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|---|---|---|
| Dept. of Applied Mechanics | HOME WORK 2 | Neptun: GBBNUL |
| 2022/23 II. | Deadline: 2023.05.29 18:00 | Late submission: □ Correction: □ |
| **Statement:** I hereby confirm that this homework is my own work and the submitted document shows my way of understanding. | | Signature: Kelle Gergő |

The formal requirements must be fulfilled! There is no option to check the results online, the results of the different methods verify each other.

## Assignment

The bending vibration of a shaft shown in the figure is investigated. The shaft is prismatic, its density is denoted by $\rho$ and its Young's modulus is $E$. A disk of mass $m$ and mass moment of inertia $\theta_{Sz}$ is attached to the shaft that is also connected to the environment through a spring of stiffness $k$.

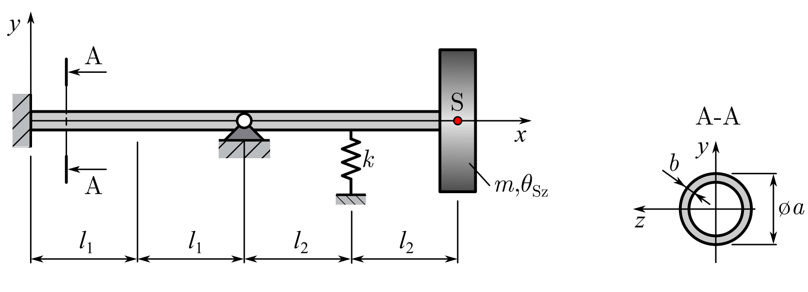


Figure: Mechanical model

**Data**
$\rho = 7800\ \mathrm{kg/m^3}$
$E = 210$ GPa
$a = 20$ mm
$b = 2$ mm
$l_1 = 95$ mm
$l_2 = 120$ mm
$m = 0.19$ kg
$\theta_{Sz} = 0.06\ \mathrm{gm^2}$
$k = 98$ kN/m

### Tasks

1. Estimate the first natural frequency of the shaft with the help of Rayleigh's Principle. Use an approximated mode shape that satisfies the boundary conditions while the disk and the spring are neglected. Sketch the applied approximating first mode shape function.
2. Determine the first couple of natural frequencies and mode shapes corresponding to the bending vibration with the help of finite element method.
(FEM software: `http://www.mm.bme.hu/edu/msc-en/advmech/Sikerez_ENG_Install.zip`)
3. Compare the results.

## Results

| $f_{n1,\mathrm{Rayleigh}}$ [Hz] | $f_{n1,\mathrm{FEM}}$ [Hz] | $f_{n2,\mathrm{FEM}}$ [Hz] | $f_{n3,\mathrm{FEM}}$ [Hz] |
|---|---|---|---|
| 127.045 | 122.3456 | 1127.319 | 2609.203 |

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# Advanced Mechanics
# II. Homework

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# Contents

# 1 Rayleigh's Principle - First natural frequency

Estimate the first natural frequency of the shaft with the help of Rayleigh's Principle. Use an approximated mode shape that satisfies the boundary conditions while the disk and the spring are neglected. Sketch the applied approximating first mode shape function.

## 1.1 Parameters

Table 1: Parameters for II. Homework

| Parameter | $\rho$ | $E$ | $a$ | $b$ | $l_1$ | $l_2$ | $m$ | $\theta_{Sz}$ | $k$ |
|---|---|---|---|---|---|---|---|---|---|
| Value | 7800 | 210 | 0.02 | 0.002 | 0.095 | 0.12 | 0.19 | 0.06 | 98 |
| Dimension | $\frac{\mathrm{kg}}{\mathrm{m}^3}$ | GPa | m | m | m | m | kg | $\frac{\mathrm{g}}{\mathrm{m}^2}$ | $\frac{\mathrm{kN}}{\mathrm{m}}$ |

