6.1

1. L_A: $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} I_{2\times2} & 2I_{2\times2} \\ 3I_{2\times2} & 4I_{2\times2} \end{bmatrix}$

2. R_A: $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}$

3. $L_AR_A = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix}$ $\quad R_AL_A = \begin{bmatrix} A^T & 2A^T \\ 3A^T & 4A^T \end{bmatrix}$

4. $L_AR_A \cdot X = L_A(XA) = A(XA)$

$R_AL_A \cdot X = R_A(AX) = (AX)A$ $\quad$ ($A(XA) = (AX)A$) law of associativity

5. new basis: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_1 & x_1 \\ x_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x_1 \\ x_1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & x_2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} x_2 & x_2 \\ 0 & x_2 \end{bmatrix} - \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix} \to \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \end{bmatrix} + \begin{bmatrix} 0 & x_3 \\ \cancel{x_3} & 0 \end{bmatrix} - \begin{bmatrix} x_3 & x_3 \\ 0 & x_3 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & x_4 \end{bmatrix} \to \begin{bmatrix} x_4 & 0 \\ 0 & x_4 \end{bmatrix} + \begin{bmatrix} 0 & x_4 \\ \cancel{x_4} & 0 \end{bmatrix} - \begin{bmatrix} x_4 & x_4 \\ x_4 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow$ the change of coordinate matrix $= \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

6.2

$V \xrightarrow{L} W$ $\qquad C_{ij} = B_i \to B_j$ or $e_i \to e_j$

$L_{ij} = B_i \to e_j$

✓ 1. $L_{11} = C_{21} L_{12}$

✗ 2. $\underset{e_1 \to B_1}{L_{11}^{-1}} = \underset{e_2 \to B_1}{L_{12}^{-1}} \underset{e_2 \to e_1}{C_{21}}$ $\qquad \underset{e_1 \to B_1}{L_{11}^{-1}} = \underset{e_2 \to B_1}{L_{12}^{-1}} \underset{e_1 \to e_2}{C_{12}}$

✓ 3. $\underset{B_2 \to e_2}{L_{22}} = \underset{e_1 \to e_2}{C_{12}} \underset{B_1 \to e_1}{L_{11}} \underset{B_2 \to B_1}{C_{12}^{-1}}$

4. $\underset{B_3 \to e_1}{L_{31}} = C_{21} L_{12} C_{31} C_{23} L_{21}^{-1} C_{21} \overset{B_1 \to e_2}{L_{12}} C_{21} C_{32}.$

✗ $e_1 \leftarrow B_2 \leftarrow B_2$

$\underset{B_1 \to e_2}{L_{12}} \underset{B_2 \to e_1}{L_{21}}$ ✗

6.3

1. $z \mapsto (2+3i) z$

$\overset{\shortparallel}{a+bi} \mapsto (2+3i)(a+bi)$

$= (2a-3b) + (3a+2b) i$

$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 2a-3b \\ 3a+2b \end{bmatrix}$

$\Rightarrow M_w = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$

2. let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\qquad wz = \begin{bmatrix} w_1 z_1 - w_2 z_2 \\ w_1 z_2 + w_2 z_1 \end{bmatrix}$

$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

then $M_w = \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix}$ $\qquad M_{wz} = \begin{bmatrix} w_1 z_1 - w_2 z_2 & -(w_1 z_2 + w_2 z_1) \\ w_1 z_2 + w_2 z_1 & w_1 z_1 - w_2 z_2 \end{bmatrix}$

$M_z = \begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix}$

$M_w \cdot M_z = \begin{bmatrix} w_1 z_1 - w_2 z_2 & -(w_1 z_2 + w_2 z_1) \\ w_1 z_2 + w_2 z_1 & w_1 z_1 - w_2 z_2 \end{bmatrix}$

//

3. $M_w M_z : a \mapsto z a \mapsto w(za)$

$M_{wz} : a \mapsto (wz) a$

$\Rightarrow$ law of associativity

4. $\phi : \underset{\begin{bmatrix} a \\ b \end{bmatrix}}{w} \mapsto \underset{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}{M_w}$

6.4.

