

8.1

$$1. A = BC$$

injective  
surjective

$$\Rightarrow \forall \bar{x} \in \text{domain}(C)$$

$$BC\bar{x} \in \text{Ran}(B)$$

||  
 $\bar{x}$

$$\Rightarrow \text{Ran}(B) \subseteq \text{Ran}(A)$$

Since  $C$  is surjective,

$$\text{Ran}(BC) \subseteq \text{Ran}(B)$$

$$\text{Ran}(A)$$



$$\longrightarrow \text{So } \text{Ran}(A) = \text{Ran}(B)$$

$$2. A^T = (BC)^T = C^T B^T$$

since  $B, C$  are injective, surjective

$C^T, B^T$  are injective, surjective

apply the proposition in the last question to this

$$\Rightarrow \text{Ran}(A^T) = \text{Ran}(C^T)$$

$$3. B = Q R$$

upper triangular  $\Rightarrow$  bijective

$\downarrow$   
orthonormal

$\downarrow$   
bijective

so it's same in the 1st subproblem

$$\Rightarrow \text{Ran}(B) = \text{Ran}(Q)$$

4. let  $X$  be the ONB of  $\text{Ran}(A)$

$$\Rightarrow A = X \cdot R$$

$\hookrightarrow$  upper triangular matrix

for  $R^T$ : let  $Y^T$  be the ONB of  $\text{Ran}(R^T)$

$$R^T = Y^T R'$$

$\hookrightarrow$  another upper triangular matrix

$$\text{since } A^T = R^T X^T \Rightarrow \text{Ran}(R^T) = \text{Ran}(A^T)$$

$$\text{let } T = R' \text{, then } A = X(Y^T)^T = X^T Y$$

B.2

$$1. \quad A = [\vec{a}_1 \vec{a}_2 \vec{a}_3] \quad B = [\vec{b}_1 \vec{b}_2 \vec{b}_3]$$

$$B = AE = A \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \Rightarrow \text{columns of } E \text{ form a ONB}$$

↓  
AE : orthogonal

↓  
B : orthogonal

also, since E as column operations that preserves "length"

so column vectors of B are still unit vectors

$\Rightarrow \vec{b}_1, \vec{b}_2, \vec{b}_3$  form a ONB

2.

$$B = AE = A \begin{bmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B \text{ doesn't have enough non-zero columns to form ONB}$$

perform Gram-Schmidt :

$$(a_1 + a_5) (a_1 - a_2 + a_4)$$

$$\vec{b}_1' = \vec{b}_1 = \vec{a}_1 + \vec{a}_5$$

$$\begin{aligned} \vec{b}_2' &= \vec{b}_2 - \frac{\langle \vec{b}_1', \vec{b}_2 \rangle}{\langle \vec{b}_1', \vec{b}_2 \rangle} \vec{b}_1' = \vec{a}_1 - \vec{a}_2 + \vec{a}_4 - \frac{\langle \vec{a}_1 + \vec{a}_5, \vec{a}_1 - \vec{a}_2 + \vec{a}_4 \rangle}{\langle \vec{a}_1 + \vec{a}_5, \vec{a}_1 - \vec{a}_2 + \vec{a}_4 \rangle} (\vec{a}_1 + \vec{a}_5) \\ &= \vec{a}_1 - \vec{a}_2 + \vec{a}_4 - \frac{1}{2} (\vec{a}_1 + \vec{a}_5) \end{aligned}$$

$$\begin{aligned} \vec{b}_3' &= \vec{b}_3 - \frac{\langle \vec{b}_1', \vec{b}_3 \rangle}{\langle \vec{b}_1', \vec{b}_3 \rangle} \vec{b}_1' - \frac{\langle \vec{b}_2', \vec{b}_3 \rangle}{\langle \vec{b}_2', \vec{b}_3 \rangle} \vec{b}_2' \\ &= 2\vec{a}_1 + \vec{a}_2 + \vec{a}_3 - \frac{2}{3} (\vec{a}_1 + \vec{a}_5) - \frac{0}{\frac{1}{2}} (\vec{a}_1 + \vec{a}_5) \end{aligned}$$

$$= \vec{a}_1 + \vec{a}_2 + \vec{a}_3 - \vec{a}_5 = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 - \vec{a}_5$$

8.3

$$1. \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{15} & \frac{1}{15} \\ 1 & \frac{1}{15} & \frac{2}{15} \\ 0 & 2 & \frac{2}{15} \\ 1 & \frac{2}{15} & \frac{4}{15} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \\ 0 & \frac{\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{5}}{2} \end{bmatrix}$$

