These slides are by courtesy of Prof. 李稻葵 and Prof. 郑捷.

Chapter Nineteen

Technology

Time to Look at Producers

- **1st: We work on producer's constraint, which is determined by technology;**
- **2nd: We study producer's objective function, which is the profit function;**
- **3rd: We examine producer's choice, which is his optimal production plan.**

Technology

- Technology possessed by the producer determines what the producer can do.
- Particularly, it determines how many units of output can be produced using some given amount of input.

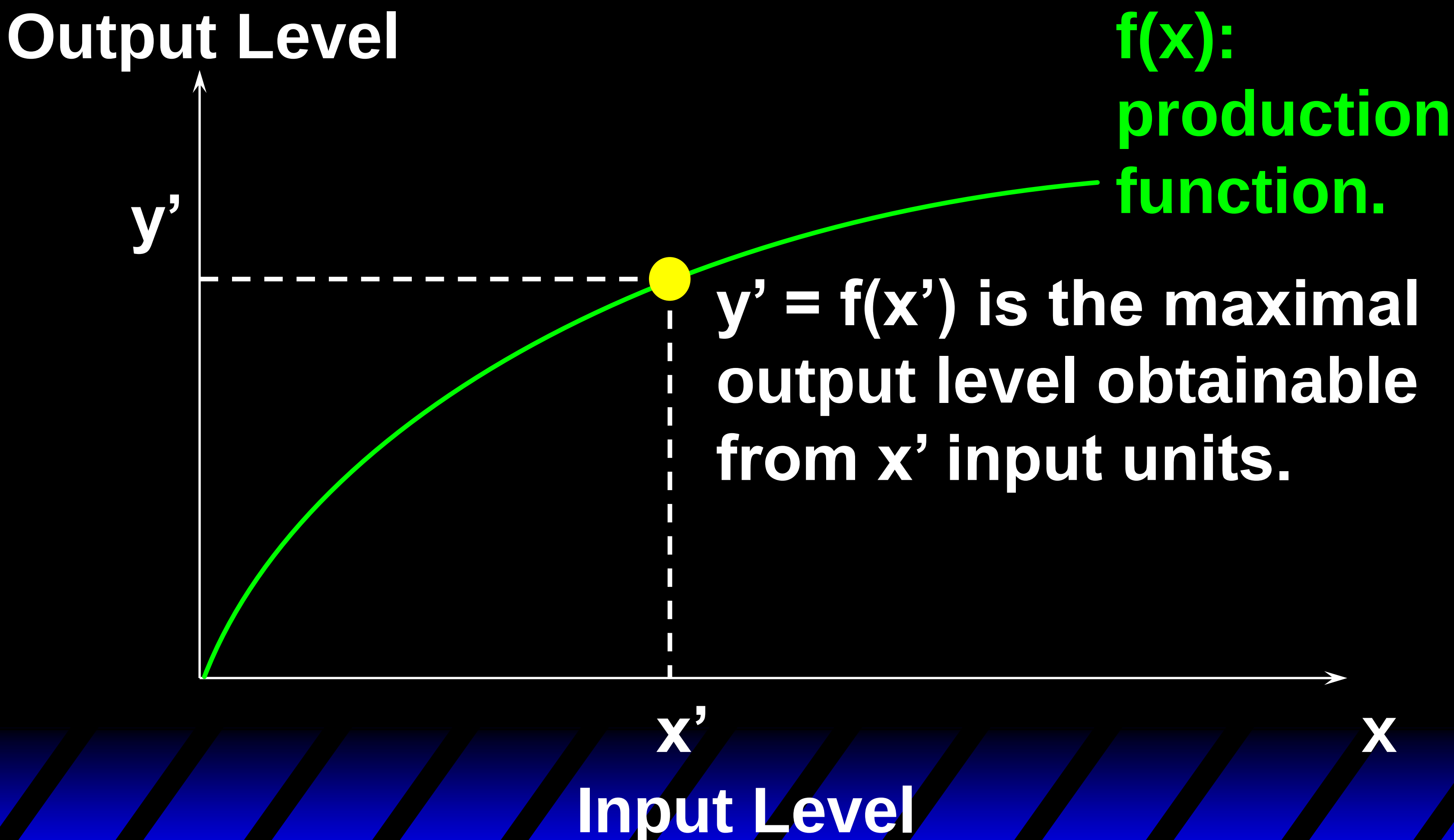
Production Functions

- Let y denote the output level.
- An **input bundle** is a vector of the input levels; $(x_1, x_2, \ldots, x_n)$.
- The technology's **production function** states the **maximum** amount of output possible from an input bundle.

$$y \le f(x_1, \cdots, x_n)$$

Production Functions

One input, one output



Technology Sets

- A **production plan** is an input bundle and an output level; $(x_1, \ldots, x_n, y)$.
- A production plan is **feasible** if

$$y \le f(x_1, \cdots, x_n)$$

- The collection of all feasible production plans is the **technology set**.

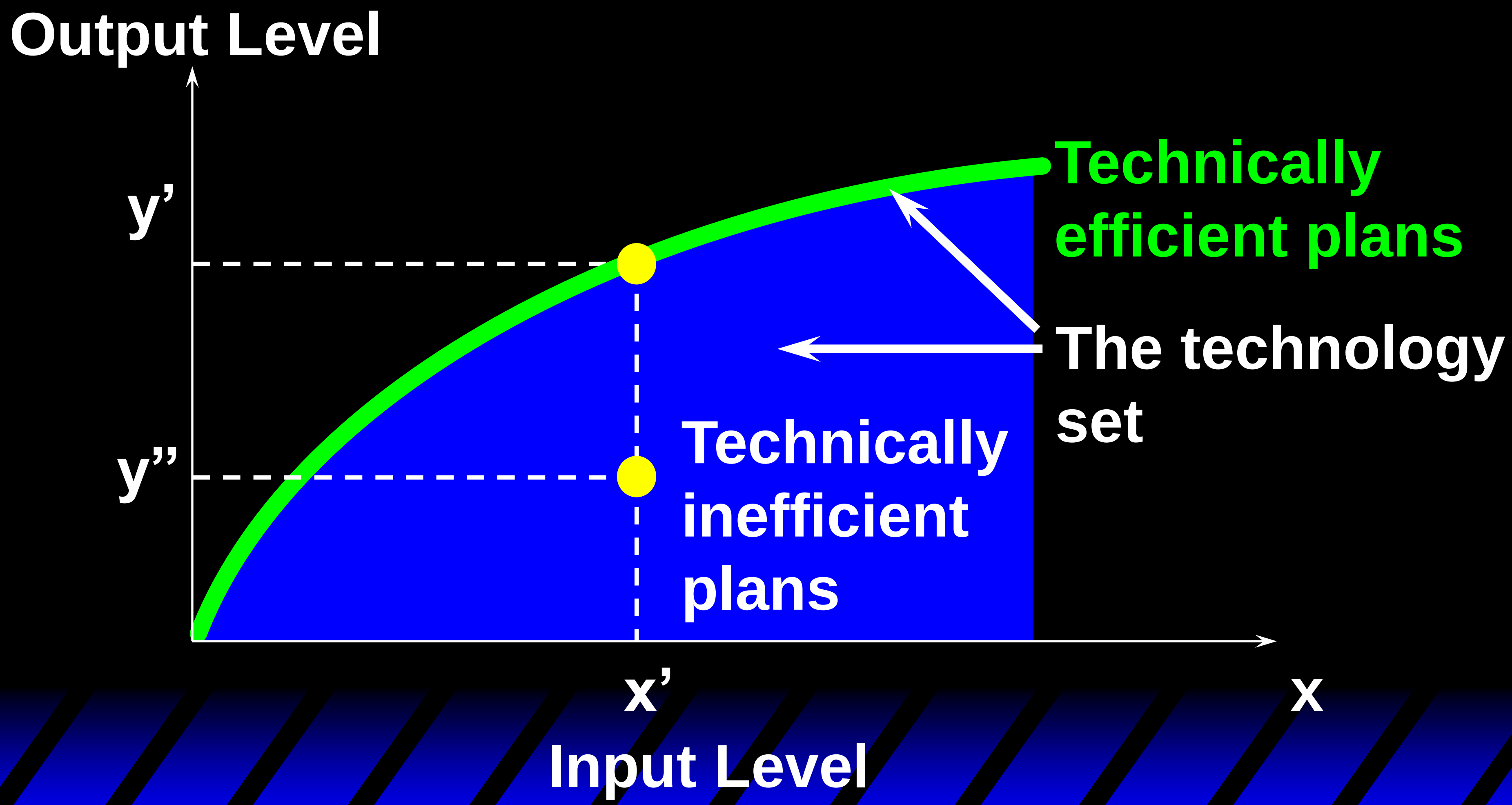
Technology Sets

The technology set is

$$T = \{(x_1, \cdots, x_n, y) \mid y \le f(x_1, \cdots, x_n) \text{ and } x_1 \ge 0, \ldots, x_n \ge 0\}.$$

