HW2

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1. (1) 解. $a=1$ 时:

$$\int_1^{+\infty} \frac{x^{a-1}}{1+x}dx = \ln(1+x)\Big|_{x=1}^{+\infty} \\ = +\infty$$

此时发散;

$a<1$ 时: 取 $0<\epsilon<1-a$

$$\lim_{x\to+\infty} \frac{\frac{x^{a-1}}{1+x}}{\frac{1}{x^{1+\epsilon}}} = \lim_{x\to+\infty} x^{a+\epsilon-1} \\ = 0$$

此时收敛;

$a>1$ 时: 取 $0<\epsilon<a-1$

$$\lim_{x\to+\infty} \frac{\frac{x^{a-1}}{1+x}}{\frac{1}{x^{1-\epsilon}}} = \lim_{x\to+\infty} x^{a-\epsilon-1} \\ = +\infty$$

此时发散.

□

(2) 解. $a=1$ 时:

$$\int_0^{1} \frac{x^{a-1}}{1+x}dx = \ln(1+x)\Big|_{x=0}^{1} \\ = +\infty$$

此时发散;

$a<0$ 时: 取 $1<p<1-a$

$$\lim_{x\to0^+} \frac{\frac{x^{a-1}}{1+x}}{\frac{1}{x^{p}}} = \lim_{x\to0^+} x^{a+p-1} \\ = +\infty$$

此时发散;

$a>0$ 且 $a\neq1$ 时: 取 $1-a<p<1$

$$\lim_{x\to0^+} \frac{\frac{x^{a-1}}{1+x}}{\frac{1}{x^{p}}} = \lim_{x\to0^+} x^{a+p-1} \\ = 0$$

此时收敛.

□

(3) 证明. 由(1)(2)知, 等号两侧的积分均收敛.

$$\int_0^1 \frac{x^{-a}}{1+x}dx$$

$$令 t = x^{-1}$$

$$= \int_{+\infty}^1 \frac{t^a}{1+t^{-1}} d(t^{-1})$$

$$= \int_1^{+\infty} \frac{t^{a-1}}{1+t}dt$$

$$= \int_1^{+\infty} \frac{x^{a-1}}{1+x}dx$$

得证. □

2. (1) 证明. 取 $0 < \epsilon < 1$:

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\exp(-ax^2-bx-c)}{\frac{1}{x^{1+\epsilon}}} &= \lim_{x\to\pm\infty} \frac{x^{1+\epsilon}}{\exp(ax^2+bx+c)} \\
&= \lim_{x\to\pm\infty} \frac{(1+\epsilon)x^\epsilon}{(2ax+b)\exp(ax^2+bx+c)} \\
&= \lim_{x\to\pm\infty} \frac{(1+\epsilon)\epsilon x^{\epsilon-1}}{(4a^2x+4abx+2a+b^2)\exp(ax^2+bx+c)} \\
&= 0
\end{aligned}$$

得证. □

(2) 解. $\lambda = 0$ 时:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

此时收敛;

$\lambda > 0$ 时: 取 $0 < \epsilon < 1$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\exp(-x^2-\lambda x^4)}{\frac{1}{x^{1+\epsilon}}} &= \lim_{x\to\pm\infty} \frac{x^{1+\epsilon}}{\exp(x^2+\lambda x^4)} \\
&= \lim_{x\to\pm\infty} \frac{(1+\epsilon)x^\epsilon}{(2x+4\lambda x^3)\exp(x^2+\lambda x^4)} \\
&= \lim_{x\to\pm\infty} \frac{(1+\epsilon)\epsilon x^\epsilon}{(2+4x^2+12\lambda x^2+16\lambda x^4+16\lambda^2x^6)\exp(x^2+\lambda x^4)} \\
&= 0
\end{aligned}$$

此时收敛;

$\lambda < 0$ 时: 取 $0 < \epsilon < 1$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\exp(-x^2-\lambda x^4)}{\frac{1}{x^{1-\epsilon}}} &= \lim_{x\to\pm\infty} \exp(-x^2-\lambda x^4)x^{1-\epsilon} \\
&= +\infty
\end{aligned}$$

此时收敛.

