5.1

1. $tr(xA+yB)=\sum_{i=1}^{n} \vec{e_i}^T (xA+yB)\vec{e_i}$

$= \sum_{i=1}^{n} (\vec{e_i}^T xA\vec{e_i} + \vec{e_i}^T yB\vec{e_i})$

$= x\sum_{i=1}^{n} \vec{e_i}^T A\vec{e_i} + y\sum_{i=1}^{n} \vec{e_i}^T B\vec{e_i}$

$= x\,tr(A) + y\,tr(B)$

2. $tr(I_n) = \sum_{i=1}^{n} \vec{e_i}^T I_n \vec{e_i} = \sum_{i=1}^{n} \vec{e_i}^T \cdot \vec{e_i} = \sum_{i=1}^{n} 1 = n$

3. $tr(A^T) = \sum_{i=1}^{n} \vec{e_i}^T A^T \vec{e_i} = \sum_{i=1}^{n} (A\vec{e_i})^T (\vec{e_i}^T)^T = \sum_{i=1}^{n} (\vec{e_i}^T A\vec{e_i})^T = \sum_{i=1}^{n} \vec{e_i}^T A\vec{e_i} = tr(A)$

4. $tr(\vec{u}\vec{u}^T) = \sum_{i=1}^{n} \vec{e_i}^T \vec{u}\vec{u}^T \vec{e_i} = \sum_{i=1}^{n} (\vec{e_i}^T\vec{u})(\vec{e_i}^T\vec{u})^T = \sum_{i=1}^{n} (\vec{e_i}^T\vec{u})^2 = |\vec{u}| = 1$

$tr(I_n - \vec{u}\vec{u}^T) = tr(I_n) - tr(\vec{u}\vec{u}^T) = n-1$

5. $tr(A^T B) = \sum_{i=1}^{2} \vec{e_i}^T A^T B\vec{e_i} = \vec{e_1}^T A^T B\vec{e_1} + \vec{e_2}^T A^T B\vec{e_2}$

$= [a_1\ a_3]\begin{bmatrix} b_1 \\ b_3 \end{bmatrix} + [a_2\ a_4] + \begin{bmatrix} b_2 \\ b_4 \end{bmatrix}$

$= \sum a_i b_i$

$tr(BA^T) = \vec{e_1}^T BA^T\vec{e_1} + \vec{e_2}^T BA^T \vec{e_2} = [b_1\ b_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + [b_3\ b_4]\begin{bmatrix} a_3 \\ a_4 \end{bmatrix}$

$= \sum a_i b_i$

6. assume $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$tr(\vec{v}\vec{w}^T) = \sum_{i=1}^{n} \vec{e_i}^T \vec{v}\vec{w}^T \vec{e_i} = \sum_{i=1}^{n} v_i \cdot w_i$ //

$tr(\vec{w}^T\vec{v}) = \vec{w}^T\vec{v} = \sum_{i=1}^{n} v_i \cdot w_i$

7. $tr(AB) = tr(\sum_{i=1}^{n} A\vec{e_i} \times \sum_{j=1}^{n} \vec{e_j}^T B) = \sum_{i,j=1}^{n} tr(A\vec{e_i}\cdot\vec{e_j}^T B) = \sum_{i,j=1}^{n} tr((A\vec{e_i})(B^T\vec{e_j})^T)$

$= \sum_{i,j=1}^{n} tr((B^T\vec{e_j})^T (A\vec{e_i}))$

$= \sum_{i,j=1}^{n} tr(\vec{e_j}^T BA\vec{e_i})$

$= tr(BA)$

8. if $AB - BA = I$

$\Rightarrow tr(AB-BA) = tr\,I = n$

$\quad\quad \|$

$tr(AB) - tr(BA) = 0$

but $n \neq 0$ $\Rightarrow$ so $AB - BA \neq I$

5.2

1. $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. $tr(P) = \sum_{i=1}^{n} (i,i)$ entry of P = the number of fixed points in P
= the number of fixed elements for the corresponding permutation

