

2.1

$$\textcircled{1} \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 4 & 5 & 6 & | & 4 \\ 7 & 8 & 9 & | & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 0 & -3 & -6 & | & 0 \\ 0 & -6 & -12 & | & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{cases} x = z + 1 \\ y = -2z \\ z \end{cases}$$

② the same solution with ①  
(due to the redundant equation of row 3)

$$\textcircled{3} \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 4 & 5 & 6 & | & 4 \\ 7 & 8 & 9 & | & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 0 & -3 & -6 & | & 1 \\ 0 & -6 & -12 & | & 1 \end{bmatrix}$$

$\Rightarrow y = 0$  and  $y = -1/6$   
contradiction!  
so no solution

2.2

$$\textcircled{1} \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 3 & b & 7 & | & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & b-6 & -2 & | & -2 \end{bmatrix}$$

$$\Rightarrow b = 6$$

$$\textcircled{2} \begin{bmatrix} 3 & 2 & | & 10 \\ 6 & 4 & | & b \end{bmatrix} \Rightarrow b \neq 20$$

$$\textcircled{3} \begin{bmatrix} 2 & 5 & 1 & | & 0 \\ 4 & b & 1 & | & 2 \\ 0 & 1 & -1 & | & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 5 & 1 & | & 0 \\ 0 & b-1 & -1 & | & 2 \\ 0 & 1 & -1 & | & 3 \end{bmatrix}$$

$$\Rightarrow b-1=1$$

$$\textcircled{4} \begin{bmatrix} b & 3 & | & 6 \\ 3 & b & | & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} b & 3 & | & 6 \\ 0 & b-9 & | & -6 \end{bmatrix}$$

$$\Rightarrow b = -3$$

$$\textcircled{5} \begin{bmatrix} 2 & b & | & 16 \\ 4 & 8 & | & c \end{bmatrix} \Rightarrow \begin{cases} \text{let's make a} \\ \text{redundant equation} \end{cases}$$

$$\Rightarrow \frac{2}{4} = \frac{b}{8} = \frac{16}{c} \Rightarrow \begin{cases} b = 4 \\ c = 32 \end{cases}$$

$$\textcircled{6} \begin{bmatrix} 1 & b & 0 & | & 0 \\ 1 & -2 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b & 0 & | & 0 \\ 0 & -(b+1) & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b & 0 & | & 0 \\ 0 & -(b+1) & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & b & 0 & | & 0 \\ 0 & b+1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b & 0 & | & 0 \\ 0 & b+1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b & 0 & | & 0 \\ 0 & b+1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

$$\Rightarrow b = -1 \text{ or } b = -2$$

$$\textcircled{7} \begin{bmatrix} b & 2 & 3 & | & 0 \\ 0 & b & 4 & | & 0 \\ b & b & b & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b & 2 & 3 & | & 0 \\ 0 & b & 4 & | & 0 \\ 0 & b-2 & b-3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} b & 2 & 3 & | & 0 \\ 0 & b & 4 & | & 0 \\ 0 & 0 & b-4 & | & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} b & 2 & 3 & | & 0 \\ 0 & b & 4 & | & 0 \\ 0 & 0 & b-4 & | & 0 \end{bmatrix}$$

$$\Rightarrow b = 0 \text{ or } b = 2 \text{ or } b = 4$$

2.3

$$\begin{bmatrix} p & 1 & 1 & \left| \begin{array}{c} 1 \\ p \\ p^2 \end{array} \right. \end{bmatrix} \Rightarrow \begin{bmatrix} -p-1 & p+2 & p+2 & \left| \begin{array}{c} p^2+p+1 \\ p \\ p^2 \end{array} \right. \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \left| \begin{array}{c} \frac{p^2+p+1}{p+2} \\ p \\ p^2 \end{array} \right. \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & \left| \begin{array}{c} \frac{p^2+p+1}{p+2} \\ \frac{p-1}{p+2} \\ \frac{p^3+p^2-p-1}{p+2} \end{array} \right. \end{bmatrix}$$

$$\textcircled{1} p = -2$$

$$\Rightarrow \begin{bmatrix} p & 1 & 1 & \left| \begin{array}{c} 1 \\ p \\ p^2 \end{array} \right. \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 & 1 & \left| \begin{array}{c} 1 \\ -2 \\ 4 \end{array} \right. \end{bmatrix}$$

so no solution

$$\textcircled{2} p \neq -2$$

$$1) p = 1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & \left| \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right. \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = 1 - y - z \\ y \\ z \end{cases}$$

$$\Rightarrow p \neq 1$$

$$\Rightarrow \begin{cases} x = \frac{-p-1}{p+2} \\ y = \frac{1}{p+2} \\ z = \frac{p^3+p^2-p-1}{p+2} \end{cases}$$

2.4

$$\textcircled{1} A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\textcircled{2} \text{ 1st column of } A \text{ is half of } \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{2} & a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & a_{13} \\ 1 & -1 & a_{23} \\ 1 & -1 & a_{33} \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow a_{13} = a_{23} = a_{33} = 0 \Rightarrow A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

2.5

$$\textcircled{1} \text{ let } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2+s \\ s \\ t \end{pmatrix}$$

$x_1$  is a dependent variable  
 $x_2$  and  $x_3$  are free variables

$$\textcircled{2} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[ \begin{array}{ccc|c} a_1 & -a_1 & 0 & 2 \\ a_2 & -a_2 & 0 & 0 \\ a_3 & -a_3 & 0 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\textcircled{3} \text{ RREF: } \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$A\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & 2 \end{array} \right]$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \Leftarrow \left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -2 & 0 & 4 \\ 1 & -1 & 0 & 2 \end{array} \right]$$

2.6

$$(1) \left[ \begin{array}{cc|c} 2 & 1 & 3 \\ 5 & 4 & 6 \end{array} \right]$$

$$C_1 \leftrightarrow C_2$$

$$C_3 \rightarrow C_3$$

C means column

$$(2) \left[ \begin{array}{cc|c} \frac{1}{2} & 2 & 3 \\ 2 & 5 & 6 \end{array} \right]$$

$$C_1 \rightarrow \frac{1}{2} C_1$$

$$C_2 \rightarrow C_2$$

$$C_3 \rightarrow C_3$$

$$(3) \left[ \begin{array}{cc|c} -1 & 2 & 3 \\ 1 & 5 & 6 \end{array} \right]$$

$$C_1 \rightarrow C_1 - C_2$$

$$C_2 \rightarrow C_2$$

$$C_3 \rightarrow C_3$$

$$(4) \left[ \begin{array}{cc|c} 1 & 2 & 4 \\ 4 & 5 & 10 \end{array} \right]$$

$$C_1 \rightarrow C_1$$

$$C_2 \rightarrow C_2$$

$$C_3 \rightarrow C_1 + C_3$$