1.

$$\ln(1+x+y+z) = x+y+z - \frac{1}{2}(x+y+z)^2 + o\left(\sqrt{x^2+y^2+z^2}\right)$$

$\exists\, \theta \in (0,1)$

$$\ln(1+x+y+z) = x+y+z - \frac{1}{2}\frac{1}{(1+\theta x+\theta y+\theta z)^2}(x+y+z)^2$$

2.

$$\arctan\frac{1+x+y}{1-x+y} = \frac{\pi}{4} + \frac{1+y}{x^2+2y+y^2+1}\Big|_{(0,0)} x + \frac{-x}{x^2+2y+y^2+1}\Big|_{(0,0)} y$$

$$+\frac{1}{2}(x\ y)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o\left(\sqrt{x^2+y^2}\right)$$

$$= \frac{\pi}{4} + x - y - xy + o\left(\sqrt{x^2+y^2}\right)$$

$\exists\, \theta \in (0,1)$

$$\arctan\frac{1+x+y}{1-x+y} = \frac{\pi}{4} + x - y + \frac{1}{2}(x\ y)\begin{pmatrix} \frac{-2\theta x(\theta y+1)}{(\theta^2x^2+\theta^2y^2+2\theta y+1)^2} & \frac{\theta^2x^2-2\theta y-\theta^2y^2-1}{(\theta^2x^2+\theta^2y^2+2\theta y+1)^2} \\ \frac{\theta^2x^2-2\theta y-\theta^2y^2-1}{(\theta^2x^2+\theta^2y^2+2\theta y+1)^2} & \frac{\theta x(2+2\theta y)}{(\theta^2x^2+\theta^2y^2+2\theta y+1)^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

3. $\forall\, P_1(x_1,y_1),\ P_2(x_2,y_2) \in D$

$$|f(x,y_1) - f(x,y_2)| = \left|\int_{y_2}^{y_1}\frac{\partial f}{\partial y}(x,y)\,dy\right| \leq \int_{y_2}^{y_1}\left|\frac{\partial f}{\partial y}(x,y)\right|dy$$

$$= \left|\frac{\partial f}{\partial y}(x, y_1+\theta(y_2-y_1))\right| \cdot |y_1-y_2|$$

$$\leq k_1|y_1-y_2|$$

同理 $|f(x_1,y)-f(x_2,y)| \leq k_2|x_1-x_2|$

$$|f(x_1,y_1) - f(x_2,y_2)| = |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)|$$

$$\leq |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)|$$

$$\leq k_1|y_1-y_2| + k_2|x_1-x_2|$$

$$\leq \sqrt{k_1^2+k_2^2}\cdot \|P_2-P_1\|$$

取 $L = \sqrt{k_1^2+k_2^2}$

4.

$$f(2h, e^{-\frac{1}{2h}}) = f(0,0) + \frac{\partial f}{\partial x}\cdot 2h + \frac{\partial f}{\partial y}\cdot e^{-\frac{1}{2h}}$$

$$+ \underbrace{\frac{1}{2}(2h \quad e^{-\frac{1}{2h}}) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \begin{pmatrix} 2h \\ e^{-\frac{1}{2h}} \end{pmatrix}}_{\frac{1}{2}\left(4h^2 \frac{\partial^2 f}{\partial x^2} + 4h e^{-\frac{1}{2h}} \frac{\partial^2 f}{\partial x \partial y} + e^{-\frac{1}{h}} \frac{\partial^2 f}{\partial y^2}\right)} + o(\sqrt{4h^2 + e^{-\frac{1}{h}}})$$

$$f(h, e^{-\frac{1}{h}}) = f(0,0) + \frac{\partial f}{\partial x}\cdot h + \frac{\partial f}{\partial y} e^{-\frac{1}{h}}$$

$$+ \underbrace{\frac{1}{2}(h \quad e^{-\frac{1}{h}}) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \begin{pmatrix} h \\ e^{-\frac{1}{h}} \end{pmatrix}}_{\frac{1}{2}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2h e^{-\frac{1}{h}} \frac{\partial^2 f}{\partial x \partial y} + e^{-\frac{2}{h}} \frac{\partial^2 f}{\partial y^2}\right)} + o(\sqrt{h^2 + e^{-\frac{2}{h}}})$$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(2h, e^{-\frac{1}{2h}}) - 2f(h, e^{-\frac{1}{h}}) + f(0,0)}{h^2}$$

