Calculus T(1) Midterm

November 5, 2022

1 填空题

1. (3 points) 设$f(x)=5(\sqrt{1+x}-1), g(x)=\frac{kln(1+x)}{x+2}(x\neq -2)$, k为常数. 若当$x\to 0$时, $f(x)$和$g(x)$为等价无穷小, 则$k=$________.

2. (3 points) $f(x)=\begin{cases} e^{\frac{-1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases}$, n为任意正整数, 则$f^{(n)}(0)=$________.

3. (3 points) 记$[x]$为不大于x的最大整数, 则极限$\lim\limits_{y\to 0} y[\frac{1}{y}]=$________.

4. (3 points) 极限$\lim\limits_{n\to +\infty}(n+\sqrt[3]{9n^2-n^3})=$________.

5. (3 points) 极限$\lim\limits_{n\to +\infty}\frac{8}{ln(n)}(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1})=$________.

6. (3 points) 设$f(x)$在点$x=1$处可导, 且$f^{'}(1)=1$, 则极限$\lim\limits_{x\to 1}\frac{f(x)-f(1)}{\sqrt{x}-1}=$________.

7. (3 points) 设$f^{'}(0)$存在, 且$\lim\limits_{x\to 0}\frac{1}{x}[f(x)-f(\frac{x}{4})]=\frac{3}{2}$, 则$f^{'}(0)=$________.

8. (3 points) 设$f(x)=\begin{cases} \frac{x^2-x}{|x|(x^2-1)}, & x\neq 0, x^2\neq 1, \\ \frac{1}{2}, & x=0, x^2=1. \end{cases}$, 则函数$f(x)$的间断点个数总共有________个.

9. (3 points) 极限$\lim\limits_{x\to 0}\frac{6x-sin2x-sin4x}{x^3}=$________.

10. (3 points) 极限$\lim\limits_{x\to 0}\frac{sinx-arctanx}{tanx-arcsinx}=$________.

11. (3 points) 设函数$f(x)$在开区间$(-1,1)$上定义, 满足$|f(x)|\leq (sinx)^2, \forall x\in(-1,1)$,则$f^{'}(0)=$________.

12. (3 points) 极限$\lim\limits_{x\to +\infty}\frac{x^{lnx}}{(lnx)^x}=$________.

13. (3 points) 设$y=y(x)$是由方程$y=1+arctan\frac{y}{x}$在点$(x,y)=(0,1)$附近确定的可导函数, 则导数$y^{'}(0)=$________.

14. (3 points) 设$f(x)=x^2sin(3x)$, 则$f^{(100)}(0)=$________.

15. (3 points) 方程$x^4+2x^3+6x^2-4x-5=0$在实轴$\mathbb{R}$上有且仅有________个根.

2 选择题

1. (3 points) 当$n \to +\infty$时, 将无穷大量$n^{100}, e^n, ln(1+n^{1000}), n!$, 按它们趋于正无穷的速度由低到高排列, 正确的顺序为:

   A. $ln(1+n^{1000}), n^{100}, e^n, n!$;

   B. $n^{100}, ln(1+n^{1000}), n!, e^n$;

   C. $n^{100}, ln(1+n^{1000}), e^n, n!$;

   D. $ln(1+n^{1000}), n^{100}, n!, e^n$.

2. (3 points) 设$f(x) = \begin{cases} ax+b, & x>1, \\ x^2, & x \le 1. \end{cases}$, 假设$f(x)$在点$x=1$处可导, 则

