1.

$$z = x^3 f(xy, \frac{y}{x})$$

$$\frac{\partial z}{\partial x} = 3x^2 f(xy, \frac{y}{x}) + x^3\left(f_1' \cdot y + f_2' \cdot \left(-\frac{y}{x^2}\right)\right)$$

$$= 3x^2 f(xy, \frac{y}{x}) + x^3 y f_1'(xy, \frac{y}{x}) - xy f_2'(xy, \frac{y}{x})$$

$$\frac{\partial z}{\partial y} = x^3\left(f_1' \cdot x + f_2' \cdot \frac{1}{x}\right)$$

$$= x^4 f_1'(xy, \frac{y}{x}) + x^2 f_2'(xy, \frac{y}{x})$$

2.

$$g'(x) = f_1' + f_2' \cdot \left(\varphi_1' \cdot 2x + \varphi_2' \cdot 3x^2\right)$$

$$g''(x) = f_{11}'' + f_{12}''\left(\varphi_1' 2x + \varphi_2' 3x^2\right)$$

$$+ \left(f_{21}'' + f_{22}''(\varphi_1' 2x + \varphi_2' 3x^2)\right)\left(2\varphi_1' + 2x(\varphi_{11}'' \cdot 2x + \varphi_{12}'' 3x^2) + 3x^2(\varphi_{21}'' \cdot 2x + \varphi_{22}'' 3x^2)\right)$$

$$g''(1) = f_{11}''(1,1) + f_{12}''(1,1)\left(2\varphi_1'(1,1) + 3\varphi_2'(1,1)\right)$$

$$+ \left(f_{21}''(1,1) + f_{22}''(1,1)\left(2\varphi_1'(1,1) + 3\varphi_2'(1,1)\right)\right) \cdot$$

$$\left(2\varphi_1'(1,1) + 4\varphi_{11}''(1,1) + 6\varphi_{12}''(1,1) + 6\varphi_{21}''(1,1) + 9\varphi_{22}''(1,1)\right)$$

3.

(1)

$$z=\left(\frac{y}{x}\right)^{\frac{x}{y}}=e^{\frac{x}{y}\ln\frac{y}{x}}$$

$$\frac{\partial z}{\partial x}=e^{\frac{x}{y}\ln\frac{y}{x}}\cdot\left(\frac{1}{y}\ln\frac{y}{x}+\frac{x}{y}\cdot\frac{x}{y}\cdot\left(-\frac{y}{x^2}\right)\right)$$

$$=\left(\frac{y}{x}\right)^{\frac{x}{y}}\cdot\left(\frac{1}{y}\ln\frac{y}{x}-\frac{1}{y}\right)$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2)}=2^{\frac{1}{2}}\cdot\left(\frac{1}{2}\ln 2-\frac{1}{2}\right)$$

$$=\frac{\sqrt{2}}{2}(\ln 2-1)$$

(2) $z=f(e^{x+y},xy)$

$$\frac{\partial z}{\partial x}=f_1'\cdot e^{x+y}+f_2'\cdot y$$

$$\frac{\partial z}{\partial y}=f_1' e^{x+y}+f_2'\cdot x$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{x+y}\left(f_{11}'' e^{x+y}+f_{12}''\cdot y\right)+x\left(f_{21}'' e^{x+y}+f_{22}'' y\right)$$

(3)

$$\frac{\partial z}{\partial x}=f'(x+y)+f'(x-y)+g(x+y)-g(x-y)$$

$$\frac{\partial^2 z}{\partial x^2}=f''(x+y)+f''(x-y)+g'(x+y)-g'(x-y)$$

$$\frac{\partial z}{\partial y}=f'(x+y)-f'(x-y)+g(x+y)+g(x-y)$$

$$\frac{\partial^2 z}{\partial y^2}=f''(x+y)+f''(x-y)+g'(x+y)-g'(x-y)$$

$$\frac{\partial^2 z}{\partial x\partial y}=f''(x+y)-f''(x-y)+g'(x+y)+g'(x-y)$$

4.

