4.2

2.

(1) $\int_L x\sqrt{(x^2-y^2)}\,dl$ , $L: r^2=a^2\cos 2\theta$ , $\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]$ , $a>0$

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\cos\theta\cdot\sqrt{r^2\cos 2\theta}\cdot r\,d\theta$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos\theta\sqrt{\cos 2\theta}\,d\theta$$

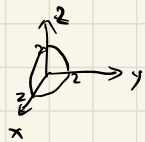
$$=a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)\,d\sin\theta$$

$$=a^3\int_{-\frac{\sqrt2}{2}}^{\frac{\sqrt2}{2}}(1-2t^2)\,dt$$

$$=a^3\left(t-\frac{2}{3}t^3\right)\Big|_{-\frac{\sqrt2}{2}}^{\frac{\sqrt2}{2}}$$

$$=\frac{2}{3}\sqrt2\,a^3$$

(4) $\int_L x\,dl$ , $L: x^2+y^2+z^2=4$ . $x,y,z>0$ 的边界




$$=2\int_0^{\frac{\pi}{2}}2\cos\theta\cdot\sqrt{(2\sin\theta)^2+(2\cos\theta)^2}\,d\theta$$

$$=2\int_0^{\frac{\pi}{2}}2\cos\theta\cdot 2\,d\theta$$

$$=8\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$$

$$=8$$

4. $m=\int_L\rho(x,y)\,dl=\int_{\sqrt3}^{\sqrt{15}}x^2\cdot\sqrt{1+(\frac{1}{x})^2}\,dx=\int_{\sqrt3}^{\sqrt{15}}x\sqrt{1+x^2}\,dx$

$$=\int_{\sqrt3}^{\sqrt{15}}(1+x^2)^{\frac12}\,d\frac12x^2=\frac12\cdot\int_3^{15}(1+t)^{\frac12}\,dt=\frac12\cdot\frac23(1+t)^{\frac32}\Big|_3^{15}$$

$$=\frac{56}{3}$$

5.

$$S=\int_L\left(a+\frac{x^2}{a}\right)dl$$

$$=\int_0^{2\pi}(a+a\cos^2\theta)\cdot a\,d\theta$$

$$=a^2\int_0^{2\pi}(1+\cos^2\theta)\,d\theta$$

$$=a^2\int_0^{2\pi}\left(\frac{3}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=\frac{a^2}{4}\int_0^{4\pi}(3+\cos\theta)\,d\theta$$

$$=\frac{a^2}{4}\cdot 12\pi$$

$$=3\pi a^2$$

6.

$$\overline{x}=\frac{1}{4a}\int_L x\,dl=\frac{1}{4a}\int_0^{\pi}a(t-\sin t)\cdot\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt$$

$$=\frac{1}{4a}\cdot a^2\int_0^{\pi}(t-\sin t)\sqrt{2-2\cos t}\,dt=\frac{a}{4}\int_0^{\pi}(t-\sin t)\cdot 2\sin\frac{t}{2}\,dt$$

$$=\frac{a}{2}\int_0^{\pi}(t-\sin t)\sin\frac{t}{2}\,dt=\frac{a}{2}\left(4-\frac{4}{3}\right)=\frac{4a}{3}$$

$$\overline{y}=\frac{1}{4a}\int_L y\,dl=\frac{1}{4a}\int_0^{\pi}a^2(1-\cos t)\cdot 2\sin\frac{t}{2}\,dt$$

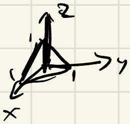
$$=\frac{a}{2}\int_0^{\pi}(1-\cos t)\sin\frac{t}{2}\,dt=\frac{a}{2}\left(2+\frac{2}{3}\right)=\frac{4a}{3}$$

$\Rightarrow$ 质心 $\left(\frac{4}{3}a,\ \frac{4}{3}a\right)$

4.3

1.

