Linear Algebra 1 (E) Homework

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Contents

1 HW 1 Due Sep 21

Problem 1.1 (Find Matrices)**.** Recall that given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, you can find its matrices as $\begin{bmatrix} f(\boldsymbol{e}_1) & \dots & f(\boldsymbol{e}_n) \end{bmatrix}$. Find the matrices of the following linear maps, or show that they are not linear (by providing a counter example). Also, are they injective? Surjective? Bijective?

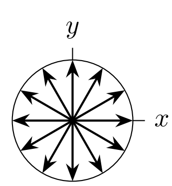
1. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection map sending every vector to its projection on the line $x + y = 0$. (The result of this projection is also a vector in $\mathbb{R}^2$.)
2. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection map about the line $x = y$.
3. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the translation map sending every $\boldsymbol{v}$ to $\boldsymbol{v} + \boldsymbol{e}_1$.
4. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map, i.e., $f(\boldsymbol{v}) = \boldsymbol{v}$.
5. $f : \mathbb{R}^2 \to \mathbb{R}$ and $f(\begin{bmatrix} x \\ y \end{bmatrix}) = \sqrt{x^2 + y^2}$.
6. $f : \mathbb{R}^2 \to \mathbb{R}$ and it sends each point $\boldsymbol{v}$ on the plane to its signed-distance to the line $x + y = 0$. Here signed distance means that it is a positive distance if $\boldsymbol{v}$ is above the line $x + y = 0$, and it is a negative distance if it is below the line $x + y = 0$.
7. $f : \mathbb{R} \to \mathbb{R}^2$ and it sends each real number k to $k \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Problem 1.2 (Matrix-Vector Multiplication)**.**

1. Let us define a ***magic matrix*** to be a 3×3 matrix whose entries are $1, \dots, 9$ in some order, such that in each row, each column, and each of the two diagonals, the three entries add up to the same number. For example, a typical magic matrix is $\begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$. For an arbitrary magic matrix M, try to find all possible values of $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
2. Let us define a ***Sudoku matrix*** to be a 9×9 matrix such that each row is made of $1, \dots, 9$, and each column is made of $1, \dots, 9$, and each of the nine 3×3 "submatrix" is also made of $1, \dots, 9$. For example, a typical Sudoku matrix is

$$\left[\begin{array}{ccc|ccc|ccc} 5 & 3 & 4 & 6 & 7 & 8 & 9 & 1 & 2 \\ 6 & 7 & 2 & 1 & 9 & 5 & 3 & 4 & 8 \\ 1 & 9 & 8 & 3 & 4 & 2 & 5 & 6 & 7 \\ \hline 8 & 5 & 9 & 7 & 6 & 1 & 4 & 2 & 3 \\ 4 & 2 & 6 & 8 & 5 & 3 & 7 & 9 & 1 \\ 7 & 1 & 3 & 9 & 2 & 4 & 8 & 5 & 6 \\ \hline 9 & 6 & 1 & 5 & 3 & 7 & 2 & 8 & 4 \\ 2 & 8 & 7 & 4 & 1 & 9 & 6 & 3 & 5 \\ 3 & 4 & 5 & 2 & 8 & 6 & 1 & 7 & 9 \end{array}\right].$$

For an arbitrary Sudoku matrix M, try to find all possible values of $M \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.



Problem 1.3. Look at this clock. There are twelve arrow vectors here. Assume tha that they are all unit vectors on the xy-plane.

1. Find the sum of all twelve vectors.

2. Find the sum of all vectors except the 2 o'clock vector.

3. Fix the end points of all vectors, but move the starting points of all vectors to the 6 o'clock location, i.e., the 12 o'clock vector is now $\begin{bmatrix}0\\2\end{bmatrix}$, and the 3 o'clock vector is now $\begin{bmatrix}1\\1\end{bmatrix}$, and so on. Now find the sum of all twelve vectors.

Problem 1.4. Write the following vector $\boldsymbol{b}$ as the result of a matrix multiplying a vector.

1. $\boldsymbol{b} = 2\begin{bmatrix}1\\1\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} + 5\begin{bmatrix}1\\0\end{bmatrix}$.

2. $\boldsymbol{b} = 5\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} + 4\begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$.

3. $\boldsymbol{b} = \begin{bmatrix}2b+a+c\\c-b\\a+b+c\\a+b\end{bmatrix}$, where a, b, c are constants.

4. $\boldsymbol{b} = f\left(\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\right)$, where $f : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear map such that $f(\boldsymbol{e}_k) = k\boldsymbol{e}_{6-k}, k = 1, \ldots, 5$.

5. ☕Suppose we know the following rule of the weathers. If it rains one day, then the next day has 0.8 chance of raining. If it does not rain one day, then the next day has 0.3 chance of raining. Suppose today has 0.5 chance of raining, let $\boldsymbol{b} = \begin{bmatrix}p\\1-p\end{bmatrix}$ where p is the chance of raining tomorrow.

Problem 1.5 (Basic geometric concepts in n-dimensional space)**.** Consider the space $\mathbb{R}^n$ and any non-zero $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ with $\boldsymbol{a} \neq k\boldsymbol{b}$. (In this class, non-zero means $\boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{0}$, i.e., the coordinates cannot ALL be zero. But some coordinates are allowed to be zero.) Think of $\boldsymbol{a}, \boldsymbol{b}$ as two distinct points in an n-dimensional space. Note that the "arrow vector" starting from $\boldsymbol{a}$ and ending in $\boldsymbol{b}$ would exactly have coordinates $\boldsymbol{b} - \boldsymbol{a}$.

A line through $\boldsymbol{a}, \boldsymbol{b}$ could be defined like this: we start at $\boldsymbol{a}$, and we move in the arrow direction $\boldsymbol{b} - \boldsymbol{a}$ by an arbitrary amount, and we get a line. So it is the set $\{\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) : t \in \mathbb{R}\}$.

1. Show that the line through $\boldsymbol{a}, \boldsymbol{b}$ is exactly the set $\{s\boldsymbol{a} + t\boldsymbol{b} : s, t \in \mathbb{R} \text{ and } s + t = 1\}$.

2. ☕Show that for any $\boldsymbol{p}$ on the line through $\boldsymbol{a}, \boldsymbol{b}$, then there is a unique pair s, t such that $\boldsymbol{p} = s\boldsymbol{a} + t\boldsymbol{b}$ and $s + t = 1$. (Hint: Suppose $s\boldsymbol{a} + t\boldsymbol{b} = s'\boldsymbol{a} + t'\boldsymbol{b}$, and show that $s = s', t = t'$. Here the fact that $\boldsymbol{a} \neq k\boldsymbol{b}$ is VERY important.)

3. Show that the line segment connecting $\boldsymbol{a}, \boldsymbol{b}$ is exactly the set $\{s\boldsymbol{a} + t\boldsymbol{b} : s, t \in \mathbb{R} \text{ and } s + t = 1 \text{ and } 0 \leq s, t \leq 1\}$. (So insides of a line segment are the "weighted averages" of the two end points.)

4. Give a similar definition of a plane through three points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $\mathbb{R}^n$.

5. Give a similar definition of a triangle with vertices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $\mathbb{R}^n$. (Here I want a subset of $\mathbb{R}^n$ whose elements are points inside the triangle, including the edges of the triangule.)

6. (Read Only) In a similar way you can define k-dimensional ***affine subspaces*** of $\mathbb{R}^n$. The word "affine" simply means "flat", i.e., no curvature. A ***line*** is a name for 1-dimensional affine subspace, a ***plane*** is a name for 2-dimensional affine subspace, and a ***hyperplane*** is a name for $n-1$-dimensional affine subspace.

Problem 1.6 (Hyperplanes and their normal vectors)**.** We know that in $\mathbb{R}^2$, a line can be represented by a "linear" equation. For example, $x + 2y = 2$ would be some line in $\mathbb{R}^2$. What is the situation in higher dimensions?

Consider the space $\mathbb{R}^3$ and a subset $H = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + 3z = 6\}$.

1. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be the intersection of H with the x, y, z-axes respectively. Find the coordinates of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

2. Using the definition in the last problem, show that H is exactly a plane through $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

3. Show that the arrow vectors $\boldsymbol{a} - \boldsymbol{b}, \boldsymbol{b} - \boldsymbol{c}, \boldsymbol{c} - \boldsymbol{a}$ (these are all arrow vectors lying on the plane) are all perpendicular to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (The last vector here is obtained by extracting the coefficients $1, 2, 3$ from the equation of our plane $x + 2y + 3z = 6$.)

4. (Read Only) In general, any "linear" equation in $\mathbb{R}^n$ would yeild a hyperplane. Specifically, if the variables are $x_1, \ldots, x_n$, then for any coefficients $a_1, \ldots, a_n \in \mathbb{R}$ and any constant $b \in \mathbb{R}$, the solutions to the equation $a_1x_1 + \cdots + a_nx_n = b$ is a hyperplane. And the normal vector to this hyperplane is exactly $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Can you see why? (Again, this is not part of the HW.)

2 HW 2 Due Sep 28

Problem 2.1 (Gaussian Eliminations)**.** Solve these by Gaussian elimination, and write out the solution set explicitly. (So don't just stop at RREF for this problem.)

1. $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$.

2. $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$.

3. $\begin{bmatrix}1 & 2\\4 & 5\\7 & 8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\4\\8\end{bmatrix}$.

Problem 2.2 (Situations of the solution set)**.** Find a constant b that satisfies the condition.

1. $\begin{bmatrix}1 & 2\\3 & b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}$ has no solution.

