6.1

1. $$A_t=\begin{bmatrix}1&1\\&1+t\end{bmatrix}=\underbrace{\begin{bmatrix}1&1\\&t\end{bmatrix}}_{X}\begin{bmatrix}1&\\&1+t\end{bmatrix}\begin{bmatrix}1&1\\&t\end{bmatrix}^{-1}\qquad \begin{bmatrix}1&1\\&t\end{bmatrix}^{-1}\to\begin{bmatrix}1&-\frac{1}{t}\\&\frac{1}{t}\end{bmatrix}$$

$$\Rightarrow f(A_t)=f\left(X\begin{bmatrix}1&\\&1+t\end{bmatrix}X^{-1}\right)$$

$$=X\begin{bmatrix}f(1)&\\&f(1+t)\end{bmatrix}X^{-1}=X\begin{pmatrix}1&\\&(1+t)^2\end{pmatrix}X^{-1}$$

$$=\begin{pmatrix}1&t+2\\&(t+1)^2\end{pmatrix}$$

$$\Rightarrow \lim f(A_t)=\begin{pmatrix}1&2\\&1\end{pmatrix}$$

2. $$A_t=\begin{pmatrix}1&1\\-t^2&1\end{pmatrix}=\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}\begin{pmatrix}1+ti&\\&1-ti\end{pmatrix}\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}\begin{pmatrix}1+ti&\\&1-ti\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2ti}\\\frac{1}{2}&\frac{-1}{2ti}\end{pmatrix}$$

$$f(A_t)=\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}\begin{pmatrix}f(1+ti)&\\&f(1-ti)\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2ti}\\\frac{1}{2}&\frac{-1}{2ti}\end{pmatrix}$$

$$=\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}\begin{pmatrix}\sqrt{1+t^2}(1+ti)&\\&\sqrt{1+t^2}(1-ti)\end{pmatrix}\begin{pmatrix}\frac{1}{2}&\frac{1}{2ti}\\\frac{1}{2}&\frac{-1}{2ti}\end{pmatrix}$$

$$=\begin{pmatrix}1&1\\ti&-ti\end{pmatrix}\begin{pmatrix}\frac{\sqrt{1+t^2}(1+ti)}{2}&\frac{\sqrt{1+t^2}(1+ti)}{2ti}\\\frac{\sqrt{1+t^2}(1-ti)}{2}&-\frac{\sqrt{1+t^2}(1-ti)}{2ti}\end{pmatrix}=\begin{pmatrix}\sqrt{1+t^2}&\sqrt{1+t^2}\\-t^2\sqrt{1+t^2}&\sqrt{1+t^2}\end{pmatrix}$$

$$\Rightarrow \lim f(A_t)=\begin{pmatrix}1&1\\&1\end{pmatrix}$$

so $f(J)$ is not well-defined

3. converge to $\begin{pmatrix}1&2\\&1\end{pmatrix}=\begin{pmatrix}f(1)&f'(1)\\&f(1)\end{pmatrix}$

6.2

1. $f(x) = \frac{1}{1-x} = \lim_{n\to\infty}(1+x+x^2+\cdots+x^n)$

$\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \Rightarrow x \in (-1,1)$

So the radius of convergence is 1

2. $(I-AB)^{-1} = f(AB) = I + AB + (AB)^2 + \cdots$

$(I-BA)^{-1} = f(BA) = I + BA + (BA)^2 + \cdots$

3. $(I-AB)^{-1} = I + AB + (AB)^2 + \cdots$

$I + A(I-BA)^{-1}B = I + A\cdot(I + BA + (BA)^2 + \cdots)B$

$= I + AB + A\cdot BA\cdot B + A\cdot(BA)^2\cdot B + \cdots$

since $A\cdot(BA)^k B = A\cdot B^k\cdot A^k\cdot B = (AB)\cdot B^{k-1} A^{k-1}(AB)$

and $A\cdot BA\cdot B = (AB)^2$

$\Rightarrow$ using induction: $A(BA)^k B = (AB)^k$

$\Rightarrow I + A(I-BA)^{-1}B = I + AB + (AB)^2 + \cdots$

$\Rightarrow (I-AB)^{-1} = I + A(I-BA)^{-1}B$

6.3

1. $f(A+tI) = f\begin{pmatrix} \lambda+t & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda+t \end{pmatrix} = \begin{pmatrix} f(\lambda+t) & f'(\lambda+t) & \frac{1}{2}f''(\lambda+t) & \cdots & \frac{1}{(n-1)!}f^{(n-1)}(\lambda+t) \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{1}{2}f''(\lambda+t) \\ & & & \ddots & f'(\lambda+t) \\ & & & & f(\lambda+t) \end{pmatrix}$

