10.1

	eigenvalue	eigenvector
1.	1	$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$
2.	1	$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$
	2	$\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$
	3	$\begin{pmatrix} x \\ 2x \\ 2x \end{pmatrix}$
3.	7	$\begin{pmatrix} 11x \\ -3x \end{pmatrix}$
	21	$\begin{pmatrix} x \\ x \end{pmatrix}$
4.	-1	$\begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}$
	1	$\begin{pmatrix} x \\ y \\ x \end{pmatrix}$
5.	$b-ai$	$\begin{pmatrix} x \\ xi \end{pmatrix}$
	$b+ai$	$\begin{pmatrix} xi \\ x \end{pmatrix}$
6.	$3i$	$\begin{pmatrix} \frac{1-3i}{4}z \\ \frac{-1-3i}{4}z \\ z \end{pmatrix}$
	$-3i$	$\begin{pmatrix} \frac{3i+1}{4}z \\ \frac{1+3i}{4}z \\ z \end{pmatrix}$
	0	$\begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix}$

10.2

1. $M_p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

characteristic polynomial of $M_p = x^2+1$

eigenvalues: $\pm i$

2. $M_p = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$

characteristic polynomial of $M_p = x^2-3x+2$

eigenvalues: 1, 2

3. eigenvalues are 1, 2 and 3

4. characteristic polynomial of $M_p = \det(xI - M_p)$

$$\det(xI - M_p) = \det\begin{bmatrix} x & & & & a_0 \\ -1 & x & & & \vdots \\ & \ddots & \ddots & & \\ & & -1 & x & a_{n-2} \\ & & & -1 & x+a_{n-1} \end{bmatrix}$$

$$= (a_{n-1}+x)\det\begin{bmatrix} x & & \\ -1 & x & \\ & \ddots & \ddots \\ & & -1 \;\; x \end{bmatrix} - a_{n-2}\begin{bmatrix} x & & & \\ -1 & x & & \\ & \ddots & \ddots & \\ & & -1 & x \\ & & & -1 \end{bmatrix} \cdots$$

$$+(-1)^{n+1} a_0 \det\begin{bmatrix} -1 & x & & \\ & \ddots & \ddots & \\ & & & x \\ & & & -1 \end{bmatrix}$$

$$= (a_{n-1}+x)\cdot x^{n-1} - a_{n-2}\cdot(-x^{n-2}) \cdots + a_0(-1)^{n+1}\cdot(-1)^{n-1}$$

$$= x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$$

so eigenvalues of M_p are roots of p

10.3

1. $P_A(x) = x^2 - bx - a$

$P_A(4) = P_A(7) = 0 \Rightarrow \begin{cases} 16 - 4b - a = 0 \\ 49 - 7b - a = 0 \end{cases}$

$\Rightarrow \begin{cases} a = -28 \\ b = 11 \end{cases}$

2. $P_A(x) = x^3 - cx^2 - bx - a$

$P_A(2) = P_A(3) = P_A(4) = 0$

$\Rightarrow \begin{cases} 8 - 4c - 2b - a = 0 \\ 27 - 9c - 3b - a = 0 \\ 64 - 16c - 4b - a = 0 \end{cases} \Rightarrow \begin{cases} a = 24 \\ b = -26 \\ c = 9 \end{cases}$

3. $\left.\begin{aligned} P_A(k) &= k^3 - (x+2)k^2 + (2x-8)k + 16 \\ P_B(k) &= k^3 - (4+y)k^2 + (4+4y)k - 4y \end{aligned}\right\}$ have same roots

$\Rightarrow \begin{cases} x = -2 \\ y = -4 \end{cases}$

4. $\det(A) = 2\det\begin{pmatrix} 2 & 1 \\ 1 & a \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

$= 2(2a-1) - (a-1) - 1 = 4a - 2 - a + 1 - 1 = 3a - 2 \neq 0$

$\Rightarrow a \neq \frac{2}{3}$

$A\vec{x} = \begin{bmatrix} b+3 \\ 2b+2 \\ a+b+1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix} \Rightarrow \frac{b+3}{1} = \frac{2b+2}{b} = \frac{a+b+1}{1}$

