Question 3A: Let $\underline{X} \sim N_p(\underline{\mu}, \underline{\Sigma})$

Show: $M_{\underline{X}}(\underline{t}) \equiv E[\exp(\underline{t}'\underline{X})] = \exp(\underline{t}'\underline{\mu} + \underline{t}'\underline{\Sigma}\,\underline{t}/2)$

Definitions / Knowns:

- For a univariate $Z_i \sim N(0,1)$: $M_{Z_i}(t_i) = E[e^{t_i z_i}]$ $\longrightarrow E[x] = \int_a^b x f(x)$

$$= \int_{-\infty}^{\infty} e^{t_i z_i} \cdot \left(\frac{e^{(-z_i^2/2)}}{\sqrt{2\pi}} \right) dz_i$$

$$= e^{(t_i)^2/2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-(z_i - t_i)^2/2} \, dz_i}_{\text{evaluates to } 1}$$

$$= e^{(t_i)^2/2}$$

- When $\underline{Z}$ is a vector of random samples: $\underline{Z} = (z_1, \dots z_n)$
 $\underline{Z} \sim N(0,1)$

$$M_{\underline{Z}}(t) = E[e^{\sum_{i=1}^n z_i t_i}] = E[\prod_{i=1}^n e^{z_i t_i}] = \prod_{i=1}^n E[e^{z_i t_i}] = \prod_{i=1}^n M_{Z_i}(t_i) = \prod_{i=1}^n e^{t_i^2/2}$$

b/c z_i is a standardized normal we can assume ind.

$$= \exp\{\sum_{i=1}^n t_i^2/2\}$$
$$= \exp\{\underline{t}^2/2\} =$$
$$= \exp\{\underline{t}\,\underline{t}'/2\}$$

- $AA' = A'A = \Sigma^{-1}$

If we define $\underline{X} = A\underline{Z} + \underline{\mu}$ ← here $\underline{X}$ is a linear combination of Z.

$M_{\underline{X}}(\underline{t}) = E[\exp\{\underline{X}\,\underline{t}'\}]$ ← sub for $\underline{X}$

$= E[\exp\{(A\underline{Z} + \underline{\mu})\,\underline{t}'\}] = E[\exp\{A\underline{Z}t' + \underline{\mu}t'\}]$ ↑ Nothing to do w/ Z so we can pull it out

$= \exp\{\underline{\mu}t'\} \cdot E[\exp\{\underline{A}\,\underline{Z}t'\}]$ ← $M_Z(At')$

$= \exp\{\underline{\mu}\,\underline{t}'\} \cdot \exp\{(At)(At)'/2\}$

$= \exp\{\underline{\mu}\,\underline{t}'\} \cdot \exp\{(AA')(tt')/2\}$

$= \exp\{\underline{\mu}\,\underline{t}'\} \cdot \exp\{\Sigma^{-1} tt'/2\} = \exp\{\underline{\mu}'\underline{t}' + \frac{1}{2}\underline{t}\Sigma^{-1}\underline{t}'\}$ ✓

Question 3b: $\underline{X} \sim N_p(\mu, \underline{\Sigma})$

If we partition $\underline{X}$ ...

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \underline{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$\underline{X}_1 = (x_1 \dots x_q)$ $\quad \underline{\mu}_1 = (\mu_1 \dots \mu_q)$

$\underline{X}_2 = (x_{q+1} \dots x_{p-q})$ $\quad \mu_2 = (\mu_{q+1} \dots \mu_{p-q})$

Show that marginal distribution of $\underline{X}_1$ is $\underline{X}_1 \sim N_q(\mu_1, \Sigma_{11})$

If we can use similar logic to part A then we can prove this by writing $\underline{X}_1$ as a linear combination of $\underline{X}$ then we can show this true.

test: $\underline{x}_1 = B\underline{X}$

In the test, we will assume B is a vector that when multiplied by $\underline{X}$ will leave $\underline{X}_1$ remaining

ex:

$$\underline{X}_1 = \overset{B}{[1\ 0]} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 1 \cdot x_1 + 0 \cdot x_2 = x_1$$

$1 \cdot 0$

$1 \times 2 \quad 2 \times 2$

Σ'

Apply:

$\underline{X}_1 = B'\underline{X}$

$E[\underline{X}_1] = E[\underbrace{B'\underline{X}}]$ → constant

$= B' E[\underline{X}]$ → this is defined above:

$= B'\underline{\mu}$ → B' is a constant that when multiplied w/ $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ μ_1 remains.

