Introductory Econometrics
Ch5 Multiple Regression Analysis: OLS Asymptotics

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Learning Goals

Key concepts of this chapter

- asymptotic properties (large sample properties)
- consistency (law of large numbers)
- asymptotic normality (central limit theorem)

Learning goals:

- Focus on intuition: what do the theorems mean?
- Put less effort into the technical details.

Asymptotic Properties

- Finite sample properties: properties hold for any sample of data.
- Examples
  - Unbiasedness of OLS
  - OLS is BLUE
  - Sampling distribution of the OLS estimators
- Asymptotic properties or large sample properties: not defined for a particular sample size; rather, they are defined as the sample size grows without bound.

- Classical linear model assumptions MLR6 Normality: $u \sim N(0, \sigma^2)$.
- Only with MLR.6, we can derive the exact sampling distribution of the OLS estimator.
- But we know this assumption is not valid under many contexts.
- We show in this chapter, that even without MLR.6, t and F statistics have approximately t and F distributions, in large sample sizes.

Outline

Consistency of the OLS Estimators

Asymptotic Distribution of the OLS Estimators

Consistency

Consistency

Let W_N be an estimator of θ based on a sample $Y_1, Y_2, ..., Y_N$ of size N. Then, W_N is a consistent estimator of θ if for every $\epsilon > 0$,

$$P(|W_N - \theta| > \epsilon) \to 0 \text{ as } N \to \infty.$$

- Equivalently, we write consistency means:

$$plim(W_N) = \theta.$$

  When W_N is consistent, we also say that θ is the probability limit of W_N.

Consistency

- Consistency involves the behavior of the sampling distribution of the estimator as the sample size gets large. To emphasize, we index the estimator by its sample size.
- Intuitively, consistency means when the sample size becomes larger, the distribution of W_N becomes more and more concentrated about θ.
- which roughly means that for larger sample sizes, W_N is less and less likely to be very far from θ.

Quiz: Consistency

Which of the following statements is correct?

1. Consistency is a desired property for an estimator.
2. All unbiased estimators are consistent.
3. All consistent estimators are unbiased.

Hint: Suppose $X \sim N(1,1)$. We draw a random sample of X with size M, and X_i is the i-th observation in the sample. Consider two estimators for $E(X) = 1$:[1]

1. $W_M = X_1$
2. $U_M = \frac{1}{M-1} \sum_{i=1}^{M} X_i$

Ans

[1] If $X_i \sim N(1,1)$, then $\sum_{i=1}^{M} X_i \sim N(M, M)$

Law of Large Numbers

- A good example of a consistent estimator is the average of a random sample:

Law of Large Numbers

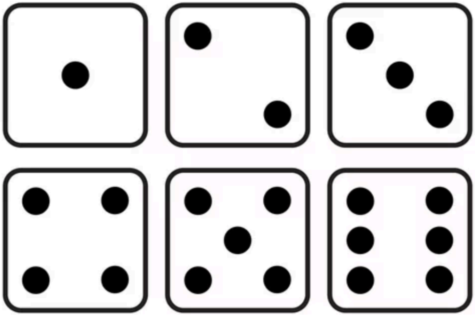
Let $Y_1, Y_2, ..., Y_N$ be independent, identically distributed random variables with mean $\mu < \infty$. $\bar{Y}_N = \frac{\sum_{i=1}^{N} Y_i}{N}$. Then

$$plim(\bar{Y}_N) = \mu$$

- The law of large numbers means one can get arbitrarily close to μ by choosing a sufficiently large sample.
- In other words, the sample average is a consistent estimator of the population expectation.

Example: Rolling a Dice

- Use X to denote the outcome number when we roll six-sided fair dice. We know $E(X) = 3.5$.
- The law of large numbers says if you rolled dice forever and kept taking the mean value you would get closer and closer to 3.5.



