4.1

1. $\begin{pmatrix} a+bi \\ c+di \end{pmatrix} \xrightarrow{A} \begin{pmatrix} (a-c)-bi \\ (-b+c)+(b-d)i \end{pmatrix}$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \xrightarrow{A} \begin{pmatrix} a-c \\ -b \\ b-c \\ b-d \end{pmatrix}$$

under this basis:

$$A = \begin{pmatrix} 1 & & -1 & \\ & -1 & & \\ & 1 & -1 & \\ & 1 & & -1 \end{pmatrix} \Rightarrow \text{eigenvalues} = -1, -1, -1, 1$$

$$\ker(A+I) = \ker \begin{pmatrix} 2 & & -1 & \\ & & & \\ & 1 & & \\ & 1 & & \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\ker(A+I)^2 = \ker \begin{pmatrix} 4 & -1 & -2 & \\ & & & \\ & & & \\ & & & \end{pmatrix} = \operatorname{span} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_1 = \operatorname{span} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 4 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \end{pmatrix} \qquad A_{V_1} = \left(\begin{array}{c|cc} -1 & & \\ \hline & -1 & 1 \\ & & -1 \end{array}\right)$$

$$\ker(A-I) = \ker \begin{pmatrix} & & -1 & \\ & -2 & & \\ & -1 & -2 & \\ & 1 & & -2 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V_2 = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A_{V_2} = [1]$$

$$\Rightarrow A = \begin{pmatrix} & 2 & 1 & 1 \\ & & 4 & \\ & 4 & & \\ 1 & 4 & & \end{pmatrix} \left(\begin{array}{c|cc|c} -1 & & & \\ \hline & -1 & 1 & \\ & & -1 & \\ \hline & & & 1 \end{array}\right) \begin{pmatrix} & 2 & 1 & 1 \\ & & 4 & \\ & 4 & & \\ 1 & 4 & & \end{pmatrix}^{-1}$$

2. $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$A(p(x)) = (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3) + a_0 + a_1 x^2$

$= (a_0 + a_1) + 2a_2 x + (a_1 + 3a_3)x^2 + 4a_4 x^3$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} a_0 + a_1 \\ 2a_2 \\ a_1 + 3a_3 \\ 4a_4 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & & & \\ & & 2 & & \\ & 1 & & 3 & \\ & & & & 4 \\ & & & & \end{pmatrix} \qquad \begin{matrix} \det A = 0 \\ tr(A) = 1 \end{matrix}$$

$\Rightarrow$ eigenvalues $= \pm\sqrt{2}, 1, 0, 0$

$$\ker(A - I) = \ker \begin{pmatrix} & 1 & & & \\ & -1 & 2 & & \\ & 1 & -1 & 3 & \\ & & & -1 & 4 \\ & & & & -1 \end{pmatrix} = span \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow V_1 = span \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ A_{V_1} = [1]$

$$\ker(A) = span \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \ker(A^2) = \ker \begin{pmatrix} 1 & 1 & 2 & & \\ & 2 & & 6 & \\ & & 2 & & 12 \\ & & & & \\ & & & & \end{pmatrix} = span \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ -6 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow V_2 = span \begin{pmatrix} 6 \\ 6 \\ -6 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 12 \\ -12 \\ 0 \\ 4 \\ 0 \end{pmatrix} \qquad A_{V_2} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$$

$$\ker(A - \sqrt{2}I) = \ker \begin{pmatrix} (1-\sqrt{2}) & 1 & & & \\ & -\sqrt{2} & 2 & & \\ & 1 & -\sqrt{2} & 3 & \\ & & & -\sqrt{2} & 4 \\ & & & & -\sqrt{2} \end{pmatrix} = span \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 1-\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_3 = span \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ -1+\frac{\sqrt{2}}{2} \\ \frac{2}{3}(\sqrt{2}-1) \\ \frac{1}{6}(-2+\sqrt{2}) \\ 0 \end{pmatrix} \qquad A_{V_3} = [\sqrt{2}]$$

$$\ker(A + \sqrt{2}I) = \ker \begin{pmatrix} (1+\sqrt{2}) & 1 & & & \\ & \sqrt{2} & 2 & & \\ & 1 & \sqrt{2} & 3 & \\ & & & \sqrt{2} & 4 \\ & & & & \sqrt{2} \end{pmatrix} = span \begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 1+\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow V_4 = span \begin{pmatrix} 1 \\ -1-\sqrt{2} \\ -1-\frac{\sqrt{2}}{2} \\ -\frac{2}{3}(\sqrt{2}+1) \\ -\frac{1}{6}(2+\sqrt{2}) \\ 0 \end{pmatrix} \qquad A_{V_4} = [-\sqrt{2}]$$