Q R

the matrix of orthogonal projection to Range:

$$QQ^T = \begin{bmatrix} \frac{3}{10} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{4}{9} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{4}{9} \\ \frac{1}{2} & \frac{4}{9} & \frac{4}{9} & \frac{16}{27} \end{bmatrix}$$

$$2. \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{5} & -\frac{1}{5} \\ 2 & -\frac{1}{5} & \frac{10}{5} \\ -1 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{10}{5} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{10} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} & \frac{10}{5} \\ \frac{-1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{10}{5} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ 0 & \frac{\sqrt{5}}{10} & -\frac{1}{10} \\ 0 & 0 & \frac{\sqrt{5}}{10} \end{bmatrix}$$

Q R

$$QQ^T = \begin{bmatrix} \frac{13}{14} & \frac{1}{2} & \frac{1}{6} & \frac{4}{21} \\ \frac{1}{2} & \frac{63}{14} & \frac{1}{9} & \frac{4}{21} \\ \frac{1}{6} & \frac{1}{9} & \frac{18}{14} & \frac{4}{21} \\ \frac{4}{21} & \frac{4}{21} & \frac{4}{21} & \frac{14}{14} \end{bmatrix}$$

8-4

1.  $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$  since all diagonal entries are positive,  
A is positive definite

$$\begin{aligned} A &= \begin{bmatrix} 2 & & \\ 2 & 2 & \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & & \\ \sqrt{2} & \sqrt{2} & \\ 0 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 1 \end{bmatrix} = L \cdot L^T \end{aligned}$$

$$\Rightarrow f(x, y, z) = (\sqrt{2}x + \sqrt{2}y)^2 + (\sqrt{2}y + \sqrt{2}z)^2 + z^2$$

2.  $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  is not positive definite

for example,  $\vec{\alpha} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$

then  $\vec{\alpha}^T A \vec{\alpha} = 0$ , but  $\vec{\alpha} \neq \vec{0}$

## 8.5

$$1. P = QQ^T$$

$$\vec{P}_i = QQ^T \vec{e}_i$$

$$\vec{P}_i^T \cdot \vec{P}_i = (\vec{e}_i^T Q Q^T) (\vec{e}_i^T Q Q^T) = \vec{e}_i^T \cdot \vec{e}_i \cdot I \cdot Q^T \vec{e}_i \\ = \vec{e}_i^T \vec{P} \vec{e}_i = (i, i) \text{ entry of } P$$

$$2. A, B \text{ orthogonal projections} \Rightarrow A^2 = A \quad A = A^T \\ B^2 = B \quad B = B^T$$

① if  $AB$  is an orthogonal projection

$$\Rightarrow (AB)^T = AB \\ \parallel \quad \parallel \\ B^T A^T = BA$$

$$\textcircled{2} \nmid AB = BA$$

$$\Rightarrow BA = B^T A^T = (AB)^T = AB$$

$$(AB)^2 = AB \cdot AB = A(BA)B = A \cdot AB \cdot B = A^2 B^2 = AB$$

$\Rightarrow$  so  $AB$  is an orthogonal projection

$$\text{to } \text{Ran}(A) \cap \text{Ran}(B)$$

3. ② if  $A+B$  is an orthogonal projection

$$\Rightarrow (A+B)^T = A+B \quad (A+B)^2 = A+B \\ \parallel \quad \parallel \\ A^2 + AB + BA + B^2 = AB + BA + A + B$$

$$\Rightarrow AB + BA = 0 \\ A + BAB^{-1} = 0$$

since  $A$  and  $BAB^{-1}$  are similar, and they're both  
trace(A) = trace(BAB<sup>-1</sup>)      idempotent  
rank(A) = rank(BAB<sup>-1</sup>) = 0      square matrices

$$\Rightarrow AB - BA = 0$$

$$\Rightarrow \text{Ran}(A) \perp \text{Ran}(B)$$

③ if  $\text{Ran}(A) \perp \text{Ran}(B)$

$$\begin{array}{c} \text{Ran}(A^T) \quad \text{Ran}(B^T) \\ \perp \quad \perp \\ \text{Ker}(A) \quad \text{Ker}(B) \end{array}$$

$$\Rightarrow AB = BA = 0$$

$$\Rightarrow (A+B)^2 = A^2 + AB + BA + B^2 \\ = A^2 + B^2 = A+B$$

$\Rightarrow A+B$  is an orthogonal projection to  $\text{Ran}(A) + \text{Ran}(B)$

8.6

$$1. A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$3. A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

8.7

$$1. \det \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \det \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = -6$$

$$2. \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{bmatrix} = -(3-2)(3-1)(3+1)(2-1)(2+1)(1+1) = -48$$

$$3. = 160$$

$$4. = 0$$

$$5. = (-1)^{n+1}$$