9.1

1. $\det(2A) = 2^3\det(A) = 40$

2. $\det(-A) = (-1)^3\det(A) = -5$

3. $\det(A^2) = \det(A)\det(A) = 25$

4. $\det(A^{-1}) = \dfrac{\det(I)}{\det(A)} = \dfrac{1}{5}$

5. $\det(A^T) = \det(A) = 5$

6. $\det\left(\begin{bmatrix} \vec{a_1}^T - \vec{a_3}^T \\ \vec{a_2}^T - \vec{a_1}^T \\ \vec{a_3}^T - \vec{a_2}^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & & -1 \\ -1 & 1 & \\ & -1 & 1 \end{bmatrix} A\right) = \det\left(\begin{bmatrix} 1 & & -1 \\ -1 & 1 & \\ & -1 & 1 \end{bmatrix}\right) * 5$

$= -5\det\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right) = -5 \times 0 = 0$

7. $\det\left(\begin{bmatrix} \vec{a_1}^T + \vec{a_3}^T \\ \vec{a_2}^T + \vec{a_1}^T \\ \vec{a_3}^T + \vec{a_2}^T \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & & 1 \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} A\right) = 5\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 5 \times 2 = 10$

9.2

1. $A = QR = [Q\vec{r_1} \;\cdots\; Q\vec{r_n}]$

$\Rightarrow \vec{a_i} = Q\vec{r_i}$

since Q is a orthogonal matrix,

Q preserves length

$\Rightarrow \|\vec{a_i}\| = \|Q\vec{r_i}\| = \|\vec{r_i}\|$

2. $\begin{cases} \det(R) = r_{11}\cdots r_{nn} \\ r_{ii} \le \|\vec{r_i}\| \end{cases}$

$\Rightarrow \det(R) \le \|\vec{r_1}\|\cdots\|\vec{r_n}\|$

3. $|\det(A)| = |\det(QR)| = |\det(Q)|\,|\det(R)|$

Since Q is a orthogonal matrix,

suppose $Q = [\vec{q_1} \cdots \vec{q_n}]$

then $\det(Q) = \|\vec{q_1}\|\cdots\|\vec{q_n}\| = 1 \times \cdots \times 1 = 1$

$\Rightarrow |\det(A)| = |\det(R)| \le \|\vec{r_1}\|\cdots\|\vec{r_n}\|$

$\le \|\vec{a_1}\|\cdots\|\vec{a_n}\|$

9.3

1. counter example:

let $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$ $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

then $AB-BA=\begin{bmatrix}1&0\\0&0\end{bmatrix}-\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\-1&-1\end{bmatrix}$

$\det(AB-BA)=-1-0=-1\neq 0$

2. counter example:

let $A=I_{2\times 2}$

$\det(-A)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}=1\neq-\det(A)=-1$

3. proof:

skew-symmetric $\Rightarrow A^T=-A \Rightarrow \underset{\det(A)}{\underset{\parallel}{\det(A^T)}}=-\det(A)$

$\Rightarrow \det(A)=0 \Rightarrow A$ is not invertible

4. proof:

$A=\begin{bmatrix}A_i&\vdots\\\cdots&\vdots\end{bmatrix}=LDU=\begin{bmatrix}L_i&0\\\cdots&\end{bmatrix}\begin{bmatrix}D_i&0\\0&d_n\end{bmatrix}\begin{bmatrix}U_i&\vdots\\0&\cdots\end{bmatrix}$

$=\begin{bmatrix}L_iD_iU_i&\vdots\\\cdots&\vdots\end{bmatrix}$

$\Rightarrow A_i=L_iD_iU_i$

$\det(A_i)=\det(L_i)\det(D_i)\det(U_i)$

since L & D are both unit triangular matrices

$\det(L_i)=\det(U_i)=1$

then $\det(A_i)=\det(D_i)=d_1d_2\cdots d_i$

$\Rightarrow \frac{\det(A_i)}{\det(A_{i-1})}=\frac{d_1d_2\cdots d_i}{d_1d_2\cdots d_{i-1}}=d_i$

9.4

1. $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = A(D - CA^{-1}B) \neq AD - BC$

2. right argument: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\right) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

$= \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$

3. this argument is right if AB & BA are both $(2k-1)\times(2k-1)$ matrices $(k=1,2,\cdots)$

if not, then $\det(-BA) = \det(-B)\det(A) = \det(B)\cdot\det(A)$
$\neq -\det(B)\det(A)$

9.5

1. $C^TA = \det(A)I = 5I$
$\det(C^TA) = \det(C^T)\det(A) = 5\det(C) = 125\det(I)$
$\Rightarrow \det(C) = 25$

