

10.1

eigenvalue eigenvector

$$1. \quad 1 \quad \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

$$2. \quad 1 \quad \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

$$2 \quad \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$$

$$3 \quad \begin{pmatrix} x \\ 2x \\ 2x \end{pmatrix}$$

$$3. \quad 7 \quad \begin{pmatrix} 11x \\ -3x \end{pmatrix}$$

$$= 1 \quad \begin{pmatrix} x \\ x \end{pmatrix}$$

$$4. \quad -1 \quad \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}$$

$$1 \quad \begin{pmatrix} x \\ y \\ x \end{pmatrix}$$

$$5. \quad b - ai \quad \begin{pmatrix} x \\ x_2 \\ x_2 \end{pmatrix}^{-1}$$

$$b + ai \quad \begin{pmatrix} x_2 \\ x \end{pmatrix}$$

$$6. \quad 3i \quad \begin{pmatrix} \frac{1+3i}{4}z \\ \frac{-1+3i}{4}z \\ z \end{pmatrix}$$

$$-3i \quad \begin{pmatrix} \frac{3i+1}{4}z \\ \frac{1+3i}{4}z \\ z \end{pmatrix}$$

$$0 \quad \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix}$$

10.2

1. $M_p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

characteristic polynomial of $M_p = x^2 + 1$

eigenvalues: $\pm i$

2. $M_p = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$

characteristic polynomial of $M_p = x^2 - 3x + 2$

eigenvalues: 1, 2

3. eigenvalues are 1, 2 and 3

4. characteristic polynomial of $M_p = \det(xI - M_p)$

$$\begin{aligned}\det(xI - M_p) &= \det \begin{bmatrix} x & a_0 & & & \\ 1 & x & \ddots & & \\ & 1 & x & \ddots & \\ & & 1 & x & a_{n-2} \\ & & & 1 & x+a_n \end{bmatrix} \\ &= (a_{n-1}+x) \det \begin{bmatrix} x & & & & \\ 1 & x & \ddots & & \\ & 1 & x & \ddots & \\ & & 1 & x & -1 \end{bmatrix} - a_{n-2} \det \begin{bmatrix} x & x & \ddots & & \\ 1 & x & \ddots & & \\ & 1 & x & \ddots & \\ & & 1 & x & -1 \end{bmatrix} - \dots \\ &\quad + (-1)^{n+1} a_0 \det \begin{bmatrix} -1 & x & & & \\ 1 & x & \ddots & & \\ & 1 & x & \ddots & \\ & & 1 & x & -1 \end{bmatrix} \\ &= (a_{n-1}+x) \cdot x^{n-1} - a_{n-2} \cdot (-x^{n-2}) \dots + a_0 (-1)^{n+1} (-1)^{n-1} \\ &= x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0\end{aligned}$$

so eigenvalues of M_p are roots of p

10.3

$$1. P_A(x) = x^2 - bx - a$$

$$P_A(4) = P_A(7) = 0 \Rightarrow \begin{cases} 16 - 4b - a = 0 \\ 49 - 7b - a = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = -28 \\ b = 11 \end{cases}$$

$$2. P_A(x) = x^3 - cx^2 - bx - a$$

$$P_A(2) = P_A(3) = P_A(4) = 0$$

$$\Rightarrow \begin{cases} 8 - 4c - 2b - a = 0 \\ 27 - 9c - 3b - a = 0 \\ 64 - 16c - 4b - a = 0 \end{cases} \Rightarrow \begin{cases} a = 24 \\ b = -2b \\ c = 9 \end{cases}$$

$$3. P_A(k) = k^3 - (x+2)k^2 + (2x-8)k + 16$$

$$P_B(k) = k^3 - (4+y)k^2 + (9+4y)k - 4y \quad \left. \right\} \text{have same roots}$$

$$\Rightarrow \begin{cases} x = -2 \\ y = -4 \end{cases}$$

$$4. \det(A) = 2\det \begin{pmatrix} 2 & 1 \\ 1 & a \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix} + \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= 2(2a-1) - (a-1) - 1 = 4a-2-a+(-1) = 3a-2$$

