7.1

1. proof:

$$|X\cup Y\cup Z| = |X\cup(Y\cup Z)| \overset{\substack{\text{inclusion- exclusion principals}\\ \text{for subsets}\\ \uparrow}}{=} |X| + |Y\cup Z| - |X\cap(Y\cup Z)|$$

also: $|Y\cup Z| = |Y| + |Z| - |Y\cap Z|$

$$|X\cap(Y\cup Z)| = |(X\cap Y)\cup(X\cap Z)| = |X\cap Y| + |X\cap Z| - |(X\cap Y)\cap(X\cap Z)|$$
$$= |X\cap Y| + |X\cap Z| - |X\cap Y\cap Z|$$

$\Rightarrow |X\cup Y\cup Z| = |X| + |Y| + |Z| - |Y\cap Z| - |X\cap Y| - |X\cap Z| + |X\cap Y\cap Z|$

2.

X: $y=x$

Y: $x=0$ $\qquad 2 \neq 1+1+1-0-0-0+0$

Z: $y=0$

3. $\dim(X+(Y+Z)) = \dim(X) + \dim(Y+Z) - \dim(X\cap(Y+Z))$

$= \dim(X) + \dim(Y) + \dim(Z) - \dim(Y\cap Z) - \dim(X\cap(Y+Z))$

$\leq \dim(X) + \dim(Y) + \dim(Z) - \dim(Y\cap Z) - \dim(X\cap Y) - \dim(X\cap Z) + \dim(X\cap Y\cap Z)$

$\Rightarrow$ only need to proof: $\dim(X\cap(Y+Z)) \geq \underline{\dim(X\cap Y) + \dim(X\cap Z) - \dim(X\cap Y\cap Z)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad \dim((X\cap Y)+(X\cap Z))$

$\Rightarrow$ only need to proof: $X\cap(Y+Z) \supseteq (X\cap Y) + (X\cap Z)$

$\forall \vec{a} \in (X\cap Y)+(X\cap Z)$,

$\exists \vec{v}\in X\cap Y$, $\vec{w}\in X\cap Z$

$\vec{a}$ is a linear combination of $\vec{v}$ and $\vec{w}$

since $\vec{v}, \vec{w}\in X \Rightarrow \vec{a}\in X$

$\vec{v}\in Y$, $\vec{w}\in Z \Rightarrow \vec{a}\in Y+Z$

$\Rightarrow \vec{a}\in X\cap(Y+Z)$

so $(X\cap Y)+(X\cap Z) \subseteq X\cap(Y+Z)$

proof is done

4. $A = C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathrm{rank}(ABC) = \mathrm{rank}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1$

$\mathrm{rank}(AB) + \mathrm{rank}(BC) - \mathrm{rank}(B) = 1 + 1 - 2 = 0$

7.2

1.

step1: $\ker(A) \subseteq \ker(A')$

$\forall \vec{x} \in \ker(A)$, $A\vec{x} = \vec{0}$

by doing Gaussian Elimination, which doesn't change $\vec{0}$

$\Rightarrow \mathrm{RREF}(A) \cdot \vec{x} = \vec{0}$

by delete and add zero rows, which also doesn't change $\vec{0}$

$\Rightarrow A' \cdot \vec{x} = 0$

step2: $\ker(A') \subseteq \ker(A)$

$\forall \vec{x} \in \ker(A')$, $A'\vec{x} = \vec{0}$

first restore all original zero-rows, which doesn't change $\vec{0}$

$\Rightarrow \mathrm{RREF}(A) \cdot \vec{x} = \vec{0}$

then do the inverse of Gaussian Elimination, which is row operations that also doesn't change $\vec{0}$

$\Rightarrow A \cdot \vec{x} = \vec{0}$

$$\Rightarrow \ker(A) = \ker(A')$$

2.