(3) $\iint_S \frac{ds}{(1+x+y)^2}$ , $S: x+y+z\le 1, x\ge 0, y\ge 0, z\ge 0$



$= \iint_{S_1} \frac{1}{(1+x+y)^2} dxdy + \iint_{S_2} \frac{1}{(1+x)^2} dxdz + \iint_{S_3} \frac{1}{(1+y)^2} dydz$

$+ \iint_{S_4} \frac{1}{(1+x+y)^2} \sqrt{3}\, dxdy$

$= (\sqrt{3}+1) \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy + \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x)^2} dz + \int_0^1 dy \int_0^{1-y} \frac{1}{(1+y)^2} dy$

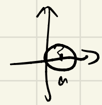
$= (\sqrt{3}+1) \int_0^1 dx \cdot \left(\frac{1}{1+x} - \frac{1}{2}\right) + 2\int_0^1 dx \frac{1-x}{(1+x)^2}$

$= (\sqrt{3}+1)(\ln 2 - \frac{1}{2}) + 2 \cdot \left[\frac{-2}{1+x} - \ln(x+1)\right)\Big|_0^1$

$= (\sqrt{3}+1)\ln 2 - \frac{\sqrt{3}+1}{2} + 2 - 2\ln 2$

$= (\sqrt{3}-1)\ln 2 + \frac{3-\sqrt{3}}{2}$

(4) $\iint_S (xy+yz+xz)\, ds$ , $S: z=\sqrt{x^2+y^2}$ , $x^2+y^2=2ax$



$= \iint_S \left(xy + \sqrt{x^2+y^2}(x+y)\right) \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\, dxdy$

$\begin{cases} x = r\cos\theta & \theta\in[-\frac{\pi}{2}, \frac{\pi}{2}] \\ y = r\sin\theta & r\in[0, 2a\cos\theta] \end{cases}$

$= \sqrt{2}\iint_S \left(xy + \sqrt{x^2+y^2}(x+y)\right) dxdy$

$= \sqrt{2}\iint_S \sqrt{x^2+y^2}(x+y)\, dxdy$ S关于x轴对称,

$= \sqrt{2}\iint_S x\sqrt{x^2+y^2}\, dxdy$

$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} r^2\cos\theta\, r dr$

$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot \frac{1}{4}\cos\theta \cdot (2a\cos\theta)^4$

$= 4\sqrt{2}a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot \cos^5\theta$

$= 4\sqrt{2}a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)^2\, d\sin\theta$

$= 4\sqrt{2}a^4 \cdot \left(t - \frac{2}{3}t^3 + \frac{1}{5}t^5\right)\Big|_{-1}^{1} = \frac{64\sqrt{2}}{15}a^4$

(5) $\iint_S x\,dS = \iint_S u\cos v\cdot\sqrt{u^2+a^2-1}\,du\,dv$

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = av \end{cases}$$

$= \int_0^r du\int_0^{2\pi} u\sqrt{u^2+a^2-1}\cos v\,dv$

$= \int_0^r du\cdot 0 = 0$

3. $m = \iint_S \sigma\,dS = \iint_S z\,dS = \iint_S \frac{x^2+y^2}{2}\cdot\sqrt{1+x^2+y^2}\,dx\,dy$

$= \frac{1}{2}\iint_S (x^2+y^2)\sqrt{1+x^2+y^2}\,dx\,dy$ $\quad\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad r\in[0,\sqrt{2}]$

$= \frac{1}{2}\int_0^{\sqrt{2}} dr\int_0^{2\pi} d\theta\cdot r^2\sqrt{1+r^2}\cdot r$

$= \frac{1}{2}\int_0^{\sqrt{2}} dr\cdot 2\pi r^3\sqrt{1+r^2}$

$= \pi\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,d\frac{1}{2}r^2$

$= \frac{\pi}{2}\int_0^2 t\sqrt{1+t}\,dt$

$= \frac{\pi}{2}\left(\frac{2}{5}(1+t)^{\frac{5}{2}} - \frac{2}{3}(1+t)^{\frac{3}{2}}\right)\Big|_0^2 = \frac{2}{15}\pi + \frac{4}{5}\sqrt{3}$

6. 第一象限:

$\bar{x} = \bar{y} = \bar{z} = \dfrac{\iint_S x\,dS}{\iint_S dS}$

$= \frac{1}{\frac{1}{2}\pi a^2}\iint_S a\sin\theta\cos\varphi\cdot(a^2\sin\theta\,d\theta\,d\varphi)$

$= \frac{2a}{\pi}\iint_S \sin^2\theta\cos\varphi\,d\theta\,d\varphi$

$= \frac{2a}{\pi}\int_0^{\frac{\pi}{2}} d\varphi\int_0^{\frac{\pi}{2}} d\theta\,\sin^2\theta\cos\varphi$

$= \frac{2a}{\pi}\int_0^{\frac{\pi}{2}} d\varphi\,\frac{\pi}{4}\cos\varphi$

$= \frac{2a}{\pi}\times\frac{\pi}{4} = \frac{a}{2}$ $\Rightarrow$ 质心 $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

