problem 1

1. $A\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1/4\\3/4\\0\\0\end{pmatrix}$ $A^2\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1/16\\6/16\\9/16\\0\end{pmatrix}$ $A^3\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1/64\\9/64\\27/64\\27/64\end{pmatrix}$

$$A=\begin{pmatrix}1/4&&&\\3/4&1/4&&\\&3/4&1/4&\\&&3/4&1\end{pmatrix}$$

2. $\lambda = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1$

Jordan canonical form of $A=\begin{pmatrix}1/4&1&&\\&1/4&1&\\&&1/4&\\&&&1\end{pmatrix}$

minimal polynomial of $A = (x-1)(x-\frac{1}{4})^3$

3. $$A=\left[\begin{array}{ccc|c}1/4&&&\\3/4&1/4&&\\&3/4&1/4&\\\hline&&3/4&1\end{array}\right]=\begin{bmatrix}A_1&\\B&A_2\end{bmatrix}=\begin{bmatrix}I&\\X&I\end{bmatrix}\cdot\begin{bmatrix}A_1&\\B-XA_1&A_2\end{bmatrix}$$

$$=\begin{bmatrix}I&\\X&I\end{bmatrix}\cdot\begin{bmatrix}A_1&\\B+A_2X-XA_1&A_2\end{bmatrix}\cdot\begin{bmatrix}I&\\X&I\end{bmatrix}^{-1}$$

$\Rightarrow B = XA_1 - A_2X$

$(0\ 0\ 3/4) = X\cdot\begin{pmatrix}1/4&&\\3/4&1/4&\\&3/4&1/4\end{pmatrix}-[1]\cdot X$

suppose $X=\begin{pmatrix}a\\b\\c\end{pmatrix}^T$

$\Rightarrow \begin{cases}0=\frac{a}{4}+\frac{3b}{4}-a\\0=\frac{b}{4}+\frac{3c}{4}-b\\\frac{3}{4}=\frac{c}{4}-c\end{cases} \Rightarrow X=(-1\ -1\ -1)$

4. $X_1=\begin{pmatrix}1&&&\\&1&&\\&&1&\\-1&-1&-1&1\end{pmatrix}$ $X_1^{-1}AX_1=\left(\begin{array}{ccc|c}1/4&&&\\3/4&1/4&&\\&3/4&1/4&\\\hline&&&1\end{array}\right)$ A'

$$A'=\begin{pmatrix}1/4&&\\3/4&1/4&\\&3/4&1/4\end{pmatrix}=\begin{pmatrix}&&1\\&3/4&\\9/16&&\end{pmatrix}\cdot\begin{pmatrix}1/4&1&\\&1/4&1\\&&1/4\end{pmatrix}\begin{pmatrix}&&1\\&3/4&\\9/16&&\end{pmatrix}^{-1}$$

$$\Rightarrow X=\begin{pmatrix}1&&&\\&1&&\\&&1&\\-1&-1&-1&1\end{pmatrix}\cdot\begin{pmatrix}&&1&\\&3/4&&\\9/16&&&\\&&&1\end{pmatrix}=\begin{pmatrix}&&1&\\&3/4&&\\9/16&&&\\-9/16&-3/4&-1&1\end{pmatrix}$$

5. $A\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}1/4\\3/4\\0\\-1\end{pmatrix}\in\text{span}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$

$A^2\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}1/16\\6/16\\9/16\\-1\end{pmatrix}\in\text{span}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$

$A^3\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}1/64\\9/64\\27/64\\-37/64\end{pmatrix}\in\text{span}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$

$\Rightarrow$ basis for V: $\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\Rightarrow \vec{v}_1=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \vec{v}_2=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

$\vec{v}_3=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$

$(A-I)_V = A_V - I_{3\times3}$

$=\begin{pmatrix}1/4&&\\3/4&1/4&\\&3/4&1/4\end{pmatrix}-I_{3\times3}$

$=\begin{pmatrix}-3/4&&\\3/4&-3/4&\\&3/4&-3/4\end{pmatrix}$

$\Rightarrow (A-I)_V^{-1}=\begin{pmatrix}-4/3&&\\-4/3&-4/3&\\-4/3&-4/3&-4/3\end{pmatrix}$

$\Rightarrow \vec{e_4}\,(A-I)^{-1}\cdot(\underbrace{\vec{e_1}-\vec{e_4}}_{=\vec{v}_1})$

