$\begin{aligned}\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) &= \sum_{\alpha} m_{\alpha} \left( \ddot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial \dot{q}_i} + \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_i} \right) \\ &= \sum_{\alpha} \mathbf{F}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial \dot{q}_i} + \frac{\partial T}{\partial \dot{q}_i} \\ &= F_i + \frac{\partial T}{\partial \dot{q}_i}\end{aligned}$$

Therefore, we have the following important relation:

$$F_i = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial \dot{q}_i}$$

If we know the kinetic energy  $T(q_i, \dot{q}_i, t)$ , we can obtain the generalized force without directly calculating constraint forces! This provides a powerful generalization of  $\mathbf{F} = m\mathbf{a}$  to a generic degree of freedom.

## 2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equations, and Hamiltonians

### 2.1 Lagrangian Mechanics and the Euler-Lagrange Equations

We now consider a simplification of the previous lecture: we assume that the forces  $\mathbf{F}_\alpha$  acting on the system are conservative.

**Definition 2.1** (Conservative Force). A force  $\mathbf{F}_\alpha$  is said to be **conservative** if the line integral of the force over any closed path is zero:

$$\oint \mathbf{F}_\alpha \cdot d\mathbf{r}_\alpha = 0$$

This implies that the work done by the force in moving a particle between two points is independent of the path taken. A conservative force can be expressed as the negative gradient of a scalar potential function  $V$ :

$$\begin{aligned} \mathbf{F}_\alpha &= -\nabla_\alpha V(\mathbf{r}_1, \dots, \mathbf{r}_\alpha) \\ &= -\frac{\partial}{\partial \mathbf{r}_\alpha} V(\mathbf{r}_1, \dots, \mathbf{r}_\alpha) \end{aligned}$$

The work done in moving a particle from  $\mathbf{r}_\alpha$  to  $\mathbf{r}'_\alpha$  is given by  $V(\mathbf{r}_\alpha) - V(\mathbf{r}'_\alpha)$ . In this course, we will primarily consider systems with conservative forces.

Since  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$ , the potential energy  $V(\mathbf{r}_\alpha)$  can also be written as a function of the generalized coordinates:

$$V(\mathbf{r}_\alpha) = V(q_i, t)$$

Using the chain rule, we obtain:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -F_i$$

Where  $F_i$  is the generalized force corresponding to the generalized coordinate  $q_i$ . From the previous lecture, we know that

$$F_i = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i}$$

Therefore, for a conservative force, we can write:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Since the potential energy  $V$  is not a function of  $\dot{q}_i$ , we have:

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

We can now define the **Lagrangian** as  $L = T - V$ , where  $L = L(q_i, \dot{q}_i, t)$ . The equation of motion can then be rewritten as:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

These are the **Euler-Lagrange Equations**. This set of equations provides a powerful means to obtain the equations of motion for a mechanical system using the Lagrangian.

These are  $N$  differential equations, one for each degree of freedom. Typically, these are second-order ordinary differential equations (ODEs) for  $q_i$ .

In summary, given a system with  $M$  particles and  $N$  degrees of freedom, the following steps should be followed to determine the equations of motion using Lagrangian Mechanics:

1. Identify the generalized coordinates  $q_i$  that specify the system's configuration, and express the position vectors of the particles as  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$ , where  $\alpha = 1, 2, \dots, M$  and  $i = 1, 2, \dots, N$ .

2. Calculate the kinetic energy  $T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha}$  as a function of  $q_i$  and  $\dot{q}_i$ .
3. Compute the potential energy  $V = V(\mathbf{r}_{\alpha}) = V(q_i, t)$ .
4. Construct the Lagrangian  $L = T - V$ .
5. Apply the Euler-Lagrange equations:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$ .

