2020F Final

Tuesday, December 13, 2022 14:12

1. Consider the matrix $A = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$.

(a) (2 points) How many terms does the big formula for det A has? Also find this determinant.

non-zero

$$\det\begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} = (-1)^2 1 \times 3 \times 3 \times 1 = 9$$

1 term

Kac-Sylvester

(b) (5 points) Find the characteristic polynomial of A, and find all eigenvalues and eigenvectors of A.

$(x-3)(x-1)(x+1)(x+3)$

all 3x3 principal sub are not invertible

$tr(A)=0$

$$\det(xI-A) = \det\begin{pmatrix} x & -3 & & \\ -1 & x & -2 & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix} = x^4 + 0x^3 - 10x^2 + 0x + 9$$

Vieta's formula

$$\det\begin{pmatrix} & 3 \\ 1 & \end{pmatrix} + \det\begin{pmatrix} & 2 \\ 2 & \end{pmatrix} + \det\begin{pmatrix} & 1 \\ 3 & \end{pmatrix}$$

$$+\det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Cofactor:

$$\det\begin{pmatrix} x & -3 & & \\ -1 & x & -2 & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix} = x\det\begin{pmatrix} x & -2 & \\ -2 & x & -1 \\ & -3 & x \end{pmatrix} - (-1)\det\begin{pmatrix} -3 & & \\ -2 & x & -1 \\ & -3 & x \end{pmatrix}$$

$$= x(x^3 - 3x - 4x) + (-3x^2 + 9)$$

$$= x^4 - 10x^2 + 9$$

as a good habbit, check $tr(A)$, $\det(A)$ any way

Counting

Gaussian:

$$\det\begin{pmatrix} x & -3 & & \\ -1 & x & -2 & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix} = x \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ -1 & x & -2 & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix} = x \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & x-\frac{3}{x} & -2 & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix}$$

$$= x \det\begin{pmatrix} x-\frac{3}{x} & -2 & \\ -2 & x & -1 \\ & -3 & x \end{pmatrix} = x^2 \det\begin{pmatrix} x-\frac{3}{x} & -2 & \\ -2 & x & -1 \\ & -\frac{3}{x} & 1 \end{pmatrix}$$

$$= x^2 \det\begin{pmatrix} x-\frac{3}{x} & -2 & \\ -2 & x-\frac{3}{x} & \\ & -\frac{3}{x} & 1 \end{pmatrix} = x^2\left(\left(x-\frac{3}{x}\right)^2-4\right)$$

Gaussian

$$= x\left(\frac{x^2-3}{x}\right) \det\begin{pmatrix} \boxed{1} & -\frac{3}{x} & & \\ & \boxed{1} & -\frac{2x}{x^2-3} & \\ & -2 & x & -1 \\ & & -3 & x \end{pmatrix} = (x^2-3) \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & 1 & -\frac{2x}{x^2-3} & \\ & & x-\frac{4x}{x^2-3} & -1 \\ & & -3 & x \end{pmatrix}$$

$$= -(x^2-3) \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & 1 & -\frac{2x}{x^2-3} & \\ & & \boxed{-3} & x \\ & & x-\frac{4x}{x^2-3} & -1 \end{pmatrix} = 3(x^2-3) \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & 1 & -\frac{2x}{x^2-3} & \\ & & \boxed{1} & -\frac{1}{3}x \\ & & x-\frac{4x}{x^2-3} & -1 \end{pmatrix}$$

$$= 3(x^2-3) \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & 1 & -\frac{2x}{x^2-3} & \\ & & 1 & -\frac{1}{3}x \\ & & & \boxed{\frac{1}{3}x\left(x-\frac{4x}{x^2-3}\right)-1} \end{pmatrix} = \det\begin{pmatrix} 1 & -\frac{3}{x} & & \\ & 1 & -\frac{2x}{x^2-3} & \\ & & 1 & -\frac{1}{3}x \\ & & & p_A(x) \end{pmatrix}$$

$Ker(\lambda I - A)$

$= Ker\begin{pmatrix} 1 & * & & \\ & 1 & * & \\ & & 1 & * \\ & & & p_A(\lambda) \end{pmatrix}$, $p_A(\lambda)=0$

Conclusion: eigenvalues $\boxed{\pm 1, \pm 3}$

eigenvectors $Ker(A-I) = Ker\begin{pmatrix} -1 & 3 & & \\ 1 & -1 & 2 & \\ & 2 & -1 & 1 \\ & & 3 & -1 \end{pmatrix}$