(1)

① $z(x,y)=f(y) \Rightarrow \forall (x,y)\in D,\ \frac{\partial z}{\partial x}\equiv 0$

↓同时对x求导

$\forall y\in[0,b]$; $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}=0$

② $\forall (x,y)\in D,\ \frac{\partial z}{\partial x}\equiv 0 \Rightarrow z(x,y)=f(y)$

则 $z(x,y)$ 与 x 无关，仅与 y 有关

令 $f(y)=z(x,y)$ $(\forall x\in[0,a])$ 即可

(2)

① $z(x,y)=F(y)+G(x) \Rightarrow \forall (x,y)\in D,\ \frac{\partial^2 z}{\partial x\partial y}\equiv 0$

$\frac{\partial z}{\partial y}=F'(y)$, $\frac{\partial^2 z}{\partial x\partial y}=0$

② $\forall (x,y)\in D,\ \frac{\partial^2 z}{\partial x\partial y}\equiv 0 \Rightarrow z(x,y)=F(y)+G(x)$

$\forall (x,y)\in D$, $\frac{\partial z}{\partial y}$ 与 x 无关

令 $\frac{\partial z}{\partial y}=f(y)$

积分得 $z(x,y)=\int f(y)\,dy + C(x)$

至多仅与x有关

令 $\int f(y)\,dy=F(y)$

$C(x)=G(x)$

得 $z(x,y)=F(y)+G(x)$

5.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \alpha\beta \neq 1$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} + \beta \frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial v}{\partial x} + \beta \left(\frac{\partial^2 z}{\partial u \partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2} \cdot \frac{\partial v}{\partial x} \right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \beta \frac{\partial^2 z}{\partial v \partial u} + \beta \frac{\partial^2 z}{\partial u \partial v} + \beta^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial z}{\partial y} = \alpha \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \alpha \left(\frac{\partial^2 z}{\partial u^2} + \beta \frac{\partial^2 z}{\partial v \partial u} \right) + \frac{\partial^2 z}{\partial u \partial v} + \beta \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \alpha^2 \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} + \alpha \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\Rightarrow A \frac{\partial^2 z}{\partial x^2} + 2B \frac{\partial^2 z}{\partial x \partial y} + C \frac{\partial^2 z}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} (A + 2B\alpha + C\alpha^2) + \frac{\partial^2 z}{\partial v \partial u} (A\beta + 2B\alpha\beta + C)$$

$$+ \frac{\partial^2 z}{\partial u \partial v} (A\beta + 2B + C\alpha) + \frac{\partial^2 z}{\partial v^2} (A\beta^2 + 2B\beta + C) = 0$$

$$\Rightarrow \begin{cases} A + 2B\alpha + C\alpha^2 = 0 \\ A\beta^2 + 2B\beta + C = 0 \end{cases} \Rightarrow \begin{cases} \beta \cdot A = \alpha \cdot C \\ 2\beta \cdot B + (\alpha\beta + 1) C = 0 \end{cases}$$

6.

$$u'_x(x,2x) = x^2$$

$$\downarrow$$

$$\frac{\partial u'_x(x,2x)}{\partial x} = u''_{xx}(x,2x) + 2u''_{xy}(x,2x) = 2x$$

$$\Downarrow$$

$$u''_{xx} + 2u''_{xy} = 2x$$

$$u(x,2x) = x$$

$$\downarrow$$

$$\frac{\partial u(x,2x)}{\partial x} = u'_x(x,2x) + 2u'_y(x,2x) = 1$$

$$\Downarrow \frac{\partial}{\partial x}$$

$$u''_{xx} + 2u''_{xy} + 2u''_{yx} + 4u''_{yy} = 0$$

$$\Downarrow$$

$$u''_{xx} + 4u''_{xy} + 4u''_{yy} = 0$$

$$\Rightarrow \begin{cases} u''_{xx} + 2u''_{xy} = 2x \\ u''_{xx} + 4u''_{xy} + 4u''_{yy} = 0 \\ u''_{xx} - u''_{yy} = 0 \end{cases} \Rightarrow \begin{cases} u''_{xx} = -\frac{4}{3}x \\ u''_{xy} = \frac{5}{3}x \\ u''_{yy} = -\frac{4}{3}x \end{cases}$$

7.

① $f(x,y)$ 为n次齐次函数 $\Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$

$\Downarrow$

$$f(tx,ty)=t^n f(x,y)$$

$\downarrow$ 对t求导

$$xf_1'(tx,ty)+yf_2'(tx,ty)=nt^{n-1}f(x,y)$$

若令 $t=1$

得 $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$

② $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf \Rightarrow f(x,y)$ 为n次齐次函数

$\Downarrow$

$$txf_1'(tx,ty)+tyf_2'(tx,ty)=nf(tx,ty)$$

$$=nt^n f(x,y)$$

$$\Rightarrow xf_1'(tx,ty)+yf_2'(tx,ty)=nt^{n-1}f(x,y)$$

$\downarrow$ 对t积分

$$f(tx,ty)=t^n f(x,y)$$

$\Rightarrow f(x,y)$ 为n次齐次函数

8.