$=\vec{e_4}\cdot(-\frac{4}{3}\vec{v}_1-\frac{4}{3}\vec{v}_2-\frac{4}{3}\vec{v}_3)$

$=\vec{e_4}\cdot(-\frac{4}{3}\vec{e_1}-\frac{4}{3}\vec{e_2}-\frac{4}{3}\vec{e_3}+4\vec{e_4})$

$=4$

problem 2

1. $A = X \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} X^{-1}$

$\Rightarrow B = X \begin{pmatrix} \pm\sqrt{\lambda_1} & & & \\ & \pm\sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \pm\sqrt{\lambda_n} \end{pmatrix} X^{-1}$

every diagonal entry has 2 choices ($\pm$)

$\Rightarrow$ there's 2^n square roots of A

2. suppose $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow B^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{cases} a^2+bc = 1 \\ bc+d^2 = 1 \\ b(a+d) = c(a+d) = 0 \end{cases} \Rightarrow \underset{\uparrow}{b=c=0}$ or $a+d=0$

$a^2=1,\ d^2=1$

$\Rightarrow B = \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} \pm 1 & \\ & \mp 1 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$ $(a, b \in \mathbb{C})$

3. all eigenvalues of A are zero

$\Rightarrow$ all eigenvalues of B are zero

$\Rightarrow \underline{B = X \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} X^{-1}}$ or $X \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right) X^{-1}$ or $X \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} X^{-1}$ $\leftarrow 0 \rightarrow$ wrong

$\Rightarrow B^2 = X \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} X^{-1}$ or $0 \rightarrow$ wrong

$\Rightarrow X \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} X^{-1} = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}$

$\Rightarrow X \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} X$

suppose $X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & d \\ 0 & 0 & g \end{pmatrix} = \begin{pmatrix} d & e & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{cases} a = f \\ d = e = g = 0 \end{cases} \Rightarrow X = \begin{pmatrix} a & b & c \\ 0 & 0 & a \\ 0 & h & i \end{pmatrix} \Rightarrow B = \begin{pmatrix} 0 & \frac{b}{a} - \frac{i}{h} & \frac{a}{h} \\ 0 & 0 & 0 \\ 0 & \frac{h}{a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p & \frac{1}{q} \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}$

4. ☆ all eigenvalues of a matrix are arranged in ascending order $\Rightarrow$ spectral decomposition is unique

$$A = X\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}X^*, \quad X^{-1}=X^* \text{ and } \lambda_i > 0 \quad (i=1,2,\cdots,n)$$

$$\Rightarrow B = X\begin{pmatrix}\sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n}\end{pmatrix}X^* \Rightarrow \sqrt{\lambda_i}\in\mathbb{R} \text{ and } \sqrt{\lambda_i}\geq 0$$

$\Rightarrow B$ is Hermitian positive-definite

suppose $B_1^2 = A$, $B_2^2 = A$

their eigenvalues are λ_{i1}, λ_{i2} $(i=1,2,\cdots,n)$

$$\Rightarrow B_1 = X_1\begin{pmatrix}\lambda_{11} & & & \\ & \lambda_{21} & & \\ & & \ddots & \\ & & & \lambda_{n1}\end{pmatrix}X_1^* \qquad B_2 = X_2\begin{pmatrix}\lambda_{12} & & & \\ & \lambda_{22} & & \\ & & \ddots & \\ & & & \lambda_{n2}\end{pmatrix}X_2^*$$

$$B_1^2 = B_2^2 = X_1\begin{pmatrix}\lambda_{11}^2 & & & \\ & \lambda_{21}^2 & & \\ & & \ddots & \\ & & & \lambda_{n1}^2\end{pmatrix}X_1^* = X_2\begin{pmatrix}\lambda_{12}^2 & & & \\ & \lambda_{22}^2 & & \\ & & \ddots & \\ & & & \lambda_{n2}^2\end{pmatrix}X_2^* = X\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}X^*$$

$$\Leftrightarrow \begin{cases}\lambda_{i1}^2 = \lambda_{i2}^2 = \lambda_i \\ X_1 = X_2 = X\end{cases}$$

, since B_1, B_2 are positive-definite

$\Rightarrow \lambda_{i1} = \lambda_{i2} = \lambda_i \geq 0 \Rightarrow B_1 = B_2$

5.