Method 1: $\forall \vec{v}$, $\vec{v} \perp \begin{pmatrix} -1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \end{pmatrix}$, so $\boxed{\vec{v} = \begin{pmatrix} 3s \\ s \\ t \\ 3t \end{pmatrix}}$

but $\vec{v} \perp \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, so $\begin{cases} 3s-s+2t=0 \\ 2s-t+3t=0 \end{cases} \Rightarrow s+t=0$

$\boxed{\text{So } \vec{v} = \begin{pmatrix} 3s \\ s \\ -s \\ -3s \end{pmatrix}}$, $Ker(A-I) = span\begin{pmatrix} 3 \\ 1 \\ -1 \\ -3 \end{pmatrix}$

$\{3$ $span(\vdots)$

Method 2: For $\lambda=1$, REF is $\begin{pmatrix} 1 & -3 & & \\ & 1 & 1 & \\ & & 1 & -\frac{1}{3} \\ & & & 0 \end{pmatrix}$

free column

$Ker = \left\{ \begin{pmatrix} -t \\ -\frac{1}{3}t \\ \frac{1}{3}t \\ t \end{pmatrix} ; t \in \mathbb{R} \right\}$

$$2s - t + 3t = 0 \quad \rightarrow \quad s + t = 0$$

$$s = \vec{v} = \begin{pmatrix} 3s \\ s \\ -s \\ -3s \end{pmatrix}, \quad \mathrm{Ker}(A - \bar{I}) = \mathrm{span}\begin{pmatrix} 3 \\ 1 \\ -1 \\ -3 \end{pmatrix}$$

$$\lambda = \begin{cases} 3 \\ 1 \\ -1 \\ -3 \end{cases} \Longrightarrow \begin{matrix} \mathrm{span}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ \mathrm{span}\begin{pmatrix} 3 \\ 1 \\ -1 \\ -3 \end{pmatrix} \\ \mathrm{span}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 3 \end{pmatrix} \\ \mathrm{span}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \end{matrix}$$

may be you see this without effort at all ?

If time allow, check $A\vec{v} = \lambda\vec{v}$

(c) (2 points) Find all vectors $\boldsymbol{v} \in \mathbb{R}^4$ such that $\lim_{n\to\infty}(\frac{1}{3}A)^n \boldsymbol{v}$ exists.

$$A^n = X D^n X^{-1}$$

$$\left(\frac{1}{3}A\right)^n \text{ has } \lambda = \begin{cases} 1^n \\ (\frac{1}{3})^n \\ (-\frac{1}{3})^n \\ (-1)^n \end{cases} \Longrightarrow \left.\begin{matrix} \mathrm{span}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ \mathrm{span}\begin{pmatrix} 3 \\ 1 \\ -1 \\ -3 \end{pmatrix} \\ \mathrm{span}\begin{pmatrix} 3 \\ -1 \\ -1 \\ 3 \end{pmatrix} \end{matrix}\right\} \text{converse}$$

$$\mathrm{span}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \longrightarrow \text{not converge } (\vec{v}, -\vec{v}, \vec{v}, -\vec{v}, \ldots)$$

$$\text{So Ans} = \mathrm{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \\ 3 \end{pmatrix}\right)$$

(d) (2 points) Find a polynomial $p(x)$ such that the eigenvalues of $p(A)$ are $1, 0, 0, 0$.

A has $\lambda = 3, 1, -1, -3$

$p(A)$ has $p(3), p(1), p(-1), p(-3)$ → $1, 0, 0, 0$

Method 1:

(Polynomial interpolation), say $p(x) = \dfrac{(x-1)(x+1)(x+3)}{(3-1)(3+1)(3+3)}$

Vandermonde matrix

Method 2:

I has $1, 1, 1, 1$
A has $3, 1, -1, -3$
A^2 has $9, 1, 1, 9$
A^3 has $27, 1, -1, -27$

which linear comb gives $1, 0, 0, 0$?

$$\begin{pmatrix} 1 & 3 & 9 & 27 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -3 & 9 & -27 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 8 & 26 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -4 & 8 & -28 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 8 & 24 & 1 \\ 0 & 0 & 8 & -24 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 \\ & & 1 & -3 & 0 \\ & & & 48 & 1 \end{array}\right)$$