上半球面:

由对称性知 $\bar{x} = \bar{y} = 0$

$\bar{z} = \frac{1}{2\pi a^2}\iint_S z\,dS$

$= \frac{1}{2\pi a^2}\iint_S a\cos\theta\,(a^2\sin\theta\,d\theta\,d\varphi)$

$= \frac{a}{2\pi}\int_0^{2\pi} d\varphi\int_0^{\frac{\pi}{2}} d\theta\,\sin\theta\cos\theta$

$= \frac{a}{2}$

$\Rightarrow$ 质心 $\left(0, 0, \frac{a}{2}\right)$

11. $(x,y,z) \longrightarrow (u,v,w)$

其中 $u = \frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}}$ $\quad u^2+v^2+w^2 \le 1 \Rightarrow \begin{array}{l} u=u \\ v=\sqrt{1-u^2}\cos\theta \\ w=\sqrt{1-u^2}\sin\theta \end{array}$

$\Rightarrow ds = ds'$

$\Rightarrow \iint_S f(ax+by+cz)\,ds = \iint_S f(\sqrt{a^2+b^2+c^2}\,u)\,ds'$

$= \iint f(\sqrt{a^2+b^2+c^2}\,u)\cdot\sqrt{1+(\frac{\partial w}{\partial u})^2+(\frac{\partial w}{\partial v})^2}\,du\,dv$

$= \int_{-1}^{1} du\, f(\sqrt{a^2+b^2+c^2}\,u)\int_0^{2\pi} d\theta$

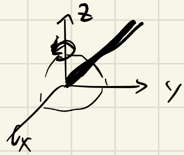
$= 2\pi\int_{-1}^{1} f(\sqrt{a^2+b^2+c^2}\,t)\,dt$

4.4

2.

(3) $\oint_{L^+} \frac{dx+dy}{|x|+|y|}$   $L^+$: 

$= 0$ (由对称性知)

(5) $\int_{L^+} xyz\,dz$ , $L: \begin{cases} x^2+y^2+z^2=1 \\ z=y \end{cases}$ 

$= \int_0^{2\pi} \cos\theta \cdot \frac{1}{2}\sin^2\theta \cdot \frac{1}{\sqrt{2}}\cos\theta \, d\theta \quad \Rightarrow \begin{cases} x=\cos\theta \\ y=z=\frac{1}{\sqrt{2}}\sin\theta \end{cases} \theta: 0 \to 2\pi$

$= \frac{1}{2\sqrt{2}} \int_0^{2\pi} \sin^2\theta \cos^2\theta \, d\theta$

$= \frac{1}{8\sqrt{2}} \int_0^{2\pi} \sin^2 2\theta \, d\theta$

$= \frac{1}{8\sqrt{2}} \int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta$

$= \frac{\sqrt{2}}{16}\pi$

4. $\vec{F} = (-x, -y)$   $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$

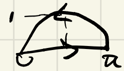
(1) $W = \int_{L^+} \vec{F} \cdot d\vec{l} = \int_{L^+} -x\,dx - y\,dy = -\int_0^{\frac{\pi}{2}} d\theta \cdot a\cos\theta \cdot (-a\sin\theta) + b^2\sin\theta\cos\theta$

$= \int_0^{\frac{\pi}{2}} d\theta \sin\theta\cos\theta \, (a^2-b^2) = \frac{a^2-b^2}{2}$

(2) $W' = 0$ (由对称性知)

4.6

1. (4) $\oint_{L^+} e^x[(1-\cos y)dx-(y-\sin y)dy]$ $\quad x\in[0,\pi],\ y\in[0,\sin x]$



$$=\iint_D (e^x(\sin y-y)-e^x(0+\sin y))dxdy$$

$$=\iint_D -e^x\cdot y\,dxdy$$

$$=\int_0^\pi dx\int_0^{\sin x}dy\,(-e^x\cdot y)$$

$$=\int_0^\pi dx\,(-\frac{1}{2}e^x\sin^2 x)$$

$$=-\frac{1}{2}\int_0^\pi e^x\sin^2 x\,dx$$

$$=-\frac{1}{2}\times\frac{2}{5}(e^\pi-1)$$

$$=\frac{1}{5}(1-e^\pi)$$

$$\oint_{\partial D} X dx+Y dy=\iint_D\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)dxdy$$

