Classical Mechanics

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1 Lecture 1: Introduction, Degrees of Freedom, and Lagrangian Dynamics

1.1 Introduction

The objective of this course is to develop a deep understanding of classical dynamical systems using the framework of Lagrangian mechanics. This formulation elegantly generalizes Newtonian mechanics and provides powerful methods for dealing with complex, constrained systems.

Overview of Coordinate Systems

The description of a physical system can vary significantly with the choice of coordinate system. Below are some common coordinate representations:

Cartesian Coordinates. For a particle in three-dimensional space, the position, velocity, and acceleration are expressed as:

$$\begin{aligned} \mathbf{r} &= (x,\, y,\, z), \\ \mathbf{v} &= \dot{\mathbf{r}}, \\ \mathbf{a} &= \ddot{\mathbf{r}}. \end{aligned} \tag{1.1}$$

Polar Coordinates. For planar motion, using polar coordinates (r, θ):

$$\begin{aligned} \frac{d\hat{\mathbf{e}}_r}{dt} &= \dot{\theta}\,\hat{\mathbf{e}}_\theta, \\ \frac{d\hat{\mathbf{e}}_\theta}{dt} &= -\dot{\theta}\,\hat{\mathbf{e}}_r, \end{aligned} \tag{1.2}$$

so that the position and velocity become:

$$\begin{aligned} \mathbf{r} &= r\,\hat{\mathbf{e}}_r, \\ \mathbf{v} &= \dot{r}\,\hat{\mathbf{e}}_r + r\,\dot{\theta}\,\hat{\mathbf{e}}_\theta. \end{aligned} \tag{1.3}$$

Spherical Coordinates. For a particle in three-dimensional space described by spherical coordinates (r, θ, ϕ):

$$\begin{aligned} \mathbf{r} &= (r,\, \theta,\, \phi), \\ \mathbf{v} &= \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\phi}\,\sin\theta\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{1.4}$$

1.2 Degrees of Freedom and Generalized Coordinates

In classical mechanics, the *degrees of freedom* (DOF) represent the number of independent parameters required to specify the configuration of a system.

Definition 1.1 (Degrees of Freedom)**. Mechanic definition:** The number of independent parameters needed to define a system's configuration.

Kinetic definition: For a rigid body, the DOF equals the number of independent movements (typically 3 translational and 3 rotational in three dimensions).

For a system of M particles in three-dimensional space with no constraints, the total degrees of freedom is $3M$. When N holonomic (i.e., integrable) constraints are present, the number of independent generalized coordinates reduces to:

$$\text{DOF} = 3M - N. \tag{1.5}$$

If there are additional k nonholonomic (non-integrable) constraints, these restrict the allowable velocities but do not reduce the configuration space's dimension. Thus, under the kinetic viewpoint, the effective degrees of freedom become:

$$\text{DOF}_{\text{kinetic}} = 3M - N - k. \tag{1.6}$$

Nevertheless, the number of generalized coordinates remains $3M - N$.

Examples of Constrained Systems

- **Two particles connected by a rod:**
  The rod fixes the distance between the particles, leading to:

$$\text{DOF} = 3 \times 2 - 1 = 5. \tag{1.7}$$

- **Four particles connected by rods with fixed angles:**
  With three constraints fixing the rod lengths and three additional constraints fixing the angles, the system behaves as a rigid body:

$$\text{DOF} = 3 \times 4 - 3\,(\text{length constraints}) - 3\,(\text{angular constraints}) = 6. \tag{1.8}$$

- **Three particles connected by rods with one unfixed angle:**
  Removing one angular constraint results in:

$$\text{DOF} = 3 \times 3 - 2 = 7. \tag{1.9}$$

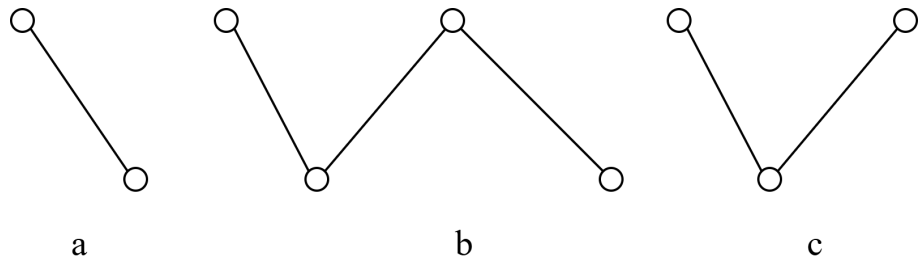

Figure 1.1: Examples of Constrained Systems

The Simple Pendulum

Consider the simple pendulum (see Figure 1.2). The motion of the pendulum is completely described by the angular displacement θ, thus it has one degree of freedom.