   A. $(a,b) = (2,1)$;

   B. $(a,b) = (-2,1)$;

   C. $(a,b) = (-2,-1)$;

   D. $(a,b) = (2,-1)$.

3. (3 points) 函数x^2cosx的100阶导函数$(x^2cosx)^{100}$为

   A. $x^2cosx + 200xsinx + 9900cosx$;

   B. $x^2cosx - 200xsinx + 9900cosx$;

   C. $x^2cosx + 200xsinx - 9900cosx$;

   D. $x^2cosx - 200xsinx - 9900cosx$.

4. (3 points) 函数$\frac{1}{cosx}$在$x=0$处带Peano余项的四阶Taylor展式为

   A. $\frac{1}{cosx} = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^4)$;

   B. $\frac{1}{cosx} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^4)$;

   C. $\frac{1}{cosx} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$;

   D. $\frac{1}{cosx} = 1 + \frac{1}{2}x^2 - \frac{5}{24}x^4 + o(x^4)$.

5. (3 points) 记(x_0, y_0)为旋轮线$x = t - sint, y = 1 - cost (0 \le t \le 2\pi)$上对应参数$t = \frac{\pi}{2}$的点, 则旋轮线在点$(x_0, y_0)$处的切线方程为

   A. $y = x - \frac{\pi}{2} + 2$;

   B. $y = 2$;

   C. $y = \frac{\pi}{2}x$;

   D. $y = x + \frac{\pi}{2}$.

6. (3 points) 设$f(x)$在实轴$\mathbb{R}$上可导, 则下列说法哪一个是错误的.

   A. 若$f(x)$是偶函数, 则$f'(x)$是奇函数;

B. 若$f(x)$是周期函数, 则$f^{'}(x)$是周期函数;

C. 若$f(x)$在$\mathbb{R}$上有界, 则$f^{'}(x)$在$\mathbb{R}$上有界;

D. 若$f(x)$是奇函数, 则$f^{'}(x)$是偶函数.

7. (3 points) 设$f(x)$在$(-\infty,+\infty)$上单调有界, x_n为一数列, 则下列命题正确的是

A. 若$\{x_n\}$收敛, 则$\{f(x_n)\}$收敛;

B. 若$\{f(x_n)\}$收敛, 则$\{x_n\}$收敛;

C. 若$\{x_n\}$单调, 则$\{f(x_n)\}$收敛;

D. 若$\{f(x_n)\}$单调, 则$\{x_n\}$收敛.

8. (3 points) 函数$x^x(x>0)$的导函数为

A. $x^{\frac{1}{x}-2}$;

B. $x^{\frac{1}{x}-2}(1-lnx)$;

C. $x^{\frac{1}{x}}$;

D. $x^{\frac{1}{x}-1}$.

9. (3 points) 函数$xln(1+x)$在$x=0$处的泰勒(Taylor)多项式为

A. $\sum_{k=2}^{n}\frac{(-1)^k x^k}{k}$;

B. $\sum_{k=2}^{n}\frac{(-1)^{k-1} x^k}{k}$;

C. $\sum_{k=2}^{n}\frac{(-1)^k x^k}{k-1}$;

D. $\sum_{k=2}^{n}\frac{(-1)^{k-1} x^k}{k-1}$.

10. (3 points) 极限$\lim\limits_{x\to 0}(\frac{2^x+3^x+4^x}{3})^{\frac{3}{x}}$等于

A. 4;

B. 2;

C. 24;

D. $+\infty$.

3 解答题

11. (10 points) (a) 证明函数$f(x)=x+arctanx$在整个实轴$\mathbb{R}$上存在反函数, 记作$x=g(y), y\in\mathbb{R}$, 并且反函数$g(y)$为二次连续可微.

   (b) 计算$g^{''}(y)$.

12. (10 points) 设$n\geq 2$为正整数.

   (a) 证明方程$x+x^2+\ldots+x^{n-1}+x^n=1$在开区间$(\frac{1}{2},1)$内有且仅有一个实根, 记作$x_n$;

(b) 证明$\lim\limits_{n\to+\infty} x_n = \frac{1}{2}$.

13. (5 points) 设$f(x)$在$[0,1]$上二阶连续可导.

(a) 若$f(0) = f(1)$, 且$max_{0\le x\le 1}|f^{''}(x)| \le 2$, 证明$max_{0\le x\le 1}|f^{'}(x)| \le 1$.

(b) 构造一个$[0,1]$上的二阶连续可微函数$f(x)$, 使得$f(0) = f(1)$, $max_{0\le x\le 1}|f^{'}(x)| = 1$, 以及$max_{0\le x\le 1}|f^{''}(x)| = 2$.