11.1

1. $(k\vec{v})\otimes\vec{w}\,(\alpha,\beta)=\alpha(k\vec{v})\beta(\vec{w})=k\alpha(\vec{v})\beta(\vec{w})$

$\vec{v}\otimes(k\vec{w})(\alpha,\beta)=\alpha(\vec{v})\beta(k\vec{w})=k\alpha(\vec{v})\beta(\vec{w})$

$k(\vec{v}\otimes\vec{w})=k\cdot\alpha(\vec{v})\beta(\vec{w})$

$\Rightarrow (k\vec{v})\otimes\vec{w}=\vec{v}\otimes(k\vec{w})=k(\vec{v}\otimes\vec{w})$

2. $(\vec{v_1}+\vec{v_2})\otimes\vec{w}\,(\alpha,\beta)=\alpha(\vec{v_1}+\vec{v_2})\beta(\vec{w})$

$=(\alpha(\vec{v_1})+\alpha(\vec{v_2}))\beta(\vec{w})$

$=\alpha(\vec{v_1})\beta(\vec{w})+\alpha(\vec{v_2})\beta(\vec{w})$

$=\vec{v_1}\otimes\vec{w}(\alpha,\beta)+\vec{v_2}\otimes\vec{w}(\alpha,\beta)$

$\Rightarrow(\vec{v_1}+\vec{v_2})\otimes\vec{w}=\vec{v_1}\otimes\vec{w}+\vec{v_2}\otimes\vec{w}$

3.
$\vec{v}\otimes\vec{w}\,([x_1,y_1],[x_2,y_2])=(x_1+y_1)(x_2+2y_2)$

$\vec{w}\otimes\vec{v}\,([x_1,y_1],[x_2,y_2])=(x_1+2y_1)(x_2+y_2)$

$\Rightarrow \vec{v}\otimes\vec{w}\neq\vec{w}\otimes\vec{v}$

4. $(\vec{v_1}\otimes\vec{w_1}-\vec{w_1}\otimes\vec{v_1})(\alpha,\beta)$

$=\alpha(\vec{v_1})\beta(\vec{w_1})-\alpha(\vec{w_1})\beta(\vec{v_1})$ ①

$(\vec{v_2}\otimes\vec{w_2}-\vec{w_2}\otimes\vec{v_2})(\alpha,\beta)$

$=\alpha(\vec{v_2})\beta(\vec{w_2})-\alpha(\vec{w_2})\beta(\vec{v_2})$ ②

if $[\vec{v_1}\ \vec{w_1}]=[\vec{v_2}\ \vec{w_2}]\begin{bmatrix}a&b\\c&d\end{bmatrix}$
where $ad-bc=1$

$\Rightarrow$ ① $=\alpha(a\vec{v_2}-c\vec{w_2})\beta(b\vec{v_2}-d\vec{w_2})-\alpha(b\vec{v_2}-d\vec{w_2})\beta(a\vec{v_2}-c\vec{w_2})$

$=(ab-ba)\alpha(\vec{v_2})\beta(\vec{v_2})+(ad-bc)\alpha(\vec{v_2})\beta(\vec{w_2})+(bc-ad)\alpha(\vec{w_2})\beta(\vec{v_2})$
$+(cd-cd)\alpha(\vec{w_2})\beta(\vec{w_2})$

$=\alpha(\vec{v_2})\beta(\vec{w_2})-\alpha(\vec{w_2})\beta(\vec{v_2})$

$=$ ②

11.2

1. $X = 2\vec{e}_2^T \otimes \vec{e}_1 + \vec{e}_1^T \otimes \vec{e}_2$

$Y = \vec{e}_1^T \otimes \vec{e}_1 + \vec{e}_2^T \otimes \vec{e}_2 + 2\vec{e}_3^T \otimes \vec{e}_3$

2. $X \otimes Y = 2\vec{e}_2^T \otimes \vec{e}_1 \otimes \vec{e}_1^T \otimes \vec{e}_1 + 2\vec{e}_2^T \otimes \vec{e}_1 \otimes \vec{e}_2^T \otimes \vec{e}_2 + 4\vec{e}_2^T \otimes \vec{e}_1 \otimes \vec{e}_3^T \otimes \vec{e}_3$

$+ \vec{e}_1^T \otimes \vec{e}_2 \otimes \vec{e}_1^T \otimes \vec{e}_1 + \vec{e}_1^T \otimes \vec{e}_2 \otimes \vec{e}_1^T \otimes \vec{e}_1 + 2\vec{e}_1^T \otimes \vec{e}_2 \otimes \vec{e}_3^T \otimes \vec{e}_3$

3.
$$X \otimes Y = \begin{array}{c} \begin{array}{cccccc} 1,1 & 1,2 & 1,3 & 2,1 & 2,2 & 2,3 \end{array} \\ \begin{pmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 2 \\ 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \end{pmatrix} \end{array} = \begin{pmatrix} & Y \\ 2Y & \end{pmatrix} \rightarrow \text{arrange } Y \text{ like entries of } X$$

11.3

1. $(\alpha \otimes \beta \otimes \gamma)_{ijk} = \alpha_i \cdot \beta_j \cdot \gamma_k$

