1.

(1) $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$

令 $y=tx$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\dfrac{t^2x^4}{t^2x^4+(t-1)^2x^2}$ 与 t 有关，二重极限不存在

$\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=0$　　$\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)=0$

(2) $D=\{(x,y)\mid x+y\neq0\}$　$f(x,y)=\dfrac{x-y}{x+y}$, $(x,y)\in D$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ 不存在 $\Rightarrow$ 令 $y=tx$, $f(x,tx)=\dfrac{1-t}{1+t}$; 与 t 有关

$\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=-1$　　$\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)=1$

(3) $f(x,y)=\begin{cases} x\sin\frac{1}{y} & xy\neq0 \\ 0 & xy=0 \end{cases}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

$\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=0$　　$\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)$ 不存在 $\Rightarrow$ 由于 $\sin\frac{1}{y}$

2.

(1) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} f(x,y)$：x、y 以任意路径趋近于 ∞ 时，$f(x,y)$ 所趋近的值

$\forall \varepsilon>0, \exists M>0, |x|,|y|>M, |f(x,y)-A|<\varepsilon$

$\Rightarrow \lim\limits_{\substack{x\to\infty\\ y\to\infty}} f(x,y)=A$

$\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x+y}{x^2-2xy+y^2}$　　$x=y$，此极限不存在

(2) $\lim\limits_{\substack{x\to+\infty\\ y\to-\infty}} f(x,y)$：$\forall \varepsilon>0, \exists M>0, x>M, y<-M, |f(x,y)-A|<\varepsilon$

$\Rightarrow \lim\limits_{\substack{x\to+\infty\\ y\to-\infty}} f(x,y)=A$

$f(x,y)=e^{x^2-y^2}\sin(2xy)$

$\lim\limits_{\substack{x\to+\infty\\ y\to-\infty}} f(x,y)$ 不存在（$x=y$ 时，$f(x,y)=\sin(2x^2)$）

$\lim\limits_{y\to-\infty}\lim\limits_{x\to+\infty} f(x,y)=+\infty$　　$\lim\limits_{x\to+\infty}\lim\limits_{y\to-\infty} f(x,y)$ 不存在

（$\lim\limits_{y\to-\infty} f(x,y)=\sin(-\infty)$ ✗）

(3) $\alpha,\beta\geq 0$，$\alpha+\beta>2$

设 $f(x,y)=\dfrac{|x|^\alpha|y|^\beta}{x^2+y^2}=\dfrac{|x|^\alpha|y|^\beta}{|x|^2+|y|^2}\leq\dfrac{|x|^\alpha|y|^\beta}{2|x||y|}=\dfrac{|x|^{\alpha-1}|y|^{\beta-1}}{2}$

① $\alpha\geq1$ 且 $\beta\geq1$

$0\leq\lim\limits_{(x,y)\to(0,0)} f(x,y)\leq 0$

$\|$

0

② $\alpha<1$ 且 $\beta>1$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$

$=\lim\limits_{r\to0} |r|^{\alpha+\beta-2}\cdot|\cos\theta|^\alpha\cdot|\sin\theta|^\beta$

$=0$

③ $\beta>1$ 且 $\alpha<1$

同②

3.

(1) $\lim\limits_{(x,y)\to(1,0)} \boxed{(x+y)^{\frac{x+y+1}{x+y-1}}} = e^2$

$\parallel$

$f(x,y) = e^{\ln(x+y)\cdot\frac{x+y+1}{x+y-1}} \xrightarrow[(x,y)\to(1,0)]{\lim} e^{(x+y-1)\cdot\frac{x+y+1}{x+y-1}} = e^2$

(2) $\lim\limits_{(x,y)\to(0,0)} (x+y)\ln(x^2+y^2) \le \lim\limits_{(x,y)\to(0,0)} (|x|+|y|)\ln(|x|+|y|)$

$= 0$

$\le$

$-\lim\limits_{(x,y)\to(0,0)} (|x|+|y|)\ln(|x|+|y|) = 0$

(3) $xy>0$:

$\lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{x^2-xy+y^2} \le \lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{xy} = 0$

$\le$

$\lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{(x+y)^2} = 0$

$xy<0$，不等式号相反，同理

$\Rightarrow \lim\limits_{\substack{x\to\infty\\y\to\infty}} \frac{x+y}{x^2-xy+y^2} = 0$

(4) $\lim\limits_{\substack{x\to+\infty\\y\to-\infty}} (x^2+y^2)\, e^{y-x}$

$= \lim\limits_{\substack{x\to+\infty\\t\to+\infty}} (x^2+t^2)\, e^{-(t+x)}$ $\quad \begin{cases} x = r\cos\theta \\ t = r\sin\theta \end{cases} \quad r\to+\infty$

$= \lim\limits_{r\to+\infty} r^2 e^{-r(\cos\theta+\sin\theta)}$

$= 0$

4. $f(x,y)=\begin{cases}\dfrac{\ln(1+xy)}{x} & x\neq 0\\ y & x=0\end{cases}$

$$\lim_{x\to 0} f(x,y)=\lim_{x\to 0}\frac{\ln(1+xy)}{x}=\lim_{x\to 0}\frac{\frac{y}{1+xy}}{1}=y$$
$$=f(0,y)$$

$\Rightarrow f(x,y)$ 在其定义域内连续

5.

