4.5

5. $\iint_{\partial\Omega} \vec{V}\,d\vec{S} = \iint_{\partial\Omega} xy\,dy\wedge dz + yz\,dz\wedge dx + zx\,dx\wedge dy$

$= \iint_{\partial\Omega} (x\,dy+y\,dx)\wedge dy\wedge dz + (y\,dz+z\,dy)\wedge dz\wedge dx + (z\,dx+x\,dz)\wedge dx\wedge dy$

$= \iint_{\partial\Omega} y\,dx\wedge dy\wedge dz + z\,dy\wedge dz\wedge dx + x\,dz\wedge dx\wedge dy$

$= \iiint_{\Omega} (x+y+z)\,dx\,dy\,dz$

$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$ $= \iiint_{\Omega} r(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta)\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi$

$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr\cdot r^3(\sin^2\theta\cos\varphi + \sin^2\theta\sin\varphi + \sin\theta\cos\theta)$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi\,(\sin^2\theta\cos\varphi + \sin^2\theta\sin\varphi + \sin\theta\cos\theta)$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta\left(\frac{\pi}{2}\sin\theta\cos\theta + 2\sin^2\theta\right)$

$= \frac{3}{16}\pi$

$S_1^+ \begin{cases} z = 0 \\ x \geq 0,\ y \geq 0 \\ x^2+y^2 \leq 1 \\ \vec{n} = (0,0,-1) \end{cases} \Rightarrow \iint_{S_1^+} xy\,dy\wedge dz + yz\,dz\wedge dx + zx\,dx\wedge dy$

$= 0$

$= \iint_{S_2^+}(\cdots) + \iint_{S_3^+}(\cdots)$

$\Rightarrow \iint_{S^+} \vec{v}\cdot d\vec{S} = \frac{3}{16}\pi$

6. $\iint_{S^+} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

(1)

$= 3V - \iint_{\substack{z=h \\ x^2+y^2\leq h^2}} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

$= 3\times\frac{1}{3}\times \pi h^2\times h - \iint_{x^2+y^2\leq h} h\cdot dx\,dy$

$= \pi h^3 - h\cdot \pi h^2$

$= 0$

(2) $= \pi h^3$ 在(1)已计算过

(3) $= \pi h^3$ 在(1)已计算过

7. $\iint_{S^+} (x^2+y^2)dx\wedge dy + y^2 dy\wedge dz + z^2 dz\wedge dx$

$= \iint_{S^+} u^2 \cdot \frac{\partial(x,y)}{\partial(u,v)} dudv + u^2\sin^2 v \cdot \frac{\partial(y,z)}{\partial(u,v)} \cdot dudv + a^2 v^2 \cdot \frac{\partial(z,x)}{\partial(u,v)} dudv$

$= \iint_{S^+} (u^3 + au^2\sin^3 v + a^2v^2\cdot(-a\cos v))dudv$

$= \int_0^{2\pi} dv \int_0^1 du\, (u^3 + au^2\sin^3 v - a^3 v^2 \cos v)$

$= \int_0^{2\pi} dv\, (\frac{1}{4} + \frac{1}{3}a\sin^3 v - a^3 v^2 \cos v)$

$= \frac{\pi}{2} - 4\pi a^3$

4.7

2. (3) $= \frac{1}{\varepsilon^3} \oiint_{S^+} x dy\wedge dz + y dz\wedge dx + z dx\wedge dy$

$= \frac{1}{\varepsilon^3} \iiint_V 3\, dxdydz$

$= \frac{1}{\varepsilon^3} \cdot 4\pi\varepsilon^3 = 4\pi$

3. (2) $\oiint_{S^+} (x-y)dx\wedge dy + x(y-z)dy\wedge dz$

$= \iiint_V 0 + (y-z)dxdydz$

$= \iiint_V (r\sin\theta - z)\cdot r dr d\theta dz$

$= \int_0^{2\pi} d\theta \int_0^1 dr \int_0^1 dz \cdot (r^2\sin\theta - rz)$

$= \int_0^{2\pi} d\theta \int_0^1 dr \cdot (r^2\sin\theta - \frac{1}{2}r)$

$= \int_0^{2\pi} d\theta \cdot (\frac{1}{3}\sin\theta - \frac{1}{4})$

$= -\frac{\pi}{2}$

5. (1) $\oint_{L^+} y dx + z dy + x dz$

$= \iint_{S^+} \begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= \iint_{S^+} -dy\wedge dz - dz\wedge dx - dx\wedge dy$

