李昊伦 2022011545

1.

(a) on the same row

$$A = \begin{bmatrix} & 1 & & 1 & \\ 1 & & & & 1 \\ 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 & \\ & 1 & & 1 & \end{bmatrix}$$

so there's no non-zero terms in the big formula

$\Rightarrow \det(A) = 0$

(b) $\operatorname{rank}(A) = 3$

$\Rightarrow \dim\ker(A) = 2$

since $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \in \ker(A)$, and they're linearly independent

so they form a basis for $\ker(A)$

(c) let $\vec{v} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \Rightarrow A\vec{v} = \lambda\vec{v}$

$$\begin{pmatrix} b+d \\ a+e \\ 0 \\ b+c+d \\ b+d \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \\ \lambda e \end{pmatrix}$$

$\Rightarrow c = 0$

$$\begin{pmatrix} b+d \\ a+e \\ 0 \\ b+d \\ b+d \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ 0 \\ \lambda d \\ \lambda e \end{pmatrix} \Rightarrow d = e = a = \frac{1}{\lambda}(b+d)$$

(d) ~~$\det(A-\lambda I) = \lambda^4(1-\lambda)$~~

eigenvalues of A:	-1	0	2
algebraic multiplicities:	1	3	1
geometric ------ :	1	2	1
eigenvectors:	$\operatorname{span}\begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}$	$\operatorname{span}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$	$\operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

A can't be diagonalized.

2.

(a) $\mathrm{Ran}(A)$: $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ & $\begin{pmatrix}1\\1\\3\end{pmatrix}$ $\qquad$ $\mathrm{Ker}(A^T)$: $\begin{pmatrix}-7\\4\\1\end{pmatrix}$

$\mathrm{Ker}(A)$: $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$ $\qquad$ $\mathrm{Ran}(A^T)$: $\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

(b)

since $A\vec{x}=\vec{0}$ & $AR\vec{x}=\vec{0}$ have same solution set

so $\forall \vec{x}\in \mathrm{Ker}(A)$, $R\vec{x}\in \mathrm{Ker}(A)$

$\Rightarrow$ $\mathrm{Ker}(A)$ is R-invariant

Also, $\mathrm{Ran}(A^T) \perp \mathrm{Ker}(A)$

so $\mathrm{Ran}(A^T)$ is also R-invariant

$\mathrm{Ran}(A) \neq \mathrm{Ran}(RA)$

$\underset{\dim \doteq 2}{\downarrow}$

so $\mathrm{Ran}(A)$ is not R-invariant

Also, $\mathrm{Ran}(A) \perp \mathrm{Ker}(A^T)$

so $\mathrm{Ker}(A^T)$ is also not R-invariant

(c)

$\mathrm{Ker}(A)$: $Q_1 = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}$

$P_1 = Q_1Q_1^T = \begin{pmatrix}\frac12 & \frac12 & 0 & 0\\ \frac12 & \frac12 & 0 & 0\\ 0 & 0 & \frac12 & -\frac12\\ 0 & 0 & -\frac12 & \frac12\end{pmatrix}$

$\mathrm{Ran}(A^T)$: $Q_2 = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ \frac{-1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$

$P_2 = Q_2Q_2^T = \begin{pmatrix}\frac12 & -\frac12 & 0 & 0\\ -\frac12 & \frac12 & 0 & 0\\ 0 & 0 & \frac12 & \frac12\\ 0 & 0 & \frac12 & \frac12\end{pmatrix}$

$P_1+P_2 = \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix} = I_{4\times4}$

(d) $\begin{cases} R^2-I \text{ is invertible} \\ R^4-I=0 \end{cases} \Rightarrow \begin{cases} \lambda^2-1\neq 0 \\ \lambda^4-1=0 \end{cases}$

$\Rightarrow \lambda^2=-1$

$\lambda = \pm i$

R is a 4×4 matrix and $\det(R) = \lambda_1\lambda_2\lambda_3\lambda_4 = 1$

$\Rightarrow \lambda$: $i, i, -i, -i$

~~$R = X\begin{pmatrix}i & & & \\ & i & & \\ & & -i & \\ & & & -i\end{pmatrix}X^{-1}$ for some invertible 4x4 X~~

since $\mathrm{tr}(R)=0 \Rightarrow 2+2\cos\theta=0$

$\boxed{\cos\theta=-1}$

$\theta=\pi$

$\boxed{\sin\theta=0}$

compose them, then get all possible R

$\boxed{R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1\end{pmatrix} \text{ or } \begin{pmatrix}\cos\theta & & -\sin\theta & \\ & 1 & & \\ \sin\theta & & \cos\theta & \\ & & & 1\end{pmatrix} \text{ or } \begin{pmatrix}\cos\theta & & & -\sin\theta\\ & 1 & & \\ & & 1 & \\ \sin\theta & & & \cos\theta\end{pmatrix} \text{ or } \begin{pmatrix}1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1\end{pmatrix} \text{ or } \begin{pmatrix}1 & & & \\ & 1 & & \\ & & \cos\theta & -\sin\theta\\ & & \sin\theta & \cos\theta\end{pmatrix}}$

3.

