Nonlinear Vibrations

Lecture Notes
2023/24/2

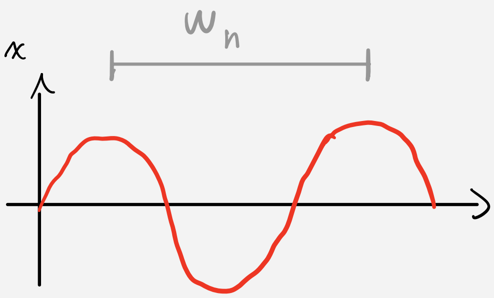
Mechanical vibration

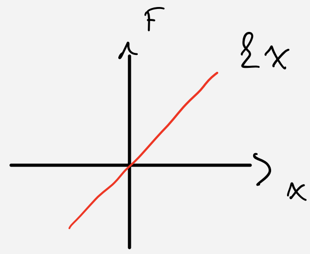
Vibration is periodic process of oscillations with respect to an equlibrium point.

Linear vibration

A vibratory system is linear, if all the basic component has linear characteristics (superposition).

↳ Harmonic oscillations, where the vibration signal is a pure sin or cos function.





linear spring characteristic

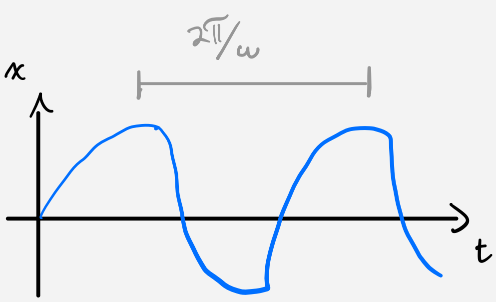
k: stiffness $[N/m]$

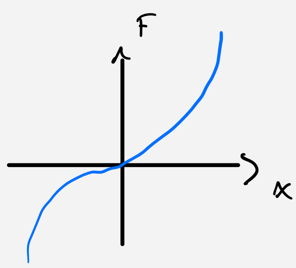
Nonlinear vibration

A vibratory system is nonlinar, if 1 or more basic components have nonlinear characteristics.

↳ Nonharmonic oscillations, where the vibration signal is a sum of sin or cos function.

nonharmonic but it can still be periodic



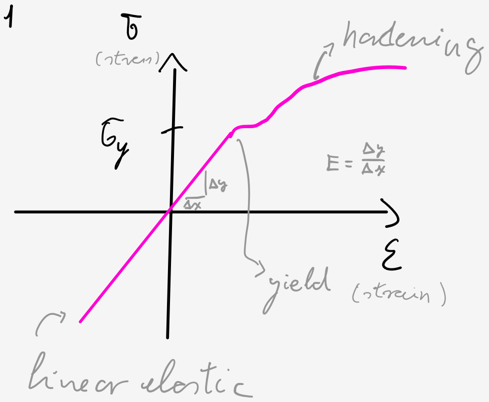


nonlinear spring characteristic

Types of nonlinearities

1. Material properties

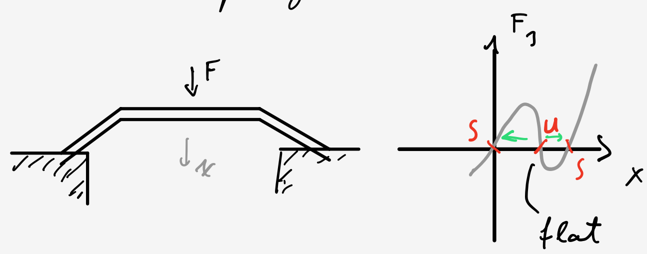
1.1



~ The material properties depend on the state of the body

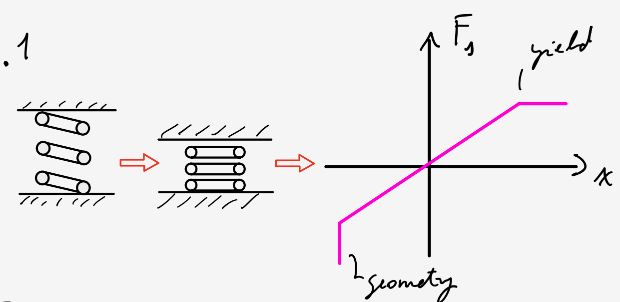
2. Geometric nonlinearities

2.1 Plate spring

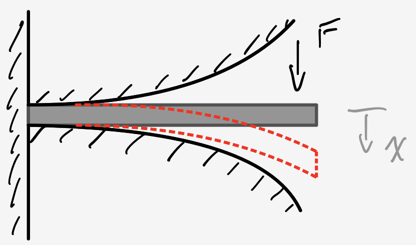


3. Structural nonlinearities

3.1



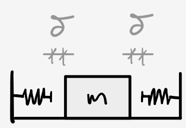
3.2



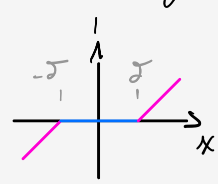
~ The spring changes the connection with the environment

4. Mechanical nonlinearities

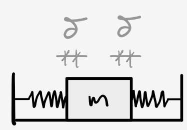
4.1 Free play



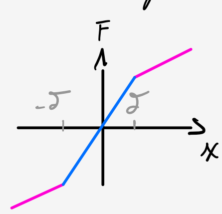
Hardening



4.2 Overlap



Softening



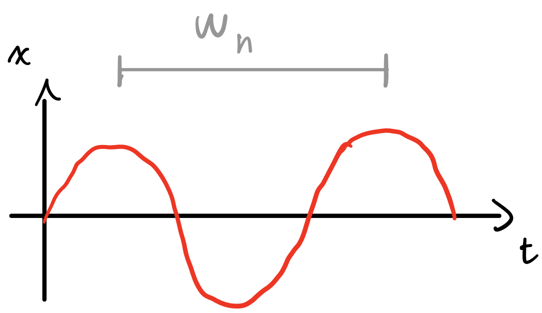
δ : gap

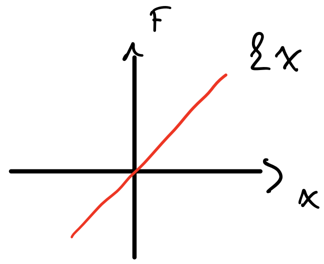
~ The spring behaves nonlinearly due to its tolerances connecting to the mass

I. Lecture

Nonlinear vibration

~ wrong terminology ⇒ the vibration signal is always nonlinear





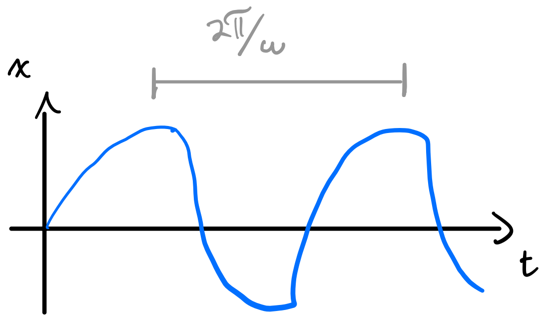
linear spring characteristic

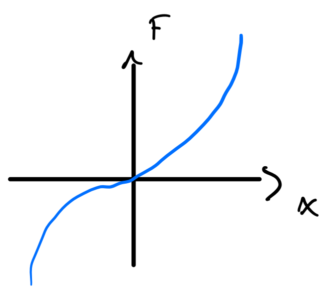
k: stiffness [N/m]

~ IF linear vibration characteristics ⇒ harmonic oscillation / vibration
↳ pure sin / cos

↳ Linear vibration theory ⇒ valid only for small vibration !!

