Problem 1:

1.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}_B \text{ fight with } \begin{bmatrix}0\\0\\1\end{bmatrix}_B$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix}2\\0\\1\end{bmatrix}_C \qquad\qquad \begin{bmatrix}0\\1\\5\end{bmatrix}_C$$

The first vector will win. Because its Attack is 2, higher than the third vector's Defense, and also the third vector's Attack is 0, which means the first vector can't lose HP.

2. $L_{B\to C} = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\1 & 0 & 5\end{bmatrix}$

$$L_{C\to B} = L_{B\to C}^{-1} = \begin{bmatrix}\frac{10}{21} & \frac{-5}{21} & \frac{1}{21}\\ \frac{1}{21} & \frac{10}{21} & \frac{-2}{21}\\ \frac{-2}{21} & \frac{1}{21} & \frac{4}{21}\end{bmatrix}$$

3. suppose a warrior $\vec{\omega}_B = \begin{bmatrix}s\\d\\10-s-d\end{bmatrix}$ $\quad s, d \in \mathbb{N}^*$

$s, d \in [0, 10]$, $s+d \le 10$

$$\Rightarrow \vec{\omega}_C = L_{B\to C}\vec{\omega}_B = \begin{bmatrix}2s+d\\10-s+d\\50-4s-5d\end{bmatrix} \ge 1$$

$d \le 9$

Attack $\in [0, 20]$

Defense $\in [0, 19]$

HP $\in [1, 50]$

3Attack + 2Defense + HP = 70

for the undefeatable warrior $\vec{\omega}_0$:

HP $\ge 1 \Rightarrow 4s+5d \le 49$

he will never defeat himself $\Rightarrow 2s+d \le 10-s+d$

$s \le \frac{10}{3}$

$\Rightarrow s \le 3$, then maximize its Defense

$$\Rightarrow \begin{cases}s=0\\d=9\end{cases} \Rightarrow \vec{\omega}_{0C} = \begin{bmatrix}9\\19\\5\end{bmatrix}$$

As long as other warriors satisfy this constraint, if their Attack ≤ 19, which means they'll never deal any damage to $\vec{\omega}_0$. So $\vec{\omega}_0$ is undefeatable, else if Attack $=20$, $\vec{\omega}_C = \begin{bmatrix}20\\0\\20\end{bmatrix}$, will be easily defeated

4.

$$R_{D\to D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \to \begin{bmatrix} a \\ b \\ c-d+b \\ d \\ e \\ f-a+e \end{bmatrix}$$

As row operation:

$$r_3 \to r_3 - r_4 + r_2$$

$$r_6 \to r_6 - r_1 + r_5$$

As column operation:

$$c_1 \to c_1 - c_6$$

$$c_2 \to c_2 + c_3$$

$$c_4 \to c_4 - c_3$$

$$c_5 \to c_5 + c_6$$

Problem 2 :

1.

$$x_{t+1} - x_t = -b y_t \qquad y_{t+1} - y_t = -a x_t$$

$$\underset{(n+1)\times(n+1)}{D} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$ (diagonal entries are -1, entries on their left are 1)

$$\underset{(n+1)\times(n+1)}{E} = \begin{bmatrix} 0 & & & & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{bmatrix} = \begin{bmatrix} \vec{0}^T & 0 \\ I_{n\times n} & \vec{0} \end{bmatrix}$$

2.

$$M = \begin{bmatrix} D & bE \\ aE & D \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} x_0 \vec{e_1} \\ y_0 \vec{e_1} \end{bmatrix}$$

3.

$$D = I - E$$

$$\underset{(n+1)\times(n+1)}{D^{-1}} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} = I + E + E^2 + E^3 + \cdots + E^n$$

$$E D^{-1} E = E(I + E + \cdots + E^n) E$$

$$= E^2 + E^3 + \cdots + E^{n+2} = E^2 + E^3 + \cdots E^n$$

$$(E^{n+1} = E^{n+2} = 0)$$

$$E^{n+1} = 0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

4.

