Homework V

January 16, 2024

Name **Student's ID**

I hope everyone enjoys this assignment and learns something from it. The questions may be more flexible than what you typically encounter in textbooks, designed to exercise your ability to understand the model in a more flexible manner. Don't worry too much about grades of this assignment. Please send emails to *lijch@sem.tsinghua.edu.cn* if you find any errors in the questions.

Oligopoly model with complementary goods

Consider a commodity J that has 2 essential components, each of which is produced by a separate firm. We assume that the component 1 is produced by firm 1 and component 2 is produced by firm 2. Both firms has the same marginal cost $\frac{C}{2}$ for producing its component, and hence the total production cost of the commodity (with the mentioned 2 parts) is C. In the market consumers value the good as a bundle of perfect complementary goods. Denote the prices of the two components as p_1 and p_2 respectively, and the total price of commodity J is p_J. The market demand function hence is $Q(p_J) = Q(p_1 + p_2)$. The following subproblems are designed you to discover the market equilibrium price p_J step by step.

(a) Write the profit function of each firm.

Answers:

$$\pi_i = Q(p_i + p_j) \times (p_i - \frac{C}{2}).$$

(b) What are the best pricing response of each firm?

Answers:

WLOG, we consider a firm i who seeks to maximize profit function $\pi_i(p_i, p_j)$ defined in (a). Firm i maximize its profits by setting its marginal revenue eqaulling marginal cost (i.e. first order condition).

$$\frac{\partial \pi_i}{\partial p_i} = Q + \frac{\partial Q}{\partial (p_1 + p_2)}(p_i - \frac{C}{2}) = 0.$$

Because there are 2 firms in our problem, the corresponding equations are

$$Q + \frac{\partial Q}{\partial (p_1 + p_2)}(p_1 - \frac{C}{2}) = 0,$$

and

$$Q + \frac{\partial Q}{\partial (p_1 + p_2)}(p_2 - \frac{C}{2}) = 0.$$

The 2 equations give rise to

$$p_1 = \frac{C}{2} - \frac{Q(p_1+p_2)}{\frac{\partial Q}{\partial(p_1+p_2)}},$$

and

$$p_2 = \frac{C}{2} - \frac{Q(p_1+p_2)}{\frac{\partial Q}{\partial(p_1+p_2)}}$$

(c) Finding the pricing strategy in equilibrium of the 2 firms by combining your solutions in (b). And write out the price p_J of the commodity J.

Answers: Since the two firms are compleltely symmetric in the context, the equilibrium would be reached by letting $p_1(p_2) = p_2(p_1)$, yielding

$$\frac{p_J}{2} = \frac{C}{2} - \frac{Q(p_J)}{\frac{\partial Q}{\partial p_J}}$$

$$p_J = C - 2\frac{Q(p_J)}{\frac{\partial Q}{\partial p_J}}$$

Therefore

$$\frac{p_J - c}{2} = -\frac{Q(p_J)}{\frac{\partial Q}{\partial p_J}}$$

(d) Suppose commodity J consists of n identical complementary components produced by n different firms. Each firm has same constant marginal production cost $\frac{c}{n}$. What is the final price p_J of commodity J?

Answers:

$$\frac{p_J - c}{n} = -\frac{Q(p_J)}{\frac{\partial Q}{\partial p_J}}$$

(e) Notably, in our lecture slides, we learned that the product's price in the n firms Cournot model with substitues is $n(p-c) = -\frac{Q(p)}{Q'(p)}$, implying that market competition leads to lower price. However, in this problem with complementary goods, what's your observation?

Answers:

More production firms give rise to higher price.

P.S.

The formula $n(p-c) = -\frac{Q(p)}{Q'(p)}$ stems from a industry with n identical firms producing perfect substitutes. Denote the quantity produced by firm j as q_j. The partial equilibrium price under total supply $\sum q_j$ by n firms is written as $p(\sum q_j)$. Suppose the marginal cost is a constant term c for each firm. The profit function of a specific firm i is expressed as follows:

$$\pi_i = [p(\sum q_j) - c] \times q_i.$$

You can take first order derivative regarding q_i on the profit function to find out the production plan of i. That is, $n(p-c) = -\frac{Q(p)}{Q'(p)}$

Exchange economy

Consider a standard Edgeworth box economy with two consumers Miss.Hu and Mr.Jin, where Miss.Hu's utility function is given by $U_H(a_H, b_H) = a_H - (8 - h_H)^2$ and Mr.Jin's utility function is given by $U_J(a_J, b_J) = a_J - (8 - h_J)^2$. Notation a and b represent the consumption of apples and bananas for Miss.Hu and Mr.Jin. What's more Miss.Hu has an endowment of 58 apples and 5 bananas, while Mr.Jin has 154 apples and 3 bananas.

(1) Find ALL Pareto optimal points and show them in the Edgeworth box. (Hint: try to figure out how indifference curves behave at the Pareto optimal points)

(2) Find the core and show it in the Edgeworth box. (Hint: Core is related to the initial endowments, with no agent blocking the allocation.)