Technology Sets

One input, one output



Technologies with Multiple Inputs

- What does a technology look like when there is more than one input?
- The two input case: Input levels are x_1 and x_2. Output level is y.
- E.g.:

$$y = f(x_1, x_2) = 2x_1^{1/3}x_2^{1/3}.$$

Technologies with Multiple Inputs

- The **isoquant** that corresponds to output level y is the set of all input bundles x such that f(x) = y.
- The complete collection of isoquants is the **isoquant map**.

Cobb-Douglas Technologies

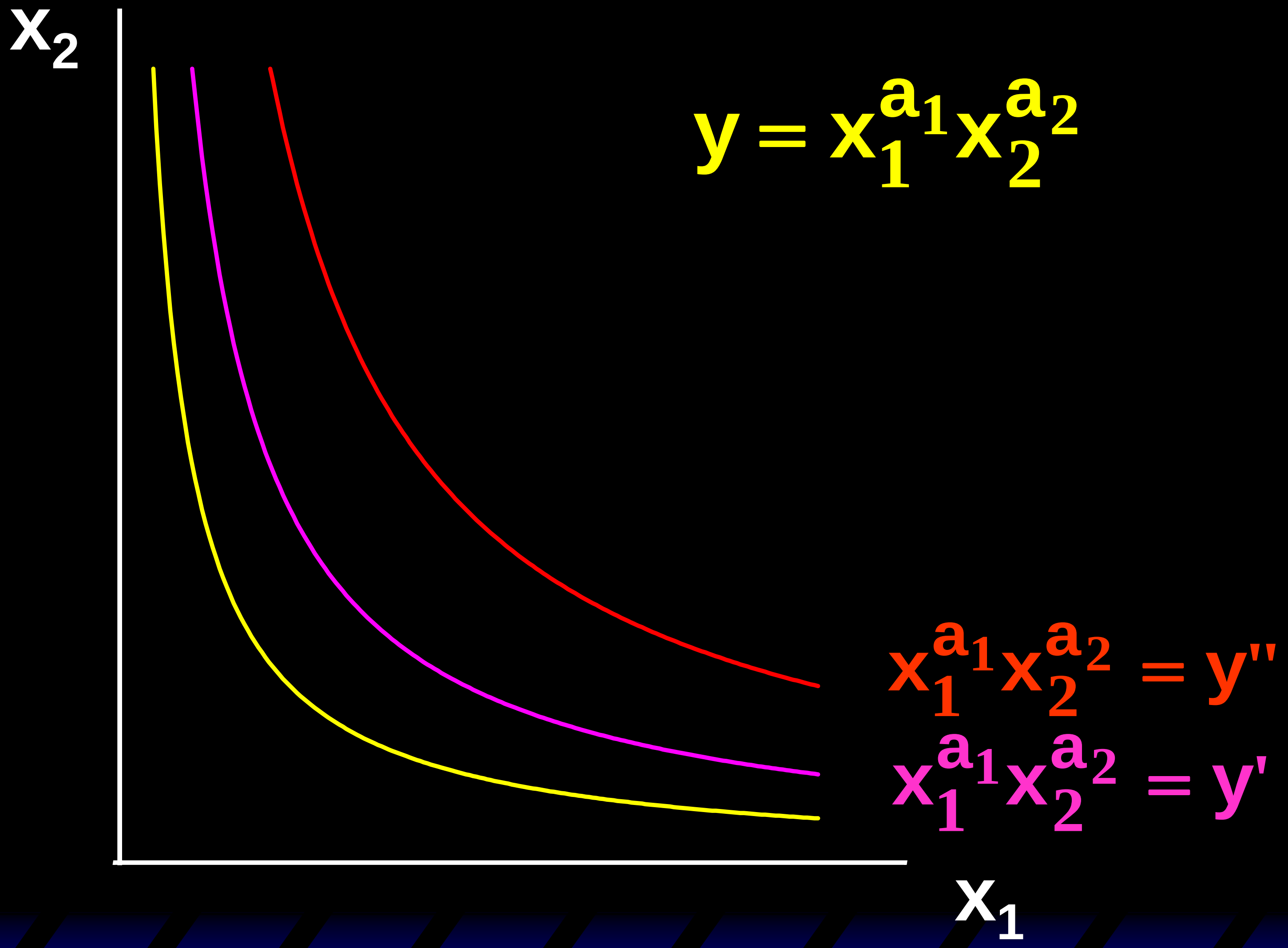
- A Cobb-Douglas production function is of the form

$$y = A x_1^{a_1} x_2^{a_2} \times \cdots \times x_n^{a_n}.$$

- E.g.

$$y = x_1^{1/3} x_2^{1/3}$$

Cobb-Douglas Technologies



Fixed-Proportions Technologies

- A fixed-proportions production function is of the form

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

- E.g.

