1.[20 points] Indifferent curves

(1) [5 points] Please draw the indifferent curves of Perfect Complements.

(2) [5 points] Please draw the indifferent curve for the following utility function:

$$U = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

(3) [5 points] Please calculate the MRS (marginal rate-of-substitution) of the utility function above at the point $(x_1, x_2) = (1,1)$.

(4) [5 points] Suppose the utility function is

$$U = \frac{x_1 + x_2 + x_3 - \min\{x_1, x_2, x_3\}}{2}$$

Let $x_1 = 6$ be fixed. Please draw the indifferent curve that goes through the point $(x_2, x_3) = (5,7)$ on the plane of x_2 and x_3. (Let's set the horizontal axis as x_2 and the vertical axis as x_3)

2. [20 Points] Consider the following utility function (Stone-Geary Utility Function):

$$u(x_1, \cdots, x_n) = \prod_{i=1}^{n} (x_i - a_i)^{b_i}$$

where x_i are the goods, $a_i \geq 0$ is the parameter that represents the necessary level of goods i for each household to survive, and $b_i \geq 0$ is the parameter s.t. $\sum_{i=1}^{n} b_i = 1$. Denote the price of x_i as p_i. The total income of the consumer is y, and let's assume that $y > \sum_{i=1}^{n} p_i a_i$.

(1) [5 Points] Please write down the budget constraint and the survival constraint of the consumer.

Let $x = (x_1, x_2, \cdots, x_n)$ be the consumption bundle chosen by the consumer under the budget constraint and the survival constraint. Now we define "discretionary income"（可自由支配收入） as $y^d = y - \sum_{i=1}^{n} p_i a_i$, and the "excess consumption"（超额消费） of good i as $c_i^e = x_i^* - a_i$.

Claim: For all i,

$$\frac{p_i c_i^e}{y^d} = b_i$$

Please show that the Claim above is true in the following cases:

(2) [4 Points] $n = 1, b_1 = 1$

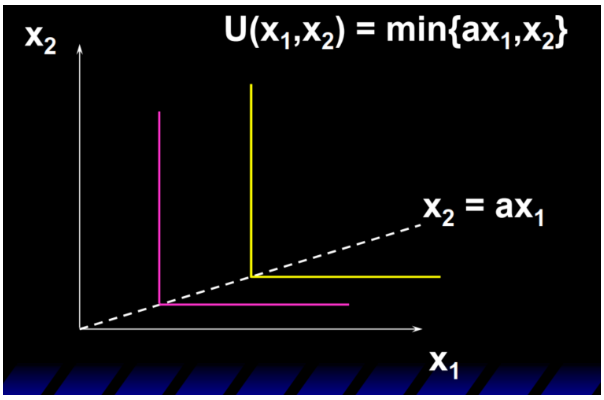
(3) [5 Points] $n = 2, a_1 = a_2 = 0, b_1 = b_2 = \frac{1}{2}$

(4) [4 Points] $n = 2$

(5) [2 Points] n in general.

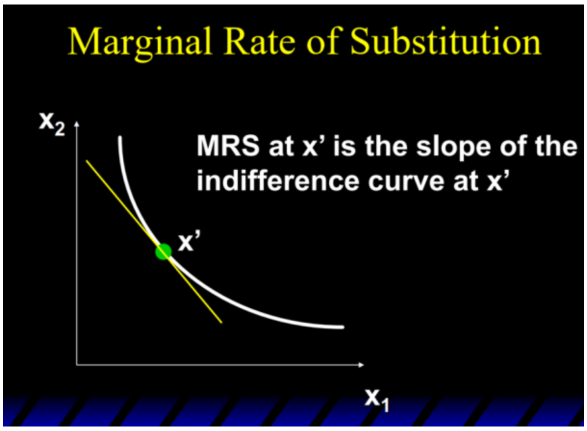
1.[20 points] Indifferent curves

(1) [5 points] Please draw the indifferent curves of Perfect Complements.



