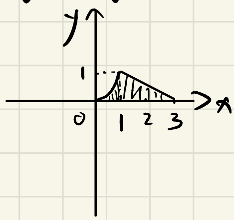
1.

(1) $\int_0^1 dx \int_0^{x^2} f(x,y)\,dy + \int_1^3 \int_0^{\frac{1}{2}(3-x)} f(x,y)\,dy$



$= \int_0^1 dy \int_{\sqrt{y}}^{3-2y} f(x,y)\,dx$

(2) $V = \iiint\limits_{\Omega} dx\,dy\,dz \quad = \int_0^1 dz \int_0^{(z-z^4)^{\frac{1}{4}}} r\,dr \int_0^{2\pi} d\theta$

$S: (x^2+y^2)^2 = z - z^4 \geq 0 \qquad = 2\pi \int_0^1 dz \int_0^{(z-z^4)^{\frac{1}{4}}} r\,dr$

$\Rightarrow z \in [0,1] \qquad = \pi \int_0^1 dz \sqrt{z-z^4}$

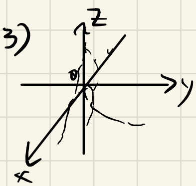
$(x^2+y^2)^2 \in [0, \frac{3}{8} 2^{\frac{1}{3}}] \quad t = z^{\frac{3}{2}} \quad = \pi \int_0^1 d\,t^{\frac{2}{3}} \cdot \sqrt{t^{\frac{2}{3}} - t^{\frac{8}{3}}}$

$= \pi \int_0^1 t^{\frac{1}{3}} \sqrt{1-t^2} \cdot \frac{2}{3} t^{-\frac{1}{3}}\,dt$

$= \frac{2\pi}{3} \int_0^1 \sqrt{1-t^2}\,dt$

$t = \sin\theta \quad = \frac{2\pi}{3} \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{2\pi}{3} \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta = \frac{2\pi}{3} \cdot \frac{\pi}{4} = \frac{\pi^2}{6}$

(3)



$\iiint\limits_{\Omega} \sqrt{x^2-y}\,dx\,dy\,dz$

$= \int_0^1 dx \int_0^{x^2} \sqrt{x^2-y}\,dy \int_0^{1-x} dz$

$= \int_0^1 dx \int_0^{x^2} \sqrt{x^2-y}\,dy \cdot (1-x)$

$= \int_0^1 dx \left(-\frac{2}{3}\right)(x^2-y)^{\frac{3}{2}} \Big|_0^{x^2} (1-x)$

$= \frac{2}{3} \int_0^1 dx\,(x^2)^{\frac{3}{2}} (1-x) = \frac{2}{3} \int_0^1 dx \cdot (x^3 - x^4)$

$= \frac{2}{3} \left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right) \Big|_0^1$

$= \frac{1}{30}$

2.

(1) $\begin{cases} u = x+y \\ v = x-y \end{cases}$ $\left|\det \frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\cancel{\det}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = 2$

$$\iint\limits_{\substack{0\le x+y\le \pi \\ 0\le x-y\le \pi}} (x+y)\sin(x-y)\,dxdy = \iint\limits_{\substack{0\le u\le \pi \\ 0\le v\le \pi}} u\sin v\cdot\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_0^\pi du\int_0^\pi dv\; u\sin v$$

$$= \frac{1}{2}\int_0^\pi du\cdot 2u = u^2\Big|_0^\pi = \pi^2$$

(2) $\begin{cases} u = x+y \\ v = y \end{cases}$ $\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right| = 1$

$$\iint\limits_{\substack{x+y\le 1 \\ x\ge 0, y\ge 0}} e^{\frac{y}{x+y}}\,dxdy = \iint\limits_{\substack{0\le u\le 1 \\ 0\le v\le u}} e^{\frac{v}{u}}\,du\,dv = \int_0^1 du\int_0^u e^{\frac{v}{u}}\,dv$$

$$= \int_0^1 du\cdot u(e-1)$$

$$= \frac{e-1}{2}$$

(3) $\begin{cases} u = 3x-y-z \\ v = -x+3y-z \\ w = -x-y+3z \end{cases}$ $\left|\det\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \left|\det\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}\right| = 16$

$$V = \iiint\limits_{\Omega} dxdydz = \frac{1}{16}\int_{-1}^1 du\int_{-1}^1 dv\int_{-1}^1 dw = \frac{1}{16}\times 2\times 2\times 2 = \frac{1}{2}$$

(4) $\begin{cases} u = \frac{x^2}{y} \\ v = \frac{y^2}{x} \end{cases}$ $\left|\det\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{pmatrix}\right| = 3$

$$\iint\limits_D \frac{x^2}{y}\sin(xy)\,dxdy = \frac{1}{3}\int_a^b du\int_p^q dv\cdot u\sin(uv)$$

$$= \frac{1}{3}\int_a^b du\cdot\left(-\cos(uv)\right)\Big|_p^q$$

$$= \frac{1}{3}\int_a^b du\left(\cos(pu)-\cos(qu)\right)$$

$$= \frac{1}{3}\frac{1}{p}\sin(pu) - \frac{1}{3}\frac{1}{q}\sin(qu)\Big|_a^b$$

$$= \frac{1}{3}\left(\frac{1}{p}\sin(pb) - \frac{1}{p}\sin(pa) + \frac{1}{q}\sin(qa) - \frac{1}{q}\sin(qb)\right)$$

