CmSc 175 Discrete Mathematics

Lesson 02: Tautologies and Contradictions. Logical Equivalences. De Morgan's Laws

1. Tautologies and Contradictions

A propositional expression is a **tautology** if and only if for all possible assignments of truth values to its variables its truth value is **T**

Example: $P \vee \neg P$ is a tautology

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

A propositional expression is a **contradiction** if and only if for all possible assignments of truth values to its variables its truth value is **F**

Example: $P \wedge \neg P$ is a contradiction

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Usage of tautologies and contradictions - in proving the validity of arguments; for rewriting expressions using only the basic connectives.

Definition: Two propositional expressions P and Q are logically equivalent, if and only if $P \leftrightarrow Q$ is a tautology. We write $P \equiv Q$ or $P \Leftrightarrow Q$.

Note that the symbols $\equiv$ and $\Leftrightarrow$ are **not logical connectives**

Exercise:

a) Show that $P \to Q \leftrightarrow \neg P \vee Q$ is a tautology, i.e. $P \to Q \equiv \neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$	$P \to Q$	$P \to Q \leftrightarrow \neg P \vee Q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

b) Show that $(P \leftrightarrow Q) \leftrightarrow ((P \land Q) \lor (\neg P \land \neg Q))$ is a tautology

i.e. $\mathbf{P \leftrightarrow Q \equiv (P \land Q) \lor (\neg P \land \neg Q)}$

c) Show that $(P \oplus Q) \leftrightarrow ((P \land \neg Q) \lor (\neg P \land Q))$ is a tautology

i.e. $\mathbf{P \oplus Q \equiv (P \land \neg Q) \lor (\neg P \land Q)}$

2. Logical equivalences

Similarly to standard algebra, there are **laws** to manipulate logical expressions, given as logical equivalences.

1. Commutative laws	$P \vee Q \equiv Q \vee P$	
	$P \wedge Q \equiv Q \wedge P$	
2. Associative laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	
	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
3. Distributive laws:	$(P \vee Q) \wedge (P \vee R) \equiv P \vee (Q \wedge R)$	
	$(P \wedge Q) \vee (P \wedge R) \equiv P \wedge (Q \vee R)$	
4. Identity	$P \vee F \equiv P$	
	$P \wedge T \equiv P$	
5. Complement properties	$P \vee \neg P \equiv T$	(excluded middle)
	$P \wedge \neg P \equiv F$	(contradiction)
6. Double negation	$\neg(\neg P) \equiv P$	
7. Idempotency (consumption)	$P \vee P \equiv P$	
	$P \wedge P \equiv P$	
8. De Morgan's Laws	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	
	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	
9. Universal bound laws (Domination)	$P \vee T \equiv T$	
	$P \wedge F \equiv F$	
10. Absorption Laws	$P \vee (P \wedge Q) \equiv P$	
	$P \wedge (P \vee Q) \equiv P$	
11. Negation of T and F:	$\neg T \equiv F$	
	$\neg F \equiv T$	

For practical purposes, instead of $\equiv$, or $\Leftrightarrow$, we can use $=$.
Also, sometimes instead of $\neg$, we will use the symbol ~.

3. Negation of compound expressions

In essence, we use De Morgan's laws to negate expressions.

1. If the expression A is an atomic expression, then the negation is ¬A.
2. If the expression is ¬A, then its negation is ¬(¬A) = A (by law 6: double negation)
3. If the expression A contains the connectives →, ↔, and ⊕, rewrite the expression so that it contains only the basic connectives AND, OR and NOT.
4. Represent A as a disjunction P V Q or a conjunction P Λ Q.

   Example: Let A = B ⊕ C. Then, A can be represented as (B Λ ¬C) V (¬B Λ C)

   This is a disjunction of the form P V Q, where P = (B Λ ¬C) and Q = (¬B Λ C)
5. Apply De Morgan's laws: ¬(P V Q) = ¬P Λ ¬Q; ¬(P Λ Q) = ¬P V ¬Q.
6. If both P and Q are atomic expressions, stop.
7. Otherwise repeat the above steps to obtain the negations of P and/or Q

Example:

~(B ⊕ C) = ~ ((B Λ ~C) V (~B Λ C)) =

apply De Morgan's Laws

= ~ (B Λ ~C) Λ ~(~B Λ C) =

apply De Morgan's laws to each side

= (~B V ~(~C)) Λ (~(~B) V ~C) =

apply double negation

= (~B V C) Λ (B V ~C) =

apply distributive law

= (~B Λ B) V (~B Λ~C) V (C Λ B) V (C Λ ~C) =

apply complement properties

= F V (~B Λ~C) V (C Λ B) V F =

apply identity laws

= (~B Λ~C) V (C Λ B) =

apply commutative laws

= (C Λ B) V (~B Λ~C) =

apply commutative laws

= (B Λ C) V (~B Λ~C) = B ↔ C

Exercises

Use the equivalence $A \rightarrow B = \sim A \vee B$, and the equivalence laws.

1. Show that $(A \rightarrow B) \wedge A$ is equivalent to $A \wedge B$

2. Show that $(A \rightarrow B) \wedge B$ is equivalent to B

3. Show that $(A \rightarrow B) \wedge (B \rightarrow A)$ is equivalent to $A \leftrightarrow B$

4. Show that $\sim((A \rightarrow B) \wedge (B \rightarrow A))$ is equivalent to $A \oplus B$

5. Show that $\neg ((P \rightarrow Q) \rightarrow P) \wedge P$ is a contradiction

6. Replace the conditions in the following if statements with equivalent conditions without using the logical operators || and &&

a) `if ((a > 0 && b > 0) || (b > 0))  c = a*b;`

b) `if (( a > 0 || b > 0 ) && (b > 0)) c = a*b;`