$$A = X\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}X^* \quad (\lambda_i\geq 0)$$

$$(A^{\frac{1}{2}})^{-1} = \left(X\begin{pmatrix}\sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n}\end{pmatrix}X^*\right)^{-1} = X\begin{pmatrix}\frac{1}{\sqrt{\lambda_1}} & & & \\ & \frac{1}{\sqrt{\lambda_2}} & & \\ & & \ddots & \\ & & & \frac{1}{\sqrt{\lambda_n}}\end{pmatrix}X^*$$

$$A^{-1} = X\begin{pmatrix}\frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n}\end{pmatrix}X^* \rightarrow \text{Hermitian positive-definite}$$

$$\Rightarrow (A^{-1})^{\frac{1}{2}} = X\begin{pmatrix}\sqrt{\frac{1}{\lambda_1}} & & & \\ & \sqrt{\frac{1}{\lambda_2}} & & \\ & & \ddots & \\ & & & \sqrt{\frac{1}{\lambda_n}}\end{pmatrix}X^* = (A^{\frac{1}{2}})^{-1}$$

6.

$$A = \begin{pmatrix}1 & \\ 1 & 2\end{pmatrix} \quad B = \begin{pmatrix}1 & \\ 1 & 3\end{pmatrix} \Rightarrow AB = \begin{pmatrix}1 & \\ 3 & 6\end{pmatrix}$$

$$A^{\frac{1}{2}} = \begin{pmatrix}1 & \\ \sqrt{2}-1 & \sqrt{2}\end{pmatrix} \quad B^{\frac{1}{2}} = \begin{pmatrix}1 & \\ \frac{\sqrt{3}-1}{2} & \sqrt{3}\end{pmatrix} \Rightarrow A^{\frac{1}{2}}B^{\frac{1}{2}} = \begin{pmatrix}1 & \\ \frac{\sqrt{2}+\sqrt{6}}{2}-1 & \sqrt{6}\end{pmatrix}$$

$$\Rightarrow (A^{\frac{1}{2}}B^{\frac{1}{2}})^2 = \begin{pmatrix}1 & \\ \frac{\sqrt{2}-\sqrt{6}}{2}+\sqrt{3}+\sqrt{2} & 6\end{pmatrix} \neq AB$$

problem 3

1. $(B^{-\frac{1}{2}} X B^{-\frac{1}{2}})^2 = B^{-\frac{1}{2}} X (B^{-\frac{1}{2}} \cdot B^{-\frac{1}{2}}) X B^{-\frac{1}{2}}$

$$= B^{-\frac{1}{2}} X (B^{\frac{1}{2}} \cdot B^{\frac{1}{2}})^{-1} X B^{-\frac{1}{2}}$$

$$= B^{-\frac{1}{2}} X B^{-1} X B^{-\frac{1}{2}}$$

$$= B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$$

$$\Rightarrow B^{-\frac{1}{2}} X B^{-\frac{1}{2}} = (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}}$$

$$B^{-\frac{1}{2}} X = (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}}$$

$$X = B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}}$$

2. $A \# A = X \Rightarrow X A^{-1} X = A$

since $X = A$ satisfy the equation above
and X is unique

$$\Rightarrow A \# A = A$$

$$(A \# B)^{-1} = \left(B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}}\right)^{-1} = B^{-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{-\frac{1}{2}} B^{-\frac{1}{2}}$$

$$A^{-1} \# B^{-1} = (B^{-1})^{\frac{1}{2}} \left((B^{-1})^{-\frac{1}{2}} A^{-1} (B^{-1})^{-\frac{1}{2}}\right)^{\frac{1}{2}} (B^{-1})^{\frac{1}{2}}$$

$$= B^{-\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\frac{1}{2}} B^{-\frac{1}{2}}$$

$$= B^{-\frac{1}{2}} \left((B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{-1}\right)^{\frac{1}{2}} B^{-\frac{1}{2}}$$

$$= B^{-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{-\frac{1}{2}} B^{-\frac{1}{2}}$$

$$\Rightarrow (A \# B)^{-1} = A^{-1} \# B^{-1}$$

3.

