Note on Slutsky Equation

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1 Exogenous Income (Chapter 8)

We are interested in the effect of a price change on a consumer's demand. More formally, assume that the consumer's preference relation is monotonic and strictly convex, and let the consumer's most preferred consumption bundle be $x(p, y)$ under the price vector p and income level y. Note that x and p are n-dimensional, where n is the number of goods in the model. Consider a particular price vector p^0 and income level y^0, and we are interested in how $x(p, y^0)$ changes as price p moves around p^0. In particular, the partial derivative

$$\frac{\partial x_i}{\partial p_j}(p^0, y^0)$$

describes how sentitive the demand for the i-th good is to a change in the price of the j-th good when price p moves around p^0.

Assuming that the consumer's preference relation is monotonic and strictly convex, and that her demand function x is differentiable, we can verify that

$$\frac{\partial x_i}{\partial p_j}(p^0, y^0) = \frac{\partial}{\partial p_j} x_i(p, px^0)|_{p=p^0} - x_j^0 \cdot \frac{\partial x_i}{\partial y}(p^0, y^0)$$

where $x^0 = x(p^0, y^0)$ is the optimal consumption bundle under the price p^0 and income y^0. The first term on the right-hand side $\frac{\partial}{\partial p_j} x_i(p, px^0)|_{p=p^0}$ is known as the substitution effect and the second term $-x_j^0 \cdot \frac{\partial x_i}{\partial y}(p^0, y^0)$ is known as the income effect.

To understand the notations above, $\frac{\partial x_i}{\partial p_j}$ (or $\frac{\partial x_i}{\partial y}$) are the partial derivative of the function $x(p, y)$ with respect to p_j (or y), and in the equation above it is evaluated at the point (p^0, y^0). For the substitution effect $\frac{\partial}{\partial p_j} x_i(p, px^0)|_{p=p^0}$, we consider $x_i(p, px^0)$ as a function in p, and take its derivative and then evaluate it at $p = p^0$.

The decomposition equation can be verified by applying the chain rule to the substitution effect term:

$$\begin{aligned}\frac{\partial}{\partial p_j}x_i(p,px^0)|_{p=p^0} &= [\frac{\partial x_i}{\partial p_j}(p,px^0)+\frac{\partial x_i}{\partial y}(p,px^0)\cdot\frac{\partial}{\partial p_j}(px^0)]|_{p=p_0}\\ &= \frac{\partial x_i}{\partial p_j}(p^0,p^0x^0)+\frac{\partial x_i}{\partial y}(p^0,p^0x^0)\cdot x_j^0\\ &= \frac{\partial x_i}{\partial p_j}(p^0,y^0)+\frac{\partial x_i}{\partial y}(p^0,y^0)\cdot x_j^0,\end{aligned}$$

where the last equality is because $p^0x^0 = y^0$, since the preference relation is assumed to be monotonic. Moving the last term $\frac{\partial x_i}{\partial y}(p^0,y^0)\cdot x_j^0$ to the left-hand side of the equation gives us the decomposition equation we want to verify.

To interpret the substitution effect term $\frac{\partial}{\partial p_j}x_i(p,px^0)|_{p=p^0}$, note that as price p changes, the income px^0 is adjusted such that the previously optimal bundle x^0 is still just affordable. To interpret the income effect term $-x_j^0\cdot\frac{\partial x_i}{\partial y}(p^0,y^0)$, note that $-x_j^0$ is the effect of an increase in p_j on the consumer's "effective" income, and $\frac{\partial x_i}{\partial y}(p^0,y^0)$ is the effect of an income increase on the consumer's demand for good i.

2 Endogenous Income (Chapter 9)

Now let's consider the model in which income comes from endowment, and therefore relies on the price vector. In that case, a price change will have an additional income effect through affecting the value of the consumer's endowment.

More formally, fix some endowment vector ω^0 and price vector p^0, and we are interested in the derivative

$$\frac{\partial}{\partial p_j}x_i(p,p\omega^0)|_{p=p^0},$$

which describes how sentitive the demand for the i-th good is to a change in the price of the j-th good when price p moves around p^0.