$\Rightarrow \begin{cases} a = 2 \\ b = 1 \text{ 或 } -2 \end{cases}$

10.4

1. $n=1$; $A_1^2=[1]=1\times I_{1\times 1}$

suppose ; $n=k$, $H_n^2=nI$

then $n=2k$: $H_{2n}^2=\begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}^2$

$$=\begin{bmatrix} 2H_n^2 & 0 \\ 0 & 2H_n^2 \end{bmatrix}$$

$$=\begin{bmatrix} 2nI & \\ & 2nI \end{bmatrix}=2nI$$

so $H_n^2=nI$

$\Rightarrow$ eigenvalue of nI is n

so eigenvalues of H_n^2 are $\pm\sqrt{n}$

2. $tr(H_n)=tr\begin{bmatrix} H_{\frac{n}{2}} & H_{\frac{n}{2}} \\ H_{\frac{n}{2}} & -H_{\frac{n}{2}} \end{bmatrix}=0$

so $\sum$ all eigenvalues of $H_n=0$

$\Rightarrow$ algebraic multiplicity of all eigenvalues are $\frac{n}{2}$

3. $\frac{1}{\sqrt{2}}H_2=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, $(\frac{1}{\sqrt{2}}H_2)^2=\frac{1}{2}H_2^2=I$

so $\frac{1}{\sqrt{2}}H_2$ is a reflection

$\Rightarrow$ 2 linearly independent eigenvectors : $\begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} \sqrt{2}-1 \\ -1 \end{bmatrix}$

4. ① suppose $H_{2n}\cdot\begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix}=\sqrt{2}\lambda\begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix}$

$$\begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}\begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix}$$

$$\begin{bmatrix} x\lambda\vec{v}+y\lambda\vec{v} \\ x\lambda\vec{v}-y\lambda\vec{v} \end{bmatrix}$$

$\Rightarrow y=(\sqrt{2}-1)x$

then let $x=1$, $y=\sqrt{2}-1$

$\Rightarrow \begin{bmatrix} \vec{v} \\ (\sqrt{2}-1)\vec{v} \end{bmatrix}$ is the answer

② similar process as ①

$\Rightarrow \begin{cases} x+y=-\sqrt{2}x \\ x-y=-\sqrt{2}y \end{cases}$

$\Rightarrow \begin{cases} (\sqrt{2}+1)x=-y \\ (1-\sqrt{2})y=x \end{cases}$

then let $x=1$, $y=-1-\sqrt{2}$

$\Rightarrow \begin{bmatrix} \vec{v} \\ -(1+\sqrt{2})\vec{v} \end{bmatrix}$ is the answer

10.5

$$A=\begin{bmatrix}4&-1&-1&-1\\-1&4&-1&-1\\-1&-1&4&-1\\-1&-1&-1&4\end{bmatrix}\qquad H_4=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$$

1. $$AH_4=\begin{bmatrix}1&5&5&5\\1&-5&5&-5\\1&5&-5&-5\\1&-5&-5&5\end{bmatrix}$$

2. $$H_4^{-1}AH_4=\frac{1}{4}H_4AH_4=\begin{bmatrix}1&&&\\&5&&\\&&5&\\&&&5\end{bmatrix}$$
$(H_4^2=4I)$

3. $$\det(A)=\frac{\det(H_4^{-1}AH_4)}{\frac{1}{4^4}[\det(H_4)]^2}=\frac{125}{1}=125$$

$$A^{-1}=H_4(H_4^{-1}AH_4)^{-1}H_4^{-1}=H_4\cdot\begin{bmatrix}\frac{1}{4}&&&\\&\frac{1}{20}&&\\&&\frac{1}{20}&\\&&&\frac{1}{20}\end{bmatrix}\frac{1}{4}H_4$$
$$=\begin{pmatrix}0.4&0.2&0.2&0.2\\0.2&0.4&0.2&0.2\\0.2&0.2&0.4&0.2\\0.2&0.2&0.2&0.4\end{pmatrix}$$

10.6

1. $$AF_4=\begin{bmatrix}10&-2-2i&-2&-2+2i\\10&2-2i&2&2+2i\\10&2+2i&-2&2-2i\\10&-2+2i&2&-2-2i\end{bmatrix}$$

