HW4

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1. (1) 证明. 由于 $\ln x$ 上凸, 有:

$$\frac{b-t}{b-a}\ln a + \frac{t-a}{b-a}\ln b \le \ln t \quad (a \le t \le b)$$

令 $a = m$, $b = m+1$, $t = x$, 得:

$$(m+1-x)\ln m + (x-m)\ln(m+1) = f(x) \le \ln x \quad (m \le x \le m+1)$$

接下来再证 $g(x) \ge \ln x$: 易知

$$t - 1 \ge \ln t \quad (t > 0)$$

令 $t = \frac{x}{m} \in [1 - \frac{1}{2m}, 1 + \frac{1}{2m})$, 则有:

$$\frac{x}{m} - 1 \ge \ln\frac{x}{m}$$

化简后有:

$$\frac{x-m}{m} + \ln m = g(x) \ge \ln x \quad (m - \frac{1}{2} \le x < m + \frac{1}{2})$$

得证. □

(2) 解.

$$\begin{aligned}\int_1^n f(x)\mathrm{d}x &= \sum_{m=1}^{n-1}\int_m^{m+1}\left[(m+1-x)\ln m + (x-m)\ln(m+1)\right]\mathrm{d}x \\ &= \sum_{m=1}^{n-1}\ln m + \frac{1}{2}\ln\frac{m+1}{m} \\ &= \ln(n-1)! + \frac{1}{2}\ln n\end{aligned}$$

$$\begin{aligned}\int_1^n g(x)\mathrm{d}x &= \sum_{m=2}^{n-1}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\left(\frac{x}{m} + \ln m - 1\right)\mathrm{d}x + \int_1^{\frac{3}{2}}\left(\frac{x-1}{1} + \ln 1\right)\mathrm{d}x + \int_{n-\frac{1}{2}}^{n}\left(\frac{x-n}{n} + \ln n\right)\mathrm{d}x \\ &= \sum_{m=2}^{n-1}\ln m + \frac{1}{2}\ln n + \frac{n-1}{8n} \\ &= \ln(n-1)! + \frac{1}{2}\ln n + \frac{n-1}{8n}\end{aligned}$$

□

(3) 证明.

$$\int_1^n \ln x\mathrm{d}x = n\ln n - n + 1$$

$$
\begin{aligned}
\int_1^n \ln x \mathrm{d}x \geq \int_1^n f(x)\mathrm{d}x &\Rightarrow n\ln n - n + 1 \geq \ln(n-1)! + \frac{1}{2}\ln n \\
&\Rightarrow \left(n+\frac{1}{2}\right)\ln n - n + 1 \geq \ln(n-1)! + \ln n \\
&\Rightarrow 1 \geq \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n
\end{aligned}
$$

$$
\begin{aligned}
\int_1^n \ln x \mathrm{d}x \leq \int_1^n g(x)\mathrm{d}x &\Rightarrow n\ln n - n + 1 \leq \ln(n-1)! + \frac{1}{2}\ln n + \frac{n-1}{8n} \\
&\Rightarrow \left(n+\frac{1}{2}\right)\ln n - n + 1 \leq \ln(n!) + \frac{n-1}{8n} \\
&\Rightarrow 1 - \frac{n-1}{8n} \leq \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n \\
&\Rightarrow \frac{7}{8} \leq \frac{7n+1}{8n} \leq \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n
\end{aligned}
$$

得证. □

(4) 证明. 由于 e^x 单增, 可以对(3)的结论取指数, 得:

$$
\exp\left(\frac{7}{8}\right) \leq \exp\left(\ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n\right) \leq e
$$

而

$$
\begin{aligned}
\exp\left(\ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n\right) &= \frac{\exp(\ln(n!)+n)}{\exp\left(\left(n+\frac{1}{2}\right)\ln n\right)} \\
&= \frac{n!e^n}{n^n\cdot\sqrt{n}} \\
&= \frac{n!}{(n/e)^n\cdot\sqrt{n}}
\end{aligned}
$$

得证. □

2. 证明.

$$
(1-x)^{-\alpha} = 1 + \sum_{n=1}^{\infty}\frac{(-\alpha)(-\alpha-1)\dots(-\alpha-n+1)}{n!}(-x)^n
$$