$$D - aED^{-1} \cdot bE = D - abED^{-1}E$$

$$= I - E - ab(E^2 + \cdots E^n) = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ -ab & -1 & 1 & & \\ -ab & -ab & -1 & \ddots & \\ \vdots & \vdots & \ddots & \ddots & \\ -ab & \cdots & -ab & -1 & 1 \end{bmatrix}$$

so A and its Schur complement are both invertible

$\Rightarrow M$ is invertible

5. Since it's a block row operation, so it should be $E\begin{bmatrix} A & B \\ X & Y \end{bmatrix}$

$$E = \begin{bmatrix} I_{n\times n} & 0 \\ -YB^{-1} & I_{m\times m} \end{bmatrix} \Rightarrow E\begin{bmatrix} A & B \\ X & Y \end{bmatrix} = \begin{bmatrix} A & B \\ -YB^{-1}A+X & 0 \end{bmatrix}$$

Problem 3:

1. $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \vec{b}$

$$\Rightarrow \left[\begin{array}{cc|c} D & I & 0 \\ & & 0 \\ & & 0 \\ & & 1 \\ I & D & 0 \\ & & 1 \end{array}\right] = \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_5 + x_6 \\ 1 - x_5 \\ 1 - x_5 - x_6 \\ 1 - x_6 \\ x_5 \\ x_6 \end{bmatrix}$$ let $x_5 = x_6 = 0$ $\hookrightarrow \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

So, just press the middle tile in the 1st row, then press upper right and lower left tile

2. $D^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} I & 0 \\ -D^{-1} & I \end{bmatrix} A = \begin{bmatrix} D & I \\ 0 & -D^{-1}+D \end{bmatrix}$

$D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad \Big\downarrow$ multiply 1st row by D^{-1}

$D - D^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & D^{-1} \\ 0 & D - D^{-1} \end{bmatrix}$

$\Downarrow$

rank of $A = 4$

3.

$$A = \begin{bmatrix} D & I \\ I & D \end{bmatrix}$$

$$\text{RREF of } A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \dim(\ker(A)) = \dim(\text{domain}(A)) - \dim(\text{Ran}(A))$

$= 6 - \text{rank}(A) = 6 - 4 = 2$

$\Rightarrow$ basis of $\ker(A)$ are : $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

"button-pressing" vectors :

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. $\forall \vec{x} \in \mathbb{F}_2^6 \quad A\vec{x} \in \text{Ran}(A)$

$\forall \vec{y} \in \ker(A)$, $(A\vec{x})^T \cdot \vec{y} = \vec{x}^T \cdot A^T \cdot \vec{y} = \vec{x}^T (A \cdot \vec{y}) = \vec{x}^T \cdot \vec{0} = 0$

$\Downarrow$

$A\vec{y} = \vec{0}$ Also, $\dim(\ker(A)) + \dim(\text{Ran}(A)) = \dim(\text{domain}(A))$

$\Rightarrow$ $\text{Ran}(A)$ and $\ker(A)$ are orthogonal complements

the basis for $\text{Ran}(A)$ are : $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

5. $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not solvable, because it's not a linear combination of basis of $\text{Ran}(A)$

The number of elements in $\text{Ran}(A) = 2 \times 2 \times 2 \times 2 = 16$

$\Rightarrow$ the number of elements of unsolvable vectors $= 2^6 - 16 = 48$

Problem 4:

1. After computation, we know $R_{\vec{x}}^{\frac{2\pi}{3}} \cdot \vec{e_1} = \vec{e_2}$, $R_{\vec{x}}^{\frac{2\pi}{3}} \vec{e_2} = \vec{e_3}$. $R_{\vec{x}}^{\frac{2\pi}{3}} \vec{e_3} = \vec{e_1}$

$$\Rightarrow R_{\vec{x}}^{\frac{2\pi}{3}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2.

