8.1

1. $A = \underset{\text{injective}}{B}\,\underset{\text{surjective}}{C}$

$\Rightarrow \forall \vec{x} \in \text{domain}(C)$

$BC\vec{x} \in \text{Ran}(B)$

$\;\; \| $

$A\vec{x}$

$\Rightarrow \text{Ran}(B) \subseteq \text{Ran}(A)$

Since C is surjective,

$\text{Ran}(BC) \subseteq \text{Ran}(B)$

$\quad \|$

$\text{Ran}(A)$

$\downarrow$

$\longrightarrow$ So $\text{Ran}(A) = \text{Ran}(B)$

2. $A^T = (BC)^T = C^T B^T$

since B, C are injective, surjective

C^T, B^T are injective, surjective

apply the proposition in the last question to this

$\Rightarrow \text{Ran}(A^T) = \text{Ran}(C^T)$

3. $B = Q\,R$ $\rightarrow$ upper triangular $\Rightarrow$ bijective

Q $\downarrow$ orthonormal $\Downarrow$ bijective

So it's same in the 1^{st} subproblem

$\Rightarrow \text{Ran}(B) = \text{Ran}(Q)$

4. let X be the ONB of $\text{Ran}(A)$

$\Rightarrow A = X \cdot R$ $\hookrightarrow$ upper triangular matrix

for R^T: let Y^T be the ONB of $\text{Ran}(R^T)$

$R^T = Y^T \cdot R'$ $\hookrightarrow$ another upper triangular matrix

since $A^T = R^T X^T \Rightarrow \text{Ran}(R^T) = \text{Ran}(A^T)$

let $T^T = R'$, then $A = X(Y^T T)^T = XTY$

8.2

1.

$A = [\vec{a_1}\ \vec{a_2}\ \vec{a_3}]$ $B = [\vec{b_1}\ \vec{b_2}\ \vec{b_3}]$

$B = AE = A\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ $\Rightarrow$ columns of E form a OGB

$\Downarrow$

AE : orthogonal

$\Downarrow$

B : orthogonal

also, since E as column operations that preserves "length"

so column vectors of B are still unit vectors

$\Rightarrow$ $\vec{b_1}, \vec{b_2}, \vec{b_3}$ form a ONB

2.

$B = AE = A\begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ B doesn't have enough non-zero columns to form ONB

perform Gram–Schmidt :

$(a_1+a_5)(a_1-a_2+a_4)$

$\vec{b_1}' = \vec{b_1} = \vec{a_1} + \vec{a_5}$

$\vec{b_2}' = \vec{b_2} - \frac{\langle \vec{b_1}', \vec{b_2} \rangle}{\langle \vec{b_1}', \vec{b_1}' \rangle} \vec{b_1}' = \vec{a_1} - \vec{a_2} + \vec{a_4} - \frac{\langle \vec{a_1}+\vec{a_5}, \vec{a_1}-\vec{a_2}+\vec{a_4} \rangle}{\langle \vec{a_1}+\vec{a_5}, \vec{a_1}+\vec{a_5} \rangle}(\vec{a_1}+\vec{a_5})$

$= \vec{a_1} - \vec{a_2} + \vec{a_4} - \frac{1}{2}(\vec{a_1}+\vec{a_5})$

$= \frac{1}{2}\vec{a_1} - \vec{a_2} + \vec{a_4} - \frac{1}{2}\vec{a_5}$

$\vec{b_3}' = \vec{b_3} - \frac{\langle \vec{b_1}', \vec{b_3} \rangle}{\langle \vec{b_1}', \vec{b_1}' \rangle} \vec{b_1}' - \frac{\langle \vec{b_2}', \vec{b_3} \rangle}{\langle \vec{b_2}', \vec{b_2}' \rangle} \vec{b_2}'$

$= 2\vec{a_1} + \vec{a_2} + \vec{a_3} - \frac{2}{2}(\vec{a_1}+\vec{a_5}) - \frac{0}{\frac{5}{2}} \vec{b_2}' = \vec{a_1} + \vec{a_2} + \vec{a_3} - \vec{a_5}$

8.3

1.
$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & \frac{7}{15} \\ 1 & \frac{1}{2} & \frac{2}{5} \\ 0 & 2 & -\frac{11}{15} \\ 1 & \frac{3}{2} & \frac{8}{15} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & \frac{13}{15} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underset{Q}{\begin{bmatrix} \frac{2}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & -\frac{7}{15}\sqrt{\frac{5}{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} & \frac{2}{5}\sqrt{\frac{5}{6}} \\ 0 & \sqrt{\frac{8}{15}} & -\frac{11}{15}\sqrt{\frac{5}{6}} \\ \frac{1}{\sqrt{6}} & \sqrt{\frac{3}{10}} & \frac{8}{15}\sqrt{\frac{5}{6}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{6} & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & \sqrt{\frac{15}{2}} & \frac{13}{15}\sqrt{\frac{15}{2}} \\ 0 & 0 & \sqrt{\frac{6}{5}} \end{bmatrix}}$$

the matrix of orthogonal projection to Range:

$$QQ^T = \begin{bmatrix} \frac{53}{54} & \frac{1}{9} & \frac{1}{54} & \frac{-2}{27} \\ \frac{1}{9} & \frac{1}{3} & \frac{-1}{9} & \frac{4}{9} \\ \frac{1}{54} & \frac{-1}{9} & \frac{53}{54} & \frac{2}{27} \\ \frac{2}{27} & \frac{4}{9} & \frac{2}{27} & \frac{19}{27} \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} & -\frac{6}{5} \\ 2 & -\frac{1}{3} & \frac{13}{10} \\ -1 & \frac{2}{3} & \frac{7}{5} \\ 0 & 1 & \frac{11}{10} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{10} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{4}{3}\sqrt{\frac{3}{10}} & -\frac{6}{5}\sqrt{\frac{10}{63}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & \frac{13}{10}\sqrt{\frac{10}{63}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{3}\sqrt{\frac{3}{10}} & \frac{7}{5}\sqrt{\frac{10}{63}} \\ 0 & \sqrt{\frac{3}{10}} & \frac{11}{10}\sqrt{\frac{10}{63}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{6} & -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ 0 & \sqrt{\frac{10}{3}} & -\frac{1}{\sqrt{30}} \\ 0 & 0 & \sqrt{\frac{63}{10}} \end{bmatrix}}$$

$$QQ^T = \begin{bmatrix} \frac{13}{14} & \frac{-1}{21} & \frac{-1}{6} & \frac{4}{21} \\ \frac{-1}{21} & \frac{61}{63} & \frac{-1}{9} & \frac{8}{63} \\ \frac{-1}{6} & \frac{-1}{9} & \frac{17}{18} & \frac{4}{9} \\ \frac{4}{21} & \frac{8}{63} & \frac{4}{9} & \frac{31}{63} \end{bmatrix}$$

8-4

1.

$$A=\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

since all diagonal entries are positive, A is positive definite

$$A=\begin{bmatrix} 2 & & \\ 2 & 2 & \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \sqrt{2} & & \\ \sqrt{2} & \sqrt{2} & \\ 0 & \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ & \sqrt{2} & \sqrt{2} \\ & & 1 \end{bmatrix} = L \cdot L^T$$

$$\Rightarrow f(x,y,z) = (\sqrt{2}x+\sqrt{2}y)^2 + (\sqrt{2}y+\sqrt{2}z)^2 + z^2$$

2.

$$A=\begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is not positive definite

for example, $\vec{\alpha}=\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$

then $\vec{\alpha}^T A \vec{\alpha} = 0$, but $\vec{\alpha} \neq \vec{0}$

8.5

1. $P = QQ^T$

$\vec{p_i} = QQ^T \vec{e_i}$

$\vec{p_i}^T \cdot \vec{p_i} = (\vec{e_i}^T Q Q^T)(QQ^T\vec{e_i}) = \vec{e_i}^T \cdot Q \cdot I \cdot Q^T \vec{e_i}$

$= \vec{e_i}^T P \vec{e_i} = (i, i)$ entry of P

2. $A \cdot B$ orthogonal projections $\Rightarrow$ $A^2 = A$ $\quad A = A^T$

$B^2 = B \quad B = B^T$

① if AB is an orthogonal projection

$\Rightarrow (AB)^T = AB$

$\overset{\|}{B^T A^T} = \overset{\|}{BA}$

② if $AB = BA$

$\Rightarrow BA = B^T A^T = (AB)^T = AB$

$(AB)^2 = AB \cdot AB = A(BA)B = A \cdot AB \cdot B = A^2 B^2 = AB$

$\Rightarrow$ so AB is an orthogonal projection

to $\operatorname{Ran}(A) \cap \operatorname{Ran}(B)$

3. ① if $A+B$ is an orthogonal projection

$\Rightarrow (A+B)^T = A+B$, $(A+B)^2 = A+B$

$A^2 + AB + BA + B^2 = AB + BA + A + B$

$\Rightarrow AB + BA = 0$

$A + BAB^{-1} = 0$

since A and BAB^{-1} are similar, and they're both idempotent square matrix

$\operatorname{trace}(A) = \operatorname{trace}(BAB^{-1})$

$\operatorname{rank}(A) = \operatorname{rank}(BAB^{-1}) = 0$

$\Rightarrow AB = BA = 0$

$\Rightarrow \operatorname{Ran}(A) \perp \operatorname{Ran}(B)$

② if $\operatorname{Ran}(A) \perp \operatorname{Ran}(B)$

$\operatorname{Ran}(A^T) \quad \operatorname{Ran}(B^T)$

$\perp \quad\quad \perp$

$\ker(A) \quad \ker(B)$

$\Rightarrow AB = BA = 0$

$\Rightarrow (A+B)^2 = A^2 + AB + BA + B^2$

$= A^2 + B^2 = A + B$

$\Rightarrow A+B$ is an orthogonal projection to $\operatorname{Ran}(A) + \operatorname{Ran}(B)$

8.6

1. $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

8.7

1. $\det \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \det \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} = -6$

2. $\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{bmatrix} = -(3-2)(3-1)(3+1)(2-1)(2+1)(1+1) = -48$

3. $= 160$

4. $= 0$

5. $= (-1)^{n+1}$