又易验证, $n=0$ 时, $\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}x^n = 1$, 接下来只需证:

$$
\begin{aligned}
\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} &= (-\alpha)(-\alpha-1)\dots(-\alpha-n+1)(-1)^n \\
&= \alpha(\alpha+1)\dots(\alpha+n-1)
\end{aligned}
$$

考虑 Γ 函数的性质: $\Gamma(x+1) = x\Gamma(x)$, 有:

$$
\begin{aligned}
\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} &= \frac{(n-1+\alpha)\Gamma(n-1+\alpha)}{\Gamma(\alpha)} \\
&= \dots \\
&= \frac{(n-1+\alpha)(n-2+\alpha)\dots\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \\
&= (n-1+\alpha)(n-2+\alpha)\dots\alpha
\end{aligned}
$$

得证. □

3. 证明. 令

$$f(x)=\frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)$$

要证明结论, 只需要验证 f 是 Γ 函数即可. 根据 Bohr–Mollerup theorem 分条验证:

(a)

$$\begin{aligned}f(x+1)&=\frac{2^{x}}{\sqrt{\pi}}\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{x+2}{2}\right)\\&=\frac{2^{x}}{\sqrt{\pi}}\Gamma\left(\frac{x+1}{2}\right)\frac{x}{2}\Gamma\left(\frac{x}{2}\right)\\&=xf(x)\end{aligned}$$

(b)

$$f(1)=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)\Gamma(1)=1$$

(c) 要想验证 $\ln f$ 的下凸性, 只要证: $f(\alpha x+\beta y)\leq f^{\alpha}(x)f^{\beta}(y)$, 即:

$$\begin{aligned}\frac{2^{\alpha x+\beta y-1}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha x+\beta y}{2}\right)\Gamma\left(\frac{\alpha x+\beta y+1}{2}\right)&\leq\frac{2^{\alpha x-\alpha}}{\pi^{\frac{\alpha}{2}}}\Gamma^{\alpha}\left(\frac{x}{2}\right)\Gamma^{\alpha}\left(\frac{x+1}{2}\right)\frac{2^{\beta y-\beta}}{\pi^{\frac{\beta}{2}}}\Gamma^{\beta}\left(\frac{y}{2}\right)\Gamma^{\beta}\left(\frac{y+1}{2}\right)\\&=\frac{2^{\alpha x+\beta y-(\alpha+\beta)}}{\pi^{\frac{\alpha+\beta}{2}}}\Gamma^{\alpha}\left(\frac{x}{2}\right)\Gamma^{\beta}\left(\frac{y}{2}\right)\Gamma^{\alpha}\left(\frac{x+1}{2}\right)\Gamma^{\beta}\left(\frac{y+1}{2}\right)\\&(\alpha+\beta=1)\\&=\frac{2^{\alpha x+\beta y-1}}{\pi^{\frac{1}{2}}}\Gamma^{\alpha}\left(\frac{x}{2}\right)\Gamma^{\beta}\left(\frac{y}{2}\right)\Gamma^{\alpha}\left(\frac{x+1}{2}\right)\Gamma^{\beta}\left(\frac{y+1}{2}\right)\end{aligned}$$

而由于 Γ 函数本身满足 $\ln\Gamma$ 的下凸性, 因此有

$$\Gamma^{\alpha}\left(\frac{x}{2}\right)\Gamma^{\beta}\left(\frac{y}{2}\right)\Gamma^{\alpha}\left(\frac{x+1}{2}\right)\Gamma^{\beta}\left(\frac{y+1}{2}\right)\geq\Gamma^{\alpha}\left(\frac{\alpha x+\beta y}{2}\right)\Gamma^{\beta}\left(\frac{\alpha x+\beta y+1}{2}\right)$$

从而得证.