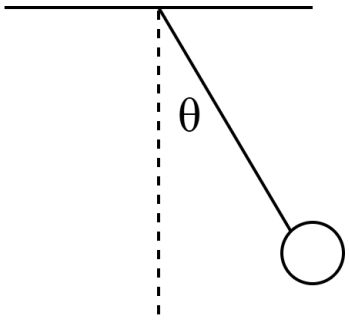

Figure 1.2: Simple Pendulum

Generalized Coordinates in Constrained Systems

For systems with constraints, we introduce generalized coordinates $\{q_i\}$ (with $i = 1, 2, \ldots, N$) such that the position of each particle α is expressed as:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, q_2, \ldots, q_N, t). \tag{1.10}$$

When constraints explicitly depend on time, they are termed *rheonomous*; otherwise, they are *scleronomous*. Moreover, if a constraint can be written as

$$f(q_1, q_2, \ldots, q_N, t) = 0,$$

(i.e., it does not involve velocities), it is a *holonomic* constraint; constraints involving velocities are classified as *nonholonomic*.

Nonholonomic systems frequently occur in practice. For example, consider a box moving on the surface of a sphere in figure 1.3: when the box loses contact with the surface, the effective degrees of freedom increase.

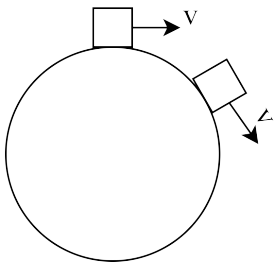

Figure 1.3: Example of a Nonholonomic System

1.3 Lagrangian Mechanics

Lagrangian mechanics reformulates the equations of motion for a system in terms of generalized coordinates. Consider a system described by generalized coordinates q_i $(i = 1, 2, \ldots, N)$; the positions of its constituent particles can be expressed as:

$$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t). \tag{1.11}$$

The primary task is to determine the evolution of $q_i(t)$, governed by a set of N differential equations, known as the *Euler–Lagrange equations*.

From Newton to Lagrange

In the traditional Newtonian approach, one must:

1. Compute the forces $\mathbf{F}_\alpha$ acting on each particle.
2. Apply Newton's second law:
$$\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha, \tag{1.12}$$
yielding a set of second-order differential equations.
3. Express $\mathbf{r}_\alpha$ in terms of the generalized coordinates q_i.

This process can become cumbersome, particularly for systems with constraints.

Generalized Forces and Virtual Work

Consider an infinitesimal virtual displacement $\delta\mathbf{r}_\alpha$ of the particle positions, while time is held fixed. The corresponding virtual work done by the forces is given by:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta\mathbf{r}_\alpha. \tag{1.13}$$

Since the particle positions depend on the generalized coordinates, a variation in q_i induces:

$$\delta\mathbf{r}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\delta q_i. \tag{1.14}$$

Substituting into the virtual work expression,

$$\delta W = \sum_i \left(\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \right) \delta q_i, \tag{1.15}$$

we define the *generalized force* F_i as:

$$F_i = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.16}$$

Kinetic Energy and its Variations

The kinetic energy of a system is

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha, \tag{1.17}$$

which can be expressed as a function of the generalized coordinates and their time derivatives, $T(q_i, \dot{q}_i, t)$. Since

$$\dot{\mathbf{r}}_\alpha = \sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\alpha}{\partial t}, \tag{1.18}$$

it follows that

$$\frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.19}$$

Thus, the partial derivative of the kinetic energy with respect to q_i and $\dot{q}_i$ is:

$$\frac{\partial T}{\partial q_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i} \tag{1.20}$$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}. \tag{1.21}$$

Taking the time derivative and using Newton's second law:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_\alpha m_\alpha \left(\ddot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial q_i}\right) \tag{1.22}$$

$$= \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} + \frac{\partial T}{\partial q_i} \tag{1.23}$$

$$= F_i + \frac{\partial T}{\partial q_i} \tag{1.24}$$

One obtains the important relation:

$$F_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}. \tag{1.25}$$

This result forms the foundation for deriving the Euler-Lagrange equations, which succinctly encapsulate the dynamics of the system.