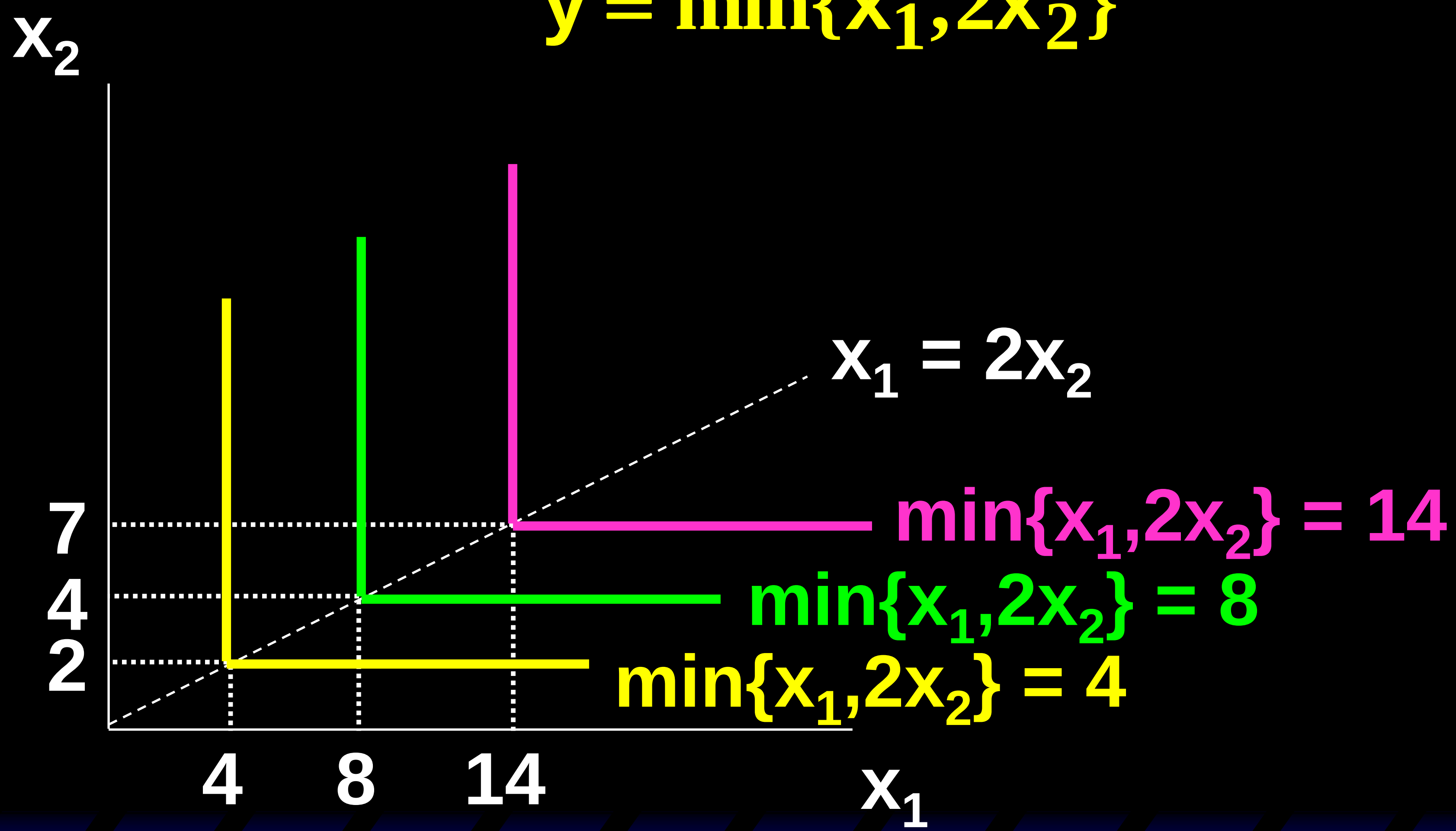
$$y = \min\{x_1, 2x_2\}$$

with

$$n = 2,\ a_1 = 1 \text{ and } a_2 = 2.$$

Fixed-Proportions Technologies

$y = \min\{x_1, 2x_2\}$



Perfect-Substitutes Technologies

- A perfect-substitutes production function is of the form

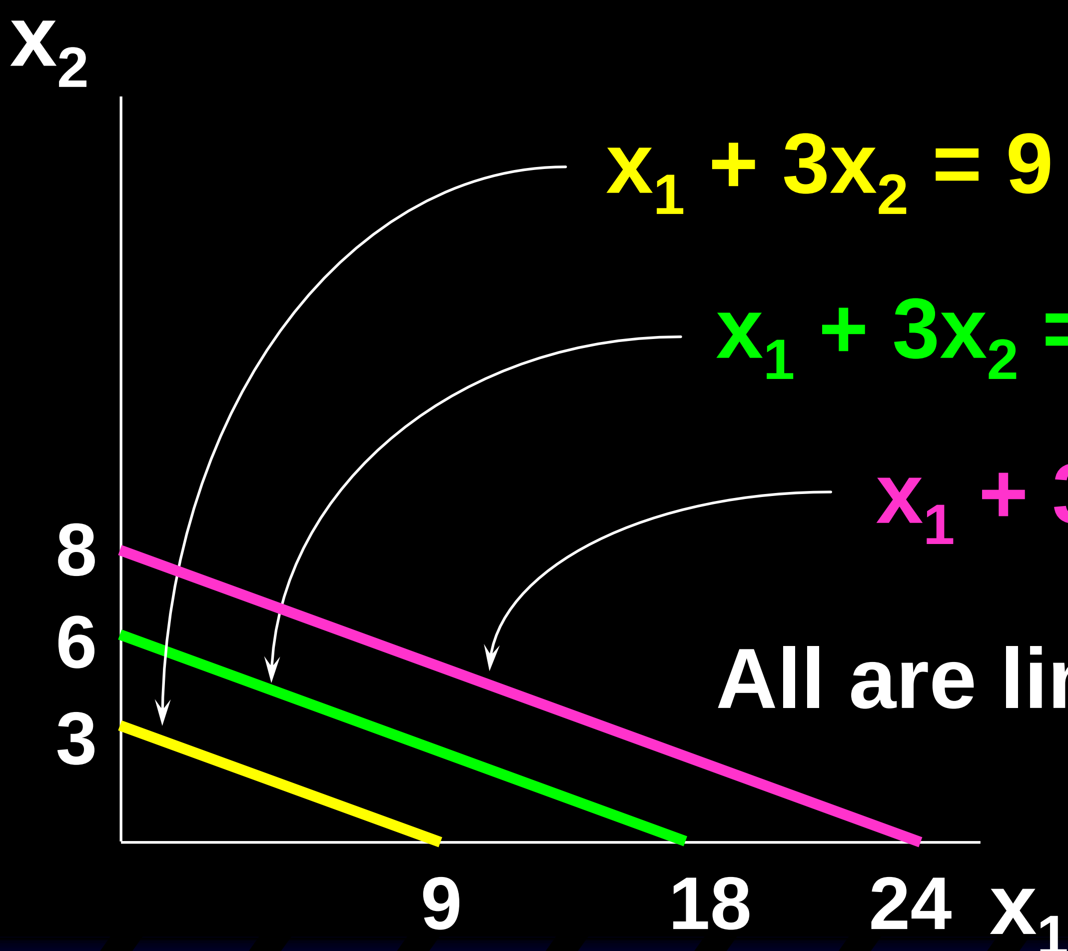
$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

- E.g.

$$y = x_1 + 3x_2$$

Perfect-Substitution Technologies

$y = x_1 + 3x_2$



All are linear and parallel

Marginal Products

$$y = f(x_1, \cdots, x_n)$$

- The **marginal product of input i is the rate-of-change of the output level as the level of input i changes, holding all other input levels fixed.**
- That is,

$$MP_i = \frac{\partial y}{\partial x_i}$$

Marginal Products

E.g. if

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

then the marginal product of input 1 is

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

Marginal Products

- The marginal product of input i is **diminishing** if it becomes smaller as the level of input i increases. That is, if

$$\frac{\partial MP_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right) = \frac{\partial^2 y}{\partial x_i^2} < 0.$$

Marginal (Physical) Products

E.g. if $y = x_1^{1/3} x_2^{2/3}$ then

$MP_1 = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$ and $MP_2 = \frac{2}{3} x_1^{1/3} x_2^{-1/3}$

so

$$\frac{\partial MP_1}{\partial x_1} = -\frac{2}{9} x_1^{-5/3} x_2^{2/3} < 0$$

and

$$\frac{\partial MP_2}{\partial x_2} = -\frac{2}{9} x_1^{1/3} x_2^{-4/3} < 0.$$

Both marginal products are diminishing.

Returns-to-Scale

- **Marginal products describe the change in output level as a single input level changes.**
- **Returns-to-scale describes how the output level changes as all input levels change in proportion (*e.g.* all input levels doubled, or halved).**

