Kelsey Cameron
NID: ke110861
PID: k3593775

Problem 1 (10 pts)

Find the inverse of 233 modulo 360 in the interval $[0, 360)$ using the Extended Euclidean Algorithm.

gcd(233, 360)
360 = 233 * 1 + 127
233 = 127 * 1 + 106
127 = 106 * 1 + 21
106 = 21 * 5 + 1
21 = 1 * 21

gcd(233, 360) = 1

1 = 106 – 21 * 5
1 = 106 – (127 – 106 * 1) * 5 = 106 * 2 – 127
(233 – 127*1) *2 – 127 = 2 * 233 – 3 * 127
= 2 * 233 – 3 * (360 – 233*1)
= 5 * 233 – 3 * 360.

Inverse of 233 modulo 360 = 5.

Problem 2 (10 pts)

Prove by mathematical induction that, for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$

Base Case: n = 1
$\sum_{i=1}^{1} 2^{1-1} = 2^1 - 1$
1 = 2-1
Thus, the base case holds.

I.H. Assume: $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$
I.S. Prove: $\sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1$

$$\begin{aligned}\sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^{k+1-1} \\ &= 2^k - 1 + 2^k \\ &= 2^{2k} - 1\end{aligned}$$

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Base Case: k = 1
$\sum_1^1 2^{1-1} = 2^1-1$
= 1 = 1
Thus, the Base Case Holds.
Now I will assume that $\sum_1^k 2^{k-1} = 2^k - 1$ is true, since the Base Case holds.
Accordingly, I will plug in k+1.
Prove: $\sum_1^{k+1} 2^{k+1-1} = 2^{k+1} - 1$

$\sum_1^{k+1} 2^{k+1-1} = \sum_1^k 2^{k-1} + 2^{k+1} - 1$
$= 2^k - 1 + 2^{k+1} - 1$

Now we can plug in our base case.
And since we know that the first part is true (the base case) and the second part is also true, since k can be any arbitrary number, we must have that $\sum_1^n 2^{i-1} = 2^n-1$

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Problem 3 (10 pts)

Prove by induction that, for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} (i)(i!) = (n+1)! - 1$$

Base Case: n=1
$\sum_{i=1}^{1}$ (i)(i!) = (1+1)!-1
(1)(1!) = (2)! – 1
1 = 1
Thus, the base case holds.
I.H: Since the base case holds, I will assume that $\sum_{i=1}^{n}$ (i)(i!) = (n+1)!-1 is true.
I.S.: Prove: $\sum_{i=1}^{n+1}$(i)(i!) = ((n+1)+1)! – 1 = (n+2)! - 1

$\sum_{i=1}^{n+1}$(i)(i!) = $\sum_{i=1}^{n}$(i)(i!) + (n+1)(n+1)!
$\sum_{i=1}^{n+1}$(i)(i!) = (n+1)! – 1 + (n+1)(n+1)!

= (n+1)! + (n+1)!(n+1) – 1
= (n+1)!(1 + n + 1) – 1
= (n+1)!(n+2) – 1
= (n+2)! – 1.

Thus, according to our base case, and the fact that this holds for k+1, the premise that $\sum_{i=1}^{n}$ (i)(i!) = (n+1)!-1 must be true!

Problem 4 (20 pts)

This problem has two parts. You may use the result of the first part in the proof of the second.

1. Prove that the product of a rational number and an irrational number is irrational. When the rational number is not equal to 0.
2. Let a be an irrational number where a^2 is a rational number. Prove by mathematical induction or generalized mathematical induction that a^n is an irrational number for all odd integers $n \geq 1$.

1.
Proof by Contradiction:
Assume that the product of a rational number and an irrational number is rational. A rational number means a number that can be written as some fraction a/b.
This means that it can be written as follows.
a/b (x) = c/d.
x = cb / ad
And since cb / ad is a rational number, this contradicts the idea that the product of a rational number and irrational number is rational.

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2. I will use Generalized Induction - Let a = π.

Base Case: n = 1 – must be an irrational number for a^n.

π^1 = π, and π is irrational.

π^3 = π * π * π, and since our proof in step 1 proves that the product of a rational number and an irrational number is rational, it holds that the product of an irrational number and an irrational number is irrational, because of the contrapositive.

Thus, π^3 is irrational.

I.H. Assume a^k is an irrational number, k>=1, for all odd k.

I.S. Prove a^{k+1} is an irrational number, k>=1, for all odd k.

$a^{k+1} = a^k a$.

Well, we know based on our inductive step, that a^k is an irrational number, and based on our proof from number 1, we know that the product of two irrational numbers is irrational.

Problem 5 (10 pts)

For $n \geq 0$ let F_n denote the n^th Fibonacci number. Prove by induction that:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

(Note: For Fibonacci numbers, $F_1 = F_2 = 1$, $F_0 = 0$ and $F_n = F_{n-1} + F_{n-2}$.)

Base Case:

N = 0

$\sum_{i=0}^{n} F_0 = F_2 - 1$

0 = 1 – 1

Thus, the base case holds.

Since the base case holds, we will assume that this is true for all n, and then use that to prove this is true for all n+1.

I.H. Assume: $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

I.S. Prove: $\sum_{i=0}^{n+1} F_i = F_{(n+1)+2} - 1$.

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We know that $\sum_{i=0}^{n+1} F_i = \sum_{i=0}^{n} F_i + F_{n+1}.$

Prove: $\sum_{i=0}^{n} F_i + F_{n+1} = F_{(n+3)} - 1.$

Now I will plug in my assumption.

$F_{n+2} - 1 + F_{n+1} = F_{n+3} – 1.$

Thanks to the small note that Sean gave us, we also know that $F_n = F_{n-1} + F_{n-2}$.

Accordingly, $F_{n+3} = F_{(n+3-1)} + F_{(n+3-2)} = F_{n+2} + F_{n+1}$.

So I will plug in F_{n+3}.

$F_{n+2} - 1 + F_{n+1} = (F_{n+2} + F_{n+1}) – 1.$

And since this holds true by the communative property, the premise that $\sum_{i=0}^{n} F_i = F_{n+2} – 1$ must hold!

Problem 6 (10 pts)

Prove by strong induction that any postage of 20 cents or greater can be made from a combination of 3-cent, 4-cent, and 11-cent stamps.

Observation: If I want to show P(k+1) is true (I can make postage of k+1 cents with these denominations of stamps, then if I just knew P(k-2), I can add a three-cent stamp, and call it a day.

Base Case: k = 20.

Take one 11 cent stamp, and three 3-cent stamps.

Base Case: k = 21.

Take one 11 cent stamp, two 3-cent stamps, and one 4-cent stamp.

11 = 4+4+3

I.H. Assume that P(k) is true for 20 <= k <= m, where m >= 22.

I.S. Prove: P(k+1) is true.

Since P(k-2) is true, we can add a 3-cent stamp, to get to k+1 cents of postage.