2. $\begin{bmatrix}3 & 2\\6 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}10\\b\end{bmatrix}$ has no solution.

3. $\begin{bmatrix}2 & 5 & 1\\4 & b & 1\\0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix}$ has no solution.

4. $\begin{bmatrix}b & 3\\3 & b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}6\\-6\end{bmatrix}$ has infinitely many solutions.

5. $\begin{bmatrix}2 & b\\4 & 8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}16\\c\end{bmatrix}$ has infinitely many solutions. (For this problem, find constants b, c that satisfy the condition.)

6. $\begin{bmatrix}1 & b & 0\\1 & -2 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has a non-zero solution.

7. $\begin{bmatrix}b & 2 & 3\\b & b & 4\\b & b & b\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has a non-zero solution. (For this problem, find three possible values for b that satisfies the condition.)

Problem 2.3 (Solving a hardcore linear system)**.** ☕For each real value of p, find all solutions to the following system:

$$\begin{bmatrix}p & 1 & 1\\1 & p & 1\\1 & 1 & p\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\p\\p^2\end{bmatrix}.$$

(Hint: first replace the first row by the sum of three rows.)

Problem 2.4 (Find the system given a solution set)**.** For a matrix A, suppose the solution set to $A\boldsymbol{x} = \begin{bmatrix}2\\4\\2\end{bmatrix}$ is $\{\begin{bmatrix}2\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix} \mid s, t \in \mathbb{R}\}$.

1. Can you find the values of $A\begin{bmatrix}2\\0\\0\end{bmatrix}, A\begin{bmatrix}1\\1\\0\end{bmatrix}, A\begin{bmatrix}0\\0\\1\end{bmatrix}$ by simply observing the solution set?

2. What does the subproblem above tell you about the columns of A and their relations? Find A.

Problem 2.5 (Alternative method to the last problem)**.** For a matrix A, suppose the solution set to $A\boldsymbol{x} = \begin{bmatrix}2\\4\\2\end{bmatrix}$ is $\{\begin{bmatrix}2\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix} \mid s, t \in \mathbb{R}\}$. Let us try an alternative method.

1. From the solution set's description, how many free variables do you have? How many dependent variable?

2. Write out RREF of the augmented matrix of this system by simply reading at the solution set.

3. Find A by doing row operations to RREF.

Problem 2.6 (Column operations and variable substitution)**.** Given a linear system $\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$, a change of variables would give rise to a new system. For example, if $x' = x + 2y$ and $y' = 4x + 5y$, then the equations $x + 2y = 3$ and $4x + 5y = 6$ would be equivalent to the equations $x' = 3$ and $y' = 6$. So the new system is $\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$. The augmented matrix will change from $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$ to $\begin{bmatrix}1 & 0 & 3\\0 & 1 & 6\end{bmatrix}$.

Find the elementary COLUMN operations on the augmented matrix that corresponds to the following change of variable.

1. $x' = y, y' = x$.

2. $x' = 2x, y' = y$.

3. $x' = x, y' = x + y$.

4. $x' = x + 1, y' = y$.

3 HW 3 Due Oct 05

Problem 3.1 (Rank-one matrices)**.** If a matrix has all rows parallel, must it also have all columns parallel? (As a convention, we treat $\mathbf{0}$ as parallel to everything.) Give a counter-example or prove why.

Problem 3.2 (Arithmetic sequences in Matrices)**.** For a given matrix, suppose the columns are all arithmetic sequences. (For example, something like $\begin{bmatrix}1 & 2 & 4\\ 3 & 3 & 3\\ 5 & 4 & 2\\ 7 & 5 & 1\end{bmatrix}$.)

1. Assume that the first row of such a matrix is $\boldsymbol{a}^{\mathrm{T}}$ and the second row is $\boldsymbol{b}^{\mathrm{T}}$. Can you deduce all lower rows?

2. Show that such a matrix must have rank at most 2.

Problem 3.3 (Dimensions in matrix multiplications)**.** Say A, B, C are $3\times 5, 5\times 3, 3\times 1$ matrices respectively, then which of $BA, AB, ABAB, BAC, BABC$ are well-defined?

Problem 3.4 (Powers of a matrix)**.** Calculate the following matrix powers.

1. $\begin{bmatrix}11 & 6\\ -20 & -11\end{bmatrix}^2$.

2. $\begin{bmatrix}11 & 6\\ -20 & -11\end{bmatrix}^3$.

3. $\begin{bmatrix}11 & 6\\ -20 & -11\end{bmatrix}^{2020}$.

4. $\begin{bmatrix}11 & 6\\ -20 & -11\end{bmatrix}^{-1}$.

5. $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}^2$.

6. $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}^3$.

7. $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}^{2020}$.

8. $\begin{bmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}^2$.

9. $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^3.$

10. $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{2020}.$

11. $\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{-1}.$

Problem 3.5 (Commutativity trouble)**.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C = A - B$.

1. Calculate $(A+B)^2, A^2 + 2AB + B^2$. Are they the same?
2. Calculate $(AB)^2, A^2B^2$. Are they the same?

Problem 3.6 (Elementary shearings)**.** In $\mathbb{R}^4$, let $X_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}$ be the corresponding 4 by 4 matrix. (Note that the elementary shearing matrix is $E_{ij}^k = I + kX_{ij}$.)

1. Write out X_{13}, X_{32}, X_{12}.
2. Calculate $X_{13}X_{32}$ and $X_{32}X_{13}$. (See if you can do this via the fact that $X_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j^{\mathrm{T}}$, and use associativity. Dot products between these standard basis vectors are super easy.)
3. Verify that $X_{ij}^2 = O$ when $i \neq j$. (Not part of the problem, but you may realize that this implies that $(E_{ij}^1)^k = (I + X_{ij})^k = I + kX_{ij} = E_{ij}^k$.)
4. For any two square matrices A, B, show that $AB = BA$ if and only if $(A-I)(B-I) = (B-I)(A-I)$. (Hence to study the commutativity behavior between elementary shearings, we only need to study the commutativity of these X_{ij}.)

Problem 3.7 (Shifting operators)**.** Let $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

1. Going from A to JA, would J be shifting things up? down? left? right?
2. Going from A to AJ, would J be shifting things up? down? left? right?
3. Given the Pascal's symmetric matrix $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$, calculate $PJ + J^{\mathrm{T}}P$. Can you explain the "coincidence"?
4. Calculate J, J^2, J^3, J^4. (Also note that their appearances is exactly how they would shift the identity matrix.)
5. Show that $I - J$ has inverse $I + J + J^2 + J^3$.

6. Calculate $(J+I)^2, (J+I)^3, (J+I)^4$. (Not part of the problem, but can you see a pattern for the entries? Can you prove the pattern for $(J+I)^k$?)

7. ☕Describe the set of all matrices that commutes with J.

Problem 3.8 (Constraints through matrix multiplications)**.** In each cases, find a matrix B that satisfies the given condition.

1. $AB = 4A$ for all 3 by 3 matrices A.

2. $BA = 4A$ for all 3 by 3 matrices A.

3. Every row of BA is the first row of A for all 3 by 3 matrices A.

4. Every entry of AB is the average of the entries in the corresponding row of A for all 3 by 3 matrices A.

5. $B^2 \neq O$, $B^3 = O$. (Just find any example of such a matrix is enough.)

6. $B\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}B$. Find all possible B for this problem.

Problem 3.9 (Polynomials of a matrix)**.** Suppose for a matrix A we have $A^2 = A$.

1. Simplify the polynomials $A^3 + 2A^2 - A - I$ and $A^2 + 3A + 4I$ into the format $sA + tI$ for some constants $s, t \in \mathbb{R}$. (Read only: note that you can use this strategy to simplify all possible polynomials of A into the format $sA + tI$.)

2. ☕Show that $I + 2A$ is invertible by finding its inverse. (Hint: The inverse is also a polynomial of A.)

4 HW 4 Due Oct 12

Problem 4.1 (Matrix multiplications and row/column operations)**.** Calculate the following matrix multiplications

1. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$

2. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$

3. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$

4. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^k$ for an integer k. Find the formula depending on k.

Problem 4.2 (Gaussian elimination to find inverse)**.** Find inverse matrices. You may start with $\begin{bmatrix} A & I_n \end{bmatrix}$ and work your way to $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$, or you may do whatever that works.

1. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$

2. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$ (Compare this with above. Merely a single-entry difference, yet the inverses look drastically different.)

3. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$ (Horses move like this in chess.)

4. $\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$

5. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$

6. $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$

7. $\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Problem 4.3 (Weird "triangular" matrices)**.** An $n \times n$ matrix is called a ***northwest*** matrix if all entries below the anti-diagonal from $(1, n)$ entry to $(n, 1)$ entry are zero. Similarly, we can define a ***southeast*** matrix. Now, if A is northwest, what about $A^{\mathrm{T}}, A^{-1}, A^2$? What if we multiply a northwest matrix with a southeast matrix?

Problem 4.4 (The place where SM formula can be used)**.** Suppose A has inverse $\begin{bmatrix} 1 & 9 & 0 \\ 1 & 0 & 8 \\ 0 & 1 & 1 \end{bmatrix}$. If I increase the $(1, 3)$ entry of A by 1, what is the new inverse? (Hint: It is easier to use Sherman-Morrison formula.)

Problem 4.5 (Block Operations)**.** Calculate the inverse of the following matrices. (Try not to use the ugly formula. Try do block operations from scratch for practice.)

1. $\begin{bmatrix} I_n & O \\ A & I_m \end{bmatrix}$. (This is lower triangular.)