Returns-to-Scale

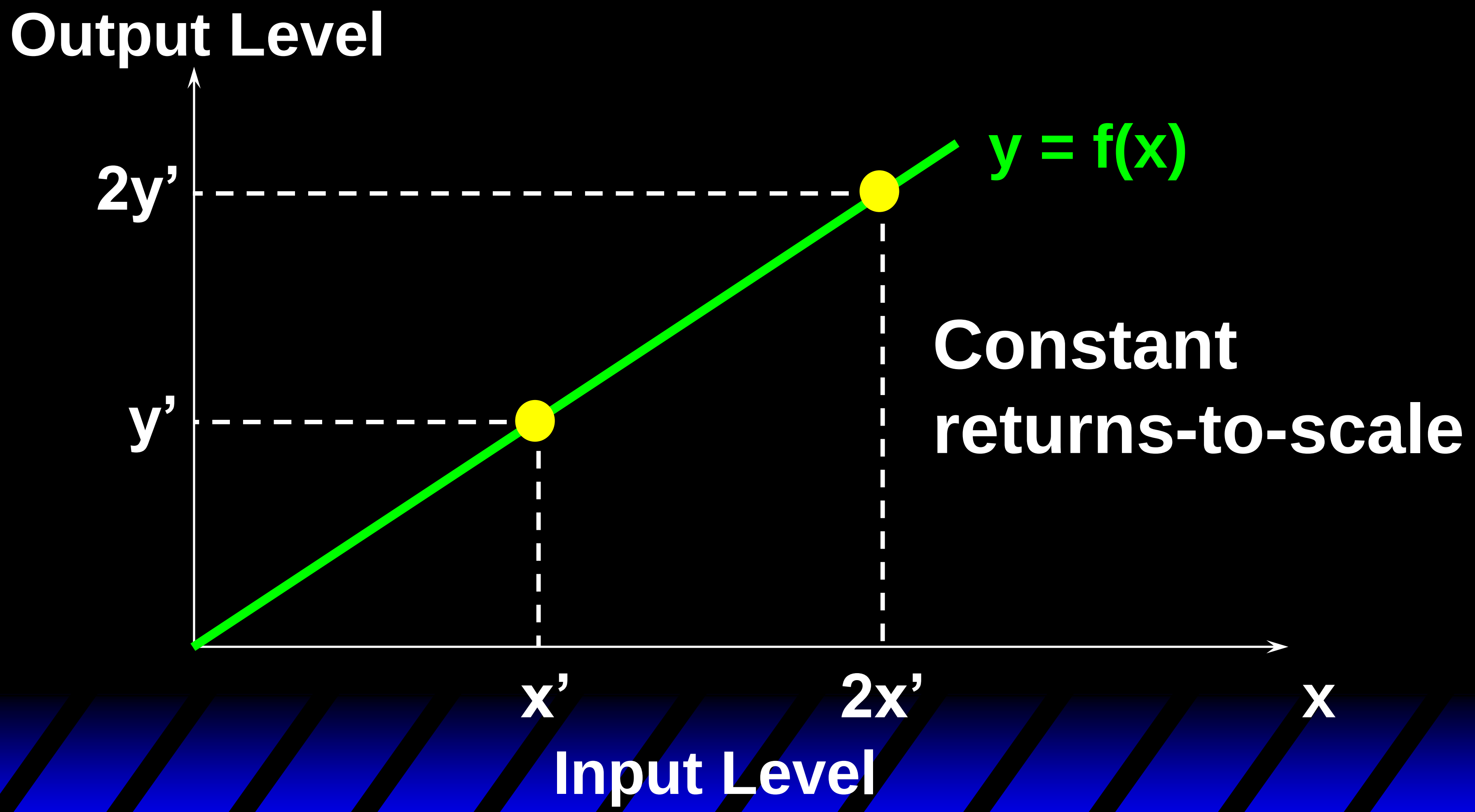
If, for *any* input bundle $(x_1,\ldots,x_n)$ and *any* k,

$$f(kx_1,kx_2,\cdots,kx_n) = kf(x_1,x_2,\cdots,x_n)$$

then the technology described by the production function f exhibits **constant returns-to-scale**.

Returns-to-Scale

One input, one output



Returns-to-Scale

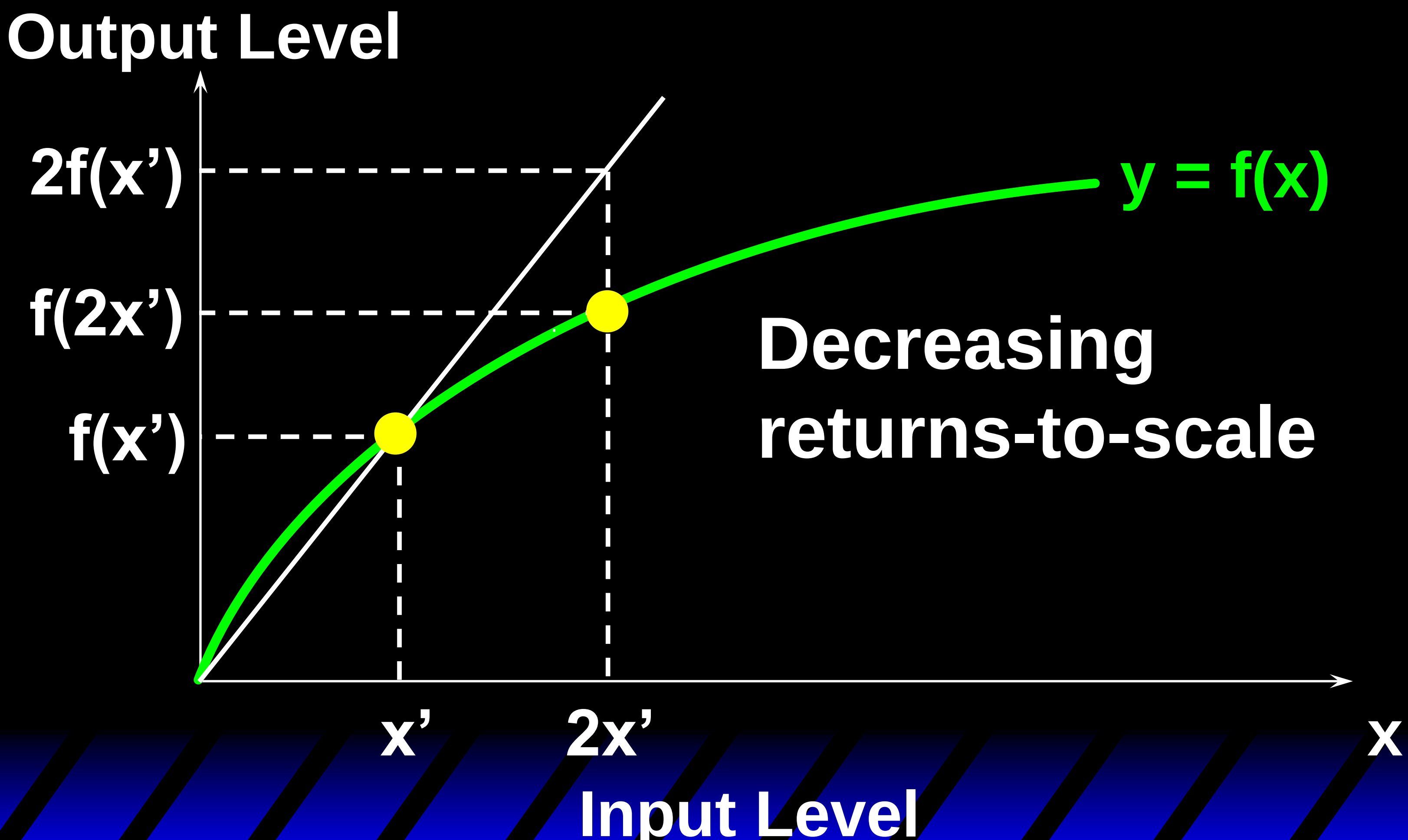
If, for ***any*** input bundle $(x_1,\ldots,x_n)$ and ***any*** $k>1$,

$$f(kx_1, kx_2, \cdots, kx_n) < kf(x_1, x_2, \cdots, x_n)$$

then the technology exhibits **decreasing returns-to-scale**.