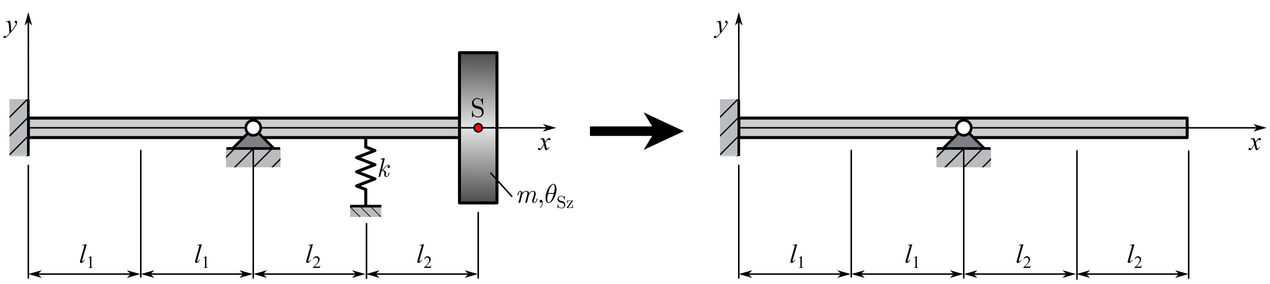


Figure 1: System with neglected spring and disk

Later calculations require the definition of some basic parameters which are the following:

$$A = \frac{a^2\pi}{4} - \frac{(a-2b)^2\,\pi}{4} = 0.000113097 \;\left[m^2\right] \tag{1}$$

$$I = \frac{\pi}{4}\left(\left(\frac{a}{2}\right)^4 - \left(\frac{a-2b}{2}\right)^4\right) = 4.637\cdot 10^{-9} \;\left[m^4\right] \tag{2}$$

## 1.2 Rayleigh's Principle

The deformation ($y$) depends on two quantities, the rod's center of gravity in the $x$-directional coordinate and time.

$$y(x,t) = Y(x)\cos(\omega_1\ t) \tag{3}$$

| | | | |
|---|---|---|---|
| Where | Y(x) | : | first mode shape function |
| | om | : | first natural frequency |

To determine the first eigenfrequency of the beam using the Rayleigh's principle, we begin by approximating the mode shape of the beam as a polynomial function, which satisfies the boundary conditions. Since the Rayleigh method is based on kinematic conditions, we can only use boundary constraints that restrict the kinematics of the system.

Table 2: Boundary conditions

| Type of constrain | Boundary condition |
|---|---|
| Fixed rod end | $Y(0) = 0$ (4)<br>$Y'(0) = 0$ (5) |
| Free rod end | $Y''(2l_1 + 2l_2) = 0$ (6)<br>$Y'''(2l_1 + 2l_2) = 0$ (7) |
| Hinged Mount in the center of the rod | $Y(2l_1) = 0$ (8) |

In this case, we need to fit a fifth-degree polynomial on a boundary condition of 5 showed in Table 2.

$$Y(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{9}$$

One of the coefficients can be chosed manually to be able to solve the problem, therefor $a_5 = 1$. Solving the problem with *Python* then plotting it with *Tikz* gives us the following solution:

$$Y(x) = x^5 - 1.5215x^4 + 0.7681x^3 - 0.09786x^2 \tag{10}$$

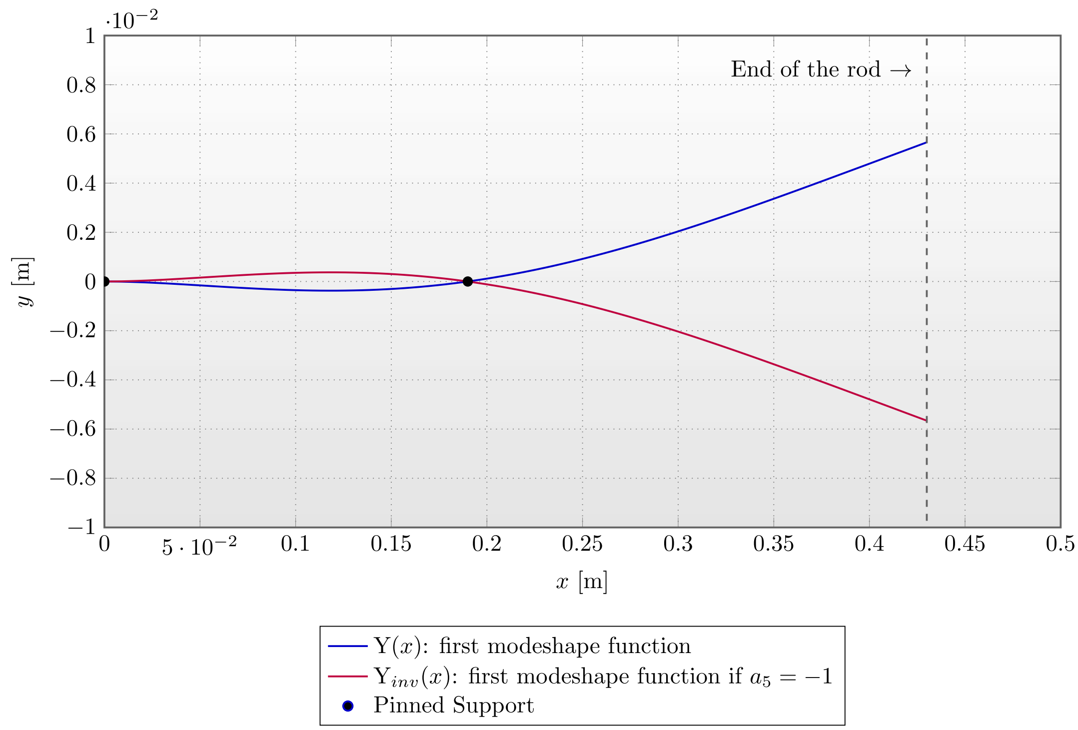


Figure 2: The approximated first modeshape function

By considering the maximum values of both the $\sin^2(\omega_n\ t)$ and $\cos^2(\omega_n\ t)$ functions, which are equal to 1, we can derive the following formulas to approximate the first eigenfrequency of the system. The total potential energy in this case is described by the following equation:

$$U = \frac{1}{2}\left(\int_0^{4L_1} I\ E\ Y''^2\ \mathrm{d}x + k\ Y^2(2l_1 + l_2)\right) \tag{11}$$

The kinetic energy can be calculated with the following integral:

$$T = \frac{1}{2}\left(\int_0^{4L_1} \rho\ AY^2(x)\ \mathrm{d}x + m\ Y^2(2l_1 + 2l_2) + \theta_{Sz}\ Y'^2(2l_1 + 2l_2)\right) \tag{12}$$

Based on the conservation of energy, Raylaigh's principle states that the maximum of the potential and kinetic energies are equal:

$$T_{max} = U_{max} \tag{13}$$

$$\omega_1 = \sqrt{\frac{I\ E \int_0^{2l_1+2l_2} Y''^2(x)\mathrm{d}x + kY^2(2l_1 + l_2)}{\rho\ A \int_0^{2l_1+2l_2} Y^2(x)\mathrm{d}x + mY^2(2l_1 + 2l_2) + \theta_{Sz}\ Y'^2(2l_1 + 2l_2)}} \tag{14}$$

By calculating this equation we get the following results:

$$\omega_1^R = 798.246 \left[\frac{\mathrm{rad}}{\mathrm{s}}\right] \tag{15}$$

$$f_{n,1}^R = \frac{\omega_1}{2\pi} = 127.045\ \ [\mathrm{Hz}] \tag{16}$$

By applying Rayleigh's principle, we've obtained an expression for the first eigenfrequency of the beam. Once we have the first eigenfrequency, we can verify the finite element model by comparing the calculated eigenfrequency with the one obtained from the finite element analysis. If they match, it confirms that the FE model accurately represents the behavior of the beam.