(2) [5 points] Please draw the indifferent curve for the following utility function:

$$U = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$



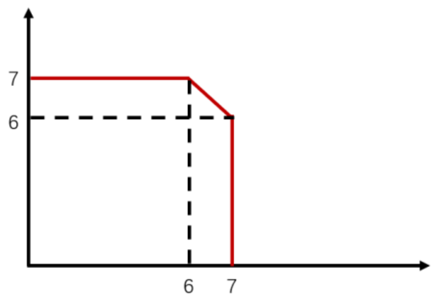
(3) [5 points] Please calculate the MRS (marginal rate-of-substitution) of $U = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$ at $(x_1, x_2) = (1,1)$.

$$MRS = -\frac{1}{2}$$

(4) [5 points] Suppose the utility function is

$$U = \frac{x_1 + x_2 + x_3 - \min\{x_1, x_2, x_3\}}{2}$$

Let $x_1 = 6$ be fixed. Please draw the indifferent curve that goes through the point $(x_2, x_3) = (5,7)$ on the plane of x_2 and x_3. (Let's set the horizontal axis as x_2 and the vertical axis as x_3)



2. [20 Points] Consider the following utility function (Stone-Geary Utility Function):

$$u(x_1, \cdots, x_n) = \prod_{i=1}^{n} (x_i - a_i)^{b_i}$$

where x_i are the goods, $a_i \geq 0$ is the parameter that represents the necessary level of goods i for each household to survive, and $b_i \geq 0$ is the parameter s.t. $\sum_{i=1}^{n} b_i = 1$. Denote the price of x_i as p_i. The total income of the consumer is y, and let's assume that $y > \sum_{i=1}^{n} p_i a_i$.

(1) [5 Points] Please write down the budget constraint and the survival constraint of the consumer.
Budget constraint: $y = \sum_{i=1}^{n} p_i x_i$
Survival constraint: $x_i \geq a_i, \forall\, i \in \{1, \cdots, n\}$

Let $x = (x_1, x_2, \cdots, x_n)$ be the consumption bundle chosen by the consumer under the budget constraint and the survival constraint. Now we define "discretionary income"（可自由支配收入）as $y^d = y - \sum_{i=1}^{n} p_i a_i$, and the "excess consumption"（超额消费）of good i as $c_i^e = x_i^* - a_i$.
Claim: For all i,

$$\frac{p_i c_i^e}{y^d} = b_i$$

Please show that the Claim above is true in the following cases:
(2) [4 Points] $n = 1, b_1 = 1$

$$x_1^* = \frac{Y}{p_1}$$

$$c_1^e = \frac{Y}{p_1} - a_1$$

$$Y^d = Y - p_1 a_1$$

$$\Longrightarrow p_1 c_1^e = Y^d$$

(3) [5 Points] $n = 2, a_1 = a_2 = 0, b_1 = b_2 = \frac{1}{2}$

$$U = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$s.t. p_1 x_1 + p_2 x_2 = Y$$

$$x_1^* = \frac{Y}{2p_1}, x_2^* = \frac{Y}{2p_2}$$

$$c_1^e = \frac{Y}{2p_1}, c_2^e = \frac{Y}{2p_2}$$

$$Y^d = Y$$

$$\frac{p_1 c_1^e}{Y^d} = \frac{p_2 c_2^e}{Y^d} = \frac{1}{2}$$

(4) [4 Points] $n = 2$

$$U = (x_1 - a_1)^{b_1} (x_2 - a_2)^{b_2}$$

$$= (x_1 - a_1)^{b_1} \left(\frac{Y - p_1 x_1}{p_2} - a_2 \right)^{b_2}$$

$$\frac{\partial U}{\partial x_1} = -b_2 \frac{p_1}{p_2} (x_1 - a_1)^{b_1} \left(\frac{Y - p_1 x_1}{p_2} - a_2 \right)^{b_2 - 1} + b_1 (x_1 - a_1)^{b_1 - 1} \left(\frac{Y - p_1 x_1}{p_2} - a_2 \right)^{b_2}$$

When $\frac{\partial U}{\partial x_1} = 0$, $x_1^* = \frac{b_1 Y + a_1 b_2 p_1 - b_1 a_2 p_2}{(b_1+b_2)p_1} = \frac{b_1 Y + a_1 b_2 p_1 - b_1 a_2 p_2}{p_1}$.

$$c_1^e = x_1^* - a_1 = \frac{b_1(Y - a_1 p_1 - a_2 p_2)}{p_1}$$

Thus, $\frac{p_1 c_1^e}{Y^d} = b_1$. Similarly, we can prove that $\frac{p_2 c_2^e}{Y^d} = b_2$.

