通过一个例子简析傅里叶级数与傅里叶变换的关系

注: 笔者由于对信号与系统了解甚少, 故本文仅从数学角度进行分析, 如果有误, 恳请批评指正。

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega t}$$

其中,

$$C_n = rac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} \mathrm{d}t$$

Q Question

例:半波整流,在电子学中表示一个交变电压 $E\sin\omega t$ 经整流后所得到的周期波形。在一个周期内的表达式为

$$f\left(t
ight) = egin{cases} 0 & -rac{\pi}{\omega} \leq t < 0 \ E\sin\omega t & 0 \leq t \leq rac{\pi}{\omega} \end{cases}.$$

求 f复数形式的Fourier展开式.

= Answer

$$\begin{split} C_n &= \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) e^{-jn\omega t} \mathrm{d}t \\ &= \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t e^{-jn\omega t} \mathrm{d}t \\ &= \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t (\cos n\omega t - j \sin n\omega t) \mathrm{d}t \\ &= \frac{E\omega}{2\pi} \left[\int_0^{\pi/\omega} \sin \omega t \cos n\omega t \mathrm{d}t - j \int_0^{\pi/\omega} \sin \omega t \sin n\omega t \mathrm{d}t \right] \\ &= \frac{E\omega}{2\pi} \left[\int_0^{\pi/\omega} \frac{\sin(1+n)\omega t + \sin(1-n)\omega t}{2} \mathrm{d}t - j \int_0^{\pi/\omega} \frac{\cos(1-n)\omega t - \cos(1+n)\omega t}{2} \mathrm{d}t \right] \end{split}$$

接下来需要分类讨论,

$$egin{split} C_1 &= rac{E\omega}{2\pi} \Bigg[\int_0^{\pi/\omega} rac{\sin 2\omega t}{2} \mathrm{d}t - j \int_0^{\pi/\omega} rac{1 - \cos 2\omega t}{2} \mathrm{d}t \Bigg] \ &= rac{E\omega}{2\pi} \Bigg[-rac{\cos 2\omega t}{4\omega} igg|_0^{\pi/\omega} - j (rac{t}{2} - rac{\sin 2\omega t}{4\omega}) igg|_0^{\pi/\omega} \Bigg] \ &= rac{E\omega}{2\pi} \cdot (-jrac{\pi}{2\omega}) \ &= -jrac{E}{4} \end{split}$$

$$egin{aligned} C_{-1} &= rac{E\omega}{2\pi} \left[\int_0^{\pi/\omega} rac{\sin 2\omega t}{2} \mathrm{d}t - j \int_0^{\pi/\omega} rac{\cos 2\omega t - 1}{2} \mathrm{d}t
ight] \ &= rac{E\omega}{2\pi} \left[-rac{\cos 2\omega t}{4\omega} igg|_0^{\pi/\omega} - j (rac{\sin 2\omega t}{4\omega} - rac{t}{2}) igg|_0^{\pi/\omega}
ight] \ &= rac{E\omega}{2\pi} \cdot (jrac{\pi}{2\omega}) \ &= jrac{E}{4} \end{aligned}$$

当 $n \neq \pm 1$ 时

$$egin{split} C_n &= rac{E\omega}{2\pi} \left(\left[-rac{\cos(1+n)\omega t}{2\omega(1+n)} - rac{\cos(1-n)\omega t}{2\omega(1-n)}
ight]_0^{\pi/\omega} - j \left[rac{\sin(1-n)\omega t}{2\omega(1-n)} - rac{\sin(1+n)\omega t}{2\omega(1+n)}
ight]_0^{\pi/\omega}
ight) \ &= rac{E\omega}{2\pi} \cdot \left[rac{1-\cos(1+n)\pi}{2\omega(1+n)} + rac{1-\cos(1-n)\pi}{2\omega(1-n)}
ight] \ &= rac{E}{4\pi} \cdot \left[rac{1-(-1)^{n-1}}{1+n} + rac{1-(-1)^{n-1}}{1-n}
ight] \ &= rac{E}{2\pi} \cdot rac{1-(-1)^{n-1}}{1-n^2} \end{split}$$

此时关于n的奇偶需要分类讨论,

$$egin{cases} n=2k-1 & C_n=C_{2k-1}=0 (k\in (-\infty,+\infty) oxed{l} k
eq 0$$
 與 1 $n=2k$ $C_n=C_{2k}=rac{E}{\pi}\cdotrac{1}{1-4k^2}(k\in (-\infty,+\infty))$

综上所述,f(t)的傅里叶级数展开式为

$$egin{aligned} f(t) &= -jrac{E}{4}e^{j\omega t} + jrac{E}{4}e^{-j\omega t} + \sum_{k=-\infty}^{+\infty}rac{E}{\pi}\cdotrac{1}{1-4k^2}e^{2k\omega tj} \ &= -jrac{E}{4}(\cos\omega t + j\sin\omega t) + jrac{E}{4}(\cos\omega t - j\sin\omega t) + \sum_{k=-\infty}^{+\infty}rac{E}{\pi}\cdotrac{1}{1-4k^2}e^{2k\omega tj} \ &= rac{E}{2}\sin\omega t + \sum_{k=-\infty}^{+\infty}rac{E}{\pi}\cdotrac{1}{1-4k^2}e^{2k\omega tj} \end{aligned}$$