2. $\begin{bmatrix} O & I_m \\ I_n & A \end{bmatrix}$. (This is block-southeast.)

3. $\begin{bmatrix} O & A \\ B & O \end{bmatrix}$ where A, B are invertible.

4. $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ where A, B are invertible. (This is block upper triangular.)

Problem 4.6. A matrix A is ***skew symmetric*** if $A^{\mathrm{T}} = -A$.

1. If A is 3×3 and skew symmetric, show that $A\boldsymbol{x} = \boldsymbol{v} \times \boldsymbol{x}$ for some $\boldsymbol{v}$ depending only on A. (Here cross product is defined as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$. In $\mathbb{R}^3$, one can verify that the vector $\boldsymbol{v} \times \boldsymbol{w}$ is always perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$. Feel free to do the dot product and check this.)

2. Find a non-zero matrix A such that $\begin{bmatrix} O & A \\ A & O \end{bmatrix}$ is skew symmetric but not symmetric.

3. Find a $2n \times 2n$ skew symmetric matrix A such that $A^2 = -I$ for each n. (Remark: For real numbers, you may never have $x^2 = -1$. But for real matrices, $A^2 = -I$ is possible.)

4. Show that for any square matrix A, you can write it as the sum of a symmetric matrix and a skew symmetric matrix.

Problem 4.7. Suppose the RREF of A is R. Find the RREF of the following matrices.

1. $\begin{bmatrix} A & 2A \end{bmatrix}$.

2. $\begin{bmatrix} A \\ 2A \end{bmatrix}$.

3. $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

4. $\begin{bmatrix} A & A \\ O & A \end{bmatrix}$, where the rank of A is equal to the number of rows of A.

Problem 4.8 (We can study affine maps using matrices as well)**.** A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an ***affine map*** if you can find an $n \times n$ matrix A and a vector $\boldsymbol{b} \in \mathbb{R}^n$ such that $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. (It does not fix the origin, so it is not linear, merely affine.)

1. Show that $\begin{bmatrix} A & \boldsymbol{b} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{x}) \\ 1 \end{bmatrix}$. Let us define M_f as $\begin{bmatrix} A & \boldsymbol{b} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$, and call this the matrix for the affine map.
2. Show that $M_f M_g = M_{f \circ g}$.
3. Show that f is invertible iff A is invertible, and $M_f^{-1} = M_{f^{-1}}$. Also find the block inverse M_f^{-1}.

Problem 4.9 (Hidden Block Diagonal Matrix)**.** Define $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \triangle \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix}$

1. Find a matrix X such that for all A, B, $X(A \triangle B)X^{-1} = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$.
2. Show that $(A_1 \triangle A_2)(B_1 \triangle B_2) = (A_1 B_1) \triangle (A_2 B_2)$. (The first sub-problem might help.)
3. Show that when A, B are invertible, $(A \triangle B)^{-1} = A^{-1} \triangle B^{-1}$. (The first sub-problem might help.)

5 HW5 Due Oct 19

These problems are a bit harder than usual. Better start early.

Problem 5.1. The ***trace*** of an $n \times n$ matrix A is the sum of all of its diagonal entries. We denote it as $\operatorname{tr}(A)$. If you like, you can write $\operatorname{tr}(A) = \sum_{i=1}^n \boldsymbol{e}_i^{\mathrm{T}} A \boldsymbol{e}_i$.

1. Show that $\operatorname{tr}(xA + yB) = x \operatorname{tr}(A) + y \operatorname{tr}(B)$.
2. Show that $\operatorname{tr}(I_n) = n$.
3. Show that $\operatorname{tr}(A) = \operatorname{tr}(A^{\mathrm{T}})$.
4. For a unit vector $\boldsymbol{u}$, show that $\operatorname{tr}(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) = 1$ and $\operatorname{tr}(I_n - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}) = n - 1$. (For projections, trace simply tells you the dimension of the projection range.)
5. Show that, if $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then $\operatorname{tr}(A^{\mathrm{T}} B) = \operatorname{tr}(B A^{\mathrm{T}}) = \sum a_i b_i$. (This is a "dot product" for matrices.)
6. Show that $\operatorname{tr}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}) = \operatorname{tr}(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{v})$.
7. Can you prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ in general? Here A is $m \times n$ and B is $n \times m$. (Hint: break down A into columns and B into rows, and use sub-problem 6. Alternatively, exploit the symmetry of the "dot product" in sub-problem 5, but you will need to generalize the formula in sub-problem 5 to arbitrary non-square matrices first.)
8. Prove that for any square matrices A, B, then $AB - BA$ CANNOT be the identity matrix. (However, in infinite dimensional spaces, $AB - BA = I$ is related to the famous Heisenberg uncertainty principle in physics.) (Hint: Obviously use previous subproblems.)

Problem 5.2. We know permutation matrices corresponds to permutations. For example, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to the permutation $(1, 2, 3) \mapsto (2, 1, 3)$. We see that 3 is unchanged, so this is a ***fixed point*** of our permutation. In this examlpe, we have one fixed point.

1. Find a 5×5 permutation matrix P such that $P^k \neq I$ for all $1 \leq k \leq 5$, but $P^6 = I$.
2. For a permutation matrix P, prove that $\operatorname{tr}(P)$ is the number of fixed elements for the corresponding permutation.
3. Consider the compositions of two permutations $P_1 P_2$ and $P_2 P_1$. Show that they always have the same number of fixed points. (Hint: Obviously previous sub-problems should make your life easier.)

Problem 5.3 (Vector spaces and basis)**.** For the following, if it is not a vector space, just point out which axiom of vector spaces is failed by what vectors. If it is a vector space, you don't need to verify axioms. Just find its dimension instead, and find a basis if it is finite dimensional. (All spaces are over $\mathbb{R}$ unless specifically mentioned.)

1. V is the set of all discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ with its usual addition and scalar multiplication.
2. The solution set of $x^2 + y^2 \leq z^2$ in $\mathbb{R}^3$. (This is the pair of cones but we also include the insides of the two cones.)

3. ☕The set of subsets of $X = \{1, 2, 3\}$, where we define $S + T$ for subsets S, T as the ***symmetric difference***, i.e., $S + T = \{x \in X \mid x \in S \text{ but not } T, \text{ or } x \in T \text{ but not } S\}$.

   This is a vector space defined over $\mathbb{F}_2$, and scalar-vector multiplication is simply $0S = \varnothing$ and $1S = S$ for all subsets S.

   (Here $\mathbb{F}_2 = \{0, 1\}$ such that $0 = 0+0 = 1+1 = 0\times 1 = 1\times 0 = 0\times 0$ and $1 = 0+1 = 1+0 = 1\times 1$. You can also think "even" and "odd" in place of 0 and 1. Then all these calculations would make sense.)

4. $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and V is the space of all matrices that commutes with J. (You can use the results of previous HW directly.)

5. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and V is the space of all matrices that commutes with A.

6. ☕V is the space of all 3×3 magic matrices. (I.e., matrices like the zero matrix, or like $M = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$, where each row, each column, and each of the two diagonal adds up to the same number.)

   (Hint: Flip all columns of M. Flip all rows of M. Take transpose of M. Take reflection of M along the anti-diagonal. Take differences of these things. Take linear combinations of these things. Try until you figure this out.)

Problem 5.4 (Solution space to a differential equation)**.** Consider the space V of all real functions f that solves the differential equation $f''' - 6f'' + 11f' - 6f = 0$. You may use the fact that V is spanned by $\mathrm{e}^x, \mathrm{e}^{2x}, \mathrm{e}^{3x}$.

1. Show that the matrix $A = \begin{bmatrix} \mathrm{e}^0 & \mathrm{e}^0 & \mathrm{e}^0 \\ \mathrm{e}^1 & \mathrm{e}^2 & \mathrm{e}^3 \\ \mathrm{e}^2 & \mathrm{e}^4 & \mathrm{e}^6 \end{bmatrix}$ is invertible.

2. Show that $\mathrm{e}^x, \mathrm{e}^{2x}, \mathrm{e}^{3x}$ are linearly independent, so that they form a basis for V. (Hint: use the previous sub-problem.)

3. ☕Given any $f \in V$, find the coordinates of f under this basis in terms of $f(0), f(1), f(2)$. (Hint: use the matrix A in the first sub-problem. You may simply write A^{-1} without calculating the inverse matrix.)

Problem 5.5 (Quotient Space)**.** Recall that a line in $\mathbb{R}^3$ is a subset $L = \{\boldsymbol{p} + t\boldsymbol{v} : t \in \mathbb{R}\}$ for fixed $\boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^3$. This is a line through $\boldsymbol{p}$ and in the direction of $\boldsymbol{v}$.

Consider the space $V = \mathbb{R}^3$, and fix a line W through the origin. Let V/W be the space of all lines in V parallel to W. For any two lines $L_1, L_2 \in V/W$, we define $L_1 + L_2 := \{\boldsymbol{p} + \boldsymbol{q} : \boldsymbol{p} \in L_1, \boldsymbol{q} \in L_2\}$. And for any scalar $k \in \mathbb{R}$ and any line $L \in V/W$, we define $kL := \{k\boldsymbol{p} : \boldsymbol{p} \in L\}$.

1. Show that for any two lines $L_1, L_2 \in V/W$, we still have $L_1 + L_2 \in V/W$. And for any line $L \in V/W$ and any $k \in \mathbb{R}$, we still have $kL \in V/W$.

2. Show that for any two lines $L_1, L_2 \in V/W$ and any $k \in \mathbb{R}$, we have $k(L_1 + L_2) = kL_1 + kL_2$. (You don't need to verify the other axioms. But in fact we also have other axioms as well, so V/W is a vector space.)