REF

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & & -\frac{1}{48} \\ & 1 & & & -\frac{1}{48} \\ & & 1 & & \frac{1}{16} \\ & & & 1 & \frac{1}{48} \end{array}\right) \to \left(\begin{array}{c|c} I & \begin{matrix} -\frac{1}{16} \\ -\frac{1}{48} \\ \frac{1}{16} \\ \frac{1}{48} \end{matrix} \end{array}\right)$$

Answer not unique, 4 answers in total
(but any one is enough)

2. We have points $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}$ on the plane $\mathbb{R}^2$. Together they form a data matrix $A = \begin{bmatrix}1 & -2 & 1\\1 & 1 & -2\end{bmatrix}$. (Note that the points are already centered.)

(a) (4 points) Find the spectral decomposition of AA^{T}. (Note that the eigenvector for the largest eigenvalue is the direction of the best fit line.)

$\boxed{\text{must be sym}}$

$$\boxed{AA^T} = \begin{pmatrix}1 & -2 & 1\\1 & 1 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 1\\1 & -2\end{pmatrix} = \begin{pmatrix}6 & -3\\-3 & 6\end{pmatrix}$$

→ see eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$ for $\lambda = 3$

by spectral thm

the other eigenvector is $\begin{pmatrix}1\\-1\end{pmatrix}$

$AA^T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}9\\-9\end{pmatrix}$, so $\lambda = 9$

→ search online for formulas for eigenvectors of 2x2 matrices

$$\text{So } U = \begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\end{pmatrix}$$

must be unit vectors

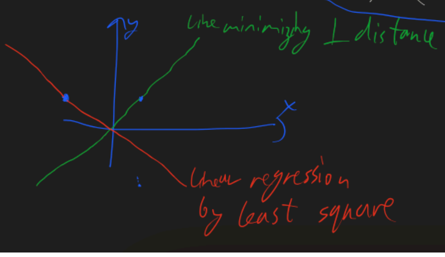
$$AA^T = U\begin{pmatrix}3 & \\ & 9\end{pmatrix}U^T$$

(b) (3 points) Find the $\mathrm{LDL}^{\mathrm{T}}$ decomposition of AA^{T}. (Note that the bottom left entry of L is the slope for the line from linear regression.)

$$\begin{pmatrix}6 & -3\\-3 & 6\end{pmatrix} = \begin{pmatrix}1 & \\ -\frac12 & 1\end{pmatrix}\begin{pmatrix}6 & -3\\0 & 4.5\end{pmatrix} = \begin{pmatrix}1 & \\ -\frac12 & 1\end{pmatrix}\begin{pmatrix}6 & 0\\0 & 4.5\end{pmatrix}\begin{pmatrix}1 & -\frac12\\ & 1\end{pmatrix}$$

row col

$\xleftarrow{\text{cholesky}}$ $\underbrace{L\begin{pmatrix}\sqrt6 & \\ & \sqrt{4.5}\end{pmatrix}}_{R}\underbrace{\begin{pmatrix}\sqrt6 & \\ & \sqrt{4.5}\end{pmatrix}L^T}_{R^T}$



(c) (4 points) Find all singular values and all left and right singular vectors of A. (Note that there might be singular vectors for the singular value zero.)

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}_{2\times 3}$$

$$\boxed{\begin{aligned} \sigma_i v_i &= A^T u_i \\ \sigma_i u_i &= A v_i \end{aligned}}$$

$$\underset{\sqrt{3}}{\sigma_1} \vec{v}_1 = A^T \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

so $\vec{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$, similarly $3\vec{v}_2 = \frac{1}{\sqrt{2}} A^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$

So $\vec{v}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\text{So } V = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

why so common?

because it is the easiest non-trivial 3x3 orth matrix

(d) (2 points) Find the maximum and minimum Rayleigh quotient $\frac{v^T S v}{v^T v}$ for $S = (A^T A)^2 - 2A^T A + 2I$.

