

3.1 Yes.

proof: let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

all rows parallel $\Rightarrow \text{REF} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

so REF of A has all column parallel

Then, do row operation to recover A from its REF

recover 2nd row: $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$

\because row parallel

$$\therefore \frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \cdots = \frac{a_{1n}}{a_{2n}}$$

\therefore all columns are still parallel

\Rightarrow recover all rows, for the same reason,
all columns are still parallel

3.2

$$1. \vec{a}^T = [a_1 \ \cdots \ a_n]$$

$$\vec{b}^T = [b_1 \ \cdots \ b_n]$$

so the i-th row of the matrix is

$$[a_{11} + (b_1 - a_1)(i-1) \ \cdots \ a_{1n} + (b_n - a_n)(i-1)]$$

-1

2. Do row operations as follow:

$$r_2 \rightarrow r_2 - r_1$$

$$r_i \ (i \geq 3) \rightarrow r_i - (i-1)r_1$$

$$r_3 \ (i \geq 3) \rightarrow r_3 - r_1$$

$$\text{so, } \vec{a}_2^T = \vec{0}$$

REF of the matrix:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ b_1 - a_1 & \cdots & b_n - a_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

so the rank ≤ 2

3.3

well-defined: $BA \cdot AB \cdot \underbrace{ABAB}_{\substack{\downarrow \\ \text{associative}}} \cdot \underbrace{BABC}_{\substack{\leftarrow \\ \text{law}}}$

not well-defined:

BAC :

$$(BA)C \rightarrow (5 \times 5) \times (3 \times 1)$$

3.4

$$1. = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2. = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 20 & -11 \end{bmatrix}$$

$$3. = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{1010} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$4. \text{ from 1. we know } \begin{bmatrix} 1 & 6 \\ -20 & -11 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 6 \\ -20 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 \\ -20 & -11 \end{bmatrix}$$

$$5. = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$$6. = \begin{bmatrix} 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \end{bmatrix}$$

$$7. = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$\underbrace{\quad}_{4}$

$$8. = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$9. = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & -4 & -4 \\ 4 & 4 & -4 & -4 \\ -4 & 4 & 4 & -4 \\ -4 & -4 & -4 & 4 \end{bmatrix}$$

$$10. = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^{1010}$$

$$= \begin{bmatrix} 2^{2000} & 0 & 0 & 0 \\ 0 & 2^{2000} & 0 & 0 \\ 0 & 0 & 2^{2000} & 0 \\ 0 & 0 & 0 & 2^{2000} \end{bmatrix}$$

$$11. \text{ from 8. } \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^2 = I$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$$

3.5

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$1. (A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 2 \times \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix} \end{aligned}$$

$$\Rightarrow (A+B)^2 \neq A^2 + 2AB + B^2$$

$$2. (AB)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^2 B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow (AB)^2 \neq A^2 B^2$$

3.6

$$1. X_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} 2. X_{13} X_{32} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_{32} X_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \\ &\quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \\ &\quad (\vec{e}_1 \cdot \vec{e}_3^\top) (\vec{e}_3 \cdot \vec{e}_2^\top) \quad (\vec{e}_3 \cdot \vec{e}_2^\top) (\vec{e}_1 \cdot \vec{e}_2^\top) \quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \\ &\quad \stackrel{\text{II}}{=} \vec{e}_1 \cdot (\vec{e}_3^\top \vec{e}_3) \vec{e}_2^\top \quad = \vec{e}_3 \cdot 0 \cdot \vec{e}_3^\top = \vec{0} \quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \\ &\quad = \vec{e}_1 \cdot \vec{e}_2^\top = X_{12} \quad \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \end{aligned}$$

$$3. X_{ij}^2 = (\vec{e}_i \vec{e}_j^\top)^2 = \vec{e}_i \cdot (\vec{e}_j^\top \vec{e}_i) \vec{e}_j^\top$$

$$\therefore i=j$$

$$\therefore \vec{e}_j^\top \cdot \vec{e}_i = 0$$

$$\therefore X_{ij}^2 = 0$$

$$\begin{aligned} &\therefore (I + \sum_{ij} x_{ij})(I + \sum_{ij} y_{ij}) \\ &= (I + \sum_{ij} x_{ij})(I + \sum_{ij} x_{ij}) \\ &\therefore (I + \sum_{ij} x_{ij} + y_{ij} + \sum_{ij} x_{ij}y_{ij}) \\ &= I^2 + \sum_{ij} x_{ij} + y_{ij} + \sum_{ij} x_{ij}y_{ij} \\ &\therefore (\sum_{ij} x_{ij})(\sum_{ij} y_{ij}) = (\sum_{ij} y_{ij})(\sum_{ij} x_{ij}) \\ &\therefore (A-I)(B-I) = (B-I)(A-I) \\ &\therefore B(A-I)(B-I) = (B-I)(A-I)(A-I) \\ &\therefore AB - IB - AI + I^2 = BA - IA - BI + I^2 \\ &\therefore AB - (A+B) + I^2 = BA - (A+B) + I^2 \\ &\therefore AB = BA \end{aligned}$$

3.7

1. up 2. right

$$3. P\bar{J} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{bmatrix} \quad \bar{J}^T P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \end{bmatrix}$$

$$\Rightarrow P\bar{J} + \bar{J}^T P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Because $P\bar{J}$ is shifting things right, and $\bar{J}^T P$ is shifting things down. Meanwhile, each entry in P is the sum of its up entry and left entry. So this's not a coincidence.

$$4. J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J^4 = \vec{0}$$

$$5. -(I - J)(I + J + J^2 + J^3)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$6. (J+I)^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (J+I)^3 = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(J+I)^4 = \begin{bmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (J+I)^k = \begin{bmatrix} 1 & K & \frac{K(K-1)}{2} & \frac{K(K-1)(K-2)}{3} \\ 0 & 1 & K & \frac{K(K-1)}{2} \\ 0 & 0 & 1 & K \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$7. A = \begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$$

3.8

$$1. \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$2. \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$3. \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$4. \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$5. \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$6. \text{ let } B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \endbmatrix B = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

$$\Rightarrow \begin{cases} a+b = a+c \\ a+b = b+d \\ c+d = a+c \\ c+d = b+d \end{cases} \Rightarrow \begin{cases} b=c \\ a=d \\ d=a \\ b=c \end{cases} \Rightarrow B = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad (a, b \in \mathbb{R})$$

3.9

$$\begin{aligned} 1. \quad & A^3 + 2A^2 - A - I & A^2 + 3A + 4I \\ & = A^2 + 2A - A - I & = 4A + 4I \\ & = 2A - I \end{aligned}$$

$$2. \text{ let the answer} = sA + tI \quad (s, t \in \mathbb{R})$$

$$(I + 2A)(sA + tI)$$

$$\begin{aligned} &= sI \cdot A + 2sA^2 + tI^2 + 2tA \cdot I \\ &= (s + 2s + 2t)A + tI = I \\ \Rightarrow & \begin{cases} 3s + 2t = 0 \\ t = 1 \end{cases} \Rightarrow \text{answer is } (-\frac{2}{3}A + I) \end{aligned}$$