$E[\underline{X}_1] = \mu_1$

Similarly for the $Var(\underline{X}_1)$

$r(\underline{X}_1) = Var(B'\underline{X})$

$= B' Var(\underline{X})$ → Σ covariance matrix

$= B\underline{\Sigma}$

Question 3c: Show $x_1 | x_2 \sim N_q(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2), \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

Show: $\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}(x_2-\mu_2)$ $\quad var(x_1|x_2) = \Sigma_{11} - \Sigma_{12}\cdot\Sigma_{22}^{-1}\cdot\Sigma_{21}$

Knowns: $f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)}$

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix}$$

$$\Sigma^{11} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\cdot\Sigma_{12}(\Sigma_{22} - A_{12}'\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12})^{-1}\Sigma_{12}'\,\Sigma_{11}^{-1}$$

$$\Sigma^{12} = (\Sigma^{21})' = -\Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22}-\Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{12})^{-1}$$

$$\Sigma^{22} = (\Sigma_{22}-\Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{12})^{-1} =$$

$$= \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\cdot\Sigma_{12}'(\Sigma_{11}-\Sigma_{12}\cdot\Sigma_{22}^{-1}\cdot\Sigma_{12}')^{-1}\cdot\Sigma_{12}\cdot\Sigma_{22}^{-1}$$

$$f(x_1,x_2) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}\cdot \exp\{-\tfrac{1}{2}\underbrace{(\underline{x}-\underline{\mu})'\cdot\Sigma^{-1}\cdot(\underline{x}-\underline{\mu})}\}$$

$$R(x_1,x_2) = (\underline{x}-\underline{\mu})^T\Sigma^{-1}(\underline{x}-\underline{\mu})$$

$$= (x_1-\mu_1)' \, (x_2-\mu_2)' \begin{bmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= (x_1-\mu_1)'\Sigma^{11}(x_1-\mu_1) + 2(x_1-\mu_1)'\Sigma^{12}(x_2-\mu_2) + (x_2-\mu_2)'\Sigma^{22}(x_2-\mu_2)$$

Plug in for Σ^{-1}:

$$R(x_1,x_2) = (x_1-\mu_1)'[\Sigma_{11}^{-1} + \Sigma_{11}^{-1}\cdot\Sigma_{12}(\Sigma_{22}-A_{12}'\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12})^{-1}\cdot\Sigma_{12}'\cdot\Sigma_{11}^{-1}](x_1-\mu_1)$$

$$-2(x_1-\mu_1)'[\Sigma_{11}^{-1}\cdot\Sigma_{12}(\Sigma_{22}-\Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{12})^{-1}]\cdot(x_2-\mu_2) +$$

$$+ (x_2-\mu_2)'[(\Sigma_{22}-\Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{21})^{-1}]\cdot(x_2-\mu_2)$$

$R(x_1, x_2) = (x-\mu)^T \Sigma^{-1}(x-\mu)$

$\cancel{A} =$

$V = x_2 - \mu_2$

$R(x_1, x_2) =$

$$= \left((x_1-\mu_1)\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12} - (x_2-\mu_2)\right)^T (\Sigma_{22}\cdot A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}\cdot$$

$$\left((x_1-\mu_1)\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12} - (x_2-\mu_2)\right)$$

$(x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1)$

$$(x_1-\mu_1)\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12}^T(\Sigma_{22}\cdot A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1} - (x_2-\mu_2)^T\cdot(\Sigma_{22}\cdot A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}$$

$$\cdot(x_1-\mu_1)\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12} \qquad \cdot(x_2-\mu_2)$$

$$R(x_1, x_2) = \begin{cases} R_1(x_1, x_2) = (x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1) + (x_1-\mu_1)\cdot\Sigma_{11}^{-1}\cdot\Sigma_{12}^T(\Sigma_{22}\cdot A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1} \cdot (x_1-\mu_1)\Sigma_{11}^{-1}\Sigma_{12} \\ R_2(x_2) = -(x_2-\mu_2)^T(\Sigma_{22} - A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}\cdot x_2-\mu_2 \end{cases}$$