Returns-to-Scale

One input, one output



Returns-to-Scale

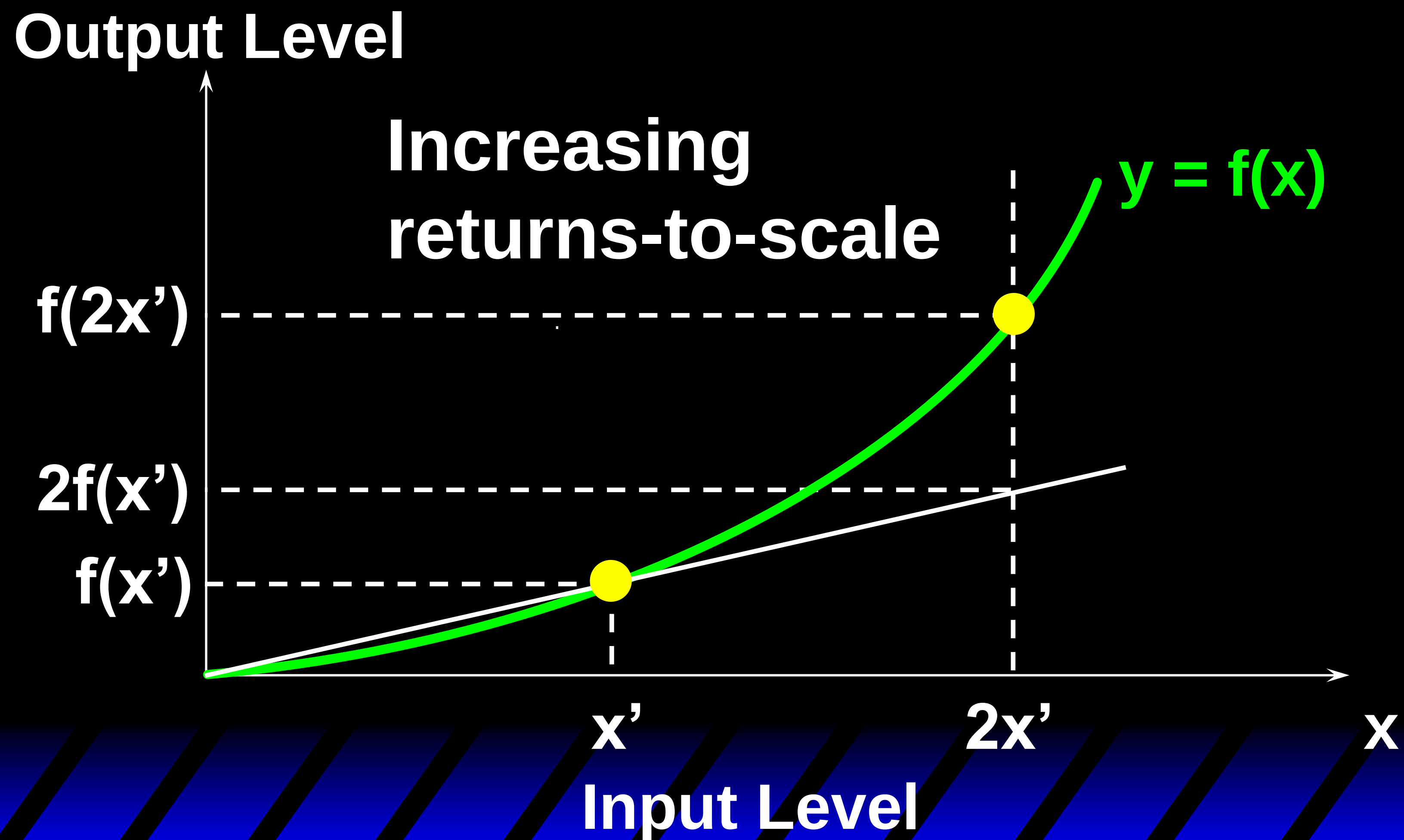
If, for ***any*** input bundle $(x_1,\ldots,x_n)$ and ***any*** $k>1$,

$$f(kx_1,kx_2,\cdots,kx_n) > kf(x_1,x_2,\cdots,x_n)$$

then the technology exhibits **increasing returns-to-scale**.

Returns-to-Scale

One input, one output

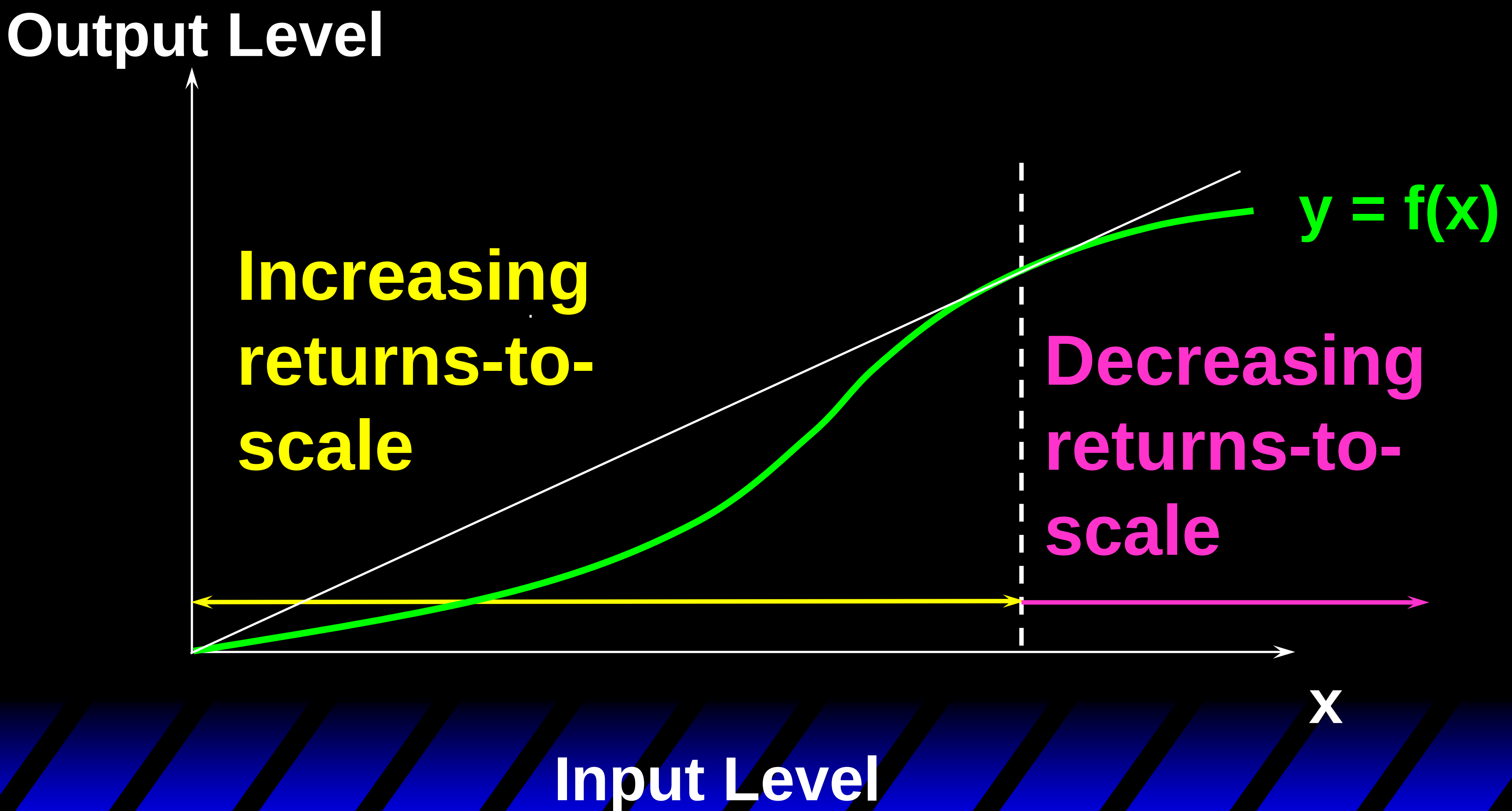


Local Notion of Returns-to-Scale

- We may also define ‘local’ returns-to-scale, by considering k slightly greater than 1.

Returns-to-Scale

One input, one output



Examples of Returns-to-Scale

The perfect-substitutes production function is

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

We have:

$$\begin{aligned} & a_1(kx_1) + a_2(kx_2) + \cdots + a_n(kx_n) \\ &= k(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= ky. \end{aligned}$$

The perfect-substitutes production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The perfect-complements production function is

$$y = \min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}.$$

We have:

$$\begin{aligned}
&\min\{a_1 (kx_1), a_2 (kx_2), \cdots, a_n (kx_n)\} \\
&= k(\min\{a_1 x_1, a_2 x_2, \cdots, a_n x_n\}) \\
&= ky.
\end{aligned}$$

The perfect-complements production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

We have

$$(kx_1)^{a_1}(kx_2)^{a_2}\cdots(kx_n)^{a_n}$$

$$= k^{a_1}k^{a_2}\cdots k^{a_n}x^{a_1}x^{a_2}\cdots x^{a_n}$$

$$= k^{a_1+a_2+\cdots+a_n}x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$$

$$= k^{a_1+\cdots+a_n}y.$$

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

$$(kx_1)^{a_1}(kx_2)^{a_2} \cdots (kx_n)^{a_n} = k^{a_1 + \cdots + a_n} y.$$

The Cobb-Douglas technology's returns-to-scale is

constant if $a_1 + \ldots + a_n = 1$

increasing if $a_1 + \ldots + a_n > 1$

decreasing if $a_1 + \ldots + a_n < 1$.

Comparing Returns-to-Scale with Marginal Product

- **Q: Can a technology exhibit increasing returns-to-scale even if all of its marginal products are diminishing?**
- **A: Yes.**
- **E.g.** $y = x_1^{2/3} x_2^{2/3}.$

Comparing Returns-to-Scale with Marginal Product (Cont.)

$$y = x_1^{2/3} x_2^{2/3} = x_1^{a_1} x_2^{a_2}$$

$a_1 + a_2 = \frac{4}{3} > 1$ so this technology exhibits increasing returns-to-scale.