1.4 The Principle of Virtual Work and d'Alembert's Principle

Principle of Virtual Work

Applied Forces. These are the external forces actively acting on the system. They can include gravity, friction (if it does work), external loads, etc. Applied forces are usually responsible for driving the motion or deformation of the system.

Constraint Forces. These forces arise from the constraints imposed on the system—such as rods, surfaces, or other geometric restrictions—that limit the motion of the system's parts. A key aspect of ideal constraints is that the constraint forces do no work during any *virtual displacement*.

The *principle of virtual work* states that for a system in equilibrium, the total virtual work done by the applied forces during any virtual displacement (consistent with the constraints) is zero:

$$\delta W = \sum_\alpha \mathbf{F}_\alpha \cdot \delta \mathbf{r}_\alpha = 0. \tag{1.26}$$

d'Alembert's Principle

d'Alembert's principle extends the principle of virtual work to dynamics. It introduces the concept of *inertial forces* (or *d'Alembert forces*) to reformulate Newton's second law into a form that is analogous to static equilibrium:

$$\sum_\alpha (\mathbf{F}_\alpha - m_\alpha \ddot{\mathbf{r}}_\alpha) \cdot \delta \mathbf{r}_\alpha = 0. \tag{1.27}$$

Here, the term $-m_\alpha \ddot{\mathbf{r}}_\alpha$ can be interpreted as an inertial force. By including these forces, the dynamic problem is converted into a statics-like problem where the total virtual work (including contributions from both applied and inertial forces) vanishes.

2 Lecture 2: Lagrangian Mechanics, Euler-Lagrange Equations, and Hamiltonians

2.1 Lagrangian Mechanics and the Euler-Lagrange Equations

We now consider a simplification of the previous lecture: we assume that the forces $\mathbf{F}_\alpha$ acting on the system are conservative.

Definition 2.1 (Conservative Force)**.** A force $\mathbf{F}_\alpha$ is said to be **conservative** if the line integral of the force over any closed path is zero:

$$\oint \mathbf{F}_\alpha \cdot d\mathbf{r}_\alpha = 0 \tag{2.1}$$

This implies that the work done by the force in moving a particle between two points is independent of the path taken. A conservative force can be expressed as the negative gradient of a scalar potential function V:

$$\mathbf{F}_\alpha = -\nabla_\alpha V(\mathbf{r}_1,\ \ldots,\ \mathbf{r}_\alpha) \tag{2.2}$$

$$= -\frac{\partial}{\partial \mathbf{r}_\alpha} V(\mathbf{r}_1,\ \ldots,\ \mathbf{r}_\alpha) \tag{2.3}$$

The work done in moving a particle from $\mathbf{r}_\alpha$ to $\mathbf{r}'_\alpha$ is given by $V(\mathbf{r}_\alpha) - V(\mathbf{r}'_\alpha)$. In this course, we will primarily consider systems with conservative forces.

Since $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, the potential energy $V(\mathbf{r}_\alpha)$ can also be written as a function of the generalized coordinates:

$$V(\mathbf{r}_\alpha) = V(q_i, t) \tag{2.4}$$

Using the chain rule, we obtain:

$$\frac{\partial V}{\partial q_i} = \sum_\alpha \frac{\partial V}{\partial \mathbf{r}_\alpha} \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -\sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} = -F_i \tag{2.5}$$

Where F_i is the generalized force corresponding to the generalized coordinate q_i. From the previous lecture, we know that

$$F_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \tag{2.6}$$

Therefore, for a conservative force, we can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{2.7}$$

Since the potential energy V is not a function of $\dot{q}_i$, we have:

$$\frac{\partial V}{\partial \dot{q}_i} = 0 \tag{2.8}$$

We can now define the **Lagrangian** as $L = T - V$, where $L = L(q_i, \dot{q}_i, t)$. The equation of motion can then be rewritten as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.9}$$

These are the **Euler-Lagrange Equations**. This set of equations provides a powerful means to obtain the equations of motion for a mechanical system using the Lagrangian.