$$\Rightarrow \frac{d}{dt} f(A+tI) = \begin{pmatrix} f'(\lambda+t) & f''(\lambda+t) & \cdots & \frac{f^{(n)}(\lambda+t)}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & f''(\lambda+t) \\ & & & f'(\lambda+t) \end{pmatrix}$$

$$// \quad f'(A+tI)$$

$$\Rightarrow \frac{d}{dt} f(A+tI) = f'(A+tI)$$

2. if A is a single Jordan block matrix

$\Rightarrow A = X(J+tI)X^{-1}$

$$\Rightarrow \frac{d}{dt} f(A+tI) = \frac{d}{dt} \cdot X f(J+tI) X^{-1} = X f'(J+tI) X^{-1}$$

$$f'(A+tI) = X f'(J+tI) X^{-1}$$

$\Rightarrow \frac{d}{dt} f(A+tI) = f'(A+tI)$

then if A has 2 or 2^+ Jordan blocks, since all blocks behave independently, it's same as single Jordan block

$\Rightarrow \frac{d}{dt} f(A+tI) = f'(A+tI)$ is true in general

3. $$f(A+tB) \overset{t\to 0}{=} f(A) + f'(A)\cdot tB + \frac{1}{2} f''(A)\cdot (tB)^2 + \cdots$$

$$\frac{d}{dt}(A+tB) = 0 + f'(A)B + \frac{1}{2} f''(A)\cdot 2t B^2 + \cdots$$

$$\frac{d}{dt}(A+tB)\Big|_{t=0} = f'(A)B$$

6.4

1. $f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \begin{bmatrix} 2A & A \\ & 2A \end{bmatrix} \cdot \begin{bmatrix} 4A^2 & 4A^2 \\ & 4A^2 \end{bmatrix}$

$$= \begin{bmatrix} 8A^3 & 12A^3 \\ & 8A^3 \end{bmatrix}$$

2. suppose $A = [a]$

$$f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = f\left(\begin{bmatrix} 2a & a \\ & 2a \end{bmatrix}\right) = f\left(\lim_{t \to 0} \begin{bmatrix} 2a & a \\ & 2a+t \end{bmatrix}\right)$$

$$= \lim_{t \to 0} \begin{bmatrix} 1 & a \\ & t \end{bmatrix} \begin{bmatrix} f(2a) & \\ & f(2a+t) \end{bmatrix} \begin{bmatrix} 1 & -\frac{a}{t} \\ & \frac{a}{t} \end{bmatrix} = \lim_{t \to 0} \begin{bmatrix} f(2a) & \frac{f(2a+t)-f(2a)}{t} a \\ & f(2a+t) \end{bmatrix}$$

$$= \begin{bmatrix} f(2A) & f'(2A) \cdot A \\ & f(2A) \end{bmatrix}$$

3. $X_t = \begin{pmatrix} I & A \\ & tI \end{pmatrix}$

4. $f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \lim_{t \to 0} f(M_t)$

$$= \lim_{t \to 0} X_t \, f\left(\begin{bmatrix} 2A & \\ & 2A+tI \end{bmatrix}\right) X_t^{-1}$$

$$= \lim_{t \to 0} \begin{bmatrix} I & A \\ & tI \end{bmatrix} \cdot \begin{bmatrix} f(2A) & \\ & f(2A+tI) \end{bmatrix} \cdot \begin{bmatrix} I & -\frac{1}{t}A \\ & \frac{1}{t}I \end{bmatrix}$$

$$= \lim_{t \to 0} \begin{bmatrix} I & A \\ & tI \end{bmatrix} \cdot \begin{bmatrix} f(2A) & -\frac{1}{t} f(2A) \cdot A \\ & \frac{1}{t} f(2A+tI) \end{bmatrix}$$

$$= \lim_{t \to 0} \begin{bmatrix} f(2A) & A \frac{f(2A+tI)-f(2A)}{t} \\ & f(2A+tI) \end{bmatrix}$$

$$= \begin{bmatrix} f(2A) & f'(2A) \cdot A \\ & f(2A) \end{bmatrix}$$