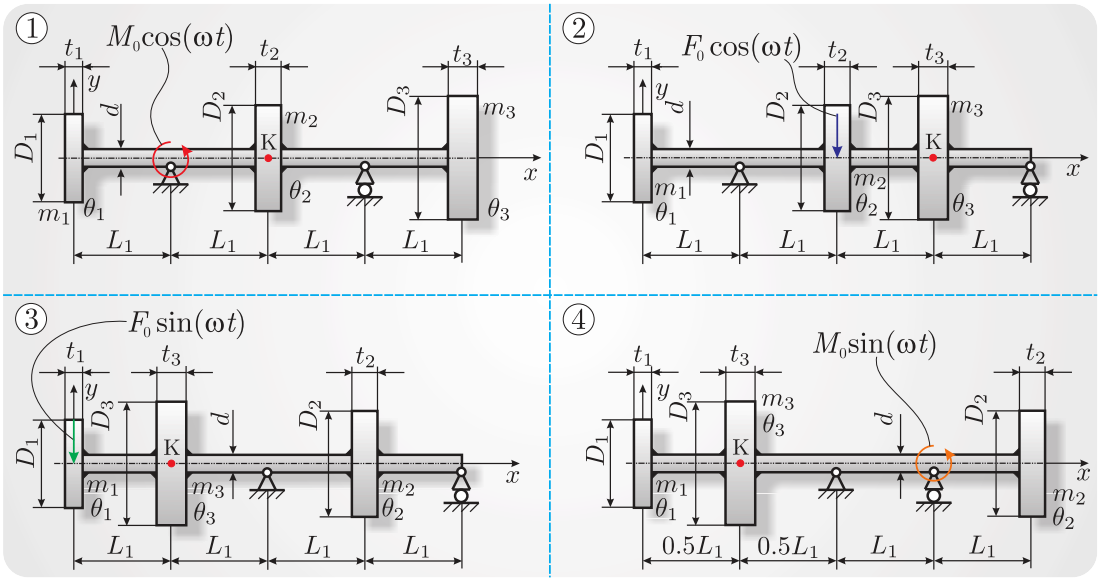
BME Faculty of Mechanical Engineering Department of Applied Mechanics	FINITE ELEMENT ANALYSIS 3rd **Homework**	Name: Gergő Kelle
MSc Program SOLID MECHANICS specialization		Neptun ID: GBBNUL

Solution of an externally excited shaft with disks by direct time integration

Total: (20) p.

Code ① (figure)	Code ②	Code ③
1	2	1

The shafts above are subjected to three rigid disks. The material of the shafts is linear elastic, homogeneous and isotropic. The disks are rigid and have a d diameter center bore. ***We consider only the bending vibration of the structure!*** Data: $\omega = \text{const}$, $E = 210$ GPa, $\varrho = 7860$ kg/m^3, $D_2 = 0.8D_1$, $D_3 = 1.3D_1$, $t_1 = D_1/10$, $t_2 = D_2/15$, $t_3 = D_3/12$.



Data table

	Code ②			Code ③		
	L_1 [m]	D_1 [mm]	ω [rad/s]	d [mm]	F_0 [N], M_0 [Nm]	Scheme
1	0.3	300	420	60	100, 10	Constant acceleration
2	0.28	350	370	52	80, 8	Wilson-θ
3	0.38	200	380	48	90, 7	Fox-Goodwin
4	0.45	250	295	65	110, 7	Constant acceleration
5	0.4	400	455	45	95, 12	Wilson-θ

Tasks:

1. *Create the FE model of the structure in MAXIMA using BEAM1D elements and rigid disks using the minimally required number of elements! Determine the structural matrices of the structure and calculate the first four eigenfrequencies of the system!* (7) p.
2. *Apply the Rayleigh-quotient, determine the first eigenfrequency approximately and verify the finite element model!* (4) p.
3. *Determine the response of the system by direct time integration! Determine the required time step size! Apply the assigned method and plot the transverse displacement and the angle of rotation at cross section K! Apply 2500-3000 time steps!* (9) p.
4. *Apply the freeware MAXIMA code for the calculations, refer to the MAXIMA 7.1-7.3 sheets! Create a detailed report on the results in PDF! Submit Your work as "**surname_firstname_HW3.pdf**"!*

Budapest University of Technology and Economics
Faculty of Mechanical Engineering

Finite Element Analysis
III. Homework

Gergő Kelle

May 19, 2023

Contents

1 Equation of motion on a multi-DoF system

Create the *FE* model of the structure in *MAXIMA* using *BEAM1D* elements and rigid disks using the minimally required number of elements! Determine the structural matrices of the structure and calculate the first four eigenfrequencies of the system!