$$\frac{\partial f}{\partial \vec{v}^*}(P_0) = \frac{\partial f}{\partial \vec{i}^*} \cdot \frac{\partial \vec{i}^*}{\partial \vec{v}^*}(P_0) + \frac{\partial f}{\partial \vec{j}^*} \cdot \frac{\partial \vec{j}^*}{\partial \vec{v}^*}(P_0) \qquad \begin{cases} \vec{j} = \vec{v} + \vec{u} \\ \vec{i} = 2\vec{v} + \vec{u} \end{cases}$$

$$= 2\frac{\partial f}{\partial \vec{i}^*}(x_0, y_0) + \frac{\partial f}{\partial \vec{j}^*}(x_0, y_0) = 2$$

$$\frac{\partial f}{\partial \vec{u}^*}(P_0) = \frac{\partial f}{\partial \vec{i}^*} \cdot \frac{\partial \vec{i}^*}{\partial \vec{u}^*}(P_0) + \frac{\partial f}{\partial \vec{j}^*} \cdot \frac{\partial \vec{j}^*}{\partial \vec{u}^*}(P_0)$$

$$= \frac{\partial f}{\partial \vec{i}^*}(x_0, y_0) + \frac{\partial f}{\partial \vec{j}^*}(x_0, y_0) = 1$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial \vec{i}^*}(x_0, y_0) = 1 \\ \frac{\partial f}{\partial \vec{j}^*}(x_0, y_0) = 0 \end{cases}$$

$$\Rightarrow f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + \Delta x + o(\sqrt{\Delta x^2 + \Delta y^2})$$

9. $f(x,y)=x^2-xy+y^2$. $P_0(1,1)$

设 $\vec{l}=a\vec{i}+b\vec{j}$ $a^2+b^2=1$

$$\frac{\partial f(P_0)}{\partial \vec{l}}=\lim_{t\to 0}\frac{f(1+ta,1+tb)-f(1,1)}{t}$$

$$=\lim_{t\to 0}\frac{(1+2ta+t^2a^2)-(1+ta+tb+t^2ab)+(1+2tb+t^2b^2)-1}{t}$$

$$=\lim_{t\to 0}\frac{2t(a+b)+t^2(a^2+b^2)-t(a+b)-t^2ab}{t}$$

$$=\lim_{t\to 0}(a+b)+t(a^2-ab+b^2)$$

$$=a+b$$

① 最大值

$$\left.\frac{\partial f(P_0)}{\partial \vec{l}}\right|_{max}=a+b\Big|_{max}=\sqrt{2(a^2+b^2)}=\sqrt{2}$$

$(a=b=\frac{\sqrt{2}}{2})$

② 最小值

$$\left.\frac{\partial f(P_0)}{\partial \vec{l}}\right|_{min}=-\left.\frac{\partial f(P_0)}{\partial \vec{l}}\right|_{max}=-\sqrt{2}$$

③ 零值

$$\begin{cases}a+b=0\\a^2+b^2=1\end{cases}\Rightarrow\begin{cases}a=\frac{\sqrt{2}}{2}\\b=\frac{-\sqrt{2}}{2}\end{cases}\text{ or }\begin{cases}a=\frac{-\sqrt{2}}{2}\\b=\frac{\sqrt{2}}{2}\end{cases}$$

10.

设 $\vec{l} = (\cos\theta, \sin\theta)$

$$\frac{\partial z}{\partial \vec{l}} = \lim_{t \to 0} \frac{z(3+\cos\theta t, 4+\sin\theta t) - z(3,4)}{t}$$

$$= \lim_{t \to 0} \frac{2+a(3+\cos\theta t)^2 + b(4+\sin\theta t)^2 - 2 - a\cdot 3^2 - b\cdot 4^2}{t}$$

$$= 6a\cos\theta + 8b\sin\theta \qquad \cos\varphi = \frac{8b}{\sqrt{36a^2+64b^2}}$$

$$= \sqrt{36a^2+64b^2}\,\sin(\theta+\varphi)$$

$$\le \sqrt{36a^2+64b^2} = 10$$

$$\Rightarrow \begin{cases} \dfrac{8b}{\sqrt{36a^2+64b^2}} = -\dfrac{3}{5} = \cos\theta \\ \dfrac{6a}{\sqrt{36a^2+64b^2}} = -\dfrac{4}{5} = \sin\theta \\ \sqrt{36a^2+64b^2} = 10 \end{cases} \Rightarrow \begin{cases} a = -\dfrac{4}{3} \\ b = -\dfrac{3}{4} \end{cases}$$

(1)

$$\begin{cases} \dfrac{\partial f(x,y)}{\partial x} = \displaystyle\int \dfrac{\partial f(x,y)}{\partial x \partial y}\, dy \\ \dfrac{\partial f(x,y)}{\partial y} = \displaystyle\int \dfrac{\partial f(x,y)}{\partial x \partial y}\, dx \end{cases}$$

令 $g(x,y) = \dfrac{\partial f(x,y)}{\partial x \partial y}$

$$\Rightarrow \int g(x,y)\, dx = \int g(x,y)\, dy$$

$$\Rightarrow g(x,y) \equiv 0, \ \forall (x,y) \in \mathbb{R}^2$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0$$

$\Rightarrow f(x,y)$ 为常值函数

由于 $f(x,0) > 0$

$\Rightarrow f(x,y) > 0$

12.