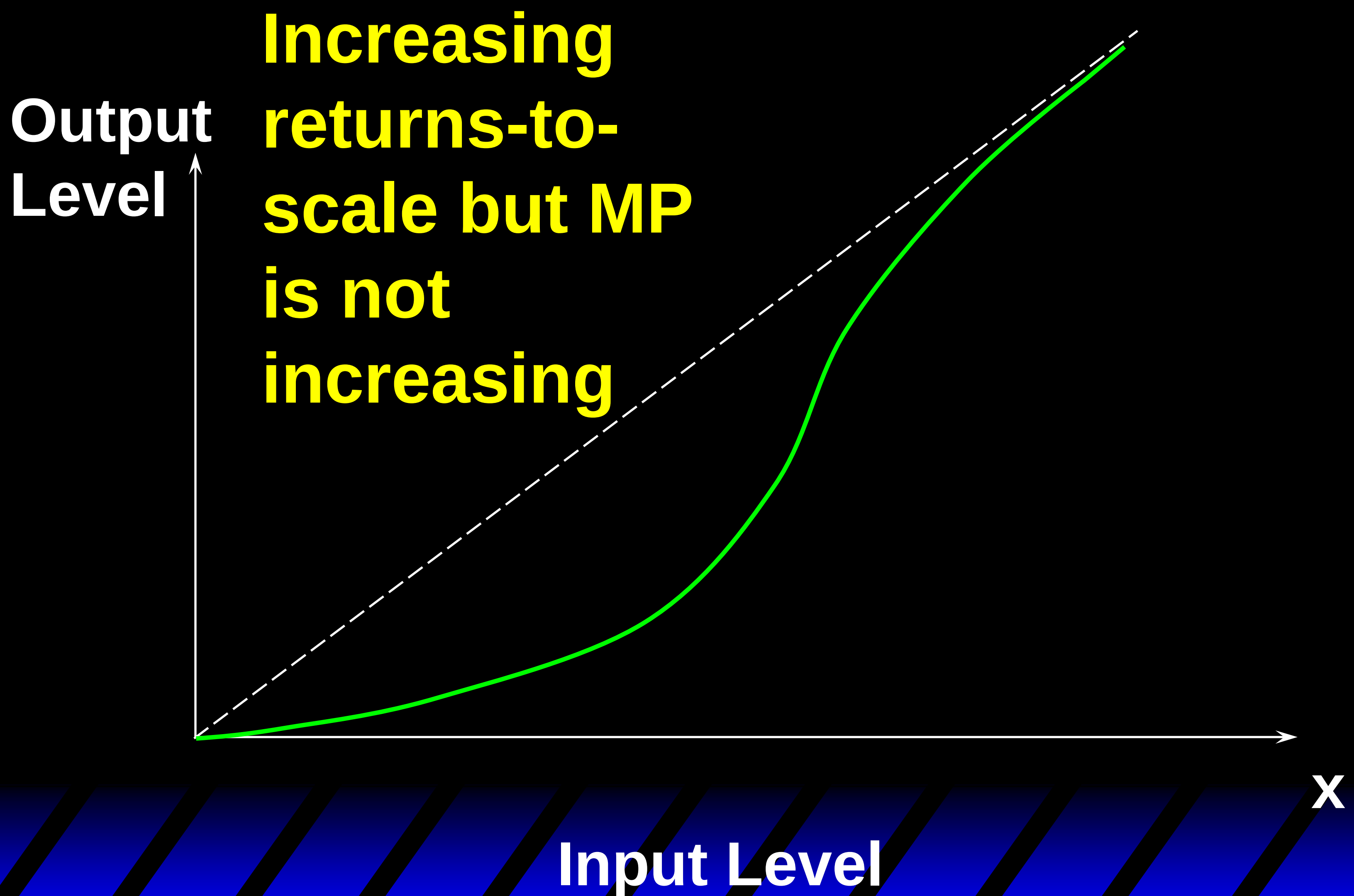
But $MP_1 = \frac{2}{3} x_1^{-1/3} x_2^{2/3}$ diminishes as x_1 increases and

$MP_2 = \frac{2}{3} x_1^{2/3} x_2^{-1/3}$ diminishes as x_1 increases.

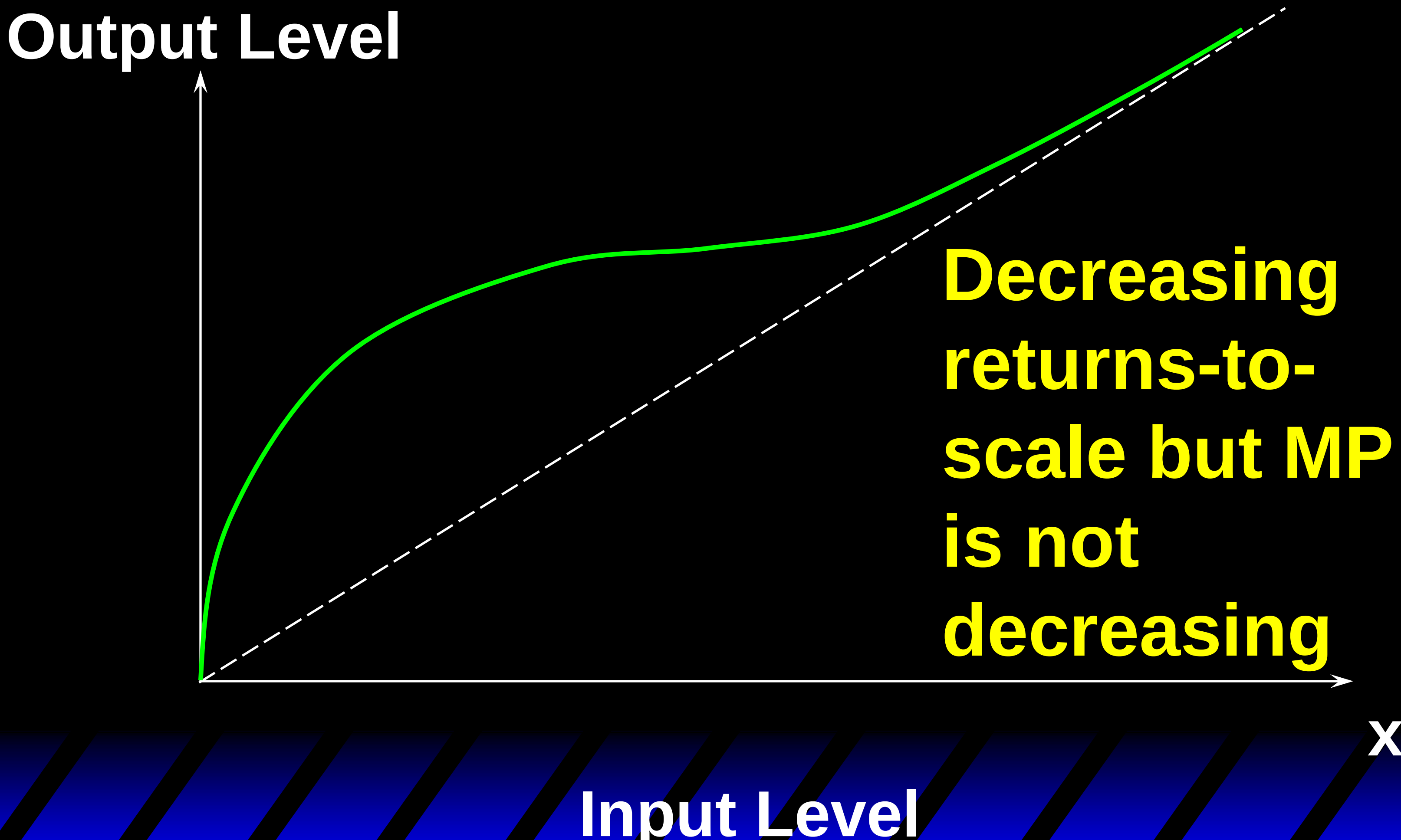
Increasing Returns to Scale Does NOT Imply Increasing MP

- Marginal product is the rate-of-change of output as **one** input level increases, holding all other input levels fixed.
- Even with increasing returns to scale, MP may still diminish because the other inputs are fixed, so an additional unit of input has less and less other inputs to work with.