3. Which line in V/W is the "zero vector"?

4. Find the dimension of V/W and find a basis for V/W.

Problem 5.6 (Addition and Scalar multiplication on polynomials mod $p(x)$)**.** (☕This problem is a bit more advanced and abstract.) Let V be the space of all polynomials. Fix a polynomial $p(x) = x^2 + 3x + 2$.

1. For any finitely many polynomials, can you always find another polynomial that is NOT in their span? (This means V is infinite dimensional.)

2. Let W be the collections of all polynomials that contain $p(x)$ as a factor. (By convention, the zero polynomial contains all other polynomial as factors.) Is W a subspace?

3. ☕For any $r(x) \in V$, we write $[r(x)]$ as the subset $\{r(x) + p(x)q(x) : q(x) \in V\}$, i.e., all polynomials whose remainder after divided by $p(x)$ is the same as $r(x)$. We define $[r_1(x)] + [r_2(x)]$ as the subset $\{f_1(x)+f_2(x) : f_1(x) \in [r_1(x)]$ and $f_2(x) \in [r_2(x)]\}$. For $k \neq 0$, we define $k[r(x)]$ as the subset $\{kf(x) : f(x) \in [r(x)]\}$ for scalar $k \in \mathbb{R}$, and we define $0[r(x)] = [0]$. Show that $[r_1(x)]+[r_2(x)] = [r_1(x)+r_2(x)]$ and $k[r(x)] = [kr(x)]$.

4. ☕We use V/W to denote the set of all subsets $[r(x)]$ with addition and scalar multiplication as specified above. This is a finite dimensional vector space (you don't need to verify this, but feel free to do so). Find its dimension and find a basis.

6 HW6 Due Oct 26

Problem 6.1 (Space of matrices)**.** Consider $M_{2\times 2}$ be the space of 2×2 matrices. We fix a basis $X_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, X_2 = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, X_3 = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, X_4 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$. We also fix a matrix $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

1. Consider the linear map $L_A : M_{2\times 2} \to M_{2\times 2}$ that sends X to AX. Write out the matrix for this linear map under the given basis. (Preferably as a block matrix. The block form would make it easier to imagine generalizations to higher dimensions.)

2. Consider the linear map $R_A : M_{2\times 2} \to M_{2\times 2}$ that sends X to XA. Write out the matrix for this linear map under the given basis. (Preferably as a block matrix. The block form would make it easier to imagine generalizations to higher dimensions.)

3. Calculate the 4×4 matrix multiplication $L_A R_A$ and $R_A L_A$. (You may use block matrix multiplication to facilitate the process.)

4. Can you see why $L_A R_A = R_A L_A$ without any calculation? (Hint: Think about the meaning of these linear maps. L_A means multiplying A to X from the left, while R_A means multiplying A to X from the right. Which law of matrix multiplication is this?)

5. Find a basis of $M_{2\times 2}$ made of invertible matrices, and find the change of coordinate matrix to this new basis.

Problem 6.2 (Chang of basis and linear maps)**.** Let V be a space with three bases $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and W be a space with two bases $\mathcal{C}_1, \mathcal{C}_2$, and a bijective linear map $L : V \to W$. We use C_{ij} to mean a change of coordinate map from $\mathcal{B}_i$ to $\mathcal{B}_j$ or from $\mathcal{C}_i$ to $\mathcal{C}_j$, and L_{ij} to mean the matrix for L from $\mathcal{B}_i$ to $\mathcal{C}_j$. Which of the following formula about matrices L_{ij}, C_{ij} are true? (No proof needed for correct ones. But for wrong ones, just briefly point out what is wrong with it.)

1. $L_{11} = C_{21} L_{12}$.

2. $L_{11}^{-1} = L_{12}^{-1} C_{21}$.

3. $L_{22} = C_{12} L_{11} C_{12}^{-1}$.

4. $L_{31} = C_{21} L_{12} C_{31} C_{23} L_{21}^{-1} C_{21} L_{12} C_{21} C_{32}$.

Problem 6.3 (Complex numbers)**.** Consider the complex plane $\mathbb{C}$ as a vector space over $\mathbb{R}$. We know it is a two dimensional real vector space with a basis $1, \mathrm{i}$.

1. For the complex number $w = 2 + 3\mathrm{i}$, consider the map $M_w : \mathbb{C} \to \mathbb{C}$ that maps input z to output wz. Find the matrix for M_w under the basis $1, \mathrm{i}$.

2. For any two complex numbers w, z, do the matrix multiplication to verify that $M_w M_z = M_{wz}$.

3. Can you see without computation that $M_w M_z = M_{wz}$? Which law of complex number multiplication is this? (Hint: Think about the meaning of these linear maps.)

4. Consider $\mathbb{F} = \{\begin{bmatrix}a & -b\\ b & a\end{bmatrix} \mid a, b \in \mathbb{R}\}$, and find a bijective map $\phi : \mathbb{C} \to \mathbb{F}$ that satisfies $\phi(w+z) = \phi(w) + \phi(z), \phi(wz) = \phi(w)\phi(z)$. (Hint: surely the previous sub-problems are building up to something, yes?)

5. (Reading only) ϕ above is called a field isomorphism. It shows that $\mathbb{F}$ is also a field, and $\mathbb{F}$ and $\mathbb{C}$ are essentially the same field, just with different names for each element. Working on the field $\mathbb{C}$ is identical to working on the field $\mathbb{F}$. In a sense, you may think of "complex numbers" as a specific kind of 2×2 real matrices in disguise.

Problem 6.4 (Equation space and its subspace)**.** let V be the space of linear equations on three variables, with elements such as $ax + by + cz = d$ for arbitrary real numbers a, b, c, d. So the vectors here are again equations. We can do linear combination of these equations, i.e., $2(2x + 3y - z = 3) + 3(x - y = 2) = (7x + 3y - 2z = 12)$.

(Note that if $x = 1$ and $x = 2$ are both elements of V, and V is a vector space, then V must include its difference, i.e., $0 = 1$. This equation $0 = 1$ is also an element of V, and it is indeed of the form $ax+by+cz = d$ where $a = b = c = 0$ and $d = 1$.)

1. Find a basis of V, and find the dimension of V.
2. Suppose we have a basis made of equations $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. What is the solution set when all equations $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are satisfied?
3. Fix a point $\boldsymbol{p}$ in $\mathbb{R}^3$. Let W be the subset of V, made of all equations that contains this point $\boldsymbol{p}$ in its solution set. Is W a subspace? Prove or give counter examples. (Hint: to show that something is a subspace, you just need to show that $\boldsymbol{v} + \boldsymbol{w}$ and $k\boldsymbol{v}$ are still in W when $\boldsymbol{v}, \boldsymbol{w}$ is in W.)
4. Consider $x = 1, y = 2, z = 3, x + y + z = 7$. Are they linearly independent? Prove or provide a linear relation. (Note: you might want to try this proof for fun. First prove that the first three are linearly independent. For the last equation, take the subspace of all equations whose solution set contains the point $\begin{bmatrix}1\\2\\3\end{bmatrix}$. Then this subspace contains $x = 1, y = 2, z = 3$, but fails to contain the last equation $x + y + z = 7$. This gives you a way to use independence extension lemma.)
5. Consider $x = 1, x = 2, x = 3$. Are they linearly independent? Prove or provide a linear relation.
6. (Not part of the HW) If you are curious, you may consider ways to generalize the statements here. What if we fix a line L, and let W be made of all equations that contains this line L in its solution set? What if we include more variables and go to higher dimensions?

Problem 6.5 (Effective number of chemical reactions)**.** We have three chemical reactions when burning charcoals, $C + O_2 = CO_2$, $2C + O_2 = 2CO$, and $2CO + O_2 = 2CO_2$. Note that these are NOT really equations. Rather, they reflect the process that the left hand side is transformed into the right hand side during the chemical reaction involved. So we don't solve them. Rather, we do their linear combinations according to how much of each reaction occurred. Then we shall see what's converted into what in total.

For example, $(2C + O_2 = 2CO) + (2CO + O_2 = 2CO_2) = 2(C + O_2 = CO_2)$, and this means first we burn C to get CO, and then all the CO are burned and converted into CO_2. The total effect of combining $2C + O_2 = 2CO$ and $2CO + O_2 = 2CO_2$ is to do $C + O_2 = CO_2$ twice.

Let W be the space of linear combinations of the molecules C, CO, CO_2, O_2.

1. The chemical reaction $C + O_2 = CO_2$ is a map sending each $\boldsymbol{w} \in W$ to $\boldsymbol{w} - C - O_2 + CO_2$. Is this map linear?
2. Consider the maps corresponding to the chemical reactions $C + O_2 = CO_2$, $2C + O_2 = 2CO$, and $2CO+O_2 = 2CO_2$. We define their linear combination in the sense of $(2C+O_2 = 2CO)+(2CO+O_2 = 2CO_2) = 2(C + O_2 = CO_2)$. Are they linearly independent? What is the dimension of their span?
3. Let $M = \begin{bmatrix}-1 & -2 & 0\\ 0 & 2 & -2\\ 1 & 0 & 2\\ -1 & -1 & -1\end{bmatrix}$, i.e., the three columns corresponds to the changes induced by the three chemical reactions on W with basis C, CO, CO_2, O_2. Find invertible matrices R, C such that RMC is in rank normal form, and find the rank r of M. (So we see that the three equations $C + O_2 = CO_2$, $2C + O_2 = 2CO$, and $2CO + O_2 = 2CO_2$ only have r effective equations.)