$= -3 S_{圆}$

$= -3 \cdot \frac{1}{\sqrt{3}}\pi a^2$

$= -\sqrt{3}\pi a^2$

9.

(1) $$\operatorname{div}(u\vec{v}) = \frac{\partial(u\vec{v})}{\partial x} + \frac{\partial(u\vec{v})}{\partial y} + \frac{\partial(u\vec{v})}{\partial z}$$

$$= \frac{\partial u}{\partial x}\vec{v} + u\frac{\partial \vec{v}}{\partial x} + \frac{\partial u}{\partial y}\vec{v} + u\frac{\partial \vec{v}}{\partial y} + \frac{\partial u}{\partial z}\vec{v} + u\frac{\partial \vec{v}}{\partial z}$$

$$= \operatorname{grad} u \cdot \vec{v} + u \operatorname{div}(\vec{v})$$

(2) $$\operatorname{rot}(u\vec{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ uA_x & uA_y & uA_z \end{vmatrix} = \begin{pmatrix} \frac{\partial u}{\partial y}A_z + u\frac{\partial A_z}{\partial y} - \frac{\partial u}{\partial z}A_y - u\frac{\partial A_y}{\partial z} \\ \frac{\partial u}{\partial z}A_x + u\frac{\partial A_x}{\partial z} - \frac{\partial u}{\partial x}A_z - u\frac{\partial A_z}{\partial x} \\ \frac{\partial u}{\partial x}A_y + u\frac{\partial A_y}{\partial x} - \frac{\partial u}{\partial y}A_x - u\frac{\partial A_x}{\partial z} \end{pmatrix}$$

$$= u\cdot\begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} + \operatorname{grad} u \times \vec{A}$$

$$= u \operatorname{rot}\vec{A} + \operatorname{grad} u \times \vec{A}$$

(3) $$\operatorname{div}(\vec{A}\times\vec{B}) = \operatorname{div}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \operatorname{div}\begin{pmatrix} A_yB_z - A_zB_y \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{pmatrix}$$

$$= \frac{\partial(A_yB_z - A_zB_y)}{\partial x} + \frac{\partial(A_zB_x - A_xB_z)}{\partial y} + \frac{\partial(A_xB_y - A_yB_x)}{\partial z}$$

$$\vec{B}\cdot\operatorname{rot}\vec{A} - \vec{A}\cdot\operatorname{rot}\vec{B} = (B_x\ B_y\ B_z)\cdot\begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} - (A_x\ A_y\ A_z)\begin{pmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{pmatrix}$$

$$= \frac{\partial(A_yB_z - A_zB_y)}{\partial x} + \frac{\partial(A_zB_x - A_xB_z)}{\partial y} + \frac{\partial(A_xB_y - A_yB_x)}{\partial z}$$

$$\Rightarrow \operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{rot}\vec{A} - \vec{A}\cdot\operatorname{rot}\vec{B}$$

(4) $$\operatorname{rot}(\operatorname{grad} u) = \operatorname{rot}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial z\partial y} - \frac{\partial u}{\partial y\partial z} \\ \frac{\partial u}{\partial x\partial z} - \frac{\partial u}{\partial z\partial x} \\ \frac{\partial u}{\partial y\partial x} - \frac{\partial u}{\partial x\partial y} \end{pmatrix} = \vec{0}$$

(5) $$\operatorname{div}(\operatorname{rot}\vec{A}) = \operatorname{div}\begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} = \frac{\partial A_z}{\partial x\partial y} - \frac{\partial A_y}{\partial x\partial z} + \frac{\partial A_x}{\partial y\partial z} - \frac{\partial A_z}{\partial x\partial y} + \frac{\partial A_y}{\partial x\partial z} - \frac{\partial A_x}{\partial y\partial z}$$

$$= 0$$

10.

