8.1

1. $$M_L = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} \Rightarrow \text{Vandermonde matrix}$$

2. $$\det(M_L) = \prod_{1\le j<i\le n} (a_i - a_j) \ne 0$$

$\Longleftrightarrow a_i \ne a_j$ for all $j<i$

$\Longleftrightarrow a_1, \cdots, a_n$ are distinct

3. $a_1, \cdots, a_n$ are distinct

$\Longleftrightarrow M_L$ is invertible

$\Longleftrightarrow$ columns of M_L^T form a basis for V^*, which are $ev_{a_1}, ev_{a_2}, \cdots, ev_{a_n}$

$$M_L^T = \begin{pmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ a_1^2 & \cdots & a_n^2 \\ \vdots & & \vdots \\ a_1^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$$

$\Longleftrightarrow ev_{a_1}, ev_{a_2}, \cdots, ev_{a_n}$ for a basis for V^*

4. $ev_{-1}: p \mapsto p(-1)$ pick basis: $1, x, x^2$

$ev_0: p \mapsto p(0)$

$ev_1: p \mapsto p(1)$

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

$\Rightarrow p_{-1} = -\frac{1}{2}x + \frac{1}{2}x^2$

$p_0 = 1 - x^2$

$p_1 = \frac{1}{2}x + \frac{1}{2}x^2$

5. under the basis $(1, x, x^2, x^3)$

$ev_{-2} = (1 \ -2 \ 4 \ -8)$

$ev_{-1} = (1 \ -1 \ 1 \ -1)$

$ev_0 = (1 \ 0 \ 0 \ 0)$

$ev_1 = (1 \ 1 \ 1 \ 1)$

$ev_2 = (1 \ 2 \ 4 \ 8)$

$ev_{-2} + ev_2 = (2 \ 0 \ 8 \ 0)$

$ev_{-1} + ev_1 = (2 \ 0 \ 2 \ 0)$

$\Rightarrow (ev_{-2} + ev_2) - 4(ev_{-1} + ev_1) = (-6 \ 0 \ 0 \ 0)$

$\Rightarrow ev_{-2} + ev_2 - 4ev_{-1} - 4ev_1 + 6ev_0 = 0$

8.2

1. pick a basis for V: $1, x, x^2$

$$M=\begin{pmatrix} 2 & 1 & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \end{pmatrix}\begin{matrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{matrix} \qquad M^{-1}=\begin{array}{c}\begin{pmatrix} \frac{9}{2} & -18 & 15 \\ -18 & 96 & -90 \\ 15 & -90 & 90 \end{pmatrix}\\ \begin{matrix}\downarrow & \downarrow & \downarrow \\ p_0 & p_1 & p_2\end{matrix}\end{array}$$

$$\Rightarrow p_0=\frac{9}{2}-18x+15x^2$$
$$p_1=-18+96x-90x^2$$
$$p_2=15-90x+90x^2$$

2.
$$\alpha_0: f(x)\longmapsto \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$
$$\alpha_1: f(x)\longmapsto \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$
$$\alpha_2: f(x)\longmapsto \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$
$$\alpha_3: f(x)\longmapsto \frac{1}{\pi}\int_0^{2\pi} f(x)\sin 2x\,dx$$
$$\alpha_4: f(x)\longmapsto \frac{1}{\pi}\int_0^{2\pi} f(x)\cos 2x\,dx$$

3. $ev_{\vec{v}_1}, ev_{\vec{v}_2}, \cdots, ev_{\vec{v}_n}$

8.3

$$\nabla f(\vec{v})=\lim_{t\to 0}\frac{f(\vec{p}+t\vec{v})-f(\vec{p})}{t} = f_x'(\vec{p})\cdot(\vec{v}\cdot\vec{e}_1)+f_y'(\vec{p})\cdot(\vec{v}\cdot\vec{e}_2)$$

$$\nabla f(\vec{v}+\vec{w})=f_x'(\vec{p})\cdot(\vec{v}+\vec{w})\cdot\vec{e}_1+f_y'(\vec{p})\cdot(\vec{v}+\vec{w})\cdot\vec{e}_1$$
$$=\left(f_x'(\vec{p})\,\vec{v}\cdot\vec{e}_1+f_y'(\vec{p})\cdot\vec{v}\cdot\vec{e}_2\right)+\left(f_x'(\vec{p})\,\vec{w}\cdot\vec{e}_1+f_y'(\vec{p})\cdot\vec{w}\cdot\vec{e}_2\right)$$
$$=\nabla f(\vec{v})+\nabla f(\vec{w})$$

$$\nabla f(k\vec{v})=f_x'(\vec{p})\cdot k\vec{v}\cdot\vec{e}_1+f_y'(\vec{p})\cdot k\vec{v}\cdot\vec{e}_2$$
$$=k\left(f_x'(\vec{p})\cdot\vec{v}\cdot\vec{e}_1+f_y'(\vec{p})\cdot\vec{v}\cdot\vec{e}_2\right)=k\,\nabla f(\vec{v})$$