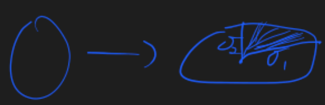
$\underset{2\times 2}{AA^T}$ eigen 3, 9

So $\underset{3\times 3}{A^TA}$ eigen 3, 9, 0

So S eigen $\underbrace{5, 65}_{\text{max}}, \underbrace{2}_{\text{min}}$

AB

BA



3. We have vectors $\boldsymbol{v}_1 = \begin{bmatrix}1\\3\\2\\4\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}1\\4\\4\\7\end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix}2\\7\\6\\11\end{bmatrix}$ in the space $\mathbb{R}^4$. We wish to find all vectors $\boldsymbol{x}$ perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(a) (4 points) Find a 3×4 matrix A such that $\boldsymbol{x}$ is perpendicular to $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ if and only if $A\boldsymbol{x} = \mathbf{0}$.

$$A = \begin{pmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \\ \vec{v}_3^{\,T} \end{pmatrix}$$

$$A\vec{x} = \vec{0}$$

(b) (4 points) Find a basis for $\mathrm{Ker}(A)$.

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 1 & 4 & 4 & 7 \\ 2 & 7 & 6 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -4 & -5 \\ & 1 & 2 & 3 \\ & & & 0 \end{pmatrix}, \quad Ker = \left\{ \begin{pmatrix} 4s+5t \\ -2s-3t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -3 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\} \longrightarrow \text{basis}$$

$$\begin{pmatrix} \boxed{1} & 0 & -4 & -5 \\ & \boxed{1} & 2 & 3 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} - I = \text{basis}$$

$$\begin{pmatrix} -4 \\ 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \\ 0 \\ -1 \end{pmatrix}$$

(c) (4 points) Find all solutions of $A\boldsymbol{x} = \begin{bmatrix}4\\5\\9\end{bmatrix}$.

$$\left\{ \begin{pmatrix} 1+4s+5t \\ 1-2s-3t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

Method 1: spot $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution

So solution set is $\boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Ker(A)}$

Method 2

Method 2: redo Gaussian

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 4 \\ 1 & 4 & 4 & 7 & 5 \\ 2 & 7 & 6 & 11 & 9 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 4 \\ & 1 & 2 & 3 & 1 \\ & 1 & 2 & 3 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} \boxed{1} & 0 & -4 & -5 & 1 \\ & \boxed{1} & 2 & 3 & 1 \\ & & & 0 & 0 \end{array}\right)$$

So solution set $= \left\{ \begin{pmatrix} 1+4s+5t \\ 1-2s-3t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$

(d) (4 points) Let $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find all points perpendicular to $B\mathbf{v}_1, B\mathbf{v}_2, B\mathbf{v}_3$.

not $B\vec{e}_1, B\vec{e}_2, B\vec{e}_3$!!!

Method 1: $\boxed{\text{Brute force}}$ $\longrightarrow$ you should know how.

Method 2: $\vec{x} \perp B\vec{v}_1, B\vec{v}_2, B\vec{v}_3$

iff $\begin{pmatrix} \vec{v}_1^T B^T \\ \vec{v}_2^T B^T \\ \vec{v}_3^T B^T \end{pmatrix} \vec{x} = \vec{0}$

iff $\boxed{A}\, B^T \vec{x} = \vec{0}$

iff $B^T \vec{x} \in \operatorname{span}\left(\begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right)$

$\in \operatorname{Ker}(A)$

iff $\vec{x} \in \operatorname{span}\left((B^T)^{-1} \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, (B^T)^{-1} \begin{pmatrix} 5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right)$

$(B^T)^{-1} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix}$

iff $\vec{x} \in \operatorname{span}\left(\begin{pmatrix} 4 \\ -6 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ -8 \\ 3 \\ 1 \end{pmatrix} \right)$

Ans

$$\begin{pmatrix} XC_1 \\ XC_2 \\ XC_3 \end{pmatrix} = \begin{pmatrix} X & & \\ & X & \\ & & X \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$(XC_1, XC_2, XC_3) = X(C_1, C_2, C_3)$$

$\begin{pmatrix} 1 & \text{all } 1 \\ \text{all } 0 & 1 \end{pmatrix}$ integration

$^{-1} = \begin{pmatrix} 1 & & \\ -1 & \ddots & \\ & -1 & 1 \end{pmatrix}$ derivative

$\begin{pmatrix} 1 & & \\ -1 & \ddots & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$

$\begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_0 \\ a_0 + a_1 \\ \vdots \end{pmatrix}$

rank 2

Ans

rank 2

4. Consider $A = \begin{bmatrix}1 & 1\\ 2 & 5\\ 2 & 8\end{bmatrix}$. We aim to find the orthogonal projection matrix to $\mathrm{Ran}(A)$.