$$A \# B = B \# A$$

$$\Leftrightarrow B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}$$

$$\Leftrightarrow A^{-\frac{1}{2}} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}}$$

$$\Leftrightarrow \left(A^{-\frac{1}{2}} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}}\right)^2 = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$$

$$\Leftrightarrow A^{-\frac{1}{2}} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$$

$$\Leftrightarrow B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} = B$$

$$\Leftrightarrow (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{\frac{1}{2}} = I$$

let $B^{-\frac{1}{2}} A B^{-\frac{1}{2}} = C$

$\Leftrightarrow C^{\frac{1}{2}} (C^{-1}) C^{\frac{1}{2}} = I \leftarrow$ it's obvious   so $A \# B = B \# A$

4.

$$\frac{A+B}{2} - A\#B = \frac{1}{2}A + \frac{1}{2}B - B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}$$

$$= \frac{1}{2}A + B^{\frac{1}{2}}\left(\frac{1}{2}I - (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}\right)B^{\frac{1}{2}}$$

$$= B^{\frac{1}{2}}\left(\frac{1}{2}B^{-\frac{1}{2}}AB^{-\frac{1}{2}} + \frac{1}{2}I - (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}\right)B^{\frac{1}{2}}$$

$$= B^{\frac{1}{2}} \cdot \frac{1}{2}\left((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}} - I\right)^2 B^{\frac{1}{2}}$$

Since A, B are Hermitian matrices

$\Rightarrow \left((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}} - I\right)^2 \geq 0$

and $(B^{\frac{1}{2}})^* = B^{\frac{1}{2}}$

$\Rightarrow B^{\frac{1}{2}}\left((B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}} - I\right)^2 B^{\frac{1}{2}} \geq 0$

$\Rightarrow \frac{A+B}{2} - A\#B \geq 0$

$\frac{A+B}{2} \geq A\#B$

5.

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} = \begin{bmatrix} I & \\ XA^{-1} & I \end{bmatrix} \begin{bmatrix} A & X \\ & B-XA^{-1}X \end{bmatrix}$$

$$= \begin{bmatrix} I & \\ XA^{-1} & I \end{bmatrix} \begin{bmatrix} A & \\ & B-XA^{-1}X \end{bmatrix} \begin{bmatrix} I & A^{-1}X \\ & I \end{bmatrix} \geq 0$$

$$\Rightarrow B - XA^{-1}X \geq 0$$

$$\Rightarrow A^{-\frac{1}{2}} B A^{-\frac{1}{2}} - \underbrace{A^{-\frac{1}{2}} X A^{-1} X A^{-\frac{1}{2}}}_{= (A^{-\frac{1}{2}} X A^{-\frac{1}{2}})^2} \geq 0$$

$$\Rightarrow \underbrace{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}}_{= A^{-\frac{1}{2}}(B\#A)A^{-1}(B\#A)A^{-\frac{1}{2}} = (A^{-\frac{1}{2}}(B\#A)A^{-\frac{1}{2}})^2} \geq (A^{-\frac{1}{2}} X A^{-\frac{1}{2}})^2$$

let $C = A^{-\frac{1}{2}}(B\#A)A^{-\frac{1}{2}}$, $D = A^{-\frac{1}{2}} X A^{-\frac{1}{2}}$

$$= (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}}$$

$$\Rightarrow C^2 \geq D^2$$

$$C^{-1} C^2 C^{-1} \geq C^{-1} D^2 C^{-1}$$

$$I - C^{-1} D^2 C^{-1} \geq 0$$

$$I - (C^{-1}D) \cdot (C^{-1}D)^* \geq 0$$

since $(C^{-1}D)(C^{-1}D)^*$ has eigenvalues that are not larger than 1

$\Rightarrow$ $C^{-1}D$ has eigenvalues that are not larger than 1

$$\Rightarrow I - C^{-1}D \geq 0$$

$$I - A^{\frac{1}{2}}(B\#A)^{-1}A^{\frac{1}{2}} A^{-\frac{1}{2}} X A^{-\frac{1}{2}} \geq 0$$

$$I \geq A^{\frac{1}{2}}(B\#A)^{-1} X A^{-\frac{1}{2}}$$

$$\Rightarrow I \geq (B\#A)^{-1} X$$

$$\Rightarrow X \leq B\#A = A\#B$$

collaborate with

金昱考. 么达一

problem 4

1. $\frac{d}{dt}\vec{r}(t) = \vec{v}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

$$\underset{\begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}}{\underset{\|}{\frac{d}{dt}\vec{v}(t)}} = \frac{\vec{F}(t)}{m} = \vec{F}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{r}(t) = \begin{pmatrix} x(t)+y(t) \\ y(t) \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & 1 & & \\ & 1 & & \end{pmatrix}$$

