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Check 3,7,10

Problem 1 (10 pts)

Let $a, b \in \mathbb{Z}$. Prove or disprove $a \mid b^2 \Rightarrow a \mid b$

Counter Example:
a = 8, b = 4.

8 | 16 – Yes, this is true.
8 | 4 – This can't possibly be true, because a number cannot evenly divide into a number smaller than it.

Problem 2 (10 pts)

Let $x, y, z \in \mathbb{Z}$, where $12 \mid (7x + 3y)$ and $12 \mid (2z)$. Prove that $12 \mid (-10x + 6y - 10z + 48)$.

Since 12 | (7x + 3y), we know that 12 = m(7x+3y), for some integer m. Since 12 | (2z), we know that 12 = n(2z), for some integer n.
So, we can effectively split the equation we want to prove into two parts, then fill in 12c everywhere 7x+3y or 2z occurs.

-10x + 6y = (7x+3y) +(7x+3y) + 24x
= (12c) + (12c) + 24
12(c+c+24)
Thus, 12 | -10x + 6y.

-10z + 48 = -5(2z) + 48
= -5(12c) + 48
= 12(-5c + 4)
Thus, 12 | -10z + 48.

Finally, based on Theorem 4.1.1, and definition of divisibility if a | b and a | c, then a | b+c. So 12 | (-10x + 6y – 10z + 48)

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Problem 3 (20 pts)

The n^{th} triangle number is given by the equation:

$$\sum_{i=1}^{n} i$$

Prove that the n^{th} triangle number is odd if $n \equiv 1 \ (mod\ 4)$ or $n \equiv 2 \ (mod\ 4)$.

Hint: Use the following identity to aid in your proof:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We know that since n is logically equivalent to 1, when modded by 4, we get the equation n = 4x+1. Likewise, n could also be equal to n = 4x + 2.

Now, let's plug in n for both scenarios.

Left:
(4x+1)(4x+2) / 2
This is equal to $16x^2$+ 12x + 2 / 2 = $8x^2$+6x+1
Now let's factor out a 2. 2($4x^2$+3x) + 1.
Since the entire equation can be created by multiplying by 2, but with an extra plus 1 at the end, the number must be odd.

Right:
(4x+2)(4x+3) /2 = ($16x^2$ + 20x + 6) / 2
= $8x^2$+ 10x + 3 = $8x^2$ + 10x + 2 + 1
= 2($4x^2$+5x+1) + 1.
Since the entire equation can be created by multiplying by 2, but with an extra plus 1 at the end, the number must be odd, and the statement holds.

Problem 4 (10 pts)

Prove or disprove that $n \in \mathbb{Z}^+$ is a perfect square if and only if all the exponents in n's prime factorization are even.

Note: An integer n is a perfect square if and only if $n = v^2$ for some $v \in \mathbb{Z}$.

By the definition of a perfect square, n = v^2.

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We want to prove if all exponents in n's prime factorization are even, then n is a perfect square.

If all of n's exponents are even, n can be written such that $n = x^{2k}$, of $n= (x^k)^2$.
Since , the number can be expressed as a number squared, I know that if x is a perfect square, it's exponents must be even, by definition of perfect square.

We also want to prove that n is a perfect square if it's exponents are all even.

Let us plug in both odd powers and even powers to see if this holds true.
$n = v^{2k}$
$n = (v^k)^2$
Thus, if the exponents are even, the number can be expressed as a number squared, and thus it must be a perfect square.

Now for the odd numbers:

$n = v^{2k+1}$
This can be expressed as $n = (v^k)^2(v)$.
This means that no matter what odd number we plug in, there will always be an extra variable hanging off of the end, so we cannot express n as a number squared, therefore all odd numbered powers cannot be perfect squares.

Problem 5 (10 pts)

Let $a, b \in \mathbb{Z}^+$. Prove or disprove that if $\gcd(a, b) = 1$ then a cannot divide b.

Let a = 1, and b = 1. Since gcd(1,1) = 1, and 1 divides into 1, this must be false.
B can actually be any number in this case, because 1 divides into any number. Thus, the above statement is false.

Problem 6 (10 pts)

Let $a, b \in \{x \in \mathbb{Z} \mid x > 1\}$. Prove or disprove that if $\gcd(a, b) = 1$ then a cannot divide b.

If the gcd of two numbers = 1, by definition, this means that the largest integer d such that d | a and d | b, is called the greatest common divisor of a and b is equal to 1. This means that a and b are relatively prime, and since neither value can be one, they cannot divide into each other.

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Mathematically, according to Euclid's algorithm, you can do the following:

Gcd(a,b) = b%a.
By definition of modulus division, if you divide out one number by another, then grab the remainder, the number you divided out cannot possibly be the remainder.

Problem 7 (15 pts)

Prove or disprove for an arbitrary prime number p there exists some composite number q where $\gcd(p, p+q) > 1$.

Since 1 is prime, let p = 1, let q = any composite number.

Well, the gcd(1, any composite number) = 1, since that is the only prime factor that both of them share. Thus, this statement must be false.

Problem 8 (10 pts)

Find the smallest integer that is divisible by all numbers in the set $\{x \in \mathbb{Z} \mid 1 \leq x \leq 10\}$. Show how you derived your result.

The set: $\{1,2,3,4,5,6,7,8,9,10\}$

Let's prime factorize each composite number, and we shall leave the others since they are prime.
Prime factorized set: $\{1,2,3,2^2,5,3^1 2^1, 7, 2^3, 3^2, 2^1 5^1)$
Thus, we need a minimum of $2^3 (3^2)\, 5(7) =$ **2520**
I was able to do this because I know that the number must contain at least the factors in each of the sets, for 1, it has at least one 1, for 4, it has at least 2^2, etc etc.

Problem 9 (10 pts)

Say we have the number $n = 2{,}246{,}142{,}360$ and know that the number $n = \mathrm{lcm}(a, b)$.
We also know $a = 68{,}064{,}920$. How many possible values of b exist if b is a positive integer?

(Hint: All of n's prime factors are ≤ 41)

9.
n = 2,246,142,360 = (2^3)$(3^1)(5^1)(7^3)(11^3)(41)$
a = 68,064,920 = (2^3)$(5)(7^3)(11^2)(41)$

To get LCM(a, b), pick the greater power of each. B is a max of $(3^1)(11^3)$
There are two powers for three (0 and 1) and 4 powers for 11 (0,1,2,3).
Thus, there are 2 * 4 possibilities = 8 possibilities.

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Problem 10 (15 pts)

Prove for all integers $a, b \in \mathbb{Z}^+$ and $a \geq b$, that $\gcd(a, b) = \gcd(a - b, b)$.

Bezout's theorem states that ax+by=gcd(a,b).

Let's plug this in.
I'm going to change the y to a z to make things simpler for the top equation.
gcd(a, b) = ax + bz
gcd(a-b, b) = (a-b)x + by.
This equals ax – bx + by
= ax + b (y-x)

Now, since we have our two equations:

gcd(a,b) = ax + bz, gcd(a-b, b) = ax + b(y-x).
Let's divide the gcd's on both sides.

1 = (ax+by) / gcd(a,b) 1 = ((a-b)x) + by) / gcd(a-b,b).

Now let's substitute z = y-x. Since we know these are equal to 1, we can set them equal.

(ax+b(y-x) / gcd(a,b) = (a-b)x + by / gcd(a-b, b).

(ax-bx + by) / gcd(a,b) = ax-bx + by / gcd(a-b, b).

Thus, since both of the numerators are equal, the denominators must also be equal, and gcd(a,b) must equal gcd(a-b,b).
Q.E.D