These are N differential equations, one for each degree of freedom. Typically, these are second-order ordinary differential equations (ODEs) for q_i.

In summary, given a system with M particles and N degrees of freedom, the following steps should be followed to determine the equations of motion using Lagrangian Mechanics:

1. Identify the generalized coordinates q_i that specify the system's configuration, and express the position vectors of the particles as $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i, t)$, where $\alpha = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$.
2. Calculate the kinetic energy $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha$ as a function of q_i and $\dot{q}_i$.
3. Compute the potential energy $V = V(\mathbf{r}_\alpha) = V(q_i, t)$.
4. Construct the Lagrangian $L = T - V$.
5. Apply the Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

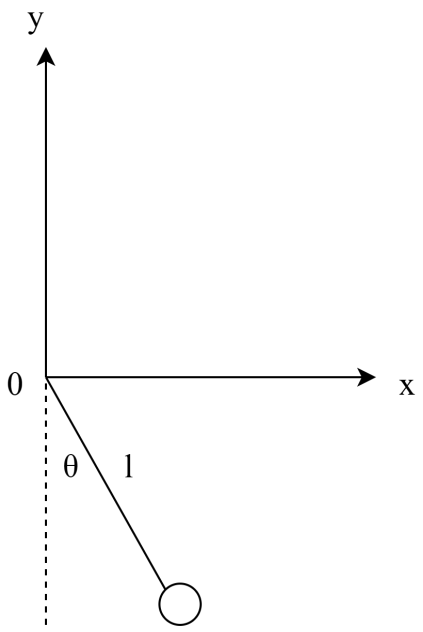

Figure 2.1: Simple Pendulum Example

Example 2.1. In Figure 2.1, we have a simple pendulum with a mass m and length l. Compute the equations of motion for the pendulum using Lagrangian mechanics.

As an example, let's consider the simple pendulum depicted in Figure 2.1. The position of the pendulum bob can be written as:

$$x = l\sin\theta \tag{2.10}$$
$$y = -l\cos\theta \tag{2.11}$$

The time derivatives of the coordinates are:

$$\dot{x} = l\cos\theta\cdot\dot{\theta} \tag{2.12}$$
$$\dot{y} = l\sin\theta\cdot\dot{\theta} \tag{2.13}$$

Thus, the kinetic energy is:

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) = \frac{1}{2}ml^2\dot{\theta}^2 \tag{2.14}$$

The potential energy is:

$$V = -mgy = -mgl\cos\theta \tag{2.15}$$

The Lagrangian for the pendulum is:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{2.16}$$

Applying the Euler-Lagrange equation we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) = ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{2.17}$$

Thus the equation of motion of the simple pendulum is:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{2.18}$$

This result is consistent with what is obtained using Newton's laws of motion.

2.2 Hamiltonian Mechanics

While Lagrangian mechanics is conceptually valuable, we can take a further step by introducing the **Hamiltonian**. For a system with dynamical variables q_i and a Lagrangian $L(q_i,\dot{q}_i,t)$, we define the Hamiltonian H as:

$$H = \sum_i \dot{q}_i\frac{\partial L}{\partial\dot{q}_i} - L \tag{2.19}$$

Let's compute the total time derivative of the Hamiltonian:

$$\frac{dH}{dt} = \sum_i\left(\ddot{q}_i\frac{\partial L}{\partial\dot{q}_i} + \dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\right) - \sum_i\left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial\dot{q}_i}\ddot{q}_i\right) - \frac{\partial L}{\partial t} \tag{2.20}$$
$$= \sum_i\dot{q}_i\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial t} \tag{2.21}$$

Using the Euler-Lagrange equation, we have:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.22}$$

Thus, the equation simplifies to:

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{2.23}$$

The total time derivative of the Hamiltonian is the negative of the partial time derivative of the Lagrangian. If L has no explicit time dependence, then:

$$\frac{dH}{dt} = 0 \tag{2.24}$$

This implies that H is a conserved quantity.