3.(3) 辅助线：$L_1$：$x=3$, $y:3e\to1$，$L_2$：$y=1$, $x:3\to1$

$$\int_{L^++L_1+L_2}(\ln\frac{y}{x}-1)dx+\frac{x}{y}dy \quad ,L:(1,1)\to(3,3e)$$

$$=\iint_D\left(\frac{1}{y}-\frac{x}{y}\cdot\frac{1}{x}\right)dxdy=0$$

$$\Rightarrow\int_{L^+}(\ln\frac{y}{x}-1)dx+\frac{x}{y}dy=-\int_{L_1+L_2}(\ln\frac{y}{x}-1)dx+\frac{x}{y}dy$$

$$=-\int_{3e}^{1}\frac{3}{y}dy-\int_3^1(\ln\frac{1}{x}-1)dx$$

$$=3\ln y\Big|_1^{3e}+x\ln\frac{1}{x}\Big|_1^{3}=(3\ln3+3)+3\ln\frac{1}{3}=3$$

5.(1)

$$\oint_{L^+}f(xy)(ydx+xdy)=\iint_D(f(xy)+xyf'(xy))-(f(xy)+yf'(xy))dxdy$$

$$=\iint_D 0\,dxdy=0$$

8.

(1) $\oint_{\partial D}\frac{\partial u}{\partial \vec{n}}dl=\oint_{\partial D}\left(\frac{\partial u}{\partial x}\cos\langle\vec{n},\vec{i}\rangle+\frac{\partial u}{\partial y}\cos\langle\vec{n},\vec{j}\rangle\right)dl \xlongequal{\text{Green公式}} \iint_D \Delta u\,dxdy$

(2) $\oint_{\partial D}v\frac{\partial u}{\partial \vec{n}}dl=\oint_{\partial D}\left(v\frac{\partial u}{\partial x}\cos\langle\vec{n},\vec{i}\rangle+v\frac{\partial u}{\partial y}\cos\langle\vec{n},\vec{j}\rangle\right)dl$

$=\oint_{\partial D}\left(v\frac{\partial u}{\partial x}\cos\langle\vec{\tau},\vec{j}\rangle-v\frac{\partial u}{\partial y}\cos\langle\vec{\tau},\vec{i}\rangle\right)dl$

$=\oint_{\partial D}\left(v\frac{\partial u}{\partial x}dy-v\frac{\partial u}{\partial y}dx\right)$

$=\iint_D\left(\partial\left(v\frac{\partial u}{\partial x}\right)/\partial x+\partial\left(v\frac{\partial u}{\partial y}\right)/\partial y\right)dxdy=\iint_D v\Delta u\,dxdy+\iint_D\nabla u\nabla v\,dxdy$

(3) $\oint_{\partial D}\begin{vmatrix}\frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix}dl=\iint_D\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}dxdy$

$=\oint_{\partial D}\left(v\frac{\partial u}{\partial \vec{n}}-u\frac{\partial v}{\partial \vec{n}}\right)dl$

$=\left(\iint_D v\Delta u\,dxdy+\iint_D\nabla u\nabla v\,dxdy\right)-\left(\iint_D u\Delta v\,dxdy+\iint_D\nabla v\nabla u\,dxdy\right)$

$=\iint_D v\Delta u\,dxdy-\iint_D u\Delta v\,dxdy$

$=\iint_D\begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}dxdy$

10.

(1) $(x^2-y)dx-(x+\sin^2 y)dy=0$

$\Rightarrow\frac{1}{3}x^3-xy+\frac{y}{2}-\frac{1}{4}\sin 2y+C=0$

(2) $xe^y-y^2=C$

(3) $\frac{xdx+ydy}{\sqrt{x^2+y^2}}=\frac{ydx-xdy}{x^2}$

$\sqrt{x^2+y^2}+C=\frac{y}{x}$

$\sqrt{x^2+y^2}-\frac{y}{x}+C=0$

(4) $(\cos x+\frac{1}{y})dx+(\frac{1}{y}-\frac{x}{y^2})dy=0$

$\sin x+\frac{x}{y}+\ln y+C=0$

11.