(5) [2 Points] For general $n \in N^+$,

$$MRS_{ij} = \frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

$$\Longrightarrow \frac{b_i c_j^e}{b_j c_i^e} = \frac{p_i}{p_j}$$

$$\Longrightarrow \frac{p_i c_i^e}{b_i} = \lambda$$

where λ is a constant.

$$Y^d = \sum_{i=1}^{n} p_i c_i^e = \sum_{i=1}^{n} b_i \lambda = \lambda$$

$$\Longrightarrow \frac{p_i c_i^e}{Y^d} = b_i$$

Also, we can use Lagrange multipliers to solve the problem.

$$L(x_1, \cdots, x_n) = \prod_{i=1}^{n} (x_i - a_i)^{b_i} + \lambda\left(y - \sum_{i=1}^{n} p_i x_i\right)$$

$$\frac{\partial L(x_1, \cdots, x_n)}{\partial x_i} = \frac{b_i \prod_{i=1}^{n} (x_i - a_i)^{b_i}}{x_i - a_i} - \lambda p_i$$

Let $\frac{\partial L(x_1, \cdots, x_n)}{\partial x_i} = 0$. That is,

$$\frac{p_i(x_i - a_i)}{b_i} = \frac{\prod_{i=1}^{n} (x_i - a_i)^{b_i}}{\lambda}$$

$$\Rightarrow \sum_{i=1}^{n} p_i(x_i - a_i) = \sum_{i=1}^{n} b_i \frac{\prod_{i=1}^{n} (x_i - a_i)^{b_i}}{\lambda} = \frac{\prod_{i=1}^{n} (x_i - a_i)^{b_i}}{\lambda}$$

$$\Rightarrow \frac{p_i(x_i - a_i)}{b_i} = \sum_{i=1}^{n} p_i(x_i - a_i) = y - \sum_{i=1}^{n} p_i a_i = y^d$$

Besides, using mathematical induction（数学归纳法）is difficult when solving this question.

3. (20 pts in total) Peter works for McDonald's and is paid \$4 an hour. He also gets \$16 a day from his parents. Let y be his total income a day. He has 16 hours a day being awake, and so his leisure time r is the rest of the 16 hours after work. His utility function is

$$U(y, r) = yr$$

   (a) (5 pts) Please solve for the optimal choice for Peter.

   Answer:

$$\max_r U = [4(16 - r) + 16]r$$
$$\Rightarrow \quad r = 10$$

   Thus, he chooses to work 6 hours (rest for 10 hours) and receives a total income of \$40.

   (b) (15 pts) Now McDonald's raises Peter's salary to \$5 per hour, please solve for his new optimal choice and do the Slutsky decomposition for this change. Please be clear about the budget constraint in each step, and note that the income effect contains two parts.

   Answer: (r, y) changes from (10,40) to (9.6,48).

   The Slutsky decomposition is: Substitution effect from budget constraint 1 to budget constraint 2; Ordinary income effect from budget constraint 2 to budget constraint 3; Endowment income effect from budget constraint 3 to budget constraint 4, which is the new budget constraint.

