Question 3: Derive the $\underline{LRT}$

Let $\underline{X}_1, \underline{X}_2 \dots \underline{X}_n \sim N_p(\underline{\mu}, \underline{\Sigma})$

$H_0: \Sigma = \sigma^2 I$, $\Lambda \leq c_\alpha$

$H_1: \Sigma \neq \sigma^2 I$?

$$\Lambda^{2/np} = \frac{\frac{1}{p}\sum_{i=1}^{p} \hat{\lambda}_i}{\prod_{i=1}^{p} \hat{\lambda}_i}$$

The loglikelihood for μ_i & Σ_i given the Observations, $\underline{X}_i$

$$l(\mu_i, \Sigma_i) = -\frac{n_i}{2} \log |\Sigma_i| - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \mu_i)' \Sigma_i^{-1} (x_{ij} - \mu_i)$$

For the ~~unrestricted model~~ numerator (restricted) $\Sigma = \sigma^2 \cdot I$ ... For denominator: $\Sigma = \Sigma$

- The above function is maximized w/ $\mu_i = \bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$ for $i = 1 \dots p$
- To maximize for Σ first plug in for Sigma:

$$l(\mu_i\ \sigma_i^2 \cdot I) = -\frac{n_i}{2} \log |\sigma_i^2 I| - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)' (\sigma_i^2 \cdot I)(x_{ij} - \bar{x}_i)$$

Take derivative w/ respect to σ_i^2:

$$= -\frac{n_i}{2} (\sigma_i^2 I)^{-1} - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} -(\sigma_i^2 \cdot I)^{-1} \cdot (x_{ij} - \bar{x}) \cdot (x_{ij} - \bar{x}_i)' \cdot (\sigma_i^2 \cdot I)$$

$$= -\frac{n_i}{2} (\sigma_i^2 \cdot I)^{-1} + \frac{1}{2} (1 (\sigma_i^2 \cdot I))^{-1} \cdot (\sigma_i^2 \cdot I)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)(x_{ij} - x_i)'$$

$$0 = \cancel{(-\sigma_i^2 \cdot I)^{-1}} \left(-\frac{n_i}{2} + \frac{1}{2} (\sigma_i^2 \cdot I)^{-1} \right) \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)(x_{ij} - x_i)'$$

$$\frac{n_i}{\cancel{2}} = \frac{1}{\cancel{2}} (\sigma_i^2 \cdot I)^{-1} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)(x_{ij} - x_i)'$$

$$\hat{\sigma}_i^2 I = \frac{1}{n_i} \sum_{i=1}^{m} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)(X_{ij} - x_i)'$$

Alternative

$$\Lambda = \frac{|\hat{\Sigma}|^{-n_i/2} + \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\hat{\mu})'\Sigma^{-1}(x_{ij}-\hat{\mu})\right\}}{(\hat{\sigma}_i^2 \cdot I)^{-n_i/2} + \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\hat{\mu})'(\hat{\sigma}_i^2 \cdot I)^{-1}(x_{ij}-\hat{\mu})\right\}}$$

null model

I am not entirely sure I know what to do here. The way I understood is that for our null model $\Sigma = \sigma_i \cdot I$ so we can replace all of our Σ's w/ this value, but then we don't have an MLE for σ_i^2 so I tried to find that. The LRT is the ratio of the maximized likelihood functions. I am not sure if I can reduce much further.