(3) Find the competitive equilibrium. (Hint: You need to specify the value of a_w, b_w, a_J, b_J, and the relative price of apples to bananas P_a/P_b in the equilibrium).

Answers: $(a_w^*, b_w^*) = (66, 4)$, $(a_J^*, b_J^*) = (146, 4)$, $p_a/p_b = 1/8$.

Externality

An airport is located next to a large tract of land owned by a housing developer. The developer would like to build houses on this land, but noise from the airport reduces the value of the land. The more planes that fly, the lower is the amount of profits that the developer makes. Let X be the number of planes that fly per day and let Y be the number of houses that the developer builds. The airport's total profits are $48X - X^2$ and the developer's total profits are $60Y - Y^2 - XY$. Let us consider the outcome under various assumptions about institutional rules and about bargaining between the airport and the developer.

"Free to Choose with No Bargaining"

(a) Suppose that no bargains can be struck between the airport and the developer and that each can decide on its own level of activity. No matter how many houses the developer builds, what is the number of planes per day that maximizes profits for the airport? Write out the profits of the airport.

Answers: 24, 576.

(b) Given that the airport is landing this number of planes, what is the number of houses that maximizes the developer's profits? Write our the profits of the developer.

Answers: 18, 324.

"Lawyer's Paradise"

(c) Suppose that a law is passed that makes the airport liable for all damages to the developer's property values. Since the developer's profits are $60Y - Y^2 - XY$ and his profits would be $60Y - Y^2$ if no planes were flown, the total amount of damages awarded to the developer will be XY. Therefore if the airport flies X planes and the developer builds Y houses, then the airport's profits after it has paid damages will be $48X - X^2 - XY$. The developer's profits including the amount he receives in payment of damages will be $60Y - Y^2 - XY + XY = 60Y - Y^2$. Then how many houses will be built by the developer? How many planes will be flown by the airport? How many profits will they make?

Answers: 30, 9, 900, 81.

Fair and exchange economy

Roger and Gordon have identical utility functions, $U(x, y) = x^2 + y^2$. There are 10 units of x and 10 units of y to be divided between them.

(a) Draw an Edgeworth box showing some of their indifference curves and mark the Pareto optimal allocations with bold black ink. (Hint: Notice that the indifference curves are nonconvex.)

(b) What are the fair allocations in this case?

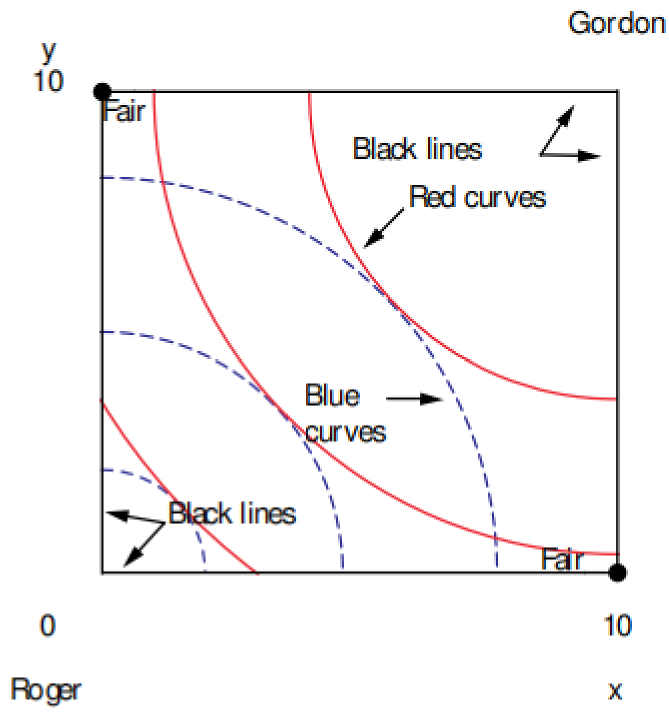

Figure 1: The answer of this problem.

Vertical Differentiation

Suppose the consumer's utility is given by $\theta s - p$, where θ is the consumer's type, s is the quality of the product, which the consumer can choose to buy or not buy, and p is the price of that product. θ is assumed to be uniformly distributed between $[\underline{\theta}, \overline{\theta}]$, and $\overline{\theta} = \underline{\theta} + 1$. Consider the case where there are two firms, indexed by $i = 1, 2$, each of which provides a product of quality s_i. Without loss of generality, let's assume $s_2 \geq s_1$. The marginal cost is c for both firms. Denote $\Delta s \equiv s_2 - s_1$.

Assumption 1 $\overline{\theta} \geq 2\underline{\theta}$

Assumption 2 $c + \frac{\overline{\theta} - 2\underline{\theta}}{3}\Delta s \leq \underline{\theta} s_1$

(a) Derive the demand for each firm, i.e., $q_i(p_1, p_2), i = 1, 2$.