$$R = R_{\vec{x}}^{\frac{2\pi}{3}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R' = R^{\frac{\pi}{2}}_{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$RR' = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \neq R'R = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

3. suppose $R_{\vec{u}}^{\theta} = [\vec{c_1} \ \vec{c_2} \ \vec{c_3}]$

$$\Rightarrow \begin{aligned} \vec{c_1} &= R_{\vec{u}}^{\theta} \vec{e_1} \\ \vec{c_2} &= R_{\vec{u}}^{\theta} \vec{e_2} \\ \vec{c_3} &= R_{\vec{u}}^{\theta} \vec{e_3} \end{aligned}$$

after the $R_{\vec{u}}^{\theta}$ (3D rotation), these 3 vectors' length don't change, also, their relative position don't change

$\Rightarrow$ so, $\vec{c_1}, \vec{c_2}, \vec{c_3}$ are still unit vectors and are still mutually orthogonal to each other

4. we already know: $\vec{v} \perp \vec{u}$

since $\vec{w} = R_{\vec{v}}^{\frac{\pi}{2}} \vec{u}$:

for any θ, $(R_{\vec{v}}^{\theta} \vec{u}) \perp \vec{v} \Rightarrow \vec{w} \perp \vec{v}$

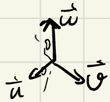
for any vector $\vec{x}$, $(R_{\vec{x}}^{\frac{\pi}{2}} \vec{u}) \perp \vec{u} \Rightarrow \vec{w} \perp \vec{u}$

$\Rightarrow$ so $\vec{u}, \vec{v}, \vec{w}$ are mutually orthogonal to each other

$\Rightarrow$ they're linearly independent

since it's in $\mathbb{R}^3$, $\Rightarrow$ they form a basis

5. $\vec{u} \perp \vec{v} \perp \vec{w}$



basis$(\vec{v},\vec{u},\vec{w})$

$$\left.\begin{aligned} R^{\theta}_{\vec{v}}\vec{v} &= \vec{v} \\ R^{\theta}_{\vec{v}}\vec{u} &= \cos\theta\,\vec{u} - \sin\theta\,\vec{w} \\ R^{\theta}_{\vec{v}}\vec{w} &= \cos\theta\,\vec{w} + \sin\theta\,\vec{u} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} M\begin{bmatrix}1\\0\\0\end{bmatrix} &= \begin{bmatrix}1\\0\\0\end{bmatrix} \\ M\begin{bmatrix}0\\1\\0\end{bmatrix} &= \begin{bmatrix}0\\ \cos\theta\\ -\sin\theta\end{bmatrix} \\ M\begin{bmatrix}0\\0\\1\end{bmatrix} &= \begin{bmatrix}0\\ \sin\theta\\ \cos\theta\end{bmatrix} \end{aligned}\right.$$

$$\Rightarrow M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$X = [\vec{v}, \vec{u}, \vec{w}]^{-1}$$

6. $\exists$ a matrix N of $R^{\theta}_{\vec{v}}$ under the basis $(\vec{v},\vec{u},\vec{w})$, $N = \begin{bmatrix} 1 & \vec{0}^{T} \\ \vec{0} & R^{\theta} \end{bmatrix}$

$\Rightarrow R^{\theta}_{\vec{v}} = [\vec{v},\vec{u},\vec{w}]^{-1} N [\vec{v},\vec{u},\vec{w}]$

$M R^{\theta}_{\vec{v}} M^{-1} = ([\vec{v},\vec{u},\vec{w}]M^{-1})^{-1} N ([\vec{v},\vec{u},\vec{w}]M^{-1})$

$\exists$ a matrix N' of $R^{\theta}_{M\vec{v}}$ under the basis $M[\vec{v},\vec{u},\vec{w}]$, $N' = \begin{bmatrix} 1 & \vec{0}^{T} \\ \vec{0} & R^{\theta} \end{bmatrix}$