(1) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{f(x,y)-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=a$

$\displaystyle\Rightarrow \lim_{(x,y)\to(0,0)}\frac{f(x,y)-(a+1)\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=0$

$\Rightarrow f(x,y)-(a+1)\sqrt{x^2+y^2}=o(\sqrt{x^2+y^2})\to 0$

$\Rightarrow f(x,y)\to f(0,0)=0$ $(x,y)\to(0,0)$ 时

$\Rightarrow f$ 在 $(0,0)$ 连续

(2)

① $\displaystyle\lim_{(x,y)\to(0,0)}\frac{f(x,y)-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=a=-1$

$\displaystyle\Rightarrow \lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}=0$

$f(x,y)-f(0,0)=o(\sqrt{x^2+y^2})$

② $\begin{cases}\displaystyle\lim_{(x,y)\to(0,0)}\frac{f(x,y)-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=a \Rightarrow f(x,y)-f(0,0)=(a+1)\sqrt{x^2+y^2}+o(\sqrt{x^2+y^2})\\ f(x,y)-f(0,0)=f'_x(0,0)\cdot x+f'_y(0,0)\cdot y+o(\sqrt{x^2+y^2})\end{cases}$

$\Rightarrow a=-1$

且 $f'_x(0,0)=f'_y(0,0)=0$

6. $f(x,y) = \begin{cases} \dfrac{\sqrt{|xy|}}{x^2+y^2} \sin(x^2+y^2) & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

$\overset{(x,y)\to(0,0)}{=} \dfrac{\sqrt{|xy|}}{x^2+y^2}(x^2+y^2)+o(x^2+y^2)$

$= \sqrt{|xy|} = 0$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \Rightarrow$ 连续

$f(x,y) = \dfrac{\sqrt{|xy|}}{x^2+y^2} = \dfrac{r|\sin\theta\cos\theta|}{r^2} = \dfrac{|\sin\theta\cos\theta|}{r} \to$ 不可能是 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ 的线性组合

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - L(x,y)\begin{pmatrix} x \\ y \end{pmatrix}}{\sqrt{x^2+y^2}} \neq 0$

$\Rightarrow$ 不可微

7. $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

证：$\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\dfrac{1}{x^2+y^2} < \lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

$\Rightarrow f(x,y) = f(0,0) + o(\sqrt{x^2+y^2})$

$\Rightarrow f$ 在 $(0,0)$ 可微

$f_x'(x,y) = (2x+y^2)\sin\dfrac{1}{x^2+y^2} + (x^2+y^2)\cos\left(\dfrac{1}{x^2+y^2}\right) \cdot -\dfrac{2x}{(x^2+y^2)}$

$\hookrightarrow$ 在 $(0,0)$ 不连续

$f_y'(x,y) = (x^2+2y)\sin\dfrac{1}{x^2+y^2} + (x^2+y^2)\cos\left(\dfrac{1}{x^2+y^2}\right) \cdot \dfrac{-2y}{x^2+y^2}$

$\Rightarrow$ 得证

8.

设 $\lim\limits_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2} = A$

$\Rightarrow f(0,0)=0$

$(x,y)\to(0,0)$ 时：

$\lim\limits_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} A\sqrt{x^2+y^2} = 0$

$\Rightarrow f(x,y) = f(0,0) + o(\sqrt{x^2+y^2})$

9. $f(t,t) = f(0,0) + f_x'(0,0)t + f_y'(0,0)t + o(t)$

$\Rightarrow \lim\limits_{t\to0} \frac{f(t,t)-f(0,0)}{t} = 7$

10.

(1) $f(x,y) = \int \frac{\partial f}{\partial x}\,dx = x\sin y - \frac{1}{y}\ln|xy-1| + C$

代入 $f(1,y)=\sin y$

$\Rightarrow \sin y - \frac{1}{y}\ln(y-1) + C = \sin y$

$C = \frac{1}{y}\ln(y-1)$

$\Rightarrow f(x,y) = x\sin y + \frac{1}{y}\ln\left|\frac{y-1}{xy-1}\right|$

(2) $f = e^{xy}\sin x + C$

代入 $f(0,0)=1 \Rightarrow C=1 \quad f(x,y) = e^{xy}\sin x + 1$