1.1 Parameters and discretization

Table 1: Parameters for Homework ID 121

Parameter	Value	Dimension
L_1	0.28	m
D_1	0.35	m
ω	370	$\frac{\text{rad}}{\text{s}}$
d	0.06	m
F_0	100	N
M_0	10	Nm
E	210	GPa
ρ	7860	$\frac{\text{kg}}{\text{m}^3}$
Scheme:	Constant acceleration	

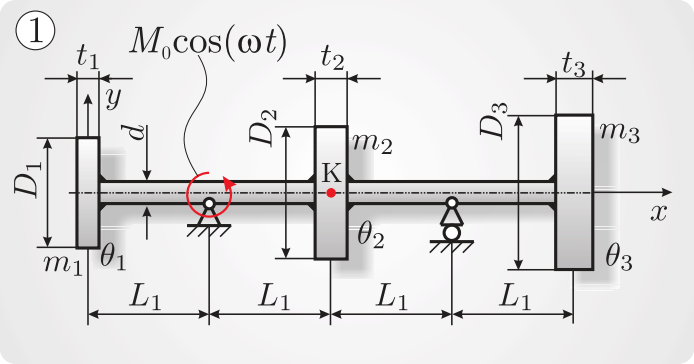


Figure 1: Your Figure Caption

Before any further calculations, some base parameters need to be determined.

$$D_2 = 0.8\ D_1 = 0.28\ [\text{m}] \tag{1}$$

$$D_3 = 1.3\ D_1 = 0.455\ [\text{m}] \tag{2}$$

$$t_1 = \frac{D_1}{10} = 0.035\ [\text{m}] \tag{3}$$

$$t_2 = \frac{D_1}{15} = 0.0187\ [\text{m}] \tag{4}$$

$$t_3 = \frac{D_1}{12} = 0.0379\ [\text{m}] \tag{5}$$

Based on Table **??** values and the determined parameters above the disk's mass and the moment of inertia can be determined.

$$m_1 = \pi\ \rho\ t_1 \frac{(D_1^2 - d^2)}{4} = 25.67\ [\text{kg}] \tag{6}$$

$$m_2 = \pi\ \rho\ t_2 \frac{(D_2^2 - d^2)}{4} = 8.62\ [\text{kg}] \tag{7}$$

$$m_3 = \pi\ \rho\ t_3 \frac{(D_3^2 - d^2)}{4} = 47.62\ [\text{kg}] \tag{8}$$

$$\theta_1 = \frac{1}{4}m_1\left(\frac{D_1^2}{4}+\frac{d^2}{4}\right)+\frac{1}{12}m_1\ t_1^2 = 0.2051\ [\text{kgm}]^2 \tag{9}$$

$$\theta_2 = \frac{1}{4}m_2\left(\frac{D_2^2}{4}+\frac{d^2}{4}\right)+\frac{1}{12}m_2\ t_2^2 = 0.0444\ [\text{kgm}]^2 \tag{10}$$

$$\theta_3 = \frac{1}{4}m_3\left(\frac{D_3^2}{4}+\frac{d^2}{4}\right)+\frac{1}{12}m_2\ t_3^2 = 0.6325\ [\text{kgm}]^2 \tag{11}$$

For the mass and stiffness matrices the area and second moment of inertia has to be determined.

$$A = \frac{d^2\pi}{4} = 0.0028274\ [\text{m}]^2 \tag{12}$$

$$I_z = \frac{d^4\pi}{64} = 6.3617251\cdot 10^{-7}\ [\text{m}]^4 \tag{13}$$

Due to the circular shape of the beam's cross section remaining constant along the X-axis, the area and second moment of inertia remains consistent throughout the beam. The information about discretization showed in Table **??**.