□

(3) 解.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \exp(-ax^2 - bx - c)dx &= \int_{-\infty}^{+\infty} \exp(-ax^2 - bx - c)dx \\
&= \exp\left(\frac{b^2}{4a} - c\right)\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} \exp(-\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2)d\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right) \\
&= \exp\left(\frac{b^2}{4a} - c\right)\frac{1}{\sqrt{a}}I \\
&= \frac{I}{\sqrt{a}}\exp\left(\frac{b^2}{4a} - c\right)
\end{aligned}$$

□

3. (1) 证明. $\forall x = 0, 1, 2, \ldots, n$, 有:

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{a_i x^i \exp(-x^2)}{\frac{1}{x^2}} &= a_i \lim_{x\to\pm\infty} \frac{x^{1+i+\epsilon}}{\exp(x^2)} \\
&= \frac{a_i(1+i+\epsilon)}{2} \lim_{x\to\pm\infty} \frac{x^{i+\epsilon-1}}{\exp(x^2)} \\
&= \ldots \\
&= 0
\end{aligned}$$

再根据积分的可加性可知 $f(x)$ 的每一项按照题干中的式子积分后均收敛,

因此 $\int_{-\infty}^{+\infty} f(x)\exp(-x^2)dx$ 收敛. □

(2) 证明. 如果能够证明 $\int_{-\infty}^{+\infty} ia_i x^{i-1}\exp(-x^2)dx = \int_{-\infty}^{+\infty} 2a_i x^{i+1}\exp(-x^2)dx$,

那么就能使得命题等号两侧的每一项都相等, 自然得证.

接下来证明 $\int_{-\infty}^{+\infty} ia_i x^{i-1}\exp(-x^2)dx = \int_{-\infty}^{+\infty} 2a_i x^{i+1}\exp(-x^2)dx$:

等价于证明 $i\int_{-\infty}^{+\infty} x^{i-1}\exp(-x^2)dx = 2\int_{-\infty}^{+\infty} x^{i+1}\exp(-x^2)dx$, $i = 1, 2, \ldots, n$.

当 i 是偶数时, 因为被积函数是奇函数, 等号两侧均为 0 ;

当 i 是奇数时, 对等号左侧式子进行分部积分:

$$\begin{aligned}
i\int_{-\infty}^{+\infty} x^{i-1}\exp(-x^2)dx &= \int_{-\infty}^{+\infty} \exp(-x^2)d(x^i) \\
&= \exp(-x^2)x^i \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} x^i d\exp(-x^2) \\
&= -\int_{-\infty}^{+\infty} x^i \exp(-x^2)(-2x)dx \\
&= 2\int_{-\infty}^{+\infty} x^{i+1}\exp(-x^2)dx
\end{aligned}$$

综上, 得证. □

(3) 解. m 是奇数时, 被积函数是奇函数, 值为 0 ;

m 是偶数时, 使用(2)中的结论:

$$\begin{aligned}
\int_{-\infty}^{+\infty} x^m \exp(-x^2)dx &= \frac{m-1}{2}\int_{-\infty}^{+\infty} x^{m-2}\exp(-x^2)dx \\
&= \frac{(m-1)(m-3)}{4}\int_{-\infty}^{+\infty} x^{m-4}\exp(-x^2)dx \\
&= \dots \\
&= \frac{(m-1)!!}{2^{\frac{m}{2}}}\int_{-\infty}^{+\infty}\exp(-x^2)dx \\
&= \frac{(m-1)!!}{2^{\frac{m}{2}}}\sqrt{\pi}
\end{aligned}$$

□

4. (1) 当 $x \geq 1$ 时, 上述反常积分收敛.

证明.

$$\begin{aligned}
\Gamma(x) &= \int_0^{+\infty} t^{x-1}\exp(-t)dt \\
&= \int_0^1 t^{x-1}\exp(-t)dt + \int_1^{+\infty} t^{x-1}\exp(-t)dt
\end{aligned}$$

因此需要对 0 和 $+\infty$ 两处进行检验:

取 $0 < p < 1$:

$$\begin{aligned}
\lim_{t\to 0^+}\frac{t^{x-1}e^{-t}}{\frac{1}{t^p}} &= \lim_{t\to 0^+} t^{x+p-1} \\
&= 0
\end{aligned}$$