(1) $(y\cos x - x\sin x)dx + (y\sin x + x\cos x)dy = 0$

$$\frac{\partial P}{\partial y} = \cos x \qquad \frac{\partial Q}{\partial x} = y\cos x + \cos x - x\sin x$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{-P} = 1$$

$$\Rightarrow \mu = e^{\int 1 dy} = e^{y+C}$$

$$e^y(y\cos x - x\sin x)dx + e^y(y\sin x + x\cos x)dy = 0$$

$\Rightarrow$ 解：$e^y y\sin x + e^y(x\cos x - \sin x) + C = 0$

(2) $(x+y)dx + (y-x)dy = 0$

$$(x+y)dx = (x-y)dy$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

令 $u = \frac{y}{x}$

$$\Rightarrow u + x\frac{du}{dx} = \frac{1+u}{1-u}$$

$$\frac{1+u}{1-u^2}du = \frac{dx}{x}$$

$$\arctan u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$$

$$\Rightarrow \arctan\frac{y}{x} - \frac{1}{2}\ln(1+u^2) - \ln|x| + C = 0$$

(3) $(3x^3+y)dx + (2x^2y - x)dy = 0$

$$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 4xy - 1$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2-4xy}{2x^2y - x} = \frac{-2}{x}$$

$$\Rightarrow \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x| + C} = \frac{C}{x^2}$$

$$\Rightarrow (3x + \frac{y}{x^2})dx + (2y - \frac{1}{x})dy = 0$$

$\Rightarrow$ 解：$\frac{3}{2}x^2 - \frac{y}{x} + y^2 + C = 0$

(4) $(x+y)(dx - dy) = dx + dy$

$$d(x-y) = \frac{dx+dy}{x+y} = d(\ln|x+y|)$$

$$\Rightarrow x - y - \ln|x+y| + C = 0$$

(5) $(x^2 - \sin^2 y)dx + x\sin 2y\, dy = 0$

$$\frac{\partial P}{\partial y} = -2\sin y\cos y \quad \frac{\partial Q}{\partial x} = \sin 2y$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{-2\sin 2y}{x\sin 2y} = \frac{-2}{x}$$

$$\Rightarrow \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x| + C} = \frac{C}{x^2}$$

$$\Rightarrow (1 - \frac{1}{x^2}\sin^2 y)dx + \frac{1}{x}\sin 2y\, dy = 0$$

$$x + \frac{1}{x}\sin^2 y = C$$

# 第4章总复习题

9. (1) $u(x_0,y_0)=\frac{1}{2\pi}\oint_{\partial D}\left(u\frac{\partial \ln r}{\partial n}-\ln r\frac{\partial u}{\partial n}\right)dl$

由作业前题知：

$$\oint_{\partial D}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)dl=\iint_D(v\Delta u-u\Delta v)\,dx\,dy$$

令 $v=\ln r$

$\Rightarrow$ 原式 $=\frac{1}{2\pi}\iint_D(u\Delta(\ln r)-\ln r\,\Delta u)\,dx\,dy$

$=\frac{1}{2\pi}\iint_D u\Delta(\ln r)\,dx\,dy$

$$\frac{\partial \ln r}{\partial x}=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\cdot\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$=\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}$$

$$\frac{\partial^2(\ln r)}{\partial x^2}=\frac{1\cdot r^2-(x-x_0)\cdot 2(x-x_0)}{r^4}=\frac{(y-y_0)^2-(x-x_0)^2}{r^4}$$

$$\frac{\partial^2(\ln r)}{\partial y^2}=\frac{(x-x_0)^2-(y-y_0)^2}{r^4}$$

$\Rightarrow \Delta(\ln r)=0\quad (x\neq x_0$ 且 $y\neq y_0)$

取 $(x_0,y_0)$ 一个 $\varepsilon$-邻域：

原式 $=\frac{1}{2\pi}\oint_{\partial B_\varepsilon(x_0,y_0)}\left(u\frac{\partial \ln r}{\partial n}-\ln r\frac{\partial u}{\partial n}\right)dl$

$\varepsilon\to 0$：

$=\frac{1}{2\pi}\cdot 2\pi u(x_0,y_0)=u(x_0,y_0)$

(2) $u(x_0,y_0)=\frac{1}{2\pi R}\oint_L u(x,y)\,dl$

右侧 $=\frac{1}{2\pi R}\int_0^{2\pi}u(x_0+R\cos\theta,y_0+R\sin\theta)\sqrt{R^2\sin^2\theta+R^2\cos^2\theta}\,d\theta$

$=\frac{1}{2\pi}\int_0^{2\pi}u(x_0+R\cos\theta,y_0+R\sin\theta)\,d\theta$

对 $R$ 求偏导 $\frac{\partial u}{\partial R}=0$

令 $R=0$

$\Rightarrow$ 原式右侧 $=\frac{1}{2\pi}\int_0^{2\pi}u(x_0,y_0)\,d\theta$

$=u(x_0,y_0)$