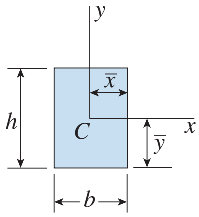
D

Properties of Plane Areas

Notation: A = area
$\bar{x}, \bar{y}$ = distances to centroid C
I_x, I_y = moments of inertia with respect to the x and y axes, respectively
I_{xy} = product of inertia with respect to the x and y axes
$I_P = I_x + I_y$ = polar moment of inertia with respect to the origin of the x and y axes
I_{BB} = moment of inertia with respect to axis B-B

1



Rectangle (Origin of axes at centroid)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0 \qquad I_P = \frac{bh}{12}(h^2 + b^2)$$

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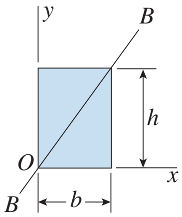

Rectangle (Origin of axes at corner)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3} \qquad I_{xy} = \frac{b^2h^2}{4} \qquad I_P = \frac{bh}{3}(h^2 + b^2)$$

$$I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

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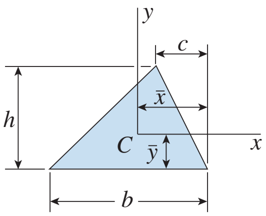
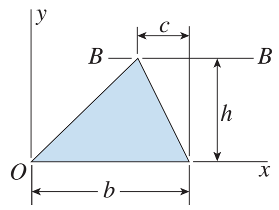


Triangle (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

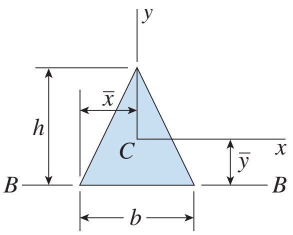
$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_P = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

4 

Triangle (Origin of axes at vertex)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

5 

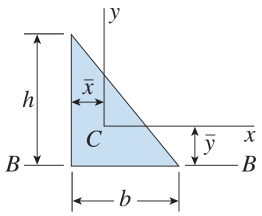
Isosceles triangle (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_P = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(*Note:* For an equilateral triangle, $h = \sqrt{3}\, b/2$.)

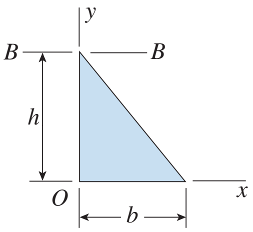
6 

Right triangle (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

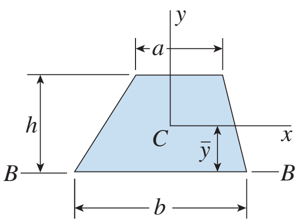
$$I_P = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

7 

Right triangle (Origin of axes at vertex)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

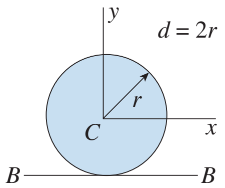
$$I_P = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

8 

Trapezoid (Origin of axes at centroid)

$$A = \frac{h(a + b)}{2} \qquad \bar{y} = \frac{h(2a + b)}{3(a + b)}$$

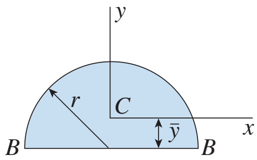
$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

9 

Circle (Origin of axes at center)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

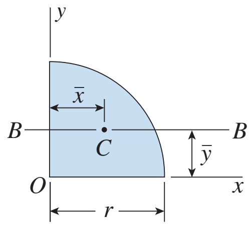
$$I_{xy} = 0 \qquad I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

10 

Semicircle (Origin of axes at centroid)

$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

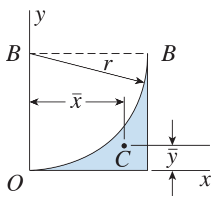
$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_y = \frac{\pi r^4}{8} \qquad I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

11 

Quarter circle (Origin of axes at center of circle)

$$A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$$

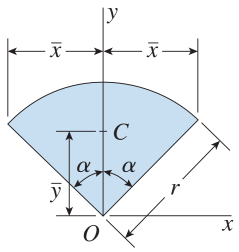
$$I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8} \qquad I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488r^4$$

12 

Quarter-circular spandrel (Origin of axes at point of tangency)

$$A = \left(1 - \frac{\pi}{4}\right)r^2 \qquad \bar{x} = \frac{2r}{3(4 - \pi)} \approx 0.7766r \qquad \bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)} \approx 0.2234r$$

$$I_x = \left(1 - \frac{5\pi}{16}\right)r^4 \approx 0.01825r^4 \qquad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right)r^4 \approx 0.1370r^4$$

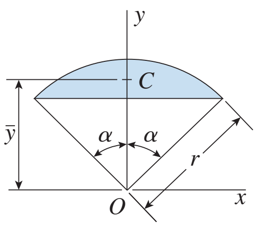
13 

Circular sector (Origin of axes at center of circle)

