3.1

1. $X_{\vec{p}}(fg) = (X(fg))(\vec{p})$

$$= (fX(g))(\vec{p}) + (gX(f))(\vec{p})$$

$$= f(\vec{p})\, X(g)(\vec{p}) + g(\vec{p})\, X(f)(\vec{p})$$

$$= f(\vec{p}) \cdot X_{\vec{p}}(g) + g(\vec{p})\, X_{\vec{p}}(f)$$

$\Rightarrow$ so X_p is a derivation at $\vec{p}$

2. $df(X)(\vec{p}) = \sum \frac{\partial f}{\partial x_i} \cdot \underset{\substack{\uparrow \\ \text{i-th coordinate of } \vec{p}}}{p_i}$

$$X(f)(\vec{p}) = \underset{\substack{\uparrow \\ \text{row vector}}}{\left(\frac{\partial f}{\partial x_i}\right)} \vec{p} = \sum \frac{\partial f}{\partial x_i} p_i$$

$\Rightarrow df(X)(\vec{p}) = X(f)(\vec{p}),\ \forall \vec{p} \in M$

$\Rightarrow df(X) = X(f)$

3. $(X\circ Y - Y\circ X)(fg) = X\circ Y(fg) - Y\circ X(fg)$

$$= X(Y(fg)) - Y(X(fg)) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f))$$

$$= \cancel{X(f)Y(g)} + f\cdot X\circ Y(g) + \cancel{X(g)Y(f)} + g\, X\circ Y(f)$$

$$- \left(\cancel{Y(f)X(g)} + f\, Y\circ X(g) + \cancel{Y(g)X(f)} + g\, Y\circ X(f)\right)$$

$$= f\cdot (X\circ Y - Y\circ X)(g) + g\,(X\circ Y - Y\circ X)(f)$$

4. $\begin{cases} A = -A^{ad} \\ B = -B^{ad} \end{cases}$ $\quad (AB - BA)^{ad} = (AB)^{ad} - (BA)^{ad}$

$$= B^{ad}A^{ad} - A^{ad}B^{ad}$$

$$= (-B)(-A) - (-A)(-B) = BA - AB$$

9.2

1. bilinear: $\langle k\vec{v}, \vec{w}\rangle = (k\vec{v})^T A\vec{w} = k(\vec{v}^T A\vec{w}) = k\langle \vec{v}, \vec{w}\rangle$

$\langle \vec{v}, k\vec{w}\rangle = \vec{v}^T A(k\vec{w}) = k(\vec{v}^T A\vec{w}) = k\langle \vec{v}, \vec{w}\rangle$

symmetric: $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w} = (\vec{v}^T A\vec{w})^T = \vec{w}^T A^T \vec{v}$

$= \vec{w}^T A \vec{v}$

$= \langle \vec{w}, \vec{v}\rangle$

positive-definite: $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T \underset{\substack{\uparrow \\ \text{positive} \\ \text{-definite}}}{A} \vec{w} \underset{\substack{\uparrow \\ \text{iff } \vec{v}=\vec{w}=\vec{0} \text{ for the "="}}}{\geq} 0$

2. $\vec{v}^T$ to $A^{-1}\vec{v}$ (according to the definition)

3. to $\vec{v}^T A$

4. to $A^{-1}\vec{v}$

9.3

1. $S^*(ev_5)(p) = ev_5(S(p)) = ev_5\left(\int_0^1 p(x,y)dx\right) = \int_0^1 p(x,5)dx$

pick basis for V: $(x^2, xy, y^2, x, y, 1)$

$\Rightarrow S^*(ev_5) = \left(\frac{1}{3} \quad \frac{5}{2} \quad 25 \quad \frac{1}{2} \quad 5 \quad 1\right)$

2. $S^*(\int_0^1)(p) = \int_0^1 dy \int_0^1 p(x,y)dx$

pick basis for V: $(x^2, xy, y^2, x, y, 1)$

$\Rightarrow S^*(\int_0^1) = \left(\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{1}{2} \quad 1\right)$

3. pick basis for V: $(x^2, xy, y^2, x, y, 1)$, W: $(y^2, y, 1)$

$S^*BS(2xy) = S^*BS\begin{pmatrix}0\\2\\0\\0\\0\\0\end{pmatrix} = S^*B\begin{pmatrix}0\\1\\0\end{pmatrix} = S^*\left(\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}\right)$

$\int_0^1 y(ay^2+by+c)dy$
$= \frac{1}{4}a + \frac{1}{3}b + \frac{1}{2}c$

$\Rightarrow S^*\left(\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}\right)(p) = \left(\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}\right)\cdot(S(p))$

$= \left(\frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}\right)\begin{pmatrix} & & 1 & & & \\ & \frac{1}{2} & & & 1 & \\ \frac{1}{3} & & & \frac{1}{2} & & 1\end{pmatrix} p$

$= \left(\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2}\right)\cdot p$

coordinates of $S^*BS(2xy)$

$\text{rank}(S^*BS) = 3$