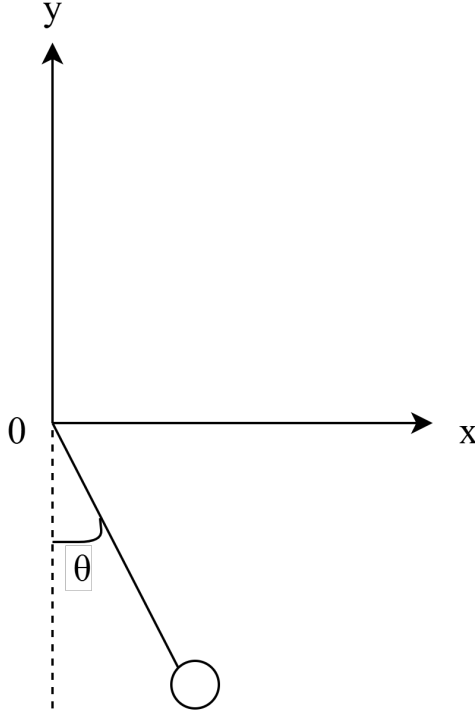


Figure 4: Simple Pendulum Example

As an example, let's consider the simple pendulum depicted in Figure 4. The position of the pendulum bob can be written as:

$$\begin{aligned} x &= l \sin \theta \\ y &= -l \cos \theta \end{aligned}$$

The time derivatives of the coordinates are:

$$\begin{aligned} \dot{x} &= l \cos \theta \cdot \dot{\theta} \\ \dot{y} &= l \sin \theta \cdot \dot{\theta} \end{aligned}$$

Thus, the kinetic energy is:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\theta}^2$$

The potential energy is:

$$V = -mgy = -mgl \cos \theta$$

The Lagrangian for the pendulum is:

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$



Applying the Euler-Lagrange equation we get:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt} (ml^2 \ddot{\theta}) - (-mgl \sin \theta) = ml^2 \ddot{\theta} + mgl \sin \theta = 0$$

Thus the equation of motion of the simple pendulum is:

$$ml^2 \ddot{\theta} + mgl \sin \theta = 0$$

This result is consistent with what is obtained using Newton's laws of motion.

## 2.2 Hamiltonian Mechanics

While Lagrangian mechanics is conceptually valuable, we can take a further step by introducing the **Hamiltonian**. For a system with dynamical variables  $q_i$  and a Lagrangian  $L(q_i, \dot{q}_i, t)$ , we define the Hamiltonian  $H$  as:

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Let's compute the total time derivative of the Hamiltonian:

$$\begin{aligned} \frac{dH}{dt} &= \sum_i \left( \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right) - \sum_i \left( \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) - \frac{\partial L}{\partial t} \\ &= \sum_i \dot{q}_i \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial t} \end{aligned}$$

Using the Euler-Lagrange equation, we have:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)$$

Thus, the equation simplifies to:

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t}$$

The total time derivative of the Hamiltonian is the negative of the partial time derivative of the Lagrangian. If  $L$  has no explicit time dependence, then:

$$\frac{dH}{dt} = 0$$

This implies that  $H$  is a conserved quantity.

What is the **Hamiltonian** physically? If the constraints are time-independent, then  $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$ , and we have:

$$T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha}$$

We can then compute

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \dot{\mathbf{r}}_{\alpha}}{\partial \dot{q}_i} = \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_i}$$

$$\begin{aligned} \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} &= \sum_i \dot{q}_i \left( \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \frac{\partial \mathbf{r}_{\alpha}}{\partial q_i} \right) \\ &= \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \left( \sum_i \frac{\partial \mathbf{r}_{\alpha}}{\partial q_i} \dot{q}_i \right) \\ &= \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} \\ &= 2T \end{aligned}$$

So, in this case:

$$\begin{aligned} H &= \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \\ &= 2T - (T - V) \\ &= T + V \end{aligned}$$

Therefore,  $H$  represents the total energy of the system (when the constraints are time independent and the potential only depends on coordinates)!

From the above analysis, we can draw the following conclusions:

1. If the **Lagrangian**  $L$  does not depend explicitly on time, then the total energy of the system, represented by the Hamiltonian  $H$ , is conserved.
2. If the **Lagrangian**  $L$  does not depend explicitly on time, then the system possesses time-translation symmetry.

Thus, energy conservation is associated with time-translation symmetry.