(a)

$$1 \xrightarrow{L} 0$$

$$x \qquad x$$

$$x^2 \qquad 2x^2+2x$$

$$x^3 \qquad 3x(x+1)^2 = 3x(x^2+2x+1) = 3x^3+6x^2+3x$$

$$\Rightarrow L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(b) $p_L(x) = \det(L - xI) = -x(1-x)(2-x)(3-x)$

eigenvalues of L: $0, 1, 2, 3$

eigen-polynomials: $\underset{p_1}{1}, \underset{p_2}{x}, \underset{p_3}{2x+x^2}, \underset{p_4}{\frac{15}{2}x+6x^2+x^3}$

(c) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ under $\mathcal{C} \Rightarrow p_1+p_2+p_3+p_4 = 1+x+2x+x^2+\frac{15}{2}x+6x^2+x^3$

$$= 1+\frac{21}{2}x+7x^2+x^3$$

$$\Rightarrow \begin{pmatrix} 1 \\ \frac{21}{2} \\ 7 \\ 1 \end{pmatrix} \text{ under } \mathcal{B}$$

(d) change of coordinate matrix$_{\mathcal{C}\to\mathcal{B}}$: $X = \cancel{\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{15}{2} & 2 & 1 & 0 \\ 6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ & 1 & 2 & \frac{15}{2} \\ & & 1 & 6 \\ & & & 1 \end{pmatrix}$ $X^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & -2 & \frac{9}{2} \\ & & 1 & -6 \\ & & & 1 \end{pmatrix}$

$$\Rightarrow T_{\mathcal{C}} = X^{-1} T_{\mathcal{B}} X = \begin{pmatrix} 1 & & & \\ & 1 & -2 & \frac{9}{2} \\ & & 1 & -6 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 2 & 10 & 75 \\ & & 5 & 60 \\ & & & 10 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 5 & \\ & & & 10 \end{pmatrix}$$

(e)

$$1 \xrightarrow{T} 1$$

$$x \qquad 2x$$

$$2x+x^2 \qquad 5(2x+x^2)$$

$$\frac{15}{2}x+6x^2+x^3 \qquad 10(\frac{15}{2}x+6x^2+x^3)$$

T: $\cancel{f(x)} \to \cancel{2x f'(x)+f(x)}$

4.

(a) $G^n = (A^TA)^n = A^T(AA^T)^{n-1}A$ $\qquad AA^T = \begin{bmatrix}10 & 0\\ 0 & 34\end{bmatrix}$

$$= \begin{bmatrix}-2 & 4\\ -1 & 1\\ 0 & 0\\ 1 & 1\\ 2 & 4\end{bmatrix} \times \begin{pmatrix}10^{n-1} & \\ & 34^{n-1}\end{pmatrix} \times \begin{bmatrix}-2 & -1 & 0 & 1 & 2\\ 4 & 1 & 0 & 1 & 4\end{bmatrix}$$

(b) $$\det\begin{pmatrix}I & A\\ A^T & 2G-I\end{pmatrix} = \det\begin{pmatrix}I & 0\\ A^T & 2G-I-A^TA\end{pmatrix} = \det\begin{pmatrix}I_{2\times2} & \\ A^T & G-I_{5\times5}\end{pmatrix} = \det(I)\det(G-I)$$

$$= \det(G-I) = -\det(I-G)$$

$$= -\det(I-A^TA) = -\det(I+(-A^T)\cdot A) = -\det(I+A\cdot(-A^T))$$

$$= -\det(I-AA^T) = -\det\begin{pmatrix}-9 & \\ & -33\end{pmatrix} = -9\times33 = -297$$

$$\begin{pmatrix}I & A\\ A^T & 2G-I\end{pmatrix} = \begin{pmatrix}I & \\ A^T & 2G-I\end{pmatrix}\begin{pmatrix}I & A\\ & I\end{pmatrix} = \begin{pmatrix}I & \\ A^T & I\end{pmatrix}\begin{pmatrix}I_{2\times2} & \\ & G-I_{5\times5}\end{pmatrix}\begin{pmatrix}I & A\\ & I\end{pmatrix}$$

so $\begin{pmatrix}I & A\\ A^T & 2G-I\end{pmatrix}$ is positive semi-definite $\Longleftrightarrow$ $G-I$ is positive semi-definite