$\Rightarrow \nabla f$ is a dual vector in $(\mathbb{R}^2)^*$

the "coordinates" are coordinates of $\vec{v}$ under the basis $(\vec{e}_1\ \vec{e}_2)$

8.4

1. $c(x,y,z)=c_0$

$$v(c\cdot c) = v(c)$$
$$= c(\vec{p})\,v(c) + c(\vec{p})\,v(c)$$
$$= 2c_0 \cdot v(c)$$
$$\Rightarrow v(c) = 0$$

2. $$v((x-p_1)f) = v(xf) - p_1 v(f)$$
$$= x(\vec{p})\,v(f) + f(\vec{p})\,v(x) - p_1 v(f)$$
$$= f(\vec{p})\,v(x)$$
$$v((y-p_2)f) = v(yf) - p_2 v(f)$$
$$= y(\vec{p})\,v(f) - p_2 v(f) + f(\vec{p})\,v(y)$$
$$= f(\vec{p})\,v(y)$$
$$v((z-p_3)f) = v(zf) - p_3 v(f)$$
$$= z(\vec{p})\,v(f) + f(\vec{p})\,v(z) - p_3 v(f)$$
$$= f(\vec{p})\,v(z)$$

3. using the proposition proved in the last subproblem

$$v((x-p_1)^a(y-p_2)^b(z-p_3)^c)$$
$$= v((x-p_1)^a(y-p_2)^b(z-p_3)^c \cdot 1)$$
$$= \left((x-p_1)^{a-1}(y-p_2)^b(z-p_3)^c\right)(\vec{p}) \cdot v(x)$$
$$= 0 \cdot v(x) = 0$$

4. $$v(f) = v(f(\vec{p}) + f'_x(x-p_1) + f'_y(y-p_2) + f'_z(z-p_3))$$
$$= 0 + f'_x(\vec{p})\,v(x) + f'_y(\vec{p})\,v(y) + f'_z\,v(z)$$
$$= \frac{\partial f}{\partial x}(\vec{p})\,v(x) + \frac{\partial f}{\partial y}(\vec{p})\,v(y) + \frac{\partial f}{\partial z}(\vec{p})\,v(z)$$

5. using the basis $\left(\frac{\partial f}{\partial x}(\vec{p}), \frac{\partial f}{\partial y}(\vec{p}), \frac{\partial f}{\partial z}(\vec{p})\right)$

$$\Rightarrow \quad v = \begin{bmatrix} v(x) \\ v(y) \\ v(z) \end{bmatrix} = \nabla_v$$

8.5

1. $\forall w \in \ker(L^*) \subseteq W^*$

$\Leftrightarrow L^*(w) = 0 \in V^*$

$\Leftrightarrow \forall v \in V, \ (L^*(w'))(v) = 0$

$\Leftrightarrow w'(L(v)) = 0$

$\Leftrightarrow w'$ kills $\mathrm{Ran}(L)$

$\Leftrightarrow \ker(L^*) = \{w' \mid \forall w \in \mathrm{Ran}(L), \ w'(w) = 0\}$

2. $\forall v' \in \mathrm{Ran}(L^*) \subseteq V^*$

$\Leftrightarrow \exists w' \in W^*, \ L^*(w') = v'$

$\Leftrightarrow \forall v \in V, \ [L^*(w')](v) = v'(v) \in \mathbb{R}$

let $v \in \ker(L)$, which means $L(v) = 0 \in W$

$$\Rightarrow (L^*(w'))(v) = \underset{\underset{w'(0)=0}{\parallel}}{w'(L(v))} = v'(v)$$

$\Leftrightarrow v'(v) = 0, \ v \in \ker(L)$

$\Leftrightarrow v'$ kills $\ker(L)$

$\Leftrightarrow \mathrm{Ran}(L^*) = \{v' \mid \forall v \in \ker(L), \ v'(v) = 0\}$