**Definition 2.2** (Noether's Theorem). Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

This is an important theorem that relates symmetries and conservation laws. Let's look at another example to illustrate **Noether's Theorem**: Suppose  $L(q_i, \dot{q}_i, t)$  is independent of  $q_i$  (though it could still depend on  $\dot{q}_i$  and  $t$ ). Then we have  $L = L(\dot{q}_i, t)$ , and

$$\frac{\partial L}{\partial q_i} = 0$$

Then from the Euler-Lagrange equation we have:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

So  $\frac{\partial L}{\partial \dot{q}_i}$  is a conserved quantity. This is defined as the **momentum** conjugate to  $q_i$ .

**Definition 2.3** (Generalized Momentum). The **generalized momentum**  $p_i$  conjugate to  $q_i$  is defined as  $p_i = \frac{\partial L}{\partial \dot{q}_i}$ . This momentum is conserved if the Lagrangian  $L$  does not depend on the corresponding coordinate  $q_i$ .

As a final example, consider a particle moving in a circle of radius  $R$ . The equations of motion can be written as:

$$\begin{aligned} x &= R \cos(\theta) \\ y &= R \sin(\theta) \end{aligned}$$

The time derivatives are:

$$\begin{aligned} \dot{x} &= -R \sin(\theta) \dot{\theta} \\ \dot{y} &= R \cos(\theta) \dot{\theta} \end{aligned}$$

The Lagrangian is thus:

$$L = T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m R^2 \dot{\theta}^2$$

Note that  $L$  is independent of  $\theta$ . The generalized momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$$

This is recognized as the angular momentum.

In summary, a linear translational symmetry leads to conservation of linear momentum, a rotational symmetry leads to conservation of angular momentum, and time translation symmetry leads to conservation of energy.

### 3 Lecture 3: The Action Principle and the Calculus of Variations

Let's begin by recalling Fermat's principle, which governs the path of light.

**Definition 3.1** (Fermat's Principle). Light travels along the path that minimizes the travel time between two points.

Equivalently, one can say that light takes a path that minimizes the optical path length, given by  $L = \int ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}$ . This principle demonstrates that light minimizes a particular quantity in its trajectory.

This prompts the question: does a similar minimization principle apply to the trajectories of mechanical systems? The answer is yes. Consider the set of all possible paths  $q_i(t)$  that a system could take through configuration space.

For a given path  $q_i(t)$ , we define the **action** of the path, denoted by  $S[q_i(t)]$ , as

$$S[q_i(t)] = \int_{t_{\text{initial}}}^{t_{\text{final}}} L(q_i, \dot{q}_i, t) dt$$

where  $L$  is the Lagrangian of the system. The path that a mechanical system takes through configuration space is the path that extremizes the action. This principle is known as the **Principle of Least Action** or **Hamilton's Principle**. The action  $S[q_i(t)]$  is a function of a function; it's called a **functional**. We use square brackets, such as  $S[f]$ , to denote the dependence of functionals, rather than parentheses  $f(x)$  used for functions.

In single-variable (or multi-variable) calculus, we are accustomed to minimizing a function of one (or  $N$ ) variable(s). Here, we need to minimize a functional, which can be thought of as a function of an infinite number of variables (all the points of a curve). This requires the use of the **Calculus of Variations**.

Let's consider a general problem: given a function  $F(y(x), \frac{dy}{dx}, x)$ , we define a functional as  $I[y(x)] = \int_{x_0}^{x_1} dx F(y(x), \frac{dy}{dx}, x)$ . Here,  $y(x)$  is defined on the interval  $x_0 \leq x \leq x_1$ . Our goal is to find the function  $y(x)$  that extremizes  $I[y(x)]$ .

For a function of a single variable  $I(y)$ , the condition for an extremum is simply  $\frac{dI}{dy} = 0$ . This means that if we were to deform  $y$  to  $y + \delta y$ , then from the Taylor expansion, we would get:

$$I(y + \delta y) = I(y) + \left. \frac{dI}{dy} \right|_y \delta y + \mathcal{O}(\delta y^2)$$

The extremum occurs when the term linear in the variation  $\delta y$  vanishes. We now apply similar principles to functionals.