(1) $\operatorname{div}\vec{V} = 2x+2y+2z\big|_P = 4$

$\operatorname{rot}\vec{V} = \vec{0}$

(2) $\operatorname{div}\vec{V} = 2xyz+2xyz+2xyz\big|_P = -36$

$$\operatorname{rot}\vec{V} = \begin{pmatrix} xz^2-xy^2 \\ x^2y-yz^2 \\ y^2z-x^2z \end{pmatrix}_P = \begin{pmatrix} 5 \\ -9 \\ 16 \end{pmatrix}$$

(3) $\operatorname{div}\vec{V} = 6xz+2y+3xy\big|_P = 28$

$$\operatorname{rot}\vec{V} = \begin{pmatrix} 3xz+3x \\ 3x^2-3yz \\ -3z-2 \end{pmatrix}_P = \begin{pmatrix} 12 \\ 15 \\ -11 \end{pmatrix}$$

11.

(3) $\operatorname{div}(r^n\vec{a}) = r^n \underbrace{\operatorname{div}(\vec{a})}_{=0} + \operatorname{grad}(r^n)\cdot\vec{a} = 0$

$\Rightarrow \operatorname{grad}(r^n) = 0$

$\Rightarrow n = 0$

(4) $\operatorname{div}(f(r)\cdot\vec{r}) = \underbrace{f(r)\operatorname{div}\vec{r}}_{f(r)\cdot 3} + \underbrace{\operatorname{grad}f(r)\cdot\vec{r}}_{f'(r)\cdot\left(\frac{x^2}{\sqrt{x^2+y^2+z^2}}+\frac{y^2}{\sqrt{x^2+y^2+z^2}}+\frac{z^2}{\sqrt{x^2+y^2+z^2}}\right) = f'(r)\cdot r} = 0$

$\Rightarrow 3f(r)+f'(r)\cdot r = 0$

$f(r) = \dfrac{C}{r^3}$

(7) $$\operatorname{rot}(f(r)\vec{r}) = \operatorname{rot}\begin{pmatrix} f(r)x \\ f(r)y \\ f(r)z \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} = \begin{pmatrix} \frac{\partial f(r)}{\partial y}z - \frac{\partial f(r)}{\partial z}y \\ \frac{\partial f(r)}{\partial z}x - \frac{\partial f(r)}{\partial x}z \\ \frac{\partial f(r)}{\partial x}y - \frac{\partial f(r)}{\partial y}x \end{pmatrix} = \vec{0}$$

(9) $$\operatorname{rot}(\vec{r}\times f(r)\vec{a}) = \operatorname{rot}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ f(r)a_x & f(r)a_y & f(r)a_z \end{vmatrix} = \operatorname{rot}\begin{pmatrix} f(r)ya_z - f(r)za_y \\ f(r)za_x - f(r)xa_z \\ f(r)xa_y - f(r)ya_z \end{pmatrix} \leftarrow \text{设为 } \vec{E}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ E_x & E_y & E_z \end{vmatrix} = \vec{0}$$

第十章总复习：

2. (1) $\iint\limits_{S^+} F(x,y,z)\,dy\wedge dz$

$= \iint\limits_{S^+} F(x,y,f(x,y))\ dy\wedge df(x,y)$

$= \iint\limits_{S^+} F(x,y,f(x,y))\ dy\wedge\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$

$= \iint\limits_{S^+} F(x,y,f(x,y))\ dy\wedge\left(\frac{\partial f}{\partial x}dx\right)$

$= -\iint\limits_{S^+} F(x,y,f(x,y))\cdot\frac{\partial f}{\partial x}(dx\wedge dy)$

$= -\iint\limits_{D} F(x,y,f(x,y))\cdot\frac{\partial f}{\partial x}\,dxdy$

8.