α = angle in radians $(\alpha \le \pi/2)$

$$A = \alpha r^2 \qquad \bar{x} = r \sin \alpha \qquad \bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

$$I_x = \frac{r^4}{4}(\alpha + \sin \alpha \cos \alpha) \qquad I_y = \frac{r^4}{4}(\alpha - \sin \alpha \cos \alpha) \qquad I_{xy} = 0 \qquad I_P = \frac{\alpha r^4}{2}$$

14 

Circular segment (Origin of axes at center of circle)

α = angle in radians $(\alpha \le \pi/2)$

$$A = r^2(\alpha - \sin \alpha \cos \alpha) \qquad \bar{y} = \frac{2r}{3}\left(\frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha}\right)$$

$$I_x = \frac{r^4}{4}(\alpha - \sin \alpha \cos \alpha + 2 \sin^3 \alpha \cos \alpha) \qquad I_{xy} = 0$$

$$I_y = \frac{r^4}{12}(3\alpha - 3 \sin \alpha \cos \alpha - 2 \sin^3 \alpha \cos \alpha)$$

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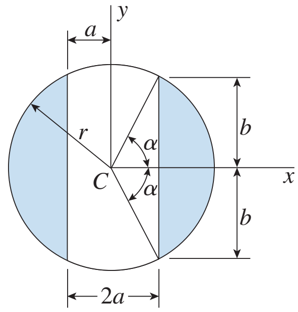

Circle with core removed (Origin of axes at center of circle)

α = angle in radians $\qquad (\alpha \leq \pi/2)$

$$\alpha = \arccos\frac{a}{r} \qquad b = \sqrt{r^2 - a^2} \qquad A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$$

$$I_x = \frac{r^4}{6}\left(3\alpha - \frac{3ab}{r^2} - \frac{2ab^3}{r^4}\right) \qquad I_y = \frac{r^4}{2}\left(\alpha - \frac{ab}{r^2} + \frac{2ab^3}{r^4}\right) \qquad I_{xy} = 0$$

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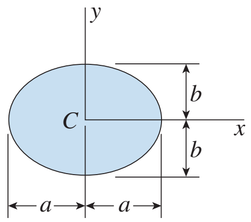

Ellipse (Origin of axes at centroid)

$$A = \pi ab \qquad I_x = \frac{\pi ab^3}{4} \qquad I_y = \frac{\pi ba^3}{4}$$

$$I_{xy} = 0 \qquad I_P = \frac{\pi ab}{4}(b^2 + a^2)$$

$$\text{Circumference} \approx \pi[1.5(a + b) - \sqrt{ab}\,] \qquad (a/3 \leq b \leq a)$$

$$\approx 4.17b^2/a + 4a \qquad (0 \leq b \leq a/3)$$

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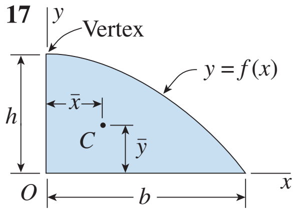

Parabolic semisegment (Origin of axes at corner)

$$y = f(x) = h\left(1 - \frac{x^2}{b^2}\right)$$

$$A = \frac{2bh}{3} \qquad \bar{x} = \frac{3b}{8} \qquad \bar{y} = \frac{2h}{5}$$

$$I_x = \frac{16bh^3}{105} \qquad I_y = \frac{2hb^3}{15} \qquad I_{xy} = \frac{b^2h^2}{12}$$

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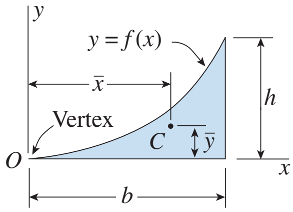


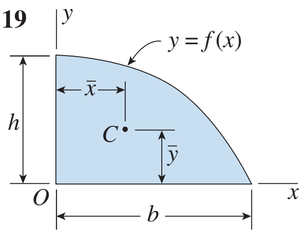
Parabolic spandrel (Origin of axes at vertex)

$$y = f(x) = \frac{hx^2}{b^2}$$

$$A = \frac{bh}{3} \qquad \bar{x} = \frac{3b}{4} \qquad \bar{y} = \frac{3h}{10}$$

$$I_x = \frac{bh^3}{21} \qquad I_y = \frac{hb^3}{5} \qquad I_{xy} = \frac{b^2h^2}{12}$$

19



Semisegment of *n*th degree (Origin of axes at corner)

$$y = f(x) = h\left(1 - \frac{x^n}{b^n}\right) \qquad (n > 0)$$

$$A = bh\left(\frac{n}{n + 1}\right) \qquad \bar{x} = \frac{b(n + 1)}{2(n + 2)} \qquad \bar{y} = \frac{hn}{2n + 1}$$

$$I_x = \frac{2bh^3n^3}{(n + 1)(2n + 1)(3n + 1)} \qquad I_y = \frac{hb^3n}{3(n + 3)} \qquad I_{xy} = \frac{b^2h^2n^2}{4(n + 1)(n + 2)}$$

