2021F Final

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1. We have points $\begin{bmatrix}-2\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\end{bmatrix}$ in the space $\mathbb{R}^3$. Together they form a data matrix $A = \begin{bmatrix}-2 & 1 & 1\\-1 & 2 & -1\\1 & 1 & -2\end{bmatrix}$. (Note that the points are already centered.)

   (a) (3 points) Find a singular value decomposition of A.

Idea 1: Brute force

Step 1. Find A^TA

Method 1.

$$A^TA = \begin{pmatrix} g_{ij} = \langle i\text{-th col}, j\text{-th col}\rangle \\ \& \text{ sym} \end{pmatrix} = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}$$

Method 2:

$$A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = DA'$$

$$A^TA = (A')^T D^T D A' = (A')^2 = (X - 3I)^2 \text{ where } X = (\text{all } 1)$$

$$X^2 = 3X$$

$$= X^2 - 6X + 9I = 9I - 3X = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}$$

Step 2

$-3 \quad -3 \quad 6$

Step 2. Spectral of A^TA

Method 1:

$$\boxed{\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}} \qquad \text{char poly } p(x) = \det\begin{pmatrix} x-6 & 3 & 3 \\ 3 & x-6 & 3 \\ 3 & 3 & x-6 \end{pmatrix}$$

Gaussian
Cofactor
Big formula
Vieta

$$= x^3 - 18x^2 + 81x$$

$$= \boxed{(x-9)^2 x}$$

Gaussian or by sight

$$\ker(A^TA - 0I) = \boxed{\operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

or take orth compl

$$\ker(A^TA - 9I) = \ker(\text{all } -3) = \boxed{\operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)}$$

$$A^TA = VDV^T, \quad V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & & \\ & 9 & \\ & & 9 \end{pmatrix}$$

$$\cancel{V = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}}$$

Method 2:

$A^TA = \boxed{9I - 3X}$, X eigen $0, 0, 3$, eigen space $\operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)$ & $\operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

so $9I - 3X$ has eigen $9, 9, 0$

Step 3. $A^TA = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}}_{V} \begin{pmatrix} 4 & & \\ & 4 & \\ & & 0 \end{pmatrix} V^T$

$\Sigma = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 0 \end{pmatrix}$

$$\boxed{\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1} = \frac{1}{3}\frac{1}{\sqrt{2}} A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{3\sqrt{2}} \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\boxed{\vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2} = \frac{1}{3}\frac{1}{\sqrt{6}} A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{3\sqrt{6}} \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

$\vec{u}_3$ is arbitrary $\perp \vec{u}_1, \vec{u}_2$, say $\vec{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

$$A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & & \\ & 3 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Idea 2:

$$A = DA' = \underbrace{\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}}_{\text{orth}} \underbrace{\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}}_{\text{sym}}$$

polar decomp

$$= D \underbrace{Q \Sigma Q^T}_{\text{Spectral of } A'}$$

$QA = QU\boxed{\Sigma}V^T$

$U^TAV = \Sigma$

Takeaway: $\forall$ SVD problem, you can simplify A using orth matrices

(b) (3 points) Find two mutually orthogonal lines of best fit.

$\boxed{9, 9}$

direction of lines are $\vec{u}$ for largest σ

A

But $\sigma_1 = \sigma_2$ both largest

So Ans = lines with directions $\boxed{\vec{u_1}, \vec{u_2}}$

$= \text{span}(\vec{u_1})$, $\text{span}(\vec{u_2})$

(c) (2 points) Find the plane of best fit.

$\text{span}(\vec{u_1}, \vec{u_2})$

(d) (2 points) Find the maximum and minimum Rayleigh quotient $\frac{v^T S v}{v^T v}$ for $S = (A^T A)^2 + 2A^T A + 3I$.

$A^T A$ has eigen $9, 9, \boxed{0}$

S has eigen $102, 102, 3$

(e) (3 points) Find a polynomial $p(x)$ such that $(A^TA + 2I)^{-1} = p(A^TA)$.

