Last question on Math Exam

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1 Vieta's formulae for quadratics

1.1 Motivation for Vieta's

Skip if you already now

Sometimes finding the roots of polynomials can be annoying, with square roots, fractions and random junk everywhere. It can be hard to directly solve for the roots of a polynomial. However, sometimes you don't always need to solve for the roots, and it may suffice for you to just know the sum of the roots of a polynomial, or the product of the roots of the polynomial. Vieta's formulae presents you with a way to solve for the sum of all roots or the product of all roots extremely quickly just from the standard form $(ax^2 + bx + c)$ of a polynomial.

1.2 Derivation

Consider a quadratic equation, and let its two roots be $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. Given these 2 roots, the quadratic can therefore be expressed as follows:

$$f(x) = ka(x - x_1)(x - x_2)$$

to guarantee that $f(x_1) = f(x_2) = 0$. $k \in \mathbb{R}$ here is an arbitrary constant since we are allowed to vertically dilate or contract the function $f(x)$ and still maintain the two roots x_1 and x_2. Converting to standard form (x is bolded for clarity):

$$\begin{aligned} f(x) &= k(\mathbf{x} - x_1)(\mathbf{x} - x_2) \\ &= k\left(\mathbf{x}^2 - x_1\mathbf{x} - x_2\mathbf{x} + x_1x_2\right) \\ &= k\mathbf{x}^2 - k\left(x_1 + x_2\right)\mathbf{x} + kx_1x_2 \end{aligned}$$

Comparing coefficients between the above equation and the standard form $f(x) = a\mathbf{x}^2 + b\mathbf{x} + c$:

$$a = k \tag{1}$$

$$b = -k\left(x_1 + x_2\right) \tag{2}$$

$$c = kx_1x_1 \tag{3}$$

Divide eqn 2 by eqn 1 yields

$$\frac{b}{a} = -\left(x_1 + x_2\right)$$
$$x_1 + x_2 = -\frac{b}{a} \tag{4}$$

Divide eqn 3 by eqn 1 yields

$$\frac{c}{a} = x_1 x_2$$
$$x_1 x_2 = \frac{c}{a} \tag{5}$$

Note that equation 4 allows you to quickly compute the sum of the 2 roots $x_1 + x_2$ just using the coefficients a, b, c without even having to solve any quadratic equation, and equation 5 allows you to compute the product of the 2 roots $x_1 x_2$ in a similar way.

These 2 equations are known the special case for Vieta's formulas for the second order polynomial. Similar results exists for higher order polynomials, and their derivations are anaglous to the one shown above. Proof for high order cases are left as an exercise to the reader.

2 Question

I don't exactly remember the subparts to the question, but I remember the train of thought the question was trying to go off of. The final goal after everything is to find the value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ using some dodgy and slow method that the teachers decided on.

The key quadratic in question is $f(x) = x^2 + 2\cot 2\theta x - 1$. The teachers commenced by asking you to show that the roots of this are $\tan x$ and $-\cot x$. Brute force solve:

$$x^2 + 2\cot 2\theta x - 1 = 0$$

Compound angle identity for $\cot\theta$

$$x^2 + \left(2 \times \frac{1 - \tan^2\theta}{2\tan\theta}\right) x - 1 = 0$$
$$x^2 + \frac{1 - \tan^2\theta}{\tan\theta} x - 1 = 0$$

Split the fraction

$$x^2 + \frac{1}{\tan\theta} x - \frac{\tan^2\theta}{\tan\theta} x - 1 = 0$$
$$x^2 + \left(\cot\theta - \tan\theta\right) x - 1 = 0$$

$1 = \tan\theta\cot\theta$:

$$x^2 + (\cot\theta - \tan\theta)\,x - \tan x \cot x = 0$$

factorise

$$\begin{aligned}(x - \tan\theta)\,(x + \cot x) &= 0 \\ x &= \tan\theta \ \textbf{OR} \ -\cot\theta\end{aligned}$$

If we let x_1 and x_2 be the two roots of $f(x)$, then

$$\begin{aligned}x_1 &= \tan\theta \\ x_2 &= -\cot\theta\end{aligned}$$

you can swap the 2 around btw it doesnt matter which root you label as x_1 and which one you label as x_2

Now here comes the dumb part. Vieta's is usually used to speed things up, but instead, they wanted you to apply equation 4 (see vietas formuale derivation in the section above) on $f(x)$ to realise that:

$$x_1 + x_2 = -\frac{b}{a} = -2\cot 2\theta$$

By brute forcing the roots to $f(x)$, however, we also know that $x_1 + x_2 = \tan\theta - \cot\theta$. Hence:

$$\begin{aligned}\tan\theta - \cot\theta &= x_1 + x_2 \\ &= -2\cot 2\theta\end{aligned}$$

Sub $\theta = \frac{\pi}{24}$:

$$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$$

the value of $\cot\frac{\pi}{12}$ was already found in some previous question (i can send explanations of that if you want), but I think the solution should be sufficienctly obvious from here

My biggest comment on this question is that it tests conceptual understanding of Vieta's really well, but the application of Vieta's is a bit artifical. This is cuz Vieta's is only remotely useful if you dont know the roots of a polynomial themselves, and the entire trig thing can be proven without even touching quadratics (in fact the bulk of the difficulty of the 'proof' was just related to the brute force solving of the quadratic and vieta's, and not the actual trig)