Classical Mechanics (McGill University)

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1 Lecture 1: Introduction, Degrees of Freedom & Lagrangian Dynamics

1.1 Introduction

Our goal is to study the dynamics in classical systems ("dynamical systems"). For example, consider a particle moving in 3D, a dynamical system with a dynamical variable $\mathbf{r}$.

$$\begin{aligned}\mathbf{r} &= (x_1, x_2, x_3) = \text{position}\\ \dot{\mathbf{r}} &= \mathbf{v}\\ \ddot{\mathbf{r}} &= \mathbf{a}\end{aligned}$$

Definition 1.1 (Dynamical Variables)**.** A set of continuous parameters which uniquely specify the state of the system.

For example, consider the motion of a system, which is uniquely specified by $\mathbf{r}(t)$: M particles with $3M$ variables $\mathbf{r}_\alpha(t)$, $\alpha = 1, 2, ..., M$.

However, we will be interested in systems where these positions are constrained, i.e., $\mathbf{r}_\alpha$ obey some relations.

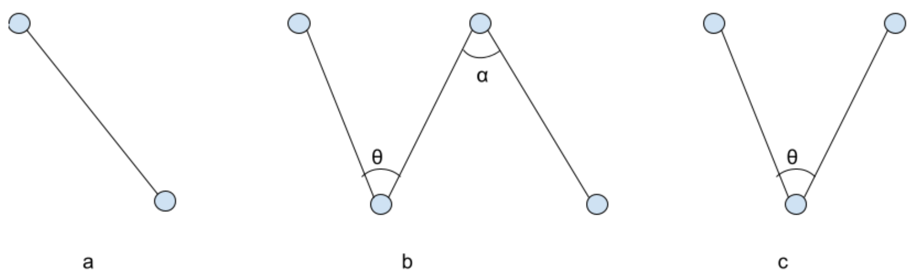

Figure 1: Rigid Body

1.2 Degrees of Freedom

Definition 1.2 (Degrees of Freedom)**.** Number of variables required to uniquely specify the system.

For example, if we have a 3D object which consists of M moving parts, then we have:

$$\#\text{ degrees of freedom} = 3M - N$$

where N is the number of constraints in this system.

Let's take a look at the Figure 1. For a,

$$\#\text{ degrees of freedom} = 3 \times 2 - 1 = 5\text{ DOF}$$

For b (all angles are fixed),

$$\#\text{ degrees of freedom} = 3 \times 4 - 3\text{ lengths} - 3\text{ angles} = 3\text{ COM} + 3\text{ orientations} = 6\text{ DOF}$$

For c (the angle is not fixed),

$$\#\text{ degrees of freedom} = 3 \times 3 - 2\text{ lengths} = 7\text{ DOF}$$

What needs to be noticed is that dynamic variables don't have to be the usual Cartesian coordinates.

$$\mathbf{r} = (x, y, z) = (r, \theta, \phi)...$$

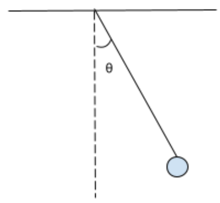

Figure 2: Pendulum Example

Consider the pendulum example in Figure 2. There is only 1 DOF, so you can choose x, y, or θ to depict the motion of the pendulum.

Let's introduce the concept of Generic Degrees of Freedom $q_i, i = 1, 2, ..., N$, where N is the number of degrees of freedom. In this way, for a constrained system, the position of any part of the system will be a function of q_i.

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t),\ \alpha = \#\text{ parts}$$

Here we allow any part of the system to have explicit dependence on time. If we can write $\mathbf{r}_\alpha(q_i, t)$ for a system, then the system (or sometimes we say the constraints of the system) is **holonomic**. Otherwise, the system is **nonholonomic**. For these systems, if the relations are time independent, then the system is **scleronomic**. Otherwise, the system is **rheonomic**.