Table 2: Discretization

Node	x [mm]	y [mm]
1	0	0
2	280	0
3	560	0
4	840	0
5	1120	0

(a) Node coordinates

Element	Local node 1	Local node 2
1	1	2
2	2	3
3	3	4
4	4	5

(b) Element connectivity

1.2 Determination of the first four eigenfrequencies

Table 3: Main equations

Parameter	**Equation**	
Consistent mass matrix	$\mathbf{M}_e = \frac{Al\rho}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$	(14)
Material stiffness matrix	$\mathbf{K}_e = \frac{I_z E l^3}{12}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$	(15)

The stiffness and mass matrices of all four discretized elements are identical due to the beam's uniform cross-section and isotropic material properties. Thus the global matrices are obtained by summing the identical mass or stiffness matrices with regard to it's DoF. This way we get 10×10 matrices that has to be condensed based on the boundary conditions of $v_2 = 0$ and $v_4 = 0$ resulting the ellimination of the 3rd and 7th rows and columns of the structural matrices. Condensed matrices will be 8×8 matrices.
Another thing to point out is the mass matrix for the whole structure, because we have to consider the disk. Thus before the structural mass condensation, we should substitute the beam's parameters into $\mathbf{M}$ then calculate the global structural mass matrix of the beam $\mathbf{M}_{beam}$ and add the disk's mass matrix Mdisk leaving us with the following global mass matrix:

$$\mathbf{M} = \mathbf{M}_{beam} + \mathbf{M}_{disk} \tag{16}$$

Where $M_{disk} =$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_3 \end{bmatrix}$$

After the condensation we can write out the equation of motion of the system as

$$\mathbf{M}_c \, \ddot{\mathbf{u}}_c + \mathbf{K}_c \, \mathbf{u}_c = \mathbf{F}_c \tag{17}$$

To calculate the natural angular freqency we can use the following formula

$$\det\left(-\omega_n^2 \mathbf{M}_c + \mathbf{K}_c\right) = 0. \tag{18}$$

By calculating the first four solution for ω_n we get the following results:

Table 4: Eigenfreqencies

i	$\omega_{n,i}\left[\frac{\text{rad}}{\text{s}}\right]$	$f_{n,i}$ [Hz]
1	300.08	47.76
2	484.22	77.07
3	1669.45	265.70
4	2429.42	386.65

Where $f_n = \dfrac{\omega_n}{2\pi}$

2 The first eigenfrequency using Rayleigh's Principle

Apply the Rayleigh-quotient, determine the first eigenfrequency approximately and verify the finite element model!

To determine the first eigenfrequency of the beam using the Rayleigh-quotient method, we begin by approximating the mode shape of the beam as a polynomial function, which satisfies the boundary conditions. Since the Rayleigh method is based on kinematic conditions, we can only use boundary constraints that restrict the kinematics of the system. At $x = L_1$, the beam has a pinned support, so the deformation in the y direction should be zero.

$$Y(L_1) = 0 \tag{19}$$

Similarly, at $x = 3L_1$, the beam has a pinned-roller support, so the deformation in the y direction should be zero.

$$Y(3L_1) = 0 \tag{20}$$

Based on these two boundary conditions, we can estimate the mode shape function as a second-order polynomial of the form

$$Y(x) = x^2 + a_1 x + a_0, \tag{21}$$

where a_i, $i = 0, 1$ are constant parameters. We also require the first and second derivatives of $Y(x)$ with respect to x, which are

$$Y'(x) = 2x + a_1 \tag{22}$$

and

$$Y''(x) = 2, \tag{23}$$

respectively.

By substituting the boundary conditions into the polynomial, we can obtain a system of equations for the constants a_0 and a_1. Solving these equations yields

$$a_0 = 3L_1^2 = 0.2352 \ [m] \tag{24}$$

and

$$a_1 = -4L_1 = -1.12 \ [1]. \tag{25}$$

Using these constants, we can calculate the approximate mode shape function of the beam, which is