2. $\det(xI - A) = \det\begin{pmatrix} x & & -1 & \\ & x & & -1 \\ -1 & -1 & x & \\ & -1 & & x \end{pmatrix} = (x+1)^2(x-1)^2$

$\Rightarrow \lambda = \pm 1$

$$\ker(A-I) = \ker\begin{pmatrix} -1 & & 1 & \\ & -1 & & 1 \\ 1 & 1 & -1 & \\ & 1 & & -1 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\ker(A-I)^2 = \ker\begin{pmatrix} 2 & 1 & -2 & \\ & 2 & & -2 \\ -2 & -2 & 2 & 1 \\ & -2 & & 2 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$\Rightarrow V_1 : \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

$A_{V_1} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$$\ker(A+I) = \ker\begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & 1 & 1 & \\ & 1 & & 1 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\ker(A+I)^2 = \ker\begin{pmatrix} 2 & 1 & 2 & \\ & 2 & & 2 \\ 2 & 2 & 2 & 1 \\ & 2 & & 2 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix}$$

$\Rightarrow V_2 : \mathrm{span}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \\ -2 \end{pmatrix}$

$A_{V_2} = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$

$$\Rightarrow J = \left(\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & -1 & 1 \\ & & & -1 \end{array}\right), \quad X = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & -1 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

3.

$$\left.\begin{aligned} p(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ p(-t) &= a_0 - a_1 t + a_2 t^2 - a_3 t^3 \end{aligned}\right\} \Rightarrow e^t - e^{-t} = 2a_1 t + 2a_3 t^3$$

$$\left.\begin{aligned} p'(t) &= a_1 + 2a_2 t + 3a_3 t^2 \\ p'(-t) &= a_1 - 2a_2 t + 3a_3 t^2 \end{aligned}\right\} \Rightarrow e^t - e^{-t} = 4a_2 t$$

$$a_2 = \frac{e^t - e^{-t}}{4t}$$

$$2a_1 t + 6a_3 t^3 = t(e^t + e^{-t}) \longrightarrow a_3 = \frac{(t-1)e^t + (t+1)e^{-t}}{4t^3}$$

$$\Rightarrow a_1 = \frac{3}{4t}(e^t - e^{-t}) - \frac{1}{4}(e^t + e^{-t})$$

$$\Rightarrow p(x) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \frac{1}{4}te^t - \frac{1}{2}te^{-t} + x\cdot\left(\frac{3}{4t}(e^t - e^{-t}) - \frac{1}{4}(e^t + e^{-t})\right) + x^2\cdot\frac{e^t - e^{-t}}{4t} + x^3\cdot\frac{(t-1)e^t + (t+1)e^{-t}}{4t^3}$$

4.

$$e^{At} = e^{\begin{pmatrix} & & t & \\ & & & t \\ t & t & & \\ & t & & \end{pmatrix}} = a_0 I + a_1 \begin{pmatrix} & & t & \\ & & & t \\ t & t & & \\ & t & & \end{pmatrix} + a_2 \begin{pmatrix} t^2 & t^2 & & \\ & t^2 & & \\ & & t^2 & t^2 \\ & & & t^2 \end{pmatrix} + a_3 \begin{pmatrix} & & t^3 & t^3 \\ & & & t^3 \\ t^3 & 2t^3 & & \\ & t^3 & & \end{pmatrix}$$

$$= \begin{pmatrix} a_0 + a_2 t^2 & a_2 t^2 & a_1 t + a_3 t^3 & a_3 t^3 \\ & a_0 + a_2 t^2 & & a_1 t + a_3 t^3 \\ a_1 t + a_3 t^3 & a_1 t + 2a_3 t^3 & a_0 + a_2 t^2 & a_2 t^2 \\ & a_1 t + a_3 t^3 & & a_0 + a_2 t^2 \end{pmatrix}$$

problem 5

1. we can't directly plug in $x=B$, $y=A$ into the definition of f
we should first calculate the formula of f, expressed by using x, y,
then plug in $x=B$, $y=A$

2. $f(x,y) = \det\begin{pmatrix} -2y & x-3y \\ & -2y \end{pmatrix} = 4y^2$

$f(B,A) = 4A^2 = 0$

3. $f(x,y) = \det(xA-yB) = \det(A)\cdot\det(xI - yA^{-1}B)$

$y^n P_c(y^{-1}x) = y^n \det(y^{-1}xI - A^{-1}B)$ $\qquad \det(A)\cdot y^n \det(y^{-1}xI - A^{-1}B)$