2. expand the last row

$\underbrace{\det\begin{bmatrix} 1 & 1 & \cdots & \cdots \\ \vdots & 2 & & \\ \vdots & & 3 & \\ & & \ddots & n \end{bmatrix}}_{A_n} = (-1)^{n+1}\det\begin{bmatrix} 1 & 1 & \cdots & \\ 2 & & & \\ & 3 & \ddots & \\ & & n-1 & 0 \end{bmatrix} + (-1)^{2n}\cdot n\det\begin{bmatrix} 1 & 1 & \cdots & \\ \vdots & 2 & & \\ \vdots & & 3 & \\ \vdots & & \ddots & n-1 \end{bmatrix}$

expand the last column

$= -(n-1)! + n\det(A_{n-1})$

and $A_2 = 1$

$\Rightarrow A_3 = -2 + 3\times 1 = 1$

$A_4 = -6 + 4\times 1 = -2$

$A_5 = -24 + 5\times(-2) = -34$

4.6

1. $\det\begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$

coefficient for x^4: $= 2$

coefficient for x^3: $a_{21} \cdot a_{12} \cdot a_{33}\, a_{44} \times (-1) \Rightarrow -x^3$

$\| \; -1$

2. for each term in the big formula: $\operatorname{sign}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(4),4}$

will change into $24 \times \operatorname{sign}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(4),4}$

$\det(A) \longrightarrow \det\left(\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix} A\right) = \det\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix} \det(A) = 24 \det(A)$

3. $\det(A) = acfh - bceh - adfg + bdeg$ → 4 non-zero terms

$= af(ch - dg) - be(ch - dg) = (af - be)(ch - dg)$

9.7

1. $\det(P_n) = \det(L_n)\det(U_n) = \det(L_n)^2 = 1$

2. $\det(P_n) = \sum_{i=1}^{n} P_{ni} \cdot \det(P_{ni}) = 1$

(expand the n-th row)

$\det(A_n) = \sum_{i=1}^{n} a_{ni} \det(A_{ni}) = \sum_{i=1}^{n} a_{ni} \det(P_{ni})$

and $a_{ni} = \begin{cases} P_{ni} & (i \neq n) \\ P_{nn} - 1 & i = n \end{cases}$

So $\det(A_n) = \det(P_n) - \det(P_{n-1}) = 1 - 1 = 0$

$\Rightarrow \det(A_n) = 0$

9.8

1. $\det(H_n) = \det \begin{bmatrix} 2 & 1 & \cdots & 1 & 0 \\ 1 & \ddots & \ddots & & \vdots \\ & \ddots & \ddots & 1 & 0 \\ & & \ddots & 2 & -1 \\ & & & 1 & 1 \end{bmatrix} \overset{\text{expand the last column}}{=} 1 \times \det(H_{n-1}) - (-1)\det \left[\begin{array}{c|c} H_{n-2} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline 0 \cdots 0 \; 1 & 1 \end{array}\right]$

$= \det(H_{n-1}) + \det(H_{n-2})$

2. $\det(S_n) = 3\det(S_{n-1}) - 1 \times \det \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & S_{n-2} \end{array}\right] = 3\det(S_{n-1}) - \det(S_{n-2})$

it's actually a half of the Fibonacci sequence

9.9

1. $\frac{\partial f}{\partial a} = \frac{d}{ad-bc}$ $\frac{\partial f}{\partial b} = \frac{-c}{ad-bc}$ $\frac{\partial f}{\partial c} = \frac{-b}{ad-bc}$ $\frac{\partial f}{\partial d} = \frac{a}{ad-bc}$

2. $\begin{bmatrix} a & d \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \\ \frac{\partial f}{\partial c} & \frac{\partial f}{\partial d} \end{bmatrix}^T$

3. $\forall$ $n \times n$ invertible matrix A:

$$\frac{\partial \ln(\det(A))}{\partial a_{ij}} = \frac{\partial \ln(a_{1j}\det A_{1j} + \cdots + a_{ij}\det A_{ij} + \cdots + a_{nj}\det\det A_{nj})}{\partial a_{ij}}$$

$$= \frac{\det(A_{ij})}{\det(A)}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial \ln(\det(A))}{\partial a_{11}} & \cdots & \vdots \\ \vdots & \ddots & \\ \cdots & & \frac{\partial \ln(\det(A))}{\partial a_{nn}} \end{bmatrix}^T = \frac{1}{\det(A)} C^T$$

since $C^T A = \det(A) I$

then $\frac{1}{\det(A)} C^T = I A^{-1} = A^{-1}$