$R^{\theta}_{M\vec{v}} = (M[\vec{v},\vec{u},\vec{w}])^{-1} N' (M[\vec{v},\vec{u},\vec{w}])$

then just proof: $[\vec{v},\vec{u},\vec{w}]M^{-1} = M[\vec{v},\vec{u},\vec{w}]$

$\Downarrow$ rotate clockwise then change basis $\Downarrow$ change basis, then rotate counter-clockwise

they're the same thing with different sequences of operations

$\Rightarrow$ so $M R^{\theta}_{\vec{v}} M^{-1} = R^{\theta}_{M\vec{v}}$ $(N = N')$

7. $R^{\theta}_{\vec{v}} R^{\phi}_{\vec{u}} = R^{\phi}_{\vec{u}} R^{\theta}_{\vec{v}} \Rightarrow R^{\theta}_{\vec{v}} R^{\phi}_{\vec{u}} R^{-\theta}_{\vec{v}} = R^{\phi}_{\vec{u}} R^{\theta}_{\vec{v}} R^{-\theta}_{\vec{v}} \Rightarrow R^{\phi}_{R^{\theta}_{\vec{v}}\vec{u}} = R^{\phi}_{\vec{u}}$

$\Rightarrow R^{\theta}_{\vec{v}} \cdot \vec{u} = k\vec{u}$ $(k>0)$ $\Rightarrow$ so $\vec{u}$ and $\vec{v}$ are parallel

if $\vec{u}$ and $\vec{v}$ are NOT parallel

then $R^{\theta}_{\vec{v}} \cdot \vec{u}$ and $\vec{u}$ are NOT parallel

$\Rightarrow R^{\theta}_{\vec{v}} \cdot \vec{u} \neq k\vec{u}$ $(\forall k>0)$, which is wrong

Problem 5:

1. $R_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$

2. $n=2$: suppose $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $R\vec{u} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\Rightarrow \tan\theta = -\frac{y}{x}$, so $\exists R$, such that $R\vec{u} = \vec{e_1}$

suppose when $n=k$, $\forall \vec{u_k}$, $\exists R_k$ such that $R_k\vec{u_k} = \vec{e_1}$

then when $n=k+1$:

$\forall a \in \mathbb{R}$, $\vec{u_{k+1}} = \begin{bmatrix} \sqrt{1-a^2}\,\vec{u_k} \\ a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} R_k & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} \vec{u_{k+1}} = \begin{bmatrix} \sqrt{1-a^2} \\ 0 \\ \vdots \\ a \end{bmatrix}$

$\Rightarrow \underbrace{\begin{bmatrix} \cos\theta & & -\sin\theta \\ & 1 \ddots 1 & \\ \sin\theta & & \cos\theta \end{bmatrix} \begin{bmatrix} R_k & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix}}_{R_{k+1}} \vec{u_{k+1}} = \vec{e_1}$, $\tan\theta = -\frac{a}{\sqrt{1-a^2}}$

So $\forall \vec{u_{k+1}}$, $\exists R_{k+1}$ such that $R_{k+1}\vec{u_{k+1}} = \vec{e_1}$

3. $A = RU \Rightarrow R^{-1}A = R^{-1}RU = U \in \{\text{upper triangular matrix}\}$

suppose $\underset{(n\times n)}{A_n} = \begin{bmatrix} a_{11} & \vec{a_1}'^T \\ \vec{a_1} & A_n' \end{bmatrix}$ $\underset{(n\times n)}{R_n^{-1}} = \begin{bmatrix} r_{11} & \vec{r_1}'^T \\ \vec{r_1} & R_n^{-1\prime} \end{bmatrix}$

from the last subproblem, we know $\exists R_n^{-1}$, such that $R_n^{-1}\begin{bmatrix} a_{11} \\ \vec{a_1} \end{bmatrix} = \begin{bmatrix} a_n \\ \vec{0} \end{bmatrix}$

then let $R_{n-1}^{-1} = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & R_{n-1}^{-1\prime} \end{bmatrix}$, which satisfies: $R_{n-1}^{-1} \times (A_{n-1} \times \vec{e_1}) \parallel \vec{e_1}$

and so on, we get $\underbrace{R_1^{-1} R_2^{-1} \cdots R_n^{-1}}_{R^{-1}} A = U$