(a) (4 points) Find the LL^{T} decomposition of $A^{\mathrm{T}}A$.

$L = R^T$

Method 1: sight QR
$$A = \begin{pmatrix}1 & 1\\ 2 & 5\\ 2 & 8\end{pmatrix} = \begin{pmatrix}1 & -2\\ 2 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\ & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{2}{3}\end{pmatrix}\underbrace{\begin{pmatrix}3 & 9\\ & 3\end{pmatrix}}_{L^T}$$

9 27

Method 2:
$$A^TA = \begin{pmatrix}9 & 27\\ 27 & 90\end{pmatrix} = \begin{pmatrix}1 & \\ 3 & 1\end{pmatrix}\underbrace{\begin{pmatrix}9 & \\ & 9\end{pmatrix}}_{D = \sqrt{D}\sqrt{D}}\begin{pmatrix}1 & 3\\ & 1\end{pmatrix}$$
$$= \begin{pmatrix}3 & \\ 9 & 3\end{pmatrix}\underbrace{\begin{pmatrix}3 & 9\\ & 3\end{pmatrix}}_{R}$$

(b) (4 points) Find the QR decomposition of A. (For simplicity, we want R to be upper triangular, while Q can be non-square but has orthonormal columns.)

then $Q = AR^{-1}$

(c) (3 points) Find the 3×3 matrix of orthogonal projection to $\mathrm{Ran}(A)$.

$$QQ^T = \left(\frac{1}{3}\begin{pmatrix}1 & -2\\ 2 & -1\\ 2 & 2\end{pmatrix}\right)\left(\frac{1}{3}\begin{pmatrix}1 & 2 & 2\\ -2 & -1 & 2\end{pmatrix}\right) = \frac{1}{9}\begin{pmatrix}5 & 4 & -2\\ 4 & 5 & 2\\ -2 & 2 & 8\end{pmatrix}$$

or $A\boxed{(A^TA)^{-1}}A$ I guess

$A^TA = QDR^T$

$A = U\Sigma V^T$

(d) (2 points) Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the left singular vectors of A for its two singular values σ_1, σ_2. Can you find $[\boldsymbol{u}_1 \;\; \boldsymbol{u}_2]\begin{bmatrix}\boldsymbol{u}_1^{\mathrm{T}}\\ \boldsymbol{u}_2^{\mathrm{T}}\end{bmatrix}$? (Hint: You may not need to calculate anything. But in that case you would need to show your arguments still.)

$\vec{u}_1, \vec{u}_2$ is ONB for $\mathrm{Ran}(A)$

Q_0

So $(\vec{u}_1, \vec{u}_2)\begin{pmatrix}\vec{u}_1^T\\ \vec{u}_2^T\end{pmatrix}$ is same as

ON basis for Ran(A)

or $A(A^TA)^{-1}A$ ← guess

(d) (2 points) Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ be the left singular vectors of A for its two singular values σ_1, σ_2. Can you find $[\boldsymbol{u}_1 \ \ \boldsymbol{u}_2]\begin{bmatrix}\boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T\end{bmatrix}$? (Hint: You may not need to calculate anything. But in that case you would need to show your arguments still.)

$A^TA = QDQ^T$

$A = U\Sigma V^T$

$\vec{u}_1, \vec{u}_2$ is ONB for $\text{Ran}(A)$

Q_0

so $(\vec{u}_1, \vec{u}_2)\begin{pmatrix}\vec{u}_1^T \\ \vec{u}_2^T\end{pmatrix}$ is same as

$Q_0 Q_0^T$

ON basis for $\text{Ran}(A)$

Or brute force

$U = (\vec{u}_1 \ \ \vec{u}_2 \ \ \vec{u}_3)$ $\boxed{\text{Ker}(A^T)}$

$\sigma_1 \quad \sigma_2$

$V = (\vec{v}_1 \ \ \vec{v}_2)$

ON basis $\text{Ran}(A^T)$

5. Consider the real matrix $M = \begin{bmatrix} A & B \\ B^T & C\end{bmatrix}$ where A, C are symmetric, and A, B, C are all 2×2 real matrices.

(a) (3 points) If B is invertible, find a formula for $\boxed{\det(M)}$ in terms of determinants of 2×2 matrices.