Answers: The consumer i in this economy chooses one of the three options to maximize the consumer's utility: commodities produced by firm 1, commodities produced by firm2, or buying nothing.

$$\max\{\theta_i s_1 - p_1, \theta_i s_2 - p_2, 0\}$$

The demand function of firm i is a measure of the share of consumers who find the commodity produced by firm i fits them best. We will take the support of θ into consideration later.

$$\begin{aligned} q_1(p_1, p_2) &= Pr(\theta s_1 - p_1 \geq 0, \theta s_1 - p_1 \geq \theta s_2 - p_2) \\ &= Pr(\frac{p_1}{s_1} \leq \theta \leq \frac{p_2 - p_1}{s_2 - s_1}) \end{aligned} \tag{0.1}$$

$$\begin{aligned} q_2(p_1, p_2) &= Pr(\theta s_2 - p_2 \geq 0, \theta s_2 - p_2 \geq \theta s_1 - p_1) \\ &= Pr(\theta \geq \max\{\frac{p_2}{s_2}, \frac{p_2 - p_1}{s_2 - s_1}\}) \end{aligned} \tag{0.2}$$

Denote $C_1 = \frac{p_1}{s_1}$, $C_2 = \frac{p_2 - p_1}{s_2 - s_1}$, $C_3 = \max\{\frac{p_2}{s_2}, \frac{p_2 - p_1}{s_2 - s_1}\}$. By definition, $C_3 \geq C_2$. But the relation between C_1 and C_2 is unclear. Recall that $\theta \sim U[\underline{\theta}, \bar{\theta}]$.

For firm 1

When $C_1 > C_2$, $q_1(p_1, p_2) = 0$. The following discussion is based on the assumption that $C_1 \leq C_2$.

When $C_1 > \bar{\theta}$, $q_1(p_1, p_2) = 0$.

When $\underline{\theta} \leq C_1 \leq \bar{\theta}$ and $C_2 > \bar{\theta}$,

$$q_1(p_1, p_2) = \bar{\theta} - \frac{p_1}{s_1}. \tag{0.3}$$

When $\underline{\theta} \leq C_1 \leq \bar{\theta}$ and $\underline{\theta} \leq C_2 \leq \bar{\theta}$,

$$q_1(p_1, p_2) = \frac{C_2 - C_1}{\bar{\theta} - \underline{\theta}} = C_2 - C_1. \tag{0.4}$$

When $C_1 < \underline{\theta}$, $C_2 > \bar{\theta}$, $q_1(p_1, p_2) = \bar{\theta} - \underline{\theta} = 1$.

When $C_1 < \underline{\theta}$, and $\underline{\theta} \leq C_2 \leq \bar{\theta}$

$$q_1(p_1, p_2) = C_2 - \underline{\theta}. \tag{0.5}$$

When $C_1 < \underline{\theta}$, and $C_2 < \underline{\theta}$, $q_1(p_1, p_2) = 0$

For firm 2

When $C_3 > \bar{\theta}$, $q_2(p_1, p_2) = 0$.

When $\underline{\theta} \leq C_3 \leq \bar{\theta}$, $q_2(p_1, p_2) = \bar{\theta} - C_3$.

When $C_3 < \underline{\theta}$, $q_2(p_1, p_2) = 1$.

(b) Write the profits function for firm i.

The following discussion is based on the assumption that $C_1 \leq C_2$.
For firm 1
When $C_1 > \bar{\theta}$,

$$\pi_1 = 0$$

When $\underline{\theta} \leq C_1 \leq \bar{\theta}$ and $C_2 > \bar{\theta}$,

$$\pi_1 = \left(\bar{\theta} - \frac{p_1}{s_1}\right)(p_1 - c)$$

When $\underline{\theta} \leq C_1 \leq \bar{\theta}$ and $\underline{\theta} \leq C_2 \leq \bar{\theta}$,

$$\pi_1 = (C_2 - C_1)(p_1 - c).$$

When $C_1 < \underline{\theta}$, $C_2 > \bar{\theta}$,

$$\pi_1 = (p_1 - c)(\bar{\theta} - \underline{\theta}) = p_1 - c$$

When $C_1 < \underline{\theta}$, and $\underline{\theta} \leq C_2 \leq \bar{\theta}$

$$\pi_1 = (C_2 - \underline{\theta})(p_1 - c).$$

When $C_1 < \underline{\theta}$, and $C_2 < \underline{\theta}$,

$$\pi_1 = 0$$

For firm 2
When $C_3 > \bar{\theta}$,

$$\pi_2 = 0$$

When $\underline{\theta} \leq C_3 \leq \bar{\theta}$,

$$\pi_2 = (\bar{\theta} - C_3)(p_2 - c)$$

When $C_3 < \underline{\theta}$,

$$\pi_2 = p_2 - c$$

(c) Derive each firm's best-response function, that is, its optimal price, taking as given the (arbitrary) price of the competing firm.