IF Nonlinear





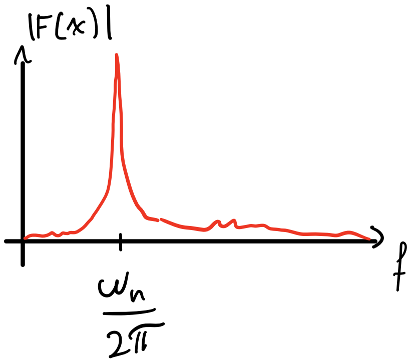
nonlinear spring characteristic

nonharmonic but it can be still periodic

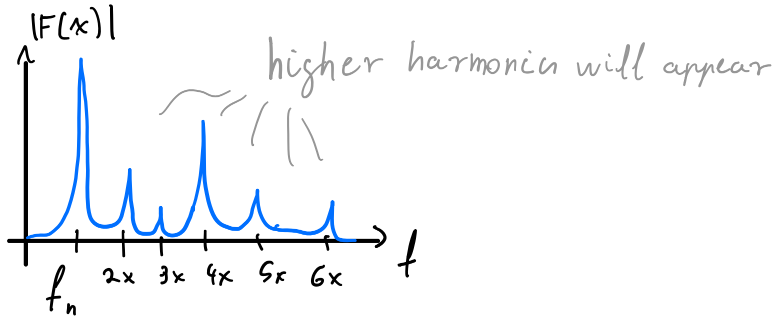
Using FFT to analyze

~The main difference between the harmonic and non-harmonic signals can be observed in frequency domain

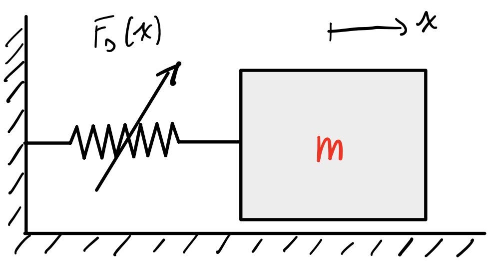
Linear



Nonlinear



Nonlinear springs



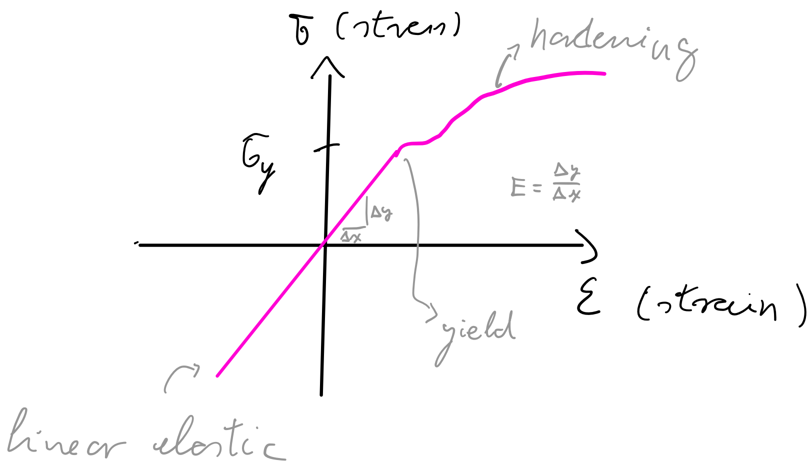
conservative => potential function can be developed

$m\ddot{x} + F_s(x) = 0$

Typical nonlinearities

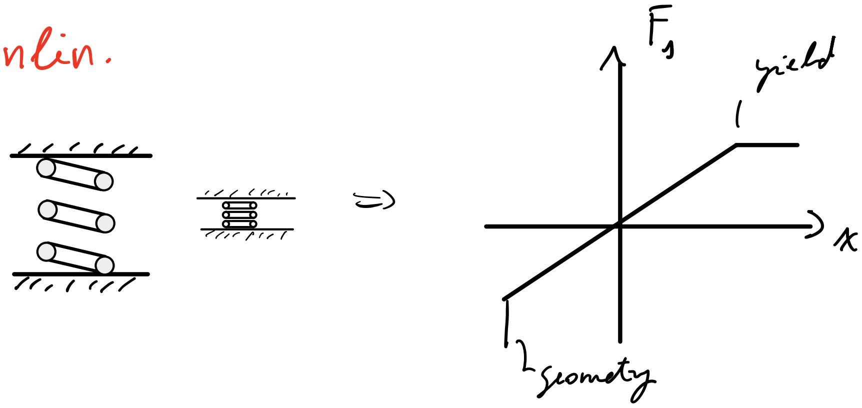
$\boxed{1.}$ Material

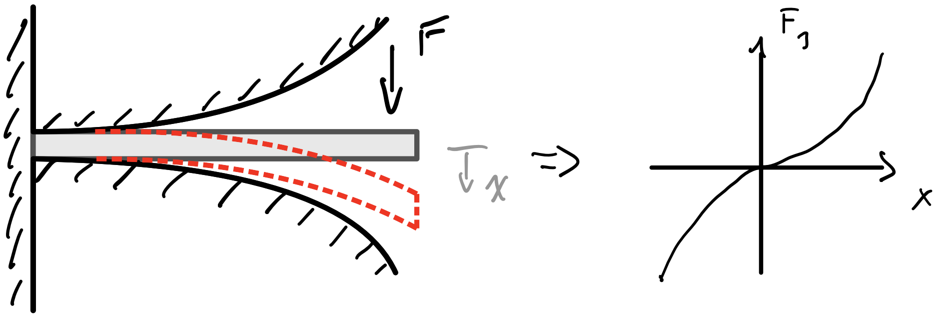
Nonlinear properties of the material



$\boxed{2.}$ Geometric nonlin.

- helical spring

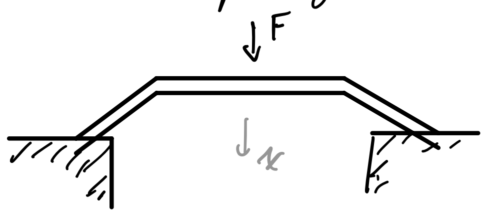


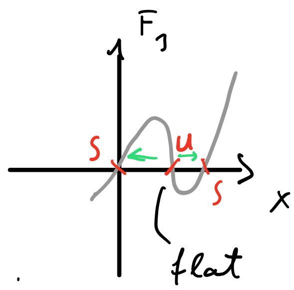


~ can be called "Structural nonlinearity", spring changes the connection with the environment

- Bibi clutch

- Plate spring





Example:

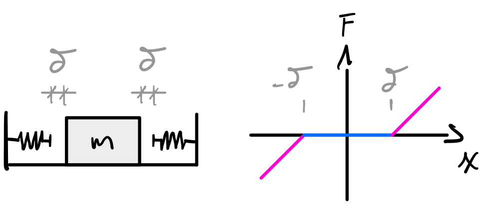
hungarian bow

hard to pull at the beginning, but then easy to hold

↳ good aim

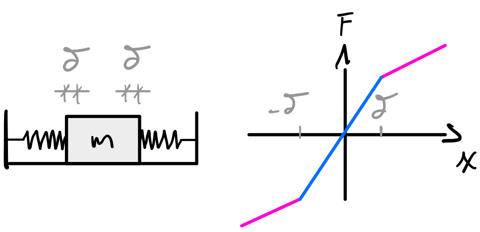
$\boxed{3.}$ Engineering nonlinearity due to tolerance

