Steel members come in a wide variety of shapes; the properties of the cross section are needed for analysis and design.
(Bob Scott/Getty Images)

10

Review of Centroids and Moments of Inertia

CHAPTER OVERVIEW

Topics covered in Chapter 10 include centroids and how to locate them (Sections 10.2 and 10.3), moments of inertia (Section 10.4), parallel-axis theorems (Section 10.5), polar moments of inertia (Section 10.6), products of inertia (Section 10.7), rotation of axes (Section 10.8), and principal axes (Section 10.9). Only plane areas are considered. There are numerous examples within the chapter and problems at the end of the chapter available for review.

A table of centroids and moments of inertia for a variety of common geometric shapes is given in Appendix D, available online, for convenient reference.

Chapter 10 is organized as follows:

10.1 INTRODUCTION

This chapter is a review of the definitions and formulas pertaining to centroids and moments of inertia of plane areas. The word "review" is appropriate because these topics are usually covered in earlier courses, such as mathematics and engineering statics, and therefore most readers will already have been exposed to the material. However, since centroids and moments of inertia are used repeatedly throughout the preceding chapters, they must be clearly understood by the reader and the essential definitions and formulas must be readily accessible.

The terminology used in this and earlier chapters may appear puzzling to some readers. For instance, the term "moment of inertia" is clearly a misnomer when referring to properties of an area, since no mass is involved. Even the word "area" is used inappropriately. When we say "plane area," we really mean "plane surface." Strictly speaking, area is a measure of the *size* of a surface and is not the same thing as the surface itself. In spite of its deficiencies, the terminology used in this book is so entrenched in the engineering literature that it rarely causes confusion.

10.2 CENTROIDS OF PLANE AREAS

The position of the centroid of a plane area is an important geometric property. To obtain formulas for locating centroids, we will refer to Fig. 10-1, which shows a plane area of irregular shape with its centroid at point C. The xy coordinate system is oriented arbitrarily with its origin at any point O. The **area** of the geometric figure is defined by the following integral:

$$A = \int dA \tag{10-1}$$

in which dA is a differential element of area having coordinates x and y (Fig. 10-1) and A is the total area of the figure.

The **first moments** of the area with respect to the x and y axes are defined, respectively, as follows:

$$Q_x = \int y\, dA \qquad Q_y = \int x\, dA \tag{10-2a,b}$$

Thus, the first moments represent the sums of the products of the differential areas and their coordinates. First moments may be positive or negative, depending upon the position of the xy axes. Also, first moments have units of length raised to the third power; for instance, mm^3.

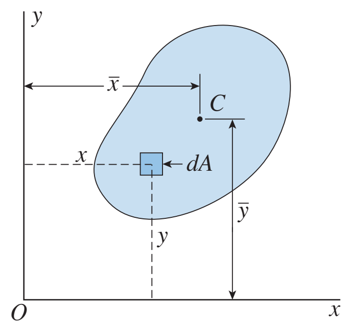


FIG. 10-1 Plane area of arbitrary shape with centroid C

The coordinates $\bar{x}$ and $\bar{y}$ of the **centroid** C (Fig. 10-1) are equal to the first moments divided by the area:

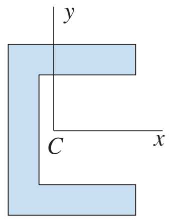


FIG. 10-2 Area with one axis of symmetry

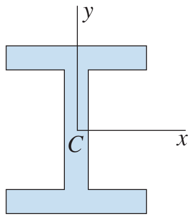


FIG. 10-3 Area with two axes of symmetry

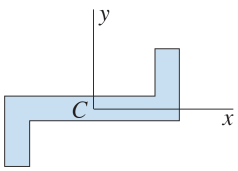


FIG. 10-4 Area that is symmetric about a point

The centroid of wide-flange steel sections lies at the intersection of the axes of symmetry (Photo courtesy of Louis Geschwinder)

$$\bar{x} = \frac{Q_y}{A} = \frac{\int x\,dA}{\int dA} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\int y\,dA}{\int dA} \tag{10-3a,b}$$

If the boundaries of the area are defined by simple mathematical expressions, we can evaluate the integrals appearing in Eqs. (10-3a) and (10-3b) in closed form and thereby obtain formulas for $\bar{x}$ and $\bar{y}$. The formulas listed in Appendix D were obtained in this manner. In general, the coordinates $\bar{x}$ and $\bar{y}$ may be positive or negative, depending upon the position of the centroid with respect to the reference axes.

If an area is **symmetric about an axis**, the centroid must lie on that axis because the first moment about an axis of symmetry equals zero. For example, the centroid of the singly symmetric area shown in Fig. 10-2 must lie on the x axis, which is the axis of symmetry. Therefore, only one coordinate must be calculated in order to locate the centroid C.

If an area has **two axes of symmetry**, as illustrated in Fig. 10-3, the position of the centroid can be determined by inspection because it lies at the intersection of the axes of symmetry.

An area of the type shown in Fig. 10-4 is **symmetric about a point**. It has no axes of symmetry, but there is a point (called the **center of symmetry**) such that every line drawn through that point contacts the area in a symmetrical manner. The centroid of such an area coincides with the center of symmetry, and therefore the centroid can be located by inspection.

If an area has **irregular boundaries** not defined by simple mathematical expressions, we can locate the centroid by numerically evaluating the integrals in Eqs. (10-3a) and (10-3b). The simplest procedure is to divide the geometric figure into small finite elements and replace the integrations with summations. If we denote the area of the ith element by ΔA_i, then the expressions for the summations are

$$A = \sum_{i=1}^{n} \Delta A_i \qquad Q_x = \sum_{i=1}^{n} \bar{y}_i \Delta A_i \qquad Q_y = \sum_{i=1}^{n} \bar{x}_i \Delta A_i \tag{10-4a,b,c}$$

in which n is the total number of elements, $\bar{y}_i$ is the y coordinate of the centroid of the ith element, and $\bar{x}_i$ is the x coordinate of the centroid of the ith element. Replacing the integrals in Eqs. (10-3a) and (10-3b) by the corresponding summations, we obtain the following formulas for the coordinates of the centroid:

$$\bar{x} = \frac{Q_y}{A} = \frac{\sum_{i=1}^{n} \bar{x}_i\,\Delta A_i}{\sum_{i=1}^{n} \Delta A_i} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\sum_{i=1}^{n} \bar{y}_i\,\Delta A_i}{\sum_{i=1}^{n} \Delta A_i} \tag{10-5a,b}$$

The accuracy of the calculations for $\bar{x}$ and $\bar{y}$ depends upon how closely the selected elements fit the actual area. If they fit exactly, the results are exact. Many computer programs for locating centroids use a numerical scheme similar to the one expressed by Eqs. (10-5a) and (10-5b).

Example 10-1

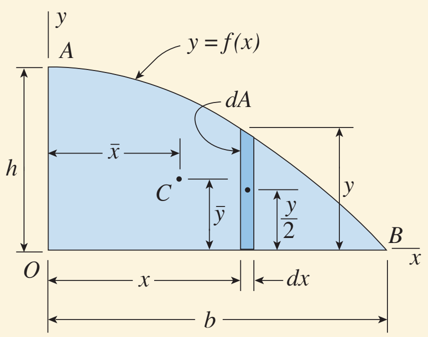


FIG. 10-5 Example 10-1. Centroid of a parabolic semisegment

A parabolic semisegment OAB is bounded by the x axis, the y axis, and a parabolic curve having its vertex at A (Fig. 10-5). The equation of the curve is

$$y = f(x) = h\left(1 - \frac{x^2}{b^2}\right) \tag{a}$$

in which b is the base and h is the height of the semisegment.

Locate the centroid C of the semisegment.

Solution

To determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C (Fig. 10-5), we will use Eqs. (10-3a) and (10-3b). We begin by selecting an element of area dA in the form of a thin vertical strip of width dx and height y. The area of this differential element is

$$dA = y\,dx = h\left(1 - \frac{x^2}{b^2}\right)dx \tag{b}$$

Therefore, the area of the parabolic semisegment is

$$A = \int dA = \int_0^b h\left(1 - \frac{x^2}{b^2}\right)dx = \frac{2bh}{3} \tag{c}$$

Note that this area is 2/3 of the area of the surrounding rectangle.

The first moment of an element of area dA with respect to an axis is obtained by multiplying the area of the element by the distance from its centroid to the axis. Since the x and y coordinates of the centroid of the element shown in Fig. 10-5 are x and $y/2$, respectively, the first moments of the element with respect to the x and y axes are

$$Q_x = \int \frac{y}{2}\,dA = \int_0^b \frac{h^2}{2}\left(1 - \frac{x^2}{b^2}\right)^2 dx = \frac{4bh^2}{15} \tag{d}$$

$$Q_y = \int x\,dA = \int_0^b hx\left(1 - \frac{x^2}{b^2}\right)dx = \frac{b^2h}{4} \tag{e}$$

in which we have substituted for dA from Eq. (b).

We can now determine the coordinates of the centroid C:

$$\bar{x} = \frac{Q_y}{A} = \frac{3b}{8} \qquad \bar{y} = \frac{Q_x}{A} = \frac{2h}{5} \tag{f,g}$$

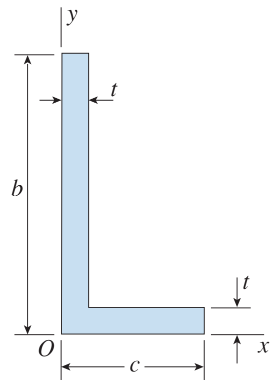
These results agree with the formulas listed in Appendix D, Case 17.

Notes: The centroid C of the parabolic semisegment may also be located by taking the element of area dA as a horizontal strip of height dy and width

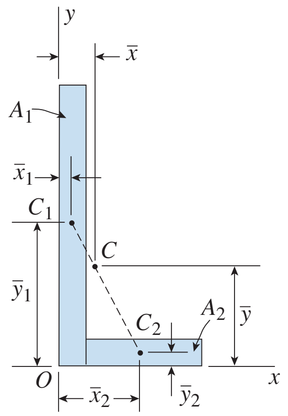
$$x = b\sqrt{1 - \frac{y}{h}} \tag{h}$$

This expression is obtained by solving Eq. (a) for x in terms of y.

10.3 CENTROIDS OF COMPOSITE AREAS



(a)



(b)

FIG. 10-6 Centroid of a composite area consisting of two parts

In engineering work we rarely need to locate centroids by integration, because the centroids of common geometric figures are already known and tabulated. However, we frequently need to locate the centroids of areas composed of several parts, each part having a familiar geometric shape, such as a rectangle or a circle. Examples of such **composite areas** are the cross sections of beams and columns, which usually consist of rectangular elements (for instance, see Figs. 10-2, 10-3, and 10-4).