What is the **Hamiltonian** physically? If the constraints are time-independent, then $\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_i)$, and we have:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \tag{2.25}$$

We can then compute

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \dot{\mathbf{r}}_\alpha}{\partial \dot{q}_i} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i} \tag{2.26}$$

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \left(\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\right) \tag{2.27}$$

$$= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \left(\sum_i \frac{\partial \mathbf{r}_\alpha}{\partial q_i}\dot{q}_i\right) \tag{2.28}$$

$$= \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \tag{2.29}$$

$$= 2T \tag{2.30}$$

So, in this case:

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{2.31}$$

$$= 2T - (T - V) \tag{2.32}$$

$$= T + V \tag{2.33}$$

$$\tag{2.34}$$

Therefore, H represents the total energy of the system (when the constraints are time independent and the potential only depends on coordinates)!

From the above analysis, we can draw the following conclusions:

1. If the **Lagrangian** L does not depend explicitly on time, then the total energy of the system, represented by the Hamiltonian H, is conserved.

2. If the **Lagrangian** L does not depend explicitly on time, then the system possesses time-translation symmetry.

Thus, energy conservation is associated with time-translation symmetry.

Definition 2.2 (Noether's Theorem)**.** Every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

This is an important theorem that is related to symmetries and conservation laws. Let's look at another example to illustrate **Noether's Theorem**: Suppose $L(q_i, \dot{q}_i, t)$ is independent of q_i (though it could still depend on $\dot{q}_i$ and t). Then we have $L = L(\dot{q}_i, t)$, and

$$\frac{\partial L}{\partial q_i} = 0 \tag{2.35}$$

Then from the Euler-Lagrange equation we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{2.36}$$

So $\frac{\partial L}{\partial \dot{q}_i}$ is a conserved quantity. This is defined as the **momentum** conjugate to q_i.

Definition 2.3 (Generalized Momentum)**.** The **generalized momentum** p_i conjugate to q_i is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$. This momentum is conserved if the Lagrangian L does not depend on the corresponding coordinate q_i.

As a final example, consider a particle moving in a circle of radius R. The equations of motion can be written as:

$$x = R\cos(\theta) \tag{2.37}$$
$$y = R\sin(\theta) \tag{2.38}$$
$$\tag{2.39}$$

The time derivatives are:

$$\dot{x} = -R\sin(\theta)\dot{\theta} \tag{2.40}$$
$$\dot{y} = R\cos(\theta)\dot{\theta} \tag{2.41}$$
$$\tag{2.42}$$

The Lagrangian is thus:

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mR^2\dot{\theta}^2 \tag{2.43}$$

Note that L is independent of θ. The generalized momentum is:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{2.44}$$

This is recognized as the angular momentum.

In summary, a linear translational symmetry leads to conservation of linear momentum, a rotational symmetry leads to conservation of angular momentum, and time translation symmetry leads to conservation of energy.

3 Lecture 3: The Action Principle and the Calculus of Variations

We begin by considering **Fermat's principle**, which governs the path of light.

Definition 3.1 (Fermat's Principle)**.** Light travels along the path that minimizes the travel time between two points.

This can also be expressed by stating that light follows a path that minimizes the optical path length, given by

$$L = \int ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}. \tag{3.1}$$

This principle demonstrates that light minimizes a particular quantity along its trajectory. This naturally leads us to ask: does a similar principle apply to the trajectories of mechanical systems? The answer is yes.

Consider all possible paths $q_i(t)$ that a mechanical system could take through configuration space. For a given path, we define the **action** of the path, denoted by $S[q_i(t)]$, as:

$$S[q_i(t)] = \int_{t_{\text{initial}}}^{t_{\text{final}}} L(q_i, \dot{q}_i, t)\, dt \tag{3.2}$$

where L is the Lagrangian of the system. It turns out that the path that a mechanical system actually takes through configuration space is the one that **extremizes** the action. This is known as the **Principle of Least Action** or **Hamilton's Principle**. The action $S[q_i(t)]$ is a *functional*, which is a function of a function. We denote functionals using square brackets, e.g., $S[f]$, to distinguish them from functions like $f(x)$.

In standard calculus, we minimize functions of a finite number of variables. Here, we are tasked with minimizing a *functional*, which depends on an entire curve. This requires a new set of tools, which fall under the domain of the **Calculus of Variations**.