3. since $tr(P_1P_2) = tr(P_2P_1)$
so the conclusion is proved

5.3

1. × $f(x) = [x]$ $g(x) = x - [x]$
$f, g \in V$, $f+g \notin V$

2. × $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin V$

3. √ subset of X: $\boxed{\emptyset, \{1\}, \{2\}, \{3\}} \Rightarrow$ basis
$\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
dimension: 3

4. √ In HW3, we know matrices like $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$ commute with J
(upper triangle matrix & anti-symmetric)

$A \in V$, $A_1 + A_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ 0 & a_1+a_2 & b_1+b_2 \\ 0 & 0 & a_1+a_2 \end{bmatrix} \in V$. $\lambda A = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda a \end{bmatrix} \in V$

dimension: 3
basis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. √ matrices like $B = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$ commute with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
dimension: 2 basis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ 4 & 0 & -4 \\ -2 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix}$

6. √ let $S_M = \sum_{i=1}^{3} \vec{e_i} M \vec{e_i}$

if $M \in V$, $\lambda \in \mathbb{R} \Rightarrow \lambda M \in V$ (because $S_{\lambda M} = \lambda S_M$)

if $M_1, M_2 \in V \Rightarrow M_1 + M_2 \in V$
$S_{M_1+M_2} = S_{M_1} + S_{M_2}$

dimension: 3
basis: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

5.4

1. ① A is injective

so $A\vec{x}=\vec{0}$ solution is only $\vec{0}$

suppose we have a $\vec{y}\neq\vec{0}$, $A\vec{y}=\vec{0}$

$\Rightarrow$ let $\vec{y}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $A\vec{y}=\begin{bmatrix} y_1+y_2+y_3 \\ ey_1+e^2y_2+e^3y_3 \\ e^2y_1+e^4y_2+e^6y_3 \end{bmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} y_1+y_2+y_3=0 \\ y_1+ey_2+e^2y_3=0 \\ y_1+e^2y_2+e^4y_3=0 \end{cases}$ $\Rightarrow y_1=-y_2-y_3=(e^2+e-1)y_3$

$\Rightarrow (e^2-e)y_2+(e^4-e^2)y_3=0$

$y_2+(e^2+e)y_3=0$

so $[e^2+e-1-(e^3+e^2)+e^2]y_3=0$

$\Rightarrow y_1=y_2=y_3=0$, $\vec{y}=\vec{0}$

$\Rightarrow$ contradition

so A is injective

② A is surjective

$\Rightarrow A^T=\begin{pmatrix} e^0 & e^1 & e^2 \\ e^0 & e^2 & e^4 \\ e^0 & e^3 & e^6 \end{pmatrix}$ is injective

so $A^T\vec{x}=\vec{0}$ has only one solution: $\vec{0}$

suppose $\vec{y}=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\neq\vec{0}$, $A^T\vec{y}=\vec{0}$

$\left(\begin{array}{ccc|c} 1 & e & e^2 & 0 \\ 1 & e^2 & e^4 & 0 \\ 1 & e^3 & e^6 & 0 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & e & e^2 & 0 \\ 0 & e^2-e & e^4-e^2 & 0 \\ 0 & e^3-e & e^6-e^2 & 0 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & e & e^2 & 0 \\ 0 & 1 & e^2+e & 0 \\ 0 & 1 & e^3+e & 0 \end{array}\right)$

$=\left(\begin{array}{ccc|c} 1 & e & e^2 & 0 \\ 0 & 1 & e^2+e & 0 \\ 0 & 0 & e^3-e^2 & 0 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & e & e^2 & 0 \\ 0 & 1 & e^2+e & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)=\left(\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{array}\right)$

so $\vec{y}=\vec{0}$

so A^T is injective $\Rightarrow$ A is surjective

all in all. A is bijective $\Rightarrow$ A is invertible

2. since A is invertible, A is row & column linearly independent

if e^x. e^{2x}. e^{3x} are linearly dependent,

then e^1. e^2. e^3 are ------

e^2 e^4 e^6 are ------

then A is row linearly depedent $\rightarrow$ contradition

so e^x. e^{2x}. e^{3x} are linearly independent

3. the coordinates of f: $A^{-1}\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$