$$\Rightarrow A = \begin{pmatrix} 1 & 12 & 6 & 1 & 1 \\ & -12 & 6 & \sqrt{2}-1 & -1-\sqrt{2} \\ & & -6 & 1-\frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} \\ & 4 & -2 & & \\ & & 1 & & \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & & 1 & & \\ & & & & \\ & & & \sqrt{2} & \\ & & & & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 12 & 6 & 1 & 1 \\ & -12 & 6 & \sqrt{2}-1 & -1-\sqrt{2} \\ & & -6 & 1-\frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} \\ & 4 & -2 & & \\ & & 1 & & \end{pmatrix}^{-1}$$

3.

characteristic poly of $A = (x^2 - a_1a_4)(x^2 - a_2a_3)$

eigenvalues $= \pm\sqrt{a_1a_4}, \pm\sqrt{a_2a_3}$

(maybe complex numbers)

① $a_1a_2a_3a_4 \neq 0$

$$\ker(A - \sqrt{a_1a_4}I) = \ker\begin{pmatrix} -\sqrt{a_1a_4} & & & a_1\\ & -\sqrt{a_1a_4} & a_2 & \\ & a_3 & -\sqrt{a_1a_4} & \\ a_4 & & & -\sqrt{a_1a_4}\end{pmatrix} = \text{span}\begin{pmatrix}\sqrt{a_1}\\0\\0\\\sqrt{a_4}\end{pmatrix}$$

$$\ker(A + \sqrt{a_1a_4}I) = \ker\begin{pmatrix} \sqrt{a_1a_4} & & & a_1\\ & \sqrt{a_1a_4} & a_2 & \\ & a_3 & \sqrt{a_1a_4} & \\ a_4 & & & \sqrt{a_1a_4}\end{pmatrix} = \text{span}\begin{pmatrix}\sqrt{a_1}\\0\\0\\-\sqrt{a_4}\end{pmatrix}$$

$$\ker(A - \sqrt{a_2a_3}I) = \ker\begin{pmatrix} -\sqrt{a_2a_3} & & & a_1\\ & -\sqrt{a_2a_3} & a_2 & \\ & a_3 & -\sqrt{a_2a_3} & \\ a_4 & & & -\sqrt{a_2a_3}\end{pmatrix} = \text{span}\begin{pmatrix}0\\\sqrt{a_2}\\\sqrt{a_3}\\0\end{pmatrix}$$

$$\ker(A + \sqrt{a_2a_3}I) = \ker\begin{pmatrix} \sqrt{a_2a_3} & & & a_1\\ & \sqrt{a_2a_3} & a_2 & \\ & a_3 & \sqrt{a_2a_3} & \\ a_4 & & & \sqrt{a_2a_3}\end{pmatrix} = \text{span}\begin{pmatrix}0\\\sqrt{a_2}\\-\sqrt{a_3}\\0\end{pmatrix}$$

$$\Rightarrow A = X\begin{pmatrix}\sqrt{a_1a_4} & & & \\ & -\sqrt{a_1a_4} & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3}\end{pmatrix}X^{-1}$$

$$X = \begin{pmatrix}\sqrt{a_1} & \sqrt{a_1} & & \\ & & \sqrt{a_2} & \sqrt{a_2}\\ & & \sqrt{a_3} & -\sqrt{a_3}\\ \sqrt{a_4} & -\sqrt{a_4} & & \end{pmatrix}$$

② $a_1a_4 = 0, a_2a_3 \neq 0$

WLOG, suppose $a_4 = 0$

$$\ker(A^2) = \ker\begin{pmatrix}a_1a_4 & & & \\ & a_2a_3 & & \\ & & a_2a_3 & \\ & & & a_1a_4\end{pmatrix} = \ker\begin{pmatrix}0 & & & \\ & a_2a_3 & & \\ & & a_2a_3 & \\ & & & 0\end{pmatrix} = \text{span}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\Rightarrow V_1 = \text{span}\begin{pmatrix}a_1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad Av_1 = \begin{pmatrix}0 & 1\\ & 0\end{pmatrix}$$

$$\Rightarrow A = X\begin{pmatrix}0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3}\end{pmatrix}X^{-1}, \quad X = \begin{pmatrix}a_1 & & & \\ & & \sqrt{a_2} & \sqrt{a_2}\\ & & \sqrt{a_3} & -\sqrt{a_3}\\ & 1 & & \end{pmatrix}$$