$\Rightarrow$ so $f(x,y) = \det(A)\cdot y^n P_c(y^{-1}x)$

4. $f(B,A) = \det(A)\cdot B^n \cdot P_{A^{-1}B}(A^{-1}B)$

It's obvious that $P_{A^{-1}B}(A^{-1}B) = 0$

$\Rightarrow f(B,A) = 0$

5. $A_t = A + tI \Rightarrow$ find all $t \to 0$ that makes A_t invertible $(\lim A_t = A)$,
which means $t \neq \lambda_i$ for all $i = 1, 2, \cdots, n$

$\Rightarrow A_t B = AB + tB$

$BA_t = BA + tB$

since $AB = BA$

$\Rightarrow A_t B = BA_t$ for all A_t

problem 6

1. since X_n has no purely imaginary eigenvalue

$\Rightarrow X_n = X J X^{-1}$, with eigenvalues λ_i $(i=1,2,\cdots,n)$

$\Rightarrow X_n^{-1} = X J^{-1} X^{-1}$, with eigenvalues λ_i^{-1} $(i=1,2,\cdots,n)$

$\Rightarrow X_n + X_n^{-1} = X(J+J^{-1})X^{-1}$

$\hookrightarrow$ eigenvalues: $\lambda_i + \frac{1}{\lambda_i}$

suppose $\lambda_i = a+bi$ $(a\in\mathbb{R}\setminus\{0\},\ b\in\mathbb{R})$

$$\Rightarrow \lambda_i + \frac{1}{\lambda_i} = a+bi+\frac{a-bi}{a^2+b^2} = \underbrace{\left(a+\frac{a}{a^2+b^2}\right)}_{\neq 0} + \left(b-\frac{b}{a^2+b^2}\right)i$$

$\Rightarrow X_{n+1}$ has no purely imaginary eigenvalue

2. suppose $A=[a+bi]$

$$X_0 = a+bi \qquad \frac{X_n-1}{X_n+1} = \frac{X_{n-1}-2+\frac{1}{X_{n-1}}}{X_{n-1}+2+\frac{1}{X_{n-1}}} = \left(\frac{X_{n-1}-1}{X_{n-1}+1}\right)^2$$

let $y_n = \frac{X_n-1}{X_n+1} \Rightarrow y_n = y_{n-1}^2$, $y_0 = \frac{a-1+bi}{a+1+bi} \Rightarrow y_n = \left(\frac{a-1+bi}{a+1+bi}\right)^{(2^n)}$

$$= \left(\frac{\sqrt{(a-1)^2+b^2}\, e^{i\theta_1}}{\sqrt{(a+1)^2+b^2}\, e^{i\theta_2}}\right)^{(2^n)}$$

$\Rightarrow$ if $a>0$: $|a-1|<|a+1| \Rightarrow \lim_{n\to+\infty} y_n = 0 \Rightarrow \lim_{n\to+\infty}(X_n-1)=0 \Rightarrow \lim_{n\to+\infty} X_n = 1$

if $a<0$: $|a-1|>|a+1| \Rightarrow \lim_{n\to+\infty} y_n = +\infty \Rightarrow \lim_{n\to+\infty}(X_n+1)=0 \Rightarrow \lim_{n\to+\infty} X_n = -1$

$\Rightarrow \lim_{n\to+\infty} X_n = \mathrm{sign}(A)$

3. $A = XDX^{-1} = X_0$

$$(X_n - I)\cdot(X_n + I)^{-1} = \frac{1}{4}(X_{n-1} + X_{n-1}^{-1} - 2I)\cdot(X_{n-1} + X_{n-1}^{-1} + 2I)^{-1}$$

$$= \frac{1}{4}(X_{n-1} - X_{n-1}^{-1})^2 \cdot ((X_{n-1} + X_{n-1})^2)^{-1}$$

$$= \frac{1}{4} X\cdot(D_{n-1} - D_{n-1}^{-1})^2 \cdot ((D_{n-1} + D_{n-1}^{-1})^{-1})^2 \cdot X^{-1}$$

suppose $D_k = \begin{pmatrix} \lambda_{k1} & & & \\ & \lambda_{k2} & & \\ & & \ddots & \\ & & & \lambda_{kn} \end{pmatrix}$