$X = $ (all 1)

Method 1: $A^TA + 2I = 9I - 3X + 2I = \boxed{11I - 3X}$ $\quad \boxed{X^2 = 3X}$

Say $(A^TA + 2I)^{-1} = aI + bX$

Then $I = (11I - 3X)(aI + bX)$

$= -3bX^2 + (11b - 3a)X + 11aI$

$= (2b - 3a)X + 11aI$

So $a = \frac{1}{11}$, $b = \frac{3}{22}$

So $(A^TA + 2I)^{-1} = \frac{1}{11}I + \frac{3}{22}X = \frac{1}{11}I + \frac{3}{22}\left(3I - \frac{1}{3}A^TA\right)$

$= \frac{1}{2}I - \frac{1}{22}A^TA$, $\quad p(x) = -\frac{1}{22}x + \frac{1}{2}$

Method 2: A^TA eigen $9, 9, 0$ & $\boxed{\text{corresponding eigenspace}}$

$(A^TA + 2I)^{-1}$ eigen $\frac{1}{11}, \frac{1}{11}, \frac{1}{2}$

So we need $p(x)$: $\boxed{9 \mapsto \frac{1}{11}}$, $\boxed{0 \mapsto \frac{1}{2}}$

interpolation $p(x) = \frac{1}{11}\frac{x-0}{9-0} + \frac{1}{2}\frac{x-9}{0-9}$

$\begin{pmatrix} 9 \\ 0 \end{pmatrix} \xmapsto{\frac{1}{22}x} \begin{pmatrix} \frac{9}{22} \\ 0 \end{pmatrix} \xmapsto{x - \frac{1}{2}} \begin{pmatrix} -\frac{1}{11} \\ -\frac{1}{2} \end{pmatrix} \xmapsto{-x} \begin{pmatrix} \frac{1}{11} \\ \frac{1}{2} \end{pmatrix}$

A^TA must satisfy

$(A^TA - 9I)^2 A^TA = 0$

$\boxed{A^3 + 2A^2 + 4A + I = 0}$

$\boxed{(A + 3I)^{-1} = p(A)}$

$\boxed{(aA^2 + bA + cI)}(A + 3I) = I$

$\boxed{aA^3} + (b + 3a)A^2 + (c + 3b)A + 3cI = I$

$\underbrace{(b + a)}_{=0}A^2 + \underbrace{(c + 3b - 4a)}_{=0}A + \underbrace{(3c - a)}_{=1}I = I$

$\boxed{M}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{u}$; Ans M = coef, $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

RREF M

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Method 2: A^TA eigen $9, 9, 0$
$(A^TA+2I)^{-1}$ eigen $\frac{1}{11}, \frac{1}{11}, \frac{1}{2}$ & corresponding eigenspace

So we need $p(x)$: $9 \mapsto \frac{1}{11}$, $0 \mapsto \frac{1}{2}$

interpolation $p(x) = \frac{1}{11}\frac{x-0}{9-0} + \frac{1}{2}\frac{x-9}{0-9}$

$$\begin{pmatrix} 9 \\ 0 \end{pmatrix} \xmapsto{\frac{1}{22}x} \begin{pmatrix} \frac{9}{22} \\ 0 \end{pmatrix} \xmapsto{x-\frac{1}{2}} \begin{pmatrix} -\frac{1}{11} \\ -\frac{1}{2} \end{pmatrix} \xrightarrow{-x} \begin{pmatrix} \frac{1}{11} \\ \frac{1}{2} \end{pmatrix}$$

So $p(x) = -\left(\left(\frac{1}{22}x\right) - \frac{1}{2}\right)$

$$\frac{1}{2} - \frac{1}{11} = \frac{9}{22}$$

2. In the xy-plane, we have a line $ax + y = 1$ for some unknown constant $a \in \mathbb{R}$. Suppose we also know that the line must go through the points $\begin{bmatrix} 1 \\ b \end{bmatrix}, \begin{bmatrix} 2 \\ -b \end{bmatrix}, \begin{bmatrix} -2 \\ 4b \end{bmatrix}$ for some unknown constant $b \in \mathbb{R}$. We wish to find all possible a, b.

(a) (2 points) Find a 3×2 matrix A and a vector $\boldsymbol{u}$ such that a, b is a possible solution to the problem above if and only if $A\begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{u}$.

$$\begin{cases} a + b = 1 \\ 2a - b = 1 \\ -2a + 4b = 1 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) (2 points) Show that $A\begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{u}$ above has no solution.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -2 & 4 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 6 & 3 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & \boxed{1} \end{array}\right)$$

So no solution

(c) (4 points) Find the QR decomposition of A, where Q is 3×2 with orthonormal columns and R is 2×2 and upper triangular with positive diagonal entries.

$$\begin{pmatrix}1 & 1\\ 2 & -1\\ -2 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\\ 2 & 1\\ -2 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}3 & -3\\ & 3\end{pmatrix}$$

$9 \quad -9$

(d) (2 points) Find the least square solution to $A\begin{bmatrix}a\\ b\end{bmatrix} = \boldsymbol{u}$.