(1) $\oiint\limits_{\partial\Omega}\frac{\partial u}{\partial \vec{n}}dS = \oiint\limits_{\partial\Omega}\left(\frac{\partial u}{\partial x}\cdot\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right)dS$

$= \oiint\limits_{\partial\Omega}\left[\frac{\partial u}{\partial x}\ \ \frac{\partial u}{\partial y}\ \ \frac{\partial u}{\partial z}\right]d\vec{S}$

$= \iiint\limits_{\Omega}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)dxdydz$

$= \iiint\limits_{\Omega}\Delta u\,dxdydz$

(2) $\oiint\limits_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}}dS = \oiint\limits_{\partial\Omega}\left(v\frac{\partial u}{\partial x}\cos\alpha + v\frac{\partial u}{\partial y}\cos\beta + v\frac{\partial u}{\partial z}\cos\gamma\right)dS$

$= \oiint\limits_{\partial\Omega}\left[v\frac{\partial u}{\partial x}\ \ v\frac{\partial u}{\partial y}\ \ v\frac{\partial u}{\partial z}\right]d\vec{S}$

$= \iiint\limits_{\Omega}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial z} + v\Delta u\right)dxdydz$

$= \iiint\limits_{\Omega} v\Delta u\,dxdydz + \iiint\limits_{\Omega}\nabla u\cdot\nabla v\,dxdydz$

(3) $\oiint\limits_{\partial\Omega}\begin{vmatrix}\frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix}dS \overset{(2)}{=} \iiint\limits_{\Omega}(v\Delta u + \nabla u\cdot\nabla v)dxdydz - \iiint\limits_{\Omega}(u\Delta v + \nabla u\cdot\nabla v)dxdydz$

$= \iiint\limits_{\Omega}\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}dxdydz$

10.

(1) 不妨设 $\vec{r_0}$ 为原点。

$$\frac{1}{4\pi}\iint_{\partial\Omega}\left(u\frac{\cos\langle\vec{r},\vec{n}\rangle}{r^2}+\frac{1}{r}\frac{\partial u}{\partial\vec{n}}\right)ds$$

$$=\frac{1}{4\pi}\iint_{\partial\Omega}\left(\frac{u\vec{r}}{r^3}+\frac{1}{r}\left[\frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial u}{\partial z}\right]\right)\cdot\vec{n}\,ds$$

$$=\frac{1}{4\pi}\iint_{\partial\Omega}\left(\frac{u}{r^3}[x\ y\ z]+\frac{1}{r}\left[\frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial u}{\partial z}\right]\right)d\vec{S}$$

$$=\frac{1}{4\pi}\iiint_{\Omega}0\cdot dxdydz$$

$$=0$$

$\Rightarrow$ 取原点一邻域：

$$=\frac{1}{4\pi}\iiint_{r\le\varepsilon}\frac{3u(\vec{0})}{r^3}dxdydz$$

$$=\frac{3u(\vec{0})}{4\pi}\times\frac{4}{3}\pi=u(\vec{0})$$

$\Rightarrow$ 得证

(2) 不妨设 $\vec{M}$ 为原点

$$\iint_{\partial\Omega-\partial\Omega_0}\left(\frac{1}{r}\frac{\partial u}{\partial\vec{n}}-u\frac{\partial\frac{1}{r}}{\partial\vec{n}}\right)ds$$

$$=\iint_{\partial\Omega-\partial\Omega_0}\left(\frac{1}{r}\left[\frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial u}{\partial z}\right]+\frac{u}{r^3}[x\ y\ z]\right)d\vec{S}$$

$$=0\quad(\text{散度为}0)$$

$$\Rightarrow\iint_{\partial\Omega}\left(\frac{1}{r}\frac{\partial u}{\partial\vec{n}}-u\frac{\partial\frac{1}{r}}{\partial\vec{n}}\right)ds=\iint_{\partial\Omega_0}\left(\frac{1}{r}\frac{\partial u}{\partial\vec{n}}-u\frac{\partial\frac{1}{r}}{\partial\vec{n}}\right)ds$$

$\Rightarrow$ 得证

5.1

6.

