11.1

	A	eigenvalues	Ma	Mg	eigenvectors	diagonalizable?
1.	$\begin{bmatrix} a & b & \cdots & b \\ b & \ddots & & \vdots \\ \vdots & & \ddots & b \\ b & \cdots & b & a \end{bmatrix}_{n\times n}$	$a-b$	$n-1$	$n-1$	$\ker\begin{bmatrix} b & \cdots & b \\ \vdots & \ddots & \vdots \\ b & \cdots & b \end{bmatrix}$	✓
		$a+(n-1)b$	1	1	$\mathrm{span}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$	

$$\Rightarrow A = \begin{bmatrix} a & b & \cdots & b \\ b & \ddots & & \vdots \\ \vdots & & \ddots & b \\ b & \cdots & b & a \end{bmatrix} = X \begin{bmatrix} a-b & & & \\ & \ddots & & \\ & & a-b & \\ & & & a+(n-1)b \end{bmatrix} X^{-1}$$

where the first $(n-1)$ columns of X is in $\ker\begin{bmatrix} b & \cdots & b \\ \vdots & \ddots & \vdots \\ b & \cdots & b \end{bmatrix}$

the last column of X is $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

	A	eigenvalues	Ma	Mg	eigenvectors	diagonalizable?
2.	$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$	-1	2	1	$\mathrm{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	X

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1}$$

11.2

1. $A^2 = I \Rightarrow \begin{cases} (A+I)(A-I)=0 \\ (A-I)(A+I)=0 \end{cases}$

$\begin{cases} \dim \ker(A+I) \geq r(A-I) \\ \dim \ker(A-I) \geq r(A+I) \end{cases}$

$\Rightarrow r(A-I)+r(A+I) \leq n$

also $r(A-I)+r(A+I) \geq r(A-I+A+I)=n$

$\Rightarrow r(A-I)+r(A+I)=n$

$= \dim \ker(A+I) + \dim \ker(A-I)$

also $\ker(A+I) \cap \ker(A-I) = \{\vec{0}\}$

$\Rightarrow \underbrace{\ker(A+I)}_{\text{eigenspace for } -1} + \underbrace{\ker(A-I)}_{\text{eigenspace for } 1} = \mathbb{C}^n$

so A is diagonalizable

2. $A^2 = A \Rightarrow \begin{cases} A(A-I)=0 \\ (A-I)A=0 \end{cases}$

$\Leftarrow$ similar with last subquestion using $r(A)+r(I-A) \geq r(I)=n$ and rank-nullity

also $\ker(A) \cap \ker(A-I) = \{\vec{0}\}$

$\Rightarrow \underbrace{\ker(A)}_{\text{eigenspace for } 0} + \underbrace{\ker(A-I)}_{\text{eigenspace for } 1} = \mathbb{C}^n$

so A is diagonalizable

3.

$aA(A-I)\vec{v} + bA(A+I)\vec{v} + c(A-I)(A+I)\vec{v}$

$= aA^2\vec{v} - aA\vec{v} + bA^2\vec{v} + bA\vec{v} + cA^2\vec{v} - c\vec{v}$

$= (a+b+c)A^2\vec{v} + (b-a)A\vec{v} - c\vec{v} = \vec{v}$

$\Rightarrow \begin{cases} a=b=\frac{1}{2} \\ c=-1 \end{cases}$

4. $A^3 = A \Rightarrow \begin{cases} A(A+I)(A-I)=0 \\ (A+I)(A-I)A=0 \end{cases}$

similar with 1 & 2

$\Rightarrow \underbrace{\ker(A)}_{\text{eigenspace for } 0} + \underbrace{\ker(A+I)}_{\text{eigenspace for } -1} + \underbrace{\ker(A-I)}_{\text{eigenspace for } 1} = \mathbb{C}^n$