(5)

$$z' = \frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}} \qquad \left|\det\left(\frac{\partial(x',y',z')}{\partial(x,y,z)}\right)\right| = 1$$

$$x' = \frac{bx-ay+cz}{h}$$

$$y' = \frac{cx+by-az}{h}$$

$$\Rightarrow \iiint\limits_{x^2+y^2+z^2\le 1} f(ax+by+cz)\,dxdydz = \iiint\limits_{x'^2+y'^2+z'^2\le 1} f(hz')\,dx'dy'dz'$$

$$= \int_{-1}^{1} dz' f(hz') \cdot \iint_S dx'dy'$$

$$= \pi \int_{-1}^{1} dz' \,(1-z'^2) f(hz')$$

$$= \pi \int_{-1}^{1} (1-t^2) f(ht)\,dt$$

(6) $0 \le x^2+y^2 \le R^2$

$\sqrt{x^2+y^2} \le z \le \sqrt{R^2-x^2-y^2}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint_V (x^2+y^2+z^2)\,dxdydz = \int_0^R r\,dr \int_0^{2\pi} d\theta \cdot \int_r^{\sqrt{R^2-r^2}} dz \cdot (r^2+z^2)$$

$$= 2\pi \int_0^R r\,dr \cdot \left(r^2(\sqrt{R^2-r^2}-r) + \frac{1}{3}(R^2-r^2)^{\frac{3}{2}} - \frac{1}{3}r^3\right)$$

$$= 2\pi \int_0^R dr \cdot \left(r^3\sqrt{R^2-r^2} - \frac{4}{3}r^4 + \frac{1}{3}r(R^2-r^2)^{\frac{3}{2}}\right)$$

$$= 2\pi \left(-\frac{1}{15}(R^2-r^2)^{\frac{3}{2}}(3r^2+2R^2) - \frac{4}{15}r^5 - \frac{1}{15}(R^2-r^2)^{\frac{5}{2}}\right)\Bigg|_0^R$$

$$= 2\pi\left(0 - \frac{4}{15}R^5 - 0\right) - 2\pi\left(-\frac{1}{15}R^3 \cdot 2R^2 - 0 - \frac{1}{15}R^5\right)$$

$$= -\frac{8\pi}{15}R^5 - 2\pi\left(-\frac{3}{15}R^5\right)$$

$$= -\frac{2\pi}{15}R^5$$

(7)

(i) ∃ 可逆矩阵 M, $\det(M)=1$

s.t. $A = M\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}M^{-1}$ $\quad H(\vec{x}) = M\begin{pmatrix}x'\\y'\\z'\end{pmatrix}^T\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}M^{-1}$

$\det A = \lambda_1\lambda_2\lambda_3 > 0$ $\quad = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$

$$V = \iiint_{\Omega} dx\,dy\,dz = \iiint_{\lambda_1 x'^2+\lambda_2 y'^2+\lambda_3 z'^2 \le 1} dx'\,dy'\,dz'$$

$$\begin{cases} x'' = \sqrt{\lambda_1}\, x' \\ y'' = \sqrt{\lambda_2}\, y' \\ z'' = \sqrt{\lambda_3}\, z' \end{cases}$$

$$\Rightarrow V = \iiint_{x''^2+y''^2+z''^2\le 1} \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\, dx''dy''dz'' = \frac{4\pi}{3\sqrt{\lambda_1\lambda_2\lambda_3}} = \frac{4\pi}{3\sqrt{\det A}}$$

(ii)

$$I = \iiint_{\lambda_1 x'^2+\lambda_2 y'^2+\lambda_3 z'^2\le 1} e^{\sqrt{\lambda_1 x'^2+\lambda_2 y'^2+\lambda_3 z'^2}}\, dx'\,dy'\,dz'$$

$$= \iiint_{x''^2+y''^2+z''^2\le 1} e^{\sqrt{x''^2+y''^2+z''^2}} \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\, dx''\,dy''\,dz'' = \frac{1}{\sqrt{\det A}} \int_0^1 r^2 dr \int_0^{2\pi} |\sin\theta|\, d\theta \int_0^{\pi} d\varphi \cdot e^r$$

$$= \frac{2\pi}{\sqrt{\det A}} \int_0^1 r^2 e^r dr = \frac{2\pi(e-2)}{\sqrt{\det A}}$$

(8) $\begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = \dfrac{y}{x^2+y^2} \end{cases}$ $\left|\det\left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)\right| = \left|\det\begin{pmatrix} \dfrac{y^2-x^2}{(x^2+y^2)^2} & \dfrac{-2xy}{(x^2+y^2)^2} \\ \dfrac{-2xy}{(x^2+y^2)^2} & \dfrac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}\right| = \left|\dfrac{-(x^2+y^2)^2}{(x^2+y^2)^4}\right| = \dfrac{1}{(x^2+y^2)^2}$

$$\Rightarrow I = \iint_D \frac{1}{xy}\,dxdy = \iint_D \frac{(x^2+y^2)^2}{xy} \cdot \frac{1}{(x^2+y^2)^2}\,dxdy$$

$$= \iint_{D'} \frac{1}{uv} \cdot dudv$$

$$= \int_2^4 du \int_2^4 dv \frac{1}{uv} = \int_2^4 \ln 2 \cdot \frac{1}{u} \cdot du = (\ln 2)^2$$