12.1

1. $T = \vec{p_1} \otimes \vec{e_1} + \vec{p_2} \otimes \vec{e_2} + \vec{p_3} \otimes \vec{v}$

$= -\vec{e_1} \otimes \vec{e_1} + (-\vec{e_2}) \otimes \vec{e_2} + \frac{\sqrt{2}}{2}(\vec{e_1} + \vec{e_2}) \otimes (-\vec{e_1} - \vec{e_2})$

$= -\vec{e_{11}} - \vec{e_{22}} - \frac{\sqrt{2}}{2}(\vec{e_{11}} + \vec{e_{12}} + \vec{e_{21}} + \vec{e_{22}})$

$= (-1 - \frac{\sqrt{2}}{2})\vec{e_{11}} + (-1 - \frac{\sqrt{2}}{2})\vec{e_{22}} - \frac{\sqrt{2}}{2}\vec{e_{12}} - \frac{\sqrt{2}}{2}\vec{e_{21}}$

2. $\vec{x} = \frac{\sqrt{2}}{2}\vec{e_1} + \frac{\sqrt{2}}{2}\vec{e_2}$

$\vec{y} = \frac{\sqrt{2}}{2}\vec{e_1} - \frac{\sqrt{2}}{2}\vec{e_2}$

$\Rightarrow T = (-1-\sqrt{2})\vec{x} \otimes \vec{x} - \vec{y} \otimes \vec{y}$

3. along $\vec{x}$ and $\vec{y}$

$\uparrow$ long axis $\quad \uparrow$ short axis

4. $\vec{x} = \cos\theta\, \vec{e_1} + \sin\theta\, \vec{e_2}$

$\Rightarrow \vec{x} \otimes \vec{x} = \cos^2\theta\, \vec{e_{11}} + \sin\theta\cos\theta\, \vec{e_{12}} + \sin\theta\cos\theta\, \vec{e_{21}} + \sin^2\theta\, \vec{e_{22}}$

$T = \sum a_i \vec{x_i} \otimes \vec{x_i}$

so for T, coefficients for $\vec{e_{12}}$ and $\vec{e_{21}}$ are the same

$\Rightarrow$ pick $T = \vec{e_{12}}$, which can't be obtained via squeezing the circle perpendicularly

12.2

1. for $\mathrm{Alt}(\vec{u}\otimes\vec{v}\otimes\vec{w})$, $\vec{u}, \vec{v}, \vec{w}$ are equivalent, so just verify $\vec{u}$ is enough and since "$\otimes$" operator is bilinear for its left and right input. so $(k\vec{u})$ and $(\vec{u}_1 + \vec{u}_2)$ as input can be distribute by "$\otimes$", which ensures multilinearity for Alt( )

2. if swapping any pair of inputs,

   like $\vec{a}^T, \vec{b}^T, \vec{c}^T \to \vec{b}^T, \vec{a}^T, \vec{c}^T$

   then for $\mathrm{sign}(\sigma)\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}, \vec{v}_{\sigma(3)}$, output:

$$\mathrm{sign}(\sigma)(\vec{a}^T\vec{v}_{\sigma(1)})\cdot(\vec{b}^T\vec{v}_{\sigma(2)})\cdot(\vec{c}^T\vec{v}_{\sigma(3)}) \to \mathrm{sign}(\sigma)(\vec{b}^T\vec{v}_{\sigma(1)})(\vec{a}^T v_{\sigma(2)})(\vec{c}^T v_{\sigma(3)})$$

   which is the same as change permutation for $\vec{v}$s, and that'll negate the output

3. since T is already alternating:

   for even permutation: $(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}, \vec{v}_{\sigma(3)}) = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$, $\mathrm{sign}(\sigma) = 1$

   for add permutation $(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}, \vec{v}_{\sigma(3)}) = -(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, $\mathrm{sign}(\sigma) = -1$

so $\mathrm{Alt}(T) = \frac{1}{k!}\sum_\sigma \vec{v}_1\otimes\vec{v}_2\otimes\vec{v}_3 = \vec{v}_1\otimes\vec{v}_2\otimes\vec{v}_3 = T$