① $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 0 \\ 4 \\ -1 \end{bmatrix}$ are linearly independent

② $\dim(\ker(A)) = \dim(\mathrm{domain}(A)) - \mathrm{rank}(A)$

$$= 4 - 2 = 2$$

since there's only 2 non-zero columns,

$\Rightarrow \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 4 \\ -1 \end{bmatrix}$ form a basis for $\ker(A)$

3.

row operation $DA' = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$

column operation $A'D = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$

4.

$$(A')^2 = A' \cdot A' = A' \cdot (DA') = (A'D)A' = DA' = A'$$

$$\Rightarrow A'(A' - I) = (A')^2 - A' = A' - A' = 0$$

so $\forall \vec{x} \in \mathrm{Ran}(A' - I)$, $\vec{x} \in \ker(A)$

$$\Rightarrow \mathrm{Ran}(A' - I) \subseteq \ker(A)$$

5.

suppose $\underset{n\times n}{U} = \begin{bmatrix} U_1 & B \\ & U_1' \end{bmatrix}$

$$\Rightarrow \mathrm{Rank}(U) \geq \mathrm{rank}(U_1) + \mathrm{Rank}(U_1')$$

$$U_1 = \begin{bmatrix} U_2 & B_1 \\ & U_2' \end{bmatrix}$$

the same for all U_i'

$$\Rightarrow \mathrm{Rank}(U_1) \geq \mathrm{rank}(U_2) + \mathrm{Rank}(U_2')$$

$\vdots$

$$U_n = [U_{11}] \Rightarrow \mathrm{rank}(U_n) = \begin{cases} 1 & (U_{11} \neq 0) \\ 0 & (U_{11} = 0) \end{cases}$$

$$\Rightarrow \mathrm{Rank}(U) \geq \underbrace{1 + 1 + \cdots + 1}_{k \text{ times}} + \underbrace{0 + \cdots + 0}_{(n-k) \text{ times}} = k$$

7.3

1.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ & 1 & 4 & 0 & 5 \\ & & & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ & 1 & 4 & 0 & 5 \\ & & 0 & 0 & 0 \\ & & & 1 & 6 \\ & & & & 0 \end{bmatrix} \xrightarrow{-I} \begin{bmatrix} 0 & 0 & 2 & 0 & 3 \\ & 0 & 4 & 0 & 5 \\ & & -1 & 0 & 0 \\ & & & 0 & 6 \\ & & & & -1 \end{bmatrix}$$

$\Rightarrow$ basis : $\begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \\ 6 \\ -1 \end{bmatrix}$

2. basis : $\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -c \\ b \end{bmatrix}$

3. ① basis for $\ker(A)$;

$$\text{RREF of } A = \begin{bmatrix} 1 & 0 & \frac{b-n}{r-n} & \frac{q-n}{r-n} & \frac{k-n}{r-n} & \frac{b-n}{r-n} & 0 & 1 \\ & 1 & \frac{r-b}{r-n} & \frac{r-q}{r-n} & \frac{r-k}{r-n} & \frac{r-b}{r-n} & 1 & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = A'$$

$$A' - I = \begin{bmatrix} \frac{b-n}{r-n} & \frac{q-n}{r-n} & \frac{k-n}{r-n} & \frac{b-n}{r-n} & 0 & 1 \\ \frac{r-b}{r-n} & \frac{r-q}{r-n} & \frac{r-k}{r-n} & \frac{r-b}{r-n} & 1 & \\ -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix} \Rightarrow \text{basis for } \ker(A)$$

② basis for $\ker(A^T)$

$$A^T = \begin{bmatrix} r & p & 0 & 0 & 0 & 0 & p & r \\ n & p & & & & & p & n \\ b & p & & & & & p & b \\ q & p & & & & & p & q \\ k & p & & & & & p & k \\ b & p & & & & & p & b \\ n & p & & & & & p & n \\ r & p & & & & & p & r \end{bmatrix} \Rightarrow \text{RREF of } A^T = \begin{bmatrix} 1 & & & & & & & 1 \\ & 1 & & & & & 1 & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} = A'$$

$$\Rightarrow A' - I = \begin{bmatrix} 0 & & & & & & 1 \\ 0 & & & & & 1 & \\ -1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & & & & & -1 & \end{bmatrix} \Rightarrow \text{basis for } \ker(A^T)$$