5.5

1. ① let $L_1=\{\vec{p_1}+t_1\vec{v_1}: t_1\in\mathbb{R}\}$

$L_2=\{\vec{p_2}+t_2\vec{v_2}: t_2\in\mathbb{R}\}$

$(L_1//L_2$, so $\vec{v_1}=\vec{v_2}=\vec{v})$

$\forall\, \vec{q}\in(L_1+L_2)$

$\vec{q}=\vec{p_1}+\vec{p_2}+t_1\vec{v_1}+t_2\vec{v_2}$

$=(\vec{p_1}+\vec{p_2})+(t_1+t_2)\vec{v}$

so $L_1+L_2\in V/W$

② $L=\{\vec{p}+t\vec{v}: t\in\mathbb{R}\}$

$kL=\{k\vec{p}+(kt)\vec{v}: t\in\mathbb{R}\}=\{\vec{p'}+t'\vec{v}: t'\in\mathbb{R}\}\in V/W$

2. $k(L_1+L_2)=\{k\vec{p_1}+k\vec{p_2}+kt_1\vec{v}+kt_2\vec{v}: t_1.t_2\in\mathbb{R}\}$

$=\{k(\vec{p_1}+\vec{p_2})+k(t_1+t_2)\vec{v}: t_1.t_2\in\mathbb{R}\}$

$\|$

$kL_1+kL_2=\{k\vec{p_1}+kt_1\vec{v}+k\vec{p_2}+kt_2\vec{v}: t_1.t_2\in\mathbb{R}\}$

3. $L_0=\{t\vec{v}: t\in\mathbb{R}\}$

4. dimension: 2

basis: $L_1=\{\vec{p}+t\vec{v}\}$

$L_2=\{\vec{q}+t\vec{v}\}$

$(\vec{p}\neq k\vec{q}\ \ \forall k\in\mathbb{R})$

5.6

1. Yes. Because for any finitely many polynomials, we can find a maximum degree: n

so just find an another polynomial, whose degree is larger than n

2. $\forall q(x) \in V \Rightarrow p(x) \cdot q(x) \in W$

① $\forall q_1(x), q_2(x) \in V \Rightarrow p(x) q_1(x) + p(x) q_2(x) = p(x)(q_1(x) + q_2(x)) \in W$

② $\forall q(x) \in V \Rightarrow k p(x) q(x) = p(x) \cdot (k q(x)) \in W$

So W is a subspace

3. ① $\forall g(x) \in [r_1(x)] + [r_2(x)]$

$g(x) = r_1(x) + r_2(x) + p(x)(q_1(x) + q_2(x)) \quad \forall q_1(x), q_2(x) \in V$

$= \underline{r_1(x) + r_2(x) + p(x) q(x)} \quad \forall q(x) \in V$

$\forall h(x) \in [r_1(x) + r_2(x)]$

$h(x) = \underline{(r_1(x) + r_2(x)) + p(x) q(x)} \quad \forall q(x) \in V$

same format

so $[r_1(x)] + [r_2(x)] = [r_1(x) + r_2(x)]$

② $\forall g(x) \in k[r(x)]$

$g(x) = k(r(x) + p(x) q(x)) = k r(x) + p(x) \cdot (k q(x))$

$= \underline{k r(x) + p(x) \cdot q'(x)} \quad \forall q(x), q'(x) \in V$

$\forall h(x) \in [k r(x)]$

$h(x) = \underline{k r(x) + p(x) q(x)} \quad \forall q(x) \in V$

same $\Rightarrow$ so $k[r(x)] = [k r(x)]$

4. dimension: 2

basis: $\{1 + p(x) q(x)\}$ $\{x + p(x) q(x)\}$