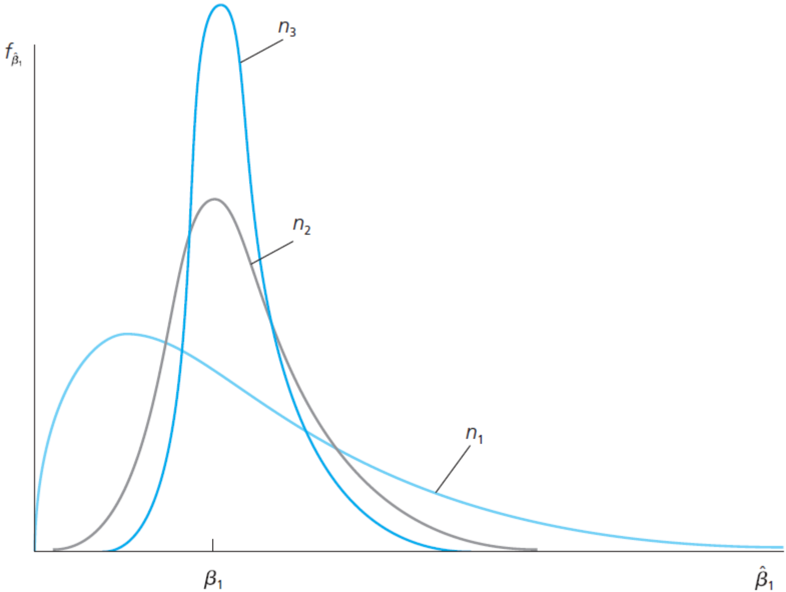
The Consistency of the OLS Estimator

Consistency of OLS

Under Assumptions MLR.1 through MLR.4, the OLS estimator $\hat{\beta}_j$ is consistent for β_j, for all $j = 0, 1, ..., k$.

- When the sample size is larger, the OLS estimator is centered around the true parameter closer and closer.

What it means is that as N gets large $\hat{\beta}_j$ gets closer and closer to β_j:



Sketch of Proof[2]

In a simple linear regression model:

$$\hat{\beta}_1 = \beta_1 + \frac{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})u_i}{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}.$$

Using the law of large numbers:

$$\begin{aligned} plim[\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})u_i] &= E[(x_i - \bar{x})u_i] \\ &= Cov(x, u) \\ plim[\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2] &= E[(x_i - \bar{x})^2] \\ &= Var(x) \end{aligned}$$

[2] not required

We then have[3]

$$plim(\hat{\beta}_1) = \beta_1 + \frac{cov(x, u)}{var(x)}.$$

Under $E(u|x) = 0$, $cov(x, u) = 0$.[4] So $\hat{\beta}_1$ is consistent.

[3] If $plim(T_n) = \alpha$ and $plim(U_n) = \gamma$, then $plim(T_n/U_n) = \alpha/\gamma$.

[4] $cov(x, u) = E[(x - \bar{x})u] = E[E((x - \bar{x})u|x)] = E[(x - \bar{x})E(u|x)] = 0$.

Outline

Consistency of the OLS Estimators

Asymptotic Distribution of the OLS Estimators

Large Sample Inference

- Consistency shows that the OLS estimators are getting closer to the population value as the sample size grows.
- OLS estimators are consistent, another reason why we prefer them.
- To test hypotheses about the parameters, we also need to know the asymptotic distribution of the OLS estimators, i.e., when the sample size becomes larger and larger, what is the approximate distribution of OLS estimators?

Central Limit Theorem

- **Asymptotic normality**: Use the notation

$$\hat{\theta}_N \overset{a}{\sim} N(0, \sigma^2)$$

to mean that as the sample size N gets larger, $\hat{\theta}_N$ is approximately normally distributed with mean 0 and variance σ^2.

Central Limit Theorem

Let $\{Y_1, Y_2, ..., Y_N\}$ be a random sample with mean μ and variance σ^2. Then,

$$Z_N = \frac{\bar{Y}_N - \mu}{\sigma/\sqrt{N}} \overset{a}{\sim} N(0, 1).$$

Central Limit Theorem (CLT)

- The variable Z_N is the standardized version of the sample average $\bar{Y}_N$
- Intuitively, CLT means when the sample size gets larger, the distribution of the **sample average** is closer to a normal distribution, regardless of the population distribution of Y
- Regardless of the population distribution of Y, the entire distribution of Z_N gets arbitrarily close to the standard normal distribution as N gets large.

Asymptotic Normality of OLS

Asymptotic Normality of OLS

Under the Gauss-Markov Assumptions MLR.1 through MLR.5, for each $j = 0, 1, ..., k$

$$\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \overset{a}{\sim} Normal(0, 1).$$

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \overset{a}{\sim} Normal(0, 1).$$

- OLS estimators are approximately normally distributed in large enough sample sizes.
- The asymptotic normality of the OLS estimators also implies that the F statistics have approximate F distributions in large sample sizes.

How to Understand?

- The theorem implies that we can relax the normality assumption of u.
- No matter how u is distributed, the OLS estimators are approximately normal when the sample size is large enough, under MLR.1 through MLR.5.
- We can use the same methods in Chapter 4 to construct confidence intervals and conduct t and F tests even without the normality assumption, being aware that this is a good approximation with a large sample size.

Summary

- Under MLR.1-MLR.4, OLS estimators are
  - unbiased
  - consistent
- Under MLR.1-MLR.5, OLS estimators
  - are BLUE (best linear unbiased estimators)
  - have an asymptotic normal distribution
- Under MLR.1-MLR.6, OLS estimators
  - are BUE (best unbiased estimators)
  - have a normal sampling distribution

Answer to quiz

- The first statement is correct. Statements 2 and 3 are incorrect: in the example, W_M is unbiased and inconsistent, while U_M is biased yet consistent. Consistency and unbiasedness are two different concepts and neither implies the other.

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