## 2 Finite element method - Natural frequencies & mode shapes

Determine the first couple of natural frequencies and mode shapes corresponding to the bending vibration with the help of finite element method.

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Table 3: Discretization

| Node | x [mm] | y [mm] |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 95 | 0 |
| 3 | 190 | 0 |
| 4 | 310 | 0 |
| 5 | 430 | 0 |

(a) Node coordinates

| Element | Local node 1 | Local node 2 |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 2 | 3 |
| 3 | 3 | 4 |
| 4 | 4 | 5 |

(b) Element connectivity

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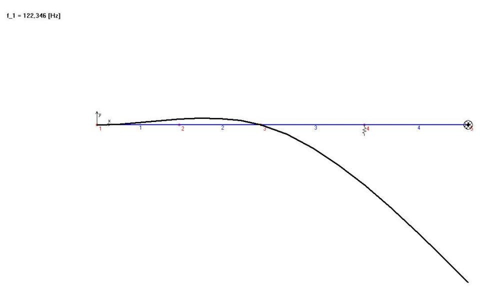
The parameters for the beam were set according to the (1) and (2) equations and Table **??**. The boundary conditions were set according to the Table 2 and while the spring is added onto the fourth node and the mass is added to the fifth node. The resulting first four natural freqencies are the following:

$$\omega_1^{\mathrm{FE}} = 768.7203 \text{ [rad/s]} \quad \rightarrow \quad f_1^{FE} = 122.3456 \text{ [Hz]} \tag{17}$$

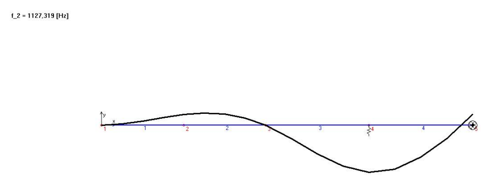
$$\omega_2^{\mathrm{FE}} = 7083.155 \text{ [rad/s]} \quad \rightarrow \quad f_2^{FE} = 1127.319 \text{ [Hz]} \tag{18}$$

$$\omega_3^{\mathrm{FE}} = 16394.11 \text{ [rad/s]} \quad \rightarrow \quad f_3^{FE} = 2609.203 \text{ [Hz]} \tag{19}$$

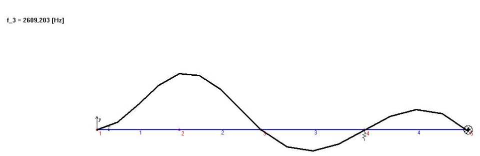
$$\omega_4^{\mathrm{FE}} = 21977.83 \text{ [rad/s]} \quad \rightarrow \quad f_4^{FE} = 3497.88 \text{ [Hz]} \tag{20}$$



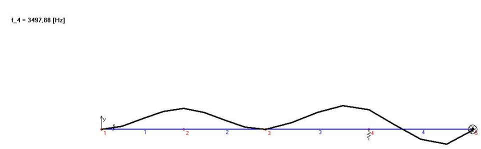

(a) 1st modeshape




(b) 2nd modeshape




(c) 3rd modeshape




(d) 4th modeshape

Figure 3: Modeshapes for the first couple of eigenfrequencies

# 3 Result comparison

Compare the results.

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Table 4: Comparison of different methods

| i | $\omega_i^{\mathrm{R}}$ [rad/s] | $\omega_i^{\mathrm{FE}}$ [rad/s] |
|---|---|---|
| 1 | 798.246 | 768.7203 |
| 2 | - | 7083.155 |
| 3 | - | 16394.11 |
| 4 | - | 21977.83 |

From the first natural frequency defined with the Rayleigh's Principle, it can be generally stated that it is considered to be a conservative method that estimates the solution from above. The (21) equation shows that the difference between the two method is $\leq 10$ [%] which makes it -depending on the area of use- a reliable approximation method.

$$\frac{\omega_1^{\mathrm{R}} - \omega_1^{\mathrm{FE}}}{\omega^{\mathrm{FE}}} = 3.84 \text{ [\%]} \tag{21}$$