12.3

1. $e^1 \otimes e^2 (e_1 \otimes e_2) = e^1 e_1 \cdot e^2 e_2 = 1$

2. $\mathrm{Alt}(e^1 \otimes e^2)(e_1 \otimes e_2) = \frac{1}{2}(e^1 \otimes e^2)(e_1 \otimes e_2) - \frac{1}{2}(e^2 \otimes e^1)(e_1 \otimes e_2)$
$= \frac{1}{2} - 0 = \frac{1}{2}$

3. $e^1 \otimes e^2 (\mathrm{Alt}(e_1 \otimes e_2)) = \frac{1}{2} e^1 \otimes e^2 (e_1 \otimes e_2) - \frac{1}{2} e^1 \otimes e^2 (e_2 \otimes e_1)$
$= \frac{1}{2} - 0 = \frac{1}{2}$

4. $\mathrm{Alt}(e^1 \otimes e^2)(\mathrm{Alt}(e_1 \otimes e_2))$
$= \frac{1}{4} e^1 \otimes e^2 (e_1 \otimes e_2) - \frac{1}{4} e^1 \otimes e^2 (e_2 \otimes e_1) - \frac{1}{4} e^2 \otimes e^1 (e_1 \otimes e_2) + \frac{1}{4} e^2 \otimes e^1 (e_2 \otimes e_1)$
$= \frac{1}{4} - 0 - 0 + \frac{1}{4}$
$= \frac{1}{2}$

5. $e^1 \wedge e^2 (\mathrm{Alt}(e_1 \otimes e_2)) = (e^1 \otimes e^2 - e^2 \otimes e^1)(\mathrm{Alt}(e_1 \otimes e_2))$
$= 2\,\mathrm{Alt}(e^1 \otimes e^2)(\mathrm{Alt}(e_1 \otimes e_2))$
$= 2 \times \frac{1}{2} = 1$

6. $e^1 \wedge e^2 (e_1 \otimes e_2) = 2\,\mathrm{Alt}(e^1 \otimes e^2)(e_1 \otimes e_2)$
$= 2 \times \frac{1}{2}$
$= 1$

7. $e^1 \otimes e^2 (e_1 \wedge e_2) = 2 e^1 \otimes e^2 (\mathrm{Alt}(e_1 \otimes e_2))$
$= 2 \times \frac{1}{2}$
$= 1$

8. $e^1 \wedge e^2 (e_1 \wedge e_2) = 4\,\mathrm{Alt}(e^1 \otimes e^2)(\mathrm{Alt}(e_1 \otimes e_2)) = 4 \times \frac{1}{2} = 2$

9. $\langle e_1 \otimes e_2, e_1 \otimes e_3 \rangle = \langle e_1, e_1 \rangle \langle e_2, e_3 \rangle = 0$

$\langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle = \langle e_1 \otimes e_2, e_1 \wedge e_3 \rangle - \langle e_2 \otimes e_1, e_1 \wedge e_3 \rangle$
$= \langle e_1 \otimes e_2, e_1 \otimes e_3 \rangle - \langle e_1 \otimes e_2, e_3 \otimes e_1 \rangle$
$- \langle e_2 \otimes e_1, e_1 \otimes e_3 \rangle + \langle e_2 \otimes e_1, e_3 \otimes e_1 \rangle$
$= 0 - 0 - 0 + 0 = 0$

10. $\| e_1 \wedge e_2 \wedge e_3 \|^2 = \langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3 \rangle = 6$
$\Rightarrow \| e_1 \wedge e_2 \wedge e_3 \| = \sqrt{6}$
$\| \mathrm{Alt}(e_1 \otimes e_2 \otimes e_3) \|^2 = \| \frac{1}{6} e_1 \wedge e_2 \wedge e_3 \|^2 = \frac{1}{36} \times 6 = \frac{1}{6}$
$\Rightarrow \| \mathrm{Alt}(e_1 \otimes e_2 \otimes e_3) \| = \frac{\sqrt{6}}{6}$
$\| \mathrm{Alt}(e_1 \otimes e_2) \wedge e_3 \| = \frac{1}{2} \| (e_1 \wedge e_2) \wedge e_3 \| = \frac{\sqrt{6}}{2}$