- Free play



Hardening

- Overlap



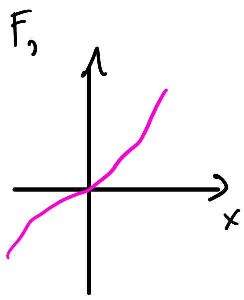
Softening

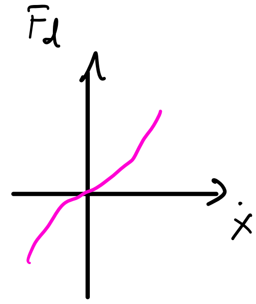
Too loose or overlap

↳ δ : gap

II. Lecture

$$m\ddot{x} + F_d(\dot{x}) + F_s(x) = 0$$





Resistance of environment

$$F_d = c|\dot{x}|^n \operatorname{sgn} \dot{x} \quad \Rightarrow c = ?\ ;\ n = ?$$

its power is (often) negative

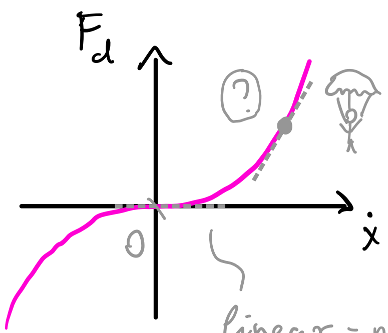
$$P_d = \underline{F}_d \cdot \underline{v} \leq 0$$

$$|\underline{F}_d| \cdot |\underline{v}| \cos\left(\sphericalangle(\underline{F}_d, \underline{v})\right)$$

angle

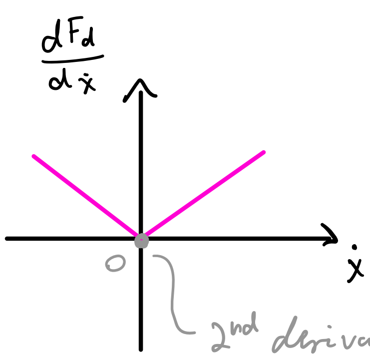
$\underline{F}_d$ and $\underline{v}$ are "opposite"

1.) gas $\Rightarrow n = 2 \rightarrow F_d = c\dot{x}^2 \operatorname{sgn} \dot{x}$



- For small oscillation around 0 is negligible

linear-part in 0 -> negligible (elhanyagolható)



2nd derivative does not exist at 0

$\Rightarrow$

needs to be 4 times differentiable

$\Downarrow$

it will cause problems later in the Hopf bifurcation theorem

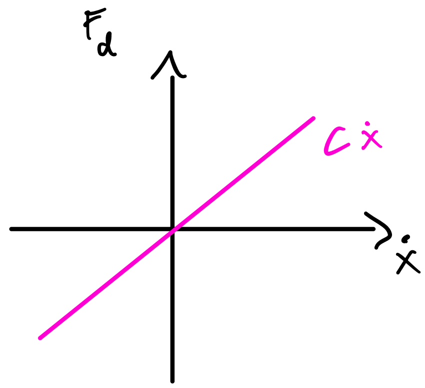
2.) liquid $\Rightarrow$ $R_e > R_{e_{cr}}$ $\Rightarrow$ turbulent flow

$n = \frac{7}{4} = 1.75 \approx 2$

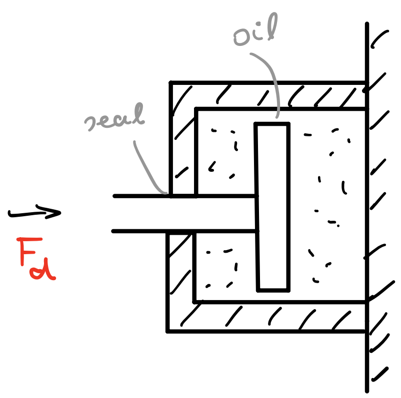
$\rightarrow$ $R_e < R_{e_{cr}}$

$R_e = \dfrac{\text{velocity} \cdot \text{geometry}}{\text{viscosity}}$ $\rightarrow$ laminar flow

$n = 1$ $\Rightarrow$ linear



$c \left[\frac{Ns}{m}\right]$: damping factor

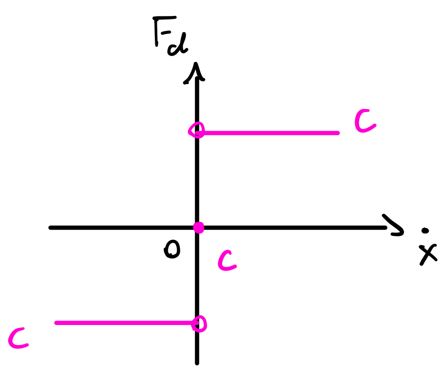


narrow channel
high viscosity } low R_e number $\rightarrow$ linear!

Thin oil film on metal surfaces $\nearrow$ against rust
$\searrow$ nice damping

3.) Solid $\Rightarrow$ $n = 0$ $\Rightarrow$ $F_d = C \operatorname{sgn} \dot{x}$

C: coulomb friction force
$C = \mu N$



correct math modelling:
differential inclusion

$$F_d = \begin{cases} = C \operatorname{sgn} \dot{x}, & \text{if } \dot{x} \neq 0 \\ \in [-C, C], & \text{if } \dot{x} = 0 \end{cases}$$

$\Rightarrow \ddot{x} \in f(\dot{x})$

=> "infinite" derivative

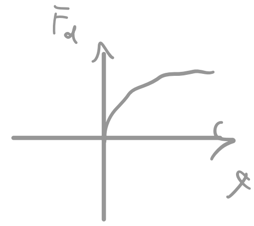
↳ Lipschitz condition violated

Schauder theorem -> no uniqueness!

