

$$\text{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

$$F(0) = 0$$

$$F(1) = \frac{\phi^1 - \psi^1}{\sqrt{5}} = \left[ \left( \frac{1+\sqrt{5}}{2} \right) - \left( \frac{1-\sqrt{5}}{2} \right) \right] \cdot \frac{1}{\sqrt{5}} = \frac{2\sqrt{5}}{2} \cdot \frac{1}{\sqrt{5}} = 1$$

$$F(k) = \frac{\phi^k - \psi^k}{\sqrt{5}}$$

$$F(k-1) = \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

$$F(k+1) = F(k) + F(k-1)$$

$$= \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

$$= \frac{\phi^k + \phi^{k-1}}{\sqrt{5}} - \frac{\psi^k + \psi^{k-1}}{\sqrt{5}}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k + \left( \frac{1+\sqrt{5}}{2} \right)^{k-1}}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^k + \left( \frac{1-\sqrt{5}}{2} \right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k \left( 1 + \frac{1}{\left( \frac{1+\sqrt{5}}{2} \right)} \right)}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^k \left( 1 + \frac{1}{\left( \frac{1-\sqrt{5}}{2} \right)} \right)}{\sqrt{5}}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k \left( 1 + \frac{2}{1+\sqrt{5}} \right)}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^k \left( 1 + \frac{2}{1-\sqrt{5}} \right)}{\sqrt{5}}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k \left( \frac{3+\sqrt{5}}{1+\sqrt{5}} \right)}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^k \left( \frac{3-\sqrt{5}}{1-\sqrt{5}} \right)}{\sqrt{5}}$$

$$\therefore \frac{3+\sqrt{5}}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{-2-2\sqrt{5}}{-4} = \frac{-2(1+\sqrt{5})}{-4} = \frac{1+\sqrt{5}}{2}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^k \left( \frac{1+\sqrt{5}}{2} \right)}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^k \left( \frac{1-\sqrt{5}}{2} \right)}{\sqrt{5}}$$

$$= \frac{\left( \frac{1+\sqrt{5}}{2} \right)^{k+1}}{\sqrt{5}} - \frac{\left( \frac{1-\sqrt{5}}{2} \right)^{k+1}}{\sqrt{5}}$$

$$\text{since } \phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

$$\therefore \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$