so A is diagonalizable

11.3

1. $X = \begin{bmatrix} I & M \\ & I \end{bmatrix}$

2. $X = \begin{bmatrix} i & 1 \\ & i \end{bmatrix}$

3. $X = \begin{bmatrix} 0 & 3 & 1 \\ & 3 & \\ & & 1 \end{bmatrix}$

4. $X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

5. $X = \begin{bmatrix} I & 2I \\ I & 3I \end{bmatrix}$

6. $X = \begin{bmatrix} iI & I \\ I & iI \end{bmatrix}$

7. $\begin{bmatrix} I_2 & M \\ & I_3 \end{bmatrix}$

11.4

1. $\left.\begin{matrix} AX = XB \\ n = 1 \end{matrix}\right\} \Rightarrow \xrightarrow{B = [b]} Ax = bX$

so X is a eigenvector of B

corresponding eigenvalue is b

2. $\vec{v} \in Ran(X) \Rightarrow$ let $X\vec{e} = \vec{v}$

then $A\vec{v} = AX\vec{e} = XB\vec{e} \in Ran(X)$

3. $\forall\, a \in \mathbb{C},\ n \in \mathbb{N}$

$(aA^n)X = aA^{n-1}XB$

$= aA^{n-2}XB^2 = \cdots$

$= aXB^n = X(aB^n)$

so any power of A is ok

$\Rightarrow p(A)X = Xp(B)$

4. $\forall$ eigenvalue of B λ_B, $\forall$ corresponding eigenvector $\vec{v}$

$XB\vec{v} = X\lambda_B\vec{v}$

$\parallel$

$AX\vec{v}$

$\Rightarrow AX = \lambda_B X$

suppose $X = [\vec{x_1} \cdots \vec{x_n}]$

$\Rightarrow A\vec{x_i} = \lambda_B \vec{x_i}$

as $\forall \lambda_B$

$\Rightarrow \vec{x_i} = \vec{0}$

$\Rightarrow X = 0$

11.5

1.
$$L(aX+bY) = A(aX+bY) - (aX+bY)B$$
$$= aAX + bAY - aXB + bYB$$
$$= a(AX - XB) + b(AY - YB)$$
$$= aL(X) + bL(Y)$$

2.
$$\ker(L) = \{X \mid AX = XB\}$$

from Problem 11.4, we know $\left.\begin{array}{l} AX = XB \\ A, B \text{ have no common eigenvalue} \end{array}\right\} \Rightarrow X = 0$

so $\ker(L)$ is trivial

3. since L is a linear map,
and $\ker(L)$ is trivial

$\Rightarrow$ L is bijective

so $L(X) = C$ always have an unique solution

11.6

1. $A \geq B$, $B \geq C$

$\Downarrow$

$$\left.\begin{array}{l}\vec{v}^*(A-B)\vec{v} \geq 0 \\ \vec{v}^*(B-C)\vec{v} \geq 0\end{array}\right\} \Rightarrow \vec{v}^*(A-B+B-C)\vec{v} \geq 0$$

$$\vec{v}^*(A-C)\vec{v} \geq 0$$

so $A \geq C$

2. $\vec{v}^*(A-B)\vec{v} = \vec{v}^*A\vec{v} - \vec{v}^*B\vec{v} \geq 0$

$\vec{v}^*(B-A)\vec{v} = \vec{v}^*B\vec{v} - \vec{v}^*A\vec{v} \geq 0$

$\Rightarrow \vec{v}^*A\vec{v} = \vec{v}^*B\vec{v}$

$A = B$

3. ① $\exists a$, s.t $A - aI$ is positive semidefinite

$A = UDU^* = UDU^{-1}$

$\Rightarrow$ all eigenvalues ≥ 0

$\Rightarrow A - aI = U(D - aI)U^{-1}$

$\Rightarrow$ all entries of $(D - aI) \geq 0 \Rightarrow a \leq \min\{d_i\}$ $\left(D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}\right)$

② $\exists b$. s.t. $bI - A$ is positive semidefinite

similar with ① $\Rightarrow$ $b \geq \max\{d_i\}$