$$R(x_1, x_2) = R_1(x_1, x_2) + R_2(x_2)$$

$$f(x_1,x_2) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(R(x_1,x_2))\right)$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma_{22}|^{1/2} \cdot |\Sigma_{11} - \Sigma_{12}^T \Sigma_{22}^{-1}\Sigma_{12}|^{1/2}} \exp\left\{-\frac{1}{2}(R(x_1,x_2))\right\}$$

$$= \frac{1}{(2\pi)^{p/2} \cdot |\Sigma_{11}-\Sigma_{12}^T\Sigma_{22}^{-1}\Sigma_{12}|^{1/2}} \cdot \exp\left\{-\frac{1}{2}\left((x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1) + (x_1-\mu_1)^T \cdot \Sigma_{11}^{-1}\Sigma_{12}^T(\Sigma_{22}-A_{12}^T\Sigma_{11}^T\Sigma_{12}) \cdot (x_1-\mu_1)\Sigma^{-1}\Sigma_{12}\right.\right.$$

$$\cdot \frac{1}{(2\pi)^{q/2} \cdot |\Sigma_{22}|^{1/2}} \exp\left\{-\frac{1}{2}\left(-(x_2-\mu_2)^T(\Sigma_{22}-A_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1} \cdot (x_2-\mu_2)\right\}\right.$$

<u>Define</u>:

$$f_2(x_2) = \int f(x_1,x_2)\,dx_1 = \frac{1}{(2\pi)^{p/2}|\Sigma_{22}|^{1/2}} \exp\left\{-\frac{1}{2}(x_2-\mu_2) \cdot (\Sigma_{11}-A_{12}^T\Sigma_{22}^{-1}\Sigma_{12})^{-1}(x_2-\mu_2)\right\}$$

$$f(x_1|x_2) = \frac{f(x_1,x_2)}{f_2(x_2)} = \frac{\exp\left\{-\frac{1}{2}((x_1-\mu_1)^T(\Sigma_{11}^{-1}+\Sigma_{11}^{-1}\Sigma_{12}^T(\Sigma_{22}-A_{12}^T\Sigma_{11}^T\Sigma_{12}) \cdot (x_1-\mu_1)\right.}{(2\pi)^{p/2} \cdot |\Sigma_{11}-\Sigma_{12}^T\Sigma_{22}^{-1}\Sigma_{12}|^{1/2}}$$

I am stopping because I'm not entirely sure I have done the math right to this point. ~~Also, I a~~ So I don't want to derive $E[x_1|x_2]$ and $Var(x_1|x_2)$ if my $f(x_1|x_2)$ isn't correct.

I struggled with this problem. I referenced website (see below); but they derived for $x_2|x_1$. I tried to go back & figure it out, but I am not sure where I went wrong.

http://fourier.eng.hmc.edu/e161/lectures/gaussianprocess/node7.html?fbclid=IwAR3ig6V9IwRY2w4QcnWw4GAuA5tzjCIpKho2iBiRG9lFGv3UE-d0eZriJyo

Question 4. Let $\underline{Z} \sim N_p(0, I_p)$

$\underline{\Gamma}$: orthogonal matrix
$\underline{\Gamma}\underline{\Gamma}' = I_p$

Show that $\underline{W} = \underline{\Gamma}\underline{Z} \sim N_p(0, I_p)$:

Here $\underline{W} = \underline{\Gamma}\underline{Z}$ is a linear combination of Z_i

I attempted 2 solutions, but I'm not sure Solution #2 is valid.

Solution #1: Similar to 3A.

