1.

$$F(x,y,z(x,y)) = (z+y)^x - x^2 y \equiv 0$$

$$\begin{cases} \left.\dfrac{\partial F}{\partial z}\right|_{(1,1)} = \left. x(z+y)^{x-1} \right|_{(1,1)} = 1\times(0+1)^{1-1} = 1 \neq 0 \\ z(1,1)=0 \end{cases}$$

$\Rightarrow$ 有唯一 $z = z(x,y)$

且连续性与 $F(x,y,z)$ 一致，均为 C^{∞}

$$\frac{\partial z}{\partial y} = -\frac{F_y'}{F_z'} = -\frac{x(z+y)^{x-1} - x^2}{x(z+y)^{x-1}} = -1 + \frac{x}{(z+y)^{x-1}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = -1 + \frac{1}{(0+1)^0} = 0$$

$$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{(1,1)} = \left.\frac{(z+y)^{x-1} - x\cdot \ln(z+y)\cdot(z+y)^{x-1}}{(z+y)^{2x-2}}\right|_{(1,1)} = 1$$

$$z(1+x,1+y) = 2x + \frac{1}{2}(x\ \ y)\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + o(x^2+y^2)$$

$$= 2x - x^2 + xy + o(x^2+y^2)$$

2.

$$F(x,y,z(x,y)) = xyz + \sqrt{x^2+y^2+z^2} - \sqrt{2} \equiv 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x'}{F_z'} = -\frac{yz + \frac{x}{\sqrt{x^2+y^2+z^2}}}{xy + \frac{z}{\sqrt{x^2+y^2+z^2}}} \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,0)} = -\frac{0+\frac{1}{\sqrt{2}}}{0+\frac{-1}{\sqrt{2}}} = 1$$

$$\frac{\partial z}{\partial y} = -\frac{F_y'}{F_z'} = -\frac{xz + \frac{y}{\sqrt{x^2+y^2+z^2}}}{xy + \frac{z}{\sqrt{x^2+y^2+z^2}}} \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,0)} = -\frac{-1+0}{0+\frac{-1}{\sqrt{2}}} = -\sqrt{2}$$

$$\Rightarrow dz\big|_{(1,0)} = dx - \sqrt{2}\,dy$$

3.

$$\begin{cases} u_x' = f_x' + f_z' \cdot z_x' + f_t' \cdot t_x' \\ g_z' \cdot z_x' + g_t' \cdot t_x' = 0 \\ h_z' \cdot z_x' + h_t' \cdot t_x' = 0 \end{cases} \Rightarrow u_x' = f_x'$$

$$\frac{\partial(g,h)}{\partial(z,t)}$$

$$\left.\begin{cases} u_y' = f_y' + f_z' z_y' + f_t' \cdot t_y' \\ g_y' + g_z' \cdot z_y' + g_t' \cdot t_y' = 0 \\ h_z' \cdot z_y' + h_t' \cdot t_y' = 0 \end{cases}\right\} \Rightarrow \begin{pmatrix} g_z' & g_t' \\ h_z' & h_t' \end{pmatrix} \begin{pmatrix} z_y' \\ t_y' \end{pmatrix} = \begin{pmatrix} -g_y' \\ 0 \end{pmatrix}$$

$$(f_z' \ f_t') \begin{pmatrix} z_y' \\ t_y' \end{pmatrix} = (f_z', f_t') \left(\frac{\partial(g,h)}{\partial(z,t)}\right)^{-1} \cdot \begin{pmatrix} -g_y' \\ 0 \end{pmatrix}$$

$$\Rightarrow u_y' = f_y' + \frac{\partial f}{\partial(z,t)} \cdot \left(\frac{\partial(g,h)}{\partial(z,t)}\right)^{-1} \cdot \begin{pmatrix} -g_y' \\ 0 \end{pmatrix}$$

4.

