

6.1

$$1. L_A : \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} I_{2x2} & 2I_{2x2} \\ 3I_{2x2} & 4I_{2x2} \end{bmatrix}$$

$$2. R_A : \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}$$

$$3. L_A R_A = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix} \quad R_A L_A = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix}$$

$$4. L_A R_A \cdot X = L_A(XA) = A(XA) \quad \text{law of associativity}$$

$$R_A L_A \cdot X = R_A(AX) = (AX)A$$

$$5. \text{ new basis: } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 & x_1 \\ x_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_2 & x_2 \\ 0 & x_2 \end{bmatrix} - \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \end{bmatrix} + \begin{bmatrix} 0 & x_3 \\ x_3 & 0 \end{bmatrix} - \begin{bmatrix} x_3 & x_3 \\ 0 & x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & x_4 \end{bmatrix} \rightarrow \begin{bmatrix} x_4 & 0 \\ 0 & x_4 \end{bmatrix} + \begin{bmatrix} 0 & x_4 \\ x_4 & 0 \end{bmatrix} - \begin{bmatrix} x_4 & x_4 \\ x_4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{the change of coordinate matrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

6.2

$$V \xrightarrow{L} W$$

$$G_{ij} = B_i \rightarrow B_j \quad \text{or} \quad C_i \rightarrow C_j$$

$$\checkmark 1. L_{11} = C_2 L_{12}$$

$$L_{1j} = B_i \rightarrow C_j$$

$$\times 2. L_{11}^{-1} = L_{12}^{-1} C_{21} \quad L_{11}^{-1} = L_{12}^{-1} C_{12}$$

$$\checkmark 3. L_{22} = C_{12} L_{11} C_{12}^{-1}$$

$$B_2 \rightarrow C_1 \quad C_1 \rightarrow B_1 \quad B_2 \rightarrow B_1$$

 $B_1 \rightarrow C_2$ 

$$4. L_{31} = C_{21} L_{12} C_{21} C_{23} L_{21}^{-1} C_{21} L_{12} C_{21} C_{32}$$

$$B_3 \rightarrow C_1$$

$$\times C_1 \leftarrow B_2 \circ B_3$$

-3

$$\begin{matrix} L_{12} L_{21} \\ B_1 \rightarrow C_2 \quad B_2 \rightarrow C_1 \end{matrix} \quad \times$$

6.3

$$1. z \mapsto (2z+3)z$$

$$\begin{aligned} & "a+bz \mapsto (2z+3)z" (at bz) \\ & = (2a-3b) + (3a+2b)z^2 \end{aligned}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 2a-3b \\ 3a+2b \end{bmatrix}$$

$$\Rightarrow M_w = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$2. \text{ let } w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad wz = \begin{bmatrix} w_1 z_1 - w_2 z_2 \\ w_1 z_2 + w_2 z_1 \end{bmatrix}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\text{then } M_w = \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix}$$

$$M_z = \begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix}$$

$$M_w \cdot M_z = \begin{bmatrix} w_1 z_1 - w_2 z_2 & -(w_1 z_2 + w_2 z_1) \\ w_1 z_2 + w_2 z_1 & w_1 z_1 - w_2 z_2 \end{bmatrix} \quad //$$

$$3. M_w M_z : a \mapsto z(a) \mapsto w(z(a))$$

$$M_w z : a \mapsto (wz)a$$

$\Rightarrow$  law of associativity

$$4. \phi : w \mapsto M_w$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

6.4.

1. basis  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  dimension = 4

2. there's no solution is the set

$$\Rightarrow \text{set} = \emptyset$$

3. suppose  $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} \in W$

suppose  $\vec{p} = \begin{pmatrix} x \\ y \\ z \\ d \end{pmatrix}$

$$\Rightarrow \begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d \end{cases}, \Rightarrow (a+a')x+(b+b')y+(c+c')z=d+d'$$

$$\Rightarrow \begin{pmatrix} a+a' \\ b+b' \\ c+c' \\ d+d' \end{pmatrix} \in W$$

$\hookrightarrow kax+kby+kcz=kd$

$$\Rightarrow k \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in W$$

so  $W$  is a subspace

4.  $\begin{cases} x=1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ y=2 \Rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \\ z=3 \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} \end{cases} \quad \begin{matrix} x+y+z=1 \\ \downarrow \\ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \end{matrix}$

for first 3 equation, it's obvious  
that they're linearly independent

$\Rightarrow$  let  $W$  be the subspace, made of all  
equations that contains  $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$  in its solution set

$\Rightarrow \forall \vec{y} \in W, \vec{y} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ , but for  $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ , there's no solution  
 $\text{for } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$

$$\Rightarrow x=1, y=2, z=3, x+y+z=7$$

are linearly independent

5.  $x=1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$   $\downarrow$  obvious, they're  
 $x=2 \Rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$  linearly independent

$\Rightarrow$  let  $W$  be the subspace of  $V$ ,  
spanned by  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \forall \vec{y} \in W, \vec{y} = \begin{pmatrix} a+b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{if } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in W$$

$$\text{then } \begin{cases} a+b=1 \\ a+b=3 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=2 \end{cases}$$

$$\Rightarrow (x=3) \in W,$$

$\Rightarrow x=1, x=2, x=3$  are NOT  
linearly independent

6.5

1. NO.  $\vec{w} \mapsto \vec{w} - C - O_2 + CO_2$   
 ~~$2\vec{w} \mapsto 2\vec{w} - C - O_2 + CO_2$~~

2. NO. dimension = 2

3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{the rank of } M = 2$$

6.6

1.  $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$  Because we just delete non-pivotal columns of  $A$ , which doesn't delete pivots in  $A$ , so the rank isn't change as well

2. REF of  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$   
rank of  $R = 2$

3.  $BR = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$

6.7

1.  $\text{Ran}(M)$ 's dimension = 3 ( $a+b+c+d$ , ref)  $\rightarrow$  the rank of  $M=3$

$$2. X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$3. Y = \begin{bmatrix} a & b & c & d & e & f \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \text{educational} \\ \text{funny} \end{array}$$

$$4. X Y = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{bmatrix}$$

$$M - XY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

↑  
rank 1

$$5. M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$7 \times 7$   $7 \times 3$   $7 \times 2$   $7 \times 3$

↓  
educational videos      ↓  
funny videos      ↓  
other factors