$$= \lim_{h\to 0^+} \frac{\frac{\partial f}{\partial y}(e^{-\frac{1}{2h}} - 2e^{-\frac{1}{h}}) + h^2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} 2h(e^{-\frac{1}{2h}} - e^{-\frac{1}{h}}) + \frac{\partial^2 f}{\partial y^2}(\frac{1}{2}e^{-\frac{1}{h}} - e^{-\frac{2}{h}})}{h^2}$$

$$= \frac{\partial^2 f}{\partial x^2}$$

6. $z = x^y = e^{y\ln x}$ $\quad (x^y \cdot \frac{y}{x},\ x^y \cdot \ln x) \Rightarrow \left(e^{y\ln x}\frac{1}{x^2}(y^2-y),\ e^{y\ln x}\frac{1}{x}\left(\frac{y}{x}+1\right),\ e^{y\ln x}\ln^2 x\right)$

$\Downarrow$

$$z(1+x, y) = 1 + 0\cdot x + 0\cdot y$$

$$+ \frac{1}{2}(x \quad y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$+ \frac{1}{6}\Big(\underset{\overset{\|}{0}}{\frac{\partial^3 f}{\partial x^3}}\cdot x^3 + 3\underset{\overset{\|}{-1}}{\frac{\partial^3 f}{\partial x^2 \partial y}}x^2 y + 3\underset{\overset{\|}{0}}{\frac{\partial^3 f}{\partial x \partial y^2}}xy^2 + \underset{\overset{\|}{0}}{\frac{\partial^3 f}{\partial y^3}}y^3\Big)$$

$$+ \frac{1}{24}\Big(\underset{\overset{\|}{0}}{\frac{\partial^4 f}{\partial x^4}}x^4 + 4\underset{\overset{\|}{2}}{\frac{\partial^4 f}{\partial x^3 \partial y}}x^3 y + 6\underset{\overset{\|}{2}}{\frac{\partial^4 f}{\partial x^2 \partial y^2}}x^2y^2 + 4\underset{\overset{\|}{0}}{\frac{\partial^4 f}{\partial x \partial y^3}}xy^3 + \underset{\overset{\|}{0}}{\frac{\partial^4 f}{\partial y^4}}y^4\Big) + o(\|\rho\|^4)$$

$$= 1 + xy - \frac{1}{2}x^2 y + \frac{1}{3}x^3 y + \frac{1}{2}x^2y^2 + o(\|\rho\|^4)$$

$$z^{(2,2)}(1,0) = 2$$

$$1.1^{0.2} = (1+0.1)^{0.2} = 1 + 0.1\times 0.2 - \frac{1}{2}0.1^2\, 0.2$$

$$+ \frac{1}{3}0.1^3\, 0.2 + \frac{1}{2}0.1^2\, 0.2^2$$

$$\approx 1.0193$$

7. 设 $A(x_1, y_1)$ $B(x_2, y_2)$ $C(x_3, y_3)$

$$d(x,y) = \frac{1}{3}\left(\sqrt{(x-x_1)^2+(y-y_1)^2} + \sqrt{(x-x_2)^2+(y-y_2)^2} + \sqrt{(x-x_3)^2+(y-y_3)^2}\right)$$

$$\begin{cases} \dfrac{\partial d}{\partial x} = \dfrac{1}{3}\left(\dfrac{x-x_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} + \dfrac{x-x_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} + \dfrac{x-x_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}}\right) = 0 \\[2ex] \dfrac{\partial d}{\partial y} = \dfrac{1}{3}\left(\dfrac{y-y_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} + \dfrac{y-y_2}{\sqrt{(x-x_2)^2+(y-y_2)^2}} + \dfrac{y-y_3}{\sqrt{(x-x_3)^2+(y-y_3)^2}}\right) = 0 \end{cases}$$

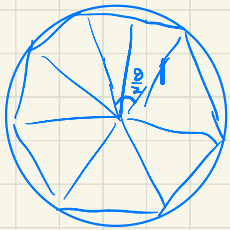
可通过最速下降法：

$$\vec{p} = -\mathrm{grad}\, d(\vec{p})$$

不断进行迭代找出解

$\Rightarrow$ 即 ΔABC 的费马点

8.