1. basis $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ dimension $=4$

2. there's no solution is the set

$\Rightarrow$ set $= \emptyset$

3. suppose $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}, \begin{pmatrix}a'\\b'\\c'\\d'\end{pmatrix} \in W$

suppose $\vec{p} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow \begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \end{cases} \Rightarrow (a+a')x+(b+b')y+(c+c')z=d+d'$

$\Rightarrow \begin{pmatrix}a+a'\\b+b'\\c+c'\\d+d'\end{pmatrix} \in W$

$\Rightarrow kax+kby+kcz=kd$

$\Rightarrow k\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \in W$

so W is a subspace

4. $x=1 \Rightarrow \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ $\quad x+y+z=7$

$y=2 \Rightarrow \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$ $\quad \Downarrow$ $\begin{pmatrix}1\\1\\1\\7\end{pmatrix}$

$z=3 \Rightarrow \begin{pmatrix}0\\0\\1\\3\end{pmatrix}$

for first 3 equation, it's obvious that they're linearly independent

$\Rightarrow$ let W be the subspace, made of all equations that contains $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in its solution set

$\Rightarrow \forall \vec{y} \in W, \vec{y} = \begin{pmatrix}a\\b\\c\\a+2b+3c\end{pmatrix}$, but for $\begin{pmatrix}1\\1\\1\\7\end{pmatrix}$, there's no solution for $\begin{pmatrix}a\\b\\c\\a+2b+3c\end{pmatrix} = \begin{pmatrix}1\\1\\1\\7\end{pmatrix}$

$\Rightarrow x=1, y=2, z=3, x+y+z=7$ are linearly independent

5. $x=1 \Rightarrow \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ obviously, they're linearly independent

$x=2 \Rightarrow \begin{pmatrix}1\\0\\0\\2\end{pmatrix}$

$\Rightarrow$ let W be the subspace of V, spanned by $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\\2\end{pmatrix}$

$\Rightarrow \forall \vec{y} \in W, \vec{y} = \begin{bmatrix}a+b\\0\\0\\a+2b\end{bmatrix}$

if $\begin{pmatrix}1\\0\\0\\3\end{pmatrix} \in W$

then $\begin{cases} a+b=1 \\ a+2b=3 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=2 \end{cases}$

$\Rightarrow (x=3) \in W$,

$\Rightarrow x=1, x=2, x=3$ are NOT linearly independent

6.5

1. NO. $\vec{w} \mapsto \vec{w} - C - O_2 + CO_2$

$2\vec{w} \not\mapsto 2(\vec{w} - C - O_2 + CO_2)$

2. NO. dimension = 2

3.
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & 2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{2\times2} & \vec{0} \\ 0 & \vec{0} \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

the rank of $M = 2$

6.6

1. $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ Because we just delete non-pivotal columns of A, which doesn't delete pivots in A, so the rank isn't change as well

2. REF of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

rank of $R = 2$

3. $BR = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$

6.7

1. Ran(M)'s dimension = 3 (a, b+c+d, e, f)

$\Rightarrow$ the rank of $M = 3$

2.

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$$

3.

$$Y = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{educational} \\ \text{funny} \end{array}$$

4.

$$XY = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M - XY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow$

rank 1

5.

$$M = \underbrace{\left[\begin{array}{cc|c} 1 & 1 & \\ 1 & 1 & \\ 1 & 1 & \\ 1 & 1 & 0_{7\times 4} \\ -1 & -1 & \\ -1 & -1 & \\ -1 & -1 & \end{array}\right]}_{\text{educational videos}} + \underbrace{\left[\begin{array}{c|ccc} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ 0_{7\times 3} & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & 1 & 1 & 1 \end{array}\right]}_{\text{funny videos}} + \underbrace{\left[\begin{array}{c|c|c} & 1 & \\ & 1 & \\ & 1 & \\ 0_{7\times 2} & 1 & 0_{7\times 3} \\ & -1 & \\ & 1 & \\ & -1 & \end{array}\right]}_{\text{other factors}}$$