$$F(x,y,z) = x^2 + y^2 + z^2 - 3xyz \qquad F(P_0) = 0 \ \checkmark$$

$$\left.\frac{\partial F}{\partial y}\right|_{(1,1,1)} = (2y - 3xz)\Big|_{(1,1,1)} = -1 \neq 0 \Rightarrow y = y(x,z) \ \checkmark$$

$$\left.\frac{\partial f(x, y(x,z), z)}{\partial x}\right|_{(1,1,1)} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x}\Big|_{(1,1,1)}$$

$$= y^2 z^3 + 2xyz^3 \frac{\partial y}{\partial x}\Big|_{(1,1,1)}$$

$$= -1$$

5. 设 $G(x,y,u)=F(x+ux^{-1},\ y+uy^{-1})$

则 $G(x_0,y_0,0)=F(x_0,y_0)=0$

$G_u'(x_0,y_0,0)=y_0^{-1}F_x'(x_0,y_0)+x_0^{-1}F_y'(x_0,y_0)$

$$=\frac{1}{x_0y_0}\left(x_0F_x'(x_0,y_0)+y_0F_y'(x_0,y_0)\right)\neq 0$$

则 $\exists\,(x_0,y_0)$ 的一邻域中，$F(x+ux^{-1},y+uy^{-1})=0$ 唯一确定了 $u=u(x,y)$ 且 $u(x_0,y_0)=0$

由于 $G(x,y,u)\in C^1$

则 $u=u(x,y)\in C^1$

6.

(1) $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = ct \end{cases}$ $(a>0, c>0)$，$M\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, \frac{\pi c}{4}\right)$ 处切线与法平面

$\Downarrow$

$t_0 = \frac{\pi}{4}$

$$\vec{n} = \begin{pmatrix} -\frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \\ c \end{pmatrix}$$

$\Rightarrow$ 切线：$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \\ c \end{pmatrix} t + \begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \\ \frac{\pi c}{4} \end{pmatrix}$

法平面：$-\frac{a}{\sqrt{2}}\left(x - \frac{a}{\sqrt{2}}\right) + \frac{a}{\sqrt{2}}\left(y - \frac{a}{\sqrt{2}}\right) + c\left(z - \frac{\pi c}{4}\right) = 0$

(2) $\begin{cases} x^2+y^2+z^2-6=0 \\ z-x^2-y^2=0 \end{cases}$ 在 $M_0(1,1,2)$ 处的切线方程

$F(x,y,z) = x^2+y^2+z^2-6$

$G(x,y,z) = z-x^2-y^2$

$\Rightarrow$ $\operatorname{grad} F(M_0) = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$

$\operatorname{grad} G(M_0) = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$

切线方向：$\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 4 \\ -2 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix}$

$\Rightarrow$ 切线：$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

7.

求曲面 $S: 2x^2-2y^2+2z=1$ 上切平面与直线 $L\begin{cases}3x-2y-z=5\\x+y+z=0\end{cases}$ 平行的切点的轨迹

$$F(x,y,z)=2x^2-2y^2+2z-1=0$$

$$\operatorname{grad}F(x,y,z)=\begin{pmatrix}4x\\-4y\\2\end{pmatrix}=\vec{s}$$

$$\vec{l}=(3,-2,-1)\times(1,1,1)=\begin{pmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-2&-1\\1&1&1\end{pmatrix}=\begin{pmatrix}-1\\-4\\5\end{pmatrix}$$

$\Rightarrow \vec{s}\perp\vec{l}$

$\Rightarrow \vec{s}\cdot\vec{l}=-4x+16y+10=0$

$\Rightarrow \begin{cases}2x-8y-5=0\\2x^2-2y^2+2z-1=0\end{cases}$，即该切点轨迹

8. 证明球面 $S_1: x^2+y^2+z^2=R^2$ 与锥面 $S_2: x^2+y^2=a^2z^2$ 正交

S_1 的法向量 $\vec{n_1}=\begin{pmatrix}x\\y\\z\end{pmatrix}$，$S_2$ 的法向量 $\vec{n_2}=\begin{pmatrix}x\\y\\-a^2z\end{pmatrix}$

$$\vec{n_1}\cdot\vec{n_2}=x^2+y^2-a^2z^2$$

由于 S_1, S_2 相交处的点同时满足两者方程

$\Rightarrow \vec{n_1}\cdot\vec{n_2}=0$

$\Rightarrow \vec{n_1}\perp\vec{n_2}$

得证

9.