cutting force

$F_c = x \cdot w h^{3/4}$

width, height

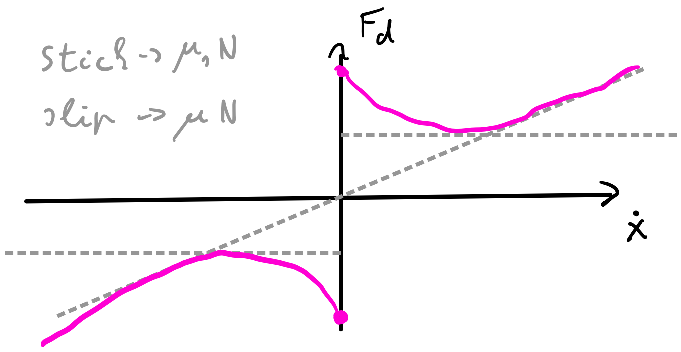


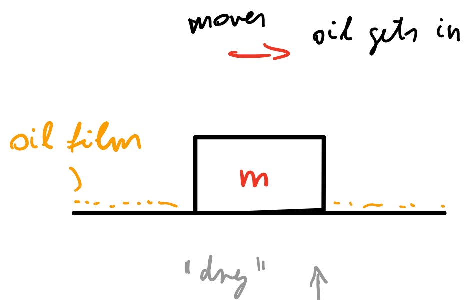
4.) Mixed friction ~ solid / fluid

=> Stribeck function

stick -> $\mu_s N$

slip -> μN





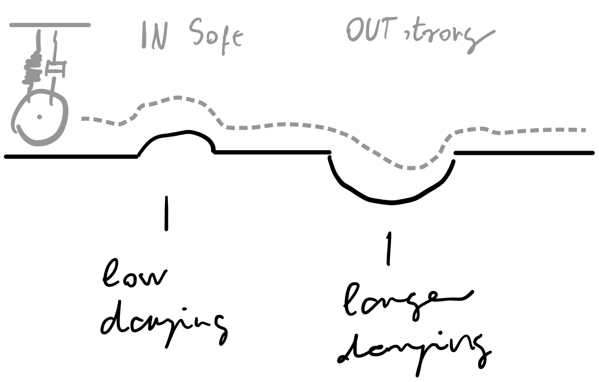
Tribology

Dry and viscous damping at once

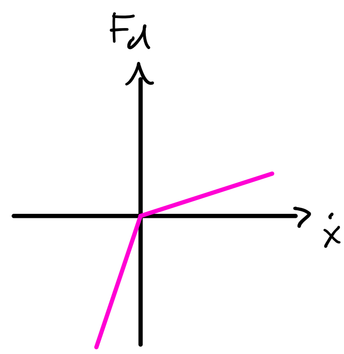
45° -> tg -> 1 -> μ

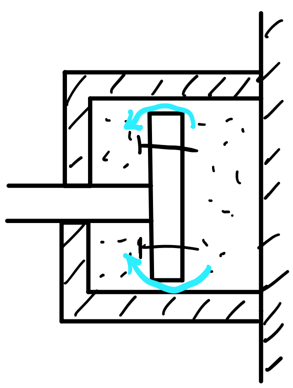
sliding -> μ -> 0.3-0.4 -> ABS

5.) Non-linear structure



=>

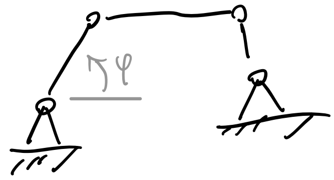




non-linear mass/inertia

rotating ? , mechanism

$\Theta(\varphi)\ddot{\varphi} + \ldots = 0$

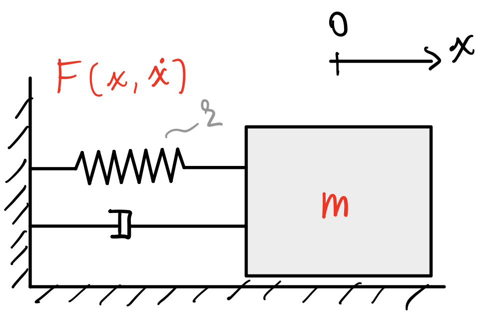


$\Theta(\varphi) > \rho$ for $\forall \varphi \in [0, 2\pi]$

Combined non-linearities

$$m(x)\ddot{x} + \underbrace{F_d(\dot{x})}_{= c\dot{x} + c_2\dot{x}^2 + c_3\dot{x}^3 + \ldots} + \underbrace{F_s(x)}_{= k x + k_2 x^2 + k_3 x^3} = 0 \quad \Big/ \frac{1}{m(x)}$$

Taylor series



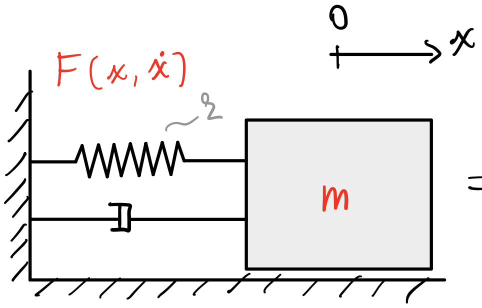
$$F(x,\dot{x}) = c\dot{x} + kx + c_2\dot{x}^2 + b_{11}x\dot{x} + k_2 x^2 + c_3\dot{x}^3 + k_3 x^3 + b_{21}x^2\dot{x} + b_{12}x\dot{x}^2$$

$$m(x)\ddot{x} + R(x,\dot{x}) = 0 \quad \Big/ \frac{1}{m(x)}$$ ~ Cauchy transformation

$$\ddot{x} = g(x,\dot{x}) \Rightarrow \begin{array}{l} \dot{x} = y \\ \dot{y} = g(x,y) \end{array} \Rightarrow \begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{array} \sim 1\,\text{DoF}$$

III. Lecture

1 DoF non-linear vibration



$\Rightarrow$

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

x: coordinate

y: velocity

Usually $\dot{x} = y$

~ Find equilibria (~ trivial solution)

$$\left.\begin{aligned} x(t) &= x^* \\ y(t) &= y^* \end{aligned}\right\} \quad \left.\begin{aligned} \dot{x}(t) &= 0 \\ \dot{y}(t) &= 0 \end{aligned}\right\} \quad \begin{aligned} f(x^*, y^*) &= 0 \\ g(x^*, y^*) &= 0 \end{aligned}$$

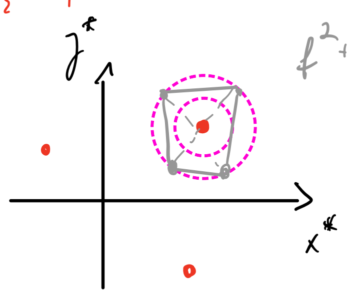
~ Algebraic solution in special cases only (like preselected fixed points...)

Generally, we used numerical algorithms
like the one based on:

$$\min_{(x^*, y^*)} \left(f^2(x^*, y^*) + g^2(x^*, y^*) \right)$$

→ many sol. (even ∞)

(x_i^*, y_i^*), $i = 1, 2 \dots n$



$f^2 + g^2 = \text{const} > 0$

Linearization about (at) equilibria:

$$\left.\begin{aligned} x(t) &= x^* + \xi(t) \\ y(t) &= y^* + \eta(t) \end{aligned}\right\}$$ where ξ and η are "small"

2D Taylor series: $f'_x = \frac{\partial f}{\partial x}$, $f'_y = \frac{\partial f}{\partial y}$

$$f(x,y) = \frac{1}{0!} f(x^*,y^*) + \frac{1}{1!} f'_x(x^*,y^*)\underbrace{(x-x^*)}_{=\xi} + \frac{1}{1!} f'_y \overbrace{(y-y^*)}^{=\eta} + \frac{1}{2!} \dots \underbrace{\xi^2 \ \eta^2}_{\approx 0}$$

$$g(x,y) = \frac{1}{0!} g(x^*,y^*) + g'_x(x^*,y^*)\xi + g'_y(x^*,y^*)\eta + \underbrace{h.o.t.}_{0}$$ (: higher order term)

$$\dot{x} = \underbrace{\dot{x}^*}_{0} + \dot{\xi}(t)$$

$$\dot{y} = \dot{\eta}$$

Linearized equation:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} f'_x(x^*,y^*) & f'_y(x^*,y^*) \\ g'_x(x^*,y^*) & g'_y(x^*,y^*) \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + h.o.t.$$

Trivial solution:

$$\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} = \underline{K} e^{\lambda t}$$, $\underline{K} \in \mathbb{C}^2$, $\lambda \in \mathbb{C}$

characteristic exponent

$$(\lambda \underline{\underline{I}} - \underline{\underline{A}}) \underline{K} e^{\lambda t} = \underline{0}$$

$$\det(.) = 0 \Leftarrow \neq 0$$

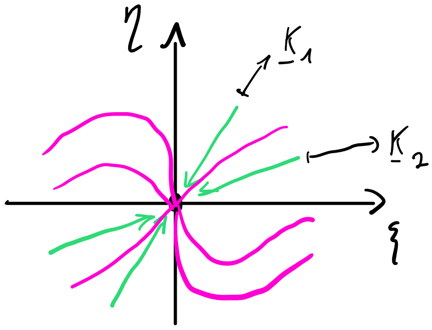
characteristic eq.: $\det(\lambda \underline{\underline{I}} - \underline{\underline{A}}) = 0$ $\Rightarrow$ $\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = 0$

$$\lambda^2 - Tr\,\underline{\underline{A}}\,\lambda + \det \underline{\underline{A}} = 0 \Rightarrow \lambda_{1,2}$$

the generic cases for the char. exponent (root)

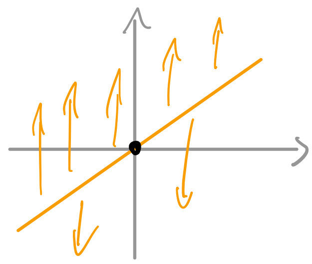
Generic cases:

1. $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 \cdot \lambda_2 > 0 \Rightarrow$ Node



$\underline{K}_1, \underline{K}_2$ eigenvecs. are real

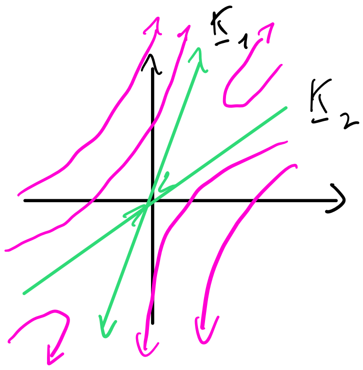
unstable: $\lambda_2 > \lambda_1 > 0$



stable node: $\lambda_{1,2} < 0$

unstable node: $\lambda_{1,2} > 0$

2. $\lambda_{1,2} \in \mathbb{R}$; $\lambda_1 \cdot \lambda_2 < 0$ ~ saddle

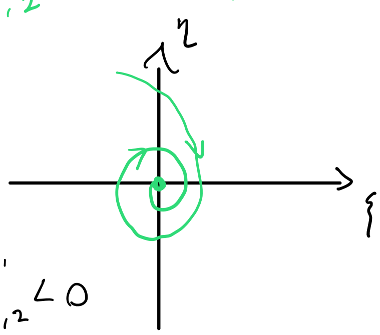


$\lambda_1 > 0$

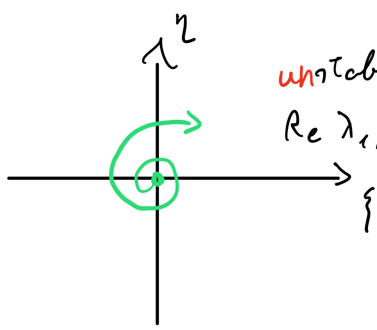
$\lambda_2 < 0$

unstable

3. $\lambda_{1,2} \in \mathbb{C}$, $\lambda_2 = \bar{\lambda}_1$ – focus spiral



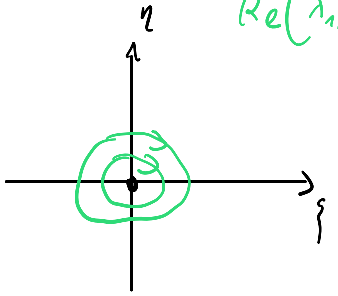
stable,
$Re\,\lambda_{1,2} < 0$



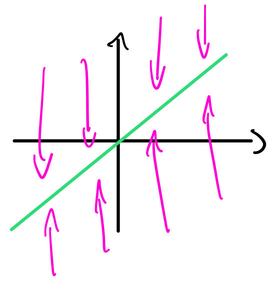
unstable,
$Re\,\lambda_{1,2} > 0$

Non generic cases

1. Centre $\lambda_{1,2} \in \mathbb{C}$, $Re(\lambda_{1,2}) = 0$



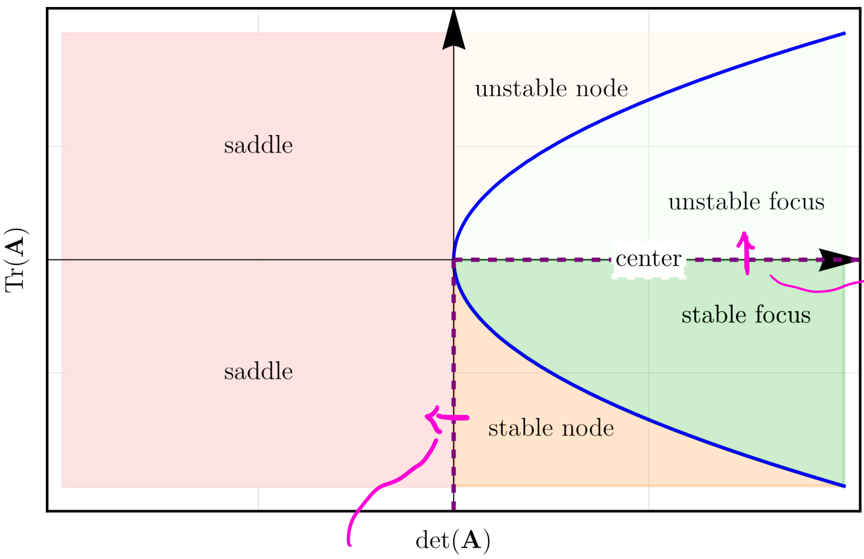
2. $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = 0$ $\lambda_2 \neq 0$



Summary: $\lambda_{1,2} = \frac{1}{2}\left(\operatorname{Tr}\underline{\underline{A}} \pm \sqrt{\operatorname{Tr}^2\underline{\underline{A}} - 4\det\underline{\underline{A}}}\right)$

TRACE-DET plane

$\operatorname{Tr}^2\underline{\underline{A}} = 4\det\underline{\underline{A}}$

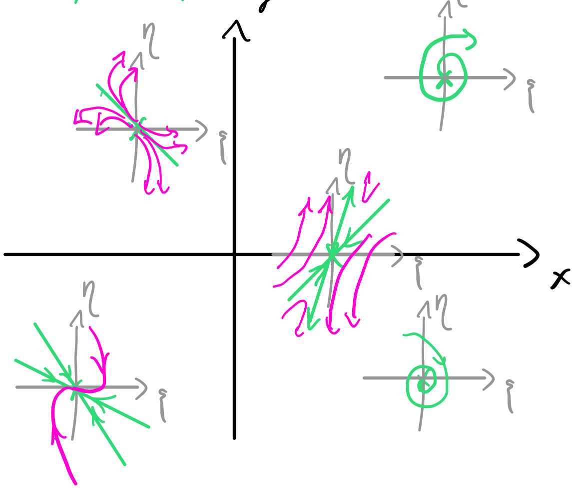


Hopf bifurcation (H)
dynamic loss of stability

saddle-node bifurcation (SN)
static loss of stability

Phase plane method for the non-lin. sys.:

x: fixed points



Global picture:

- existence & uniqueness $\Rightarrow$ trajectories cannot cross each other
- physical meaning helps to build connections...
- Euler's rule for polyhedrons

Tetrahedron:

\# faces + \# vertices = \# edges + 2

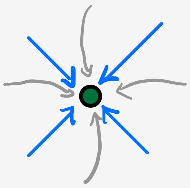
stable + unstable = saddle + 2

(node or focus) (node or focus)

Generic (and nongeneric) cases of local system behaviour

1. Nodes

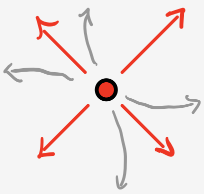
1.1 Stable Node



$\lambda_{1,2}<0$

$\lambda_{1,2}\in\mathbb{R}$

1.2 Unstable Node



$\lambda_{1,2}>0$

$\lambda_{1,2}\in\mathbb{R}$

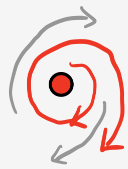
2. Spiral (or focus)

2.1 Stable Spiral

$Re(\lambda_{1,2})<0$

$\lambda_{1,2}\in\mathbb{C}$

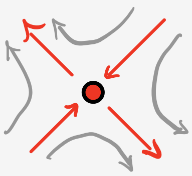
2.2 Unstable Spiral



$Re(\lambda_{1,2})>0$

$\lambda_{1,2}\in\mathbb{C}$

3. Saddle

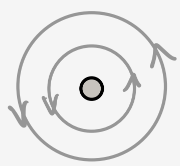


$\lambda_{1,2}\in\mathbb{R}$

$\lambda_1\lambda_2<0$

~ Saddle is always unstable

4. Center



$\lambda_{1,2}=\pm i\cdot\square$

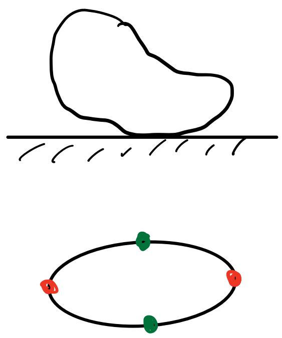
~ pure imaginary

The center is a nongeneric case of solution

IV. Lecture

The development of "Gömböc"