③ basis for $\mathrm{Ran}(A) \perp \ker(A^T)$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

④ basis for $\mathrm{Ran}(A^T) \perp \ker(A)$

$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ \frac{r-b}{r-n} \\ \frac{r-a}{r-n} \\ \frac{r-k}{r-n} \\ \frac{r-b}{r-n} \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ \frac{b-n}{r-n} \\ \frac{a-n}{r-n} \\ \frac{k-n}{r-n} \\ \frac{b-n}{r-n} \\ 0 \\ 1 \end{bmatrix}$$

4.

$$\left.\begin{aligned} \mathrm{RREF}(A) &= A \\ \mathrm{RREF}(B) &= B \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} A' &= \begin{bmatrix} I_{n\times n} & F \\ O_{n\times n} & O_{n\times n} \end{bmatrix} \Rightarrow A' - I = \begin{bmatrix} O_{n\times n} & F \\ O_{n\times n} & -I_{n\times n} \end{bmatrix} \\ B' &= \begin{bmatrix} I_{n\times n} & G \\ O_{n\times n} & O_{n\times n} \end{bmatrix} \qquad B' - I = \begin{bmatrix} O_{n\times n} & G \\ O_{n\times n} & -I_{n\times n} \end{bmatrix} \end{aligned}\right.$$

$\Rightarrow$ basis for $\ker(A)$: $\begin{bmatrix} F\vec{e_i} \\ -\vec{e_i} \end{bmatrix}$ $(i=1,2,\cdots,n)$ basis for $\ker(B)$: $\begin{bmatrix} G\vec{e_i} \\ -\vec{e_i} \end{bmatrix}$ $(i=1,2,\cdots,n)$

$$\Rightarrow \begin{bmatrix} F\vec{e_i} \\ -\vec{e_i} \end{bmatrix} = \begin{bmatrix} G\vec{e_i} \\ -\vec{e_i} \end{bmatrix} \Rightarrow F\vec{e_i} = G\vec{e_i} \Rightarrow F = G$$

7.4

1. suppose $H = [\vec{h}_1 \cdots \vec{h}_n]$

$$H^T H = \begin{bmatrix} \vec{h}_1^T \\ \vdots \\ \vec{h}_n^T \end{bmatrix} [\vec{h}_1 \cdots \vec{h}_n] = \begin{bmatrix} |\vec{h}_1|^2 & & \\ & \ddots & \\ & & |\vec{h}_n|^2 \end{bmatrix}$$

2. ① it's obvious all entries in $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ are all ± 1

② suppose $H = [\vec{h}_1 \cdots \vec{h}_n]$

$$\Rightarrow \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \left[\begin{pmatrix} \vec{h}_1 \\ \vec{h}_1 \end{pmatrix} \cdots \begin{pmatrix} \vec{h}_n \\ \vec{h}_n \end{pmatrix} \begin{pmatrix} \vec{h}_1 \\ -\vec{h}_1 \end{pmatrix} \cdots \begin{pmatrix} \vec{h}_n \\ -\vec{h}_n \end{pmatrix} \right]$$

$i, j = 1, 2, \cdots, n$ $(i \neq j)$: $\begin{pmatrix} \vec{h}_i \\ \vec{h}_i \end{pmatrix}^T \begin{pmatrix} \vec{h}_j \\ \vec{h}_j \end{pmatrix} = 2(\vec{h}_i^T \cdot \vec{h}_j) = 2 \times 0 = 0$

$i, j = 1, 2, \cdots, n$ (i can be equal to j) $\begin{pmatrix} \vec{h}_i \\ \vec{h}_i \end{pmatrix}^T \begin{pmatrix} \vec{h}_j \\ -\vec{h}_j \end{pmatrix} = \vec{h}_i^T \cdot \vec{h}_j - \vec{h}_i^T \cdot \vec{h}_j = 0$

$\Rightarrow$ all columns in $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ are mutually