(3) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n-3)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n-3}\right) = \frac{1}{2}\lim_{n\to+\infty}\left(\frac{1}{3}-\frac{1}{2n-3}\right)=\frac{1}{6}$

$\Rightarrow$ 收敛

(7) $\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2} = \sum_{n=1}^{\infty}\left(\arctan\frac{1}{2n-1}-\arctan\frac{1}{2n+1}\right) = \lim_{n\to+\infty}\left(\arctan 1-\arctan\frac{1}{2n+1}\right)$

$= \frac{\pi}{4}$

$\Rightarrow$ 收敛

5.2

1. (7) $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1} \Rightarrow \lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to+\infty} n\left(\sqrt{\frac{n+1}{n}}\,\frac{\ln(1+\frac{2}{n-1})}{\ln(1+\frac{2}{n})}-1\right)$

$= \lim_{n\to+\infty} n\left(\sqrt{\frac{n+1}{n}}\,\frac{n}{n-1}-1\right)$

$= \lim_{n\to+\infty} n\left(\frac{\sqrt{n(n+1)}}{n-1}-1\right)$

$= \lim_{n\to+\infty} n\left(\frac{n\sqrt{1+\frac{1}{n}}}{n-1}-1\right)$

$= \lim_{n\to+\infty} n\left(\frac{n+\frac{1}{2}}{n-1}-1\right)$

$= \lim_{n\to+\infty} n\cdot\frac{\frac{3}{2}}{n-1} = \frac{3}{2} > 1$

$\Rightarrow$ 收敛

2. (5) $\sum_{n=1}^{\infty}\left(\sin\left(\frac{\pi}{4}+\frac{1}{n}\right)\right)^n \Rightarrow \lim_{n\to+\infty}\sqrt[n]{u_n} = \lim_{n\to+\infty}\sin\left(\frac{\pi}{4}+\frac{1}{n}\right)=\frac{\sqrt{2}}{2}<1$

$\Rightarrow$ 收敛

3. (2) $\sum_{n=1}^{\infty}\frac{1}{3^n}\left(1+\frac{1}{n}\right)^{n^2} \Rightarrow \lim_{n\to+\infty}\sqrt[n]{u_n} = \lim_{n\to+\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3}<1$

$\Rightarrow$ 收敛

8. (1) $\sum_{n=1}^{\infty}\frac{\sqrt{n!}}{(1+\sqrt{1})(1+\sqrt{2})\cdots(1+\sqrt{n})} \Rightarrow \lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to+\infty} n\left(\frac{1+\sqrt{n+1}}{\sqrt{n+1}}-1\right) = \lim_{n\to+\infty}\frac{n}{\sqrt{n+1}} \to +\infty > 1$

$\Rightarrow$ 收敛

10. $\lim_{n\to+\infty}\frac{u_n}{u_{n+1}}$; $\lim_{n\to+\infty}\frac{\frac{u_n}{1+u_n}}{\frac{u_{n+1}}{1+u_{n+1}}} = \lim_{n\to+\infty}\frac{\frac{1}{1+\frac{1}{u_n}}}{\frac{1}{1+\frac{1}{u_{n+1}}}} = \lim_{n\to+\infty}\frac{1+\frac{1}{u_{n+1}}}{1+\frac{1}{u_n}}$

$$\left.\begin{aligned}\frac{u_n}{u_{n+1}}<1 &\Leftrightarrow \frac{1+\frac{1}{u_{n+1}}}{1+\frac{1}{u_n}}<1\\ \frac{u_n}{u_{n+1}}\geq 1 &\Leftrightarrow \frac{1+\frac{1}{u_{n+1}}}{1+\frac{1}{u_n}}\geq 1\end{aligned}\right\} \Rightarrow \text{得证}$$