$$Y(x) = x^2 - 1.12x + 0.2352. \tag{26}$$

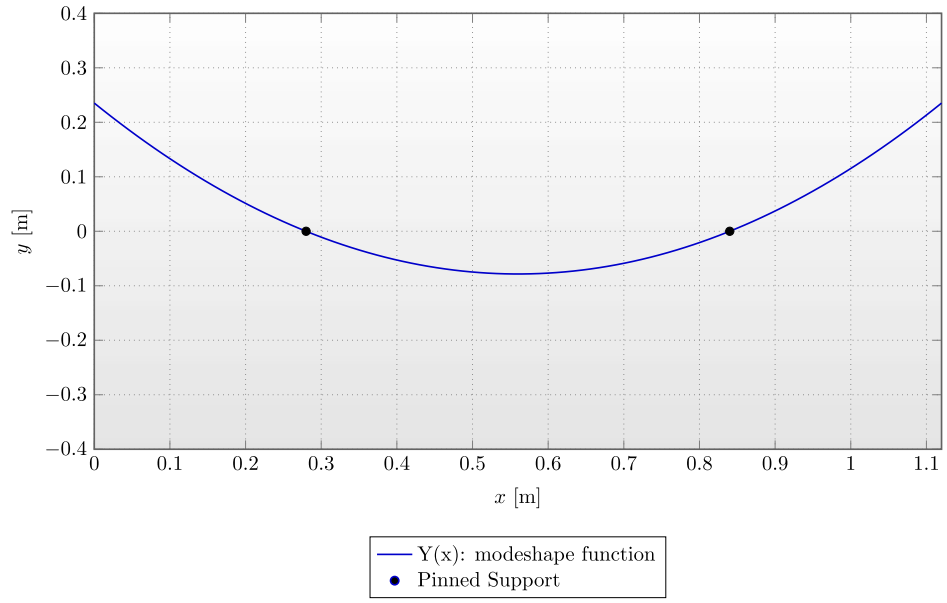


Figure 2: The $Y(x)$ approximate modeshape function

The total potential energy in this case is described by the following equation:

$$U = U_{shaft} = \frac{1}{2} I_z \; E \; \cos^2(\omega_n \; t) \; \int_0^{4L_1} Y''^2(x) \; \mathrm{d}x \tag{27}$$

The kinetic energy can be calculated with the following integral:

$$T = T_{\text{shaft}} + T_{\text{disks}} = \frac{1}{2} \Big(\rho \; A \; \int_0^{4L_1} Y(x) \; \mathrm{d}x + m_1 \; Y^2(0) + \theta_1 \; Y'^2(0) + \tag{28}$$
$$+ m_2 \; Y^2(2L_1) + \theta_2 Y'^2(2L_1) + m_3 Y^2(4L_1) + \theta_3 Y'^2(L) \Big) \; \omega_n^2 \; \sin^2(\omega_n \; t)$$

Based on the conservation of energy, Raylaigh's principle states that the maximum of the potential and kinetic energies are equal:

$$T_{max} = U_{max} \tag{29}$$

By considering the maximum values of both the $\sin^2(\omega_n \; t)$ and $\cos^2(\omega_n \; t)$ functions, which are equal to 1, we can derive the following formula to approximate the first eigenfrequency of the system.

$$\omega_{n,1}^R \leq \sqrt{\frac{I_z E \int_0^{4L_1} Y''^2(x)\, dx}{\rho A \int_0^{4L_1} Y(x)\, dx + m_1 Y^2(0) + \theta_1 Y'^2(0) + m_2 Y^2(2L_1) + \theta_2 Y'^2(2L_1) + m_3 Y^2(4L_1) + \theta_3 Y'^2(4L_1)}} \tag{30}$$

By calculating this equation we get the following results:

$$\omega_{n,1}^R \leq 333.12 \left[\frac{\text{rad}}{\text{s}} \right] \tag{31}$$

$$f_{n,1}^R = \frac{2\pi}{\omega_{n,1}} = 53.02 \;\; [\text{Hz}] \tag{32}$$

By applying the Rayleigh-quotient method, we've obtained an expression for the first eigenfrequency of the beam. Once we have the first eigenfrequency, we can verify the finite element model by comparing the calculated eigenfrequency with the one obtained from the finite element analysis. If they match, it confirms that the FE model accurately represents the behavior of the beam under bending vibrations.

$$\frac{f_{n,1}^R - f_{n,1}}{f_{n,1}} \cdot 100 = 11.01 \text{ [\%]} \tag{33}$$

The relative error in our approximation may not be very low, but it's important to consider that we used a second-order polynomial as the estimating mode shape function. In summary, we employed the Rayleigh method to estimate the first natural frequency and corresponding mode shape of the system. This method provides an upper estimate for the first eigenfrequency. Based on our results, we can conclude that the finite element model we applied is valid.

