HW1

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1. (1) 证明. 函数 $f(x)$ 的 Fourier 级数为

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

其中

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n \in \mathbb{N} \tag{2}$$

而

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{3}$$

因此

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \hat{f}(0) \tag{4}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})dx \\ &= \hat{f}(n) + \hat{f}(-n) \end{aligned} \tag{5}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ &= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})dx \\ &= i(\hat{f}(n) - \hat{f}(-n)) \end{aligned} \tag{6}$$

综上, 得证. □

(2) 证明.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{7}$$

$$\begin{aligned} \hat{f}(-n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\ &= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)e^{-in(-x)}d(-x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad (f(x)\text{是偶函数}) \\ &= \hat{f}(n) \end{aligned} \tag{8}$$

而由(1)知,

$$\begin{aligned} f(x) &\sim \hat{f}(0)+\sum_{n\geq 1}(\hat{f}(n)+\hat{f}(-n))\cos nx+i(\hat{f}(n)-\hat{f}(-n))\sin nx \\ &= \hat{f}(0)+\sum_{n\geq 1}2\hat{f}(n)\cos nx \end{aligned} \tag{9}$$

因此 f 的 Fourier 级数是余弦级数. □

(3) 证明.

$$\begin{aligned} \hat{f}(-n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\ &= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)e^{-in(-x)}d(-x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(x)e^{-inx}dx \quad (f(x)\text{是奇函数}) \\ &= -\hat{f}(n) \end{aligned} \tag{10}$$

而由(1)知,

$$\begin{aligned} f(x) &\sim \hat{f}(0)+\sum_{n\geq 1}(\hat{f}(n)+\hat{f}(-n))\cos nx+i(\hat{f}(n)-\hat{f}(-n))\sin nx \\ &= \sum_{n\geq 1}2i\hat{f}(n)\sin nx \end{aligned} \tag{11}$$

因此 f 的 Fourier 级数是正弦级数. □

(4) 证明. $\forall n=2k+1, k\in\mathbb{N}$

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x-\pi)e^{-in(x-\pi)}d(x-\pi)+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-in(x-\pi)}dx+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx+in\pi}dx+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}e^{in\pi}dx+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}(-1)dx+\frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-inx}dx \\ &= 0 \end{aligned} \tag{12}$$

□

(5) 证明.

$$\begin{aligned}
f(x) \sim \hat{f}(0) + \sum_{n\geq 1}(\hat{f}(n)+\hat{f}(-n))\cos nx + \sum_{n\geq 1} i(\hat{f}(n)-\hat{f}(-n))\sin nx \\
= Re(\hat{f}(0)) + Im(\hat{f}(0)) \\
+ \sum_{n\geq 1}(Re(\hat{f}(n)) + Im(\hat{f}(n)) + Re(\hat{f}(-n)) + Im(\hat{f}(-n)))\cos nx \\
+ \sum_{n\geq 1} i(Re(\hat{f}(n)) + Im(\hat{f}(n)) - Re(\hat{f}(-n)) - Im(\hat{f}(-n)))\sin nx
\end{aligned} \tag{13}$$

由 f 是实值函数得

$$\begin{aligned}
Im(\hat{f}(0)) &= 0 \\
Im(\hat{f}(n)) &= -Im(\hat{f}(-n)) \\
Re(\hat{f}(n)) &= Re(\hat{f}(-n))
\end{aligned} \tag{14}$$

进而得

$$\overline{\hat{f}(n)} = \hat{f}(-n) \tag{15}$$

综上, 得证. □

2. 证明. $\forall n \neq 0, n \in \mathbb{Z}$

$$\begin{aligned}
|\hat{f}(n)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| \\
&= \frac{1}{2\pi}\left|\frac{1}{-in}\int_{-\pi}^{\pi} f(x)de^{-inx}\right| \\
&= \frac{1}{2\pi|n|}\left|if(x)e^{-inx}|_{x=-\pi}^{\pi} - i\int_{-\pi}^{\pi} e^{-inx}df(x)\right| \\
&= \frac{1}{2\pi|n|}\left|i\cos n\pi(f(\pi)-f(-\pi)) - i\int_{-\pi}^{\pi} e^{-inx}f^{'}(x)dx\right| \\
&= \frac{1}{2\pi|n|}\left|0 - \frac{i}{-in}\int_{-\pi}^{\pi} f^{'}(x)de^{-inx}\right| \\
&= \frac{1}{2\pi|n|^2}\left|\int_{-\pi}^{\pi} f^{'}(x)de^{-inx}\right| \\
&= \frac{1}{2\pi|n|^2}\left|\cos n\pi(f^{'}(\pi)-f^{'}(-\pi)) - \int_{-\pi}^{\pi} f^{''}(x)e^{-inx}dx\right| \\
&= \frac{1}{2\pi|n|^2}\left|\int_{-\pi}^{\pi} f^{''}(x)e^{-inx}dx\right|
\end{aligned} \tag{16}$$