2. $k(\alpha \otimes \beta \otimes \gamma) \xrightarrow{M_E} k((\alpha E) \otimes \beta \otimes \gamma)$

$\alpha_1 \otimes \beta \otimes \gamma + \alpha_2 \otimes \beta \otimes \gamma \xrightarrow{M_E} (\alpha_1 + \alpha_2)E \otimes \beta \otimes \gamma$

$= (\alpha_1 E) \otimes \beta \otimes \gamma + (\alpha_2 E) \otimes \beta \otimes \gamma$

3. $\sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i \xrightarrow{M_E} \sum_{i=1}^{r} (\alpha_i E) \otimes \beta_i \otimes \gamma_i$

the operation doesn't affect linearly independency

4. $M = \alpha \otimes \beta \otimes \gamma \overset{\text{suppose}}{=} \alpha \otimes \sum_{i=1}^{r} \beta_i \otimes \gamma_i \rightarrow \text{rank}(M) = r$

$\overset{\text{suppose(2)}}{=} \sum_{i=1} \alpha_i \otimes \beta_i \otimes \gamma_i = \alpha_1 \otimes \sum_{j=1}^{r} \beta_j \otimes \gamma_j + \sum_{i=2} \alpha_i \otimes \beta_i \otimes \gamma_i \Rightarrow \text{rank}(M) \geq r$

$\Rightarrow M$ at least has rank r

5. $M = \frac{1}{2}[1,1] \otimes [1,1] \otimes [1,1] + \frac{1}{2}[1,-1] \otimes [1,-1] \otimes [1,-1]$

$\Rightarrow \text{rank}(M) = 2$

11.4

1. $M = \begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ 7 & 8 & 9 \end{pmatrix}$

$\qquad\qquad A_1 \qquad\qquad A_2 \qquad\qquad A_3$

$M(\vec{v},\vec{v},\vec{v}) = [\vec{v}^T A_1 \vec{v} \quad \vec{v}^T A_2 \vec{v} \quad \vec{v}^T A_3 \vec{v}]\,\vec{v}$

$= 3x^3 + 6y^3 + 9z^3$

$+ 12x^2y + 15x^2z + 15y^2x + 21y^2z + 21z^2x + 24z^2y$

$+ 36xyz$

2. $M(\vec{v_1}, \vec{v_2}, \vec{v_3}) = [\vec{v_1}^T A_1 \vec{v_2} \quad \vec{v_1}^T A_2 \vec{v_2} \quad \vec{v_1}^T A_3 \vec{v_2}]\,\vec{v_3}$

since $A_1 \sim A_3$ are symmetric $\Rightarrow$ $\vec{v_1}$ & $\vec{v_2}$ can permute in any way

then if permute $\vec{v_3}$ with $\vec{v_2}$, suppose $\vec{v_1}^T A_i = [a_i \; b_i \; c_i]$

$M(\vec{v_1}, \vec{v_2}, \vec{v_3}) = [a_1 v_{21} + b_1 v_{22} + c_1 v_{23} \quad a_2 v_{21} + b_2 v_{22} + c_2 v_{23} \quad a_3 v_{21} + b_3 v_{22} + c_3 v_{23}] \cdot \vec{v_3}$

$= (a_1 v_{21} + b_1 v_{22} + c_1 v_{23}) \cdot v_{31} + (a_2 v_{21} + b_2 v_{22} + c_2 v_{23}) \cdot v_{32} + (a_3 v_{21} + b_3 v_{22} + c_3 v_{23}) v_{33}$

$M(\vec{v_1}, \vec{v_3}, \vec{v_2}) = [a_1 v_{31} + b_1 v_{32} + c_1 v_{33} \quad a_2 v_{31} + b_2 v_{32} + c_2 v_{33} \quad a_3 v_{31} + b_3 v_{32} + c_3 v_{33}]\,\vec{v_2}$

$= (a_1 v_{31} + b_1 v_{32} + c_1 v_{33}) \cdot v_{21} + (a_2 v_{31} + b_2 v_{32} + c_2 v_{33}) v_{22} + (a_3 v_{31} + b_3 v_{32} + c_3 v_{33}) v_{23}$

$\Rightarrow M(\vec{v_1}, \vec{v_2}, \vec{v_3}) = M(\vec{v_1}, \vec{v_3}, \vec{v_2})$

So with any permutation $\sigma: \{1,2,3\} \to \{1,2,3\}$, $M(\vec{v_1}, \vec{v_2}, \vec{v_3}) = M(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}, \vec{v}_{\sigma(3)})$

3. $\text{rank} \begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{pmatrix} = 2$

$\Rightarrow \text{rank}(M) \geq 2$

and since $M(\vec{v_1}, \vec{v_2}, \vec{v_3}) = M(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}, \vec{v}_{\sigma(3)})$

so $\dim \text{Dom}(M) = 3$ (since the position of 3 inputs of M doesn't matter)

$\Rightarrow \dim \text{Ran}(M) \leq 3$

$\Rightarrow \text{rank}(M) \leq 3$

so $2 \leq \text{rank}(M) \leq 3$