HW 6: Question 2 A ii) Derive a LRT statistic for testing differences b/w mean effects across digits.

Assume the var-covariance matrices from part A.i) are the true values because they are not equal.

Definition of LRT statistic:

$$\Lambda = \frac{\max_{\{\Sigma\}} L(\mu_0, \Sigma)}{\max_{\{\mu_1, \Sigma\}} L(\mu_1, \Sigma)} = \frac{\max_{\{\Sigma\}} \left(L(\mu, \Sigma_0) \cdot L(\mu, \Sigma_1) \cdots L(\mu, \Sigma_9)\right)}{\max_{\{\mu_i, \Sigma\}} L(\mu_0, \Sigma_0) \cdot L(\mu_1, \Sigma_1) \cdots L(\mu_9, \Sigma_9)}$$

where $\underset{1\times10}{\mu_0} = (\mu, \mu, \mu \cdots \mu)$ ← means are the same

& $\underset{1\times10}{\mu_1} = (\mu_0, \mu_1, \mu_2 \cdots \mu_9)$ ← means are different

& $\Sigma = (\Sigma_0, \Sigma_1, \cdots \Sigma_i, \cdots \Sigma_9)$ where Σ_i is the true var-cov matrix for digit i.

To make this easier I am going to define the numerator & denominator separately, & the combine at the end.

Defining the denominator

$$L(\mu_1, \Sigma) = \prod_{i=1}^{9} \boxed{L(\mu_i, \Sigma_i)}$$

$$= f(X; \mu_i, \Sigma_i) = \frac{1}{2\pi^{p/2} \cdot |\Sigma_i|^{1/2}} \exp\left\{\frac{1}{2}(X_i - \mu_i)' \Sigma_i^{-1} (X_i - \mu_i)\right\}$$

$$= \prod_{j=1}^{n_i}$$

$$= (2\pi)^{-n_i p/2} \cdot |\Sigma_i|^{-n_i/2} \cdot \exp\left\{-\frac{1}{2} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)' \Sigma_i^{-1} (x_{ij} - \mu_i)\right\}$$

$$= (2\pi)^{-n_i p/2} \cdot |\Sigma_i|^{-n_i/2} \cdot \exp\left\{\frac{1}{2} \sum_{j=1}^{n_i} tr\, (x_{ij} - \mu_i)' \Sigma_i^{-1} (x_{ij} - \mu_i)\right\}$$

$$= 2\pi^{-n_i p/2} \cdot |\Sigma_i|^{-n_i/2} \cdot \exp\left\{-\frac{1}{2} tr\, \Sigma_i^{-1} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)(x_{ij} - \mu_i)'\right\}$$

$$= 2\pi^{-n_i p/2} \cdot |\Sigma_i|^{-n_i/2} \exp\left\{-\frac{1}{2} tr\, \Sigma_i^{-1} \cdot \Sigma_i n_i\right\}$$

$$= 2\pi^{-n_i p/2} \cdot |\Sigma_i|^{n_i/2} \cdot \exp\left\{-\frac{n_i p}{2}\right\}$$ Plug back in

$$= (2\pi)^{\frac{n_0 p}{2}} \cdot |\Sigma_0|^{-n_0/2} \cdot \exp\left\{-\frac{n_0 p}{2}\right\} \cdot (2\pi)^{\frac{n_1 p}{2}} \cdot |\Sigma_1|^{n_1/2} \cdot \exp\left\{-\frac{n_1 p}{2}\right\} \cdots 2\pi^{\left(\frac{n_9 p}{2}\right)} |\Sigma_9|^{-\frac{n_9}{2}} \exp\left\{-\frac{n_9 p}{2}\right\}$$

$$= 2\pi^{-\frac{(n_0 + n_1 \cdots n_9) p}{2}} \cdot \left(|\Sigma_0|^{-n_0/2} \cdots |\Sigma_9|^{-n_9/2}\right) \cdot \exp\left\{-\frac{(n_0 + n_1 + \cdots n_9) p}{2}\right\}$$

Defining the numerator:

$$L(\mu_0, \Sigma) = \prod_{i=1}^{q} L(\mu, \Sigma_i) = \prod_{i=1}^{q} \boxed{f(X, \mu, \Sigma_i)}$$

$$\to = \frac{1}{(2\pi)^{p/2} \cdot |\Sigma_i|^{1/2}} \cdot \exp\left\{-\frac{1}{2}(\underline{X_i} - \underline{\mu})' \cdot \Sigma_i^{-1} (\underline{X_i} - \underline{\mu})\right\}$$

$$= \prod_{j=1}^{n_i}$$

$$= 2\pi^{-\frac{n_i \cdot p}{2}} \cdot |\Sigma_i|^{-n_i/2} \cdot \exp\left\{-\frac{1}{2}\underbrace{\sum_{j=1}^{n_j} (X_{ij} - \mu) \cdot \Sigma_i^{-1} (X_j - \mu)}_{\text{this can't reduce this time}}\right\}$$

Plug back in.

$$= \left(2\pi^{-\frac{n_0 p}{2}} |\Sigma_0|^{-n_0/2} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_0} (X_{0j} - \mu) \cdot \Sigma_0^{-1} (X_{0j} - \mu\right\}\right) \cdot$$

$$\cdot \left(2\pi^{-\frac{n_1 p}{2}} |\Sigma_1|^{-n_1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_1} (X_{1j} - \mu) \cdot \Sigma_1^{-1} (X_{1j} - \mu\right\}\right) \cdots$$

$$\cdots \left(2\pi^{-\frac{n_q p}{2}} \cdot |\Sigma_q|^{-n_q/2} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_q} (X_{qj} - \mu) \cdot \Sigma_q^{-1} (X_{qj} - \mu\right\}\right)$$

The final LRT Statsic is the ratio of the two maximized Likihood functions. Here, I am denoting the numerater MLE as $\wedge$ & denom as $\sim$

$$\Lambda = \frac{2\pi^{-\frac{(n_0+n_1+\cdots n_q)p}{2}} \left(|\hat{\Sigma}_0|^{-n_0/2} \cdot |\hat{\Sigma}_1|^{-n_1/2} \cdots |\hat{\Sigma}_q|^{-n_q/2}\right) \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_0} (X_{0j} - \hat{\mu}) \cdot \hat{\Sigma}_0^{-1} (X_j - \hat{\mu}\right\} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_1} (X_{1j} - \hat{\mu})' \hat{\Sigma}_1^{-1} (X_1 - \hat{\mu})\right\} \cdots \exp\left\{-\frac{1}{2}\sum_{j=1}^{n_q} (X_{qj} - \hat{\mu})' \cdot \hat{\Sigma}_q^{-1} (X_q - \hat{\mu})\right\}}{2\pi^{-\frac{(n_0+n_1+\cdots n_q)p}{2}} \cdot \left(|\tilde{\Sigma}_0|^{-n_0/2} \cdot |\tilde{\Sigma}_1|^{-n_1/2} \cdots |\tilde{\Sigma}_q|^{-n_q/2}\right) \cdot \exp\left\{-\frac{(n_0+n_1+\cdots n_q)p}{2}\right\}}$$