

4.1

→ row operations      column operations

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$2. \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$$

$$3. \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 \\ 0 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$4. \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k = \begin{cases} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & (k \bmod 4 = 1) \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & (k \bmod 4 = 2) \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & (k \bmod 4 = 3) \end{cases}$$

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4.2

$$I_1: \left[ \begin{array}{cc|ccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc|ccc} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc|ccc} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$2. \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$3. \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$4. \quad \left[ \begin{array}{ccc} 1 & \dots & 1 \end{array} \right]^{-1} = \left[ \begin{array}{ccc} 1 & \dots & 1 \end{array} \right]$$

$$6. \left[ \begin{array}{ccc|cc} 3 & -1 & -1 & 1 & 0 \\ -1 & 3 & -1 & 0 & 1 \\ -1 & -1 & 3 & 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|cc} 3 & -1 & -1 & 1 & 0 \\ -3 & 9 & -3 & 0 & 3 \\ -3 & -3 & 9 & 0 & 0 \end{array} \right] = \left[ \begin{array}{ccc|cc} 1 & -1 & -1 & 1 & 0 \\ 0 & 8 & 4 & 3 & 0 \\ 0 & 4 & 8 & 0 & 3 \end{array} \right]$$

$$5. \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 3/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 & -1 & | & 100 \\ 0 & 12 & 0 & | & 363 \\ 0 & 0 & 12 & | & 336 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1/2 \cdot 1/4 \cdot - \\ 0 & 1 & 0 & | & 1/4 \cdot 1/2 \cdot 1/4 \\ 0 & 0 & 1 & | & 1/4 \cdot 1/4 \cdot 1/2 \end{bmatrix}$$

$$7. \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.3 suppose  $n=4$

$$A: \text{northwest} : \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$B: \text{southeast} : \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & x & x \\ 0 & x & x & x \\ x & x & x & x \end{bmatrix}$$

$$(c+sd=0 = bl+cp+dt = fl+gp = al+bm+cq+du = el+fm+gq = h+k+l)$$

$A^T: \text{northwest}$

$$A^{-1} : \begin{bmatrix} j & 0 & 0 & K \\ 0 & 0 & l & m \\ 0 & m & p & q \\ r & s & t & u \end{bmatrix}$$

$A^2: \text{an arbitrary matrix}$

$$\text{northwest} \times \text{southeast} = \begin{bmatrix} x & K & K & x \\ 0 & x & x & x \\ 0 & 0 & A & x \\ 0 & 0 & 0 & K \end{bmatrix}$$

4.4

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$(A + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})^{-1} = A^{-1} - \frac{(A^{-1} \vec{e}_1)(\vec{e}_3^T A^{-1})}{1 + \vec{e}_3^T \vec{e}_1}$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_3^T \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 & -1 \\ 0 & 1 & 7 \\ 0 & 1 & 1 \end{bmatrix}$$

4.5

$$1. \begin{bmatrix} I_n & 0 \\ -A & I_m \end{bmatrix}$$

$$2. \begin{bmatrix} -A & I_m \\ I_n & 0 \end{bmatrix}$$

$$3. \begin{bmatrix} 0 & B^{-1} \\ A & 0 \end{bmatrix}$$

$$4. \begin{bmatrix} A^{-1} - A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

4.6

$$1. A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$$\text{let } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, A\vec{x} = \begin{bmatrix} ax_2 + bx_3 \\ -ax_1 + cx_3 \\ -bx_1 - cx_2 \end{bmatrix} = \vec{v} \times \vec{x} = \begin{bmatrix} v_3x_3 - v_2x_2 \\ v_2x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \begin{cases} v_1 = -c \\ v_2 = b \\ v_3 = -a \end{cases}$$

2.

$$\left[ \begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{array} \right]$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} & \dots & a_{mm} \end{bmatrix} \quad a_{ij} = -a_{ji}^T (i \neq j)$$

$$3. n=1: A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$$

$$\Rightarrow a=\pm 1 \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A^2 = -I$$

$$I = -A \cdot A = A^T \cdot A = A^{-1} \cdot A$$

$$\Rightarrow A^T = A^{-1} = -A$$

$$\therefore A = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \quad (\text{the anti-diagonal entries are } 1 \text{ or } -1, \text{ others are all zero, also } (i, n+1-i) \text{ entry is the opposite of } (n+1-i, i) \text{ entry})$$

$$4. \text{ let } A = B + C \rightarrow \text{skew symmetric matrix}$$

↓  
symmetric  
matrix

$$\text{just let } b_{ii} = a_{ii}, c_{ii} = 0$$

$$(\text{if } j \neq i) \quad b_{ij} = b_{ji} = \frac{a_{ij} + a_{ji}}{2}, \quad c_{ij} = -c_{ji} = \frac{a_{ij} - a_{ji}}{2}$$

so the work is done

4.7

$$1. \begin{bmatrix} R & 2R \end{bmatrix}$$

2.

