12.1

1.

since A is diagonal

let $A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$

then AB means do row scaling to B

BA — — — column — — — —

since $a_1 \sim a_n$ are distinct

then $AB = BA$ if and only if

every row/column operation has same effect

iff every entry of B that are not on its diagonal line is zero

$\Rightarrow B$ is diagonal

2.

$AB = BA$, A has no repeated eigenvalues

$\Rightarrow (X^{-1}AX)B = B(X^{-1}AX)$

$AXB = XBX^{-1}AX$

$A(XBX^{-1}) = (XBX^{-1})A$

$\Rightarrow (X^{-1}AX)(X^{-1}BX) = (X^{-1}BX)(X^{-1}AX)$

from last subquestion, we know

if $(X^{-1}AX)$ is diagonal

then $(X^{-1}BX)$ is diagonal

so the proof is done

3. $A\vec{v} = \lambda\vec{v}$. $\forall \vec{v}$

$\Rightarrow B\vec{v} = B(\frac{1}{\lambda}A\vec{v})$

$= \frac{1}{\lambda}(BA)\vec{v}$

$= \frac{1}{\lambda}A(B\vec{v})$

$\Rightarrow A(B\vec{v}) = \lambda(B\vec{v})$

so $B\vec{v}$ is an eigenvector for eigenvalue λ of A

$\Rightarrow B\vec{v} \in \mathrm{span}(\vec{v})$

$\Rightarrow \vec{v}$ is also an eigenvector of B

4.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

$AB = BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

eigenvalues of B: 1 eigenvalues of A: 0, 3

eigenvectors of B: $\forall \vec{v} \in R$ eigenvectors of A: span$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. span$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so $\vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\notin$ span$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
$\notin$ span$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5.

$A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

6.

$AB = BA$

$A\vec{v} = \lambda \vec{v}$

then $B\vec{v} = B\left(\frac{1}{\lambda} A\vec{v}\right) = \frac{1}{\lambda} BA\vec{v} = \lambda A(B\vec{v})$

so $B\vec{v}$ is also an eigenvector of A for the same eigenvector λ)

12.2

1. $A^k = O \Rightarrow \lambda^k = 0$

so k eigenvalues of A are zero

since A is invertible iff A has no zero eigenvalue

so A is not invertible

2.

① $A^n = 0 \Longleftrightarrow$ all eigenvalues of A are zero

obvious

② since $\begin{cases} A \text{ is a } n\times n \text{ matrix} \\ \text{Cayley-Hamilton} \end{cases}$

$\Rightarrow P_A(A) = O$

then if A is nilpotent, $\underset{\underset{P_A(A)}{\parallel}}{A^k} = 0$ for some k

$A^n = A^{n-k} \cdot A^k = A^{n-k} = 0$

if $A^n = 0$, then just let $k=n$, $A^k = A^n = 0$

so A is nilpotent

3. do Schur decomposition $\Rightarrow A = UTU^{-1}$, then let D be the diagonal matrix, taking all diagonal entries of T

then let $B = UDU^{-1}$, is a normal matrix (spectral theorem)

so $C = U(T-D)U^{-1}$

$$C^k = U(T-D)^k U^{-1} = U \begin{pmatrix} 0 & \ddots & * \\ \vdots & \ddots & \\ 0 & \cdots & 0\ 0 \end{pmatrix}^k U^{-1}$$

then just let $k = n-1$, $(T-D)^k = O$, $C^k = O$

$\Rightarrow C$ is nilpotent

12.3

1. $$\frac{2xy}{x^2+y^2} = \frac{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \in [-1, 1]$$

2. $$\frac{3x^2+2xy+3y^2}{x^2+y^2} = \frac{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \in [2, 4]$$

3. $$\frac{x^2+2xy+4y^2}{x^2+y^2} = \frac{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}} \in \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right]$$

12.4

1. $A^TA = V\Sigma^T\Sigma V^T = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$

$V = \begin{bmatrix} \frac{3}{5} & 0 & 0 \\ \frac{4}{5} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$AA^T = \begin{bmatrix} 25 \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$

$U = \begin{bmatrix} 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & 0 \\ 0.8 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T$

2. $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$

3. $A^TA = \begin{bmatrix} 9 & & & \\ & 1 & & \\ & & 16 & \\ & & & 4 \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{bmatrix}$

$V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & & & \\ & 4 & & \\ & & 9 & \\ & & & 16 \end{bmatrix} \Rightarrow U = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$

$\Rightarrow \begin{bmatrix} & 1 & & \\ & & & 2 \\ 3 & & & \\ & & 4 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 4 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{bmatrix}^T$

4. $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}^T$

12.5

$$\sigma_1 = \|A\| = \max \frac{\|A\vec{v}\|}{\|\vec{v}\|}$$

if $\vec{v}$ is an eigenvector for A

then $\sigma_1 \geq \max\{\text{eigenvalues of } A\}$

so all eigenvalues are less that σ_1

12.6

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix} \begin{bmatrix} 4 \\ 20 \end{bmatrix} \qquad \bar{x} = 2, \quad \bar{y} = 9$$

$$\downarrow$$

$$A = \begin{bmatrix} -2 & -1 & 1 & 2 \\ -9 & -1 & -1 & 11 \end{bmatrix}$$

do SVD to A:

$$A^T A = \begin{bmatrix} 4 & 2 & -2 & -4 \\ 2 & 65 & 63 & 158 \\ -2 & 63 & 65 & 162 \\ -4 & 158 & 162 & 404 \end{bmatrix} \Rightarrow \sigma_1 = \frac{3159}{217}, \quad \vec{v}_1 = \begin{bmatrix} -\frac{781}{1243} \\ -\frac{181}{2242} \\ -\frac{383}{7088} \\ \frac{493}{642} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 10 & 40 \\ 40 & 204 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} \frac{643}{3309} \\ \frac{669}{682} \end{bmatrix}$$

$$\Rightarrow B = \sigma_1 \vec{u}_1 \vec{v}_1^T = \begin{bmatrix} -\frac{5203}{2905} & -\frac{1555}{6809} & -\frac{854}{5587} & \frac{2257}{1039} \\ -\frac{5461}{604} & -\frac{626}{543} & -\frac{223}{289} & \frac{4178}{381} \end{bmatrix}$$

$$\Rightarrow \quad k = \frac{\frac{4178}{381} + \frac{5461}{604}}{\frac{2257}{1039} + \frac{5203}{2905}} = \frac{2519}{499}$$

$$\Rightarrow y' = kx' \Rightarrow y - 9 = k(x-2)$$

$$y = kx + 9 - 2k = 5.0481x - 1.0962$$

12.7

1. $A^TA = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \Rightarrow \Sigma = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$

$\Rightarrow \sigma_1 = \sigma_2 = \sigma_3 = 1$

$\sigma_4 = 0$

eigenvalues of A: $0, 0, 0, 0$

2. $A^TA = \begin{bmatrix} 10^{-8} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \Rightarrow \sigma_1 = \sigma_2 = \sigma_3 = 1$

$\sigma_4 = 10^{-8}$

eigenvalues of A: $\pm 0.1, \pm 0.1i$