③ $a_2a_3 = 0, a_1a_4 \neq 0$

↳ same as ②

④ $a_1a_4 = a_2a_3 = 0$

WLOG, suppose $a_3 = a_4 = 0$

$$\ker(A) = \text{span}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\ker(A^2) = \mathbb{R}^4$$

$$\Rightarrow V = \text{span}\begin{pmatrix}0\\a_2\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}a_1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\Rightarrow A = X\begin{pmatrix}0 & 1 & & \\ & 0 & & \\ & & 0 & 1\\ & & & 0\end{pmatrix}X^{-1}$$

$$X = \begin{pmatrix} & & a_1 & \\ a_2 & & & \\ & 1 & & \\ & & & 1\end{pmatrix}$$

4.2

1. suppose $n = [\dim\ker(A)] + \cdots + [\dim\ker(A^k) - \dim\ker(A^{k+1})]$

$\Rightarrow a_1 = k$

$a_2 =$ number of that ≥ 2

$\vdots$

$a_i =$ number of that $\geq i$

$n = a_1 + a_2 + \cdots + a_j = \sum \dim\ker(\cdots)$

1st dot diagram　　2nd dot diagram

$\Rightarrow$ the number of dots in i-th row in 1st

$=$ the number of dots in i-th column in 2nd

2. for any self-conjugate dot diagram:

the number of dots in i-th row

$=$ the number of dots in i-th column

$\Rightarrow$ total number of dots in i-th row & i-th column

$= 2k_i - 1$ ($k_i =$ number of dots in i-th row)

an odd number　　$k_i > k_{i+1}$

So each self-conjugate partition correspond with a partition of n into distinct odd positive integers

so the proof is done

3. $$A = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p(4=4) = p(\operatorname{rank} A = 3) = p(a \neq 0, d \neq 0, f \neq 0) = 1$$

$$A^2 = \begin{pmatrix} 0 & 0 & ad & ae+bf \\ 0 & 0 & 0 & df \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p(4=3+1) = p(4=2+1+1) = p(4=1+1+1+1) = 0$$

$p(4=4) = p(\dim\ker A = 1) = p(\operatorname{rank} A = 3)$

$= \left(\frac{2}{3}\right)^3 = \frac{8}{27} = \frac{216}{729}$

$p(4=3+1) = p(\operatorname{rank} A = 2) \times p(A^3 \neq 0)$

$= \left(1 - \frac{8}{27} - [illegible]\right) \times \left(1 - \left(\frac{1}{3}\right)^4\right) = \frac{52}{81} \cdot \frac{80}{81} = \frac{4160}{6561}$

$p(4=2+2) = p(\operatorname{rank} A = 2) \times p(A^2 = 0)$

$= \frac{52}{81} \times \frac{1}{81} = \frac{52}{6561}$

$p(4=2+1+1) = p(\operatorname{rank} A = 1)$

$=$ [illegible]

$= \frac{64}{729}$

$p(4=1+1+1+1) = p(\operatorname{rank} A = 0) = \left(\frac{1}{3}\right)^6 = \frac{1}{729}$

4.3

1. let $D=\begin{pmatrix} I & & & \\ & & I & \\ & I & & \\ & & & I \end{pmatrix}$

then $\begin{pmatrix} X & & I & \\ & Y & & I \\ & & X & \\ & & & Y \end{pmatrix} D = D \begin{pmatrix} X & I & & \\ & X & & \\ & & Y & I \\ & & & Y \end{pmatrix}$

so they're similar

2. for B: $(x-3)(x-4)$

for $A=\begin{pmatrix} 3 & & 1 & \\ & 4 & & 1 \\ & & 3 & \\ & & & 4 \end{pmatrix} \overset{\text{similar}}{\Rightarrow} \begin{pmatrix} 3 & 1 & & \\ & 4 & & \\ & & 3 & 1 \\ & & & 4 \end{pmatrix} \Rightarrow (x-3)(x-4)$

3. $A=\begin{pmatrix} B & I \\ & B \end{pmatrix} \rightarrow \text{rank}(A)=3$

$J=\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

$A^2=\begin{pmatrix} B^2 & 2B \\ & B^2 \end{pmatrix} \rightarrow \text{rank}(A^2)=1$

$A^3=0 \Rightarrow \text{rank}(A^3)=\text{rank}(A^4)=0$

4. for B: x

for A: x

5. the minimal polynomials of A & B are the same