The **areas and first moments** of composite areas may be calculated by summing the corresponding properties of the component parts. Let us assume that a composite area is divided into a total of n parts, and let us denote the area of the ith part as A_i. Then we can obtain the area and first moments by the following summations:

$$A = \sum_{i=1}^{n} A_i \qquad Q_x = \sum_{i=1}^{n} \bar{y}_i A_i \qquad Q_y = \sum_{i=1}^{n} \bar{x}_i A_i \qquad \text{(10-6a,b,c)}$$

in which $\bar{x}_i$ and $\bar{y}_i$ are the coordinates of the centroid of the ith part.

The **coordinates of the centroid** of the composite area are

$$\bar{x} = \frac{Q_y}{A} = \frac{\sum_{i=1}^{n} \bar{x}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{y} = \frac{Q_x}{A} = \frac{\sum_{i=1}^{n} \bar{y}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \text{(10-7a,b)}$$

Since the composite area is represented exactly by the n parts, the preceding equations give exact results for the coordinates of the centroid.

To illustrate the use of Eqs. (10-7a) and (10-7b), consider the L-shaped area (or angle section) shown in Fig. 10-6a. This area has side dimensions b and c and thickness t. The area can be divided into two rectangles of areas A_1 and A_2 with centroids C_1 and C_2, respectively (Fig. 10-6b). The areas and centroidal coordinates of these two parts are

$$A_1 = +bt \qquad \bar{x}_1 = \frac{t}{2} \qquad \bar{y}_1 = \frac{b}{2}$$

$$A_2 = (c - t)t \qquad \bar{x}_2 = \frac{c + t}{2} \qquad \bar{y}_2 = \frac{t}{2}$$

Therefore, the area and first moments of the composite area (from Eqs. 10-6a, b, and c) are

$$A = A_1 + A_2 = t(b + c - t)$$

$$Q_x = \bar{y}_1 A_1 + \bar{y}_2 A_2 = \frac{t}{2}(b^2 + ct - t^2)$$

$$Q_y = \bar{x}_1 A_1 + \bar{x}_2 A_2 = \frac{t}{2}(bt + c^2 - t^2)$$

Finally, we can obtain the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the composite area (Fig. 10-6b) from Eqs. (10-7a) and (10-7b):

$$\bar{x} = \frac{Q_y}{A} = \frac{bt + c^2 - t^2}{2(b + c - t)} \qquad \bar{y} = \frac{Q_x}{A} = \frac{b^2 + ct - t^2}{2(b + c - t)} \qquad \text{(10-8a,b)}$$

A similar procedure can be used for more complex areas, as illustrated in Example 10-2.

Note 1: When a composite area is divided into only two parts, the centroid C of the entire area lies on the line joining the centroids C_1 and C_2 of the two parts (as shown in Fig. 10-6b for the L-shaped area).

Note 2: When using the formulas for composite areas (Eqs. 10-6 and 10-7), we can handle the *absence* of an area by subtraction. This procedure is useful when there are cutouts or holes in the figure.

For instance, consider the area shown in Fig. 10-7a. We can analyze this figure as a composite area by subtracting the properties of the inner rectangle *efgh* from the corresponding properties of the outer rectangle *abcd*. (From another viewpoint, we can think of the outer rectangle as a "positive area" and the inner rectangle as a "negative area.")

Similarly, if an area has a hole (Fig. 10-7b), we can subtract the properties of the area of the hole from those of the outer rectangle. (Again, the same effect is achieved if we treat the outer rectangle as a "positive area" and the hole as a "negative area.")

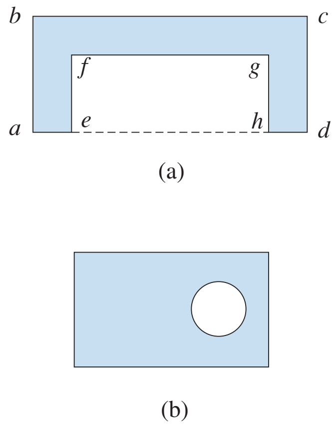
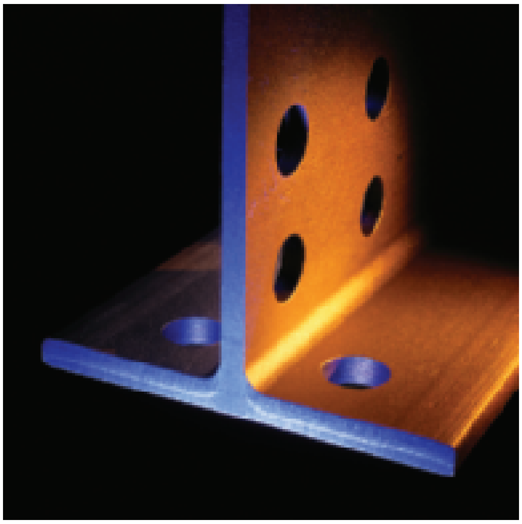


FIG. 10-7 Composite areas with a cutout and a hole



Cutouts in beams must be considered in centroid and moment of inertia calculations (Don Farrall/Getty Images)

Example 10-2

The cross section of a steel beam is constructed of a HE 450 A wide-flange section with a 25 cm × 1.5 cm cover plate welded to the top flange and a UPN 320 channel section welded to the bottom flange (Fig. 10-8).

Locate the centroid C of the cross-sectional area.

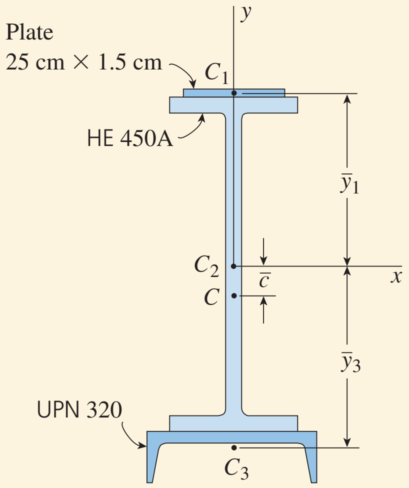


FIG. 10-8 Example 10-2. Centroid of a composite area

Solution

Let us denote the areas of the cover plate, the wide-flange section, and the channel section as areas A_1, A_2, and A_3, respectively. The centroids of these three areas are labeled C_1, C_2, and C_3, respectively, in Fig. 10-8. Note that the composite area has an axis of symmetry, and therefore all centroids lie on that axis. The three partial areas are

$$A_1 = (25 \text{ cm})(1.5 \text{ cm}) = 37.5 \text{ cm}^2 \qquad A_2 = 178 \text{ cm}^2 \qquad A_3 = 75.8 \text{ cm}^2$$

in which the areas A_2 and A_3 are obtained from Tables E-1 and E-3 of Appendix E.

Let us place the origin of the x and y axes at the centroid C_2 of the wide-flange section. Then the distances from the x axis to the centroids of the three areas are as follows:

$$\bar{y}_1 = \frac{440 \text{ mm}}{2} + \frac{15 \text{ mm}}{2} = 227.5 \text{ mm}$$

$$\bar{y}_2 = 0 \qquad \bar{y}_3 = \frac{440 \text{ mm}}{2} + 26 \text{ mm} = 246 \text{ mm}$$

continued

in which the pertinent dimensions of the wide-flange and channel sections are obtained from Tables E-1 and E-3.

The area A and first moment Q_x of the entire cross section are obtained from Eqs. (10-6a) and (10-6b) as follows:

$$A = \sum_{i=1}^{n} A_i = A_1 + A_2 + A_3$$

$$= 37.5\ \text{cm}^2 + 178\ \text{cm}^2 + 75.8\ \text{cm}^2 = 291.3\ \text{cm}^2$$

$$Q_x = \sum_{i=1}^{n} \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3$$

$$= (22.75\ \text{cm})(37.5\ \text{cm}^2) + 0 - (24.6\ \text{cm})(75.8\ \text{cm}^2) = -1012\ \text{cm}^3$$

Now we can obtain the coordinate $\bar{y}$ to the centroid C of the composite area from Eq. (10-7b):

$$\bar{y} = \frac{Q_x}{A} = \frac{-1012\ \text{cm}^3}{291.3\ \text{cm}^2} = -34.726\ \text{mm}$$

Since $\bar{y}$ is positive in the positive direction of the y axis, the minus sign means that the centroid C of the composite area is located below the x axis, as shown in Fig. 10-8. Thus, the distance $\bar{c}$ between the x axis and the centroid C is

$$\bar{c} = 34.73\ \text{mm}$$

Note that the position of the reference axis (the x axis) is arbitrary; however, in this example we placed it through the centroid of the wide-flange section because it slightly simplifies the calculations.

10.4 MOMENTS OF INERTIA OF PLANE AREAS

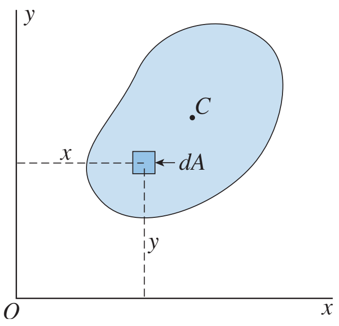


FIG. 10-9 Plane area of arbitrary shape

The **moments of inertia** of a plane area (Fig. 10-9) with respect to the x and y axes, respectively, are defined by the integrals

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA \tag{10-9a,b}$$

in which x and y are the coordinates of the differential element of area dA. Because the element dA is multiplied by the square of the distance from the reference axis, moments of inertia are also called **second moments of area**. Also, we see that moments of inertia of areas (unlike first moments) are always positive quantities.

To illustrate how moments of inertia are obtained by integration, we will consider a rectangle having width b and height h (Fig. 10-10). The x and y axes have their origin at the centroid C. For convenience, we use a differential element of area dA in the form of a thin horizontal strip of width b and height dy (therefore, $dA = b\,dy$). Since all parts of the elemental strip are the same distance from the x axis, we can express the moment of inertia I_x with respect to the x axis as follows:

$$I_x = \int y^2 dA = \int_{-h/2}^{h/2} y^2 b\,dy = \frac{bh^3}{12} \tag{a}$$

In a similar manner, we can use an element of area in the form of a vertical strip with area $dA = h\,dx$ and obtain the moment of inertia with respect to the y axis:

$$I_y = \int x^2 dA = \int_{-b/2}^{b/2} x^2 h\,dx = \frac{hb^3}{12} \tag{b}$$

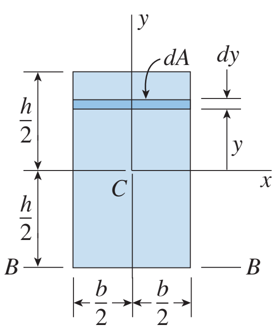


FIG. 10-10 Moments of inertia of a rectangle

If a different set of axes is selected, the moments of inertia will have different values. For instance, consider axis BB at the base of the rectangle (Fig. 10-10). If this axis is selected as the reference, we must define y as the coordinate distance from that axis to the element of area dA. Then the calculations for the moment of inertia become

$$I_{BB} = \int y^2 dA = \int_0^h y^2 b\,dy = \frac{bh^3}{3} \tag{c}$$

Note that the moment of inertia with respect to axis BB is larger than the moment of inertia with respect to the centroidal x axis. In general, the

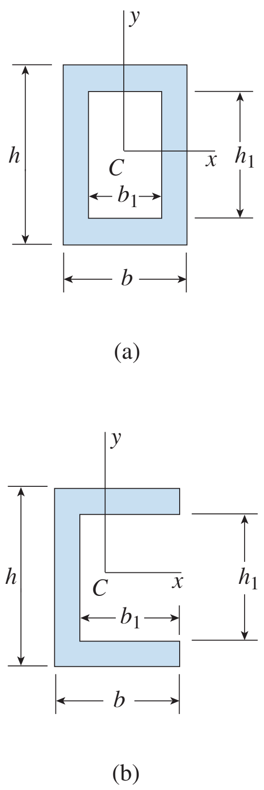


FIG. 10-11 Composite areas

moment of inertia increases as the reference axis is moved parallel to itself farther from the centroid.