□

4. 证明. 当 $\xi\neq0$ 时:

$$\begin{aligned}\hat{f}(\xi)&=\int_{-\infty}^{\infty}f(x)e^{-2\pi ix\xi}\mathrm{d}x\\&=\int_{-1}^{1}1\cdot e^{-2\pi ix\xi}\mathrm{d}x\\&=\frac{1}{-2\pi i\xi}\left(e^{-2\pi i\xi}-e^{2\pi i\xi}\right)\\&=\frac{\sin(2\pi\xi)}{\pi\xi}\end{aligned}$$

$$\begin{aligned}\hat{g}(\xi)&=\int_{-\infty}^{\infty}g(x)e^{-2\pi ix\xi}\mathrm{d}x\\&=\int_{-1}^{1}1\cdot e^{-2\pi ix\xi}\mathrm{d}x+\int_{-1}^{0}x\cdot e^{-2\pi ix\xi}\mathrm{d}x+\int_{0}^{1}(-x)\cdot e^{-2\pi ix\xi}\mathrm{d}x\\&=\frac{1}{4\pi^2\xi^2}\left(2-e^{-2\pi i\xi}-e^{2\pi i\xi}\right)\\&=\frac{1}{4\pi^2\xi^2}\left(2-2\cos(2\pi\xi)\right)\\&=\frac{\sin^2(\pi\xi)}{\pi^2\xi^2}\end{aligned}$$

当 $\xi = 0$ 时:

$$\begin{aligned}\hat{f}(0) &= \int_{-\infty}^{\infty} f(x)\mathrm{d}x \\ &= \int_{-1}^{1} 1\mathrm{d}x \\ &= 2\end{aligned}$$

$$\begin{aligned}\hat{g}(0) &= \int_{-\infty}^{\infty} g(x)\mathrm{d}x \\ &= \int_{-1}^{0} (1+x)\mathrm{d}x + \int_{0}^{1} (1-x)\mathrm{d}x \\ &= 1\end{aligned}$$

□

5. 解. 先证明

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\mathrm{d}\xi = P_y(x)$$

将等式左侧的积分以 0 为界拆成两部分,

$$\begin{aligned}\int_{0}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\mathrm{d}\xi &= \left.\frac{e^{2\pi i(x+iy)\xi}}{2\pi i(x+iy)}\right|_{0}^{\infty} \\ &= -\frac{1}{2\pi i(x+iy)}\end{aligned}$$

同理有

$$\int_{-\infty}^{0} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\mathrm{d}\xi = \frac{1}{2\pi i(x-iy)}$$

因此可得

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\mathrm{d}\xi = \frac{1}{2\pi i}\left(\frac{1}{x-iy} - \frac{1}{x+iy}\right) = \frac{y}{\pi(x^2+y^2)}$$

综上,

$$P_y(x) = \frac{y}{\pi(x^2+y^2)} \to \hat{P}_y(\xi) = e^{-2\pi|\xi|y}$$

$P_y(x)$ 的 Fourier 逆变换为 $e^{-2\pi|-\xi|y} = e^{-2\pi|\xi|y}$. □

6. 证明.

$$
\begin{aligned}
|f*g(x)| &= \left|\int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y\right| \\
&= \left|\int_{|y|\le\frac{|x|}{2}} f(x-y)g(y)\mathrm{d}y + \int_{|y|\ge\frac{|x|}{2}} f(x-y)g(y)\mathrm{d}y\right| \\
&\le \int_{|y|\le\frac{|x|}{2}} |f(x-y)g(y)|\mathrm{d}y + \int_{|y|\ge\frac{|x|}{2}} |f(x-y)g(y)|\mathrm{d}y \\
&\le \int_{|y|\le\frac{|x|}{2}} \frac{A}{1+(x-y)^2}|g(y)|\mathrm{d}y + \int_{|y|\ge\frac{|x|}{2}} |f(x-y)|\frac{B}{1+y^2}\mathrm{d}y \\
&\le \frac{1}{1+x^2/4}\left(A\int_{-\infty}^{\infty}|g(y)|\mathrm{d}y + B\int_{-\infty}^{\infty}|f(y)|\mathrm{d}y\right) \\
&\le \frac{1}{1+x^2/4}\left(A\int_{-\infty}^{\infty}\frac{B}{1+y^2}\mathrm{d}y + B\int_{-\infty}^{\infty}\frac{A}{1+y^2}\mathrm{d}y\right) \\
&= \frac{2AB}{1+x^2/4}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\mathrm{d}y \\
&= \frac{2AB}{1+x^2/4}\arctan y\Big|_{-\infty}^{\infty} \\
&= \frac{2AB\pi}{1+x^2/4} \\
&\le \frac{C}{1+x^2}
\end{aligned}
$$

□