由于 f 是 C^2 光滑函数, 因此$\exists C \geq 0$

$$\left|f^{''}(x)\right| \leq C \tag{17}$$

进而

$$\begin{aligned}
&\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f^{''}(x)e^{-inx}dx\right| \\
&\leq \frac{1}{2\pi}|2\pi C| \\
&= C
\end{aligned} \tag{18}$$

因此

$$|\hat{f}(n)| \leq \frac{C}{|n|^2} \tag{19}$$

得证. □

3. 证明.

$$f(x) \sim \hat{f}(0) + \sum_{n\geq 1}(\hat{f}(n) + \hat{f}(-n))\cos nx + i(\hat{f}(n) - \hat{f}(-n))\sin nx \tag{20}$$

由1.(3)知, 函数 f 为正弦级数且 $\hat{f}(-n) = -\hat{f}(n)$, 因此:

$$f(x) \sim \sum_{n\geq 1} 2i\hat{f}(n)\sin nx \tag{21}$$

由于 f 为奇函数, 周期为 2π 且在 $[0,\pi]$ 内, $f(x) = x(\pi - x)$, 可得:

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} x(x+\pi)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} x(\pi - x)e^{-inx}dx \\
&= \frac{2i}{n^3\pi}(\cos n\pi - 1)
\end{aligned} \tag{22}$$

因此:

$$\begin{aligned}
f(x) &\sim \sum_{n\geq 1} 2i\hat{f}(n)\sin nx \\
&= \sum_{n\geq 1} 2i\frac{2i}{n^3\pi}(\cos n\pi - 1)\sin nx \\
&= \sum_{n\geq 1} \frac{4}{n^3\pi}(1 - \cos n\pi)\sin nx
\end{aligned} \tag{23}$$

□

4. 证明. 由1.(4)知, 函数 f 为余弦级数且 $\hat{f}(-n) = \hat{f}(n)$, 因此:

$$f(x) \sim \hat{f}(0) + \sum_{n\geq 1} 2\hat{f}(n)\cos nx \tag{24}$$

由于 f 为偶函数, 周期为 2π 且在 $[-\pi,\pi]$ 内, $f(x) = |x|$, 可得:

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -xe^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}dx \\
&= \frac{1}{2\pi n^2}(-ni\pi\cos n\pi - 1 + \cos n\pi) + \frac{1}{2\pi n^2}(ni\pi\cos n\pi + \cos n\pi - 1) \\
&= \frac{1}{\pi n^2}(\cos n\pi - 1)
\end{aligned} \tag{25}$$

因此:

$$\begin{aligned}
f(x) &\sim \hat{f}(0) + \sum_{n\geq 1} 2\hat{f}(n)\cos nx \\
&= \frac{\pi}{2} + \sum_{n\geq 1} \frac{2}{\pi n^2}(\cos n\pi - 1)\cos nx
\end{aligned} \tag{26}$$

□

5. (1) 证明. $\forall x \in [-\pi,\pi]$

$$\left|\hat{f}(n)e^{inx}\right| \leq \left|\hat{f}(n)\right| \tag{27}$$

并且已知

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| \tag{28}$$

收敛, 因此由 Weierstrass 判别法知

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} \tag{29}$$

在 $[-\pi, \pi]$ 上一致收敛. □

(2) 解.

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx} \tag{30}$$

可得:

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ikx})e^{-inx}dx \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}(\hat{f}(k)\int_{-\pi}^{\pi}e^{i(k-n)x}dx) \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{n-1}(\hat{f}(k)\times 0) + \frac{1}{2\pi}\hat{f}(n)\times 2\pi + \frac{1}{2\pi}\sum_{n+1}^{\infty}(\hat{f}(k)\times 0) \\
&= \hat{f}(n)
\end{aligned} \tag{31}$$

□

(3) 证明. 令 $h = f - g$, 则

$$\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0 \tag{32}$$

由定理 2.1 知, $h = 0$, 则$f = g$. □

(4) 证明.

$$f(0) = \frac{\pi^2}{4} \tag{33}$$

令

$$g(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \tag{34}$$

则

$$\begin{aligned}
g(0) &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}\bigg|_{x=0} \\
&= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned} \tag{35}$$

由(3)知, $f = g$, 因此得

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{4} - \frac{\pi^2}{12} \\
&= \frac{\pi^2}{6}
\end{aligned} \tag{36}$$

□