The moment of inertia of a **composite area** with respect to any particular axis is the sum of the moments of inertia of its parts with respect to that same axis. An example is the hollow box section shown in Fig. 10-11a, where the x and y axes are axes of symmetry through the centroid C. The moment of inertia I_x with respect to the x axis is equal to the algebraic sum of the moments of inertia of the outer and inner rectangles. (As explained earlier, we can think of the inner rectangle as a "negative area" and the outer rectangle as a "positive area.") Therefore,

$$I_x = \frac{bh^3}{12} - \frac{b_1 h_1^3}{12} \qquad \text{(d)}$$

This same formula applies to the channel section shown in Fig. 10-11b, where we may consider the cutout as a "negative area."

For the hollow box section, we can use a similar technique to obtain the moment of inertia I_y with respect to the vertical axis. However, in the case of the channel section, the determination of the moment of inertia I_y requires the use of the parallel-axis theorem, which is described in the next section (Section 10.5).

Formulas for moments of inertia are listed in Appendix D. For shapes not shown, the moments of inertia can usually be obtained by using the listed formulas in conjunction with the parallel-axis theorem. If an area is of such irregular shape that its moments of inertia cannot be obtained in this manner, then we can use numerical methods. The procedure is to divide the area into small elements of area ΔA_i, multiply each such area by the square of its distance from the reference axis, and then sum the products.

Radius of Gyration

A distance known as the **radius of gyration** is occasionally encountered in mechanics. Radius of gyration of a plane area is defined as the square root of the moment of inertia of the area divided by the area itself; thus,

$$r_x = \sqrt{\frac{I_x}{A}} \qquad r_y = \sqrt{\frac{I_y}{A}} \qquad \text{(10-10a,b)}$$

in which r_x and r_y denote the radii of gyration with respect to the x and y axes, respectively. Since moment of inertia has units of length to the fourth power and area has units of length to the second power, radius of gyration has units of length.

Although the radius of gyration of an area does not have an obvious physical meaning, we may consider it to be the distance (from the reference axis) at which the entire area could be concentrated and still have the same moment of inertia as the original area.

Example 10-3

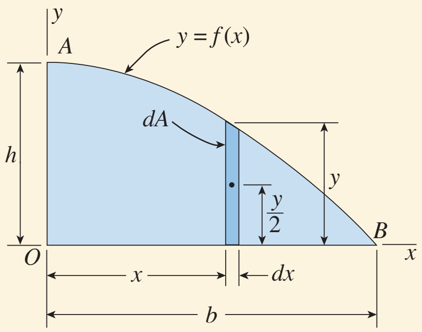


FIG. 10-12 Example 10-3. Moments of inertia of a parabolic semisegment

Determine the moments of inertia I_x and I_y for the parabolic semisegment OAB shown in Fig. 10-12. The equation of the parabolic boundary is

$$y = f(x) = h\left(1 - \frac{x^2}{b^2}\right) \tag{e}$$

(This same area was considered previously in Example 10-1.)

Solution

To determine the moments of inertia by integration, we will use Eqs. (10-9a) and (10-9b). The differential element of area dA is selected as a vertical strip of width dx and height y, as shown in Fig. 10-12. The area of this element is

$$dA = y\,dx = h\left(1 - \frac{x^2}{b^2}\right)dx \tag{f}$$

Since every point in this element is at the same distance from the y axis, the moment of inertia of the element with respect to the y axis is x^2dA. Therefore, the moment of inertia of the entire area with respect to the y axis is obtained as follows:

$$I_y = \int x^2 dA = \int_0^b x^2 h\left(1 - \frac{x^2}{b^2}\right)dx = \frac{2hb^3}{15} \tag{g}$$

To obtain the moment of inertia with respect to the x axis, we note that the differential element of area dA has a moment of inertia dI_x with respect to the x axis equal to

$$dI_x = \frac{1}{3}(dx)y^3 = \frac{y^3}{3}dx$$

as obtained from Eq. (c). Hence, the moment of inertia of the entire area with respect to the x axis is

$$I_x = \int_0^b \frac{y^3}{3}dx = \int_0^b \frac{h^3}{3}\left(1 - \frac{x^2}{b^2}\right)^3 dx = \frac{16bh^3}{105} \tag{h}$$

These same results for I_x and I_y can be obtained by using an element in the form of a horizontal strip of area $dA = x\,dy$ or by using a rectangular element of area $dA = dx\,dy$ and performing a double integration. Also, note that the preceding formulas for I_x and I_y agree with those given in Case 17 of Appendix D.

10.5 PARALLEL-AXIS THEOREM FOR MOMENTS OF INERTIA

In this section we will derive a very useful theorem pertaining to moments of inertia of plane areas. Known as the **parallel-axis theorem**, it gives the relationship between the moment of inertia with respect to a centroidal axis and the moment of inertia with respect to any parallel axis.

To derive the theorem, we consider an area of arbitrary shape with centroid C (Fig. 10-13). We also consider two sets of coordinate axes: (1) the $x_c y_c$ axes with origin at the centroid, and (2) a set of parallel xy axes with origin at any point O. The distances between the two sets of parallel axes are denoted d_1 and d_2. Also, we identify an element of area dA having coordinates x and y with respect to the centroidal axes.

From the definition of moment of inertia, we can write the following equation for the moment of inertia I_x with respect to the x axis:

$$I_x = \int (y + d_1)^2 dA = \int y^2 dA + 2d_1 \int y\, dA + d_1^2 \int dA \qquad \text{(a)}$$

The first integral on the right-hand side is the moment of inertia I_{x_c} with respect to the x_c axis. The second integral is the first moment of the area with respect to the x_c axis (this integral equals zero because the x_c axis passes through the centroid). The third integral is the area A itself. Therefore, the preceding equation reduces to

$$I_x = I_{x_c} + Ad_1^2 \qquad \text{(10-11a)}$$

Proceeding in the same manner for the moment of inertia with respect to the y axis, we obtain

$$I_y = I_{y_c} + Ad_2^2 \qquad \text{(10-11b)}$$

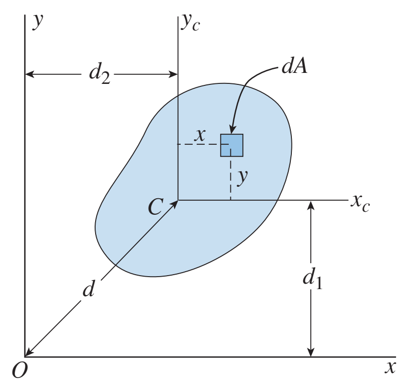


FIG. 10-13 Derivation of parallel-axis theorem

Equations (10-11a) and (10-11b) represent the **parallel-axis theorem for moments of inertia**:

> *The moment of inertia of an area with respect to any axis in its plane is equal to the moment of inertia with respect to a parallel centroidal axis plus the product of the area and the square of the distance between the two axes.*

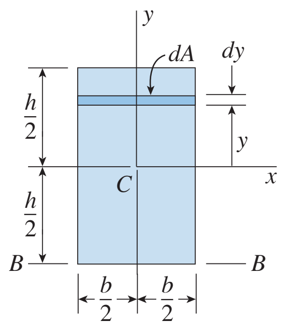


FIG. 10-10 Moments of inertia of a rectangle (Repeated)

To illustrate the use of the theorem, consider again the rectangle shown in Fig. 10-10. Knowing that the moment of inertia about the x axis, which is through the centroid, is equal to $bh^3/12$ (see Eq. a of Section 10.4), we can determine the moment of inertia I_{BB} about the base of the rectangle by using the parallel-axis theorem:

$$I_{BB} = I_x + Ad^2 = \frac{bh^3}{12} + bh\left(\frac{h}{2}\right)^2 = \frac{bh^3}{3}$$

This result agrees with the moment of inertia obtained previously by integration (Eq. c of Section 10.4).

From the parallel-axis theorem, we see that the moment of inertia increases as the axis is moved parallel to itself farther from the centroid. Therefore, the moment of inertia about a centroidal axis is the least moment of inertia of an area (for a given direction of the axis).

When using the parallel-axis theorem, it is essential to remember that one of the two parallel axes *must* be a centroidal axis. If it is necessary to find the moment of inertia I_2 about a noncentroidal axis 2-2 (Fig. 10-14) when the moment of inertia I_1 about another noncentroidal (and parallel) axis 1-1 is known, we must apply the parallel-axis theorem twice. First, we find the centroidal moment of inertia I_{x_c} from the known moment of inertia I_1:

$$I_{x_c} = I_1 - Ad_1^2 \tag{b}$$

Then we find the moment of inertia I_2 from the centroidal moment of inertia:

$$I_2 = I_{x_c} + Ad_2^2 = I_1 + A(d_2^2 - d_1^2) \tag{10-12}$$

This equation shows again that the moment of inertia increases with increasing distance from the centroid of the area.

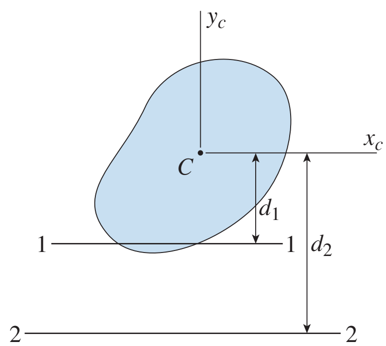


FIG. 10-14 Plane area with two parallel noncentroidal axes (axes 1-1 and 2-2)

Example 10-4

The parabolic semisegment OAB shown in Fig. 10-15 has base b and height h. Using the parallel-axis theorem, determine the moments of inertia I_{x_c} and I_{y_c} with respect to the centroidal axes x_c and y_c.

