1.

$L(x,y,z)=x^2+y^2+z^2 \qquad \varphi(x,y,z)=z^2-xy-x+y-1\equiv 0$

$F(x,y,z,\lambda)=x^2+y^2+z^2+\lambda(z^2-xy-x+y-1)$

$$\begin{cases}\frac{\partial F}{\partial x}=2x-\lambda y-\lambda=0\\ \frac{\partial F}{\partial y}=2y-\lambda x+\lambda=0\\ \frac{\partial F}{\partial z}=2z+2\lambda z=0\\ \frac{\partial F}{\partial \lambda}=z^2-xy-x+y-1=0\end{cases}$$

$\Rightarrow \lambda=2$ or $x+y=0$ $\to$ $x=\frac{\lambda}{2+\lambda},\ y=\frac{-\lambda}{2+\lambda}$

$\to z=0$

$\lambda=-1$

$\Rightarrow \begin{cases}x=-1\\ y=1\\ z<0\end{cases}$ X

$$\begin{cases}xy+x-y+1=0\\ 2x-2y-2=0\end{cases} \Rightarrow x(x-1)+x-x+1+1=0$$

$x^2-x+2=0$

X

$\Rightarrow$ 定义域内无极值点

$\Rightarrow$ 最值点处于边界上，即 $z^2=0=xy+x-y+1$ 上

$L(x,y)=x^2+y^2,\ \varphi(x,y)=xy+x-y+1\equiv 0$

$F(x,y,\lambda)=L(x,y)+\lambda\varphi(x,y)$

$\Rightarrow \begin{cases}x=\frac{\lambda}{2+\lambda}\\ y=\frac{-\lambda}{2+\lambda}\end{cases}$ 且 $xy+x-y+1=0 \Rightarrow \lambda=\frac{-4\pm\sqrt{8}}{2}=-2\pm\sqrt{2}$

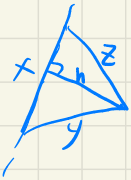
$\Rightarrow \begin{cases}x=\frac{-2\pm\sqrt{2}}{\pm\sqrt{2}}=1\pm\sqrt{2}\\ y=-1\mp\sqrt{2}\end{cases}$

$L(1+\sqrt{2},-1-\sqrt{2})=6+4\sqrt{2}$

$L(1-\sqrt{2},-1+\sqrt{2})=6-4\sqrt{2}$

最小值

2.



$$V(x,y,z) = 4\pi p(p-x)(p-y)(p-z)/3x$$

$$\varphi(x,y,z) = x+y+z-2p \equiv 0$$

$$\Rightarrow F(x,y,z,\lambda) = \frac{4\pi p}{3}\left(\frac{p}{x}-1\right)(p-y)(p-z)+\lambda(x+y+z-2p)$$

$$\begin{cases} \frac{\partial F}{\partial x} = \frac{4\pi p}{3}\left(-\frac{p(p-y)(p-z)}{x^2}\right)+\lambda \longrightarrow \lambda = \frac{\pi p^2}{3} \\ \frac{\partial F}{\partial y} = \frac{4\pi p}{3}\left(-\left(\frac{p}{x}-1\right)(p-z)\right)+\lambda \\ \frac{\partial F}{\partial z} = \frac{4\pi p}{3}\left(-\left(\frac{p}{x}-1\right)(p-y)\right)+\lambda \\ \frac{\partial F}{\partial \lambda} = x+y+z-2p = 0 \longrightarrow x = 2p-2y \to x = \frac{p}{2} \end{cases}$$

$$\left.\begin{matrix}\frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z}\end{matrix}\right\} \Rightarrow y = z$$

$$\Rightarrow \begin{cases} x = \frac{p}{2} \\ y = z = \frac{3}{4}p \end{cases}$$

绕 $\frac{p}{2}$ 长的边旋转

3. $S = \pi ab$ $a = \max L(x,y,z)$

$b = \min L(x,y,z)$

$F(x,y,z) = x^2+y^2+z^2$ $\varphi_1(x,y,z) = x+y-z \equiv 0$

$\varphi_2(x,y,z) = x^2+y^2+z^2-xy-yz-zx-1 \equiv 0$

$$\Rightarrow F(x,y) = 2x^2+2xy+2y^2,\ \varphi(x,y) = x^2-xy+y^2-1 = 0$$

$$G(x,y,\lambda) = 2x^2+2xy+2y^2+\lambda(x^2-xy+y^2-1)$$

$$\begin{cases} \frac{\partial G}{\partial x} = 4x+2y+2\lambda x-\lambda y = 0 \\ \frac{\partial G}{\partial y} = 2x+4y+2\lambda y-\lambda x = 0 \\ \frac{\partial G}{\partial \lambda} = x^2-xy+y^2-1 = 0 \end{cases}$$

$\Rightarrow \lambda = -6$ or $x+y = 0$

$x = y$ $\lambda = -\frac{2}{3}$

$x^2 = 1$ $x^2 = \frac{1}{3}$

$$\Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 1 \\ \lambda = -6 \end{cases} \text{ or } \begin{cases} x = \pm\frac{1}{\sqrt{3}} \\ y = \mp\frac{1}{\sqrt{3}} \\ \lambda = -\frac{2}{3} \end{cases}$$

$$\Rightarrow F = 6 \text{ or } \frac{2}{3}$$

$$\Rightarrow a = \sqrt{6},\ b = \sqrt{\frac{2}{3}} \Rightarrow S = 2\pi$$

4.