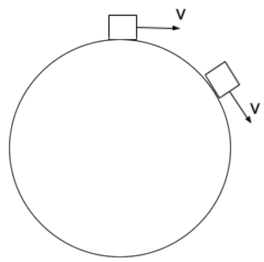

Figure 3: Rigid Body

Nonholonomic systems are common in the real world. Consider the example in Figure 3, where DOF changes from 2 to 3 if the box flies free.

1.3 Lagrangian Mechanics

Consider a dynamical system $q_i, i = 1, 2, .., \#$ DOF. For a typical mechanical system, the positions of the various parts can be written as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, and the basic problem for this system is to determine the $q_i(t)$. $q_i(t)$ satisfy a system of N differential equations known as **Equations of Motions**.

In the past, we typically used the old way of Newton's Law, which requires constraint forces:

1. Determine the force F_α on a part of the system r_α
2. Use the 2^{nd} order ordinary differential equations (ODEs) for r_α:

$$\mathbf{F}_\alpha = m\ddot{\mathbf{r}}_\alpha$$

3. Rewrite $\mathbf{r}_\alpha$ in terms of q_i, and we can get 2^{nd} order ODEs for $\mathbf{r}_\alpha$, which is easier to said than done!

Now we need to come up with a way to eliminate the need to use constraint forces: **Lagrangian Mechanics**!

If we change $\mathbf{r}_\alpha$ to $\mathbf{r}_\alpha + \delta\mathbf{r}_\alpha$, then the work done is:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \delta\mathbf{r}_\alpha$$

This raises a question: how much work is done if we change q_i to $q_i + \delta q_i$? Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, we can get (here we only consider one degree of freedom):

$$\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i$$

$$\begin{aligned}\delta W &= \sum_\alpha \mathbf{F}_\alpha \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i\right)\\ &= \sum_i \left(\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right)\delta q_i\end{aligned}$$

$$\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = F_i$$

Here we call F_i a **generalized force** associated with the variable q_i, and F_i is the force in the "allowed directions".

Now let's discuss the kinetic energy of a constrained system:

$$\begin{aligned}T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha\\ &= T(q_i, \dot{q}_i, t)\end{aligned}$$

$$\begin{aligned}\mathbf{r}_\alpha &= \mathbf{r}_\alpha(q_i, t)\\ \dot{\mathbf{r}}_\alpha &= \sum_i \frac{\partial \mathbf{r}_\alpha}{q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{t}\end{aligned}$$

Since:

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{q_i}$$

We can get:

$$\begin{aligned}\frac{\partial T}{\partial q_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\\ \frac{\partial T}{\partial \dot{q}_i} &= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) &= \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\right)\\ &= \sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i}\\ &= \mathbf{F}_i + \frac{\partial T}{\partial q_i}\end{aligned}$$

So we can get:

$$\mathbf{F}_i = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i}$$

If we know $T(q_i, \dot{q}_i, t)$, we can write down the generalized force without computing a constraint! We can get a generalization of $\mathbf{F} = m\mathbf{a}$ to a generic degree of freedom!

2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equation & Hamiltonians

2.1 Lagrangian Mechanics & Euler-Lagrange Equation

Consider the following simplification: consider the case where the force $\mathbf{F}_\alpha$ is conservative. A force $\mathbf{F}_\alpha$ is **conservative** if:

$$\oint \mathbf{F}_\alpha \, d\mathbf{r}_\alpha = 0$$

i.e., the work done to change the state of the system is independent of the path through the space of $\mathbf{r}_\alpha$. For a conservative force,

$$\begin{aligned}\mathbf{F}_\alpha &= \nabla_\alpha V(\mathbf{r}_1,\ ...,\ \mathbf{r}_\alpha) \\ &= -\frac{\partial}{\partial \mathbf{r}_\alpha} V(\mathbf{r}_1,\ ...,\ \mathbf{r}_\alpha)\end{aligned}$$

And the work done to change the state of the system from $\mathbf{r}_\alpha$ to $\mathbf{r}'_\alpha$ is $V(\mathbf{r}'_\alpha) - V(\mathbf{r}_\alpha)$. In this class, we will mostly consider conservative forces.

Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, we can write $V(\mathbf{r}_\alpha)$ as:

$$V(\mathbf{r}_\alpha) = V(q_i, t)$$

From **Chain Rule**, we can get:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\mathbf{F}_i$$

Then for a conservative force, we can get:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

Since V is not a function of q_i, we can get:

$$\frac{\partial V}{\partial \dot{q}_i} = 0$$

So we can rewrite the EOM above by defining **Lagrangian** $L = T - V$, $L = L(q_i, \dot{q}_i, t)$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

This EOM is called **Euler-Lagrange Equation**. We can see that for a general dynamic system, if we can compute $L = T - V$, then we can find the equations of motion!

This is a set of N differential equations, one for each DOF, where N is the total number of DOF. Typically, these are 2^{nd} order ODEs for q_i.

To summarize, given a system of M parts and N degrees of freedom, it is advised to follow the following steps:

1. Identify some dynamic variables q_i, and write down $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, where $\alpha = 1, 2, ..., M$, $i = 1, 2, ..., N$.
2. Compute $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha$ as a function of q_i.
3. Compute $V = V(\mathbf{r}_\alpha) = V(q_i, t)$.
4. Let $L = T - V$.
5. We can get equations of motion: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$.

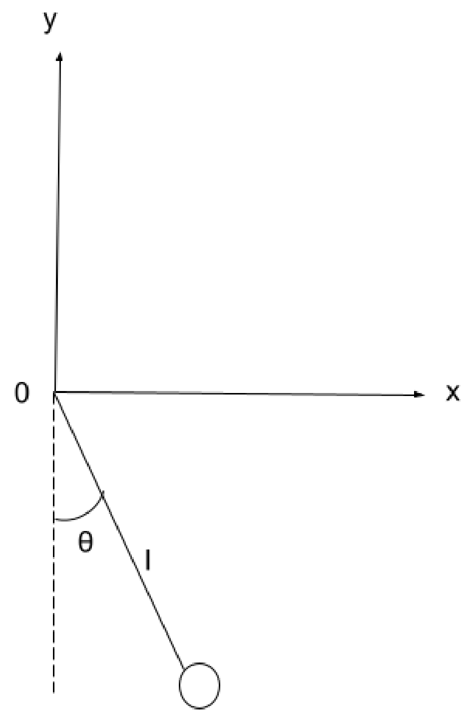

Figure 4: Pendulum Example

For example, let's take a look at the pendulum in Figure 4.

$$\begin{aligned} x &= l \sin\theta \\ y &= -l \cos\theta \end{aligned}$$

So we can get:

$$\begin{aligned} \dot{x} &= l \cos\theta \cdot \dot{\theta} \\ \dot{y} &= l \sin\theta \cdot \dot{\theta} \end{aligned}$$

So the kinetic energy can expressed as:

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2} m l^2 \dot{\theta}^2$$

The potential energy is:

$$V = -mgy = -mgl\cos\theta$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

From **Euler-Lagrange Equation** we can get:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

This is exactly the same as we get using Newton's Law!

2.2 Hamiltonian Mechanics

The Lagrangian Mechanics is conceptually useful. But what if we step further? We have dynamic variables q_i, $L(q, q_i, t)$, so we define the **Hamiltonian** as $H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$.

$$\frac{dH}{dt} = \sum_i \left(\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) - \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) - \frac{\partial q_i}{\partial t}$$

We know:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

So we can get the following equation:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

The total time derivative of H is the explicit time derivative of L! If L has no explicit time dependence, then we can get:

$$\frac{dH}{dt} = 0$$

i.e., H is conserved.

