

5.1

$$\begin{aligned} 1. \text{tr}(xA + yB) &= \sum_{i=1}^n \vec{e}_i^T (xA + yB) \vec{e}_i \\ &= \sum_{i=1}^n (\vec{e}_i^T xA \vec{e}_i + \vec{e}_i^T yB \vec{e}_i) \\ &= x \sum_{i=1}^n \vec{e}_i^T A \vec{e}_i + y \sum_{i=1}^n \vec{e}_i^T B \vec{e}_i \end{aligned}$$

$$2. \text{tr}(I_n) = \sum_{i=1}^n \vec{e}_i^T I_n \vec{e}_i = \sum_{i=1}^n \vec{e}_i^T \vec{e}_i = \sum_{i=1}^n 1 = n$$

$$3. \text{tr}(A^T) = \sum_{i=1}^n \vec{e}_i^T A^T \vec{e}_i = \sum_{i=1}^n (A \vec{e}_i)^T (\vec{e}_i^T)^T = \sum_{i=1}^n (\vec{e}_i^T A \vec{e}_i)^T = \sum_{i=1}^n \vec{e}_i^T A \vec{e}_i = \text{tr}(A)$$

$$4. \text{tr}(\vec{u} \vec{u}^T) = \sum_{i=1}^n \vec{e}_i^T \vec{u} \vec{u}^T \vec{e}_i = \sum_{i=1}^n (\vec{e}_i^T \vec{u})(\vec{e}_i^T \vec{u})^T = \sum_{i=1}^n (\vec{e}_i^T \vec{u})^2 = |\vec{u}|^2 = 1$$

$$\text{tr}(I_n - \vec{u} \vec{u}^T) = \text{tr}(I_n) - \text{tr}(\vec{u} \vec{u}^T) = n - 1$$

$$\begin{aligned} 5. \text{tr}(A^T B) &= \sum_{i=1}^2 \vec{e}_i^T A^T B \vec{e}_i = \vec{e}_1^T A^T B \vec{e}_1 + \vec{e}_2^T A^T B \vec{e}_2 \\ &= [a_1 \ a_3] \begin{bmatrix} b_1 \\ b_3 \end{bmatrix} + [a_2 \ a_4] \begin{bmatrix} b_2 \\ b_4 \end{bmatrix} \\ &= \sum a_i b_i \end{aligned}$$

$$\text{tr}(B A^T) = \vec{e}_1^T B A^T \vec{e}_1 + \vec{e}_2^T B A^T \vec{e}_2 = [b_1 \ b_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + [b_3 \ b_4] \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} = \sum a_i b_i$$

$$6. \text{assume } \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$\text{tr}(\vec{v} \vec{w}^T) = \sum_{i=1}^n \vec{e}_i^T \vec{v} \vec{w}^T \vec{e}_i = \sum_{i=1}^n v_i \cdot w_i$$

$$\text{tr}(\vec{w}^T \vec{v}) = \vec{w}^T \vec{v} = \sum_{i=1}^n v_i \cdot w_i$$

$$7. \text{tr}(AB) = \text{tr}\left(\sum_{i=1}^n A \vec{e}_i \times \sum_{j=1}^m \vec{e}_j^T B\right) = \sum_{i,j=1}^n \text{tr}(A \vec{e}_i \vec{e}_j^T B) = \sum_{i,j=1}^n \text{tr}(A \vec{e}_i)(B^T \vec{e}_j)^T$$

$$8. \text{if } AB - BA = I$$

$$\Rightarrow \text{tr}(AB - BA) = \text{tr}I = n$$

$$\text{tr}(AB) - \text{tr}(BA) = 0$$

$$\text{but } n \neq 0 \Rightarrow \text{so } AB - BA \neq I$$

$$\begin{aligned} &= \sum_{i,j=1}^n \text{tr}(B^T \vec{e}_j)^T (A \vec{e}_i) \\ &= \sum_{i,j=1}^n \text{tr}(\vec{e}_j^T B A \vec{e}_i) \\ &= \text{tr}(BA) \end{aligned}$$

5.2

$$1. P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. $\text{tr}(P) = \sum_{i=1}^n (i,i)$ entry of $P =$ the number of fixed points in P
 $=$ the number of fixed elements for the corresponding permutation