$m \quad n$

$$\det\begin{pmatrix} A & \boxed{B} \\ B^T & C\end{pmatrix} = (-1)^{\frac{mn}{2\times 2}} \det\begin{pmatrix} B & A \\ C & B^T\end{pmatrix}$$

$-CB^{-1}$

$$= \det\begin{pmatrix} B & A \\ 0 & B^T - CB^{-1}A\end{pmatrix}$$

$$= \det(B)\det(B^T - CB^{-1}A)$$

$-B^TA^{-1}$

$\begin{pmatrix} \boxed{A} & B \\ B^T & C\end{pmatrix}$

fail

$\begin{pmatrix} \boxed{0} & 1 \\ 2 & 3\end{pmatrix}$

fail

(b) (4 points) Suppose $B = 2A$ and $C = 4A$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find all eigenvalues and eigenvectors of M.

$M = \begin{pmatrix} \boxed{A} & \boxed{2A} \\ \boxed{2A} & \boxed{4A} \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, so M rank 1

parallel parallel

$$M = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 4 & 8 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 8 & 16 \end{pmatrix} \leftarrow \text{rank } 1,$$

So M has eigen $0, 0, 0, \boxed{\lambda}$

$tr(M) = tr(A) + 4\, tr(A) = 25 = \lambda$

Since $Ran(M) = span\boxed{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}} \longrightarrow$ must be eigenvector for $\lambda = 25$

$$Ker(M) = Ker\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = span\left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$M = \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes X$$

Kronecker tensor $\boxed{\text{product}}$

X & $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & M

If $X\vec{v} = \lambda\vec{v}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \mu\begin{pmatrix} e \\ f \end{pmatrix}$

$ae + bf = \mu e$
$ce + df = \mu f$

$$\begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix}\boxed{\begin{pmatrix} e\vec{v} \\ f\vec{v} \end{pmatrix}} = \begin{pmatrix} (ae+bf)\lambda\vec{v} \\ (ce+df)\lambda\vec{v} \end{pmatrix}$$

$$= \boxed{\lambda\mu}\begin{pmatrix} e\vec{v} \\ f\vec{v} \end{pmatrix}$$

(c) (2 points) (Hard) Suppose $A = C = 0$, and B has singular values $2, 1$. Find all eigenvalues of M.

$B = U \Sigma V^T$, $\Sigma = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}_{2\times 2}$

$$M = \begin{pmatrix} 0 & U\Sigma V^T \\ V\Sigma^T U^T & 0 \end{pmatrix} = \begin{pmatrix} U & \\ & V \end{pmatrix} \begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix} \begin{pmatrix} U^T & \\ & V^T \end{pmatrix}$$

inverse of each other

$M = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$

So $M \sim P \begin{pmatrix} & & 2 & \\ & & & 1 \\ 2 & & & \\ & 1 & & \end{pmatrix} P^{-1}$

swap 2,3 row

swap 2,3 col

$$\begin{pmatrix} & 2 & & \\ 2 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

eigenvalue of $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ are 1 & -1,

So eigenvalues of M are $2, -2, 1, -1$.

6. Consider the space V whose vectors are real functions of the form $(ax^2+bx+c)e^{2x}$ for constants $a, b, c \in \mathbb{R}$, and vector additions and scalar multiplications are defined in the obvious mannor.

(a) (2 points) Show that if $f \in V$, then its derivative f' is also in V. (So in particular, taking derivative is a linear map $D : V \to V$)

Method 1: direct verification

Method 2: let $p(t) = (t-2)^3$

then V = solution space to $\boxed{p\left(\frac{d}{dx}\right) f = 0}$

then $p\left(\frac{d}{dx}\right)\left(\frac{d}{dx} f\right) = \frac{d}{dx} p\left(\frac{d}{dx}\right) f = 0$

so $\frac{d}{dx} f \in V$.

$D = \frac{d}{dt}$

$M = f(x) \mapsto x f(x)$

$\mathrm{Span}(D, D^2, MD, I)$

(b) (4 points) Using basis $e^{2x}, xe^{2x}, \frac{1}{2}x^2e^{2x}$, write out the corresponding matrix for D. Is this matrix in the above subproblem diagonalizable? Why?

$$D = X\,(?)\,X^{-1}$$

$$DX = X(?)$$

$$D\left(e^{2x}, xe^{2x}, \tfrac{1}{2}x^2e^{2x}\right)$$

$$= \left(2e^{2x},\ e^{2x} + 2xe^{2x},\ xe^{2x} + x^2e^{2x}\right) = \left(e^{2x}, xe^{2x}, \tfrac{1}{2}x^2e^{2x}\right)(?)$$

$$= \left(e^{2x}, xe^{2x}, \tfrac{1}{2}x^2e^{2x}\right)\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$$