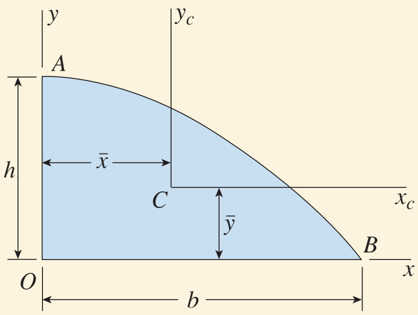


FIG. 10-15 Example 10-4. Parallel-axis theorem

Solution

We can use the parallel-axis theorem (rather than integration) to find the centroidal moments of inertia because we already know the area A, the centroidal coordinates $\bar{x}$ and $\bar{y}$, and the moments of inertia I_x and I_y with respect to the x and y axes. These quantities were obtained earlier in Examples 10-1 and 10-3. They also are listed in Case 17 of Appendix D and are repeated here:

$$A = \frac{2bh}{3} \qquad \bar{x} = \frac{3b}{8} \qquad \bar{y} = \frac{2h}{5} \qquad I_x = \frac{16bh^3}{105} \qquad I_y = \frac{2hb^3}{15}$$

To obtain the moment of inertia with respect to the x_c axis, we use Eq. (b) and write the parallel-axis theorem as follows:

$$I_{x_c} = I_x - A\bar{y}^2 = \frac{16bh^3}{105} - \frac{2bh}{3}\left(\frac{2h}{5}\right)^2 = \frac{8bh^3}{175} \qquad \text{(10-13a)}$$

In a similar manner, we obtain the moment of inertia with respect to the y_c axis:

$$I_{y_c} = I_y - A\bar{x}^2 = \frac{2hb^3}{15} - \frac{2bh}{3}\left(\frac{3b}{8}\right)^2 = \frac{19hb^3}{480} \qquad \text{(10-13b)}$$

Thus, we have found the centroidal moments of inertia of the semisegment.

Example 10-5

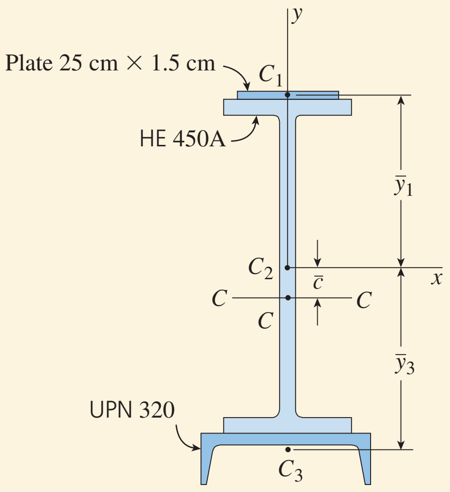


FIG. 10-16 Example 10-5. Moment of inertia of a composite area

Determine the moment of inertia I_c with respect to the horizontal axis *C-C* through the centroid *C* of the beam cross section shown in Fig. 10-16. (The position of the centroid *C* was determined previously in Example 10-2 of Section 10.3.)

Note: From beam theory (Chapter 5), we know that axis *C-C* is the neutral axis for bending of this beam, and therefore the moment of inertia I_c must be determined in order to calculate the stresses and deflections of this beam.

Solution

We will determine the moment of inertia I_c with respect to axis *C-C* by applying the parallel-axis theorem to each individual part of the composite area. The area divides naturally into three parts: (1) the cover plate, (2) the wide-flange section, and (3) the channel section. The following areas and centroidal distances were obtained previously in Example 10-2:

$$A_1 = 37.5 \text{ cm}^2 \qquad A_2 = 178 \text{ cm}^2 \qquad A_3 = 75.8 \text{ cm}^2$$

$$\bar{y}_1 = 227.5 \text{ mm} \qquad \bar{y}_2 = 0 \qquad \bar{y}_3 = 246 \text{ mm} \qquad \bar{c} = 34.73 \text{ mm}$$

The moments of inertia of the three parts with respect to horizontal axes through their own centroids C_1, C_2, and C_3 are as follows:

$$I_1 = \frac{bh^3}{12} = \frac{1}{12}(25 \text{ cm})(1.5 \text{ cm})^3 = 7.031 \text{ cm}^4$$

$$I_2 = 63720 \text{ cm}^4 \qquad I_3 = 597 \text{ cm}^4$$

The moments of inertia I_2 and I_3 are obtained from Tables E-1 and E-3, respectively, of Appendix E.

Now we can use the parallel-axis theorem to calculate the moments of inertia about axis *C-C* for each of the three parts of the composite area:

$$(I_c)_1 = I_1 + A_1(\bar{y}_1 + \bar{c})^2 = 7.031 \text{ cm}^4 + (37.5 \text{ cm}^2)(26.22 \text{ cm})^2 = 25790 \text{ cm}^4$$

$$(I_c)_2 = I_2 + A_2\bar{c}^2 = 63720 \text{ cm}^4 + (178 \text{ cm}^2)(34.73 \text{ cm})^2 = 65870 \text{ cm}^4$$

$$(I_c)_3 = I_3 + A_3(\bar{y}_3 - \bar{c})^2 = 597 \text{ cm}^4 + (75.8 \text{ cm}^2)(21.13 \text{ cm})^2 = 34430 \text{ cm}^4$$

The sum of these individual moments of inertia gives the moment of inertia of the entire cross-sectional area about its centroidal axis *C-C*:

$$I_c = (I_c)_1 + (I_c)_2 + (I_c)_3 = 1.261 \times 10^5 \text{ cm}^4$$

This example shows how to calculate moments of inertia of composite areas by using the parallel-axis theorem.

10.6 POLAR MOMENTS OF INERTIA

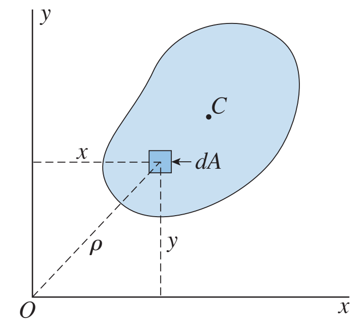


FIG. 10-17 Plane area of arbitrary shape

The moments of inertia discussed in the preceding sections are defined with respect to axes lying in the plane of the area itself, such as the x and y axes in Fig. 10-17. Now we will consider an axis *perpendicular* to the plane of the area and intersecting the plane at the origin O. The moment of inertia with respect to this perpendicular axis is called the **polar moment of inertia** and is denoted by the symbol I_P.

The polar moment of inertia with respect to an axis through O perpendicular to the plane of the figure is defined by the integral

$$I_P = \int \rho^2 dA \tag{10-14}$$

in which ρ is the distance from point O to the differential element of area dA (Fig. 10-17). This integral is similar in form to those for moments of inertia I_x and I_y (see Eqs. 10-9a and 10-9b).

Inasmuch as $\rho^2 = x^2 + y^2$, where x and y are the rectangular coordinates of the element dA, we obtain the following expression for I_P:

$$I_P = \int \rho^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA$$

Thus, we obtain the important relationship

$$I_P = I_x + I_y \tag{10-15}$$

This equation shows that the polar moment of inertia with respect to an axis perpendicular to the plane of the figure at any point O is equal to the sum of the moments of inertia with respect to *any* two perpendicular axes x and y passing through that same point and lying in the plane of the figure.

For convenience, we usually refer to I_P simply as the polar moment of inertia with respect to point O, without mentioning that the axis is perpendicular to the plane of the figure. Also, to distinguish them from **polar** moments of inertia, we sometimes refer to I_x and I_y as **rectangular** moments of inertia.

Polar moments of inertia with respect to various points in the plane of an area are related by the **parallel-axis theorem for polar moments of inertia**. We can derive this theorem by referring again to Fig. 10-13. Let us denote the polar moments of inertia with respect to the origin O and the centroid C by $(I_P)_O$ and $(I_P)_C$, respectively. Then, using Eq. (10-15), we can write the following equations:

$$(I_P)_O = I_x + I_y \qquad (I_P)_C = I_{x_c} + I_{y_c} \tag{a}$$

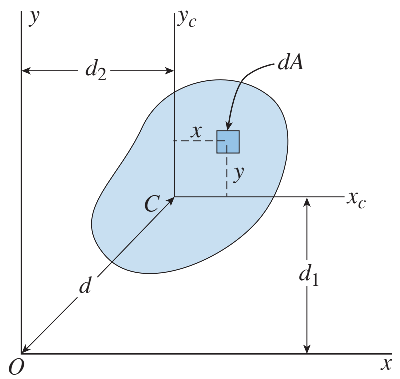


FIG. 10-13 Derivation of parallel-axis theorem (Repeated)

Now refer to the parallel-axis theorems derived in Section 10.5 for rectangular moments of inertia (Eqs. 10-11a and 10-11b). Adding those two equations, we get

$$I_x + I_y = I_{x_c} + I_{y_c} + A(d_1^2 + d_2^2)$$

Substituting from Eqs. (a), and also noting that $d^2 = d_1^2 + d_2^2$ (Fig. 10-13), we obtain

$$(I_P)_O = (I_P)_C + Ad^2 \tag{10-16}$$

This equation represents the **parallel-axis theorem** for polar moments of inertia:

The polar moment of inertia of an area with respect to any point O in its plane is equal to the polar moment of inertia with respect to the centroid C plus the product of the area and the square of the distance between points O and C.

To illustrate the determination of polar moments of inertia and the use of the parallel-axis theorem, consider a circle of radius r (Fig. 10-18). Let us take a differential element of area dA in the form of a thin ring of radius ρ and thickness $d\rho$ (thus, $dA = 2\pi\rho\, d\rho$). Since every point in the element is at the same distance ρ from the center of the circle, the polar moment of inertia of the entire circle with respect to the center is

$$(I_P)_C = \int \rho^2 dA = \int_0^r 2\pi\rho^3 d\rho = \frac{\pi r^4}{2} \tag{10-17}$$

This result is listed in Case 9 of Appendix E.

The polar moment of inertia of the circle with respect to any point B on its circumference (Fig. 10-18) can be obtained from the parallel-axis theorem:

$$(I_P)_B = (I_P)_C + Ad^2 = \frac{\pi r^4}{2} + \pi r^2(r^2) = \frac{3\pi r^4}{2} \tag{10-18}$$

As an incidental matter, note that the polar moment of inertia has its smallest value when the reference point is the centroid of the area.

A circle is a special case in which the polar moment of inertia can be determined by integration. However, most of the shapes encountered in engineering work do not lend themselves to this technique. Instead, polar moments of inertia are usually obtained by summing the rectangular moments of inertia for two perpendicular axes (Eq. 10-15).