$$\Rightarrow a = \frac{2}{3}$$

$$\vec{Ax} = \begin{bmatrix} b+3 \\ 2b+2 \\ a+b+1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix} \Rightarrow \frac{b+3}{1} = \frac{2b+2}{b} = \frac{a+b+1}{1}$$

$$\Rightarrow \begin{cases} a = 2 \\ b = 1 \end{cases}$$

10.4

$$1. n=1 : H_1^2 = [1] = 1 \times I_m$$

$$\text{suppose: } n=k, H_n^2 = kI$$

$$\text{then } n=2k: H_{2n}^2 = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}^2 \\ = \begin{bmatrix} 2H_n^2 & 0 \\ 0 & 2H_n^2 \end{bmatrix} \\ = \begin{bmatrix} 2nI & 0 \\ 0 & 2nI \end{bmatrix} = 2nI$$

$$\text{so } H_n^2 = nI$$

\Rightarrow eigenvalue of nI is n

so eigenvalues of H_n^2 are $\pm\sqrt{n}$

$$2. \text{tr}(H_n) = \text{tr} \begin{bmatrix} H_{\frac{n}{2}} & H_{\frac{n}{2}} \\ H_{\frac{n}{2}} & -H_{\frac{n}{2}} \end{bmatrix} = 0$$

$$\text{so } \sum \text{all eigenvalues of } H_n = 0$$

\Rightarrow algebraic multiplicity of all eigenvalues are $\frac{n}{2}$

$$3. \frac{1}{\sqrt{2}}H_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \left(\frac{1}{\sqrt{2}}H_2\right)^2 = \frac{1}{2}H_2^2 = I$$

so $\frac{1}{\sqrt{2}}H_2$ is a reflection

\Rightarrow 2 linearly independent eigenvectors: $\begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} \sqrt{2}-1 \\ -1 \end{bmatrix}$

$$4. \text{① suppose } H_{2n} \cdot \begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix} = \sqrt{2}\lambda \begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix}$$

$$\begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix} \begin{bmatrix} x\vec{v} \\ y\vec{v} \end{bmatrix}$$

$$\begin{bmatrix} x\lambda\vec{v} + y\lambda\vec{v} \\ x\lambda\vec{v} - y\lambda\vec{v} \end{bmatrix}$$

$$\Rightarrow y = (\sqrt{2}-1)x$$

$$\text{then let } x=1, y=\sqrt{2}-1$$

$$\Rightarrow \begin{bmatrix} \vec{v} \\ (\sqrt{2}-1)\vec{v} \end{bmatrix} \text{ is the answer}$$

② similar process as ①

$$\Rightarrow \begin{cases} x+y = -\sqrt{2}x \\ x-y = -\sqrt{2}y \end{cases}$$

$$\Rightarrow \begin{cases} (\sqrt{2}+1)x = -y \\ (\sqrt{2}-1)y = x \end{cases}$$

$$\text{then let } x=1, y=-1-\sqrt{2}$$

$$\Rightarrow \begin{bmatrix} \vec{v} \\ -(1+\sqrt{2})\vec{v} \end{bmatrix} \text{ is the answer}$$

10.5

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & 4 & -1 \\ -1 & 4 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$1. A H_4 = \begin{bmatrix} 1 & 5 & 5 & 5 \\ 1 & -5 & 5 & -5 \\ 1 & 5 & -5 & 5 \\ 1 & -5 & 5 & 5 \end{bmatrix}$$

$$2. H_4^{-1} A H_4 = H_4^{-1} H_4 A H_4 = \begin{bmatrix} 1 & 5 & 5 & 5 \\ 1 & 5 & 5 & 5 \\ 1 & 5 & 5 & 5 \\ 1 & 5 & 5 & 5 \end{bmatrix}$$

$(H_4^2 = 4I)$

$$3. \det(A) = \frac{\det(H_4^{-1} A H_4)}{\det(H_4)^2} = \frac{125}{1} = 125$$

$$\begin{aligned} A^{-1} &= H_4 (H_4^{-1} A H_4)^{-1} H_4^{-1} = H_4 \cdot \begin{bmatrix} \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} & \frac{1}{20} \end{bmatrix}^{-1} H_4 \\ &= \begin{pmatrix} 0.4 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix} \end{aligned}$$