→ not diagonalizable since Jordan block

→ $m_g(2) = 1 \neq m_a(2)$ so not diagonalizable

(c) (4 points) Write out the change of coordinate matrix from basis $e^{2x}, xe^{2x}, \frac{1}{2}x^2e^{2x}$ to basis $x^2e^{2x}, (2x^2+2x)e^{2x}, (4x^2+8x+2)e^{2x}$.

$$I_{old \leftarrow new}\, \vec{v}_{new} = \vec{v}_{old}$$

Method 1: (new basis) $= \left(x^2e^{2x},\ (2x^2+2x)e^{2x},\ (4x^2+8x+2)e^{2x}\right)$

$$= (\text{old basis})\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{pmatrix} = (\text{old basis})\, I_{old \leftarrow new}$$

$$I_{new \leftarrow old}\, M_{old \leftarrow B}$$

So $$I_{new \leftarrow old} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & \frac{1}{2} \\ -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

Say old $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$,

$$(\text{old bases})\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\text{new basis})(\text{new coord})$$

$$= (\text{old basis})\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{pmatrix}(\text{new coord})$$

So new coord $= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(d) (4 points) Using basis x^2e^{2x}, $(2x^2+2x)e^{2x}$, $(4x^2+8x+2)e^{2x}$, write out the corresponding matrix for D. What is the characteristic polynomial of this matrix?

Method 1: $D_{new\leftarrow new} = I_{new\leftarrow old}\, D_{old\leftarrow old}\, I_{old\leftarrow new}$

$$= \begin{pmatrix} 2 & -1 & \frac{1}{2} \\ -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 8 \\ 2 & 4 & 8 \end{pmatrix}$$

Method 2:

$$\left(\boxed{x^2e^{2x}},\ (2x^2+\boxed{2x})e^{2x},\ (4x^2+8x+\boxed{2})e^{2x}\right)$$

$$x^2e^{2x} \xrightarrow{\frac{d}{dx}} (2x^2+2x)e^{2x} \xrightarrow{\frac{d}{dx}} (4x^2+8x+2)e^{2x} \xrightarrow{\frac{d}{dx}}$$

$$2(4x^2+8x+2)e^{2x}$$
$$+$$
$$(8x+8)e^{2x}$$

Method 2:

$$\left(\boxed{x^2e^{2x}},\ (2x^2+\boxed{2x})e^{2x},\ (4x^2+8x+\boxed{2})e^{2x}\right)$$

$x^2e^{2x} \xrightarrow{\frac{d}{dx}} (2x^2+2x)e^{2x} \xrightarrow{\frac{d}{dx}} (4x^2+8x+2)e^{2x} \xrightarrow{\frac{d}{dx}}$

$$2(4x^2+8x+2)e^{2x}$$
$$+$$
$$(8x+8)e^{2x}$$
$$(8x^2+24x+\boxed{12})e^{2x}$$
$$\parallel$$
$$6\,(24x^2+\boxed{48x}+\boxed{12})e^{2x}$$
$$+$$
$$-12\,(-24x^2\boxed{-24x}\quad)e^{2x}$$
$$+$$
$$8\,(8x^2\qquad)e^{2x}$$

so $D_{new} = \begin{pmatrix} 0 & 0 & 8 \\ 1 & 0 & -12 \\ 0 & 1 & 6 \end{pmatrix}$

Now to find char poly

Method 1: → companion matrix,

so char poly is $\boxed{x^3-6x^2+12x-8}$

Method 2: D is derivative, & V = solution space to $p(\frac{d}{dx})f=0$

so $p(D)=0$

$p(x)=(x-2)^3 = x^3-6x^2+12x-8$

Method 3: $D_{old} = \boxed{\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}}$, so all eigenvalue 2

so $P_D(x)=(x-2)^3$

Method 4: Brute force

7. (Proof-intensiven problem) Suppose $\boxed{A^T = A^2}$ for a real matrix A. Let us investigate the possibility of such A. Note that A is a normal matrix.

(a) (2 points) For each eigenvalue λ of A, show that $\lambda^2 = \bar{\lambda}$, and find all possible $\lambda \in \mathbb{C}$ that satisfy this condition. (Hint: Spectral decomposition of A.)