Assuming that the consumer's preference relation is monotonic and strictly convex, and that her demand function x is differentiable, we can verify that

$$\frac{\partial}{\partial p_j}x_i(p,p\omega^0)|_{p=p^0} = \frac{\partial}{\partial p_j}x_i(p,px^0)|_{p=p^0} - x_j^0\cdot\frac{\partial x_i}{\partial y}(p^0,p^0\omega^0)+\omega_j^0\cdot\frac{\partial x_i}{\partial y}(p^0,p^0\omega^0),$$

where $x^0 = x(p^0,p^0\omega^0)$ is the optimal consumption bundle under the price p^0 and endowment ω^0.

The first term on the right-hand side $\frac{\partial}{\partial p_j}x_i(p,px^0)|_{p=p^0}$ is known as the substitution effect, the second term $-x_j^0\cdot\frac{\partial x_i}{\partial y}(p^0,y^0)$ is known as the ordinary income effect, and the third term $\omega_j^0\cdot\frac{\partial x_i}{\partial y}(p^0,p^0\omega^0)$ is known as the endowment income effect. The last two terms combined, $-(x_j^0-\omega_j^0)\cdot\frac{\partial x_i}{\partial y}(p^0,p^0\omega^0)$, is the total income effect.

This decomposition equation can be verified by applying the chain rule to $\frac{\partial}{\partial p_j} x_i(p, p\omega^0)|_{p=p^0}$ and $\frac{\partial}{\partial p_j} x_i(p, px^0)|_{p=p^0}$:

$$\begin{aligned}
\frac{\partial}{\partial p_j} x_i(p, p\omega^0)|_{p=p^0} &= [\frac{\partial x_i}{\partial p_j}(p, p\omega^0) + \frac{\partial x_i}{\partial y}(p, p\omega^0) \cdot \frac{\partial}{\partial p_j}(p\omega^0)]|_{p=p_0} \\
&= \frac{\partial x_i}{\partial p_j}(p^0, p^0\omega^0) + \frac{\partial x_i}{\partial y}(p^0, p^0\omega^0) \cdot \omega_j^0, \\
\frac{\partial}{\partial p_j} x_i(p, px^0)|_{p=p^0} &= [\frac{\partial x_i}{\partial p_j}(p, px^0) + \frac{\partial x_i}{\partial y}(p, px^0) \cdot \frac{\partial}{\partial p_j}(px^0)]|_{p=p_0} \\
&= \frac{\partial x_i}{\partial p_j}(p^0, p^0x^0) + \frac{\partial x_i}{\partial y}(p^0, p^0x^0) \cdot x_j^0 \\
&= \frac{\partial x_i}{\partial p_j}(p^0, p^0\omega^0) + \frac{\partial x_i}{\partial y}(p^0, p^0\omega^0) \cdot x_j^0,
\end{aligned}$$

where the last equality is due to $p^0x^0 = p^0\omega^0$, because monotonicity of the preference relation implies that the budget constraint must be binding. Comparing the two equations above gives us the decomposition equation we want to verify.

The substitution effect can be interpreted as in the previous section. The ordinary income effect term also admits the same interpretation as in the previous section, since $-x_j^0$ is the effect of an increase in p_j on the "effective" income, if the nominal income is not affected by the price increase. However, now we have a third effect, the endowment income effect, that captures the fact that a change in p_j will affect the consumer's nominal income. In particular, the term ω_j^0 is the effect of an increase in p_j on the nominal income.

Moreover, note that the total income effect is $-(x_j^0 - \omega_j^0) \cdot \frac{\partial x_i}{\partial y}(p^0, p^0\omega^0)$, and the term $-(x_j^0 - \omega_j^0)$, which is the negation of the net demand for good j, is precisely the effect of an increase in p_j on the "effective" income, taking into account the fact that the nominal income is affected by the price increase. Therefore, an increase in p_j will increase (or decrease) the "effective" income of a net seller (or a net buyer) of good j.