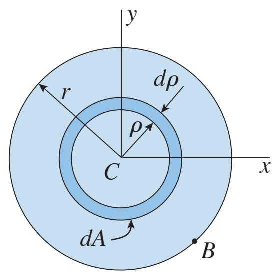


FIG. 10-18 Polar moment of inertia of a circle

10.7 PRODUCTS OF INERTIA

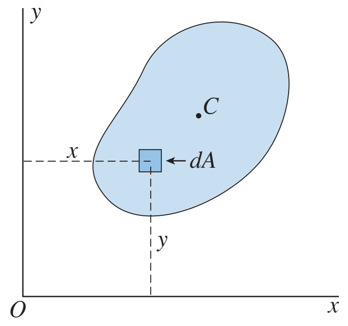


FIG. 10-19 Plane area of arbitrary shape

The product of inertia of a plane area is defined with respect to a set of perpendicular axes lying in the plane of the area. Thus, referring to the area shown in Fig. 10-19, we define the **product of inertia** with respect to the x and y axes as follows:

$$I_{xy} = \int xy \, dA \tag{10-19}$$

From this definition we see that each differential element of area dA is multiplied by the product of its coordinates. As a consequence, products of inertia may be positive, negative, or zero, depending upon the position of the xy axes with respect to the area.

If the area lies entirely in the first quadrant of the axes (as in Fig. 10-19), then the product of inertia is positive because every element dA has positive coordinates x and y. If the area lies entirely in the second quadrant, the product of inertia is negative because every element has a positive y coordinate and a negative x coordinate. Similarly, areas entirely within the third and fourth quadrants have positive and negative products of inertia, respectively. When the area is located in more than one quadrant, the sign of the product of inertia depends upon the distribution of the area within the quadrants.

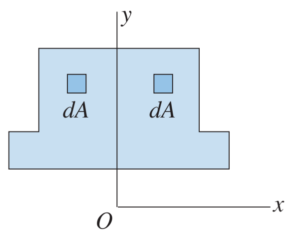


FIG. 10-20 The product of inertia equals zero when one axis is an axis of symmetry

A special case arises when one of the axes is an **axis of symmetry** of the area. For instance, consider the area shown in Fig. 10-20, which is symmetric about the y axis. For every element dA having coordinates x and y, there exists an equal and symmetrically located element dA having the same y coordinate but an x coordinate of opposite sign. Therefore, the products $xy\,dA$ cancel each other and the integral in Eq. (10-19) vanishes. Thus, *the product of inertia of an area is zero with respect to any pair of axes in which at least one axis is an axis of symmetry of the area.*

As examples of the preceding rule, the product of inertia I_{xy} equals zero for the areas shown in Figs. 10-10, 10-11, 10-16, and 10-18. In contrast, the product of inertia I_{xy} has a positive nonzero value for the area shown in Fig. 10-15. (These observations are valid for products of inertia with respect to the particular xy axes shown in the figures. If the axes are shifted to another position, the product of inertia may change.)

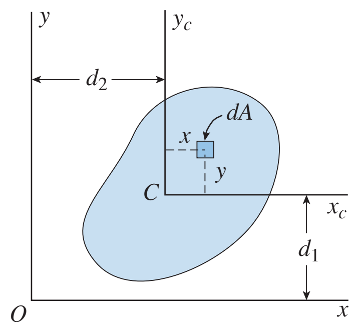


FIG. 10-21 Plane area of arbitrary shape

Products of inertia of an area with respect to parallel sets of axes are related by a **parallel-axis theorem** that is analogous to the corresponding theorems for rectangular moments of inertia and polar moments of inertia. To obtain this theorem, consider the area shown in Fig. 10-21, which has centroid C and centroidal x_cy_c axes. The product of inertia I_{xy} with respect to any other set of axes, parallel to the $x_c y_c$ axes, is

$$I_{xy} = \int (x + d_2)(y + d_1)dA = \int xy\, dA + d_1 \int x\, dA + d_2 \int y\, dA + d_1 d_2 \int dA$$

in which d_1 and d_2 are the coordinates of the centroid C with respect to the xy axes (thus, d_1 and d_2 may have positive or negative values).

The first integral in the last expression is the product of inertia $I_{x_c y_c}$ with respect to the centroidal axes; the second and third integrals equal zero because they are the first moments of the area with respect to the centroidal axes; and the last integral is the area A. Therefore, the preceding equation reduces to

$$\boxed{I_{xy} = I_{x_c y_c} + Ad_1 d_2} \qquad \text{(10-20)}$$

This equation represents the **parallel-axis theorem for products of inertia**:

The product of inertia of an area with respect to any pair of axes in its plane is equal to the product of inertia with respect to parallel centroidal axes plus the product of the area and the coordinates of the centroid with respect to the pair of axes.

To demonstrate the use of this parallel-axis theorem, let us determine the product of inertia of a rectangle with respect to xy axes having their origin at point O at the lower left-hand corner of the rectangle (Fig. 10-22). The product of inertia with respect to the centroidal $x_c y_c$ axes is zero because of symmetry. Also, the coordinates of the centroid with respect to the xy axes are

$$d_1 = \frac{h}{2} \qquad d_2 = \frac{b}{2}$$

Substituting into Eq. (10-20), we obtain

$$I_{xy} = I_{x_c y_c} + Ad_1 d_2 = 0 + bh\left(\frac{h}{2}\right)\left(\frac{b}{2}\right) = \frac{b^2 h^2}{4} \qquad \text{(10-21)}$$

This product of inertia is positive because the entire area lies in the first quadrant. If the xy axes are translated horizontally so that the origin moves to point B at the lower right-hand corner of the rectangle (Fig. 10-22), the entire area lies in the second quadrant and the product of inertia becomes $-b^2h^2/4$.

The following example also illustrates the use of the parallel-axis theorem for products of inertia.

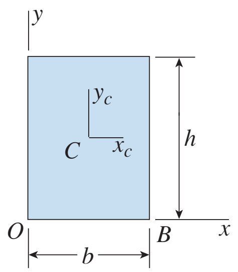


FIG. 10-22 Parallel-axis theorem for products of inertia

Example 10-6

Determine the product of inertia I_{xy} of the Z-section shown in Fig. 10-23. The section has width b, height h, and constant thickness t.

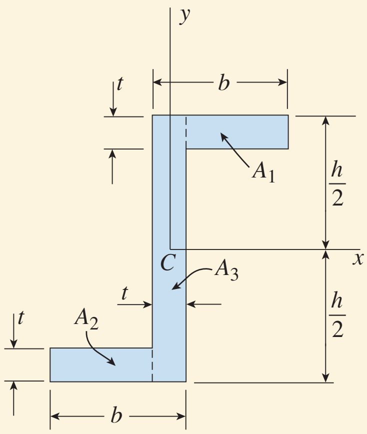


FIG. 10-23 Example 10-6. Product of inertia of a Z-section

Solution

To obtain the product of inertia with respect to the xy axes through the centroid, we divide the area into three parts and use the parallel-axis theorem. The parts are as follows: (1) a rectangle of width $b - t$ and thickness t in the upper flange, (2) a similar rectangle in the lower flange, and (3) a web rectangle with height h and thickness t.

The product of inertia of the web rectangle with respect to the xy axes is zero (from symmetry). The product of inertia $(I_{xy})_1$ of the upper flange rectangle (with respect to the xy axes) is determined by using the parallel-axis theorem:

$$(I_{xy})_1 = I_{x_c y_c} + Ad_1 d_2 \qquad \text{(a)}$$

in which $I_{x_c y_c}$ is the product of inertia of the rectangle with respect to its own centroid, A is the area of the rectangle, d_1 is the y coordinate of the centroid of the rectangle, and d_2 is the x coordinate of the centroid of the rectangle. Thus,

$$I_{x_c y_c} = 0 \qquad A = (b - t)(t) \qquad d_1 = \frac{h}{2} - \frac{t}{2} \qquad d_2 = \frac{b}{2}$$

Substituting into Eq. (a), we obtain the product of inertia of the rectangle in the upper flange:

$$(I_{xy})_1 = I_{x_c y_c} + Ad_1 d_2 = 0 + (b - t)(t)\left(\frac{h}{2} - \frac{t}{2}\right)\left(\frac{b}{2}\right) = \frac{bt}{4}(h - t)(b - t)$$

The product of inertia of the rectangle in the lower flange is the same. Therefore, the product of inertia of the entire Z-section is twice $(I_{xy})_1$, or

$$I_{xy} = \frac{bt}{2}(h - t)(b - t) \qquad \text{(10-22)}$$

Note that this product of inertia is positive because the flanges lie in the first and third quadrants.

10.8 ROTATION OF AXES

The moments of inertia of a plane area depend upon the position of the origin and the orientation of the reference axes. For a given origin, the moments and product of inertia vary as the axes are rotated about that origin. The manner in which they vary, and the magnitudes of the maximum and minimum values, are discussed in this and the following section.

Let us consider the plane area shown in Fig. 10-24, and let us assume that the xy axes are a pair of arbitrarily located reference axes. The moments and products of inertia with respect to those axes are

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA \qquad I_{xy} = \int xy\, dA \qquad \text{(a,b,c)}$$

in which x and y are the coordinates of a differential element of area dA.

The x_1y_1 axes have the same origin as the xy axes but are rotated through a counterclockwise angle θ with respect to those axes. The moments and product of inertia with respect to the x_1y_1 axes are denoted I_{x_1}, I_{y_1}, and $I_{x_1y_1}$, respectively. To obtain these quantities, we need the coordinates of the element of area dA with respect to the x_1y_1 axes. These coordinates may be expressed in terms of the xy coordinates and the angle θ by geometry, as follows:

$$x_1 = x\cos\theta + y\sin\theta \qquad y_1 = y\cos\theta - x\sin\theta \qquad \text{(10-23a,b)}$$

Then the moment of inertia with respect to the x_1 axis is

$$I_{x_1} = \int y_1^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA$$

$$= \cos^2\theta \int y^2 dA + \sin^2\theta \int x^2 dA - 2\sin\theta\cos\theta \int xy\, dA$$

or, by using Eqs. (a), (b), and (c),

$$I_{x_1} = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \qquad \text{(10-24)}$$

Now we introduce the following trigonometric identities:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

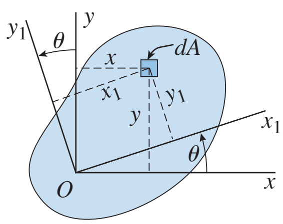


FIG. 10-24 Rotation of axes

Then Eq. (10-24) becomes

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \tag{10-25}$$

In a similar manner, we can obtain the product of inertia with respect to the x_1y_1 axes:

$$I_{x_1y_1} = \int x_1y_1\, dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA$$

$$= (I_x - I_y)\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) \tag{10-26}$$

Again using the trigonometric identities, we obtain

$$I_{x_1y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \tag{10-27}$$

Equations (10-25) and (10-27) give the moment of inertia I_{x_1} and the product of inertia $I_{x_1y_1}$ with respect to the rotated axes in terms of the moments and product of inertia for the original axes. These equations are called the **transformation equations for moments and products of inertia**.