3 Response of the system using direct time integration

Determine the response of the system by direct time integration! Determine the required time step size! Apply the assigned method and plot the transverse displacement and the angle of rotation at cross section K! Apply 2500-3000 time steps!

The direct time integration method can be employed to determine the system's time response under any given excitation. In my assigned task, I utilized the constant acceleration method, also known as the Newmark algorithm. The time integration scheme involved utilizing the α and γ parameters, as follows:

$$\alpha = \frac{1}{4} \tag{34}$$

$$\gamma = \frac{1}{2} \tag{35}$$

The constant acceleration scheme is unconditionally stable due to the α and γ parameters. A sufficiently small time step (Δt) is still needed for accurate system representation. Since we focused on the first four eigenfrequencies, I will be using $\omega_{n,4} = 2429.42 \left[\frac{\text{rad}}{\text{s}}\right]$ to determine the timestep size. The corresponding period for this frequency is:

$$T_{n,4} = \frac{2\pi}{\omega_{n,4}} = 2.5863 \cdot 10^{-3} \text{ [s]} \tag{36}$$

For the timestep size I will be using one tenth of it

$$\Delta t = \frac{T_{n,4}}{10} = 2.5863 \cdot 10^{-4} \text{ [s]}. \tag{37}$$

The discrete time scale can be determined as

$$t = n \ \Delta t \tag{38}$$

Where n is the number of steps which is choose to be $n = 3000$, meaning that t is the time of the simulation. The vector of nodal excitation at the nth timestep becomes

$$\mathbf{F}_n^T = \begin{bmatrix} 0 & 0 & M_0 \cos(\omega_n \; n \; \Delta t) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ [SI]} \tag{39}$$

The Newmark algorithm provides the following displacement and velocity vectors at the (n + 1)th time step:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \; \dot{\mathbf{u}}_n + \frac{1}{2}\Delta t \; \ddot{\mathbf{u}}_{n+\alpha} \tag{40}$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \Delta t \; \ddot{\mathbf{u}}_{n+\gamma} \tag{41}$$

The displacement for the constant acceleration scheme can be determined by analyzing the equation of motion at the n and $n+1$ timesteps

$$\mathbf{u}_{n+1} = \left(\frac{4}{\Delta t^2}\mathbf{M} \; + \mathbf{K} \right)^{-1} \left[\mathbf{F}_{n+1} + \mathbf{M} \left(\ddot{\mathbf{u}}_n + \frac{4}{\Delta t}\dot{\mathbf{u}}_n + \frac{4}{\Delta t^2}\mathbf{u}_n \right) \right]. \tag{42}$$

$$\dot{\mathbf{u}}_{n+1} = \frac{2}{\Delta t}\left(\mathbf{u}_{n+1} - \mathbf{u}_n\right) - \dot{\mathbf{u}}_n \tag{43}$$

$$\ddot{\mathbf{u}}_{n+1} = \frac{4}{\Delta t^2}\left(\mathbf{u}_{n+1} - \mathbf{u}_n\right) - \frac{4}{\Delta t}\dot{\mathbf{u}}_n - \ddot{\mathbf{u}}_n \tag{44}$$

One notable benefit of any form of the Newmark iteration scheme is its inherent self-starting capability. By providing only the initial values for nodal displacement, velocity, and acceleration, the algorithm can initiate without requiring any additional information. In this particular scenario, we assume that the nodal displacement and velocities at the initial step are both zero, denoted as

$$\mathbf{u}_0 = \mathbf{0} \tag{45}$$

and

$$\dot{\mathbf{u}}_0 = \mathbf{0}, \tag{46}$$

respectively. The initial acceleration can be determined from the equation of motion at $t = 0 (n = 0)$:

$$\mathbf{M} \; \ddot{\mathbf{u}}_0 + \mathbf{K} \; \mathbf{u}_0 = \mathbf{F}_0 \tag{47}$$

$$\downarrow$$

$$\ddot{\mathbf{u}}_0 = \mathbf{M}^{-1}\left(-\mathbf{K} \; \mathbf{u}_0 + \mathbf{F}_0\right) \tag{48}$$