Define $br_i(p_j) = \arg_{p_i}\left\{\frac{\partial \pi_i(p_i,p_j)}{\partial p_i} = 0\right\}$ as the best response (policy) function of firm i conditional on the price of firm j. The following discussion is based on the assumption that $C_1 \leq C_2$. The best responses of firm 1 and firm 2 are illustrated one by one.

Firm 1

Although C_1 and C_2 are 2 important critical points for firm 1, they will be less important in practice. Firm 1 will set $p_1 \leq \bar{\theta} s_1$ in any case, otherwise (i.e. $C_1 > \bar{\theta}$) consumers will never buy from it.

Analogously, a p_1 resulting in $C_2 <\underline{\theta}$ is also not a good choice, as $C_1 \leq C_2 <\underline{\theta}$ means nobody will buy from firm 1.

Inspired by the above discussion, we now analyze how firm 1 prices its products given p_2. The profit function of firm 1 is summarized here as

$$\max_{p_1} \pi_1 = \begin{cases} (p_1 - c)(\bar{\theta} - \frac{p_1}{s_1}), \underline{\theta} \leq \frac{p_1}{s_1} \leq \bar{\theta} \leq \frac{p_2 - p_1}{s_2 - s_1}; \\ (p_1 - c)(\frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}), \underline{\theta} \leq \frac{p_1}{s_1} \leq \frac{p_2 - p_1}{s_2 - s_1} < \bar{\theta}; \\ (p_1 - c)(\frac{p_2 - p_1}{s_2 - s_1} - \underline{\theta}), \frac{p_1}{s_1} < \underline{\theta} \leq \frac{p_2 - p_1}{s_2 - s_1} < \bar{\theta}. \end{cases}$$

We illustrate the best response functions of firm 1 in 3 scenarios one by one, as described in the profit function.

Scenario 1, $\underline{\theta} \leq \frac{p_1}{s_1} \leq \bar{\theta} \leq \frac{p_2 - p_1}{s_2 - s_1}$.

Suppose p_2 is high enough so that nobody buys from firm 2. This scenario requires $\frac{p_2 - p_1}{s_2 - s_1} \geq \bar{\theta}$, and

$$
\begin{aligned}
br_1(p_2) &= \arg\max(p_1 - c)(\bar{\theta} - \frac{p_1}{s_1}) \\
&= \frac{1}{2}c + \frac{1}{2}\bar{\theta}s_1.
\end{aligned}
$$

Substitute p_1 into $\frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$ and we have the first critical point of p_2 as $\frac{p_2^{*1}}{s_2} = \bar{\theta} + \frac{c}{2s_2} - \frac{\bar{\theta}s_1}{2s_2}$. The necessary condition for this scenario is $\frac{p_2}{s_2} \geq \frac{p_2^{*1}}{s_2}$. Firm 2 simply exits the market under such condition. What's more, $\underline{\theta} \leq \frac{p_1}{s_1} = \frac{\frac{1}{2}c+\frac{1}{2}\bar{\theta}s_1}{s_1} \leq \bar{\theta}$ is held as we assume $\bar{\theta} \geq 2\underline{\theta}$ and $c \leq \underline{\theta}s_1$ in the context.

Scenario 2, $\underline{\theta} \leq \frac{p_1}{s_1} \leq \frac{p_2-p_1}{s_2-s_1} \leq \bar{\theta}$.

The normal solution in this scenario is under the assumption that $\underline{\theta} < \frac{p_1}{s_1} < \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$. Then we have the inner solution

$$
br_1(p_2) = \frac{1}{2s_2}\left(cs_2 + p_2 s_1\right).
$$

There are 3 potential boundary solutions under scenario 2 as well. The first case is $\frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$. Putting br_1 into the identity generates another critical point of p_2 as $\frac{p_2^{*2}}{s_2} = \bar{\theta} + \frac{cs_2-\bar{\theta}s_1}{2s_2-s_1}$. When $\frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$ and $\frac{p_2}{s_2} > \frac{p_2^{*2}}{s_2}$, the corner solution of best response can be

$$
br_1(p_2) = p_2 - (s_2 - s_1)\bar{\theta}.
$$

What's more, $\frac{p_1}{s_1} = \frac{br_1(p_2)}{s_1} = \frac{p_2-(s_2-s_1)\bar{\theta}}{s_1} > \underline{\theta}$ should be satisfied.i.e. $\frac{p_2}{s_2} > \frac{p_2^{*3}}{s_2} := \bar{\theta} - \frac{\bar{\theta}-\underline{\theta}}{s_2}s_1$. And $\frac{p_1}{s_1} < \frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$.(Naturally satisfied)

The second case is $\frac{p_1}{s_1} = \frac{p_2-p_1}{s_2-s_1}$, and the identity implies $\frac{p_1}{s_1} = \frac{p_2}{s_2} = \frac{p_2-p_1}{s_2-s_1}$. It means the market shares owned by firm 1 is 0. Such case is trivial and can be dropped because firm 1 has strong incentive to deviate from it. For example, if firm 1 reduces p_1 by ε, then $\frac{p_1-\varepsilon}{s_1} < \frac{p_2-p_1+\varepsilon}{s_2-s_1}$ and firm 1 can earn profits.