$$F(x,y,z,\lambda) = z + \lambda\left(e^{-(x+y+z)} + x^2+y^2+z^2 - \frac{5}{2}\right)$$

$$\begin{cases} \frac{\partial F}{\partial x} = -\lambda e^{-(x+y+z)} + 2\lambda x = 0 \\ \frac{\partial F}{\partial y} = -\lambda e^{-(x+y+z)} + 2\lambda y = 0 \\ \frac{\partial F}{\partial z} = 1 - \lambda e^{-(x+y+z)} + 2\lambda z = 0 \\ \frac{\partial F}{\partial \lambda} = e^{-(x+y+z)} + x^2+y^2+z^2 - \frac{5}{2} = 0 \end{cases}$$

First two equations $\Rightarrow x=y$ or $\lambda = 0$ ✗ ($\rightarrow \frac{\partial F}{\partial z} = 1 \neq 0$)

$x=y \rightarrow e^{-2x-z} = 2x$

$\rightarrow 1 - \lambda e^{-2x-z} + 2\lambda z = 0$

$\rightarrow e^{-2x-z} + 2x^2 + z^2 - \frac{5}{2} = 0$

$$\Rightarrow \begin{cases} 1 - 2\lambda x + 2\lambda(-2x - \ln 2x) = 0 \\ 2x^2 + 2x + (2x + \ln 2x)^2 - \frac{5}{2} = 0 \end{cases} \Rightarrow x = \frac{1}{2} \text{ or } x \approx 0.091385636$$

$x = y$

$$\Rightarrow \begin{cases} \lambda = \frac{1}{3} \\ x = y = \frac{1}{2} \end{cases} \text{ or } \begin{cases} \lambda \approx -0.9676 \\ x = y \approx 0.09139 \end{cases}$$

$\Rightarrow z = -1$ or $z \approx 1.51675$

极小　　　　极大

5. $f'_x(x,y) = \int f''_{xy}(x,y)\,dy = (y^2+2y)e^x + C(x)$

$\Rightarrow f'_x(x,0) = 0 + C(x) = (x+1)e^x$

$\Rightarrow f'_x(x,y) = (y^2+2y+x+1)e^x$

$f(x,y) = \int f'(x,y)\,dx = (y^2+2y+1)e^x + (x-1)e^x + C(y)$

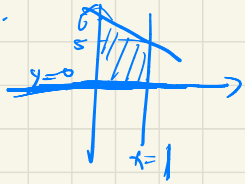
$\Rightarrow f(0,y) = y^2+2y+1 - 1 + C(y) = y^2+2y$

$\Rightarrow f(x,y) = (y+1)^2 e^x + (x-1)e^x$

$$\begin{cases} \dfrac{\partial f}{\partial x} = (y+1)^2 e^x + xe^x = 0 \\ \dfrac{\partial f}{\partial y} = (2y+2)e^x = 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=-1 \end{cases} \qquad Hf = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{正定}$$

$f(x,y)$极小值 $= f(0,-1) = -1$

6.



$D: y \geq 0,\ x \leq 1,\ y \leq 6-x$

$z = 4xy - x^2y - xy^2$

$$\begin{cases} \dfrac{\partial z}{\partial x} = 4y - 2xy - y^2 = 0 \\ \dfrac{\partial z}{\partial y} = 4x - x^2 - 2xy = 0 \end{cases} \Rightarrow y = x \ \text{ or } \ x+y=4$$

$y = x \Rightarrow x = y = 0$ or $x = y = \frac{4}{3}$

$x+y=4 \Rightarrow \begin{cases} x=4 \\ y=0 \end{cases}$ ✗ or $\begin{cases} x=0 \\ y=4 \end{cases}$ ✗

$z(0,0) = 0 \rightarrow$ 鞍点

$z(\frac{4}{3}, \frac{4}{3}) = \frac{64}{27}$

$\downarrow$ 极大值点

$$Hf(x,y) = \begin{pmatrix} -2y & 4-2x-2y \\ 4-2x-2y & -2x \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Hf(\tfrac{4}{3}, \tfrac{4}{3}) = \begin{pmatrix} -\frac{8}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{8}{3} \end{pmatrix} \text{ 负定}$$

$z(x,0) = 0 < z(\frac{4}{3}, \frac{4}{3})$

$z(1,y) = 4y - y - y^2 = -y^2 + 3y \leq \frac{9}{4} < z(\frac{4}{3}, \frac{4}{3})$
$(0 \leq y \leq 5)$

$z(x, 6-x) = 4x(6-x) - x^2(6-x) - x(6-x)^2$
$(0 \leq x \leq 1)$ $= 2x^2 - 12x \leq 0 < z(\frac{4}{3}, \frac{4}{3})$

$\Rightarrow z_{max} = z(\frac{4}{3}, \frac{4}{3}) = \frac{64}{27}$

7. $Q(x,y,z)$, $P(x_0,y_0,z_0)$

$d(x,y,z)=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$

$F(x,y,z)=0$

$$\Rightarrow \begin{cases} 2(x-x_0)+\lambda F_x'=0 \\ 2(y-y_0)+\lambda F_y'=0 \\ 2(z-z_0)+\lambda F_z'=0 \\ F=0 \end{cases}$$

$$\Rightarrow \frac{x-x_0}{F_x'}=\frac{y-y_0}{F_y'}=\frac{z-z_0}{F_z'}=-\frac{\lambda}{2}$$

$\Rightarrow \vec{PQ} \parallel \operatorname{grad} F$

$\Rightarrow \vec{PQ} \perp \vec{n}$

得证