5.1

1. $A = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ & 1 & 3 & 4 \\ \hline & & 3 & 5 \\ & & & 4 \end{array}\right)$ $\quad A_1 = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 5 \\ & 4 \end{pmatrix}$

$B = \begin{pmatrix} I & X \\ & I \end{pmatrix}, \quad A_1 X - X A_2 = -\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \begin{pmatrix} 2c-2a & -5a-3b+2d \\ -2c & -5c-3d \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} 2 & -\frac{31}{9} \\ \frac{3}{2} & -\frac{7}{6} \end{pmatrix}$

$\Rightarrow B = \begin{pmatrix} 1 & & 2 & -\frac{31}{9} \\ & 1 & \frac{3}{2} & -\frac{7}{6} \\ & & 1 & \\ & & & 1 \end{pmatrix}$

2. $\ker(A-3I) = \ker\left(B \cdot \begin{pmatrix} -2 & 2 & & \\ & -2 & & \\ & & & 5 \\ & & & 1 \end{pmatrix} \cdot B^{-1}\right)$

$= \operatorname{span}\left(B \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right) = \operatorname{span}\begin{pmatrix} 2 \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}$

$\ker(A-4I) = \ker\left(B \cdot \begin{pmatrix} -3 & 2 & & \\ & -3 & & \\ & & -1 & 5 \\ & & & \end{pmatrix} B^{-1}\right)$

$= \operatorname{span}\left(B \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}\right) = \operatorname{span}\begin{pmatrix} \frac{59}{9} \\ \frac{19}{3} \\ 5 \\ 1 \end{pmatrix}$

$\Rightarrow$ basis for $V_3 + V_4$: $\begin{pmatrix} 2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 59/9 \\ 19/3 \\ 5 \\ 1 \end{pmatrix}$

3. $(x-3)(x-4)(x-1)^2$

5.2

1. let $X = kI \ (k \in \mathbb{R})$

$\Rightarrow L(X) = L(kI) = kAI - kIA = k(A - A) = 0$

2. $L(XY) = AXY - XYA$

$$\left.\begin{aligned} L(X)Y &= AXY - XAY \\ XL(Y) &= XAY - XYA \end{aligned}\right\} \Rightarrow L(X)Y - XL(Y) = AXY - XYA = L(XY)$$

3. ① $L(kX) = kAX - kXA$

$= k(AX - XA) = L(X) \cdot k$

② $L(X)X = AX^2 - XAX = XL(X) = XAX - X^2A$

$\Rightarrow 2XAX = AX^2 + X^2A \Rightarrow 3X^2AX = \frac{3}{2}XAX^2 + \frac{3}{2}X^3A$

$\Rightarrow 2AX^2 - 2XAX = AX^2 - X^2A = L(X) \cdot 2X = L(X^2)$

using induction,

$k = 2$ ✓

$k+1$: $L(X^k) = L(X) \cdot (kX^{k-1})$ ✓

$L(X^{k+1}) = L(X) \cdot X^k + X L(X^k)$

$= L(X) \cdot X^k + X \cdot L(X) \cdot (kX^{k-1})$

$= L(X) \cdot X^k + L(X) \cdot X(kX^{k-1})$

$= L(X)(k+1)X^k$ ✓

4. $X = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, A = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \Rightarrow L(X)X = AX^2 - XAX = \begin{pmatrix} -1 & -1 \\ & 1 \end{pmatrix}$

$XL(X) = XAX - X^2A = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}$

$X^2A = \begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}$

$AX^2 = \begin{pmatrix} & 1 \\ 1 & 2 \end{pmatrix}$

$XAX = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$L(X^2) = AX^2 - X^2A = \begin{pmatrix} -2 & \\ & -2 \end{pmatrix}$