Problem 6.6 (Algorithm to find full-rank decomposition)**.** We know that, given any $m \times n$ matrix A of rank r, then $A = BC$ where B is $m \times r$ and injective, and C is $r \times n$ and surjective.

Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

1. Delete all non-pivotal columns of A, and we obtain a matrix B. Find B. Why must the rank of A equal to the rank of B?

2. Find the reduced row echelon form of A and delete all zero rows, and we obtain a matrix R. Can you find the rank of R without any further computation?

3. Compute BR, what do you have? (This gives an algorithm to perform full rank decompositon, if you ever need to.)

4. (Not part of the homework) Can you see why you have this phenomena? This process can be generalized to an arbitrary A.

Problem 6.7 (Meaning of rank-one decomposition)**.** Suppose we have seven people, A, B, ..., G, and they watched six videos on bilibili, a, b, ..., f. Suppose for each video, they can click "like" or "dislike", and this data is collected into the following matrix.

$$M = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Here 1 means the corresponding person likes the corresponding video, -1 means dislike, and 0 means no response.

Let us suppose that a, b, c are all educational videos, while c, d, e, f are all funny videos. So we have two categories for a video: educational or funny. Note that each person seems to react reasonably consistent about videos in each category. The exception is c, which is both educational and funny, and whether a person likes or dislikes seems to be influenced by other factors.

1. What is the rank of this 7×6 matrix?

2. Build a matrix X such that its two columns corresponds to the two categories of videos, and its seven rows corresponds to the seven people, and the entries records whether the corresponding person likes, dislikes or has no response to the corresponding category in general.

3. Build a matrix Y such that its two rows corresponds to the two categories of videos, and its six columns corresponds to the six videos. And an entry is 1 if the video is in this category, zero if this video is NOT in this category.

4. Check to see that $M - XY$ has rank 1. (This represents portions of M that cannot be explained by categories of the videos.)

5. ☕Write M as the sum of three rank one matrices, where the first one reflects how educational videos contributes to M, the second one reflects how funny videos contributes to M, and the third one reflects how other factors contributes to M.

7 HW7 Due Nov 9

Problem 7.1 (More Inclusion-Exclusion Principals)**.**

1. Prove that for subsets X, Y, Z, we have $|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$.
2. Find an example of subspaces X, Y, Z such that $\dim(X + Y + Z) \neq \dim(X) + \dim(Y) + \dim(Z) - \dim(X \cap Y) - \dim(X \cap Z) - \dim(Y \cap Z) + \dim(X \cap Y \cap Z)$.
3. Prove that for subspaces X, Y, Z, we have $\dim(X + Y + Z) \leq \dim(X) + \dim(Y) + \dim(Z) - \dim(X \cap Y) - \dim(X \cap Z) - \dim(Y \cap Z) + \dim(X \cap Y \cap Z)$. (Hint: Use the regular IEP for subspaces repeatedly. Prove that $(X \cap Z) + (Y \cap Z) \subseteq (X + Y) \cap Z$.)
4. Find matrices A, B, C such that $\operatorname{rank}(ABC) > \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B)$.

Problem 7.2 (Finding Kernel Algorithm)**.** Let me show you a magical way to find a basis for the kernel of a matrix. Given a matrix A, how can we find $\operatorname{Ker}(A)$? Here is an algorithm that always works.

First, we perform Gaussian elimination to get $\operatorname{RREF}(A)$, say $\operatorname{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Next, we throw away zero rows and get $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. Note that at this stage, we always have more columns than rows (can you see why?).

Next we add zero-rows so that all the pivots are on the diagonal, so we have a square matrix $A' = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Finally, I claim that non-zero columns of $A' - I = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is a basis for $\operatorname{Ker}(A)$. This should always work.

1. For any matrix A, we can obtain a square matrix A' according to the algorithm above. Show that $\operatorname{Ker}(A) = \operatorname{Ker}(A')$.
2. For $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, show that non-zero columns of $\begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is indeed a basis for $\operatorname{Ker}(A)$. (You would need to verify linear independence, then check the number of vectors, say by rank-nullity)
3. For the matrix $A' = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we pick its diagonal entries to form a diagonal matrix $D = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$. Verify that $DA' = A'$ and $A'D = D$. (No need to prove these: this is true whenever A' is upper triangular, and for each i either the i-th column of A' is $\boldsymbol{e}_i$, or the i-th row of A' is $\mathbf{0}^{\mathrm{T}}$. And this is always the case if we obtain A' from some RREF as described in the algorithm of this problem.)

4. Suppose A' is any upper triangular matrix, and its diagonal entries form a diagonal matrix D such that $DA' = A'$ and $A'D = D$, show that $\mathrm{Ran}(A' - I) \subseteq \mathrm{Ker}(A')$. (Hint: Prove that $(A')^2 = A'$ using the given conditions, and go from here to see that $A'(A' - I) = 0$.)

5. If U is an upper triangular matrices, and it has k non-zero diagonal entries, show that $\mathrm{rank}(U) \geq k$. (Hint: Repeatedly use $\mathrm{rank}\begin{bmatrix} A & B \\ & C \end{bmatrix} \geq \mathrm{rank}(A) + \mathrm{rank}(C)$.)

6. (Not part of HW) Above information should contain enough ideas to prove that our algorithm is always valid. Feel free to finish the rest of the proof if you want.

Problem 7.3 (Applications of last problem)**.** Given a subspace V of $\mathbb{R}^n$, how to find its orthogonal complement? First, we find a basis $A = \begin{bmatrix} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_k \end{bmatrix}$ for V. Then $V^\perp = \mathrm{Ran}(A)^\perp = \mathrm{Ker}(A^\mathrm{T})$. Then we use the kernel finding algorithm in the last problem to find a basis for $\mathrm{Ker}(A^\mathrm{T})$.

1. Find a basis for the kernel of $\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ & 1 & 4 & 0 & 5 \\ & & & 1 & 6 \end{bmatrix}$.

2. In $\mathbb{R}^3$, find a basis for the orthogonal complement of $\mathrm{span}(\begin{bmatrix} a \\ b \\ c \end{bmatrix})$ where $a \neq 0$. (This is a basis of the subspace with normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.)

3. The chess matrix is
$$A = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}.$$
Here r, n, b, q, k, p are distinct non-zero real numbers. Find a basis for all four fundamental subspaces of A.

4. ☕Suppose $A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}$ where F, G are both $n \times n$, and the four fundamental subspaces of A and B are identical. Prove that $F = G$.

Problem 7.4 (Orthogonal basis made of coordinates ± 1)**.** A ***Hadamard matrix*** is a matrix H whose entries are all ± 1, and the columns are mutually orthogonal (but they don't have to be unit vectors). (When n is a power of 2, then an example of Hadamard matrix would be the matrix made by the Haar wavelet basis.)

1. For any $n \times n$ Hadamard matrix, compute $H^\mathrm{T} H$.

2. If H is a Hadamard matrix, show that $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ is also a Hadamard matrix.

3. Can you find a 3×3 Hadamard matrix? Find it or show why not.

4. Let H be any Hadamard matrix. Then if we permute the rows and columns, or if we negate some rows and columns, prove that the result is still a Hadamard matrix. (If two Hadamard matrices can be obtained from each other like this, then we say they are equivalent.)

5. ☕Prove that all 4×4 Hadamard matrices are equivalent.

6. (Read only) It is conjectured that a Hadamard matrix should exist for all $(4k) \times (4k)$ matrices. However, this remains unproven to this date. For examlpe, is there a 668×668 Hadamard matrix? We don't know the answer yet, as far as I can tell. The Haar wavelet basis gives a Hadamard matrix whenever n is a power of 2. But, for examlpe, try to find a Hadamard matrix when $n = 12$, if you want a challenge. And on the question of equivalence, in general, Hadamard matrices of the same size could be inequivalent. For example, there are five inequivalent 16×16 Hadamard matrices.

Problem 7.5 (A useful "non-inner product" space)**.** Consider the space $\mathbb{R}^4$ where we define $\langle \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{bmatrix} \rangle = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2$. This is NOT an inner product, but nevertheless, this is the structure used by special relativity. This is sometimes called a ***Minkovski space-time***. We say $\boldsymbol{v} \perp \boldsymbol{w}$ if $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$ under the definition above.

1. Find a matrix D such that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^{\mathrm{T}} D \boldsymbol{w}$. Is D symmetric? Positive Definite?

2. ☕For an arbitrary subspace $W \subseteq \mathbb{R}^4$, we define its "orthogonal complement" to be $W^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^4 | \boldsymbol{v} \perp W\}$. Show that we always have $\dim W + \dim W^{\perp} = 4$. (Hint: use rank-nullity.)

3. Find a subspace strictly contained in its own "orthogonal complement". ("strictely contained" means they are not equal.)

4. Find a basis for the Minkovski spacetime made of vectors of "length" zero.

8 HW8 Due Nov 16

Problem 8.1 (Left over from the class)**.** For an $m \times n$ matrix A, let $A = BC$ be the full rank decomposition. As we shall see, the full rank decomposition is, in its essence, simply finding basis for all subspaces related to A.

1. Show that columns of B form a basis for $\mathrm{Ran}(A)$. In particular, $\mathrm{Ran}(B) = \mathrm{Ran}(A)$. (Hint: C is surjective.)

2. Show that columns of C^{T} form a basis for $\mathrm{Ran}(A^{\mathrm{T}})$. In particular, $\mathrm{Ran}(C^{\mathrm{T}}) = \mathrm{Ran}(A^{\mathrm{T}})$.