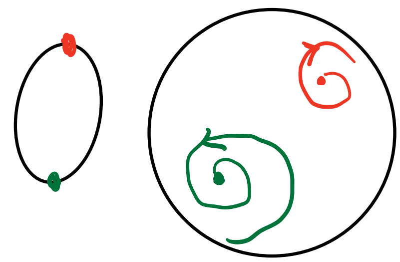
convex bodies in plane



\# equilibrium ≥ 4

$2S + 2U$

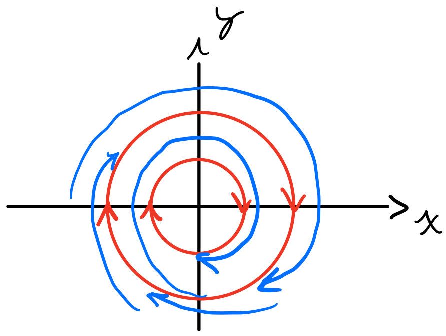
\# edges = 0



We can imagine like this

Index Theory ($\sim$ Euler rule) is only valid in those system where no limit cycle exists!

~ Limit Cycle



Def: Limit Cycles (LC)

Isolated closed trajectories on the phase plane of a nonlin. system

↳ center is not a limit cycle

The phase plane of the nonlinear systems is loaded by the existence of these limit cycles, which can't be found by linearisation

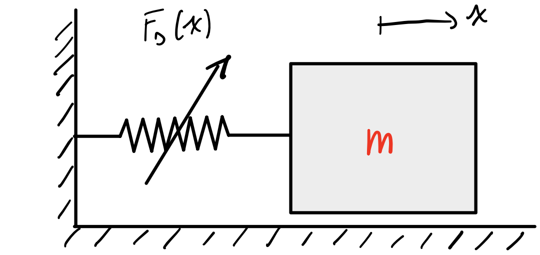
↳ The number of LC can be predicted by theorems.

↳ 2^{nd} degree nonlinearity: Based on Hilbert, the max # of LC for 2^{nd} degree nonlinearity is 3

↳ 3^{rd} degree nonlinearity: The max # of LC found so far is 7

V. Lecture

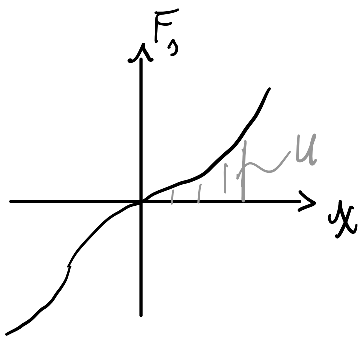
1 DoF conservative system



$$m\ddot{x} + F_s(x) = 0$$

$$|\dot{x}| = \sqrt{\frac{2}{m}}\sqrt{h - U(x)}$$

↳ h: initial mech. energy $\frac{1}{2}mv_0^2 + U(x_0)$



IC:

$x(0) = X_0$

$\dot{x}(0) = v_0$

A nonlinear system is conservative if the total mechanical energy of the system is conserved !!

$$T + U = \text{constant}$$

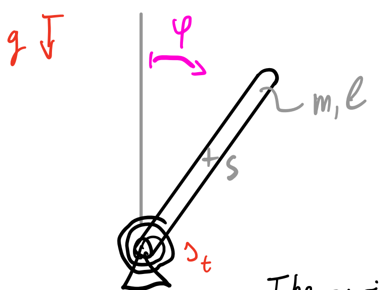
TLDR: the total mechanical energy has to be conserved

↳ proof eq. of motion

↳ $\cdot\dot{x}$; $\int_{t_0}^{t} d\tau$;

↳ $\frac{1}{2}m\dot{x}(t) + U(x) = c$

Example: Antena, mast



$$T = \frac{1}{2}\underbrace{\left(\frac{1}{3}m\ell^2\right)}\dot{\varphi}^2$$

$$\frac{1}{2}\cdot\Theta_0\cdot\omega^2$$

$$U(\varphi) = \frac{1}{2}s_t\varphi^2 + mg\frac{\ell}{2}\cos\varphi$$

The spring is linear (eq. of motion is non-lin)

$$\ddot{\varphi} + \frac{3s_t}{m\ell^2}\varphi - \frac{3g}{2\ell}\sin\varphi = 0$$

$\varphi(t) = \text{const}$, $\dot{\varphi}(t) = 0$; $\ddot{\varphi}(t) = 0$

$$\frac{3s_t}{m\ell^2}\varphi - \frac{3g}{2\ell}\sin\varphi = 0$$ ⤳ $\varphi = 0$ is a good trivial solution

⇓

Linearize around $\varphi = 0$ → $\sin\varphi \approx \varphi$

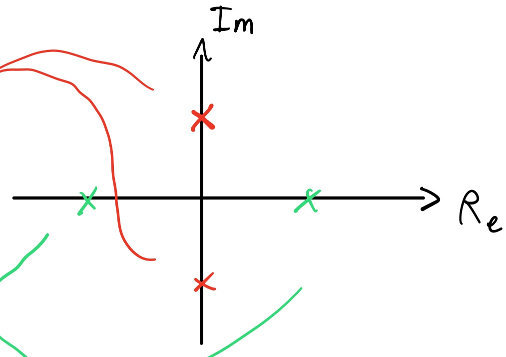
Lin. sys.: $$\ddot{\varphi} + \left(\frac{3s_t}{m\ell^2} - \frac{3g}{2\ell}\right)\varphi = 0$$

$$\varphi = A e^{\lambda t}$$

$$\lambda^2 + \left(\frac{3 s_t}{m \ell^2} - \frac{3 g}{2 \ell} \right) = 0$$

stable center

IF: $s_t > \frac{m g \ell}{2}$ positive $\Rightarrow \lambda_{1,2} = \pm \beta i$



$s_t < \frac{m g \ell}{2}$ negative $\Rightarrow \lambda_{1,2} = \pm \alpha \in \mathbb{R}$

unstable saddle

Find equlibria $\varphi(t) = \text{const}$ $\Leftarrow$ where no motion

Where the potential has extremum!

$$\frac{\partial U}{\partial \varphi} = 0$$

$$\frac{1}{2} s_t 2 \varphi - m g \frac{\ell}{2} \sin \varphi = 0$$

$$s_t \varphi - m g \frac{\ell}{2} \cos \varphi = 0 \qquad \Rightarrow \varphi = 0$$

It's stable if U has a minimum

It's unstable if U has a maximum

$$\left. \frac{\partial^2 U}{\partial \varphi^2} \right|_{\varphi = 0} \overset{?}{>} 0$$

$$\frac{\partial^2 U}{\partial \varphi^2} = \left. \left[s_t - m g \frac{\ell}{2} \cos \varphi \right) \right|_{\varphi = 0} = s_t - m g \frac{\ell}{2} > 0$$

$$s_t > m g \frac{\ell}{2}$$

same criteria

$\Rightarrow$ It is only LOCAL!

$\Rightarrow$ we need something conservative!