设每个等腰△顶角为 $\theta_i \in (0, \pi)$

$$S = \sum_{i=1}^{n} \sin\frac{\theta_i}{2}\cos\frac{\theta_i}{2} \qquad \sum_{i=1}^{n}\theta_i = 2\pi$$

$$\Rightarrow S(\theta_1, \theta_2, \cdots \theta_{n-1}) = \sin\left(\pi - \frac{\sum_{i=1}^{n-1}\theta_i}{2}\right)\cos\left(\pi - \frac{\sum_{i=1}^{n-1}\theta_i}{2}\right) + \sum_{i=1}^{n-1}\sin\frac{\theta_i}{2}\cos\frac{\theta_i}{2}$$

$\frac{1}{2}\sin(2\pi - \sum\theta_i)$

$-\frac{1}{2}\cos(2\pi - \sum\theta_i)\cdot$

$= -\frac{1}{2}\cos(\sum\theta_i)$

$$\frac{\partial S}{\partial \theta_i} = \frac{1}{2}\cos^2\frac{\theta_i}{2} - \frac{1}{2}\sin^2\frac{\theta_i}{2} + \cos^2\left(\pi - \frac{\sum_{i=1}^{n-1}\theta_i}{2}\right)\cdot\left(-\frac{1}{2}\right) - \sin^2\left(\pi - \frac{\sum_{i=1}^{n-1}\theta_i}{2}\right)\cdot\left(-\frac{1}{2}\right)$$

$i = 1, 2, \cdots, n-1$

$$= \frac{1}{2}\cos\theta_i - \frac{1}{2}\cos\left(2\pi - \sum_{i=1}^{n-1}\theta_i\right)$$

$$= \frac{1}{2}\cos\theta_i - \frac{1}{2}\cos\left(\sum_{i=1}^{n-1}\theta_i\right) = 0$$

$$\Rightarrow \theta_i + \sum_{j=1}^{n-1}\theta_j = 2\pi \quad \text{or} \quad \theta_i + \sum_{j=1}^{n-1}\theta_j = 0 \ \times$$

$$\theta_i + 2\pi - \theta_n = 2\pi$$

$$\theta_i = \theta_n$$

$$\forall i = 1, 2, \cdots, n-1$$

$$\Rightarrow \theta_1 = \theta_2 = \cdots = \theta_n = \frac{2\pi}{n}$$

$$\Rightarrow S_{极大值} = n\cdot\frac{1}{2}\sin\frac{2\pi}{n} = S_{max}$$

9. $xf_x' + yf_y' > 0$

$\theta \to 0^+$

$$f(x+\theta x, y+\theta y) = f(x,y) + \underbrace{\theta x f_x'(x,y) + \theta y f_x'(x,y)}_{\theta(xf_x' + yf_y') > 0} + o\left(\sqrt{\theta^2 x^2 + \theta^2 y^2}\right)$$

$\Rightarrow f(x+\theta x, y+\theta y) > f(x,y)$

$\Rightarrow \forall (x,y) \neq (0,0)$ $f(x,y) > f(0,0)$

$\Rightarrow f(0,0)$ 为唯一最小值点

固定 $y=0$

$$f(x,0) = f(0,0) + \underbrace{f_x'(x,0)\cdot x}_{>0} + o(x)$$

$$\Rightarrow f_x'(x,0) \begin{cases} >0 & (x>0) \\ <0 & (x<0) \end{cases}$$

又 $f \in C^1(\mathbb{R}^2)$

$\Rightarrow \lim\limits_{x\to 0} f_x'(x,0) = 0 = f_x'(0,0)$

同理 $f_y'(0,0) = 0$

$\Rightarrow f(x,y) = f(0,0) + o\left(\sqrt{x^2+y^2}\right)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0)}{\sqrt{x^2+y^2}} = 0$

11.

$$\int f''_{xy}(x,y)\,dy = 2\left(\frac{1}{2}y^2+y\right)e^x + C(x)$$
$$= y^2e^x + 2ye^x + C(x)$$

取 $y=0 \Rightarrow f'_x(x,0) = (x+1)e^x = C(x)$

$$\Rightarrow f'_x(x,y) = e^x(y^2+2y+x+1)$$

$$\int f'_x(x,y)\,dx = e^x(y^2+2y+1) + e^x(x-1) + C(y)$$

取 $x=0 \Rightarrow f(0,y) = y^2+2y = y^2+2y+1-1+C(y)$

$C(y)=0$

$$\Rightarrow f(x,y) = e^x(y^2+2y+x)$$

$\forall$ 固定 x,

$$f(x,y) \underset{y=-1}{\geq} e^x(x-1) \underset{x=0}{\geq} -1$$

$\Rightarrow f(x,y) \in [-$