PHYS 4720/6720 - Galaxies and the ISM
Prof. Cara Battersby, Spring 2023

Problem Set #4

Due: Wednesday March 29th, 2023 at 12:30pm.

Reminder: Problem Sets will be spot-graded, including points for overall completeness. Same instructions as previous PSs still apply.
4720: 22 pts total Question 1 written (+ coding for plots) 22 pts. Up to 5 pts extra credit for google colab plotting for Paper 3. **6720 and Honors: 34 pts total** Question 1 written (+ coding for plots) 22 pts, Question 3 written 12 pts. Up to 5 pts extra credit for google colab plotting for Paper 3.

1. **(22 pts) Dynamic is our Galaxy.** In this problem, we'll explore how the components of the Milky Way rotate, and see how much dark matter is needed to match the observations.

   (a) **(4 pts)** Consider the Galactic disk to have an a stellar mass **surface density**:

   $$\Sigma(r) = \Sigma_0 e^{-r/h} \tag{1}$$

   where $\Sigma(r)$ is the mass surface density (in units of $M_\odot$ kpc^{-2}), Σ_0 is the central mass surface density, and h is the radial disk scale length (assume h=3.5 kpc for this problem).

   i) 3 pts Write down an expression for the total stellar mass in the disk as a function of radius: $M(r)$. This will involve an integral in 2-dimensional polar coordinates (since we're looking at a mass surface density, we've effectively collapsed the galaxy in the z-direction, and so can disregard that here). You can solve numerically or use an integral table if you want, but please give the final analytic expression. As a reminder:

   $$\int\int dA = \int_0^{2\pi}\int_0^{r} r dr d\theta \tag{2}$$

   ii) 1 pt Solve for the normalization Σ_0 which gives the total disk mass of the Milky Way (assume this to be 6×10^{10} $M_\odot$ and R_{disk} = 25 kpc). Answer in units of $M_\odot$/kpc^2.

   (b) **(6 pts)** Now we will explore what the rotation curve for such a disk looks like.

   i) 1 pt Assuming that all objects in the disk have purely circular orbits, derive an expression for the rotational velocity as a function of the distance from the center of the disk (e.g., the radius). Give the expression in terms of $M(r)$. Take this expression and plug in your expression from part a) for $M(r)$.

   ii) 1 pt What are the units of this expression? Derive the conversion factor you need to include to ensure that the units of velocity are in km/s, but leaving quantities in earlier defined units (e.g., $M_\odot$/kpc^2, cgs units for G, r in kpc, etc). Show your work, and list your final equation for $V(r)$ [km/s].

   iii) 3 pts Plot the rotation velocity of your disk as a function of radius out to 50 kpc. You can do this in the google colab notebook I shared with you or on your own. If you do it colab, just share the completed notebook with me, if on your own, just create a PDF of your figures and code to share with me.

   iv) 1 pt What is the rotation velocity from your model of the disk at the Galactocentric radius (8 kpc), and how does it compare to the Sun's known rotational velocity of 220 km/s? How can we explain the difference?

   (c) **(5 pts)** In part b, you derived that there must be some kind of "dark matter" which is permeating the Milky Way disk, and increasing the rotational velocities of stars over what they should be when only considering the mass of the disk (awesome job!). In this part of the problem, we will consider how much dark matter we need to include.

   i) 1 pt Write down an expression for the dark matter density $\rho(r)$ in terms of the radius r and critical radius r_c we used in PS3 (Question 2b). Write down the enclosed mass of dark matter at given radius r that you derived in PS3 (Question 2b).

ii) 1 pt Using the expression for $V(r)$ you derived in part b, derive an expression for the rotational velocity $V(r)$ for this dark matter halo mass distribution (for the halo **only**, ignore the disk for now). Be sure to check your units. This should still be in terms of ρ_0 and r_c.
iii) 3 pts Now, I will give you that r_c = 1.28 kpc and ρ_0 = 7.1 $\times 10^8$ $M_\odot$ kpc^{-2}. Plot the rotation velocity of your halo as a function of radius out to 50 kpc.

(d) **(5 pts)** Now we can put both the disk and the halo together to explore the full rotation curve of our galaxy.
i) 1 pt Use the definition for circular velocity (e.g., what we showed in question b part i) for each component to show that you can combine them as: $v_{tot}^2 = v_{disk}^2 + v_{halo}^2$.
ii) 2 pts Combine your disk and halo rotation curves in this way to calculate the total rotational velocity, and plot the total versus radius, out to r=50 kpc. Please also overplot the curve for the individual components, in a different color or line style, and be sure to provide a legend to explain the curves.
iii) 1 pt What is the ratio of the velocity contributed by the dark matter halo to that by the stellar disk at the Galactocentric radius (R_0 =8 kpc)?
iv) 1 pt Is the galaxy rotation curve ever dominated by the disk component? If so, where? To illustrate your answer, make another plot of the three rotation curves (disk, halo and total), with the x-axis plotted in log.

(e) **(1 pt)** Using your expression from part c, calculate the total mass of the dark matter halo out to 50 kpc. What is the ratio of the stellar mass of the disk to the dark matter halo mass?

(f) **(1 pt)** We have ignored the bulge and the stellar halo in this problem, predominantly because the integrals are less trivial to solve. Where do you think the lack of these components matters the most?

2. **(Extra credit, up to 5 pts)** Google colab plotting for Paper 3

Extra Question for Honors and 6720 Students

3. **(12 pts)** In class, we derived the Jean's Mass for a forming protostar, by finding the critical mass where the gravitational potential of a spherical cloud is greater than its thermal energy. Please re-derive the Jean's mass for the two cases described below. For each case, calculate a Jean's mass in $M_\odot$ and compare with that using the standard assumptions (spherical cloud of uniform density balanced by isothermal gas pressure). Calculate the Jean's mass in $M_\odot$ for a cloud with a temperature, T = 10 K and gas number density, $n = 2 \times 10^5$ cm^{-3} (you will need to convert to a mass density). [1] Please give masses in $M_\odot$.

(a) **(8 pts)** a sphere with a density distribution given by $n(r) = n_f(\frac{r}{r_f})^{-p}$ where p=1.8, r_f = 1000 AU and $n_f = 10^7$ cm^{-3}, dominated by thermal pressure (as in the standard case).

(b) **(4 pts)** a sphere with uniform density (as in the standard case), however, the thermal pressure is negligible as the internal energy of the gas is dominated by turbulence. You can assume that the turbulent energy can be incorporated by replacing the sound speed of the gas with the velocity linewidth (assume $\sigma \sim 1$ kms^{-1}).[2].

[1] You may begin with the expression for internal energy from class: $E = \frac{3}{2}Nk_BT$. For part b you may use the expression for total gravitational potential energy for a uniform density sphere $U = \frac{-3}{5}\frac{GM^2}{R}$ but for part a you must derive U. Your final expression for the Jean's mass must be independent of radius.

[2] For an isothermal gas where the ideal gas equation of state, $P = \frac{\rho k_B T}{\mu m_H}$ holds, $c_s = \sqrt{k_BT/\mu m_H}$. Here μ is the mean molecular weight of the gas and m_H is the mass of Hydrogen. For a molecular gas purely composed of hydrogen, $\mu = 2$, however, for most ISM molecular gas, $\mu = 2.3$ is often used.