取 t_0，使得 $\begin{cases} x_0 = x(t_0) \\ y_0 = y(t_0) \end{cases}$

$\Rightarrow \vec{\tau} \parallel (x'(t_0), y'(t_0))$

假设 $\forall P_0 \in L$，$\dfrac{\partial f(P_0)}{\partial \vec{\tau}} \neq 0$

不妨设 $\dfrac{\partial f(P_0)}{\partial \vec{\tau}} > 0$ $(\forall P_0 \in L)$

则沿 $L: A \to B \Rightarrow t: a \to b$

$f(x(t), y(t))$ 随 t 单增

$\Rightarrow f(A) < f(B)$

与前提 $f(A) = f(B)$ 矛盾

$\Rightarrow$ 假设错误

$\Rightarrow \exists P_0 \in L$，$\dfrac{\partial f(P_0)}{\partial \vec{\tau}} = 0$

13.

(1) $z=x^{y}=e^{y\ln x}$

$$\frac{\partial z}{\partial x}=e^{y\ln x}\cdot\frac{y}{x}=\frac{y}{x}\cdot z$$
$$\frac{\partial z}{\partial y}=e^{y\ln x}\cdot\ln x=\ln x\cdot z$$

$\Rightarrow x, y, z(x,y)$ 复合成的初等函数

若不断求偏导，仍为

则 $z(x,y)$ 在 D 内为 C^{∞} 函数

(2)

① $x=0$: $z(x,y)=x^{y}=0$

无单调性，无极值

$z_{max}=z_{min}=0$

② 设直线 $y=kx-k$

$$z=x^{k(x-1)}=e^{k(x-1)\ln x}$$

$$z_x'=\underbrace{e^{k(x-1)\ln x}}_{>0}\cdot\left(k\ln x+\frac{k(x-1)}{x}\right)$$

$$=k\cdot\underbrace{x^{k(x-1)}}_{>0}\left(\ln x-\frac{1}{x}+1\right)$$

1> $k=0$

$z=1$

无单调性

无极值

$z_{max}=z_{min}=1$

2> $k>0$

$x\in(0,1)$ 单减

$x\in(1,+\infty)$ 单增

极小值 $=1$

$z_{min}=1$

无 max

3> $k<0$

$x\in(0,1)$ 单增

$x\in(1,+\infty)$ 单减

极大值 $=1$

$z_{max}=1$

无 min

14.

(1) 笛卡尔坐标下梯度: $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

球极坐标: $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\cos\theta\cos\varphi \\ z = r\sin\theta \end{cases}$ $\begin{cases} \vec{r} = \cos\theta\sin\varphi\,\vec{i} + \cos\theta\cos\varphi\,\vec{j} + \sin\theta\,\vec{k} \\ \vec{\theta} = -\sin\theta\sin\varphi\,\vec{i} - \sin\theta\cos\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \vec{\varphi} = \vec{r}\times\vec{\theta} = \cos\varphi\,\vec{i} - \sin\varphi\,\vec{j} \end{cases}$

$$\Rightarrow \nabla f = \left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x}\right)\vec{i} + \left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y}\right)\vec{j} + \left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\cdot\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial z}\right)\vec{k}$$

$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arcsin\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\frac{x}{y} \end{cases}$ $\begin{cases} \vec{i} = \cos\theta\sin\varphi\,\vec{r} - \sin\theta\sin\varphi\,\vec{\theta} + \cos\varphi\,\vec{\varphi} \\ \vec{j} = \cos\theta\cos\varphi\,\vec{r} - \sin\theta\cos\varphi\,\vec{\theta} - \sin\varphi\,\vec{\varphi} \\ \vec{k} = \sin\theta\,\vec{r} + \cos\theta\cdot\vec{\theta} \end{cases}$

$$\Rightarrow \nabla f = \frac{\partial f}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta} + \frac{1}{\cos\theta\cdot r}\frac{\partial f}{\partial \varphi}\cdot\vec{\varphi}$$

(2) $\Delta u = u_{xx} + u_{yy} + u_{zz}$