Note that these transformation equations have the same form as the transformation equations for plane stress (Eqs. 6-4a and 6-4b of Section 6.2). Upon comparing the two sets of equations, we see that I_{x_1} corresponds to σ_{x_1}, $I_{x_1y_1}$ corresponds to $-\tau_{x_1y_1}$, I_x corresponds to σ_x, I_y corresponds to σ_y, and I_{xy} corresponds to $-\tau_{xy}$. Therefore, we can also analyze moments and products of inertia by the use of **Mohr's circle** (see Section 6.4).

The moment of inertia I_{y_1} may be obtained by the same procedure that we used for finding I_{x_1} and $I_{x_1y_1}$. However, a simpler procedure is to replace θ with $\theta + 90°$ in Eq. (10-25). The result is

$$I_{y_1} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \tag{10-28}$$

This equation shows how the moment of inertia I_{y_1} varies as the axes are rotated about the origin.

A useful equation related to moments of inertia is obtained by taking the sum of I_{x_1} and I_{y_1} (Eqs. 10-25 and 10-28). The result is

$$I_{x_1} + I_{y_1} = I_x + I_y \tag{10-29}$$

This equation shows that the sum of the moments of inertia with respect to a pair of axes remains constant as the axes are rotated about the origin. This sum is the polar moment of inertia of the area with respect to the origin. Note that Eq. (10-29) is analogous to Eq. (6-6) for stresses.

10.9 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

The transformation equations for moments and products of inertia (Eqs. 10-25, 10-27, and 10-28) show how the moments and products of inertia vary as the angle of rotation θ varies. Of special interest are the maximum and minimum values of the moment of inertia. These values are known as the **principal moments of inertia**, and the corresponding axes are known as **principal axes**.

Principal Axes

To find the values of the angle θ that make the moment of inertia I_{x_1} a maximum or a minimum, we take the derivative with respect to θ of the expression on the right-hand side of Eq. (10-25) and set it equal to zero:

$$(I_x - I_y)\sin 2\theta + 2I_{xy}\cos 2\theta = 0 \tag{a}$$

Solving for θ from this equation, we get

$$\boxed{\tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y}} \tag{10-30}$$

in which θ_p denotes the angle defining a principal axis. This same result is obtained if we take the derivative of I_{y_1} (Eq. 10-28).

Equation (10-30) yields two values of the angle $2\theta_p$ in the range from 0 to 360°; these values differ by 180°. The corresponding values of θ_p differ by 90° and define the two perpendicular principal axes. One of these axes corresponds to the maximum moment of inertia and the other corresponds to the minimum moment of inertia.

Now let us examine the variation in the product of inertia $I_{x_1y_1}$ as θ changes (see Eq. 10-27). If $\theta = 0$, we get $I_{x_1y_1} = I_{xy}$, as expected. If $\theta = 90°$, we obtain $I_{x_1y_1} = -I_{xy}$. Thus, during a 90° rotation the product of inertia changes sign, which means that for an intermediate orientation of the axes, the product of inertia must equal zero. To determine this orientation, we set $I_{x_1y_1}$ (Eq. 10-27) equal to zero:

$$(I_x - I_y)\sin 2\theta + 2I_{xy}\cos 2\theta = 0$$

This equation is the same as Eq. (a), which defines the angle θ_p to the principal axes. Therefore, we conclude that *the product of inertia is zero for the principal axes*.

In Section 10.7 we showed that the product of inertia of an area with respect to a pair of axes equals zero if at least one of the axes is an axis of symmetry. It follows that if an area has an axis of symmetry, that axis and any axis perpendicular to it constitute a set of principal axes.

The preceding observations may be summarized as follows: (1) principal axes through an origin O are a pair of orthogonal axes for which the moments of inertia are a maximum and a minimum; (2) the orientation of the principal axes is given by the angle θ_p obtained from Eq. (10-30); (3) the product of inertia is zero for principal axes; and (4) an axis of symmetry is always a principal axis.

Principal Points

Now consider a pair of principal axes with origin at a given point O. If there exists a *different* pair of principal axes through that same point, then *every* pair of axes through that point is a set of principal axes. Furthermore, the moment of inertia must be constant as the angle θ is varied.

The preceding conclusions follow from the nature of the transformation equation for I_{x_1} (Eq. 10-25). Because this equation contains trigonometric functions of the angle 2θ, there is one maximum value and one minimum value of I_{x_1} as 2θ varies through a range of 360° (or as θ varies through a range of 180°). If a second maximum exists, then the only possibility is that I_{x_1} remains constant, which means that every pair of axes is a set of principal axes and all moments of inertia are the same.

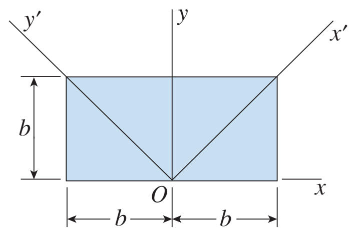


FIG. 10-25 Rectangle for which every axis (in the plane of the area) through point O is a principal axis

A point so located that every axis through the point is a principal axis, and hence the moments of inertia are the same for all axes through the point, is called a **principal point**.

An illustration of this situation is the rectangle of width $2b$ and height b shown in Fig. 10-25. The xy axes, with origin at point O, are principal axes of the rectangle because the y axis is an axis of symmetry. The $x'y'$ axes, with the same origin, are also principal axes because the product of inertia $I_{x'y'}$ equals zero (because the triangles are symmetrically located with respect to the x' and y' axes). It follows that every pair of axes through O is a set of principal axes and every moment of inertia is the same (and equal to $2b^4/3$). Therefore, point O is a principal point for the rectangle. (A second principal point is located where the y axis intersects the upper side of the rectangle.)

A useful corollary of the concepts described in the preceding four paragraphs applies to axes through the centroid of an area. Consider an area having *two different pairs* of centroidal axes such that at least one axis in each pair is an axis of symmetry. In other words, there exist two different axes of symmetry that are not perpendicular to each other. Then it follows that the centroid is a principal point.

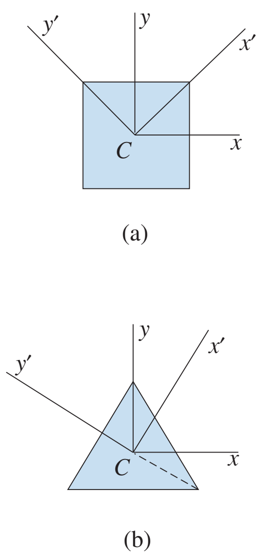


FIG. 10-26 Examples of areas for which every centroidal axis is a principal axis and the centroid C is a principal point

Two examples, a square and an equilateral triangle, are shown in Fig. 10-26. In each case the xy axes are principal centroidal axes because their origin is at the centroid C and at least one of the two axes is an axis of symmetry. In addition, a second pair of centroidal axes (the $x'y'$ axes) has at least one axis of symmetry. It follows that both the xy and $x'y'$ axes are principal axes. Therefore, every axis through the centroid C is a principal axis, and every such axis has the same moment of inertia.

If an area has *three different axes of symmetry*, even if two of them are perpendicular, the conditions described in the preceding paragraph are automatically fulfilled. Therefore, if an area has three or more axes of symmetry, the centroid is a principal point and every axis through the centroid is a principal axis and has the same moment of inertia. These conditions are fulfilled for a circle, for all regular polygons (equilateral triangle, square, regular pentagon, regular hexagon, and so on), and for many other symmetric shapes.

In general, every plane area has two principal points. These points lie equidistant from the centroid on the principal centroidal axis having the larger principal moment of inertia. A special case occurs when the two principal centroidal moments of inertia are equal; then the two principal points merge at the centroid, which becomes the sole principal point.

Principal Moments of Inertia

Let us now determine the principal moments of inertia, assuming that I_x, I_y, and I_{xy} are known. One method is to determine the two values of θ_p (differing by 90°) from Eq. (10-30) and then substitute these values into Eq. (10-25) for I_{x_1}. The resulting two values are the principal moments of inertia, denoted by I_1 and I_2. The advantage of this method is that we know which of the two principal angles θ_p corresponds to each principal moment of inertia.

It is also possible to obtain general formulas for the principal moments of inertia. We note from Eq. (10-30) and Fig. 10-27 (which is a geometric representation of Eq. 10-30) that

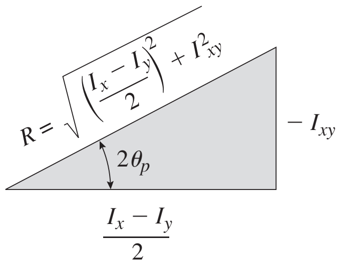


FIG. 10-27 Geometric representation of Eq. (10-30)

$$\cos 2\theta_p = \frac{I_x - I_y}{2R} \qquad \sin 2\theta_p = \frac{-I_{xy}}{R} \tag{10-31a,b}$$

in which

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{10-32}$$

is the hypotenuse of the triangle. When evaluating R, we always take the positive square root.

Now we substitute the expressions for $\cos 2\theta_p$ and $\sin 2\theta_p$ (from Eqs. 10-31a and b) into Eq. (10-25) for I_{x_1} and obtain the algebraically larger of the two principal moments of inertia, denoted by the symbol I_1:

$$I_1 = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{10-33a}$$

The smaller principal moment of inertia, denoted as I_2, may be obtained from the equation

$$I_1 + I_2 = I_x + I_y$$

(see Eq. 10-29). Substituting the expression for I_1 into this equation and solving for I_2, we get

$$I_2 = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{10-33b}$$

Equations (10-33a) and (10-33b) provide a convenient way to calculate the principal moments of inertia.

The following example illustrates the method for locating the principal axes and determining the principal moments of inertia.

Example 10-7

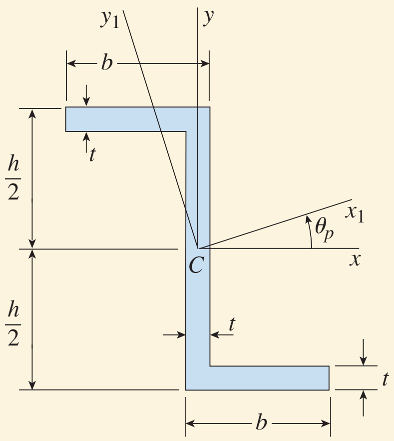


FIG. 10-28 Example 10-7. Principal axes and principal moments of inertia for a Z-section

Determine the orientations of the principal centroidal axes and the magnitudes of the principal centroidal moments of inertia for the cross-sectional area of the Z-section shown in Fig. 10-28. Use the following numerical data: height $h = 200$ mm, width $b = 90$ mm, and constant thickness $t = 15$ mm.