To understand this, let's consider a general problem. Suppose we have a function $F(y(x), y'(x), x)$, and we want to minimize the functional:

$$I[y(x)] = \int_{x_0}^{x_1} F(y(x), y'(x), x)\, dx \tag{3.3}$$

where $y(x)$ is defined on the interval $x_0 \leq x \leq x_1$ and $y'(x) = \frac{dy}{dx}$. The goal is to find the function $y(x)$ that extremizes $I[y(x)]$.

We proceed as follows:

- Fix the values of $y(x_0)$ and $y(x_1)$.
- Consider a curve $y(x)$ connecting these endpoints.
- Introduce a small perturbation to this curve: $y(x) \to y(x) + \delta y(x)$.
- Compute the variation δI due to this perturbation, and require that the term linear in δy vanishes.
- Since we have fixed the endpoints, we require $\delta y(x_0) = \delta y(x_1) = 0$.

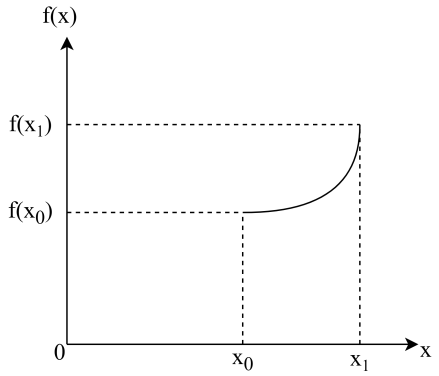

Figure 3.1: Calculus of Variations: Varying a Curve

The change in the functional due to the perturbation is:

$$\delta I = I[y(x) + \delta y(x)] - I[y(x)] = \int_{x_0}^{x_1} \left(F(y + \delta y, y' + \delta y', x) - F(y, y', x)\right) dx \quad (3.4)$$

Using a first-order Taylor expansion of F, we have:

$$\delta I \approx \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right) \quad (3.5)$$

Note that $\delta y' = \frac{d}{dx}(\delta y)$. Integrating the second term by parts (with $u = \frac{\partial F}{\partial y'}$ and $dv = \frac{d}{dx}(\delta y)\, dx$), and using the fact that $\delta y(x_0) = \delta y(x_1) = 0$, we get:

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\right) \quad (3.6)$$

Combining the terms, we obtain:

$$\delta I = \int_{x_0}^{x_1} dx \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\delta y \quad (3.7)$$

For δI to be zero for an arbitrary perturbation $\delta y(x)$, the term inside the parentheses must vanish, giving us the **Euler-Lagrange Equation**:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad (3.8)$$

The quantity

$$\frac{\delta I}{\delta y(x)} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad (3.9)$$

is called the **functional derivative** of $I[y(x)]$ with respect to $y(x)$.

The above result can be easily generalized to a functional that depends on N functions $y_i(x)$, rather than just one:

$$I[y_i(x)] = \int dx F(y_i, \frac{dy_i}{dx}, x), \quad i = 1, 2, \ldots, N \quad (3.10)$$

Following the same procedure as before, we obtain the condition for extremizing I:

$$\frac{\delta I}{\delta y_i} = \frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \tag{3.11}$$

This Euler-Lagrange equation must hold for all i.

In the context of mechanics, we have that the action is defined as $S[q_i(t)] = \int L\, dt$, where $L(q_i, \dot{q}_i, t)$ is the Lagrangian. Thus, the extremization of the action implies the following Euler-Lagrange equations:

$$\frac{\delta S}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{3.12}$$

These are the equations of motion for the system.

Let us apply the calculus of variations to an elementary problem.

Example 3.1. Prove that the shortest distance between two points is a line.