$$\begin{bmatrix} R \\ O \end{bmatrix}$$

3.

$$\begin{bmatrix} R & R \\ O & O \end{bmatrix}$$

4.

$$\begin{bmatrix} R & O \\ O & R \end{bmatrix}$$

4.8

$$1. \begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} = \begin{bmatrix} A\vec{x} + \vec{b} \\ 0+1 \end{bmatrix} = \begin{bmatrix} f(\vec{x}) \\ 1 \end{bmatrix}$$

$$2. f(\vec{x}) = A\vec{x} + \vec{b}$$

$$\text{let } g(\vec{x}) = C\vec{x} + \vec{d}$$

$$f \circ g(\vec{x}) = f(g(\vec{x}))$$

$$= A(C\vec{x} + \vec{d}) + \vec{b}$$

$$= (AC)\vec{x} + (A\vec{d} + \vec{b})$$

$$3. \text{① } f \text{ is invertible} \Rightarrow A \text{ is invertible} \& M_f^{-1} = M_{f^{-1}}$$

$$\forall \vec{y} \in \mathbb{R}^n, \exists \text{ only } \vec{x} \in \mathbb{R}^n, \vec{y} = A\vec{x} + \vec{b} \quad (f \text{ is bijective})$$

$$\text{so } (\vec{y} - \vec{b}) = A\vec{x} \Rightarrow A \text{ is bijective}$$

∴ A is invertible

$$\text{also } f^{-1}(\vec{x}) = A^{-1}(\vec{x} - \vec{b}) \Rightarrow M_{f^{-1}} = \begin{bmatrix} A^{-1} & -A^{-1}\vec{b} \\ \vec{0}^T & 1 \end{bmatrix}$$

$$= A^{-1}\vec{x} - A^{-1}\vec{b}$$

$$\begin{aligned} M_f^{-1} &= \begin{bmatrix} A & \vec{b} \\ \vec{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}\vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (I - \vec{0}^T A^{-1}\vec{b})^{-1} \end{bmatrix} \begin{bmatrix} I & \vec{0} \\ \vec{0}^T A^{-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & -A^{-1}\vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} & -A^{-1}\vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \begin{bmatrix} I & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}\vec{b} \\ \vec{0}^T & 1 \end{bmatrix} \end{aligned}$$

$$\text{so } M_f^{-1} = M_{f^{-1}}$$

$$\text{② } A \text{ is invertible} \& M_f^{-1} = M_{f^{-1}} \Rightarrow f \text{ is invertible}$$

as long as  $f^{-1}$  can be written as a condition, then  $f^{-1}$  exists, then f is invertible

$$\text{③ the block inverse } M_f^{-1} = \underline{\hspace{10em}}$$

4.9

$$1. (A \Delta B) \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X^{-1} = X$$

$$2. X((A_1 B_1) \Delta (A_2 B_2)) X^{-1} = \begin{bmatrix} A_1 B_1 & O \\ O & A_2 B_2 \end{bmatrix}$$

$$\begin{aligned} X((A_1 \Delta A_2)(B_1 \Delta B_2)) X^{-1} &= X(A_1 \Delta A_2) \cdot (X^{-1} X) \cdot (B_1 \Delta B_2) X^{-1} \\ &= (X(A_1 \Delta A_2) X^{-1}) (X(B_1 \Delta B_2) X^{-1}) \\ &= \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 B_1 & O \\ O & A_2 B_2 \end{bmatrix} \end{aligned}$$

$$\therefore (A_1 B_1) \Delta (A_2 B_2) = (A_1 \Delta A_2)(B_1 \Delta B_2)$$

$$3. X(A^{-1} \Delta B^{-1}) X^{-1} = \begin{bmatrix} A^{-1} & O \\ O & B^{-1} \end{bmatrix}$$

$$\begin{aligned} (X(A \Delta B)^{-1} X^{-1}) \cdot (X(A \Delta B) \cdot X^{-1}) &= X(A \Delta B)^{-1} (X^{-1} X) (A \Delta B)^{-1} \\ &= X((A \Delta B)^{-1} (A \Delta B)) X^{-1} \\ &= X \cdot X^{-1} = I \end{aligned}$$

$$\therefore X(A \Delta B)^{-1} X^{-1} = (X(A \Delta B) \cdot X^{-1})^{-1} = \begin{bmatrix} A & O \\ O & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & O \\ O & B^{-1} \end{bmatrix}$$

$$\therefore A^{-1} \Delta B^{-1} = (A \Delta B)^{-1}$$