So what is the **Hamiltonian**? If the constraints are time independent,

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$$

Then we can get

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}_\alpha} \cdot \dot{\mathbf{r}_\alpha}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}$$

$$\begin{aligned}
\sum_i \dot{q}_i \frac{\partial L}{\partial q_i} &= \sum_i \dot{q}_i \left(\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \\
&= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \left(\sum_i \frac{\partial r_\alpha}{\partial q_i} \dot{q}_i \right) \\
&= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\
&= 2T
\end{aligned}$$

So we can get:

$$\begin{aligned}
H &= \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - L \\
&= 2T - (T - V) \\
&= T + V
\end{aligned}$$

So H is the total energy of the system (in many cases)!

We can get the following conclusions from the analysis above:

1. If the **Lagrangian** L is time independent, then the total energy of the system is conserved.
2. If the **Lagrangian** L is time independent, the system has time translation symmetry.

So conserved energy is equivalent to time translation symmetry.

Definition 2.1 (Noether's theorem)**.** Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

Let's look at another example of **Noether's theorem**: Imagine $L(q_i, \dot{q}_i, t)$ is independent of q_i (although it could depend on q_i). In this way, we can represent $L = L(\dot{q}_i, t)$, so we can get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

So $\frac{\partial L}{\partial \dot{q}_i}$ is conserved. This is the **momentum**!

Definition 2.2 (Momentum)**.** The **momentum** p_i conjugated to q_i is defined as $\frac{\partial L}{\partial \dot{q}_i}$. This momentum is conserved if the Lagrangian L is independent of the coordinate q_i.

At last, let's take a look at a particle moving in a circle. We have the following equations of motion:

$$x = R\cos(\theta)$$
$$y = R\sin(\theta)$$

So we can get:

$$\dot{x} = -R\sin(\theta)\dot{\theta}$$
$$\dot{y} = R\cos(\theta)\dot{\theta}$$

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mR^2\dot{\theta}^2$$

L is independent of θ.

$$p_\theta = mR^2\dot{\theta} = \text{angular momentum}$$

In summary, a linear translation symmetry leads to conservation of momentum, a rotational symmetry leads to conservation of angular momentum, a time translation symmetry leads to conservation of energy.

3 Lecture 3: Action Principle & Calculus of Variations

3.1 Action Principle

For beginning, let's recall the Fermat's Principle.

Definition 3.1 (Fermat's Principle). Light travels along the shortest path between two points.

Precisely speaking, the length of a curve $L = \int ds\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}$ will be minimized for the trajectory of light.

Then we need to ask the following question: does every mechanical system obey a minimization principle of this sort? The answer is yes. So let's consider the set of all possible paths $q_i\left(t\right)$ that a system could take through configuration space.

For a given path $q_i\left(t\right)$, we define the action of this path as $S[q_i\left(t\right)]$.

$$s[q_i\left(t\right)] = \int_{t_{\text{initial}}}^{t_{\text{final}}} L\left(q_i, \dot{q}_i, t\right) dt$$

The path that a mechanical system takes through configuration space (nearly space of q_i) "minimizes" $S[q_i\left(t\right)]$ (not global, only local extremum). This is called the **Least Action Principle**, or alternatvely, **Hamiltonian's Principle**. $S[q_i\left(t\right)]$ is a function of a function, so it is called a **functional**. For functionals, we use square brackets $[f]$ to denote the dependence rather than curve bracket $f\left(x\right)$.

We are used in single (or multi) variable(s) calculus to minimizing a function of 1(N) variable(s). We need to minimize a functional, a function of ∞ number of variables, so we need to introduce the **Calculus of Variations**.

Let's take a look at a general problem: given a function $F\left(y\left(x\right), \frac{dy}{dx}, x\right)$, define the functional $I[y\left(x\right)] = \int_{x_0}^{x_1} dxF\left(y\left(x\right), \frac{dy}{dx}, x\right)$. Here $y\left(x\right)$ is defined on the domain $x_0 \leq x \leq x_1$. We want to find the function $y\left(x\right)$ that extramizes $I[y\left(x\right)]$.