$$(A^TA)\vec{x} = A^T\vec{u}$$

$$A^TA = R^TR = \begin{pmatrix}3 & \\ -3 & 3\end{pmatrix}\begin{pmatrix}3 & -3\\ & 3\end{pmatrix} = \begin{pmatrix}9 & -9\\ -9 & 18\end{pmatrix}$$

$$A^T\vec{u} = \begin{pmatrix}1\\ 4\end{pmatrix}$$

$$\left(\begin{array}{cc|c}9 & -9 & 1\\ -9 & 18 & 4\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}9 & -9 & 1\\ 0 & 9 & 5\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}9 & 0 & 6\\ 0 & 9 & 5\end{array}\right)$$

$$\longrightarrow \left(I \,\middle|\, \begin{array}{c}\frac{6}{9}\\ \frac{5}{9}\end{array}\right) \begin{array}{l}=a\\ =b\end{array}$$

FFT

Fourier matrix

(the matrix for Fourier transform)

(discrete version)

3. Consider $F = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{bmatrix}$.

(a) (2 points) Is F Hermitian or Skew-Hermitian or Unitary or none of these? Can you tell without calculation if F is diagonalizable or not?

$$F^* = \overline{F^T} = \overline{F}$$

$$\begin{pmatrix} -9 & 18 & | & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 9 & | & 5 \end{pmatrix} \to \begin{pmatrix} 0 & 9 & | & 5 \end{pmatrix}$$

$$\to \left(I \;\middle|\; \begin{matrix} \frac{6}{9} \\ \frac{5}{9} \end{matrix} \right) \begin{matrix} =a \\ =b \end{matrix}$$

3. Consider $F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$.

(a) (2 points) Is F Hermitian or skew-Hermitian or Unitary or none of these? Can you tell without calculation if F is diagonalizable or not?

$\boxed{\text{Fourier matrix}}$

(the matrix for Fourier transform)

(discrete version)

$\boxed{\text{FFT}}$

$$F^* = \overline{F^T} = \overline{F} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \neq F \text{ or } -F$$

not Herm / skew

But columns are unit vectors & mutually $\perp$,

so unitary (and thus diagonalizable)

E.G. $\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$ dot $\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \boxed{(1 \;\; i \;\; -1 \;\; -i)} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$

$= (1 \;\; -i \;\; -1 \;\; i)\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = 1 - 1 + 1 - 1 = 0$

Or verify directly $F^* F = I$

real

$\boxed{f} = \sum a_n \sin + \sum b_n \cos = \sum \boxed{e^{i\theta}}$

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda^{-1} \\ 1 & \lambda^{-1} & \lambda \end{pmatrix}$$

$$F_4 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$f = \begin{pmatrix} f(0) \\ f(\frac{\pi}{n}) \\ f(\frac{2\pi}{n}) \\ f(\frac{3\pi}{n}) \\ \vdots \end{pmatrix}$$

$$F^{-1} f = ($$

Or verify directly $F \; F = I$

(b) (4 points) Calculate F^2 and find a basis for each eigenspace of F^2

real sym

last eigen vector $\mathrm{Ran}\left(\begin{pmatrix}1&1&0\\1&0&0\\1&0&1\\1&0&0\end{pmatrix}\right)^{\perp}$

$\left(\frac{1}{2}F\right)^{-1} = \overline{\frac{1}{2}F^{T}} = \overline{\frac{1}{2}F}$

$$F^2 = \frac{1}{4}\begin{pmatrix}4&0&0&0\\0&0&0&4\\0&0&4&0\\0&4&0&0\end{pmatrix} = \begin{pmatrix}1&&&\\&&&1\\&&1&\\&1&&\end{pmatrix}$$

$\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

Method 1: By sight, $\vec{e}_1, \vec{e}_3$ are eigenvectors for 1

& F^2 is "secretly block diagonal"

$\begin{pmatrix}\boxed{1}&&\boxed{}&\\&\boxed{}&&\boxed{1}\\\boxed{}&&\boxed{1}&\\&\boxed{1}&&\boxed{}\end{pmatrix}$ and red portion has eigenvector $\begin{pmatrix}0\\\boxed{1}\\0\\\boxed{1}\end{pmatrix}$ for 1

$\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\begin{pmatrix}1\\1\end{pmatrix}$ for 1

$\begin{pmatrix}1\\-1\end{pmatrix}$ for -1

and $\begin{pmatrix}0\\\boxed{1}\\0\\\boxed{-1}\end{pmatrix}$ for -1

$\begin{pmatrix}k\\0\\0\\0\end{pmatrix} \; * \; * \; *$, then $\vec{e}_1$ is eigenvect for eigen k.