Solution

Let us use the xy axes (Fig. 10-28) as the reference axes through the centroid C. The moments and product of inertia with respect to these axes can be obtained by dividing the area into three rectangles and using the parallel-axis theorems. The results of such calculations are as follows:

$$I_x = 29.29 \times 10^6 \text{ mm}^4 \qquad I_y = 5.667 \times 10^6 \text{ mm}^4 \qquad I_{xy} = -9.366 \times 10^6 \text{ mm}^4$$

Substituting these values into the equation for the angle θ_p (Eq. 10-30), we get

$$\tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = 0.7930 \qquad 2\theta_p = 38.4° \text{ and } 218.4°$$

Thus, the two values of θ_p are

$$\theta_p = 19.2° \text{ and } 109.2°$$

Using these values of θ_p in the transformation equation for I_{x_1} (Eq. 10-25), we find $I_{x_1} = 32.6 \times 10^6$ mm^4 and 2.4×10^6 mm^4, respectively. These same values are obtained if we substitute into Eqs. (10-33a) and (10-33b). Thus, the principal moments of inertia and the angles to the corresponding principal axes are

$$I_1 = 32.6 \times 10^6 \text{ mm}^4 \qquad \theta_{p_1} = 19.2°$$

$$I_2 = 2.4 \times 10^6 \text{ mm}^4 \qquad \theta_{p_2} = 109.2°$$

The principal axes are shown in Fig. 10-28 as the x_1y_1 axes.

PROBLEMS CHAPTER 10

Centroids of Areas

The problems for Section 10.2 are to be solved by integration.

10.2-1 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a right triangle having base b and altitude h (see Case 6, Appendix D, available online).

10.2-2 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D, available online).

10.2-3 Determine the distance $\bar{y}$ to the centroid C of a semicircle of radius r (see Case 10, Appendix D, available online).

10.2-4 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a parabolic spandrel of base b and height h (see Case 18, Appendix D, available online).

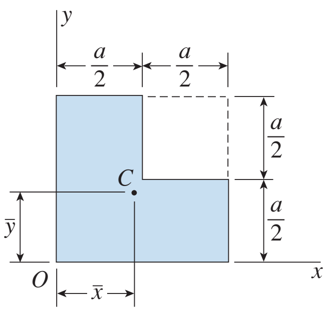
10.2-5 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a semisegment of nth degree having base b and height h (see Case 19, Appendix D, available online).

Centroids of Composite Areas

The problems for Section 10.3 are to be solved by using the formulas for composite areas.

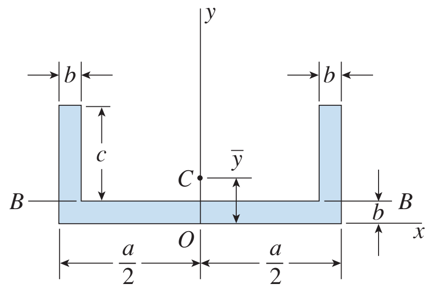
10.3-1 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix D, available online) by dividing the trapezoid into two triangles.

10.3-2 One quarter of a square of side a is removed (see figure). What are the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the remaining area?



PROBS. 10.3-2 and 10.5-2

10.3-3 Calculate the distance $\bar{y}$ to the centroid C of the channel section shown in the figure if $a = 150$ mm, $b = 25$ mm, and $c = 50$ mm.

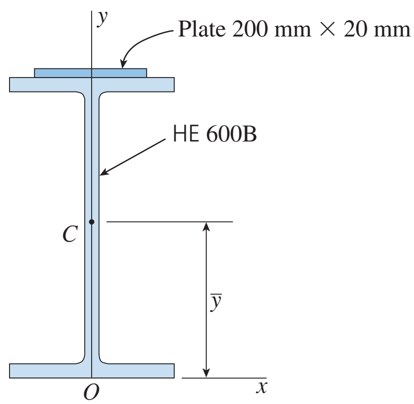


PROBS. 10.3-3, 10.3-4, and 10.5-3

10.3-4 What must be the relationship between the dimensions a, b, and c of the channel section shown in the figure in order that the centroid C will lie on line BB?

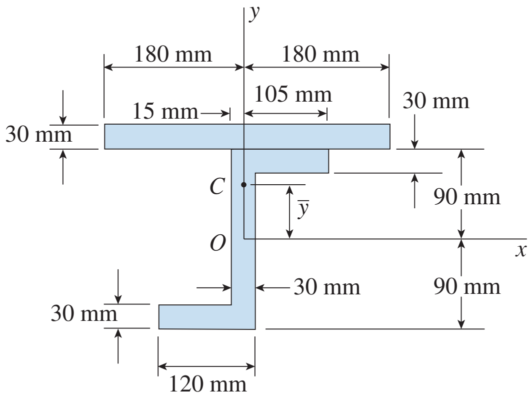
10.3-5 The cross section of a beam constructed of a HE 600 B wide-flange section with an 200 mm × 20 mm cover plate welded to the top flange is shown in the figure.

Determine the distance $\bar{y}$ from the base of the beam to the centroid C of the cross-sectional area.



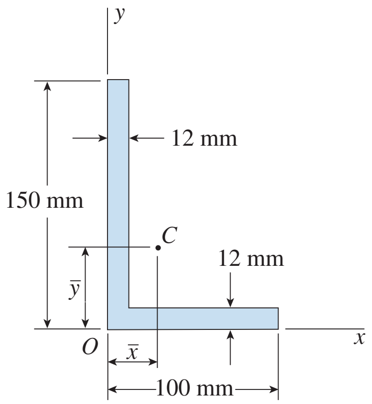
PROBS. 10.3-5 and 10.5-5

10.3-6 Determine the distance $\bar{y}$ to the centroid C of the composite area shown in the figure.



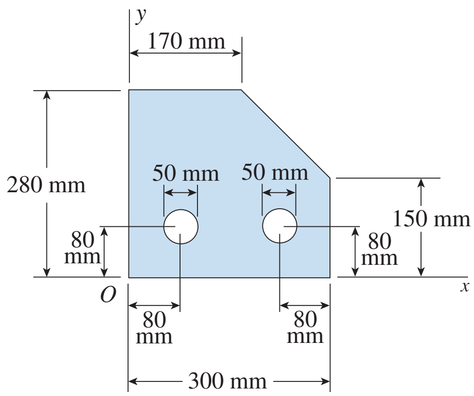
PROBS. 10.3-6, 10.5-6, and 10.7-6

10.3-7 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the L-shaped area shown in the figure.



PROBS. 10.3-7, 10.4-7, 10.5-7, and 10.7-7

10.3-8 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the area shown in the figure.



PROB. 10.3-8

Moments of Inertia

Problems 10.4-1 through 10.4-4 are to be solved by integration.

10.4-1 Determine the moment of inertia I_x of a triangle of base b and altitude h with respect to its base (see Case 4, Appendix D, available online).

10.4-2 Determine the moment of inertia I_{BB} of a trapezoid having bases a and b and altitude h with respect to its base (see Case 8, Appendix D, available online).

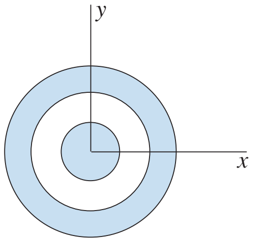
10.4-3 Determine the moment of inertia I_x of a parabolic spandrel of base b and height h with respect to its base (see Case 18, Appendix D, available online).

10.4-4 Determine the moment of inertia I_x of a circle of radius r with respect to a diameter (see Case 9, Appendix D, available online).

Problems 10.4-5 through 10.4-9 are to be solved by considering the area to be a composite area.

10.4-5 Determine the moment of inertia I_{BB} of a rectangle having sides of lengths b and h with respect to a diagonal of the rectangle (see Case 2, Appendix D, available online).

10.4-6 Calculate the moment of inertia I_x for the composite circular area shown in the figure. The origin of the axes is at the center of the concentric circles, and the three diameters are 20, 40, and 60 mm.

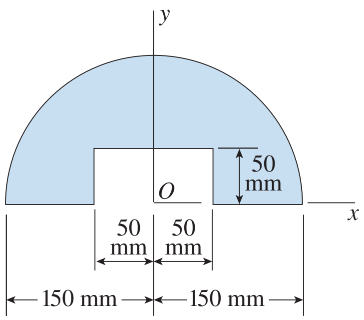


PROB. 10.4-6

10.4-7 Calculate the moments of inertia I_x and I_y with respect to the x and y axes for the L-shaped area shown in the figure for Prob. 10.3-7.

10.4-8 A semicircular area of radius 150 mm has a rectangular cutout of dimensions 50 mm × 100 mm (see figure).

Calculate the moments of inertia I_x and I_y with respect to the x and y axes. Also, calculate the corresponding radii of gyration r_x and r_y.



PROB. 10.4-8

10.4-9 Calculate the moments of inertia I_1 and I_2 of a HE 450 A wide-flange section using the cross-sectional dimensions given in Table E-l, Appendix E, available online. (Disregard the cross-sectional areas of the fillets.) Also, calculate the corresponding radii of gyration r_1 and r_2, respectively.

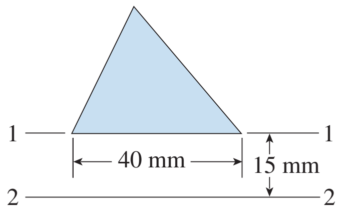
Parallel-Axis Theorem

10.5-1 Calculate the moment of inertia I_b of a HE 320 B wide-flange section with respect to its base. (Use data from Table E-l, Appendix E, available online.)

10.5-2 Determine the moment of inertia I_c with respect to an axis through the centroid C and parallel to the x axis for the geometric figure described in Prob. 10.3-2.

10.5-3 For the channel section described in Prob. 10.3-3, calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis.

10.5-4 The moment of inertia with respect to axis 1-1 of the scalene triangle shown in the figure is 90×10^3 mm^4. Calculate its moment of inertia I_2 with respect to axis 2-2.



PROB. 10.5-4

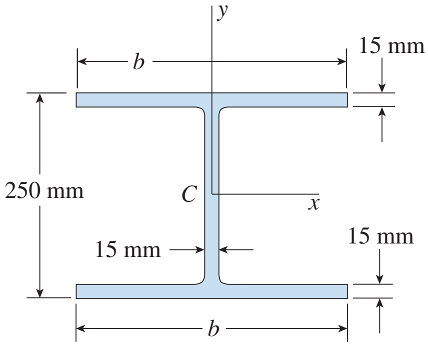
10.5-5 For the beam cross section described in Prob. 10.3-5, calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C such that the x_c axis is parallel to the x axis and the y_c axis coincides with the y axis.

10.5-6 Calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis for the composite area shown in the figure for Prob. 10.3-6.

10.5-7 Calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C and parallel to the x and y axes, respectively, for the L-shaped area shown in the figure for Prob. 10.3-7.

