3.1 Yes.

proof: let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

all rows parallel $\Rightarrow$ RREF $= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

so RREF of A has all column parallel

Then, do row operation to recover A from its RREF

recover 2^{nd} row: $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

$\because$ row parallel

$\therefore \frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \cdots = \frac{a_{1n}}{a_{2n}}$

$\therefore$ all columns are still parallel

$\Rightarrow$ recover all rows, for the same reason, all columns are still parallel

3.2

1. $\vec{a}^T = [a_1 \cdots a_n]$
$\vec{b}^T = [b_1 \cdots b_n]$

so the i-th row of the matrix is

$[a_1 + (b_1 - a_1)(i-1) \cdots a_n + (b_n - a_n)(i-1)]$

2. do row operations as follow:

$r_2 \to r_2 - r_1$

$r_i \ (i \geq 3) \to r_i - (i-1) r_2$

$r_i \ (i \geq 3) \to r_i - r_1$

so. $\vec{a}_i^T = \vec{0}$

RREF of the matrix:

$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 - a_1 & \cdots & b_n - a_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

so the rank ≤ 2

3.3

well-defined: BA . AB . $\underbrace{ABAB}$. $\underbrace{BABC}$

associative law

not well-defined:

BAC:

$(BA)C \to (5\times 5)(3\times 1)$

3.4

1. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}$

3. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{1010} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. from 1. we know $\begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} \cdot \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 11 & 6 \\ -20 & -11 \end{bmatrix}$

5. $= \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

6. $= \begin{bmatrix} 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 \end{bmatrix}$

7. $= 4\left\{\underbrace{\begin{bmatrix} 4^{2018} & \cdots & \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix}}_{4}\right.$

8. $= \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

9. $= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -4 & -4 & -4 \\ -4 & 4 & -4 & -4 \\ -4 & -4 & 4 & -4 \\ -4 & -4 & -4 & 4 \end{bmatrix}$

10. $= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^{1010}$

$= \begin{bmatrix} 2^{2020} & 0 & 0 & 0 \\ 0 & 2^{2020} & 0 & 0 \\ 0 & 0 & 2^{2020} & 0 \\ 0 & 0 & 0 & 2^{2020} \end{bmatrix}$

11. from 8, $\frac{1}{4} \times \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{2} = I$

$\Rightarrow \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}$

3.5

$$C=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

1. $$(A+B)^2=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}^2=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$$

$$A^2+2AB+B^2=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}+2\times\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}6 & 4\\ 4 & 4\end{bmatrix}$$

$$\Rightarrow (A+B)^2\neq A^2+2AB+B^2$$

2. $$(AB)^2=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}^2=\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}$$

$$A^2B^2=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}5 & 2\\ 2 & 1\end{bmatrix}$$

$$\Rightarrow (AB)^2\neq A^2B^2$$

3.6

1. $$X_{13}=\begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad X_{32}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}\quad X_{12}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

2. $$X_{13}X_{32}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\qquad X_{32}X_{13}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(\vec{e}_1\cdot\vec{e}_3^{\,T})(\vec{e}_3\cdot\vec{e}_2^{\,T}) \qquad (\vec{e}_3\cdot\vec{e}_2^{\,T})(\vec{e}_1\cdot\vec{e}_3^{\,T})$$

$$=\vec{e}_1\cdot(\vec{e}_3^{\,T}\vec{e}_3)\vec{e}_2^{\,T} \qquad =\vec{e}_3\cdot 0\cdot\vec{e}_3^{\,T}=\vec{0}$$

$$=\vec{e}_1\cdot\vec{e}_2^{\,T}=X_{12}$$

3. $$X_{ij}^2=(\vec{e}_i\vec{e}_j^{\,T})^2=\vec{e}_i\cdot(\vec{e}_j^{\,T}\vec{e}_i)\vec{e}_j^{\,T}$$

$$\because i\neq j$$

$$\therefore \vec{e}_j^{\,T}\cdot\vec{e}_i=0$$

$$\therefore X_{ij}^2=0$$

4.

① $AB=BA$

$$\therefore \Big(I+\sum_{i\neq j}x_{ij}\Big)\Big(I+\sum_{i\neq j}y_{ij}\Big)$$

$$=\Big(I+\sum_{i\neq j}y_{ij}\Big)\Big(I+\sum_{i\neq j}x_{ij}\Big)$$

$$\therefore I^2+\sum_{i\neq j}x_{ij}+y_{ij}+\Big(\sum_{i\neq j}x_{ij}\Big)\cdot\Big(\sum_{i\neq j}y_{ij}\Big)$$

$$=I^2+\sum_{i\neq j}x_{ij}+y_{ij}+\Big(\sum_{i\neq j}y_{ij}\Big)\cdot\Big(\sum_{i\neq j}x_{ij}\Big)$$

$$\therefore \Big(\sum_{i\neq j}x_{ij}\Big)\Big(\sum_{i\neq j}y_{ij}\Big)=\Big(\sum_{i\neq j}y_{ij}\Big)\Big(\sum_{i\neq j}x_{ij}\Big)$$

$$\therefore (A-I)(B-I)=(B-I)(A-I)$$

② $(A-I)(B-I)=(B-I)(A-I)$

$$\therefore AB-IB-AI+I^2=BA-IA-BI+I^2$$

$$\therefore AB-(A+B)+I^2=BA-(A+B)+I^2$$

$$\therefore AB=BA$$

3.7

1. up 2. right

3. $PJ = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{bmatrix}$ $\quad J^TP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \end{bmatrix}$$

$$\Rightarrow PJ + J^TP = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Because PJ is shifting things <u>right</u>, and J^TP is shifting things <u>down</u>. Meanwhile, each entry in P is the sum of its <u>up</u> entry and <u>left</u> entry. So this's not a coincidence.

4. $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad J^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad J^4 = \vec{0}$

5. $(I-J)(I+J+J^2+J^3)$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

6. $(J+I)^2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad (J+I)^3 = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$(J+I)^4 = \begin{bmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad (J+I)^k = \begin{bmatrix} 1 & k & \frac{k(k-1)}{2} & \frac{k(k-1)(k-2)}{6} \\ 0 & 1 & k & \frac{k(k-1)}{2} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$

3.8

1. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 4. $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 6. let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}B = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$

$\Rightarrow \begin{cases} a+b = a+c \\ a+b = b+d \\ c+d = a+c \\ c+d = b+d \end{cases} \Rightarrow \begin{cases} b=c \\ a=d \\ d=a \\ b=c \end{cases} \Rightarrow B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ $(a, b \in \mathbb{R})$

3.9

1. $A^3 + 2A^2 - A - I$
$= A^2 + 2A - A - I$
$= 2A - I$

$A^2 + 3A + 4I$
$= 4A + 4I$

2. let the answer $= sA + tI$ $(s, t \in \mathbb{R})$

$(I + 2A)(sA + tI)$
$= sI \cdot A + 2sA^2 + tI^2 + 2tA \cdot I$
$= (s + 2s + 2t)A + tI = I$
$\Rightarrow \begin{cases} 3s + 2t = 0 \\ t = 1 \end{cases} \Rightarrow$ answer is $(-\frac{2}{3}A + I)$