$\begin{pmatrix}*&0&*&*\\*&3&*&*\\*&0&*&*\\*&0&*&*\end{pmatrix}$, then $\vec{e}_2$ is eigen for 3

Method 2: F^2 is a swap matrix, $(F^2)^2 = I$

so all eigenvalues of F^2 are $\lambda = \pm 1$

$\mathrm{Ker}(F^2 - I) = \mathrm{Ker}\begin{pmatrix}0&&&\\&-1&&1\\&&0&\\&1&&-1\end{pmatrix} = \mathrm{span}(e_1, e_3, e_2 + e_4)$

$\mathrm{Ker}(F^2 + I) = $ orth compl $= \mathrm{span}(e_2 - e_4)$

In fact, $F^2 = I - 2\vec{n}\vec{n}^T$, $\vec{n} = \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$, Householder

Method 3: Brute force

(c) (3 points) Find all eigenvalues of F and their algebraic multiplicity, and find a basis for the eigenspaces of non-real eigenvalues.

Method 1: Brute force

Method 2:

If F eigenvalue $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

Then F^2 eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2$

$$\begin{array}{cccc} \lambda_1^2, & \lambda_2^2, & \lambda_3^2, & \lambda_4^2 \\ \| & \| & \| & \| \\ 1 & 1 & 1 & -1 \end{array}$$

So

$$\begin{array}{cccc} \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4 \\ \| & \| & \| & \| \\ \pm 1 & \pm 1 & \pm 1 & \pm i \end{array}$$

$\to$ only non-real one

$tr(F) = 1 + i$

So eigenvalues of F are $1, 1, -1, i$

& eigenvector of F for i = eigenvector of F^2 for i^2

only for diagonalizable matrices

So $Ker(F - iI) = span(e_2 - e_4)$

So $\ker(F - iI) = \operatorname{span}(e_2 - e_4)$

(d) (2 points) Let $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the matrix X such that $2F = X \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} P$. (This is the fundation of the famous Fast Fourier Transform algorithm, ranked as one of the top 10 algorithms of 20-th century.)

$$2F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \stackrel{?}{=} X \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix} P$$

$$= X \begin{pmatrix} 1 & & 1 & \\ 1 & & -1 & \\ & 1 & & 1 \\ & 1 & & -1 \end{pmatrix}$$

Method 1: Find inverse & solve

Method 2: col

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{c} X \\ \hline I \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & i & -2 & -i \\ 1 & -1 & 0 & -1 \\ 1 & -i & -2 & i \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & i & 1 & -i \\ 1 & -1 & 0 & -1 \\ 1 & -i & 1 & i \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & i & -i \\ 1 & 0 & -1 & -1 \\ 1 & 1 & -i & i \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

FFT

$F_{2n}\vec{v}$ $\quad O(n^2)$

$$F_{2n}\vec{v} = \begin{pmatrix} I & D \\ I & -D \end{pmatrix} \begin{pmatrix} F_n & \\ & F_n \end{pmatrix} \text{Perm}\,\vec{v}$$

$$f(2n) = 2f(n) + 2n$$

$f\left(\frac{n}{2}\right) \longrightarrow$

$$f(n) = \tilde{O}(n)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & i & -2i \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -i & 2i \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & i & i \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -i & -i \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Method 3:

$$2FP^{-1} = X \begin{pmatrix} F_2 & \\ & F_2 \end{pmatrix}$$

So

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{array}\right) = X \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & 1 \\ & & 1 & -1 \end{array}\right)$$

$F_{2n} =$

So

$$X = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{array}\right) \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} & \\ & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \end{pmatrix}$$

$$= \left(\begin{array}{c|cc} I & 1 & \\ & & i \\ \hline I & -1 & \\ & & -i \end{array}\right)$$

4. Consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{bmatrix}$. We aim to find the orthogonal projection matrix to Ran(A).

(a) (3 points) Find the reduced row echelon form of A^T, and find a basis for Ker(A^T).

Ran(A), Q, QQ^T

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\mathrm{Ker}(A^T) = \mathrm{span}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$Q\sqrt{6}$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

(b) (2 points) Find the orthogonal projection matrix P_1 to Ker(A^T).

$$P_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}^T$$

$$A \quad (A^TA)^{-1} A^T$$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

P & $I-P$

(c) (2 points) Find the orthogonal projection matrix P_2 to Ran(A).