3. Let $B = QR$ be the QR decomposition where Q has orthonormal columns, and R is upper triangular with positive diagonal entries. Show that $\mathrm{Ran}(A) = \mathrm{Ran}(Q)$. (This is the claim we used in class. You don't need to, but you can try to find three different proofs here. First perspective is to realize that this is a special case of the first subproblem. Second perspective is to think of R as a change of basis in the domain. Third perspective is to think of the QR decomposition as the Gram-Schmidt orthogonalization, so we go from a basis to a subspace to an ONB of the same subspace.)

4. ☕Given any $m \times n$ matrix A, let X be a matrix whose columns form a basis for $\mathrm{Ran}(A)$, and let Y be a matrix such that the columns of Y^{T} form a basis for $\mathrm{Ran}(A^{\mathrm{T}})$. Show that you can find invertible matrix T such that $A = XTY$. (Hint: Use the first two subproblems, and the fact that basis transition matrices are invertible. Note that this explains the non-uniqueness of full rank decomposition.)

Problem 8.2. Suppose $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ form an orthonormal basis for an inner product space V. Either prove that the followings are orthonormal basis, or perform Gram-Schmidt to them.

1. $n = 3$ and we have $\boldsymbol{b}_1 = \frac{1}{3}(2\boldsymbol{a}_1 + 2\boldsymbol{a}_2 - \boldsymbol{a}_3), \boldsymbol{b}_2 = \frac{1}{3}(2\boldsymbol{a}_1 - \boldsymbol{a}_2 + 2\boldsymbol{a}_3), \boldsymbol{b}_3 = \frac{1}{3}(-\boldsymbol{a}_1 + 2\boldsymbol{a}_2 + 2\boldsymbol{a}_3)$.

2. $n = 5$, and $\boldsymbol{b}_1 = \boldsymbol{a}_1 + \boldsymbol{a}_5, \boldsymbol{b}_2 = \boldsymbol{a}_1 - \boldsymbol{a}_2 + \boldsymbol{a}_4, \boldsymbol{b}_3 = 2\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3$.

Problem 8.3. For the following matrices, find their QR decomposition, and find the matrix of orthogonal projection to their range.

1. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Problem 8.4. Consider the following functions.

1. $f(x, y, z) = 2x^2 + 4y^2 + 3z^2 + 4xy + 4yz$. Find a symmetric matrix A such that $f(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

   Is A positive definite? If so, find the Chelosky decomposition of A, and then complete the squares for $f(x, y, z)$.

2. $f(x, y) = 4x^2 + 4xy + 2y^2 + 2y + 1$. Find a symmetric matrix A such that $f(x, y) = \begin{bmatrix} x & y & 1 \end{bmatrix} A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

   Is A positive definite? If so, find the Chelosky decomposition of A, and then complete the squares for $f(x, y, z)$.

Problem 8.5. Some interesting facts about orthogonal projections.

1. If P is an orthogonal projection, show that the length squared of its i-th column is its (i,i) entry.
2. If A, B are orthogonal projections, show that $AB = BA$ if and only if AB is an orthogonal projection. Do you know to what space? (In terms of $\mathrm{Ran}(A)$ and $\mathrm{Ran}(B)$.)
3. ☕If A, B are orthogonal projections, show that $A+B$ is an orthogonal projection iff $\mathrm{Ran}(A) \perp \mathrm{Ran}(B)$. Do you know to what space? (In terms of $\mathrm{Ran}(A)$ and $\mathrm{Ran}(B)$.)

Problem 8.6 (Linear Regression)**.** Suppose there are four points $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ on the xy-plane $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 20 \end{bmatrix}$. Consider the following models.

1. We wish to find the best line parallel to the x-axis to fit the data. I.e., our model is $y = b$, and we want to find the b to minimize $\sum_{i=1}^4 |y_i - b|^2$. Find matrix A and vector $\boldsymbol{y}$ such that $A^{\mathrm{T}}Ab = A^{\mathrm{T}}\boldsymbol{y}$ gives the best b as the solution.
2. We wish to find the best line through the origin to fit the data. I.e., our model is $y = kx$, and we want to find the k to minimize $\sum_{i=1}^4 |y_i - kx_i|^2$. Find matrix A and vector $\boldsymbol{y}$ such that $A^{\mathrm{T}}Ak = A^{\mathrm{T}}\boldsymbol{y}$ gives the best k as the solution.
3. We wish to find the best parabola to fit the data. I.e., our model is $y = ax^2+bx+c$, and we want to find the a, b, c to minimize $\sum_{i=1}^4 |y_i - ax_i^2 - bx_i - c|^2$. Find matrix A and vector $\boldsymbol{y}$ such that $A^{\mathrm{T}}A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{\mathrm{T}}\boldsymbol{y}$ gives the best a, b, c as the solution.

Problem 8.7. Calculate the following determinants.

1. $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.
2. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & -1 & 1 & -1 \end{bmatrix}$. (This is a Vandermonde matrix for $1, 2, 3, -1$. The answer should be $\pm(3-2)(3-1)(3-(-1))(2-1)(2-(-1))(1-(-1))$, i.e., plus or minus the product of all possible differences of these values.)
3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$. (This is an anti-circulant matrix. The next semester linear algebra course will talk about this.)
4. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 0 & 0 & 0 \\ 13 & 14 & 0 & 0 & 0 \\ 15 & 16 & 0 & 0 & 0 \end{bmatrix}$. (Hint: Don't calculate. Just stare at it until you see the aswer.)

5. $\begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$. The matrix is $n \times n$.

9 HW9 Due Nov 23

Problem 9.1. Let A be a 3×3 matrix $\begin{bmatrix}\boldsymbol{a}_1^{\mathrm{T}}\\ \boldsymbol{a}_2^{\mathrm{T}}\\ \boldsymbol{a}_3^{\mathrm{T}}\end{bmatrix}$ with $\det(A)=5$. Find the determinant of the matrices below.

1. $2A$.
2. $-A$.
3. A^2.
4. A^{-1}.
5. A^{T}.
6. $\begin{bmatrix}\boldsymbol{a}_1^{\mathrm{T}}-\boldsymbol{a}_3^{\mathrm{T}}\\ \boldsymbol{a}_2^{\mathrm{T}}-\boldsymbol{a}_1^{\mathrm{T}}\\ \boldsymbol{a}_3^{\mathrm{T}}-\boldsymbol{a}_2^{\mathrm{T}}\end{bmatrix}$. (Hint: Do this in two ways if you like. You can do this with row operations, or via decomposition $\det(EA)=\det(E)\det(A)$.)
7. $\begin{bmatrix}\boldsymbol{a}_1^{\mathrm{T}}+\boldsymbol{a}_3^{\mathrm{T}}\\ \boldsymbol{a}_2^{\mathrm{T}}+\boldsymbol{a}_1^{\mathrm{T}}\\ \boldsymbol{a}_3^{\mathrm{T}}+\boldsymbol{a}_2^{\mathrm{T}}\end{bmatrix}$. (Hint: Do this in two ways if you like. You can do this with row operations, or via decomposition $\det(EA)=\det(E)\det(A)$.)

Problem 9.2. Let $A=\begin{bmatrix}\boldsymbol{a}_1 & \dots & \boldsymbol{a}_n\end{bmatrix}$ be any $n\times n$ matrix. Geometrically speaking, $|\det(A)|$ is the absolute volumn of the parallelotope A, and $\|\boldsymbol{a}_i\|$ is the length of the i-th edge. As you can imagine, the volumn $|\det(A)|$ should be the product $\|\boldsymbol{a}_1\|\dots\|\boldsymbol{a}_n\|$ when all edges are mutually orthogonal. And when the edges are NOT mutualy orthogonal, then intuitively $|\det(A)|$ should be strictly less than the product $\|\boldsymbol{a}_1\|\dots\|\boldsymbol{a}_n\|$.

This is the famous Hadamard inequality $|\det(A)|\leq\|\boldsymbol{a}_1\|\dots\|\boldsymbol{a}_n\|$. We prove this inequality in this problem.

1. Suppose A is invertible and we have QR decomposition $A=QR$ where $R=\begin{bmatrix}\boldsymbol{r}_1 & \dots & \boldsymbol{r}_n\end{bmatrix}$ is upper triangular. Show that $\|\boldsymbol{r}_i\|=\|\boldsymbol{a}_i\|$.
2. Show that in the set up above, $\det(R)\leq\|\boldsymbol{r}_1\|\dots\|\boldsymbol{r}_n\|$.
3. Prove the Hadamard inequality $|\det(A)|\leq\|\boldsymbol{a}_1\|\dots\|\boldsymbol{a}_n\|$.

Problem 9.3. Prove or find counter examples.

1. We always have $\det(AB-BA)=0$ for any square matrices A,B.
2. We always have $\det(-A)=-\det(A)$ for square matrices A.
3. If n is odd and A is $n\times n$ and skew-symmetric, then A is not invertible. (Hint: take anther look at the last sub-problem.)
4. ☕Suppose we have LDU decomposition $A=LDU$. Let A_i be the upper left $i\times i$ block of A, and let d_i be the i-th diagonal entry of D. Then $d_i=\frac{\det(A_i)}{\det(A_{i-1})}$. (Hint: also write L,D,U in blocks.)

Problem 9.4 (Don't do these things)**.** What's wrong with these arguments?

1. For block matrix we have $\det\begin{bmatrix}A & B\\ C & D\end{bmatrix}=AD-BC$. (Not part of the HW: What should the correct formula be if A is invertible?)

2. We have $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}) = \det(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}) = \frac{ad-bc}{ad-bc} = 1$.