$\overset{\|}{A^TA-I} \longrightarrow$ ~~symmetric~~ symmetric

$\Updownarrow$ $\quad \vec{v}^TA^TA\vec{v} - \vec{v}^T\vec{v} \geq 0$

$\forall \vec{v},\ |A\vec{v}| \geq |\vec{v}|$

but if $\vec{v} \in \ker(A)$

then $|A\vec{v}| = 0 < |\vec{v}|$

so it's not positive semi-definite

(c) $$G-10I = \begin{pmatrix}10 & 6 & 0 & 2 & 12\\ 6 & -8 & 0 & 0 & 2\\ 0 & 0 & -10 & 0 & 0\\ 2 & 0 & 0 & -8 & 6\\ 12 & 2 & 0 & 6 & 10\end{pmatrix}$$

eigenvalues for AA^T: 10, 34

$\Rightarrow$ eigenvalues for A^TA: 10, 34, 0, 0, 0

...... $A^TA-10I = G-10I$: 0, 24, -10, -10, -10

$$\Rightarrow \frac{\vec{u}^T(G-10I)\vec{u}}{\vec{u}^T\vec{u}} \in [-10, 24]$$

min: $\vec{u} = \begin{pmatrix}1\\4\\0\\4\\1\end{pmatrix}$

max: $\vec{u} = \begin{pmatrix}4\\1\\0\\4\end{pmatrix}$

(d) $G\vec{v} = \lambda_{max}\vec{v}$ $\qquad (G-10I)\vec{v} = \overset{24}{\overset{\|}{(\lambda_{max}-10)}}\vec{v}$

$G\vec{w} = \vec{0}$ $\qquad (G-10I)\vec{w} = -10\vec{w}$

since $\vec{w} \in \ker(A)$

$\Rightarrow \vec{w} \perp \underset{\sigma_1}{\vec{v}_1}, \underset{\sigma_2}{\vec{v}_2}$

$$\Rightarrow \frac{(\vec{v}+\vec{w})^T(G-10I)(\vec{v}+\vec{w})}{(\vec{v}+\vec{w})^T(\vec{v}+\vec{w})} = 24-10 = 14$$

5.

(a)

$$A_x = \begin{pmatrix} 1 & & & \\ 1 & x-1 & & x-1 \\ 1 & 0 & x-1 & x-1 \\ 1 & x-1 & x-1 & \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ 1 & x-1 & & \\ 1 & & x-1 & x-1 \\ 1 & x-1 & x-1 & 1-x \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ 1 & x-1 & & \\ 1 & & x-1 & \\ 1 & x-1 & x-1 & 2-2x \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 1 & & & \\ & x-1 & & \\ & & x-1 & \\ & & & 2-2x \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}}_{L^T}$$

$\det(A_x) = \det(D) = -2(x-1)^3 = 2(1-x)^3$

A_x is invertible if $x \neq 1$

(b).

$$A_x = \begin{pmatrix} 1 & \frac{1-x}{2} & \frac{1-x}{2} & \frac{1-x}{2} \\ 1 & \frac{x-1}{2} & \frac{1-x}{2} & \frac{x-1}{2} \\ 1 & \frac{1-x}{2} & \frac{x-1}{2} & \frac{x-1}{2} \\ 1 & \frac{x-1}{2} & \frac{x-1}{2} & \frac{1-x}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{x+1}{2} & \frac{x+1}{2} & \frac{x+1}{2} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & x+1 & x+1 & x+1 \\ & 1-x & & \\ & & 1-x & \\ & & & 1-x \end{pmatrix}.$$

(c) eigenvalues of A_x: $\begin{cases} x-1. \\ 1-x \\ a \\ b \end{cases}$

$$\Rightarrow \begin{cases} a+b = 2x+2 \\ ab = 2(x-1) \end{cases}$$

① $x>1$ ② $x=1$ ③ $-1<x<1$

3 positive 1 positive 2 positive

④ $x=-1$ ⑤ $x<-1$

2 positive 2 positive

(d)