3. Since $\text{tr}(P_1 P_2) = \text{tr}(P_2 P_1)$
so the conclusion is proved

$$\begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 4 & 0 & -4 \\ -2 & 0 & 2 \end{pmatrix}$$

$$6. \checkmark \text{ let } S_m = \sum_{i=1}^m \vec{e}_i M \vec{e}_i^\top \quad \begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

if $M \in V$, $\lambda \in \mathbb{R} \Rightarrow \lambda M \in V$
(because $S_m = \lambda S_m$)

if $M_1, M_2 \in V \Rightarrow M_1 + M_2 \in V$
 $S_{M_1+M_2} = S_{M_1} + S_{M_2}$

dimension: 3

$$\text{basis: } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

3. \checkmark subset of \mathbb{X} : $\emptyset, \{1\}, \{2\}, \{3\} \Rightarrow \text{basis}$
 $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

dimension: 3

4. \checkmark In HW3, we know matrices like $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$ commute with J
(upper-triangular matrix & anti-symmetric)

$$AEV, A_1 + A_2 = \begin{bmatrix} a+a_2 & b_1+b_2 & c_1+c_2 \\ 0 & a+a_2 & b_1+b_2 \\ 0 & 0 & a+a_2 \end{bmatrix} EV. \quad \lambda A = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda a \end{bmatrix} EV$$

dimension: 3

$$\text{basis: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. \checkmark matrices like $B = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$ commute with $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
dimension: 2 basis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

5.4

1. ① A is injective

so $A\vec{x} = \vec{0}$ solution is only $\vec{0}$

suppose we have $\alpha \vec{y} \neq \vec{0}, A\vec{y} = \vec{0}$

$$\Rightarrow \text{let } \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, A\vec{y} = \begin{pmatrix} y_1 + y_2 + y_3 \\ -e^y + e^y y_2 + e^y y_3 \\ e^y y_1 + e^y y_2 + e^y y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1 + y_2 + y_3 = 0 \\ -e^y + e^y y_2 + e^y y_3 = 0 \\ e^y y_1 + e^y y_2 + e^y y_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = -y_2 - y_3 = (e^y - 1)y_3 \\ y_1 + e^y y_2 + e^y y_3 = 0 \\ y_1 + e^y y_2 + e^y y_3 = 0 \end{cases} \Rightarrow \begin{cases} e^y - 1 = 0 \\ y_2 + (e^y - 1)y_3 = 0 \end{cases}$$

$$\Rightarrow [e^y - 1 - (e^y - 1) + e^y] y_3 = 0$$

$$\Rightarrow y_1 = y_2 = y_3 = 0, \vec{y} = \vec{0}$$

→ contradiction

so A is injective

② A is surjective

$\Rightarrow A^T = \begin{pmatrix} e^y & e^y & e^y \\ e^y & e^y & e^y \\ e^y & e^y & e^y \end{pmatrix}$ is injective

so $A^T \vec{x} = \vec{0}$ has only one solution: $\vec{0}$

suppose $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \neq \vec{0}, A^T \vec{y} = \vec{0}$

$$\begin{pmatrix} 1 & e^y & e^{2y} & 0 \\ 1 & e^y & e^{2y} & 0 \\ 1 & e^y & e^{2y} & 0 \end{pmatrix} = \begin{pmatrix} 1 & e^y & e^{2y} & 0 \\ 0 & e^y - e^{2y} & 0 & 0 \\ 0 & e^y - e^{2y} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & e^y & e^{2y} & 0 \\ 0 & 1 & e^y & 0 \\ 0 & 1 & e^y & 0 \end{pmatrix} = \begin{pmatrix} 1 & e^y & e^{2y} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow \vec{y} = \vec{0}$

so A^T is injective $\Rightarrow A$ is surjective

all small. A is bijective $\Rightarrow A$ is invertible

2. since A is invertible, A is row & column linearly independent

if e^x, e^{2x}, e^{3x} are linearly dependent,

then e^x, e^y, e^z are $\downarrow \downarrow \downarrow$

e^x, e^y, e^z are $\cdots \cdots \cdots$

then A is row linearly dependent \rightarrow contradiction

so e^x, e^{2x}, e^{3x} are linearly independent

3. the coordinates of f: $A^{-1} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$

