3.1

$V = V_1 \oplus V_2$

V_1: diagonal matrices

V_2: matrices with all diagonal entries are zero

$$\begin{bmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{bmatrix}$$

to show matrix in a more linear way,
we define $n \times n$ matrix as a n^2-dimension "vector"

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \\ & & \ddots & a_{nn} \end{bmatrix} \Longrightarrow \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \\ a_{12} \\ a_{21} \\ a_{13} \\ a_{31} \\ \vdots \\ a_{1n} \\ a_{n1} \\ a_{23} \\ a_{32} \\ \vdots \\ a_{2n} \\ a_{n2} \\ \vdots \end{bmatrix}$$

then basis for V_1: $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow n$ vectors

basis for V_2: $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ } n zeros $\cdots\cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Rightarrow n^2 - n$ vectors

then under this basis, the matrix for T:

$$\begin{bmatrix} I_{n\times n} & & & \\ & P_{2\times 2} & & \\ & & P_{2\times 2} & \\ & & & \ddots \\ & & & & P_{2\times 2} \end{bmatrix}, \quad P_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3.2

1\. $\forall \vec{x} \in \mathbb{R}^n$, $B\vec{x} \in Ran(B)$

$$AB\vec{x} = BA\vec{x} = B(A\vec{x}) \in Ran(B)$$

$\Rightarrow A(Ran(B)) \subseteq Ran(B)$

$\forall \vec{x} \in Ker(B)$, $B\vec{x} = \vec{0}$

$$AB\vec{x} = B(A\vec{x}) = \vec{0}$$

then $A\vec{x} \in Ker(B)$

$\Rightarrow A(Ker(B)) \subseteq Ker(B)$

so, $Ker(B)$, $Ran(B)$ are both A-invariant subspaces

2\. let $p(A) = a \cdot A^k$ for any $a \in \mathbb{R}$ and $k \in \mathbb{N}$

$$A\,p(A) = a \cdot A^{k+1} = p(A)A = a \cdot A^{k+1}$$

then for any $p(x) \to p(A)$. is a linear combination of $a \cdot A^k$

$\Rightarrow A\,p(A) = p(A)A$ for all polynomials $p(x)$

3\. $\exists k \in n$

$N_\infty(A - \lambda I) = Ker\,(A - \lambda I)^k$

$R_\infty(A - \lambda I) = Ran\,(A - \lambda I)^k$

let $p(A) = (A - \lambda I)^k$

then $A\,p(A) = p(A)\,A$

then $Ker(p(A))$ and $Ran(p(A))$ are A-invariant

$\Rightarrow N_\infty(A - \lambda I)$ and $R_\infty(A - \lambda I)$ are both A-invariant

4\. $AB = BA \Rightarrow AB - \lambda B = BA - \lambda B$

$\Rightarrow (A - \lambda I)B = B(A - \lambda I)$

$\forall \vec{x} \in \mathbb{R}^n$, $(A-\lambda I)^k \vec{x} \in Ran(A - \lambda I)^k$, $B(A-\lambda I)^{k+1}\vec{x} = B(A-\lambda I)(A-\lambda I)^k \vec{x}$

$\phantom{\forall \vec{x}}$ $\parallel$ $R_\infty(A - \lambda I)$

$$= (A-\lambda I)\,B\,(A-\lambda I)^k \vec{x} \in Ran(A-\lambda I)^k$$

$\parallel$ $R_\infty(A-\lambda I)$

$\Rightarrow B(R_\infty(A - \lambda I)) \subseteq R_\infty(A - \lambda I)$

$\forall \vec{x} \in Ker(A - \lambda I)^k$, $B(A-\lambda I)^{k+1}\vec{x} = B(A-\lambda I) \cdot (A - \lambda I)^k \vec{x}$

$\parallel$ $N_\infty(A - \lambda I)$

$$= (A - \lambda I)B \cdot (A - \lambda I)^k \vec{x}$$

$$= \vec{0}$$

$\Rightarrow B(N_\infty(A - \lambda I)) \subseteq N_\infty(A - \lambda I)$

3.3

1. $\forall \vec{x} \in \ker(L')$

Since the domain of L' is V'

then $\vec{x} \in V'$

also, L' is operationally the same as L

then $\vec{x} \in \ker(L)$

$\Rightarrow \vec{x} \in V' \cap \ker(L) \Rightarrow \ker(L') = V' \cap \ker(L)$

2.

$\mathrm{Ran}(L') = L'(V')$

on domain V': $L \Leftrightarrow L'$

then $\mathrm{Ran}(L') = L'(V)$

3. $\dim L(V') = \dim \mathrm{Ran}(L') = \dim \mathrm{dom}(L') - \dim \ker(L')$

$= \dim V' - \dim(V' \cap \ker(L))$

4. A: domain$(A) \longrightarrow$ codomain(A)

A': $\mathrm{Ran}(B) \longrightarrow A(\mathrm{Ran}\, B)$

then $\underbrace{\dim A(\mathrm{Ran}(B))}_{=\ \mathrm{rank}(AB)} = \underbrace{\dim \mathrm{Ran}(B)}_{=\ \mathrm{rank}(B)} - \dim(\mathrm{Ran}(B) \cap \ker(A))$

$\Rightarrow \mathrm{rank}(B) - \mathrm{rank}(AB) = \dim(\underbrace{\mathrm{Ran}(B) \cap \ker(A)}_{\text{those in } \mathrm{Ran}(B) \text{ who are also "killed" by } A})$

3.4

$$A = X \begin{bmatrix} A_N & \\ & A_R \end{bmatrix} X^{-1}$$

1. since $A_N . A_R$ are nilpotent

$\exists K \leq n, \quad A_N^K = A_R^K = 0$

$\Rightarrow A^K = 0$

$\Rightarrow \forall \vec{v} \in V, \quad A^K \vec{v} = \vec{0}$

$\Rightarrow \vec{v} \in \ker(A^K)$

2. $\forall K \in N^*$,

since $\underline{A_N . A_R}$ are invertible

then $\downarrow$ are linear bijection

$\Rightarrow A_N^K . A_R^K$ are linear bijection

$\Rightarrow \mathrm{Ran}(A^K) = V$

$\Rightarrow \forall \vec{v} \in V, \ \vec{v} \in \mathrm{Ran}(A^K)$