Proof. The length of the path can be written as:

$$L = \int ds \sqrt{(\dot{x}^2 + \dot{y}^2)} \tag{3.13}$$

Here $\dot{x}$ represents $\frac{dx}{ds}$. The Euler-Lagrange equations give us:

$$\frac{\delta L}{\delta x(s)} = -\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.14}$$

This can be simplified to:

$$\frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.15}$$

Similarly, for y we get

$$\frac{d}{ds}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \tag{3.16}$$

From equation (3.15) we can get:

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} \tag{3.17}$$

$$\frac{d}{ds}(\log \dot{x}) = \frac{d}{ds}(\log \dot{y}) \tag{3.18}$$

which means:

$$\frac{d}{ds} \log \frac{\dot{x}}{\dot{y}} = 0 \tag{3.19}$$

i.e.,

$$\frac{dx}{dy} = \text{constant} \tag{3.20}$$

Thus the shortest path between two points is a line. ■

We have found that

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{const} \tag{3.21}$$

Likewise,

$$p_y = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \text{const} \tag{3.22}$$

So we can get:

$$\frac{p_x}{p_y} = \frac{\dot{x}}{\dot{y}} = \text{const} \tag{3.23}$$

This shows that $\frac{dx}{dy}$ is constant, which implies a straight line. Also, the quantities p_x and p_y are constants of motion.

4 Lecture 4: Lagrange Multipliers, Near Equilibrium Dynamics & Oscillators

4.1 Lagrange Multipliers: Constrained Optimization

Our standard Lagrangian approach requires expressing the positions of all particles in terms of generalized coordinates, $r_\alpha(q_i, t)$. This can be complex making it difficult to obtain an explicit form for the Lagrangian,$L(q_i, \dot{q}_i, t)$. A natural question arises: can we formulate mechanics directly in terms of the particle positions, $\mathbf{r}_\alpha$, and impose constraints directly at the level of the equations of motion, rather than at the level of the action? This is where Lagrange multipliers become a valuable tool.

4.1.1 Review of Lagrange Multipliers in Calculus

Let's first recall how Lagrange multipliers work in standard calculus. Suppose we want to minimize a function $F(x, y)$. Without constraints, this involves simply finding where the partial derivatives vanish:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \tag{4.1}$$

However, given by $g(x, y) = 0$, if we want to minimize $F(x, y)$ subject to a constraint, we introduce a Lagrange multiplier λ. We then minimize the modified function: $F(x, y) + \lambda\, g(x, y)$. This leads to the following set of equations:

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \tag{4.2}$$

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \tag{4.3}$$

$$g(x, y) = 0 \tag{4.4}$$

Notice that we now have three equations for three unknowns: x, y, and λ. The first two equations define the stationary points of F subject to the constraint, and the third equation ensures that the constraint is satisfied. This method provides a systematic way to solve constrained optimization problems.

4.1.2 Lagrange Multipliers in the Calculus of Variations

The same principle can be extended to the calculus of variations. Let's say we have M functions $y_i(x), i = 1, 2, \cdots, M$, and we want to minimize an action integral:

$$I[y_i(x)] = \int dx\ F(y_i, \dot{y}_i, x) \tag{4.5}$$

Furthermore, suppose we have $M - N$ constraints of the form $G_k[y_i] = 0, k = 1, 2, \cdots, M - N$. The traditional approach would be to express all M functions, y_i, in terms of N independent degrees of freedom, however, we can avoid this by using Lagrange multipliers. To incorporate the constraints, we modify the action integral as follows:

$$\tilde{I}[y_i, \lambda_k] = \int dx\ \left(F(y_i, \dot{y}_i, x) + \sum_k \lambda_k(x)\, G_k[y_i] \right) \tag{4.6}$$

Here, $\lambda_k(x)$ are now functions of x, the independent variable of the integral. We treat $\tilde{I}$ as the functional we want to minimize, and obtain its Euler-Lagrange equations:

$$\frac{\delta \tilde{I}}{\delta y_i} = \frac{\delta I}{\delta y_i} + \sum_k \lambda_k \frac{\delta G_k}{\delta y_i} = 0 \tag{4.7}$$

$$G_k[y_i] = 0 \tag{4.8}$$

This gives us a total of $M + (M - N)$ equations. We have M equations from the functional derivative with respect to each y_i, plus $M - N$ equations from the constraints. These equations involve the M unknowns y_i and the $M - N$ Lagrange multiplier functions, $\lambda_k(x)$. These can, in principle, be solved to obtain the solution to the constrained variational problem.

Example 4.1. Compute the equations of motion in 2.1 again, but this time using Lagrange multipliers.