$\Rightarrow A = RU$ Q.E.D

4. suppose $H = [\vec{h_1} \cdots \vec{h_n}]$ ($\vec{h_1} \sim \vec{h_n}$ all unit vectors) $\Rightarrow \vec{h_i}^T h_i = 1$

$RH = [R\vec{h_1} \cdots\cdots R\vec{h_n}]$ $\quad \vec{h_i}^T\vec{h_j} = 0$ ($i \cdot j \in [1, n]$))

first consider R is a single Givens solution

$$R = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos\theta & & -\sin\theta \\ & & \sin\theta & \ddots & \cos\theta \\ & & & & \ddots 1 \end{bmatrix} \quad (R\vec{h_i})^T \cdot R\vec{h_i} = \vec{h_i}^T \cdot R^T \cdot R\,\vec{h_i}$$

$$= \vec{h_i}^T \cdot \vec{h_i} = 1$$

$\Rightarrow R\vec{h_i}$ are still unit vectors

$$(R\vec{h_i})^T (R\vec{h_j}) = \vec{h_i}^T \cdot R^T \cdot R \cdot \vec{h_j} = \vec{h_i}^T\,\vec{h_j} = 0$$

$\Rightarrow R\vec{h_i}$ are still mutually orthogonal

then consider $R = R_k R_{k-1} \cdots R_1$ ($R_1 \sim R_k$ are all single Givens solution)

$$R^T \cdot R = (R_1^T R_2^T \cdots R_k^T) \cdot (R_k R_{k-1} \cdots R_1) = I$$

$$\Rightarrow (R\vec{h_i})^T (R\vec{h_i}) = \vec{h_i}^T R^T R \vec{h_i} = \vec{h_i}^T \cdot \vec{h_i} = 1$$

$\Rightarrow R\vec{h_i}$ are still unit vectors

$$(R\vec{h_i})^T (R\vec{h_j}) = \vec{h_i}^T R^T R \vec{h_j} = \vec{h_i}^T \vec{h_j} = 0$$

$\Rightarrow R\vec{h_i}$ are still mutually orthogonal

5. $H = [\vec{h_1} \cdots \vec{h_n}]$

from subproblem 3 $\Rightarrow$ $H = RU$, U is a upper triangular matrix

$\Downarrow$

$U = R^{-1}H$ (R^{-1} is still a rotation matrix)

from subproblem 4 $\Rightarrow$ columns in $R^{-1}H$ are unit vectors and mutually orthogonal

let $\vec{u_i} = U\vec{e_i} = \begin{bmatrix} u_{1i} \\ u_{2i} \\ \vdots \\ u_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

then $u_{1i}^2 + \cdots + u_{ii}^2 = 1$

since $\vec{u_1} \perp \vec{u_2}$, $\vec{u_1} \perp \vec{u_3}$, $\vdots$, $\vec{u_1} \perp \vec{u_n}$ $\Rightarrow \begin{cases} u_{1i} = 0 \\ (i \in [1,n]) \\ u_{11} = \pm 1 \end{cases} \Rightarrow \vec{u_2} = \begin{bmatrix} 0 \\ u_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{cases} u_{22} = \pm 1 \\ u_{2i} = 0 \\ (i \in [1,n]) \end{cases} \Rightarrow \cdots$

and so on, all $u_{ii} = \pm 1$, and other entries are all zero

so RH is a diagonal matrix with ± 1 on the diagonal