2. $X=F_4\quad D=\begin{bmatrix}10&&&\\&-2-2i&&\\&&-2&\\&&&-2+2i\end{bmatrix}$

3. $$\det(A)=\frac{\det(D)}{\det(F_4^{-1})\det(F_4)}=\frac{-160}{(\frac{5}{80}i)(-16i)}=-160$$

$$A^{-1}=XD^{-1}X^{-1}=F_4\begin{bmatrix}\frac{1}{10}&&&\\&\frac{1}{-2-2i}&&\\&&-\frac{1}{2}&\\&&&\frac{1}{-2+2i}\end{bmatrix}F_4^{-1}=\begin{bmatrix}\frac{[illegible]}{160}&\frac{[illegible]}{160}&\frac{1}{160}&\frac{1}{160}\\\frac{1}{160}&\frac{[illegible]}{160}&\frac{[illegible]}{160}&\frac{1}{160}\\\frac{[illegible]}{160}&\frac{1}{160}&\frac{[illegible]}{160}&\frac{1}{160}\\\frac{1}{160}&\frac{1}{160}&\frac{1}{160}&\frac{[illegible]}{160}\end{bmatrix}$$

10.7 det = 0 tr = 0

eigenvalues are 1 -1

eigenvectors: $\text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\text{span}\begin{pmatrix}33\\-1\\22\end{pmatrix}$

$A = \begin{bmatrix}-110 & 55 & -164\\ 42 & 21 & -62\\ 88 & 44 & -131\end{bmatrix}$

$\Rightarrow A^{2017}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$A^{2017}\begin{pmatrix}33\\-1\\22\end{pmatrix} = -\begin{pmatrix}33\\-1\\22\end{pmatrix}$

$\Rightarrow A^{2017} = A = \begin{bmatrix}-110 & 55 & -164\\ 42 & 21 & -62\\ 88 & 44 & -131\end{bmatrix}$

10.8

1. $\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix} = \begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & \\ & 5\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}^{-1}$

$\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}^{1024} = \begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2^{1024} & \\ & 5^{1024}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}^{-1}$

$= \begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{2^{1024}}{3} & -\frac{2^{1024}}{3}\\ \frac{5^{1024}}{3} & \frac{2\times 5^{1024}}{3}\end{bmatrix}$

$= \begin{bmatrix}\frac{2^{1025}+5^{1024}}{3} & 2\times\frac{5^{1024}-2^{1024}}{3}\\ \frac{5^{1024}-2^{1024}}{3} & \frac{2^{1024}+2\times5^{1024}}{3}\end{bmatrix}$

Since $\frac{5^{1024}-2^{1024}}{3} > 10^{700}$

so all entries of $\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}^{1024}$ are larger than 10^{700}

2. $\begin{bmatrix}3 & 2\\ -5 & -3\end{bmatrix} = \begin{bmatrix}3+i & 3-i\\ -5 & -5\end{bmatrix}\begin{bmatrix}i & \\ & -i\end{bmatrix}\begin{bmatrix}3+i & 3-i\\ -5 & -5\end{bmatrix}^{-1}$

$\begin{bmatrix}3 & 2\\ -5 & -3\end{bmatrix}^{1024} = \begin{bmatrix}3+i & 3-i\\ -5 & -5\end{bmatrix}\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{bmatrix}3+i & 3-i\\ -5 & -5\end{bmatrix}^{-1} = I_2$

3. $\begin{bmatrix}-5 & -7\\ 3 & 4\end{bmatrix}$

$= X\begin{bmatrix}\frac{-1+\sqrt{3}i}{2} & \\ & \frac{-1-\sqrt{3}i}{2}\end{bmatrix}X^{-1}$

$\begin{bmatrix}-5 & -7\\ 3 & 4\end{bmatrix}^{1024}$

$= X\begin{bmatrix}\left(\frac{-1+\sqrt{3}i}{2}\right)^{1024} & \\ & \left(\frac{-1-\sqrt{3}i}{2}\right)^{1024}\end{bmatrix}X^{-1}$

since $\frac{-1+\sqrt{3}i}{4}\cdot\frac{-1-\sqrt{3}i}{4} = 1$

$\left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{-1-\sqrt{3}i}{2}$

$= X\begin{bmatrix}\frac{-1+\sqrt{3}i}{2} & \\ & \frac{-1-\sqrt{3}i}{2}\end{bmatrix}X^{-1}$

$= \begin{bmatrix}-5 & -7\\ 3 & 4\end{bmatrix}$

no. 9

	eigenvalue	algebraic multiplicity	geometric multiplicity	diagonalizable?
1.	10	3	1	✗
2.	10	1	1	✓
	10.001	1	1	
	10.002	1	1	
3.	0	4	1	✗
4.	$\frac{2}{100}$	1	1	✓
	$\frac{-2}{100}$	1	1	
	$\frac{1}{100}$	1	1	
	$\frac{-1}{100}$	1	1	