3. If $AB = -BA$, then $\det(A)\det(B) = -\det(B)\det(A)$, so $2\det(A)\det(B) = 0$. So one of A, B is not invertible.

Problem 9.5. Calculate the following determinants.

1. A is 3×3 with determinant 5. Find the determinant of the cofactor matrix C for A. (Hint: What is $C^{\mathrm{T}}A$?)

2. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 5 \end{bmatrix}$. (You are not required to do the generalization of this, but feel free to try to find the determinant of $\begin{bmatrix} a_1 & 1 & \dots & 1 \\ 1 & a_2 & & \\ \vdots & & \ddots & \\ 1 & & & a_n \end{bmatrix}$. Also feel free to use two methods: Laplace expansion or low rank perturbation.)

Problem 9.6 (Leibniz formula)**.** Use Leibniz formula (the big formula) to help with the following problems.

1. Consider $\det \begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$. This is a polynomial in x. What is the coefficient for x^4? What is the coefficient for x^3?

2. Suppose we multiply each (i, j) entry of a 4×4 matrix A by j. For each term in the big formula of $\det(A)$, how would this term change? How would $\det(A)$ change?

3. Consider $A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ e & 0 & f & 0 \\ 0 & g & 0 & h \end{bmatrix}$, where all letter variables are non-zero. In the big formula of $\det(A)$, how many non-zero terms are there? Can you factorize the determinant? (Hint: This is a "hidden" block diagonal matrix.)

Problem 9.7 (Pascal Matrices)**.** Let P_n be the $n \times n$ symmetric Pascal's matrix. (E.g., the 4×4 version is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$.) Similarly, let L_n, U_n be the $n \times n$ lower triangular and upper triangular Pascal's matrix. (E.g., the 4×4 version is $L_n = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix}$, and we always have $U_n = L_n^{\mathrm{T}}$.)

For this problem, you may freely use the fact that P_n has LU decomposition $P_n = L_n U_n$.

1. Find $\det(P_n)$.

2. Let A_n be obtained by reducing the lower right entry of P_n by 1. Find $\det(A_n)$. (Hint: Can you feel how cofactor is involved here?)

Problem 9.8 (Laplace expansion)**.** Use Laplace expansion or cofactors to help with the following problems.

1. A Hessenberg matrix is a matrix that is "almost" triangular, except for an extra diagonal of entries. For example, the matrices $H_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is Hessenberg. Define the $n \times n$ matrix $H_n = \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{bmatrix}$. Show that $\det(H_n)$ is the Fibonacci sequence.

2. A tridiagonal matrix is a matrix that is simultaneously upper Hessenberg and lower Hessenberg. For examlpe, $S_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ is tridiagonal. Generalize S_4 into $n \times n$ matrices. Find the inductive formula for $\det(S_n)$. How is this related to the Fibonacci sequence?

Problem 9.9 (Determinant and inverse matrix)**.** Let $f = \ln(ad - bc)$.

1. Find $\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c}, \frac{\partial f}{\partial d}$.

2. Show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial f}{\partial c} & \frac{\partial f}{\partial d} \end{bmatrix}^{\mathrm{T}}$.

3. ☕Is this a coincidence? If not, can you generalize this to higher dimensions? (Hint: What is the derivative $\frac{\partial \ln(\det(A))}{\partial a_{ij}}$? Use chain rule and see what happens.)

10 HW10 Due Nov 30

Problem 10.1. Find all eignevalues and all corresponding eigenvectors for the following matrices. (Including complex eigenvalues and eigenvectors.)

1. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

3. $\begin{bmatrix} 10 & 11 \\ 3 & 18 \end{bmatrix}$.

4. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

5. $\begin{bmatrix} b & -a \\ a & b \end{bmatrix}$, here $,b \in \mathbb{R}$.

6. $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$.

Problem 10.2 (Companion Matrix)**.** Given a polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we can define the ***companion matrix*** to be $M_p = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & \ddots & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$. Note that this is an $n \times n$ square matrix.

1. Find the characteristic polynomial and the eigenvalues of M_p when $p(x) = x^2 + 1$.
2. Find the characteristic polynomial and the eigenvalues of M_p when $p(x) = x^2 - 3x + 2 = (x-1)(x-2)$.
3. Guess the eigenvalues of M_p when $p(x) = x^3 - 6^2 + 11x - 6 = (x-1)(x-2)(x-3)$.
4. ☕Can you prove the relation between eigenvalues of M_p and roots of p? (Hint: expand along the last column when computing $\det(xI - M_p)$.)
5. (Read Only) For computers, to find all roots of a polynomial $p(x)$, one way is to calculate the Schur decomposition of M_p.

Problem 10.3. Find all possible values of the unknown constants given the condition.

1. $A = \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}$ and the eigenvalues are $4, 7$.

2. $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$ and the eigenvalues are $2, 3, 4$.

3. $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & x & -2 \\ -2 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & & \\ & 2 & \\ & & y \end{bmatrix}$, and A, B have the same eigenvalues.

4. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a \end{bmatrix}$ is invertible, and $\boldsymbol{x} = \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix}$ is an eigenvector of A.

Problem 10.4 (Hadamard Matrix)**.** Recall that we can define the Hadamard matrix inductively as this: we set $H_1 = \begin{bmatrix} 1 \end{bmatrix}$, and $H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$. Then it is easy to see by induction that all entries of H_n are ± 1, and all columns of H_n are mutually orthogonal, and $H_n^{\mathrm{T}} = H_n$. Let us now find its eigenvalues.

Throughout this problem, n will be some power of 2.

1. Show that $H_n^2 = nI$. What does this tell you about the eigenvalues of H_n?
2. Find the algebraic multiplicity of all eigenvalues. (Hint: consider $\operatorname{tr}(H_n)$.)
3. Geometrically show that $\frac{1}{\sqrt{2}} H_2$ is a reflection, and find two linearly independent eigenvector this way.
4. If $H_n \boldsymbol{v} = \lambda \boldsymbol{v}$, find an eigenvector of H_{2n} for the eigenvalue $\sqrt{2}\lambda$ in terms of $\boldsymbol{v}$, and find an eigenvector of H_{2n} for the eigenvalue $-\sqrt{2}\lambda$ in terms of $\boldsymbol{v}$.

Problem 10.5 (Real Discrete Fourier Transform)**.** Suppose A is a 4×4 matrix with diagonal entries 4 and non-diagonal entries -1. Let H_4 be the Hadamard matrix.

1. Calculuate AH_4.
2. Calculate $H_4^{-1} A H_4$.
3. Find $\det(A), A^{-1}$.

Problem 10.6 (Discrete Fourier Transform)**.** Consider $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$. Let $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{bmatrix}$.

(This F_4 is the 4×4 Fourier matrix, and it represents a discrete version of Fourier transform. Continuous Fourier transform is useful for the study of periodic functions, whereas here we shall see that discrete Fourier transform is useful for the study of "periodic matrices".)

1. Calculuate AF_4.
2. Find the D, X in the diagonalization $A = XDX^{-1}$.
3. Find $\det(A), A^{-1}$.

Problem 10.7 (Pavel Grinfeld Problem)**.** Calculate $\begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}^{2017}$. (Hint: Stare at it, until you see two eigenvalues and their eigenvectors. Use trace to find the last eigenvalue.)

Problem 10.8 (Large power is unstable)**.** Large power of a large matrix is large? Prove the following problems. Hint: Diagonalization.

1. All entries of $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{1024}$ are larger than 10^{700}. (You may use the fact that $\frac{1}{3}(5^{1024} - 2^{1024}) > 10^{700}$.)

2. $\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}^{1024} = I_2$

3. $\begin{bmatrix} -5 & -7 \\ 3 & 4 \end{bmatrix}^{1024} = \begin{bmatrix} -5 & -7 \\ 3 & 4 \end{bmatrix}$.

4. (Read Only) You can use matrix calculator to verify that all entries of $\begin{bmatrix} -5 & -6.9 \\ 3 & 4 \end{bmatrix}^{1024}$ are less than 10^{-70} in absolute value. Furthermore, all entries of $\begin{bmatrix} -5 & -7.1 \\ 3 & 4 \end{bmatrix}^{1024}$ are larger than 10^{57} in absolute value. These things are extremely unstable....

Problem 10.9 (Eigenvalues are unstable)**.** Tiny error in the entries could have drastic effect on eigenvalues and diagonalizabilities. Find the eigenvalues of the following matricess, and their algebraic and geometric multiplicity. Are they diagonalizable?

1. $\begin{bmatrix} 10 & 1 & \\ & 10 & 1 \\ & & 10 \end{bmatrix}$.

2. $\begin{bmatrix} 10 & 1 & \\ & 10.001 & 1 \\ & & 10.002 \end{bmatrix}$. (Note that diagonalizability could be different for the slightest change in entries.)

3. $\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$.

4. $\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 0.0001 & & & 0 \end{bmatrix}$. (Note how the errors in entries is magnified in the errors in eigenvalues.)

5. (Read Only) In general, suppose you pick a random square matrix (all entries are random variables). Then it will have random eigenvalues $\lambda_1, \dots, \lambda_n$. When randome variables are continuously distributed, the chances that two random variables are equal is zero. So a random matrix A (or anything with unknown tiny error) will have n distinct eigenvalues with probability 100%. In particular, a random matrix A is diagonalizable with probability 100%. In real life, if you obtain some square matrix with potential tiny errors, then you may simply assume that it is diagonalizable.