The third case is $\underline{\theta} = \frac{p_1}{s_1}$. Obviously, the best response function is

$$
p_1 = s_1\underline{\theta}
$$

Substitute $br_1(p_2) = p_2 - (s_2 - s_1)\bar{\theta}$. into $\underline{\theta} = \frac{p_1}{s_1}$ and we have the first critical point of p_2 as $\frac{p_2^{*4}}{s_2} = 2\underline{\theta} - \frac{c}{s_1}$. When $\frac{p_2}{s_2} = \frac{p_2^{*4}}{s_2}$, the inner solution equals the boundary solution. If $\frac{p_2}{s_2} < \frac{p_2^{*4}}{s_2}$, putting $br_1(p_2)$ into $\frac{p_1}{s_1} < \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$ gives us $\underline{\theta} < \frac{p_2}{s_2} < \frac{p_2^{*5}}{s_2} := \bar{\theta} - (\bar{\theta} - \underline{\theta})\frac{s_1}{s_2}$, which is the necessary condition of such boundary solution.

But such case $p_2 = s_1\underline{\theta}$ is not the best response of firm 1. Its profit function now is $(s_1\underline{\theta} - c)(\frac{p_2-s_1\underline{\theta}}{s_2-s_1} - \underline{\theta})$, which is smaller than its profit $(p_1 - c)(\frac{p_2-p_1}{s_2-s_1} - \underline{\theta})$ when switching to scenario 3(because $s_1\underline{\theta}$ is an element of the support of p_1 in scenario 3).

Scenario 3, $\frac{p_1}{s_1} < \underline{\theta} \leq \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$.

We start from calculating the inner solution of this scenario and assume $\frac{p_1}{s_1} < \underline{\theta} < \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$. Solving the first order condition $-\frac{1}{s_1-s_2}(c - 2p_1 + p_2) - \underline{\theta} = 0$ gives us the best response function in this case.

$$
br_1(p_2) = \frac{1}{2}(p_2 + c - \underline{\theta}(s_2 - s_1)).
$$

There also have 3 potential boundary condition causing corner solutioins. First, suppose $\frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$ and put $br_1(p_2)$ into the identity and simplify it as $\frac{p_2-\frac{1}{2}(p_2+c-\underline{\theta}(s_2-s_1))}{s_2-s_1} = \bar{\theta}$, solution is: $p_2 = c - \theta s_1 + \theta s_2$, i.e.$\frac{p_2^{*6}}{s_2} = \bar{\theta} - \frac{\bar{\theta}s_1-c}{s_2}$. Like scenario 2, the best response function when $\frac{p_2}{s_2} > \frac{p_2^{*6}}{s_2}$is solved directly from the identity:

$$
br_1(p_2) = p_2 - (s_2 - s_1)\bar{\theta}.
$$

In addition, $\frac{p_1}{s_1} < \underline{\theta} < \frac{p_2-p_1}{s_2-s_1} = \bar{\theta}$ should also be satisfied, i.e. $\frac{p_2}{s_2} < \frac{p_2^{*7}}{s_2} := \bar{\theta} - (\bar{\theta} - \underline{\theta})\frac{s_1}{s_2}$.

The second boundary condition is $\underline{\theta} = \frac{p_2-p_1}{s_2-s_1}$. In fact, this condition is of no economic incentive since the market shares of firm 1 collapse to 0. Of course, we can solve it in a mathematical way. The best response function then is

$$br_1(p_2) = p_2 - (s_2 - s_1)\underline{\theta}.$$

Replace p_1 in $\frac{p_2-p_1}{s_2-s_1}$with $br_1(p_2) = \frac{1}{2}(p_2 + c - \underline{\theta}(s_2 - s_1))$ and we will have a critical point $\frac{p_2^{*8}}{s_2} = \underline{\theta} + \frac{c-\underline{\theta}s_1}{s_2}$.When $\frac{p_2}{s_2} = \frac{p_2^{*8}}{s_2}$, inner solution and corner solution coincide with each other. When $\frac{p_2}{s_2} > \frac{p_2^{*8}}{s_2}$, only the corner solution survives. In additin, $\frac{p_1}{s_1} < \underline{\theta} = \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$ should be satisfied too, i.e. $\frac{p_2^{*9}}{s_2} := \frac{p_2}{s_2} <\underline{\theta}$.