Even With Only One Input



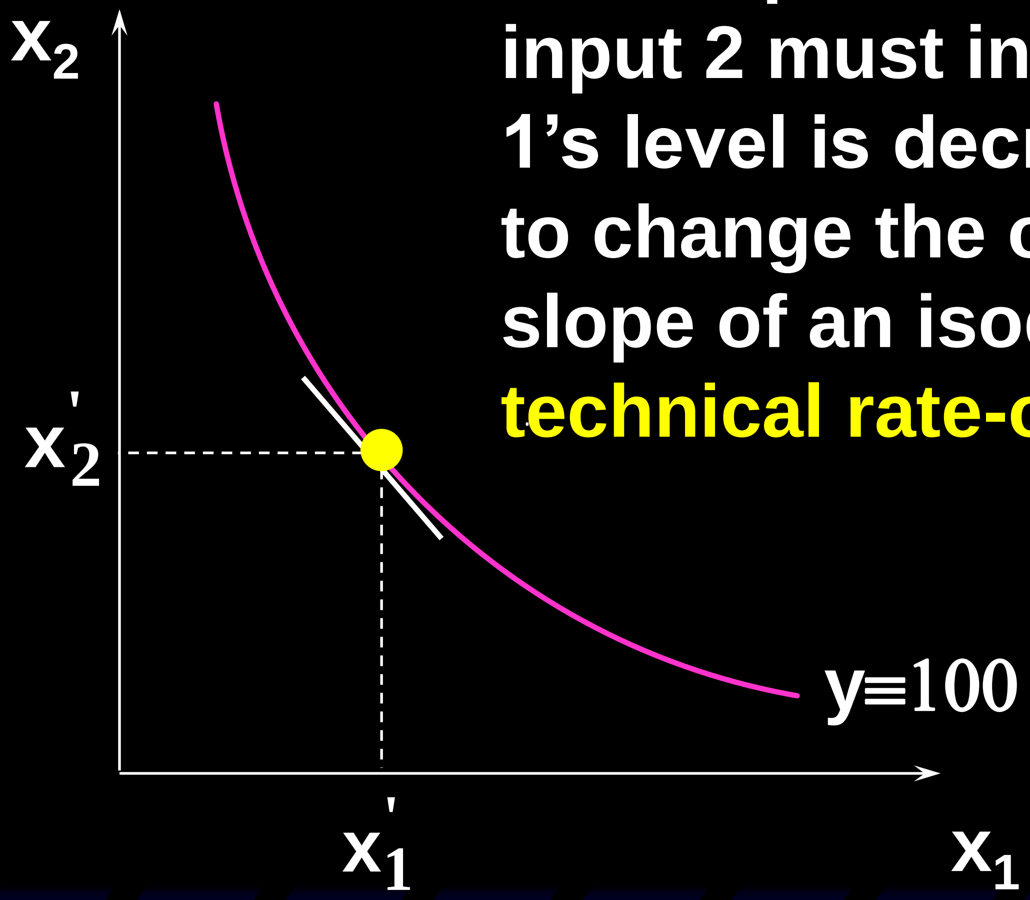
Even With Only One Input



Technical Rate-of-Substitution

- **At what rate can a firm substitute one input for another without changing its output level?**

Technical Rate-of-Substitution



The slope is the rate at which input 2 must increase as input 1's level is decreased so as not to change the output level. The slope of an isoquant is its technical rate-of-substitution.

Technical Rate-of-Substitution

- How is a technical rate-of-substitution computed?
- The production function is $y = f(x_1, x_2)$.
- A small change (dx_1, dx_2) in the input bundle causes a change to the output level of

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

Technical Rate-of-Substitution

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

No change in the output level implies $dy = 0$, so the changes dx_1 and dx_2 to the input levels must satisfy

$$0 = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

Technical Rate-of-Substitution

$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2}$$

is the rate at which input 2 must increase as input 1 decreases to keep the output level constant.
It is the slope of the isoquant.

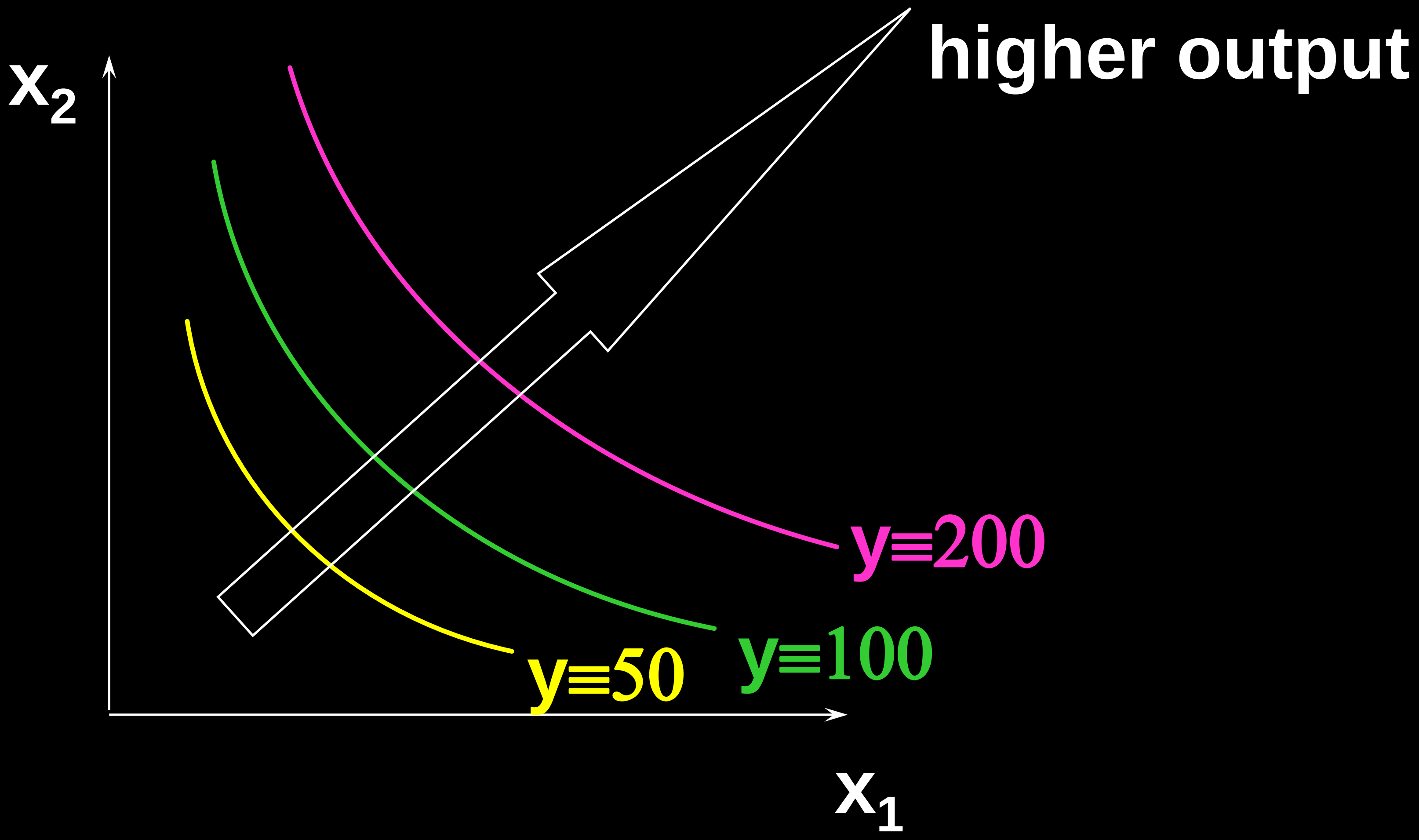
Well-Behaved Technologies

- A **well-behaved** technology is
  - **monotonic**, and
  - **convex**.