10.6

$$1. AF_4 = \begin{bmatrix} 10 & -2+2i & -2 & -2+2i \\ -2 & 2-2i & 2 & 2+2i \\ -2 & 2+2i & -2 & 2-2i \\ 10 & -2+2i & 2 & -2-2i \end{bmatrix}$$

$$2. X = F_4 \quad D = \begin{bmatrix} 10 & -2+2i & & \\ & -2-2i & & \\ & & 2 & \\ & & & -2+2i \end{bmatrix}$$

$$3. \det(A) = \frac{\det(D)}{\det(F_4) \det(F_4)} = \frac{-160}{(\frac{160}{80})(-16i)} = -160$$

$$A^{-1} = X D^{-1} X^{-1} = F_4 \left[\begin{bmatrix} \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} \end{bmatrix} \right] F_4^{-1} = \begin{bmatrix} \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} \\ \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} \\ \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} \\ \frac{1}{40} & \frac{1}{40} & \frac{1}{40} & \frac{1}{40} \end{bmatrix}$$

$$0.7 \quad \det = 0 \quad \text{tr} = 0$$

eigenvalues are 1 -1

eigenvectors: $\text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\text{span}\begin{pmatrix} 33 \\ -1 \\ 22 \end{pmatrix}$

$$A = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}$$

$$\Rightarrow A^{2017} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A^{2017} \begin{pmatrix} 33 \\ -1 \\ 22 \end{pmatrix} = - \begin{pmatrix} 33 \\ -1 \\ 22 \end{pmatrix}$$

$$\Rightarrow A^{2017} = A = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}$$

10.8

$$1. \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{1024} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2^{1024} & 5^{1024} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2^{1024}}{3} & \frac{-5^{1024}}{3} \\ \frac{1}{3} & \frac{2 \times 5^{1024}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2^{1025} + 5^{1024}}{3} & \frac{2 \times 5^{1025} - 2^{1024}}{3} \\ \frac{1}{3} & \frac{2^{1024} + 25^{1024}}{3} \end{bmatrix}$$

$$\text{Since } \frac{5^{1024} - 2^{1024}}{3} > 10^{700}$$

so all entries of $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^{1024}$ are larger than 10^{700}

$$2. \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3+i & 3-i \\ -5-i & -5+i \end{bmatrix} \begin{bmatrix} i & -i \\ -i & -i \end{bmatrix} \begin{bmatrix} 3+i & 3-i \\ -5-i & -5+i \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}^{1024} = \begin{bmatrix} 3+i & 3-i \\ -5-i & -5+i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3+i & 3-i \\ -5-i & -5+i \end{bmatrix}^{-1} = I_2$$

$$3. \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$$

$$= X \begin{bmatrix} \frac{1+\sqrt{5}i}{2} & \frac{1-\sqrt{5}i}{2} \\ \frac{1-\sqrt{5}i}{2} & \frac{1+\sqrt{5}i}{2} \end{bmatrix} X^{-1}$$

$$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}^{1024}$$

$$= X \begin{bmatrix} \frac{(1+\sqrt{5}i)^{1024}}{2} & \frac{(1-\sqrt{5}i)^{1024}}{2} \\ \frac{(1-\sqrt{5}i)^{1024}}{2} & \frac{(1+\sqrt{5}i)^{1024}}{2} \end{bmatrix} X^{-1}$$

since $\frac{1+\sqrt{5}i}{4}, \frac{1-\sqrt{5}i}{4} = 1$
 $(\frac{1+\sqrt{5}i}{2})^2 = \frac{1+\sqrt{5}i}{2}$

$$= X \begin{bmatrix} \frac{1+\sqrt{5}i}{2} & \frac{1-\sqrt{5}i}{2} \\ \frac{1-\sqrt{5}i}{2} & \frac{1+\sqrt{5}i}{2} \end{bmatrix} X^{-1}$$

$$= \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}$$

10.9

eigenvalue algebraic multiplicity geometric multiplicity diagonalizable?

1.	10	3	1	X
2.	10 10.001 10.002	1 1 1	1 1 1	✓
3.	0	4	1	X
4.	$\frac{2}{700}$ $\frac{-2}{700}$ $\frac{1}{700}$ $\frac{-1}{700}$	1 1 1 1	1 1 1 1	✓