(1) 平面 S: $e^z = xy + yz + zx$, 求 S 在 $(1,1,0)$ 处的切平面

$F(x,y,z) = e^z - xy - yz - zx$

$\mathrm{grad}\, F = \begin{pmatrix} -y-z \\ -x-z \\ e^z - y - x \end{pmatrix} \Rightarrow \mathrm{grad}\, F(1,1,0) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

$\Rightarrow$ 切平面: $(x-1) + (y-1) + z = 0$

(2) 若曲面 S 的显式方程为 $z = f(x,y)$, 求 $\mathrm{grad} f(1,1)$

$\frac{\partial f}{\partial x} = -\frac{F_x'}{F_z'} = -\frac{-y-z}{e^z - y - x} = \frac{y+z}{e^z - y - x}$

$\frac{\partial f}{\partial y} = -\frac{F_y'}{F_z'} = \frac{x+z}{e^z - y - x}$

$\Rightarrow \mathrm{grad} f(1,1) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

10.

$$F(x,y,z)=G(x^2+y^2+z^2)-ax-by-cz\equiv 0$$

$$\forall P(x_0,y_0,z_0),\ \vec{n}=\begin{pmatrix} G'(x_0^2+y_0^2+z_0^2)\cdot 2x_0-a \\ G'(x_0^2+y_0^2+z_0^2)\cdot 2y_0-b \\ G'(x_0^2+y_0^2+z_0^2)\cdot 2z_0-c \end{pmatrix}$$

$\Rightarrow$ 法线：$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}+t\vec{n}$

设定直线为 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}+t\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$\Rightarrow \forall(x_0,y_0,z_0)\exists t_1\exists t_2 \begin{pmatrix} x_1-x_0 \\ y_1-y_0 \\ z_1-z_0 \end{pmatrix}=t_1\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}+t_2\vec{n}$

取 $t_1=t_2=\dfrac{-1}{2G'(x_0^2+y_0^2+z_0^2)}$

$\begin{cases} \alpha=a \\ \beta=b \\ \gamma=c \end{cases}$，$x_1=y_1=z_1=0$ 即可

$\Rightarrow$ 定直线为 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

12. 求过直线 $\begin{cases} 3x-2y-z=-15 \\ x+y+z=10 \end{cases}$ 且与曲面 S: $x^2-y^2+z=10$ 相切的平面方程

$$\Downarrow$$

$$\vec{n}=(2x,\ -2y,\ 1)$$

$\Rightarrow$ 平面方程 $2x_0\cdot(x-x_0)-2y_0(y-y_0)+(z-z_0)=0$

将直线方程代入

$$\Rightarrow 2x_0(x-x_0)-2y_0(4x+5-y_0)+5-5x-z_0=0$$

$$x(2x_0-8y_0-5)+\left(-2x_0^2-10y_0+2y_0^2+5-z_0\right)=0$$

$$\Rightarrow \begin{cases} 2x_0-8y_0-5=0 \\ 2y_0^2-2x_0^2-10y_0-z_0+5=0 \\ x_0^2-y_0^2+z_0-10=0 \end{cases}$$

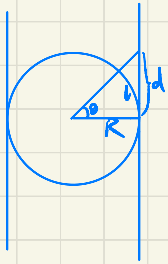
$$\Rightarrow \begin{cases} x_0=-\frac{7}{2} \\ y_0=-\frac{3}{2} \\ z_0=0 \end{cases} \quad 或 \quad \begin{cases} x_0=\frac{1}{2} \\ y_0=-\frac{1}{2} \\ z_0=10 \end{cases}$$

$\Rightarrow$ 切平面: $-7x+3y+z=20$

或 $x+y+z=10$

14.

(1)

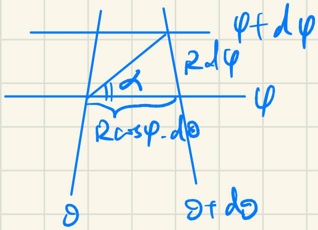


$d = R\tan\theta$

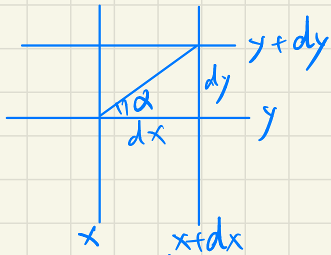
$l = R\theta$

$\frac{d}{l} = \frac{\tan\theta}{\theta}$ ⇒ 不保距，比例系数只与纬度有关

地球表面



投影面



$$\begin{cases} x = R\cos\varphi\cdot\theta \\ y = R\tan\varphi \end{cases}$$

$\tan\alpha_1 \approx \frac{d\varphi}{\cos\varphi\cdot d\theta}$　　　$\tan\alpha_2 = \frac{dy}{dx}$

$$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{dy}{d\varphi}\cdot\cos\varphi\cdot\frac{d\theta}{dx} = R\frac{1}{\cos^2\varphi}\cos\varphi\cdot\frac{1}{R\cos\varphi} = 1$$

⇒ 保角

(2) 保距则最多只能保证一个方向上距离不变

不可能完全保距