The last boundary condition is $\frac{p_1}{s_1} = \underline{\theta}$. So we have

$$p_2 = s_1\underline{\theta}$$

To get the critical value of p_2 under which the inner solution and corner solution coinside, we substitute p_1 into $\frac{p_1}{s_1}$ with $br_1(p_2) = \frac{1}{2}(p_2 + c - \underline{\theta}(s_2 - s_1))$.

$$\frac{\frac{1}{2}(p_2 + c - \underline{\theta}(s_2 - s_1))}{s_1} = \underline{\theta}.$$

Solution is: $\frac{p_2^{*10}}{s_2} = \underline{\theta} + \frac{\underline{\theta}s_1-c}{s_2}$. And when $\frac{p_2}{s_2} > \frac{p_2^{*10}}{s_2}$, inner solution is infeasible. As before, $\frac{p_1}{s_1} = \underline{\theta} < \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$ must be satisfied at the same time, implying $\underline{\theta}< \frac{p_2}{s_2} < \bar{\theta} - (\bar{\theta} - \underline{\theta})\frac{s_1}{s_2}$.

$p_2 = s_1\underline{\theta}$ is in fact a trivial solution. Similar to what we discussed in the second scenario, firm 1 will switch to scenario 2 then insist on this case.

Firm 2

Firm 2's profit maximization problem is summarized as

$$\max_{p_2} \pi_2 = \begin{cases} 0, \frac{p_2}{s_2} > \bar{\theta} OR \frac{p_2-p_1}{s_2-s_1} > \bar{\theta}; \\ (\bar{\theta} - \frac{p_2}{s_2})(p_2 - c), \frac{p_2-p_1}{s_2-s_1} < \frac{p_2}{s_2} \le \bar{\theta} \& \underline{\theta} \le \frac{p_2}{s_2}; \\ (\bar{\theta} - \frac{p_2-p_1}{s_2-s_1})(p_2 - c), \frac{p_2}{s_2} < \frac{p_2-p_1}{s_2-s_1} \le \bar{\theta} \& \underline{\theta} \le \frac{p_2-p_1}{s_2-s_1}; \\ (\bar{\theta} - \underline{\theta})(p_2 - c), \frac{p_2}{s_2} \le \underline{\theta} \& \frac{p_2-p_1}{s_2-s_1} \le \underline{\theta}. \end{cases}$$

So given p_1, best response functions of firm 2 can be organised in following scenarios.

Scenario 1, $\frac{p_2}{s_2} > \bar{\theta} OR \frac{p_2-p_1}{s_2-s_1} > \bar{\theta}$.

This is the most boring scenario. $br_2(p_1)$ is composed of a set of values.

$$br_2(p_1) = \{p_2 | \frac{p_2}{s_2} > \bar{\theta} OR \frac{p_2 - p_1}{s_2 - s_1} > \bar{\theta}\}$$

Scenario 2, $\frac{p_2-p_1}{s_2-s_1} < \frac{p_2}{s_2} \le \bar{\theta} \& \underline{\theta} \le \frac{p_2}{s_2}$.

The inner solution of the profit maximization problem gives us

$$br_2(p_1) = \frac{1}{2}c + \frac{1}{2}\bar{\theta}s_2,$$

which is feasible under necessary condition $\frac{br_2(p_1)-p_1}{s_2-s_1} < \frac{br_2(p_1)}{s_2} < \bar{\theta}$ and $\underline{\theta} \le \frac{br_2(p_1)}{s_2}$, i.e. $\frac{p_1}{s_1} > \frac{p_1^{*1}}{s_1} := \frac{\bar{\theta}}{2} + \frac{c}{2s_2}$ ($\frac{br_2(p_1)}{s_2} < \bar{\theta}$ and $\underline{\theta} \le \frac{br_2(p_1)}{s_2}$ is naturally satisfied.).

When $\frac{p_1}{s_1} \le \frac{p_1^{*1}}{s_1}$, boundary conditions will matter. Suppose $\frac{p_2}{s_2} = \bar{\theta}$ then

$$br_2(p_1) = s_2\bar{\theta}.$$

This boundary is not plossible as market shares and profits under the response function are both 0. Firm 2 can earn more simply by reducing p_2 a little bit.

Suppose $\frac{p_2-p_1}{s_2-s_1} = \frac{p_2}{s_2}$, then

$$br_2(p_1) = \frac{p_1}{s_1} s_2.$$

It requires $\frac{br_2(p_1)}{s_2} < \bar{\theta}$, i.e. $\frac{p_1}{s_1} < \bar{\theta}$.

Scenario 3, $\frac{p_2}{s_2} < \frac{p_2 - p_1}{s_2 - s_1} \leq \bar{\theta} \& \underline{\theta} \leq \frac{p_2 - p_1}{s_2 - s_1}$.

The inner solution of the profit maximization problem gives us

$$br_2(p_1) = \frac{1}{2}(c + p_1 + \bar{\theta}(s_2 - s_1)).$$

By solving $\frac{br_2(p_1)}{s_2} < \frac{br_2(p_1) - p_1}{s_2 - s_1} < \bar{\theta}$ and $\underline{\theta} < \frac{br_2(p_1) - p_1}{s_2 - s_1}$ we know the range of p_1 supporting the inner solution is $c - (s_2 - s_1)\bar{\theta} < \frac{p_1}{s_1} < \frac{c + (s_2 - s_1)\bar{\theta}}{2s_2 - s_1}$ and $\frac{p_1}{s_1} < \frac{c + (s_2 - s_1)(\bar{\theta} - \underline{\theta})}{s_1}$.