11 HW11 Due Dec 7

Problem 11.1. Find all eigenvalues, all algebraic multiplicity, all geometric multiplicity, and a description of all eigenvectors. Finally, find a diagonalization if possible, or find a Schure decomposition if it is not diagonalizable.

1. $A = \begin{bmatrix} a & b & \dots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a \end{bmatrix}$.

2. $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$.

Problem 11.2 (Polynomial identity and diagonalization)**.** Let A be an $n \times n$ complex matrix. In general, if $p(A) = 0$ and $p(x)$ is a polynomial with distinct roots, then A is diagonalizable. Here we prove some special cases.

1. Suppose $A^2 = I$. Show that $\mathrm{Ker}(A+I) + \mathrm{Ker}(A-I) = \mathbb{C}^n$, and A is diagonalizable.

2. Suppose $A^2 = A$. Show that $\mathrm{Ker}(A) + \mathrm{Ker}(A-I) = \mathbb{C}^n$, and A is diagonalizable.

3. Suppose $A^3 = A$. Find constant $a, b, c \in \mathbb{C}$ such that, for all $\boldsymbol{v} \in \mathbb{C}^n$, we have $\boldsymbol{v} = aA(A-I)\boldsymbol{v} + bA(A+I)\boldsymbol{v} + c(A-I)(A+I)\boldsymbol{v}$.

4. Suppose $A^3 = A$. Show that $\mathrm{Ker}(A+I) + \mathrm{Ker}(A-I) + \mathrm{Ker}(A) = \mathbb{C}^n$, and A is diagonalizable.

Problem 11.3. For the following matrices A, B, find X such that $A = XBX^{-1}$.

1. $A = \begin{bmatrix} MN & O \\ N & O \end{bmatrix}, B = \begin{bmatrix} O & O \\ N & NM \end{bmatrix}$. Here M, N might not be square.

2. $A = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, B = \begin{bmatrix} & -\mathrm{i} \\ \mathrm{i} & \end{bmatrix}$. (The matrices A, B and their diagonalization D are called Pauli matrices and are important in physics. It is crucial in physics that these three matrices do NOT commute, dispite having the same eigenvalues.)

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ & 1 & 3 \\ & & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$. (Hint: Try change of basis via matrices like $\begin{bmatrix} 1 & k & \\ & 1 & \\ & & 1 \end{bmatrix}$ or diagonal matrices.)

4. $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$.

5. $A = \begin{bmatrix} 4M & -2M \\ 3M & -M \end{bmatrix}, B = \begin{bmatrix} 2M & \\ & M \end{bmatrix}$. (Hint: It is related to the last sub-problem.)

6. $A = \begin{bmatrix} M & -N \\ N & M \end{bmatrix}, B = \begin{bmatrix} M + \mathrm{i}N & \\ & M - \mathrm{i}N \end{bmatrix}$. (Hint: Get hints from the cases when M, N are 1×1.)

7. $A = \begin{bmatrix} 2I_2 & M \\ & 3I_3 \end{bmatrix}, B = \begin{bmatrix} 2I_2 & \\ & 3I_3 \end{bmatrix}$. (Hint: Get hints from the cases when all blocks are 1×1.)

Problem 11.4 (Homogeneous Sylvester's equation)**.** Suppose $AX = XB$ for some $m \times m$ matrix A and $n \times n$ matrix B and $m \times n$ matrix X.

1. Suppose $n = 1$. Show that the $m \times 1$ matrix X must in fact be an eigenvector of A.

2. If $\boldsymbol{v} \in \operatorname{Ran}(X)$, show that $A\boldsymbol{v} \in \operatorname{Ran}(X)$. (This means the subspace $\operatorname{Ran}(X)$ is mapped into itself via A. Subspaces such as $\operatorname{Ran}(X)$ is called an ***invariant subspace*** of A.)

3. Show that for any polynomial $p(x)$, we have $p(A)X = Xp(B)$. (Hint: prove it first for powers of A, then note that polynomials are just linear combinations of powers.)

4. Suppose A, B has NO common eigenvalue. Show that $AX = XB$ imples $X = O$. (Hint: Why is $p_A(B)$ invertible?)

Problem 11.5 (Diagonalization via inhomogeneous Sylvester's equation)**.** Suppose we have a block matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ for some $m \times m$ matrix A and $n \times n$ matrix B and $m \times n$ matrix C. We are hoping to block diagonalize this by $\begin{bmatrix} I & X \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ & I \end{bmatrix} = \begin{bmatrix} A & C - AX + XB \\ & B \end{bmatrix}$.

In particular, we want to find the solution X to the matrix equation $AX - XB = C$. Then we shall have $\begin{bmatrix} I & X \\ & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ & I \end{bmatrix} = \begin{bmatrix} A & \\ & B \end{bmatrix}$.

A matrix equation $AX - XB = C$ for constant A, B, C and unknown X is called a Sylvester's equation, and it is extremely important in many applications such as control theory.

1. Let V be the vector space of $m \times n$ matrices. Show that $L : V \to V$ with $L(X) = AX - XB$ is a linear map.

2. Show that, if A, B has no common eigenvalues, then $\operatorname{Ker}(L)$ is trivial. (Hint: Use facts about homogeneous Sylvester's equation in the previous problem.)

3. Show that, if A, B has no common eigenvalues, then no matter what C is, the solution X to $AX - XB = C$ exists and is unique. (Consequently, we can always block diagonalize $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ using $\begin{bmatrix} I & X \\ & I \end{bmatrix}$.)

Problem 11.6. Let us write $A \geq B$ if $A - B$ is positive semidefinite.

1. Show that if $A \geq B$ and $B \geq C$, then $A \geq C$.

2. Show that if $A \geq B$ and $B \geq A$, then $A = B$.

3. Show that for any Hermitian A, we can find $a, b \in \mathbb{R}$ such that $aI \leq A \leq bI$.

12 HW12 Due Dec 14

Problem 12.1 (Commuting matrices diagonalization)**.** We explore some consequences of the identity $AB = BA$. What does it mean for two matrices to commute?

1. If $AB = BA$ and A is diagonal with distinct diagonal entries, show that B must be diagonal. (Hint: work an easy example, say $A = \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}$, to get an intuition of why non-diagonal entries of B must be zero.)
2. If $AB = BA$ and A has no repeated eigenvalues, then show that one can find an invertible matrix X such that $X^{-1}AX, X^{-1}BX$ are BOTH diagonal. (I.e., A, B are simultaneously diagonalizable. A, B are both diagonal in another basis.)
3. If $AB = BA$ and A has no repeated eigenvalues, then show that all eigenvectors of A are eigenvectors of B (for maybe different eigenvalues).
4. Find an example of $AB = BA$ where A has no repeated eigenvalues, but B has an eigenvector that is NOT an eigenvector of A. (Simple examples exist. Don't overthink.)
5. Find an example of $AB = BA$ where A is diagonalizable, but B is NOT diagonalizable. (Simple examples exist. Don't overthink.) Note how A must have repeated eigenvalues in this case.
6. If $A\boldsymbol{v} = \lambda\boldsymbol{v}$, show that $B\boldsymbol{v}$ is also an eigenvector of A for the same eigenvalue λ.
7. (No Credit, feel free to give up) $AB = BA$ might not imply simultaneous diagonalizability. However, they can be simultaneously triangularized via some unitary matrix or orthogonal matrix. (Simultaneous Schur decomposition.) Can you see why? (Hint: Start by finding a common eigenvector of A, B, and then it is a straight forward block-matrix induction.)

Problem 12.2 (Nilpotent matrices)**.** A matrix A is nilpotent if $A^k = O$ for some positive integer k.

1. If A is nilpotent, find all eigenvalues of A counting algebraic multiplicity. Why must A be non-invertible?
2. Show that a $n \times n$ matrix A is nilpotent if and only if all eigenvalues of A are zero if and only if $A^n = O$. (Hint: Cayley-Hamilton.)
3. For any matrix A, show that $A = B + C$ where B is a normal matrix and C is an nilpotent matrix. (Hint: perform Schur decomposition $A = UTU^{-1}$, and let D be the diagonal matrix taking all the diagonal entries of T. Why is $U(T - D)U^{-1}$ nilpotent?)

Problem 12.3. Given that x, y are not both zero, find the maximum value and minimum value of the $\frac{ax^2+bxy+cy^2}{x^2+y^2}$, for the following given constants $a, b, c \in \mathbb{R}$. (Hint: consider the Rayleigh quotient $\frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}$)

1. $a = 0, b = 2, c = 0$.
2. $a = 3, b = 2, c = 3$.
3. $a = 1, b = 2, c = 4$.

Problem 12.4. Find the SVD of the following matrices

1. $\begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$.
2. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$. (If you want, you could actually do this by sight alone. No calculation needed.)

4. $\begin{bmatrix} A & O \\ O & O \end{bmatrix}$ where we have SVD $A = U\Sigma V^{\mathrm{T}}$.

Problem 12.5. ☕Prove that for a square matrix A, all eigenvalues are less than the largest singular value of A. (Hint: Recall the definition of the operator norm of A.)

Problem 12.6. Consider four points on the plane $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 20 \end{bmatrix}$. Find the line $ax + by + c = 0$ that minimize the sum of squared orthogonal distances. (Feel free to use computers to speed up the calculation. Don't forget to center the data first!)

Problem 12.7 (Singular values are more stable than eigenvalues)**.** Find the singular values and eigenvalues of the following matrices. (Minor changes in entries could result in huge change in eigenvalues, but not in singular values.)

1. $\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$. (Since the matrix has rank 3, we say $\sigma_4 = 0$ by convention.)

2. $\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ 0.0001 & & & 0 \end{bmatrix}$.