$L(X)X + XL(X) = \begin{pmatrix} -2 & \\ & -2 \end{pmatrix}$

$L(X)(2X) = \begin{pmatrix} -2 & -2 \\ & 2 \end{pmatrix}$

5. if $L(X)=I$ for some X

then $L(X)X = XL(X)$

then $L(p(X)) = L(X)p'(X) = p'(X)$

$\quad\;\; \shortparallel$

$\quad A\,p(X) - p(X)A$

let $p(X)$ be the minimal polynomial of X

$\Rightarrow p'(X) = 0$

$\Rightarrow P(X) = cI \quad (c \in \mathbb{R})$

$\uparrow$ impossible

$\Rightarrow$ so $L(X) \neq I$ for any X

6. $X \in \ker(L)$ iff $L(X) = AX - XA = 0$

$\Rightarrow AX = XA$

since A is diagonalizable with distinct eigenvalues

$A = CDC^{-1}$

$\uparrow$ with distinct diagonal entries

$\Rightarrow CDC^{-1}X = XCDC^{-1}$

$C(DX)C^{-1} = C(XD)C^{-1}$

$DX = XD$

$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \quad \Rightarrow \left.\begin{aligned} x_{ij} &= \lambda_i x_{ij} \lambda_j^{-1} \\ &= \underbrace{\frac{\lambda_i}{\lambda_j}}_{\neq 1 \text{ if } i \neq j} x_{ij} \end{aligned}\right\} \Rightarrow X = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$

$\Rightarrow$ so $\dim\ker(L) = n$ (A is a $n \times n$ matrix)

7. $A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$

5.3

1. $W \xrightarrow{A} W$

then we can restrict A into W

$\Rightarrow A' : W \to W$

A' is also a linear map, which must have at least one eigenvector

obviously, this eigenvector is in W

2. $\forall \vec{v} \in \ker(A - \lambda I)$,

$A\vec{v} - \lambda\vec{v} = \vec{0} \Rightarrow B(A\vec{v} - \lambda\vec{v}) = BA\vec{v} - \lambda B\vec{v} = \vec{0}$

since $AB = BA$

$\Rightarrow AB\vec{v} - \lambda B\vec{v} = \vec{0}$

$= (A - \lambda I)B\vec{v} = \vec{0}$

$\Rightarrow B\vec{v} \in \ker(A - \lambda I)$

then $\ker(A - \lambda I)$ is B-invariant for all $\lambda \in \mathbb{C}$

3. from 1&2 $\Rightarrow B$ has an eigenvector in $\ker(A - \lambda I)$

$\Rightarrow$ same eigenvector means same eigenvalue λ

4. suppose $\vec{v}$ is the common eigenvector for A & B

$X_1 = (\vec{v}, \cdots\cdots)$

↑

arbitary find some vectors to form a basis with $\vec{v}$

then X_1 must be invertible

using schur decomposition, we can get $A = X_1 \begin{pmatrix} a_1 & * \\ & A_1 \end{pmatrix} X_1^{-1}$

$B = X_1 \begin{pmatrix} b_1 & * \\ & B_1 \end{pmatrix} X_1^{-1}$

since $AB = BA$

$\Rightarrow \begin{pmatrix} a_1 b_1 & * \\ & A_1 B_1 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & * \\ & B_1 A_1 \end{pmatrix} \Rightarrow A_1 B_1 = B_1 A_1$

5. in the last subproblem, we have done 1^{st} step of triangularization

since $A_1 B_1 = B_1 A_2 \Rightarrow$ have a common eigenvector

we can use induction:

$A_1 = X_2 \begin{pmatrix} a_2 & * \\ & A_3 \end{pmatrix} X_2^{-1}$ $\Rightarrow A_3 B_3 = B_3 A_3 \Rightarrow \cdots$

$B_1 = X_2 \begin{pmatrix} b_2 & * \\ & B_3 \end{pmatrix} X_3^{-1}$

then we get $A = X R_A X^{-1}$, $X = X_1 X_2 X_3 \cdots$

$B = X R_B X^{-1}$