$$\Rightarrow (X_n - I)\cdot(X_n + I)^{-1} = \frac{1}{4} X\cdot \sum_{i=1}^{n} (\lambda_{(n-1)i}^2 - 2 + \frac{1}{\lambda_{(n-1)i}^2})\vec{e_i}\vec{e_i}^T$$

$$\cdot \sum_{i=1}^{n} (\lambda_{(n-1)i}^2 + 2 + \frac{1}{\lambda_{(n-1)i}^2})^{-1}\vec{e_i}\vec{e_i}^T \cdot X^{-1}$$

$$= \frac{1}{4} X \sum_{i=1}^{n} \left(\frac{\lambda_{(n-1)i} - 1}{\lambda_{(n-1)i} + 1}\right)^2 \vec{e_i}\vec{e_i}^T \cdot X^{-1}$$

let $Y_n = (X_n - I)(X_n + I)^{-1}$

$\Rightarrow Y_n = Y_{n-1}^2$, $Y_0 = (A - I)\cdot(A + I)^{-1}$

$$\Rightarrow Y_n = \left[(A - I)(A + I)^{-1}\right]^{(2^n)}$$

$$= X((D - I)(D + I)^{-1})^{(2^n)} X^{-1}$$

$$= X \begin{pmatrix} \left(\frac{\lambda_1 - 1}{\lambda_1 + 1}\right)^{(2^n)} & & \\ & \ddots & \\ & & \left(\frac{\lambda_n - 1}{\lambda_n + 1}\right)^{(2^n)} \end{pmatrix} X^{-1}$$

if A positive definite: $\lim_{n\to+\infty} \left(\frac{\lambda_i - 1}{\lambda_i + 1}\right)^{(2^n)} = 0 \Rightarrow \lim_{n\to+\infty} Y_n = 0 \Rightarrow \lim_{n\to+\infty} X_n = I$

if A negative definite: $\lim_{n\to+\infty} \left(\frac{\lambda_i - 1}{\lambda_i + 1}\right)^{(2^n)} = +\infty \Rightarrow \lim_{n\to+\infty} Y_n$ doesn't exist

$\Rightarrow \lim_{n\to+\infty} X_n = -I$

$\Rightarrow \lim_{n\to+\infty} X_n = \mathrm{sign}(A)$

4. $n=1$, $A_1=[1]=X_0=\text{sign}(A)$

using induction;

suppose $X_{n-1}=\text{sign}(A_n)=I=\frac{1}{2}(X_{n-2}+X_{n-2}^{-1})$

$$A_{n+1}=\left(\begin{array}{c|c} A_n & \vec{e_n} \\ \hline & 1 \end{array}\right)=X_0' \Rightarrow \text{let } B_0=\vec{e_n}$$

$$A_{n+1}^{-1}=\begin{pmatrix} A_n^{-1} & -A_n^{-1}\vec{e_n} \\ & 1 \end{pmatrix}$$

$$X_1'=\begin{pmatrix} X_1 & \underbrace{\tfrac{1}{2}(I-A_n^{-1})\vec{e_n}}_{B_1} \\ & 1 \end{pmatrix} \quad (X_1')^{-1}=\begin{pmatrix} X_1^{-1} & -X_1^{-1}B_1 \\ & 1 \end{pmatrix}$$

$$X_2'=\begin{pmatrix} X_2 & \frac{I-X_1^{-1}}{2}B_1 \\ & 1 \end{pmatrix} \Rightarrow \cdots \Rightarrow X_n'=\begin{pmatrix} X_n & B_n \\ & 1 \end{pmatrix}$$

$$\Rightarrow B_k=\frac{1}{2}(I-X_{k-1}^{-1})B_{k-1}$$

$$=\frac{1}{2^k}(I-X_{k-1}^{-1})(I-X_{k-2}^{-1})\cdots(I-X_0^{-1})\vec{e_n}$$

$$\Rightarrow B_n=\frac{1}{2^{n-1}}(I-X_{n-1}^{-1})(I-X_{n-2}^{-1})\cdots(I-X_0^{-1})\vec{e_n}$$

since $X_n=X_{n-1}=\text{sign}(A_n)=I$

$\Rightarrow B_n=\vec{0}$

$$\Rightarrow X_n'=\begin{pmatrix} I & \vec{0} \\ & 1 \end{pmatrix}=I=\text{sign}(A_{n+1})$$