1.1

1. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

eigenvalues : 3 , -1

$$3 \downarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \qquad -1 \downarrow \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

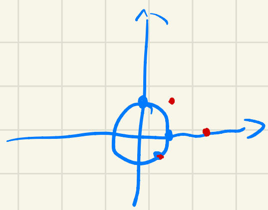
$\Rightarrow$ semi-major axis : 3

— — — minor — — — : 1

2. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

eigenvalues : 2 , 1

$$\underbrace{2 \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 1 \downarrow \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}}_{\to \text{ not orthogonal}}$$



$\Rightarrow$ NO, they're not.

1.2

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ & D - CA^{-1}B \end{bmatrix}$$

$$= \begin{bmatrix} I & \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & \\ & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ & I \end{bmatrix}$$

if $B^T = C \Rightarrow \begin{bmatrix} I & \\ CA^{-1} & I \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ & I \end{bmatrix}^T$

if A & $D - CA^{-1}B$ is positive definite $\Rightarrow M$ is positive definite

1.3

1.

$$\ker(AB) = \overbrace{\left[\{\vec{x} \mid B\vec{x} \in \ker(A),\ B\vec{x} \neq \vec{0}\}\right.}^{S} \cup \left.\ker(B)\right]$$

$\Rightarrow \dim S \le \dim \ker(A)$

$\Rightarrow \dim \ker(AB) \le \dim \ker(A) + \dim \ker(B)$

$$\underset{\displaystyle n}{\underset{\displaystyle \|}{\dim \ker(A^5)}} \le \dim \ker(A^4) + \dim \ker(A) \le \cdots \le 5 \dim \ker(A)$$

$\Rightarrow \dim \ker(A) \ge \frac{n}{5}$

example: $\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & 0 \end{bmatrix}$

2. $\ker(AA^TAAA^T) = \mathbb{R}^n$

☆: $\ker(A^TA) = \ker(A)$

$\forall \vec{x} \in \mathbb{R}^n,\ AA^TAAA^T\vec{x} = \vec{0}$

$\Rightarrow \underset{\in \operatorname{Ran}(A^T)}{A^TAAA^T\vec{x}} \in \ker(A)$

since $\operatorname{Ran}(A^T) \perp \ker(A)$

then $(A^TAAA^T)\vec{x} = \vec{0}$

$\underset{\in \operatorname{Ran}(A)}{AAA^T\vec{x}} \in \ker(A^T)$

since $\operatorname{Ran}(A) \perp \ker(A^T)$

then $AA^T\vec{x} \in \ker(A)$

$\Rightarrow AAA^T = 0$

$\Rightarrow \dim \ker(AAA^T) = n$

$\le \dim \ker(A) + \dim \ker(AA^T)$

$= \dim \ker(A) + \dim \ker(A^T)$

$= 2 \dim \ker(A)$

A is a square matrix

$\Rightarrow \dim \ker(A) \ge \frac{n}{2}$

example: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

1.4

$$A = XDX^{-1}$$

$$R_t = (XDX^{-1} - tI)^{-1}$$

$$= (X(D-tI)X^{-1})^{-1}$$

$$= X^{-1}(D-tI)^{-1}X$$

$$= X^{-1}\begin{bmatrix} \lambda_1 - t & & \\ & \ddots & \\ & & \lambda_n - t \end{bmatrix}^{-1} X$$

$$\lim_{t\to\lambda_x}(\lambda_x - t)R_t = X^{-1}\begin{bmatrix} (\lambda_1-t)^{-1}(\lambda_x-t) & & & \\ & \ddots & & \\ & & \frac{\lambda_x-t}{\lambda_x-t} & \\ & & & \ddots \; (\lambda_n-t)^{-1}(\lambda_x-t) \end{bmatrix} X$$

$$\| \qquad\qquad\qquad\qquad$$

$$B \qquad = X^{-1}\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots\, 0 & & \\ & & & 1 & \\ & & & & 0\;\ddots\; 0 \end{bmatrix} X$$

the x-th entry of the matrix

$$\Rightarrow B^2 = B = B^T$$

So proof is done

1.5

$$A = \begin{bmatrix} 3 & 2 & 1 & & \\ & 2 & & & \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$$