Note on General Equilibrium

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Model

- n goods
- I: the finite set of consumers
- J: the finite set of firms
- $\succsim^i$: preference relation of consumer i on $\mathbb{R}^n_+$, represented by u^i
- $Y^j \subset \mathbb{R}^n$: production set of firm j (a production plan $y = (y_1, \ldots, y_n)$)
- $\omega^i \in \mathbb{R}^n_+$: endowment vector of consumer i (Total endowment $\bar{\omega} = \sum_{i \in I} \omega^i$)
- $\theta^{ij} \in [0, 1]$: consumer i's share in firm j, s.t. $\sum_{i \in I} \theta^{ij} = 1$ for each $j \in J$

Definition 1. An *allocation* is (x, y), where
(1) $x = (x^i)_{i \in I}$, and $x^i \in \mathbb{R}^n_+$ is the consumption bundle of consumer i
(2) $y = (y^j)_{j \in J}$, and $y^j \in Y^j$ is the production plan of firm j
(3) (x, y) is subject to the feasibility constraint

$$\sum_{i \in I} x^i = \sum_{i \in I} \omega^i + \sum_{j \in J} y^j.$$

Two examples:

- Edgeworth box economy: $|I| = 2$, $|J| = 0$, $n = 2$
- Robinson Crusoe economy: $|I| = 1$, $|J| = 1$, $n = 2$

Definition 2. An allocation $(\hat{x}, \hat{y})$ *Pareto dominates* another allocation (x, y), if $\hat{x}^i \succsim^i x^i$ for all $i \in I$, and there exists $i_0 \in I$ s.t. $\hat{x}^{i_0} \succ^{i_0} x^{i_0}$. An allocation (x, y) is *Pareto efficient*, if it is not Pareto dominated by any other allocation.

Walrasian Equilibrium (WE)

Definition 3. The triple (p^*, x^*, y^*) is a *Walrasian equilibrium / price equilibrium / competitive equilibrium / market equilibrium*, if

(i) $p^* \in \mathbb{R}^n_{++}$,

(ii) (x^*, y^*) is an allocation,

(iii) For each $j \in J$, we have $y^{*j} \in \arg\max_{y^j \in Y^j} p^* y^j$,

(iv) For each $i \in I$, we have $x^{*i} \in \arg\max_{x^i \in \mathbb{R}^n_+} u^i(x^i)$ s.t. $p^* x^i \leq p^* \omega^i + \sum_{j \in J} \theta^{ij} p^* y^{*j}$.

Typical approach to calculate a WE:

Consider price vector $p \in \mathbb{R}^n_{++}$ as a candidate for an equilibrium price.

(1) Solve the profit maximization problem for each firm j and find $y^j(p)$.

(2) Calculate each consumer's income according to all firms' maximized profit.

(3) Solve the utility maximization problem for each consumer i, and find its solution $x^i(p)$.

(4) Check whether there exists $y^j \in y^j(p)$ and $x^i \in x^i(p)$ s.t. $\sum_{i \in I} x^i = \sum_{i \in I} \omega^i + \sum_{j \in J} y^j$.

First Welfare Theorem

Theorem 4. *Assume that $\succsim^i$ satisfies monotonicity for all consumer $i \in I$. If (p^*, x^*, y^*) is a WE, then the equilibrium allocation (x^*, y^*) is Pareto efficient.*

Proof. Suppose that (x^*, y^*) is not Pareto efficient, i.e. there exists an allocation $(\hat{x}, \hat{y})$ that Pareto dominates (x^*, y^*). So $\hat{x}^i \succsim^i x^{*i}$ for all $i \in I$, and there exists $i_0 \in I$ s.t. $\hat{x}^{i_0} \succ^{i_0} x^{*i_0}$.

For $i = i_0$, note that

$$p^* \hat{x}^i > p^* \omega^i + \sum_{j \in J} \theta^{ij} p^* y^{*j}.$$

For $i \neq i_0$, by monotonicity we have

$$p^* \hat{x}^i \geq p^* \omega^i + \sum_{j \in J} \theta^{ij} p^* y^{*j}.$$

Adding up the inequalities across all i gives us

$$\begin{aligned}
p^* \sum_{i \in I} \hat{x}^i &> p^* \sum_{i \in I} \omega^i + \sum_{i \in I} [\sum_{j \in J} \theta^{ij} p^* y^{*j}] \\
&= p^* \sum_{i \in I} \omega^i + \sum_{j \in J} [\sum_{i \in I} \theta^{ij} p^* y^{*j}] \\
&= p^* \sum_{i \in I} \omega^i + \sum_{j \in J} [p^* y^{*j} \sum_{i \in I} \theta^{ij}] \\
&= p^* \sum_{i \in I} \omega^i + \sum_{j \in J} p^* y^{*j} \\
&\geq p^* \sum_{i \in I} \omega^i + \sum_{j \in J} p^* \hat{y}^j \\
&= p^* [\sum_{i \in I} \omega^i + \sum_{j \in J} \hat{y}^j].
\end{aligned}$$

However, by definition the allocation $(\hat{x}, \hat{y})$ satisfies

$$\sum_{i \in I} \hat{x}^i = \sum_{i \in I} \omega^i + \sum_{j \in J} \hat{y}^j,$$

which contradicts to the inequality above. $\square$