7.1

1. $(A\otimes I)\,\mathrm{vec}(X) - (I\otimes A)\,\mathrm{vec}(X) = \mathrm{vec}(B)$

$$\Rightarrow M = A\otimes I - I\otimes A = \begin{pmatrix} I & 2I \\ 3I & 4I \end{pmatrix} - \begin{pmatrix} A & \\ & A \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 2 & 0 \\ -3 & -3 & 0 & 2 \\ 3 & 0 & 3 & -2 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

2. $M\,\mathrm{vec}(x) = \mathrm{vec}(B) = \begin{pmatrix} 0 \\ 4 \\ -4 \\ 0 \end{pmatrix}$

$$\mathrm{vec}(x) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} a & b \\ b & \frac{3}{2}(a+b)+2 \end{pmatrix}$$

3. $\mathrm{vec}(B) \in \mathrm{Ran}(M) = \mathrm{span}\left(\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ -3 \end{pmatrix}\right)$

$$\Rightarrow \mathrm{vec}(B) = u\cdot\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 2 \\ 3 \\ 0 \\ -3 \end{pmatrix} \quad (\forall u, v \in \mathbb{R})$$

$$= \begin{pmatrix} 2v \\ u+3v \\ -u \\ -3v \end{pmatrix}$$

$$\Rightarrow B = \begin{pmatrix} 2v & -u \\ u+3v & -3v \end{pmatrix}$$

7.2

1. $A = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right) = \begin{pmatrix} J_{3\times 3} & \\ & J_{2\times 2} \end{pmatrix}$ $\qquad B = \left(\begin{array}{ccc|cc} 0 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 \\ & & 0 & 0 & 0 \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right)$

$AB = \left(\begin{array}{ccc|cc} & & 1 & & 1 \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right) \qquad BA = \left(\begin{array}{ccc|cc} & & 1 & & 1 \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right)$

$\Rightarrow AB = BA$

2. eigenvectors of $A$: $\operatorname{span}(\vec{e_1}, \vec{e_4})$

eigenvectors of $B$: $\operatorname{span}(\vec{e_1}, \vec{e_2} - \vec{e_4})$

$\Rightarrow$ common eigenvectors: $\operatorname{span} \vec{e_1}$

$\Rightarrow$ so all eigenvectors of $C$ are multiples of $\vec{e_1}$

$\downarrow$

$C$ has only ONE Jordan block

$\Rightarrow C = X \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} X^{-1}$

3. $r(A) = 3 \qquad r(B) = 3$

$\phantom{3.}\ \ \ \| \qquad\qquad\quad \|$

$r(f(c)) \qquad r(g(c))$

$\Rightarrow (f(c))^2 = X_f \begin{pmatrix} & * \\ & \end{pmatrix} X_f^{-1}$

$(g(c))^2 = X_c \begin{pmatrix} & * \\ & \end{pmatrix} X_c^{-1}$

$\Rightarrow (f(c))^2 \cdot (g(c))^2$ has the same kernel

$\Rightarrow \ker(A^2) = \ker(B^2)$

$\ker\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & \\ & & & & \\ \hline & & & 0 & 0 \\ & & 1 & 0 & 0 \end{array}\right) \neq \ker\begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ & & & & \\ & & & & \end{pmatrix}$

$\Rightarrow$ so there's no possible $C$

4.

$$AB = (A_1 \otimes A_2)(B_1 \otimes B_2) = (A_1B_1) \otimes (A_2B_2) = (B_1A_1) \otimes (A_2B_2)$$
$$= (B_1 \otimes A_2)(A_1 \otimes B_2)$$

$$= BA = (B_1 \otimes B_2)(A_1 \otimes A_2) = (B_1A_1) \otimes (B_2A_2) = (A_1B_1) \otimes (B_2A_2)$$
$$= (A_1 \otimes B_2)(B_1 \otimes A_2)$$

$$\Rightarrow \begin{cases} A = A_1 \otimes B_2 \\ B = B_1 \otimes A_2 \end{cases} \quad \text{or} \quad \begin{cases} A = B_1 \otimes A_2 \\ B = A_1 \otimes B_2 \end{cases}$$

WLOG, we choose

$$\begin{cases} A = A_1 \otimes B_2 = A_1 \otimes A_2 \\ B = B_1 \otimes A_2 = B_1 \otimes B_2 \end{cases} \Rightarrow \begin{matrix} A_1 = A \\ B_1 = B \end{matrix} \quad A_2 = B_2 = [1]$$

5. if possible:

$$A = X J_A X^{-1} \qquad B = X J_B X^{-1}$$
$$A^2 = X J_A^2 X^{-1} \qquad B^2 = X J_B^2 X^{-1}$$
$$= X \begin{pmatrix} & 1 & \\ & & 1 \\ & & 1 \end{pmatrix} X^{-1} \qquad = X \begin{pmatrix} & 1 & \\ & & 1 \\ & & 1 \end{pmatrix} X^{-1}$$

$$\Rightarrow \ker(A^2) = \ker(B^2)$$

but actually $\ker(A^2) \neq \ker(B^2)$

So it's impossible to find $X$ that $A, B$ are both in Jordan canonical form

6.

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

$\ker: (\vec{e_1}, \vec{e_2} - \vec{e_4}) \qquad (\vec{e_1}, \vec{e_2} - \vec{e_4}, \vec{e_4}, 2\vec{e_3} - \vec{e_5})$

$B^3 = 0$

$(\vec{e_1}, \vec{e_2} - \vec{e_4}, \vec{e_4}, 2\vec{e_3} - \vec{e_5}, \vec{e_3})$

$$X = (B^2\vec{e_3}, B\vec{e_3}, \vec{e_3}, B(2\vec{e_3} - \vec{e_5}), 2\vec{e_3} - \vec{e_5})$$
$$= \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow X^{-1}AX = \begin{pmatrix} 0 & 1 & -1 & 1 & -2 \\ & 0 & 1 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad X^{-1}BX = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

7.3

1. $p \mapsto ev_5((x+1)p(x)) = 6p(5)$

$p+q \mapsto 6(p+q)(5) = 6p(5) + 6q(5)$

$kp \mapsto 6kp(5)$

$\Rightarrow$ linear ✓

2. $p \mapsto \lim_{x\to\infty} \frac{p(x)}{x}$

$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \mapsto \lim_{x\to\infty}\left(\frac{a_0}{x} + a_1 + a_2 x\right) = \infty$

$\Rightarrow$ not linear ✗

3. $p \mapsto \lim_{x\to\infty} \frac{p(x)}{x^2}$

$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \mapsto \lim_{x\to\infty}\left(\frac{a_0}{x^2} + \frac{a_1}{x} + a_2\right) = a_2$

$$\left.\begin{aligned} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} &\mapsto (a_2 + b_2) \\ k\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} &\mapsto ka_2 \end{aligned}\right\} \Rightarrow \text{linear} \checkmark$$

4. $p \mapsto p(3)\,p'(4)$

$kp \mapsto kp(3) \cdot kp'(4) = k^2\left(p(3)p'(4)\right)$

$\Rightarrow$ not linear ✗

5. $p \mapsto \deg(p)$

$kp \mapsto \deg(p) \neq k\deg(p)$

$\Rightarrow$ not linear ✗