$AA^T = A^3 = A^T A$, so $\exists$ spectral decomp

unitary

$$\boxed{A = QDQ^*}$$

then $A^T = A^* = Q D^* Q^* = Q \bar{D} Q^*$

$A^2 = Q D^2 Q^*$

so $A^T = A^2$ implies $\bar{D} = D^2$, so $\bar{\lambda} = \lambda^2$ for all λ

$\lambda^3 = \lambda \bar{\lambda} = |\lambda|^2 = 1$

$|\lambda|^3 = |\lambda|^2$, so $|\lambda| = 0$ or 1

so $\lambda = e^{i\theta}$

So $\lambda = \boxed{e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, 1, 0}$

$\forall$ normal

$A = QDQ^*$

(b) (2 points) Show that $A^4 = A$.

Method 1: $\lambda^4 = (\lambda^2)^2 = (\bar{\lambda})^2 = \overline{(\lambda^2)} = \overline{(\bar{\lambda})} = \lambda$, so $D^4 = D$

Method 2: $A^4 = (A^2)^2 = (A^T)^2 = (A^2)^T = (A^T)^T = A$

(c) (2 points) Show that $\boxed{A^2 \boldsymbol{x} = \boldsymbol{x}}$ implies that $A\boldsymbol{x} = \boldsymbol{x}$.

$A^4 \vec{x} = \vec{x}$

Method 1: If $A^2 \vec{x} = \vec{x}$,

then $A\vec{x} = A^4 \vec{x} = A^2 (A^2 \vec{x}) = A^2 \vec{x} = \vec{x}$

Method 2: If $A^2 \vec{x} = \vec{x}$, then $A^T \vec{x} = \vec{x}$

Then $(A^T)^2 \vec{x} = \vec{x}$, Then $A\vec{x} = \vec{x}$

Then $(A^T)^2 \vec{x} = \vec{x}$, then $A\vec{x}' = \vec{x}$

(d) (2 points) Show that A^3 is an orthogonal projection.

$A^3 = A^2 A = A^T A$ sym

$(A^3)^2 = A^6 = A^4 A^2 = A A^2 = A^3$ proj

$\boxed{A^3} = Q \boxed{D^3} Q^*$

all eigen are 0 or 1

$(A^3)^2 = A^3$

$(A^3)^T = (A^3)^* = Q (D^3)^* Q^*$

$= Q (D^3) Q^* = A^3$

matrix

(e) (1 point) Show that $\boxed{I + A - A^3}$ is an orthogonal and $(I + A - A^3)^3 = I$. Also prove that $\det(I + A - A^3) = 1$, so this is a rotation with period 3.

$(I + A - A^3)(I + A - A^3)^T = \boxed{(I + A - A^3)(I + A^2 - A^3)}$

$= I + A + A^2 + A^3 - 2A^3 - A^4 - A^5 + A^6$

cancel

$= I + A + A^2 - \underbrace{A^4}_{A^4 = A} - A^5 = I$

$(I + A - A^3)^2 = I + A^2 + A^6 + 2A - 2A^3 - 2A^4$

$= \boxed{I + A^2 - A^3}$

$= (I + A - A^3)^{-1}$

So $(I + A - A^3)^3 = I$

Finally, $(\det(I + A - A^3))^3 = \det((I + A - A^3)^3)$

$= \det(I) = 1$

$I + A - A^3$

$= Q \left(\boxed{I + D - D^3} \right) Q^*$

$1 + \lambda - \lambda^3$: $\lambda = 0 \longmapsto 1$

$\lambda = 1 \longmapsto 1$

$\lambda = e^{i\frac{2\pi}{3}}$ or $e^{i\frac{4\pi}{3}} \longmapsto \lambda$

$(I + A - A^3)(I + A - A^3)^T$

$Q \left[(I + D - D^3)\overline{(I + D - D^3)} \right] Q^*$

$1 \cdot 1$

or $1 \cdot 1$

or $\lambda \cdot \bar{\lambda} = 1$

$= Q I Q^*$

$= I$

Finally, $\left(\det\left(I + A - A^3\right)\right)^3 = \det\left(\left(I+A-A^3\right)^3\right)$

$= \det(I) = 1$

But $\det(I+A-A^3)$ is real

so it is real cube root of 1

So it is 1.