On Production Externalities

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In the slides for Chapter 35 on production externalities, we have an example where a fishery is affected by the pollution from a steel mill. The equilibrium analysis of this example is wrong, and I would like to thank the student who pointed out this issue in class.

After introducing 4 units of licenses for pollution and giving all the licenses to the fishery, the fishery's profit function is

$$\Pi_F(f, x) = 10f - f^2 - xf + p_x x,$$

where 10 is the exogenous market price of fish, x is the supply of licenses, and p_x is the license price endogenous in the model.

Slide#52 claims that the equilibrium price of licenses $p_x = 4$, the equilibrium fish output $f^* = 4$, and the quantity of licenses traded in equilibrium is $x^* = 2$. However, $(f^*, x^*) = (4, 2)$ in fact does not maximize $\Pi_F(f, x)$ when $p_x = 4$. To see this, given the license price $p_x = 4$, we have

$$\Pi_F(f, x) = -(f - 5 + x/2)^2 + (1 - x/2)^2 + 24.$$

Given the restriction $x \in [0, 4]$, Π_F is maximized at $(f, x) = (5, 0)$ and $(3, 4)$ with the maximum profit 25. By contrast, $\Pi_F(4, 2) = 24$. Therefore, the claimed equilibrium is actually not an equilibrium. This problem arises because $\Pi_F(f, x)$ is not a concave function (its Hessian matrix is not negative semi-definite) and therefore the first-order condition on slide#48 is not sufficient for maximization.

It turns out that there is no equilibrium in this market for licenses. To see this, note that in general

$$\Pi_F(f, x) = -(f - 5 + x/2)^2 + [x/2 - (5 - p_x)]^2 - p_x^2 + 10p_x.$$

Optimality implies $f = 5 - x/2$ no matter what value x takes in $[0, 4]$, and so the first term is always 0 at optimum. Therefore, it is sufficient to choose x to maximize $[x/2 - (5 - p_x)]^2$. Clearly, the fishery's optimal supply quantity for licenses is

$$x_S^*(p_x) = \begin{cases} 0 & p_x < 4, \\ 4 & p_x > 4, \\ 0 \text{ or } 4 & p_x = 4. \end{cases}$$

However, the steel firm's demand for licenses is[1]

$$x_D^*(p_x) = \begin{cases} 4 - p_x/2 & p_x \le 8, \\ 0 & p_x > 8. \end{cases}$$

Therefore, there exists no p_x such that $x_S^*(p_x) = x_D^*(p_x)$.

To come up with an example that works, we may consider, for example, introducing a quasi-fixed cost term $x^2/2$ to the fishery's cost function, i.e. the fishery's total cost function is given by

$$c_F(f;x) = \begin{cases} f^2 + xf + x^2/2 & f > 0, \\ 0 & f = 0. \end{cases}$$

Given $f > 0$, the fishery's profit function is

$$\begin{aligned} \Pi_F(f,x) &= 10f - f^2 - xf - x^2/2 + p_x x \\ &= -(f - 5 + x/2)^2 - x^2/4 + (p_x - 5)x + 25. \end{aligned}$$

Regardless of the value of $x \in [0,4]$, by setting $f = 5 - x/2 > 0$, the profit

$$\begin{aligned} \Pi_F(5 - x/2, x) &= -x^2/4 + (p_x - 5)x + 25 \\ &\ge -4^2/4 - 20 + 25 \\ &> 0 = \Pi_F(0,x), \end{aligned}$$

and therefore quiting the market is always sub-optimal. Furthermore, note that the profit function

$$\Pi_F(f,x) = -(f - 5 + x/2)^2 - [x/2 + (5 - p_x)]^2 + (5 - p_x)^2 + 25$$

is maximized at

$$x_S^*(p_x) = \begin{cases} 2(p_x - 5) & 5 \le p_x \le 7, \\ 0 & p_x < 5, \\ 4 & p_x > 7. \end{cases}$$

and $f^*(p_x) = 5 - x_S^*(p_x)/2$. Combined with the demand from the steel firm, we see that there is a unique market clearing price $p_x^* = 5.6$, and $x^* = 1.2$ units of licenses are traded in equilibrium. The equilibrium fish output is $f^* = 2.2$ and steel output is $s^* = 6$.

It is straightforward to verify that (s^*, f^*, x^*) maximizes the profit function of the merged firm

$$\Pi^m(s,f,x) = 12s + 10f - s^2 - (x-4)^2 - f^2 - xf - x^2/2,$$

which shows that efficiency is restored after licenses for pollution is introduced as a tradable good in the market. This justifies Coase's idea of internalizing externalities through defining property rights and allow them to be traded in a competitive market.

[1] There is no such complication in the steel firm's problem because its profit function is concave in (s,x), in which case the first-order condition is sufficient for an interior maximizer.