When $\frac{p_1}{s_1} \leq c - (s_2 - s_1)\bar{\theta}$, $\frac{p_2 - p_1}{s_2 - s_1} = \bar{\theta}$, and it implies

$$br_2(p_1) = p_1 + (s_2 - s_1)\bar{\theta}.$$

At the meanwhile, we shall have $\frac{p_1 + (s_2 - s_1)\bar{\theta}}{s_2} < \frac{p_1 + (s_2 - s_1)\bar{\theta} - p_1}{s_2 - s_1}$, i.e. $\frac{p_1}{s_1} < \bar{\theta}$.

When $\frac{p_1}{s_1} \geq \frac{c + (s_2 - s_1)\bar{\theta}}{2s_2 - s_1}$, the problem will meet another cornor solution under $\frac{p_2}{s_2} = \frac{p_2 - p_1}{s_2 - s_1}$, which is

$$br_2(p_1) = \frac{p_1}{s_1} s_2.$$

Now the additional necessary condition becomes $\frac{\frac{p_1}{s_1} s_2 - p_1}{s_2 - s_1} < \bar{\theta}$, i.e. $\frac{p_1}{s_1} < \bar{\theta}$.

Scenario 4, $\frac{p_2}{s_2} \leq \underline{\theta} \& \frac{p_2 - p_1}{s_2 - s_1} \leq \underline{\theta}$.

Firm 2 under scenario 4 maximizes its profits by choosing the highest value of p_2 within the feasible range.

$$br_2(p_1) = \min\{s_2 \underline{\theta}, p_1 + (s_2 - s_1)\underline{\theta}\}$$

(d) Solve for the Nash equilibrium profile (price, demand, profits) of each firm.

1. Only firm 1 operated in the market. Firm 2's market share is 0.

   This case is not stable. Because the marginal costs are the same for both fiirms and $s_2 > s_1$, it's feasible for firm 2 to enter the market and earn some profits no matter how firm 1 sets p_1.

2. Only firm 2 operates in the market. Firm 1's market share is 0.

   This case is not stable as well. Suppose firm 2 performs monopoly in the market, and firm 2 will strategically leave some consumers not buying from it. Firm 1 then will set a low price to eat the left profits. Contradictory to the context and assumption of case 2.

   Suppose firm 2 sets a low price so that all consumers buy from it. Then firm 2 has incentive to increase its price since the monopoly profits are higher. Whenever firm 2's price satisfy $\frac{p_2 - p_1}{s_2 - s_1} > \underline{\theta}$, firm 1 is attracted to join in. Contradictory.

3. Firm 2 operates as monopoly for its consumers. Firm 1 then sells products to left consumers. As a monopoly, $\frac{p_2 - p_1}{s_2 - s_1} < \frac{p_2}{s_2}$ and firm 2 sets $p_2 = \frac{1}{2}(c + \bar{\theta}s_2)$, sells $q_2 = \frac{1}{2}(\bar{\theta} - \frac{c}{s_2})$ and earns profit $\pi_2 = \frac{1}{4}\frac{c^2}{s_2} - \frac{1}{2}c\theta + \frac{1}{4}\theta^2 s_2$. Firm 1 would make its decision under its scenario 2 or scenario 3.

   But this case is unrealistic. The market prices shall satisfy $\frac{p_1}{s_1} < \frac{p_2 - p_1}{s_2 - s_1}$ to attract firm 1 to join in. In fact, $\frac{p_2 - p_1}{s_2 - s_1} < \frac{p_2}{s_2}$ implies $\frac{p_2}{s_2} < \frac{p_1}{s_1}$, $\frac{p_1}{s_1} < \frac{p_2 - p_1}{s_2 - s_1}$ implies $\frac{p_2}{s_2} > \frac{p_1}{s_1}$. Such conditions contradict with each other.

4. Both firms operate in market and split consumers demand. Firm 2's parameter satisfy $\frac{p_2}{s_2} < \frac{p_2-p_1}{s_2-s_1}$ But firm 1 confronts 2 scenarios:

   - Scenario 1.$\underline{\theta} \leq \frac{p_1}{s_1} \leq \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$
     Equilibrium prices are pined down by combining 2 firms' best respons functions.

     $$\begin{cases} p_1 = \frac{cs_2+p_2s_1}{2s_2} \\ p_2 = \frac{c+p_1+\bar{\theta}\Delta s}{2} \end{cases},$$

     we will have
     $$\begin{cases} p_1 = \frac{s_1}{(4s_2-s_1)}(c + \frac{2cs_2}{s_1} + \bar{\theta}(s_2-s_1)) \\ p_2 = \frac{s_2}{(4s_2-s_1)}(3c + 2\bar{\theta}(s_2-s_1)) \end{cases}.$$