For a global picture ⇒ we use mechanics approach

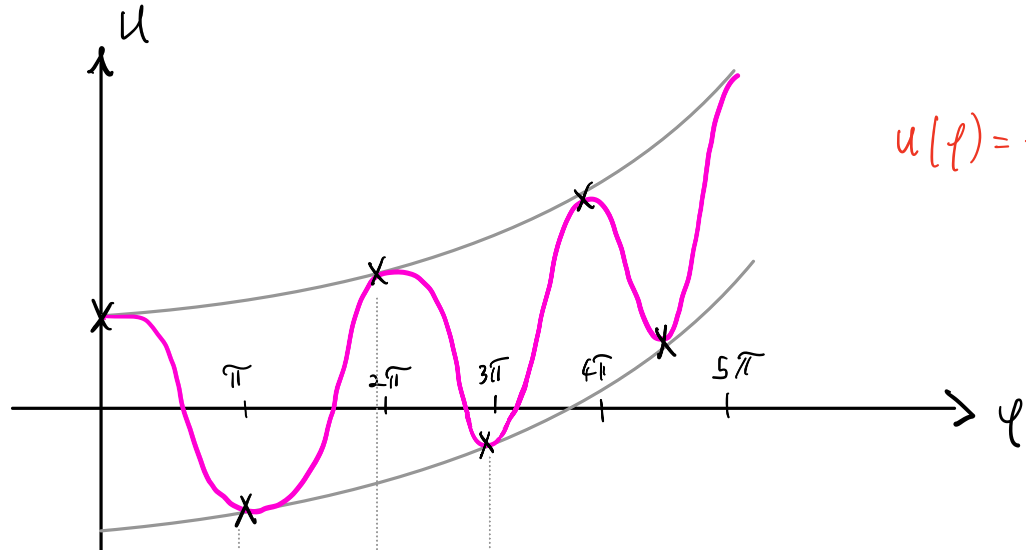
$$|\dot{\varphi}| = \sqrt{\frac{6}{m\ell^2}}\sqrt{h-U(\varphi)}$$

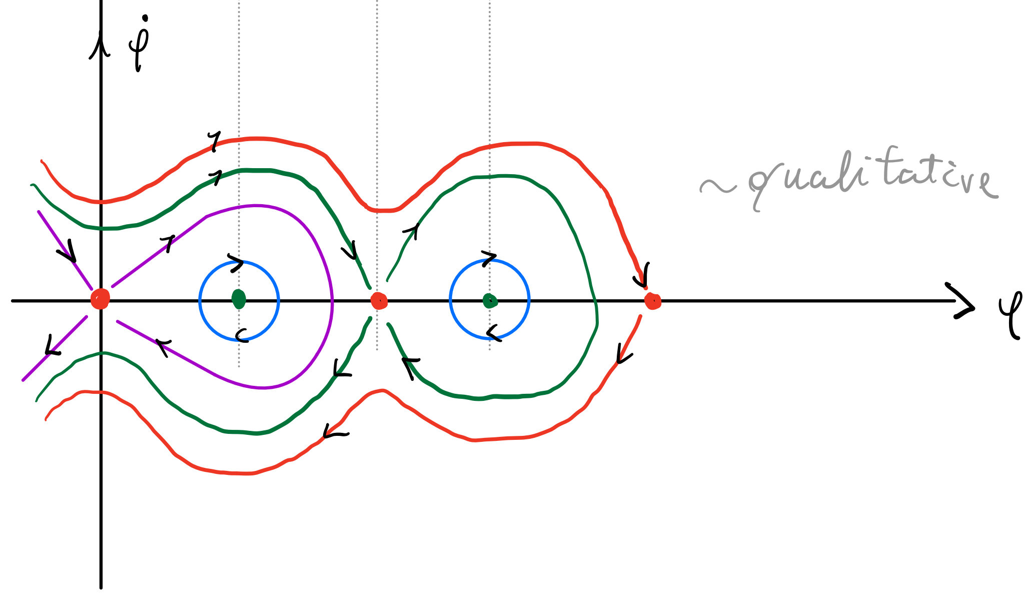
→ Construct all the trajectories when

$s_t = \frac{1}{16} mg\ell$ ⇒ unstable

$\varphi = 0$

$$u(\varphi) = \frac{1}{2} s_t \varphi^2 + mg\frac{\ell}{2}\cos\varphi$$





~ qualitative analysis

What is the physical meaning

Homoclinic orbit : starts from an eqp. → goes back to the same eqp.

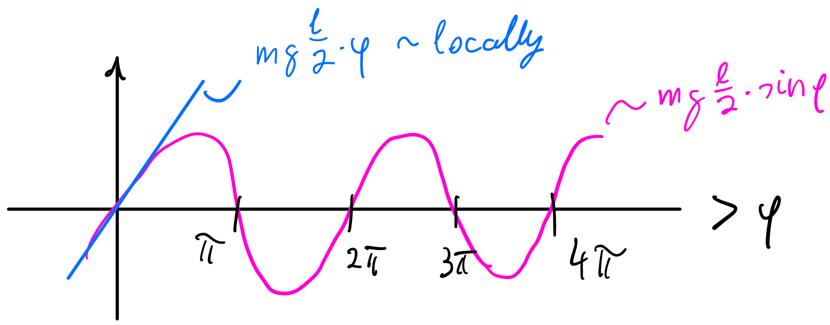
Heteroclinic orbit : starts from an eqp. → goes back to a different eqp.

- Basic idea of catastrophy theory ⇒ how the phase plane changes as the parameters (like the stiffness s_t) change?
  ↳ Most important is s_t

$$\frac{\partial U}{\partial \varphi} = s_t \cdot \varphi - mg\frac{\ell}{2}\sin\varphi = 0$$

$\varphi = 0$ is known analitically, but the others are not known in closed form

$$mg\frac{\ell}{2}\sin\varphi = s_t\,\varphi$$



Bifurcation

DEF: In dynamical systems, bifurcation occurs when a small change made to the (bifurcation) parameter values of a system causes a sudden qualitative change in its behaviour.

Types of bifurcation:

1. Hopf bifurcation (stable focus to unstable focus)
2. Saddle-node bifurcation (stable node to saddle)
3. Bogdanov-Takens bifurcation (center to saddle)

~ Bifurcation Diagram

DEF:

A bifurcation diagram shows the values equlibrium points of a system as a function of a bifurcation parameter in the system