Before we had $\underline{X} = AZ + \mu$ so

$$M_X(\underline{t}) = E[\exp\{\underline{X}\underline{t}'\}] = E[\exp\{AZ + \mu\}] = \exp\{\mu t' + \tfrac{1}{2} t \Sigma^{-1} t'\}$$

But in this case we have $\underline{W} = \Gamma Z$

$$M_W(\underline{t}) = E[\exp\{W\underline{t}'\}] = E[\exp\{\underline{\Gamma}\underline{Z}\underline{t}'\}] = M_z(\Gamma t')$$

$$\cancel{E[\exp} = \exp\{(\underline{\Gamma}\underline{t})^2/2\} = \exp\{(\underline{\Gamma}\underline{t})(\underline{\Gamma}\underline{t})/2\}$$
$$= \exp\{(\underline{\Gamma}t)'(\underline{\Gamma}t)/2\}$$
$$= \exp\{(\underline{\Gamma}\underline{\Gamma}')(tt')/2\}$$
$$= \exp\{I_p(tt')/2\}$$

There is no μ term thus the mean is 0.

usually Σ^{-1}
$I_p^{-1} = I_p$

Solution #2:

$Z \sim N_p(0, I_p)$

$\underline{\Gamma}\underline{Z} \sim N_p(\Gamma \cdot 0, \Gamma I_p \Gamma') \rightarrow$ similar to $AX \sim N(A\mu, A\Sigma A')$

$N_p(0, I_p \cdot I_p)$

$I_p \cdot I_p$ ex: $I_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 0 & 0\cdot 0 + 0\cdot 1 \\ 0\cdot 1 + 0\cdot 0 & 1\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_p$$

$\Gamma \underline{Z} \sim N_p(0, I_p)$

↳ I'm not sure if I've done enough to prove this formally though.

Question 5:

Show that the identity matrix I_p is the one among all $p \times p$ dimensional positive definite matrices B that maximizes

$$f(B) = |nB|^{n/2} \exp\{-\frac{n}{2} \operatorname{trace}(B)\}$$

How do we maximize a function? Find the MLE

Log likelihood:

$$\ln(f(B)) = \frac{n}{2}\ln(|nB|) + -\frac{n}{2}\operatorname{trace}(B)$$

estimate $\hat{B}$

$\frac{d}{d\Sigma}\ln|\Sigma| = \Sigma^{-1}$

$\frac{d}{d\Sigma}(\operatorname{trace}(a\Sigma) \rightarrow a'$

$a = I_p$ b/c $I_p \cdot B = B$

$$\frac{d}{dB}(\ln(f(B))) = \frac{n}{2} \cdot n \cdot (nB)^{-1} + -\frac{n}{2} \cdot I_p$$

Set equal to 0 & Solve for $\hat{B}$

$$0 = \frac{n^2}{2}(n\hat{B})^{-1} + -\frac{n}{2} \cdot I_p$$

$$0 = \frac{n}{2}(\cancel{n}(\cancel{n}\hat{B})^{-1} - I_p)$$

$$0 = n(n\hat{B})^{-1} - I_p$$

$$I_p = \frac{\cancel{n}}{\cancel{n}}\hat{B}$$

$n \cdot n^{-1} = 1$

Question 6: Let $\underline{X} \sim N_p(\underline{\mu}, \underline{\Sigma})$

Find the MLE of $\underline{\Sigma}$ & $\underline{\mu}$, given a sample $(X_1 \dots X_n) \sim N_p(\underline{\mu}, \underline{\Sigma})$

- use the method of derivatives of symmetric positive definite matrices & vectors

① Define likelihood:

$$\Theta = \{\underline{\mu}, \underline{\Sigma}\}$$

$$L(\Theta \mid X_1, \dots X_n) = f(X_1 \dots X_n \mid \Theta) = \prod_{i=1}^{n} f(X_i \mid \Theta) = \prod_{i=1} \frac{1}{(2\pi)^{p/2} \cdot |\Sigma|^{1/2}} \exp\{-\tfrac{1}{2}(x_i - \mu)' \Sigma^{-1} (x_i - \mu)\}$$

$$= \frac{1}{2\pi^{(np/2)} \cdot |\Sigma|^{n/2}} \prod_{i=1} \exp\{-\tfrac{1}{2}(x_i - \mu)' \Sigma^{-1} (x_i - \mu)\}$$

$$= \frac{1}{2\pi^{(np/2)} \cdot |\Sigma|^{n/2}} \cdot \exp\{-\tfrac{1}{2} \Sigma^{-1} \sum_{i=1}^{n} (x_i - \mu)' (x_i - \mu)\}$$