The numerical values for the initial acceleration is

$$\ddot{\mathbf{u}}_0 = \begin{bmatrix} 2.1569 \\ 18.1791 \\ 1150.67 \\ -4.6828 \\ 74.6986 \\ 0.842038 \\ -0.00090136 \\ 0.00447603 \end{bmatrix} \text{ [SI]}. \tag{49}$$

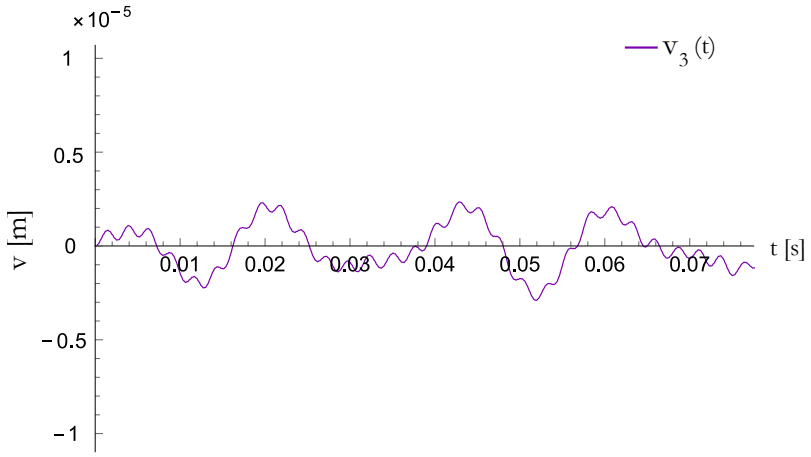


Figure 3: Timeresponse for v_3

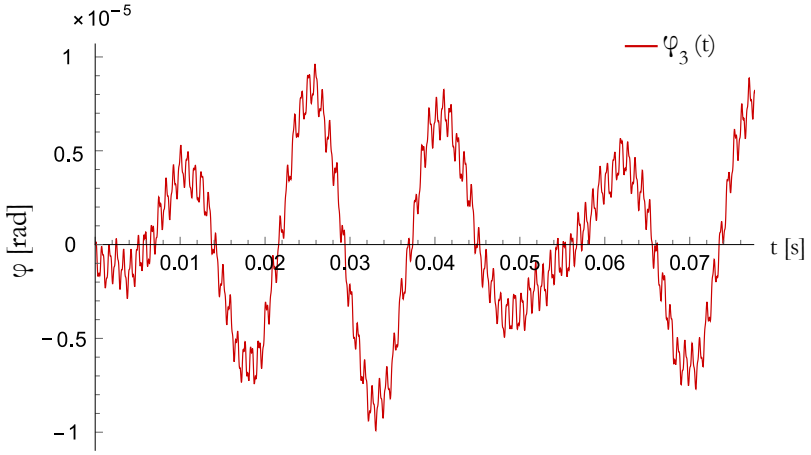


Figure 4: Timeresponse for φ_3

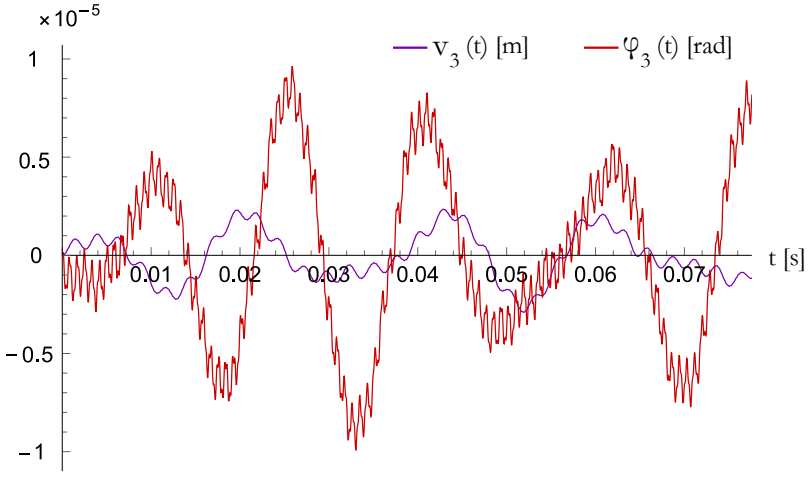


Figure 5: Timeresponse for the v_3 and φ_3 DoFs