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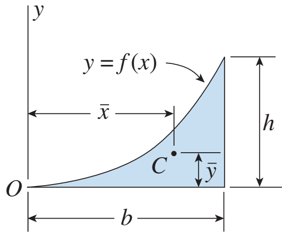


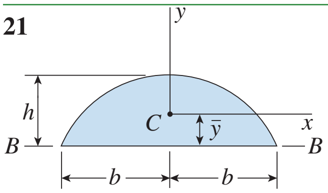
Spandrel of *n*th degree (Origin of axes at point of tangency)

$$y = f(x) = \frac{hx^n}{b^n} \qquad (n > 0)$$

$$A = \frac{bh}{n+1} \qquad \bar{x} = \frac{b(n+1)}{n+2} \qquad \bar{y} = \frac{h(n+1)}{2(2n+1)}$$

$$I_x = \frac{bh^3}{3(3n+1)} \qquad I_y = \frac{hb^3}{n+3} \qquad I_{xy} = \frac{b^2h^2}{4(n+1)}$$

21



Sine wave (Origin of axes at centroid)

$$A = \frac{4bh}{\pi} \qquad \bar{y} = \frac{\pi h}{8}$$

$$I_x = \left(\frac{8}{9\pi} - \frac{\pi}{16}\right)bh^3 \approx 0.08659bh^3 \qquad I_y = \left(\frac{4}{\pi} - \frac{32}{\pi^3}\right)hb^3 \approx 0.2412hb^3$$

$$I_{xy} = 0 \qquad I_{BB} = \frac{8bh^3}{9\pi}$$

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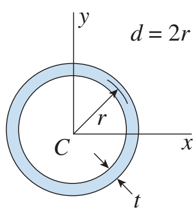

Thin circular ring (Origin of axes at center)
Approximate formulas for case when t is small

$$A = 2\pi rt = \pi dt \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$

$$I_{xy} = 0 \qquad I_P = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

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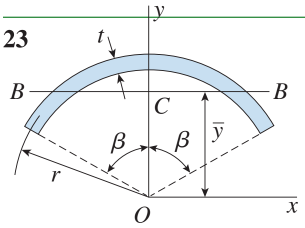

Thin circular arc (Origin of axes at center of circle)
Approximate formulas for case when t is small

β = angle in radians (*Note:* For a semicircular arc, $\beta = \pi/2$.)

$$A = 2\beta rt \qquad \bar{y} = \frac{r \sin \beta}{\beta}$$

$$I_x = r^3 t(\beta + \sin\beta\cos\beta) \qquad I_y = r^3 t(\beta - \sin\beta\cos\beta)$$

$$I_{xy} = 0 \qquad I_{BB} = r^3 t\left(\frac{2\beta + \sin 2\beta}{2} - \frac{1 - \cos 2\beta}{\beta}\right)$$

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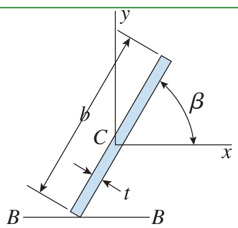

Thin rectangle (Origin of axes at centroid)
Approximate formulas for case when t is small

$$A = bt$$

$$I_x = \frac{tb^3}{12}\sin^2\beta \qquad I_y = \frac{tb^3}{12}\cos^2\beta \qquad I_{BB} = \frac{tb^3}{3}\sin^2\beta$$

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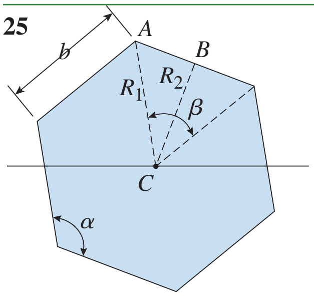

Regular polygon with *n* sides (Origin of axes at centroid)

C = centroid (at center of polygon)

n = number of sides ($n \geq 3$) b = length of a side

β = central angle for a side α = interior angle (or vertex angle)

$$\beta = \frac{360°}{n} \qquad \alpha = \left(\frac{n-2}{n}\right)180° \qquad \alpha + \beta = 180°$$

R_1 = radius of circumscribed circle (line CA) R_2 = radius of inscribed circle (line CB)

$$R_1 = \frac{b}{2}\csc\frac{\beta}{2} \qquad R_2 = \frac{b}{2}\cot\frac{\beta}{2} \qquad A = \frac{nb^2}{4}\cot\frac{\beta}{2}$$

I_c = moment of inertia about any axis through C (the centroid C is a principal point and every axis through C is a principal axis)

$$I_c = \frac{nb^4}{192}\left(\cot\frac{\beta}{2}\right)\left(3\cot^2\frac{\beta}{2} + 1\right) \qquad I_P = 2I_c$$