10.5-8 The wide-flange beam section shown in the figure has a total height of 250 mm and a constant thickness of 15 mm.

Determine the flange width b if it is required that the centroidal moments of inertia I_x and I_y be in the ratio 3 to 1, respectively.



PROB. 10.5-8

Polar Moments of Inertia

10.6-1 Determine the polar moment of inertia I_P of an isosceles triangle of base b and altitude h with respect to its apex (see Case 5, Appendix D, available online).

10.6-2 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a circular sector (see Case 13, Appendix D, available online).

10.6-3 Determine the polar moment of inertia I_P for a HE 220 B wide-flange section with respect to one of its outermost corners.

10.6-4 Obtain a formula for the polar moment of inertia I_P with respect to the midpoint of the hypotenuse for a right triangle of base b and height h (see Case 6, Appendix D, available online).

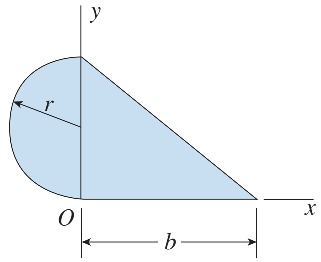
10.6-5 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a quarter-circular spandrel (see Case 12, Appendix D, available online).

Products of Inertia

10.7-1 Using integration, determine the product of inertia I_{xy} for the parabolic semisegment shown in Fig. 10-5 (see also Case 17 in Appendix D, available online).

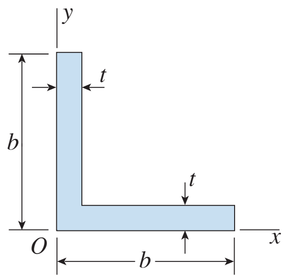
10.7-2 Using integration, determine the product of inertia I_{xy} for the quarter-circular spandrel shown in Case 12, Appendix D, available online.

10.7-3 Find the relationship between the radius r and the distance b for the composite area shown in the figure in order that the product of inertia I_{xy} will be zero.



PROB. 10.7-3

10.7-4 Obtain a formula for the product of inertia I_{xy} of the symmetrical L-shaped area shown in the figure.



PROB. 10.7-4

10.7-5 Calculate the product of inertia I_{12} with respect to the centroidal axes 1-1 and 2-2 for an L 150 × 150 × 15 mm angle section (see Table E-4, Appendix E, available online). (Disregard the cross-sectional areas of the fillet and rounded corners.)

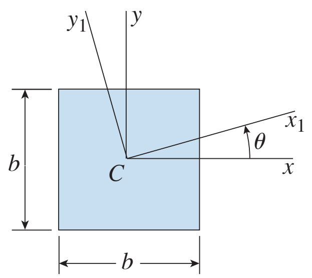
10.7-6 Calculate the product of inertia I_{xy} for the composite area shown in Prob. 10.3-6.

10.7-7 Determine the product of inertia $I_{x_c y_c}$ with respect to centroidal axes x_c and y_c parallel to the x and y axes, respectively, for the L-shaped area shown in Prob. 10.3-7.

Rotation of Axes

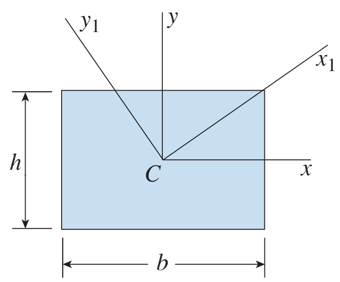
The problems for Section 10.8 are to be solved by using the transformation equations for moments and products of inertia.

10.8-1 Determine the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ for a square with sides b, as shown in the figure. (Note that the x_1y_1 axes are centroidal axes rotated through an angle θ with respect to the xy axes.)



PROB. 10.8-1

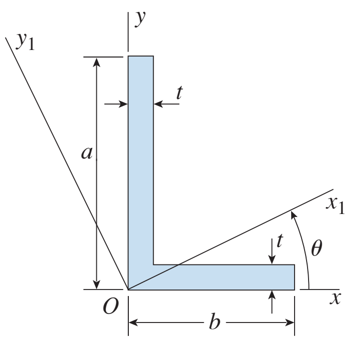
10.8-2 Determine the moments and product of inertia with respect to the x_1y_1 axes for the rectangle shown in the figure. (Note that the x_1 axis is a diagonal of the rectangle.)



PROB. 10.8-2

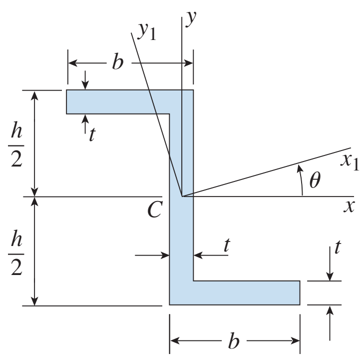
10.8-3 Calculate the moment of inertia I_d for a HE 320 A wide-flange section with respect to a diagonal passing through the centroid and two outside corners of the flanges. (Use the dimensions and properties given in Table E-1.)

10.8-4 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the L-shaped area shown in the figure if a = 150 mm, b = 100 mm, t = 15 mm, and θ = 30°.



PROBS. 10.8-4 and 10.9-4

10.8-5 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the Z-section shown in the figure if b = 75 mm, h = 100 mm, t = 12 mm, and θ = 60°.



PROBS. 10.8-5, 10.8-6, 10.9-5, and 10.9-6

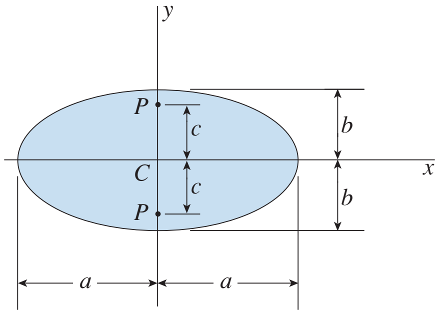
10.8-6 Solve the preceding problem if b = 80 mm, h = 120 mm, t = 12 mm, and θ = 30°.

Principal Axes, Principal Points, and Principal Moments of Inertia

10.9-1 An ellipse with major axis of length $2a$ and minor axis of length $2b$ is shown in the figure.

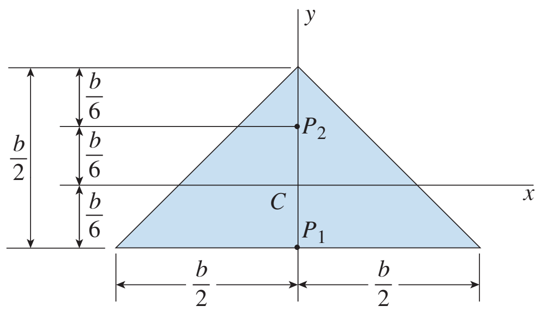
(a) Determine the distance c from the centroid C of the ellipse to the principal points P on the minor axis (y axis).

(b) For what ratio a/b do the principal points lie on the circumference of the ellipse? (c) For what ratios do they lie inside the ellipse?



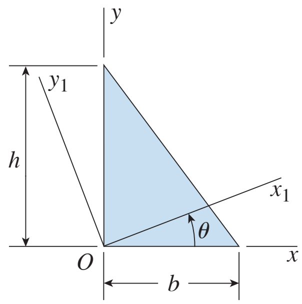
PROB. 10.9-1

10.9-2 Demonstrate that the two points P_1 and P_2, located as shown in the figure, are the principal points of the isosceles right triangle.



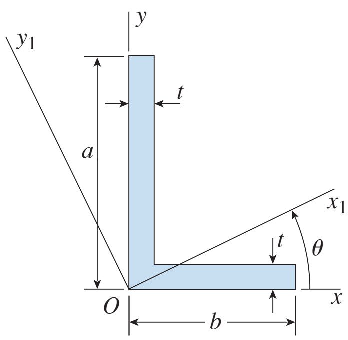
PROB. 10.9-2

10.9-3 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O for the right triangle shown in the figure if $b = 150$ mm and $h = 200$ mm. Also, calculate the corresponding principal moments of inertia I_1 and I_2.



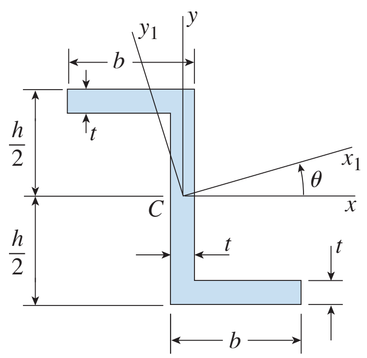
PROB. 10.9-3

10.9-4 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area described in Prob. 10.8-4 ($a = 150$ mm, $b = 100$ mm, and $t = 15$ mm).



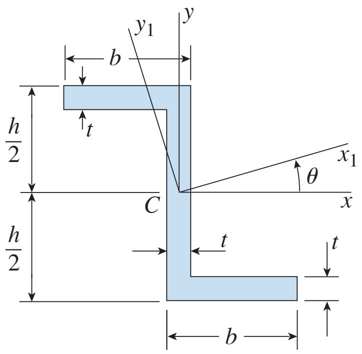
PROBS. 10.8-4 and 10.9-4

10.9-5 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C and the corresponding principal centroidal moments of inertia I_1 and I_2 for the Z-section described in Prob. 10.8-5 ($b = 75$ mm, $h = 100$ mm, and $t = 12$ mm).



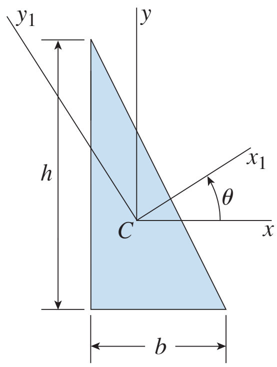
PROBS. 10.8-5, 10.8-6, 10.9-5, and 10.9-6

10.9-6 Solve the preceding problem for the Z-section described in Prob. 10.8-6 (b = 80 mm, h = 120 mm, and t = 12 mm).



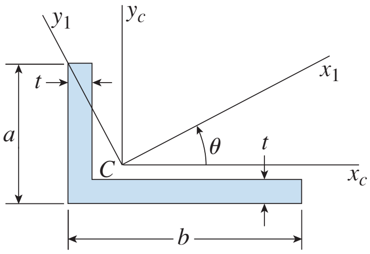
PROBS. 10.8-5, 10.8-6, 10.9-5, and 10.9-6

10.9-7 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C for the right triangle shown in the figure if $h = 2b$. Also, determine the corresponding principal centroidal moments of inertia I_1 and I_2.



PROB. 10.9-7

10.9-8 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal centroidal axes and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area shown in the figure if a = 80 mm, b = 150 mm, and t = 16 mm.



PROBS. 10.9-8 and 10.9-9

10.9-9 Solve the preceding problem if a = 75 mm, b = 150 mm, and t = 12 mm.