2.1

① $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 4 \\ 7 & 8 & 9 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x = z + 1 \\ y = -2z \\ z \end{cases}$

② the same solution with ①
(due to the redundant equation of row 3)

③ $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 7 & 8 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & -6 & 1 \end{array}\right]$

$\Rightarrow y = 0$ and $y = -1/6$
contradiction!
so no solution

2.2

① $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & b & 7 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & b-6 & -2 \end{array}\right]$

$\Rightarrow b = 6$

② $\left[\begin{array}{cc|c} 3 & 2 & 10 \\ 6 & 4 & b \end{array}\right] \Rightarrow b \neq 20$

③ $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & b & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & b-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$

$\Rightarrow b - 10 = 1$

$b = 11$

④ $\left[\begin{array}{cc|c} b & 3 & 6 \\ 3 & b & -6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} b+3 & b+3 & 0 \\ 3 & b & -6 \end{array}\right]$

$\Rightarrow b = -3$

⑤ $\left[\begin{array}{cc|c} 2 & b & 16 \\ 4 & 8 & c \end{array}\right] \Rightarrow$ let's make a redundant equation

$\Rightarrow \frac{2}{4} = \frac{b}{8} = \frac{16}{c} \Rightarrow \begin{cases} b = 4 \\ c = 32 \end{cases}$

⑥ $\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -(b+2) & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$\Downarrow$

$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & (b+2)(b+1) & 0 & 0 \\ 0 & 0 & b+1 & 0 \end{array}\right] \Leftarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & b+2 & 1 & 0 \\ 0 & 0 & b+1 & 0 \end{array}\right]$

$\Downarrow$

$\left[\begin{array}{ccc|c} (b+2)(b+1) & 0 & 0 & 0 \\ 0 & (b+2)(b+1) & 0 & 0 \\ 0 & 0 & b+1 & 0 \end{array}\right] \Rightarrow$ $b = -1$
or $b = -2$ -3

⑦ $\left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ b & b & 4 & 0 \\ b & b & b & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & b-2 & 1 & 0 \\ 0 & b-2 & b-3 & 0 \end{array}\right]$

$\Downarrow$

$\left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & (b-2)(b-4) & 0 & 0 \\ 0 & 0 & b-4 & 0 \end{array}\right] \Leftarrow \left[\begin{array}{ccc|c} b & 2 & 3 & 0 \\ 0 & b-2 & 1 & 0 \\ 0 & 0 & b-4 & 0 \end{array}\right]$

$\Downarrow$

$b = 0$ or $b = 2$ or $b = 4$

2.3

$$\left[\begin{array}{ccc|c} p & 1 & 1 & 1 \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} p+2 & p+2 & p+2 & p^2+p+1 \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{p^2+p+1}{p+2} \\ 1 & p & 1 & p \\ 1 & 1 & p & p^2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{p^2+p+1}{p+2} \\ 0 & p-1 & 0 & \frac{p-1}{p+2} \\ 0 & 0 & p-1 & \frac{p^3+p^2-p-1}{p+2} \end{array}\right]$$

① $p=-2$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & -2 \\ 1 & 1 & -2 & 4 \end{array}\right]$$

so no solution

② $p \neq -2$

1> $p=1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} x = 1-y-z \\ y \\ z \end{cases}$$

2> $p \neq 1$

$$\Rightarrow \begin{cases} x = \frac{-p-1}{p+2} \\ y = \frac{1}{p+2} \\ z = \frac{p^2+2p+1}{p+2} \end{cases}$$

2.4

① $A\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

② 1st column of A is half of $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 2 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \end{bmatrix}$$

$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 & a_{13} \\ 2 & -2 & a_{23} \\ 1 & -1 & a_{33} \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow a_{13} = a_{23} = a_{33} = 0 \Rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

2.5

① let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2+s \\ s \\ t \end{pmatrix}$

x_1 is a dependent variable

x_2 and x_3 are free variables

② $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} a_1 & -a_1 & 0 & 2 \\ a_2 & -a_2 & 0 & 0 \\ a_3 & -a_3 & 0 & 0 \end{array}\right] \xRightarrow{RREF} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

③ RREF: $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$A\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 \end{array}\right]$$

$$\Downarrow$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \Leftarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -2 & 0 & 4 \\ 1 & -1 & 0 & 2 \end{array}\right]$$

2.6

C means column

① $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 5 & 4 & 6 \end{array}\right]$ $C_1 \leftrightarrow C_2$, $C_3 \to C_3$

② $\left[\begin{array}{cc|c} \frac{1}{2} & 2 & 3 \\ 2 & 5 & 6 \end{array}\right]$ $C_1 \to \frac{1}{2} C_1$, $C_2 \to C_2$, $C_3 \to C_3$

③ $\left[\begin{array}{cc|c} -1 & 2 & 3 \\ 1 & 5 & 6 \end{array}\right]$ $C_1 \to C_1 - C_2$, $C_2 \to C_2$, $C_3 \to C_3$

④ $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 4 & 5 & 10 \end{array}\right]$ $C_1 \to C_1$, $C_2 \to C_2$, $C_3 \to C_1 + C_3$