2.1

1. $n=2$: $A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$n=4$: $A_4 = \begin{bmatrix} A_2 & \\ & A_2 \end{bmatrix}$

$n=6$: $A_6 = \begin{bmatrix} A_2 & & \\ & A_2 & \\ & & A_2 \end{bmatrix}$

$\vdots$

$n=2k$: $A_{2k} = \begin{bmatrix} A_2 & & \\ & \ddots & \\ & & A_2 \end{bmatrix}$ (block diagonal)

$\begin{pmatrix} k \in \mathbb{N} \\ k \geq 2 \end{pmatrix}$

2. $n = 2k+1$ $\quad A_{2k+1} = \begin{bmatrix} A_2 & & & & \\ & \ddots & & & \\ & & A_2 & & \\ & & & i & \\ & & & & A_2 \\ & & & & & \ddots \\ & & & & & & A_2 \end{bmatrix}$

A_2 is the smallest real solution for $A^2 = -I$

if odd n, there's always a i in A

$\Rightarrow$ there's no real solution

2.2

1. ① complex linear $\Rightarrow AB=BA$

let $k=a+bi$ $(a,b\in\mathbb{R})$

$$B(k\vec{v}) = B((a+bi)\vec{v})$$
$$= B(a\vec{v}) + B(bi\vec{v})$$
$$= aB\vec{v} + bB(i\vec{v})$$
$$= aB\vec{v} + bBA\vec{v}$$

also, since B is complex linear

so $B(k\vec{v}) = aB\vec{v} + (bi)(B\vec{v})$
$$= aB\vec{v} + bAB\vec{v}$$

so $AB=BA$

② $AB=BA \Rightarrow$ complex linear

$$B(k\vec{v}) = aB\vec{v} + bBA\vec{v}$$
$$= aB\vec{v} + bAB\vec{v}$$
$$= aB\vec{v} + (bi)B\vec{v}$$
$$= (a+bi)B\vec{v}$$
$$= kB\vec{v}$$

so B is complex linear

2. since $X\cdot X = X\cdot X = X^2 = -I$

then X is complex linear

3. let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$C_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

2.3

1.
$$P\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\qquad P\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}i\\-1\\-i\\1\end{bmatrix}$$

2.
$$D=\bar{F}_4^{-1}P\bar{F}_4$$
$$=\frac{1}{4}\bar{F}_4 P \bar{F}_4$$
$$=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{bmatrix}\begin{bmatrix}&1&&\\&&1&\\&&&1\\1&&&\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{bmatrix}$$
$$=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&1&-1\\1&i&-1&-i\end{bmatrix}\begin{bmatrix}1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\\1&1&1&1\end{bmatrix}$$
$$=\frac{1}{4}\begin{bmatrix}4&&&\\&4i&&\\&&-4&\\&&&-4i\end{bmatrix}=\begin{bmatrix}1&&&\\&i&&\\&&-1&\\&&&-i\end{bmatrix}$$

so eigenvalues of P: 1, -1, i, $-i$

eigenvectors of P: $\mathrm{span}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$

3.
$$C\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}c_0+c_1+c_2+c_3\\c_0+c_1+c_2+c_3\\c_0+c_1+c_2+c_3\\c_0+c_1+c_2+c_3\end{bmatrix}\qquad C\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}c_0-c_2+(c_1-c_3)i\\c_3-c_1+(c_0-c_2)i\\c_2-c_0+(c_3-c_1)i\\c_1-c_3+(c_2-c_0)i\end{bmatrix}$$

4.
$$C=c_1P+c_2P^2+c_3P^3+c_0P^4$$

$\Rightarrow$ eigenvalues of C: $c_0+c_1+c_2+c_3$, $c_0-c_1+c_2-c_3$, $c_0-c_2+(c_1-c_3)i$, $c_0-c_2+(c_3-c_1)i$

eigenvectors of C: $\mathrm{span}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}$, $\mathrm{span}\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$

2.4

1. counter example:

$$\begin{cases} x=0 \\ y=0 \end{cases}, \quad \begin{cases} x=0 \\ z=0 \end{cases}, \quad \begin{cases} y=0 \\ z=0 \end{cases}, \quad x=y=z$$

2. counter example:

same as above

3.

V_1+V_2, V_3+V_4 linear independent $\Rightarrow$

$\forall\, \vec{v} \in V_1+V_2,\ \vec{v} \neq \vec{0},\ \vec{v} \notin V_3+V_4$

$\forall\, \vec{v} \in V_3+V_4,\ \vec{v} \neq \vec{0},\ \vec{v} \notin V_1+V_2$

$\Downarrow$

V_1, V_2 linear independent $\Rightarrow V_1 \oplus V_2$

V_3, V_4 linear independent $\Rightarrow V_3 \oplus V_4$

$(V_1+V_2) \oplus (V_3+V_4)$

$\Downarrow$

$V_1 \oplus V_2 \oplus V_3 \oplus V_4$

2.5

1. $L(a+bx^2) = \frac{d}{dx}\left(x^3 \cdot \left(a+\frac{b}{x^2}\right)\right)$

$= \frac{d}{dx}(ax^3 + bx)$

$= (3a)x^2 + b \in \{a+bx^2 : a,b \in \mathbb{C}\}$

$L(ax+bx^3) = \frac{d}{dx}\left(x^3 \cdot \left(\frac{a}{x} + \frac{b}{x^3}\right)\right)$

$= \frac{d}{dx}(ax^2 + b)$

$= 2ax \in \{ax+bx^3 : a,b \in \mathbb{C}\}$

2. $L(1) = \frac{d}{dx} x^3 = 3x^2$

$L(x^2) = \frac{d}{dx}\left(x^3 \cdot \frac{1}{x^2}\right) = 1$

$L(x+x^3) = \frac{d}{dx}\left(x^3\left(\frac{1}{x}+\frac{1}{x^3}\right)\right) = \frac{d}{dx}(x^2+1) = 2x$

$L(x-x^3) = \frac{d}{dx}\left(x^3\left(\frac{1}{x}-\frac{1}{x^3}\right)\right) = \frac{d}{dx}(x^2-1) = 2x$

$$L = \begin{bmatrix} 0 & 1 & & \\ 3 & 0 & & \\ & & 1 & 1 \\ & & 1 & 7 \end{bmatrix}$$