$\perp$

$$P_2 = I - P_1 = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

(d) (3 points) Let $B = \begin{bmatrix} 5 & 5 & 6 \\ 5 & 8 & 9 \\ 6 & 9 & 13 \end{bmatrix}$. Show that $\begin{bmatrix} I & A^T \\ A & B \end{bmatrix}$ is positive definite. Please explicitly states the criteria of positive-definiteness that you are using. (Hint: Block LDL^T decomposition.)

Block LDU

$$\begin{pmatrix} I & A^T \\ A & B \end{pmatrix} = \begin{pmatrix} I & \\ A & I \end{pmatrix} \begin{pmatrix} I & \\ & B-AA^T \end{pmatrix} \begin{pmatrix} I & A^T \\ & I \end{pmatrix}$$

$B - AA^T = I$, so pos-def as $\begin{pmatrix} I & A^T \\ A & B \end{pmatrix}$ congruence to I.

(or pivots are positive)

$\begin{pmatrix} I & A^T \\ A & B \end{pmatrix}$ definition

when $\vec{v} \neq \vec{0}$

$\vec{v}^T \begin{pmatrix} I & A^T \\ A & B \end{pmatrix} \vec{v} > 0$ $\qquad B = AA^T + I$

$\vec{v} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$ $\quad \vec{x}^T\vec{x} + 2\vec{y}^T A\vec{x} + \vec{y}^T B \vec{y}$

$= \|\vec{x} + A\vec{y}\|^2 + \|\vec{y}\|^2 > 0$

when $\vec{x}, \vec{y}$ not both $\vec{0}$.

5. Let V be the space of homogeneous polynomials in x, y with degree two. (Homogeneous means we only have degree two terms. For example, elements of V could be $x^2+2xy+3y^2$, $4x^2-3xy-y^2$ and so on. There cannot be degree one or degree zero terms.)

Let W be the space of polynomials in x with degree at most two. Now, given any polynomial in V, say $x^2+2xy+3y^2$, we can substitute y by $x+2$, and therefore get $x^2+2x(x+2)+3(x+2)^2$. This would be an element of W. Hence "substitute y by $x+2$" is an abstract map $S : V \to W$. This is a linear map.

(a) (1 point) Verify that $S(2x^2+3xy) = 2S(x^2)+3S(xy)$.

S : function$(x, y) \to$ function(x)

S = substitute y by $x+2$

$S(2x^2+3xy) = 2x^2 + 3x(x+2)$

$= 2S(x^2) + 3S(xy)$

(b) (2 points) Pick basis x^2, xy, y^2 for V and basis $x^2, x, 1$ for W, find the matrix A for S under these basis.

$$S: \begin{array}{l} x^2 \longrightarrow x^2 \\ xy \longrightarrow x^2 + 2x \\ y^2 \longrightarrow x^2 + 4x + 4 \end{array}, \text{ so } A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

(c) (2 points) Find all possible $p \in V$ such that $S(p) = 3x^2 + 6x + 4$.

$A\vec{x} = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$ find $\vec{x}$.

$$\text{So } \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ & 2 & 4 & 6 \\ & & 4 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ & 1 & 2 & 3 \\ & & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{array}\right)$$

So $p = x^2 + xy + y^2$

$y = x + 2$

(d) (2 points) Pick basis $x^2, x(y-x), (y-x)^2$ for V and basis $x^2, x, 1$ for W, find the matrix B for S under these basis.

$$S: \begin{array}{l} x^2 \\ x(y-x) \\ (y-x)^2 \end{array} \longmapsto \begin{array}{l} x^2 \\ 2x \\ 4 \end{array} \quad \text{so } B = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 4 \end{pmatrix}$$

(e) (2 points) Let X the change of coordinate matrix in V from the basis x^2, xy, y^2 to the basis $x^2, x(y-x), (y-x)^2$. How are A, B, X related? Calculate X from this relation.

$\to \mathcal{B}$

$\to \mathcal{C}$

$\mathcal{D} = (x^2, x, 1)$ basis for W

$A = \boxed{S_{\mathcal{D}\leftarrow\mathcal{B}}}$ $I_{\mathcal{B}\leftarrow\mathcal{C}}$

$B = S_{\mathcal{D}\leftarrow\mathcal{C}}$

$X = I_{\mathcal{C}\leftarrow\mathcal{B}}$

since $S_{\mathcal{D}\leftarrow\mathcal{B}} = S_{\mathcal{D}\leftarrow\mathcal{C}} I_{\mathcal{C}\leftarrow\mathcal{B}}$

therefore $A = BX$

so $X = B^{-1}A = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix}$