     Related quantities sold and profits will be

     $$\begin{cases} q_1 = \frac{1}{4s_2-s_1}\left(2\bar{\theta}s_2 - c\right) \\ q_2 = \frac{1}{4s_2-s_1}\frac{s_2}{s_1}\left(\bar{\theta}s_1 - 2c\right) \end{cases}$$

     $$\begin{cases} \pi_1 = (p_1-c)q_1 \\ \pi_2 = (p_2-c)q_2 \end{cases}$$

     Remark: The equilibrium exists under condition $\underline{\theta} \leq \frac{p_1}{s_1}$, i.e.

     $$-\frac{1}{s_1-4s_2}\left(cs_1 + 2cs_2 + \underline{\theta}s_1^2 - \bar{\theta}s_1^2 - 4\underline{\theta}s_1s_2 + \theta s_1 s_2\right) > 0$$

     i.e.

     $$c + \frac{(\bar{\theta}-2\tau)(s_2-s_1)}{3} > \frac{2}{3}c(1-\frac{s_2}{s_1}) + \frac{\theta}{3}(s_1+2s_2)$$

     However, since $s_1 > c$, $s_2 - s_1 > c(\frac{s_2}{s_1}-1)$, i.e.$\frac{2}{3}c(1-\frac{s_2}{s_1}) + \frac{\theta}{3}(s_1+2s_2) > \underline{\theta}s_1$. So it means this scenario is contradictory to the assumption of this problem. This scenario thus will be dropped in future discussion.

   - Scenario 2.$\frac{p_1}{s_1} < \underline{\theta} \leq \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$

     $$\begin{cases} p_1 = \frac{p_2+c-\underline{\theta}(s_2-s_1)}{2} \\ p_2 = \frac{c+p_1+\bar{\theta}\Delta s}{2} \end{cases}$$

     i.e.

     $$\begin{cases} p_1 = \frac{\bar{\theta}-2\underline{\theta}}{3}(s_2-s_1) + c \\ p_2 = \frac{2\bar{\theta}-\underline{\theta}}{3}(s_2-s_1) + c \end{cases}$$

     Quantities and profits will be

     $$\begin{cases} q_1 = \frac{1}{3}\bar{\theta} - \frac{2}{3}\underline{\theta} \\ q_2 = \frac{2}{3}\bar{\theta} - \frac{1}{3}\underline{\theta} \end{cases}$$

     $$\begin{cases} \pi_1 = \left(\frac{1}{3}\bar{\theta} - \frac{2}{3}\underline{\theta}\right)^2(s_2-s_1) \\ \pi_2 = \left(\frac{2}{3}\bar{\theta} - \frac{1}{3}\underline{\theta}\right)^2(s_2-s_1) \end{cases}$$

     Remarks: The equilibrium exists under conditions $\frac{p_2}{s_2} < \frac{p_2-p_1}{s_2-s_1}$ and $\frac{p_1}{s_1} < \underline{\theta} < \frac{p_2-p_1}{s_2-s_1} < \bar{\theta}$, i.e.

     $$\begin{gathered} 3c < (2\bar{\theta}-\underline{\theta})s_1 - (\bar{\theta}-2\underline{\theta})s_2 \\ \underline{\theta} < \frac{1}{3}\bar{\theta} + \frac{1}{3}\underline{\theta} < \bar{\theta} \\ \frac{\bar{\theta}-2\underline{\theta}}{3}(s_2-s_1) + c < \underline{\theta}s_1 \end{gathered}$$

In summary, under the assumption in the context that $\theta \geq 2\tau$ and $\frac{\bar{\theta}-2\tau}{3}\Delta s + c < \tau s_1$ the N.E. profile of firm 1 is

$$(p_1, q_1, \pi_1) = (\frac{\bar{\theta} - 2\underline{\theta}}{3}\Delta s + c, \frac{1}{3}\bar{\theta} - \frac{2}{3}\underline{\theta}, \left(\frac{1}{3}\bar{\theta} - \frac{2}{3}\underline{\theta}\right)^2 \Delta s), \tag{0.6}$$

the profile of firm 2 is

$$(p_2, q_2, \pi_2) = (\frac{2\bar{\theta} - \underline{\theta}}{3}\Delta s + c, \frac{2}{3}\bar{\theta} - \frac{1}{3}\underline{\theta}, \left(\frac{2}{3}\bar{\theta} - \frac{1}{3}\underline{\theta}\right)^2 \Delta s). \tag{0.7}$$

(e) Interpret these results and their relationship to Δs.

We learn from (0.6) and (0.7) that in equilibrium the quantities produced by both firms are invariant with respect to quality gap Δs. For $s_2 > s_1$, firm 1 will set a lower price to compete with firm 2. As s_1 gets smaller , Δs gets bigger and hence low quality firm has a price and profit levels that decrease with its own quality. Δs affects customers' willingness to pay for the products. It deternines the markup rate of the two products. Larger Δs represents larger product differentiation, yet larger product differentiation renders less market competition. Consequently, both the firms can harvest more profits with larger Δs.