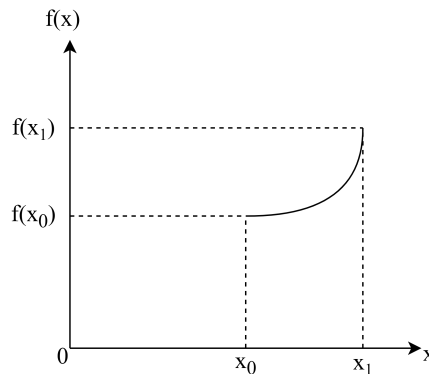


Figure 5: Calculus of Variations: Varying a Curve

To extremize the functional  $I[y(x)]$ , we proceed as follows:

- Fix the values of  $y(x_0)$  and  $y(x_1)$ .
- Consider a curve  $y(x)$  connecting these endpoints.

- Introduce a small perturbation to this curve:

$$y(x) \rightarrow y(x) + \delta y(x)$$

- Compute the variation  $\delta I$  under this perturbation, and require that the term linear in  $\delta y$  vanishes.
- Since we have fixed the end points, we require  $\delta y(x_0) = \delta y(x_1) = 0$ .

Starting with the definition of the functional:

$$I[y(x)] = \int_{x_0}^{x_1} F(y(x), y'(x), x) dx$$

The difference between the functional evaluated on the perturbed path and the original path is:

$$\delta I = I[y(x) + \delta y(x)] - I[y(x)] = \int_{x_0}^{x_1} (F(y + \delta y, y' + \delta y', x) - F(y, y', x)) dx$$

Note that:

$$\begin{aligned} y &\rightarrow y + \delta y \\ y' &\rightarrow y' + \delta y' \end{aligned}$$

and thus:

$$\delta y' = (\delta y)'$$

Using the first-order Taylor expansion of the function  $F$  and the above equation, we get:

$$\begin{aligned} \delta I &= \int_{x_0}^{x_1} dx \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) \\ &= \int_{x_0}^{x_1} dx \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) \right) \end{aligned}$$

We can integrate the second term by parts, using the fact that  $\int u dv = uv - \int v du$ , where we set  $u = \frac{\partial F}{\partial y'}$  and  $dv = \frac{d}{dx} (\delta y) dx$ :

$$\delta I = \int_{x_0}^{x_1} dx \left( \frac{\partial F}{\partial y} \delta y - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \delta y \right) + \frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1}$$

Since  $\delta y(x_0) = \delta y(x_1) = 0$ , the boundary term vanishes, and we have:

$$\delta I = \int_{x_0}^{x_1} dx \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right) \delta y \quad (3.1)$$

For  $\delta I$  to be zero for an arbitrary perturbation  $\delta y(x)$ , the term inside the parenthesis must vanish:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0 \quad (3.2)$$

This is the **Euler-Lagrange Equation**. The quantity:

$$\frac{\delta I}{\delta y(x)} = \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \quad (3.3)$$

is called the functional derivative of  $I[y(x)]$  with respect to  $y(x)$ .

Now, let's consider the generalization to a functional that depends on  $N$  functions, rather than just one:

$$I[y_i(x)] = \int dx F(y_i, \frac{dy_i}{dx}, x), \quad i = 1, 2, \dots, N \quad (3.4)$$

The variation of this functional is given by:

$$\delta I = \int dx \sum_{i=1}^N \left( \frac{\partial F}{\partial y_i} \delta y_i + \frac{\partial F}{\partial y'_i} \delta y'_i \right) \quad (3.5)$$

Following the same procedure as before, we obtain:

$$\delta I = \int dx \sum_{i=1}^N \left( \frac{\partial F}{\partial y_i} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'_i} \right) \right) \delta y_i \quad (3.6)$$

To minimize  $I$  with respect to all  $y_i(x)$ , each  $y_i(x)$  must satisfy the corresponding Euler-Lagrange equation:

$$\frac{\delta I}{\delta y_i} = \frac{\partial F}{\partial y_i} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'_i} \right) = 0 \quad (3.7)$$

for all  $i$ .

In conclusion, for a mechanical system with Lagrangian  $L(q_i, \dot{q}_i, t)$ , the condition for the action  $S[q_i(t)] = \int L dt$  to be extremized is

$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad (3.8)$$

**Example 3.1.** Prove that the shortest distance between two points is a line.

*Proof.* The length of the path can be written as:

$$L = \int ds \sqrt{\dot{x}^2 + \dot{y}^2} \quad (3.9)$$

Here  $\dot{x}$  represents  $\frac{dx}{ds}$ .

■