5.5

1. ① let $L_1 = \{ \vec{p}_1 + t_1 \vec{v}_1 : t_1 \in \mathbb{R} \}$

$L_2 = \{ \vec{p}_2 + t_2 \vec{v}_2 : t_2 \in \mathbb{R} \}$

($L_1 \cap L_2, \text{ so } \vec{v}_1 = \vec{v}_2 = \vec{0}$)

$\vec{q} \in L_1 + L_2$

$$\begin{aligned} \vec{q} &= \vec{p}_1 + \vec{p}_2 + t_1 \vec{v}_1 + t_2 \vec{v}_2 \\ &= (\vec{p}_1 + \vec{p}_2) + (t_1 + t_2) \vec{v} \end{aligned}$$

3. $L_0 = \{ t \vec{v} : t \in \mathbb{R} \}$

② $L = \{ \vec{p} + t \vec{v} : t \in \mathbb{R} \}$

$$kL = \{ k \vec{p} + (kt) \vec{v} : t \in \mathbb{R} \} = \{ \vec{p} + t' \vec{v} : t' \in \mathbb{R} \} \in V/W$$

so $L + L_0 \in V/W$

basis: $L_1 = \{ \vec{p} + t \vec{v} \}$

$L_2 = \{ \vec{q} + t \vec{v} \}$

$(\vec{p} \neq \vec{q} \cdot \vec{v} \in \text{Ker})$

$$2. K(L_1 + L_2) = \{ k \vec{p}_1 + (k+kt) \vec{v} + kt \vec{v} : t \in \mathbb{R} \}$$

$$= \{ k(\vec{p}_1 + \vec{p}_2) + k(t+t_2) \vec{v} : t, t_2 \in \mathbb{R} \}$$

$$KL_1 + KL_2 = \{ k \vec{p}_1 + kt \vec{v} + k \vec{p}_2 + kt_2 \vec{v} : t, t_2 \in \mathbb{R} \}$$

5.6

1. Yes. Because for any finitely many polynomials,
we can find a maximum degree : N

so just find one another polynomial, whose degree is larger than N

2. $\forall q(x) \in V \Rightarrow p(x) \cdot q(x) \in W$

$$\textcircled{1} \forall q_1(x), q_2(x) \in V \Rightarrow p(x)q_1(x) + p(x)q_2(x) = p(x)(q_1(x) + q_2(x)) \in W$$

$$\textcircled{2} \forall q(x) \in V \Rightarrow k p(x) q(x) = p(x) \cdot (k q(x)) \in W$$

so W is a subspace

3. $\textcircled{1} \forall g(x) \in [r_1(x)] + [r_2(x)]$

$$g(x) = r_1(x) + r_2(x) + p(x)(q_1(x) + q_2(x)) \quad \forall q_1(x), q_2(x) \in V$$

$$= r_1(x) + r_2(x) + \underline{p(x) q(x)} \quad \forall q(x) \in V$$

$$\forall h(x) \in [r_1(x) + r_2(x)]$$

$$h(x) = \underline{(r_1(x) + r_2(x)) + p(x)q(x)} \quad \forall q(x) \in V \quad \xrightarrow{\text{same format}} \quad \text{so } [r_1(x)] + [r_2(x)] = [r_1(x) + r_2(x)]$$

$\textcircled{2} \forall g(x) \in K[r(x)]$

$$g(x) = k(r(x) + p(x)q(x)) = k(r(x)) + p(x) \cdot (kq(x))$$

$$= \underline{k(r(x)) + p(x) \cdot q(x)} \quad \forall q(x) \in V$$

$$\forall h(x) \in [k(r(x))]$$

$$h(x) = \underline{k(r(x)) + p(x)q(x)} \quad \forall q(x) \in V \quad \xrightarrow{\text{same}} \quad \text{so } K[r(x)] = [k(r(x))]$$

4. dimension: 2

basis: $\{1 + p(x)q(x)\}, \{x + p(x)q(x)\}$