3. NO. for any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $x_1, x_2, x_3 \in \{\pm 1\}$, their dot product can only be ± 1 or ± 3, which means they'll never be mutually orthogonal

4. ① permute or negate columns: these operations preserve column vectors' directions, which means column vectors are still mutually orthogonal

② permute rows: $\forall \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are columns of H. $\Rightarrow \vec{a} \cdot \vec{b} = 0$, $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i \cdot b_i = 0$

permute any 2 rows of H $r_i \leftrightarrow r_j$

which doesn't change the result of $\vec{a} \cdot \vec{b}$

$\Rightarrow$ columns of H are still mutually orthogonal

③ negate rows: if negate i-th row of H, $\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + (-a_i)(-b_i) + \cdots + a_n b_n = 0$

$\Rightarrow$ columns of H are still mutually orthogonal

5. $\forall H = [\vec{h_1}\ \vec{h_2}\ \vec{h_3}\ \vec{h_4}]$

since columns in H are mutually orthogonal,

$\Rightarrow$ the number of 1 in $\vec{h_i}$ $(i=1,2,3,4)$ are either all odd (1 or 3) or all even (0 or 2 or 4)

$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\begin{bmatrix}-1\\-1\\-1\\-1\end{bmatrix}$

step 1: proof that "all odd" H are all equivalent:

① have only one "1" in $\vec{h_i}$: $\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$ $\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}$ $\begin{bmatrix}-1\\-1\\1\\-1\end{bmatrix}$ $\begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}$

$\updownarrow$ opposite

② have three "1" in $\vec{h_i}$: $\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$ $\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}$ $\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$ $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$

$\Rightarrow$ So, by permute columns or negate columns,

any columns can change into any "odd" columns,

which creates any other "all odd" H

$\Rightarrow$ so, "all odd" H are all equivalent.

Similarly, for "all even" H, they're all equivalent

step 2: $\exists$ an "all odd" H_o and an "all even" H_e, they're equivalent

suppose: $H_o = \begin{bmatrix}1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}$ $H_e = \begin{bmatrix}-1 & 1 & 1 & 1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}$

it's just negate 1^{st} row of H_o, we get H_e

$\Rightarrow \begin{cases}\text{all "all odd" } H \text{ are equivalent}\\ \text{all "all even" } H \text{ are equivalent}\\ \text{an "all odd" } H_o \text{ and an "all even" } H_e \text{ are equivalent}\end{cases}$

$\Rightarrow$ all 4×4 H are equivalent

7.5

1. $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ symmetric, not positive definite

(has a "-1" diagonal entry)

2. find a basis for $W \Rightarrow A$
and for $W^\perp \Rightarrow B$

since colums of A & B are all mutually orthogonal
$\Rightarrow [A\ B]$ form a new basis

$\dim W = \dim(\mathrm{Ran}(A))$
since A is a basis (column linearly independent), $\dim(\ker(A)) = 0$
$\Rightarrow \dim W = \dim(\mathrm{domain}(A)) - 0$
$\dim W + \dim W^\perp = \dim(\mathrm{domain}(A)) + \dim(\mathrm{domain}(B))$
$= \dim(\mathrm{domain}([A\ B]))$

if $\dim(\mathrm{dom}([A\ B])) < 4$, since $W, W^\perp \subseteq \mathbb{R}^4$, it's not spanning,
then there's must have one vector that is orthogonal
to all colums in $[A\ B]$, until $\dim(\mathrm{dom}([A\ B])) = 4$

so $\dim W + \dim W^\perp = 4$

3. $\Rightarrow W \subsetneq W^\perp \Rightarrow \dim W \leq 2$

$W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ $W^\perp = \mathbb{R}^4$

$W \subsetneq W^\perp$

$\dim W = 0$ $\dim W^\perp = 4$

4. $\|\vec{v}\| = \langle \vec{v}, \vec{v} \rangle$ (length of $\vec{v}$)
$\Rightarrow x^2 + y^2 + z^2 - t^2 = 0$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$