   The budget constraints are:

$$\begin{aligned}
&\text{Budget constraint 1:} \quad (y-16)+4(r-16)=0 \Rightarrow y+4r=80\\
&\text{Budget constraint 2:} \quad (y-40)+5(r-10)=0 \Rightarrow y+5r=90\\
&\text{Budget constraint 3:} \quad (y-80)+5(r-0)=0 \Rightarrow y+5r=80\\
&\text{Budget constraint 4:} \quad (y-16)+5(r-16)=0 \Rightarrow y+5r=96
\end{aligned}$$

4. (20 pts in total) Steve is a computer programmer who needs to go to office T days a month. He can either walk or ride a shared bicycle to the office and back home, and each one-way travel takes him 1 hour on foot or 0.5 hour by bicycle. His monthly income is 3000 Yuan, and his utility functions is

$$U(m_{net}, t) = m_{net} - Bt,$$

where m_{net} is the rest of his monthly income after spending on shared bicycles, t is the total time spent on travel between home and office in a month, and $B > 0$ is the parameter that shows how Steve dislikes spending time on traffic.

The shared bicycle has the following pricing rule: 2 Yuan each time or 30 Yuans for a month pass, which allows unlimitled rides in a month.

(a) (7 pts) Suppose that Steve finds himself strictly prefer riding a bicycle for 2 Yuan a time to traveling on foot. Please provide the range for the parameter B.

Answer:

For each trip, we have the welfare loss:

$$0.5 \times B + 2 < 1 \times B \quad \Rightarrow \quad B > 4$$

(b) (7 pts) Suppose that Steve striclty prefer buying a month pass to paying for each ride. Please provide the range for the parameter T. (Note that he has to make $2T$ one-way travels in a month.)

Answer:

Because the welfare losses caused from the travelling time are the same. The only difference is from the expenditure, then:

$$2T \times 2 > 30 \quad \Rightarrow \quad T \geqslant 8$$

(c) (6 pts) Suppose that Steve finds himself indifferent between riding a bicycle for 2 Yuan a time to traveling on foot, and $T = 12$. However, now he finds that only 50% of the times he can find a shared bicycle. If he fails, he has to go on foot. Will he buy the month pass?

Answer:

Whether he buys a month pass or not, he always fails to find a shared bicycle with the other 50% possibility. Thus, this is identical to the problem that $T = 6$ but always finds a shared bicycle. From the result above, he will not buy a month pass.

5. (20 pts in total) Consider a student who is preparing for the exams of two courses and she needs to decide how to allocate her time. It takes her T_1 minutes to study an exercise question for course 1 and T_2 minutes to study an exercise question for course 2. Suppose that studying x_i exercise questions for course i will lead to an exam score of $s_i = \sqrt{x_i}$ for course i. (To simplify the analysis, let's assume that x_i has no upper bound and is NOT restricted to integer values.)

(a) (8 pts) Suppose that the student have m minutes in total to prepare for the two exams, and she aims at maximizing her total score obtained in the two exams. How will she allocate her time?

Answer:

She will spend $\frac{T_2 m}{T_1+T_2}$ minutes on course 1 and $\frac{T_1 m}{T_1+T_2}$ minutes on course 2.

(b) (8 pts) Now suppose that the student may choose her total time m spent on preparing for the exams, and her utility function is

$$U(s_1, s_2, m) = s_1 + s_2 - Am,$$

where $A > 0$ is a parameter that reflects how costly studying is for this student. How much time will she spend on preparing for each of the two exams?

Answer:

$$\begin{aligned} U &= \sqrt{x_1} + \sqrt{x_2} - Am \\ &= \sqrt{x_1} + \sqrt{x_2} - A(T_1 x_1 + T_2 x_2) \\ &= \sqrt{x_1}(1 - AT_1\sqrt{x_1}) + \sqrt{x_2}(1 - AT_2\sqrt{x_2}) \end{aligned}$$

Thus, The optimal condition is:

$$\sqrt{x_i} = \frac{1}{2AT_i} \quad \forall i = 1, 2$$

Thus, she will spend $\frac{1}{4A^2T_1}$ minutes on course 1 and $\frac{1}{4A^2T_2}$ minutes on course 2.

(c) (4 pts) Following the previous question, suppose that the student improves her efficiency in studying exercise questions (that is, she now has a smaller T_1 and T_2). Will she spend more time or less time on preparing for the exams?

Answer: More time.