Well-Behaved Technologies - Monotonicity

- **Monotonicity: More of any input generates more output, i.e. the production function is increasing in all inputs.**

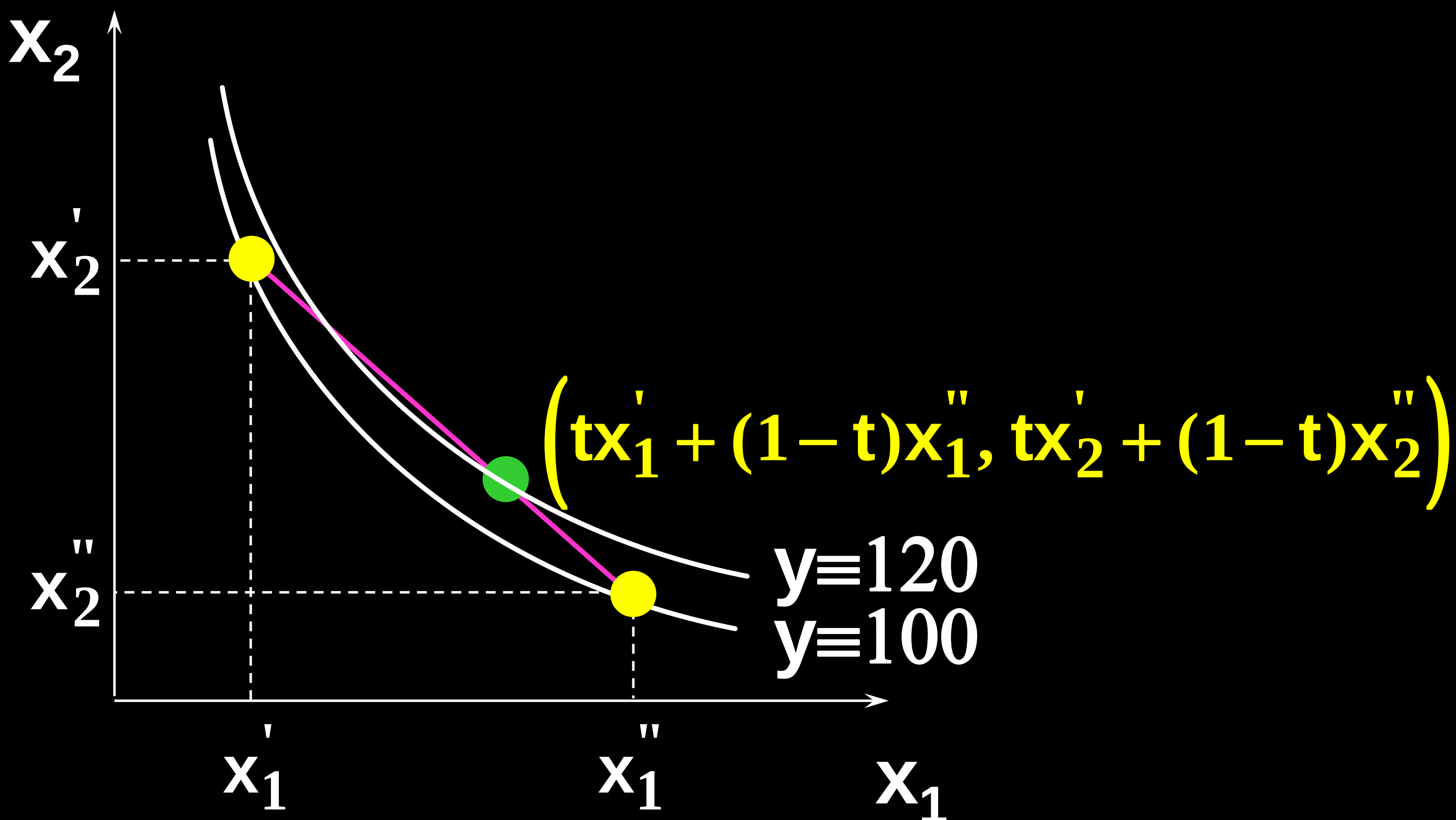
Well-Behaved Technologies
Monotonicity



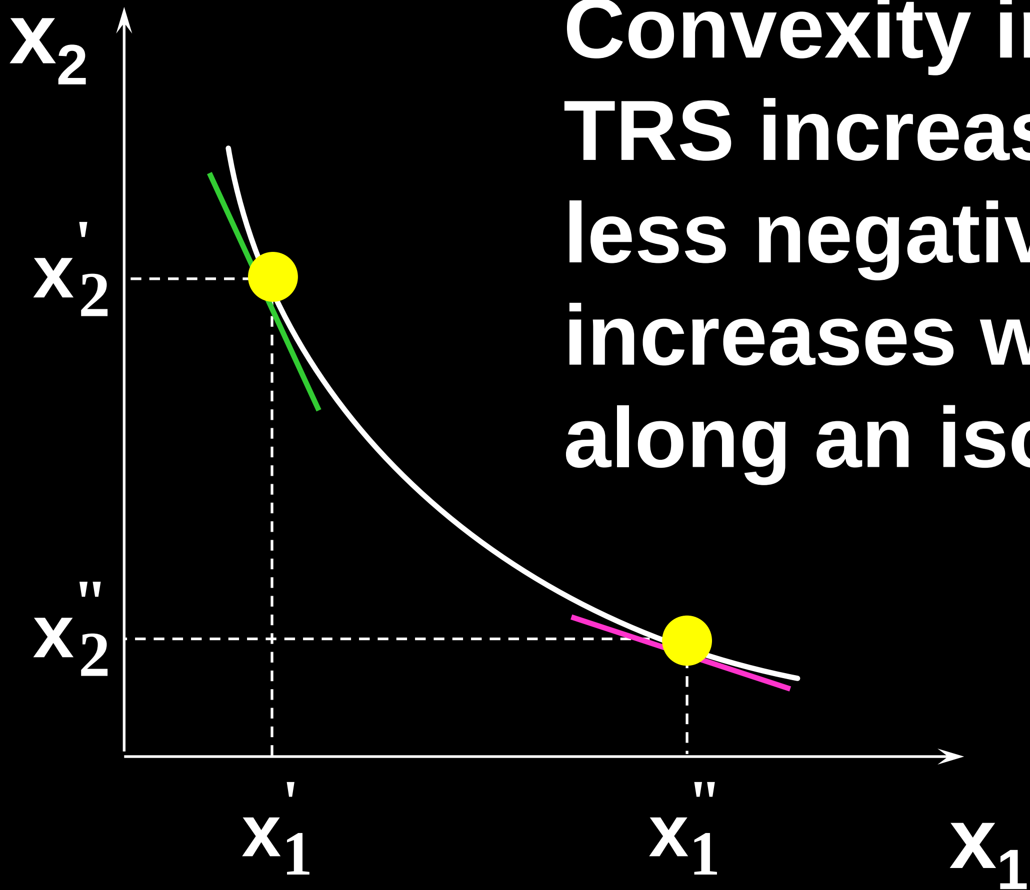
Well-Behaved Technologies - Convexity

- **Convexity: If the input bundles x' and x" both provide y units of output, then the combination tx' + (1-t)x" provides at least y units of output, for any 0 < t < 1.**

Well-Behaved Technologies - Convexity



Well-Behaved Technologies - Convexity



Convexity implies that the TRS increases (becomes less negative) as x_1 increases when moving along an isoquant.

The Long-Run and the Short-Run

- **The long-run** is the circumstance in which the firm can freely choose all input levels.
- **The short-run** is typically the circumstance in which the levels of some inputs are fixed (e.g. land, factories, machines).

The Long-Run and the Short-Run

- **What do short-run restrictions imply for a firm's technology?**
- **Suppose the short-run restriction is fixing the level of input 2.**
- **Input 2 is thus a fixed input in the short-run. Input 1 remains variable.**

The Long-Run and the Short-Run

$y = x_1^{1/3} x_2^{1/3}$ is the long-run production function (both x_1 and x_2 are variable).

Suppose x_2 cannot be easily changed in the short run, and now the firm is running at $x_2 \equiv 1$,

Then the short-run production function

$$y = x_1^{1/3} 1^{1/3} = x_1^{1/3}.$$

Summary: Key Concepts

- Production function and isoquants
- Marginal product and returns-to-scale
- Technical rate of substitution