Introduction

We will attempt to solve a second-order ODE numerically on python. We will use the classical harmonic oscillator as an example. However, we will first solve the problem analytically. Recall for a mass on a 1-D mass-less spring, the mass moves in one-dimension according to Hooke's law, and Newton's second law:

$$F_H = -kx(t) \quad Hookes'law \tag{1}$$

$$F_N = \frac{\mathrm{d}p(t)}{\mathrm{d}t} \quad Newton's\ second\ law \tag{2}$$

Where k is the force constant. The mass of the object is constant, and as such, equation (2) reduces to:

$$F = m\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{3}$$

$$= mx\ddot(t) \tag{4}$$

where $x\ddot(t)$ is the acceleration of the mass attached to the spring.

Solving the H.O. without damping

We begin by setting Hooke's law equal to Newton's law to see the ODE:

$$\begin{aligned} F_N &= F_H \\ m\ddot{x} &= -kx \\ m\ddot{x} + kx &= 0 \\ \ddot{x} + \frac{k}{m}x &= 0 \end{aligned} \tag{5}$$

where we omitted to write $x(\ddot{t})$ for clarity, but remembering that it is time dependent. We will also set the constant $\frac{k}{m} = \omega^2$.

Solving the H.O.-cont'd

Our equation can now be written as:

$$\ddot{x} + \omega^2 x = 0 \tag{6}$$

We can now ask ourselves, which function when differentiated twice, and added to itself yields zero. Both cosine and sine would work, as well as a linear combination of them.

Solving cont'd

If we choose $x(t) = Acos(\omega t) + Bsin(\omega t)$ as our guess for the solution, we see:

$$x = Acos(\omega t) + Bsin(\omega t) \tag{7}$$

$$\dot{x} = -A\omega sin(\omega t) + B\omega cos(\omega t) \tag{8}$$

$$\ddot{x} - A\omega^2 cos(\omega t) - B\omega^2 sin(\omega t) \tag{9}$$

We can in fact see, if we add our expression for $\ddot{x}$ to our guess for x, it does solve the Homogeneous equation. Now, we need to define initial conditions to fix the values of A and B.

Choosing initial conditions

Let us choose that the particle starts at rest $v(0) = 0$, and at some elongation $x(0) = x_0$

$$\dot{x} = -A\omega sin(\omega * 0) + B\omega cos(\omega * 0) = 0 \tag{10}$$

this condition is satisfied if $B = 0$, this picks out the cosine solution.

$$x(0) = Acos(\omega * 0) = x_0 \quad \rightarrow A = x_0. \tag{11}$$

Analytic solution

Hence, we see that the equation, for these sets of initial conditions, our solution simplifies to:

$$x(t) = x_0 cos(\omega t) \tag{12}$$

$$x(t) = x_0 cos\left(\sqrt{\frac{k}{m}}t\right)$$

Therefore, the mass oscillates about the position x_0 (the amplitude of the motion), with angular frequency $\omega = \sqrt{\frac{k}{m}}$.

Solving the H.O. with damping

In the case of damped motion, air(fluid) resistance or friction with a surface or both could contribute to damping the motion of the mass m. This friction force is proportional to the velocity of the mass m (for small velocities of oscillation). Hence, we can write the intervening forces as:

$$F_H = -kx;\ F_N = m\ddot{x}\ quad; F_D = -\mu\dot{x}$$

the drag coefficient μ is related to the medium/material doing the drag. It is important to remember that the spring acts on the mass with a restoring force F_H, so it acts opposite to the motion of the mass. Similarly, the drag force resists the mass' motion, so it acts in the same direction as the F_H. This fact is the reason mathematically, F_H and F_D have the same sign.

Second-order differential with constant coefficient

The equation we want to solve is the following:

$$m\ddot{x} + \mu\dot{x} + kx = 0 \tag{13}$$

$$\ddot{x} + \frac{\mu}{m} + \frac{k}{m} = 0 \tag{14}$$

We can guess a function $x(t) = e^{ct}$ as the solution and plug it back in the differential equation:

$$c^2 e^{ct} + \frac{\mu c}{m} e^{ct} + \frac{k}{m} e^{ct} = 0 \tag{15}$$

$$e^{ct}\left(c^2 + \frac{\mu c}{m} + \frac{k}{m}\right) = 0 \tag{16}$$

We realize then that we need only solve the quadratic equation for the parameter c.The general solutions of c are given by:

$$c_{1,2} = -\frac{\mu}{2m} \pm \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}} \tag{17}$$

Solving the quadratic part and finding general solutions

We can set $\omega^2 = \frac{k}{m}$ as before, in addition to setting $\delta = \frac{\mu}{2m}$. Our expression for the solutions of c is then:

$$c_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} \tag{18}$$

Depending on the value of the discriminant, we can have, for c: (i) ***two solutions*** $c_1 \neq c_2$ if $\delta^2 - \omega^2 > 0$, (ii) ***one solution*** such that $c_1 = c_2$ if $\delta^2 - \omega^2 = 0$ and (iii) ***complex solutions*** if $\delta^2 - \omega^2 < 0$. These three cases correspond to three different physical situations. If:

- $\delta^2 - \omega^2 > 0$, this referred to as an *overdamped* motion
- $\delta^2 - \omega^2 = 0$, this is referred to as a *critically damped* motion
- $\delta^2 - \omega^2 < 0$, this is referred to as the *underdamped* motion

The general solutions to three physical situations

For each type of damped motion, there are associated solutions that describe the motion of the mass:

- *overdamped* motion: $x(t) = Ae^{c_1 t} + Be^{c_2 t}$
- *critically damped* motion: $x(t) = Ae^{c_1 t} + Bte^{c_1 t}$
- *underdamped* motion:$x(t) = e^{-\delta t}\left(Ae^{i\omega t} + Be^{-i\omega t}\right)$

The values of the coefficients A and B are determined from the initial conditions. The values of m, k and μ fix the values of δ and ω, specifying which physical situation we are in, and consequently, which model to choose.