取 $\epsilon > 0$:

$$\begin{aligned}
\lim_{t\to+\infty}\frac{t^{x-1}e^{-t}}{\frac{1}{t^{1+\epsilon}}} &= \lim_{t\to+\infty}\frac{t^{x+\epsilon}}{e^t} \\
&= 0
\end{aligned}$$

综上, 得证. □

(2) 证明.

$$\begin{aligned}
x\Gamma(x) &= x\int_0^{+\infty} t^{x-1}\exp(-t)dt \\
&= \int_0^{+\infty}\exp(-t)d(t^x) \\
&= \exp(-t)t^x\,\Big|_0^{+\infty} - \int_0^{+\infty} t^x d\exp(-t) \\
&= \int_0^{+\infty} t^x\exp(-t)dt \\
&= \Gamma(x+1)
\end{aligned}$$

得证. □

(3) 证明.

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}\exp(-t)dt$$

$$\begin{aligned}
&\text{令} s^2 = t \\
&= \int_0^{+\infty} s^{2x-2}\exp(-s^2)d(s^2) \\
&= \int_0^{+\infty} s^{2x-2}\exp(-s^2)2sds \\
&= 2\int_0^{+\infty} s^{2x-1}\exp(-s^2)ds
\end{aligned}$$

得证. □

5. (1) 解.

$$\begin{aligned}
B(\alpha,\beta) &= \int_0^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy \\
&= \int_0^{1} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy + \int_1^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy
\end{aligned}$$

因此需要对 0 和 $+\infty$ 两处进行检验:

取 $0 < p < 1$:

$$\begin{aligned}
\lim_{y\to 0^+} \frac{\frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}}{\frac{1}{y^p}} &= \lim_{y\to 0^+} \frac{y^{\alpha+p-1}}{(1+y)^{\alpha+\beta}} \\
&= \lim_{y\to 0^+} y^{\alpha+p-1} \\
&= 0
\end{aligned}$$

解得 $\alpha \geq 1$;

取 $\epsilon > 0$:

$$\begin{aligned}
\lim_{y\to +\infty} \frac{\frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}}{\frac{1}{y^{1+\epsilon}}} &= \lim_{y\to +\infty} \frac{y^{\alpha+\epsilon}}{(1+y)^{\alpha+\beta}} \\
&= \lim_{y\to +\infty} \frac{y^{\alpha+\epsilon}}{y^{\alpha+\beta}} \\
&= \lim_{y\to +\infty} y^{\epsilon-\beta} \\
&= 0
\end{aligned}$$

解得 $\beta > 0$. □

(2) 证明. 对等式右侧式子进行分部积分:

$$\begin{aligned}
\frac{\alpha}{\alpha+\beta}B(\alpha,\beta) &= \frac{\alpha}{\alpha+\beta}\int_0^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy \\
&= \frac{1}{\alpha+\beta}\int_0^{+\infty} \frac{1}{(1+y)^{\alpha+\beta}}d(y^\alpha) \\
&= \frac{1}{\alpha+\beta}\left(\frac{y^\alpha}{(1+y)^{\alpha+\beta}}\Big|_0^{+\infty} - \int_0^{+\infty} \frac{y^\alpha}{(1+y)^{\alpha+\beta+1}}y^{\alpha-1}(-\alpha-\beta)dy\right) \\
&= \frac{1}{\alpha+\beta}\left(0 + (\alpha+\beta)\int_0^{+\infty} \frac{y^\alpha}{(1+y)^{\alpha+\beta+1}}dy\right) \\
&= \int_0^{+\infty} \frac{y^\alpha}{(1+y)^{\alpha+\beta+1}}dy \\
&= B(\alpha+1,\beta)
\end{aligned}$$

□

(3) 证明.

$$
\begin{aligned}
B(\alpha,\beta) &= \int_0^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}} dy \\
\text{令} x = &1 - \frac{1}{y+1} \\
&= \int_0^1 \frac{(\frac{1}{1-x}-1)^{\alpha-1}}{(\frac{1}{1-x})^{\alpha+\beta}} d(\frac{1}{1-x}-1) \\
&= \int_0^1 (\frac{1}{1-x}-1)^{\alpha-1}(1-x)^{\alpha+\beta}\frac{1}{(1-x)^2} dx \\
&= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx
\end{aligned}
$$

得证. □