PHYS 4720/6720 - Galaxies and the ISM
Prof. Cara Battersby, Spring 2023

Problem Set #3

Due: Monday February 27, 2023 at 12:30pm.

Reminder: Problem Sets will be spot-graded, including points for overall completeness. Same instructions as previous PSs still apply.
Coding Problems: 4 for 20 points total
4720: Written Problems: 3 problems for 15 points total, 6720 and honors: 3 problems for 24 points total

1. **(4 pts)** The radial velocity of a star orbiting the Galactic Center is given by:

$$v_r = \Theta\ cos(\alpha) - \Theta_0\ sin(\ell) \tag{1}$$

where ℓ is the Galactic longitude of the star, Θ_0 is the orbital velocity of the local standard of rest of the Sun system (LSR; the Sun if it had perfectly circular motion about the Galactic Center), Θ is the orbital velocity curve as a function of distance from the Galactic Center (R), and α is the angle between the line connecting the tangent and the Galactic Center and the line connecting the star and the Galactic Center (see diagram from notes below).

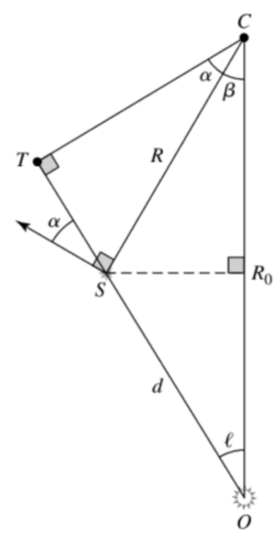

Figure 1: Top-down view of the Galaxy from the Northern Galactic Plane, described in class and used for Problem 2.

Knowing that the maximum velocity ($\Theta_{\max}$) occurs at the tangent point, show that the orbital velocity curve as a function of distance from the Galactic Center (R) can be expressed as:

$$\Theta(R) = \Theta_{\max} + \Theta_0\ sin(\ell) \tag{2}$$

You may replace Θ with V for velocity if you prefer at any point in this problem (up to you!), for a final expression:

$$V(R) = V_{\max} + V_0\ sin(\ell) \tag{3}$$

2. **(5 pts) It's a dark matter.**

(a) One simple back-of-the-envelope estimate for the mass density of dark matter suggests that it follows the following form,

$$\rho(r) = \frac{V^2}{4\pi G r^2} \tag{4}$$

where ρ is the mass density, V is the orbital velocity, G is the gravitational constant, and r is distance from the center of the Galaxy. Using this approximation, estimate the mass density of dark matter in the solar neighborhood in units of g cm^{-3}, $M_\odot pc^{-3}$, and $M_\odot AU^{-3}$. Also look up the stellar mass density in the solar neighborhood, and compare these values.

(b) One modification to this simplistic expression for the dark matter mass density has been made to force the density function to approach a constant value near the center, rather than diverge. This formula appears to be consistent with reality and takes the form:

$$\rho(r) = \frac{\rho_0}{1 + (r/a)^2} \tag{5}$$

Show that the enclosed mass of dark matter at a radius r is given by:

$$M_r = 4\pi\rho_0 a^2 \left[r - a\ \tan^{-1}\left(\frac{\mathrm{r}}{\mathrm{a}}\right)\right]. \tag{6}$$

(Feel free to use an integral table, numerical integrator, or ask me for help on this part!) Assume that there is 5.4 $\times 10^{11}$ $M_\odot$ of dark matter in the inner 50 kpc of the Galaxy and determine ρ_0 in unites of $M_\odot$ pc^{-3}. You can assume that a = 2.8 kpc

3. **(15 pts) A Disk of Stars.** Using the expression discussed in class (and found in the Schneider textbook Ch. 2), you will calculate the expression $n(R, z)$ for the number density of stars at a particular radial distance R from the Galactic center, and height z from the Galactic midplane. Note that n(0,0) is the number density of stars, in units of stars pc^{-3}, in the disk at the Galactic center (and is thus quasi-unrealistic, as directly at the center there is a supermassive black hole, but we'll ignore that for this problem).

(a) **(2 pts)** Calculate the normalization n_0, such that the number density in the Solar neighborhood is 0.2 stars pc^{-3}. Note that the Sun is at z=30 pc.

(b) **(2 pts)** At the Galactocentric radius of the Sun ($R_0 \sim 8.0$ kpc), what is the stellar density (in stars per pc^3) at z = $[0, 10, 100, 1000]$ pc? (You will need to use your normalized expression.)

(c) **(2 pts)** Let's define the "edge" of the Galactic disk radially as the point where the stellar density reaches 0.1% (10^{-3}) of the central value. Find the radial "edge" at the Galactic midplane ($z = 0$); i.e. find R where n(R,0) = 10^{-3} n(0,0).

Extra Questions for Honors and 6720 Students

(d) **(2 pts)** Do the same as question 3c but for the "edge" of the disk in height; i.e. find z where n(0,z) = 10^{-3} n(0,0).

(e) **(5 pts)** Show that the total number of stars in the Galactic disk is $\sim 10^{11}$. This will involve integrating the expression derived above in cylindrical coordinates. (You can do this analytically by parts. Feel free to use an integral table for specific parts, but compute the analytic expression.) First, give the analytic expression, then you can plug in your limits of integration; for the radial limit, use your result from part c; for the z-direction, use the result from part d. As a reminder:

$$\iiint dV = \int_0^r \int_{-z'}^{z'} \int_0^{2\pi} r dr d\theta dz \tag{7}$$

(f) **(2 pts)** Assuming that the typical mass of a star in the Milky Way is 0.5 $M_\odot$, calculate the total stellar mass of the disk.