② Take the natural log (ln):

$$\ln(L(\Theta \mid X_1, \dots X_n)) = \ln(\ )$$

$$= -n/2 (p \ln(2\pi) + \ln(|\Sigma|) + -\tfrac{1}{2} \cdot \Sigma^{-1} \cdot \sum_{i=1}^{n} (x_i - \mu)'(x - \mu)$$

③ Estimate Parameters:

$$\hat{\underline{\mu}} = \frac{d}{d\mu}\left[\ln(L(\Theta \mid X_1, \dots X_n))\right] = \frac{d}{d\mu}\left(-\frac{n}{2}(p \ln(2\pi) + \ln |\Sigma|)\right) + \frac{d}{d\mu}\left(-\tfrac{1}{2}(x - \mu)' \Sigma^{-1} (x - \mu)\right)$$

no μ = constant

$$= 0 + -\tfrac{1}{2} \cdot \Sigma^{-1} \sum_{i=1}^{n} \frac{d}{d\mu} \underbrace{(x_i - \mu) \cdot (x_i - \mu)'}$$

$$\frac{d}{d\mu} \begin{bmatrix} x_1^2 - \mu \\ \vdots \\ x_n^2 - \mu \end{bmatrix} \rightarrow \begin{bmatrix} 2x_1 - \mu \\ \vdots \\ 2x_n - \mu \end{bmatrix}$$

$$= -\cancel{\tfrac{1}{2}} \cdot \Sigma^{-1} \cdot -\cancel{2} \sum_{i=1}^{n} (x_i - \mu)$$

$$= \Sigma^{-1} \cdot \sum_{i=1}^{n} x_i - \hat{\mu} = 0 \leftarrow \text{Solve for } \hat{\mu}$$

$$\sum_{i=1}^{n} x_i - n\hat{\mu} = 0$$

$$\sum_{i=1}^{n} x_i = n\hat{\mu}$$

$$\boxed{\frac{1}{n} \sum_{i=1}^{n} x_i = \hat{\mu}}$$

$$\hat{\Sigma} = \frac{d}{d\Sigma}\left[\ln(L(\theta \mid x_1, \dots x_n))\right] = \frac{d}{d\Sigma}\left(\frac{n}{2}(p\ln(2\pi) + \ln|\Sigma|\right) + \frac{d}{d\Sigma}\left(-\frac{1}{2}\cdot\Sigma^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)\right)$$

$$= \frac{d}{d\Sigma}\left(-\frac{n}{2}p\ln(2\pi)\right) - \frac{d}{d\Sigma}\left(+\frac{n}{2}\cdot\ln|\Sigma|\right) + -\frac{1}{2}\cdot\sum_{i=1}^{n}(x_i-\mu)'(x_i-\mu)\frac{d}{d\Sigma}(\Sigma^{-1})$$

$$= 0 + \frac{n}{2}\Sigma^{-1} + -\frac{1}{2}\sum_{i=1}^{n}\left(\frac{d}{d\Sigma}\left((x_i-\mu)(x_i-\mu)'\cdot\Sigma^{-1}\right)\right)$$

$$= -\frac{n}{2}\Sigma^{-1} + -\frac{1}{2}\cdot\sum_{i=1}^{n}\cdot -\Sigma^{-1}\cdot(x_i-\mu)\cdot(x_i-\mu)'\cdot\Sigma^{-1}$$

$$= -\frac{n}{2}\Sigma^{-1} + +\frac{1}{2}\cdot+\Sigma^{-1}\cdot\Sigma^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'$$

$$= \frac{1}{2}\Sigma^{-1}\left(-n + \Sigma^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'\right)$$

Set equal to 0 & Solve for $\hat{\Sigma}$:

$$0 = \frac{1}{2}\hat{\Sigma}^{-1}\left(-n + \hat{\Sigma}^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'\right)$$

$$0 = -n + \hat{\Sigma}^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'$$

$$\hat{\Sigma}\cdot n = \hat{\Sigma}^{-1}\cdot\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'\cdot\hat{\Sigma}^{-1}$$

$$\hat{\Sigma}n = \sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'$$

$$\boxed{\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)'} \rightarrow$$

Sample variance is usually defined as:

$$S_k^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ik} - \bar{x}_k)^2$$

but sometimes the denominator is an n instead of $n-1$ which can happen if the mean is known. In this case we are fixing the mean (it is known) in order to solve for Σ.