Now we treat x & y as independent variables, and they subject to $x^2+y^2-l^2 = 0$.

$$G(x, y) = x^2 + y^2 - l^2 \tag{4.9}$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{4.10}$$

$$L + \lambda G = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mgy + \lambda \left(x^2 + y^2 - l^2\right) \tag{4.11}$$

So by taking the derivative of $L + \lambda G$ with respect to x, we can get:

$$m\ddot{x} - 2\lambda x = 0 \tag{4.12}$$

$$m\ddot{y} + mg - 2\lambda y = 0 \tag{4.13}$$

$$x^2 + y^2 = l^2 \tag{4.14}$$

From 4.12 and 4.13 we get:

$$m\frac{\ddot{x}}{x} = m\left(\frac{\ddot{x}}{x} + \frac{g}{y}\right) \tag{4.15}$$

$$\ddot{x}y - \ddot{y}x - gx = 0 \tag{4.16}$$

Previously from 2.12 and 2.13 we get the time derivatives of the coordinates are:

$$\dot{x} = l\cos\theta \cdot \dot{\theta} \tag{4.17}$$

$$\dot{y} = l\sin\theta \cdot \dot{\theta} \tag{4.18}$$

So the second time derivatives of the coordinates are:

$$\ddot{x} = l\left(\cos\theta\ \ddot{\theta} - \sin\theta\ \dot{\theta}^2\right) \tag{4.19}$$

$$\ddot{y} = l\left(\sin\theta\ \ddot{\theta} + \cos\theta\ \dot{\theta}^2\right) \tag{4.20}$$

Take 4.19 and 4.20 into 4.16, we can get:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{4.21}$$

This is the same as what we got in 2.18.

4.2 Oscillators

So far we have written down equations of motion. Now we will solve them for a simple oscillator. Consider a general 1D system, where the degree of freedom is q and the langrangian is $L(q, \dot{q})$. An equilibrium solution is one where $q = \text{const} = q_0$, and q is independent of time.

The equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{4.22}$$

is a 2^{nd} order differential equation. Solutions depend on initial values of q, $\dot{q}$ at the same time t.

For an equilibrium solution, $q = q_0$, $\dot{q} = 0$ will remain at rest only if $\ddot{q} = 0$.

$$L = T - V \to 0 = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} \tag{4.23}$$

This vanishes when $\frac{\partial V}{\partial q}\big|_{q_0} = 0$.

There are two types of equilibrium solutions, namely stable equilibrium and unstable equilibrium. Consider a small perterbation around an equilibrium point $q = q_0 = 0$, let's expand around the equilibrium point:

$$L = A + Bq + C\dot{q} + Dq^2 + E\dot{q}^2 + Fq\dot{q} + \cdots \tag{4.24}$$

The EOM will remain unchanged if you add total time derivative to L:

$$L \to L + \frac{d}{dt}J \tag{4.25}$$

$$S \to S + \sum_{t_0}^{t_1} dt\ \frac{d}{dt}J = S + J(t_1) - J(t_0) \tag{4.26}$$

I can ignore A, C and E, so we can get:

$$\frac{\partial V}{\partial q} = 0 \to B = 0 \tag{4.27}$$

$$L = F\dot{q} + Dq^2 = 2F\left(\frac{1}{2}\dot{q}^2 + \frac{D}{2F}q^2\right) \tag{4.28}$$

This is equivalent to

$$L = \frac{1}{2}\left(\dot{q}^2 + \frac{D}{F}q^2\right) \tag{4.29}$$

Let the frequency ω be:

$$\omega^2 = -\frac{D}{F} \tag{4.30}$$

So the equations of motion are:

$$\ddot{q} - \frac{D}{F}q = 0 \tag{4.31}$$

which is,

$$\ddot{q} + \omega^2 q = 0 \tag{4.32}$$

If ω is real ($\frac{D}{F} > 0$), then the solutions to the equations of motion are:

$$q = A\sin\omega t + B\cos\omega t \tag{4.33}$$

The solution oscillates around $q = 0$ with frequency ω. In this case the equilibrium is stable.

If the ω is imaginary ($\frac{D}{F} < 0$), then we can get:

$$q = Ae^{i\omega t} + Be^{-i\omega t} \tag{4.34}$$

generically (unless we choose initial condition with $A = 0$) the perterbation grows in time. The equilibrium is then unstable.