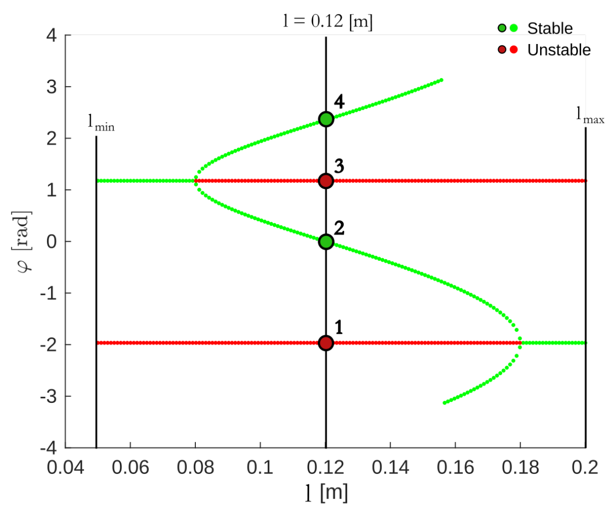
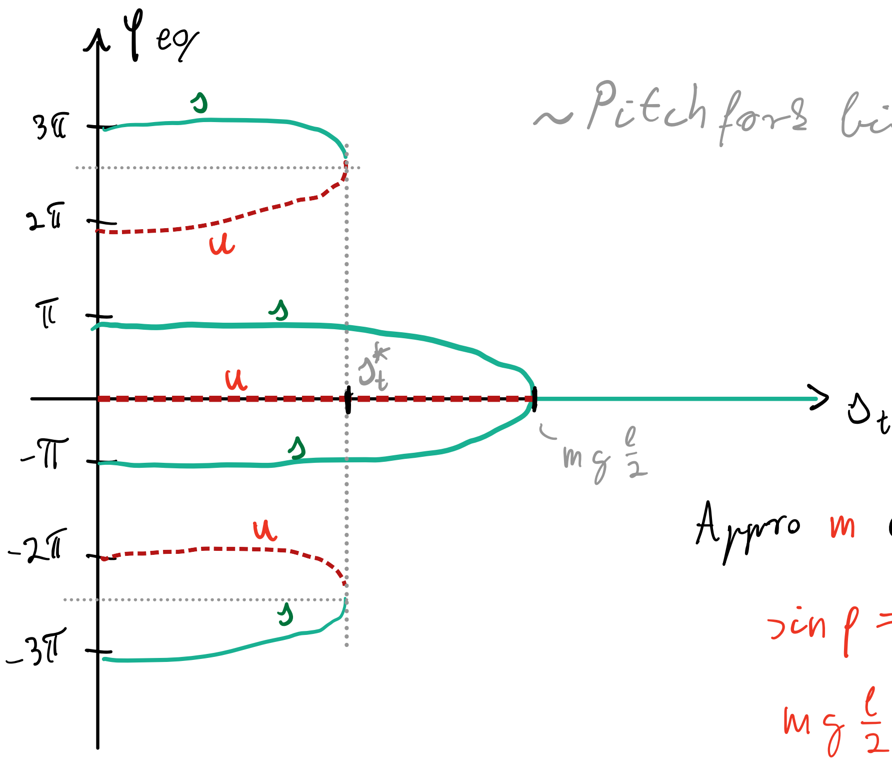


Figure from the 2nd Homework

Bifurcation Diagram



~ Pitchfork bifurcation

$s_t > mg\frac{\ell}{2}$ $\varphi_0 = 0$ ONLY

$s_t < mg\frac{\ell}{2}$ $\varphi_0 = 0\ ;\ \varphi_1 = -\varphi_2$ will merge

Approx in closed form:

$$\sin\varphi = \varphi - \frac{1}{3!}\varphi^3 + h.o.t.$$

$$mg\frac{\ell}{2}\left(\varphi - \frac{1}{6}\varphi^3\right) = s_t\varphi$$

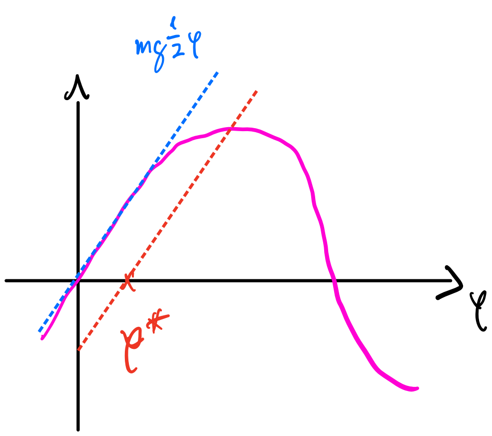
$$\varphi_{1,2} = \pm\sqrt{\frac{12}{mg\ell}\left(mg\ell - s_t\right)}$$ ~ parabola

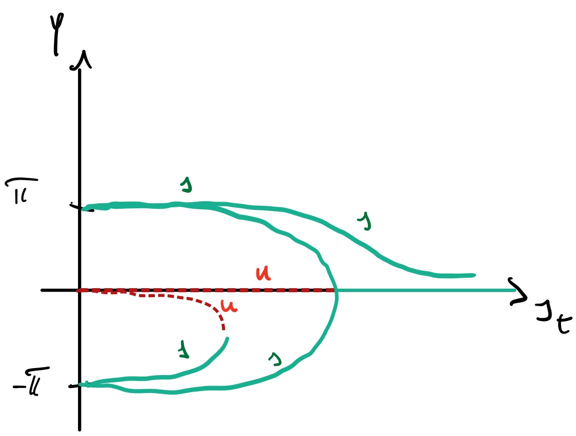
In real life the spring can't be relaxed exactly at $\varphi = 0$. There's always some error (φ^*)

$$U(\varphi) = \frac{1}{2}s_t(\varphi - \varphi^*)^2 + mg\frac{\ell}{2}\cos\varphi$$

$$\frac{\partial U}{\partial \varphi} = s_t(\varphi - \varphi^*) - mg\frac{\ell}{2}\sin\varphi = 0$$ => equilibria

$$mg\frac{\ell}{2}\sin\varphi = s_t\varphi - s_t\varphi^*$$

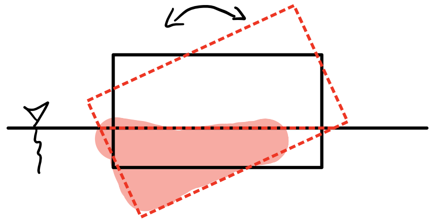




VII. Lecture

The boat problem

1 DoF

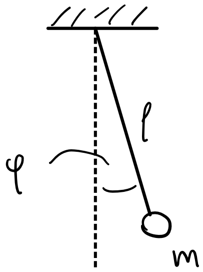


$\rightarrow$ Time period of oscillation, many pendulum next to each other video

Time period of oscillation in conservative non. lin. system

Example:

1 DoF



$T = \frac{1}{2} m \ell^2 \dot{\varphi}^2$

$U = -mg \ell \cos\varphi$

$\ddot{\varphi} + \underbrace{\frac{g}{\ell}}_{} \sin\varphi = 0$

$= \omega^2$ if φ is small

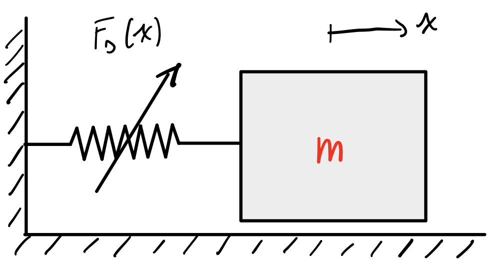
$(\varphi < 0.1 \ [rd]$ or $5°$?$)$

$\downarrow$

3 digit accuracy for time period

$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{\ell}{g}}$

If we observe it knowing the conservative sys. theory:

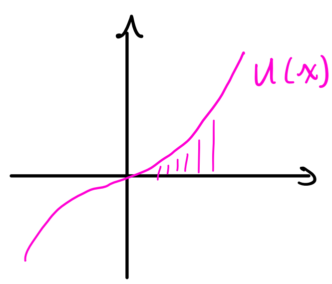


$m\ddot{x} + F_s(x) = 0$

$\frac{1}{2} m \dot{x}^2 + U(x) = 0$

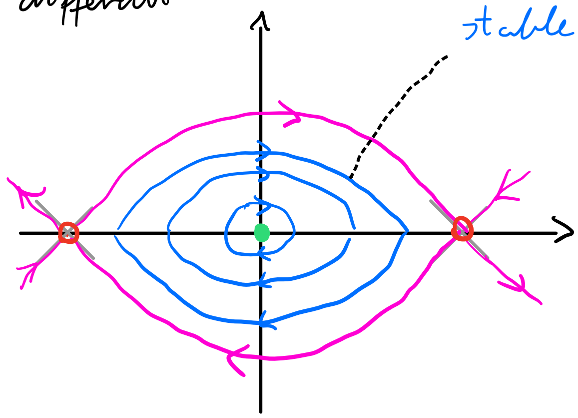
$h = \frac{1}{2} m v^2 + U(x)$

$T + U = h$



We can construct the orbits, regarding to the different energy levels:

In Ljapunov sense they are not stable but orbitally stable



Film: Gravity

How the time period changes if we are in different energy levels?

$$|\dot{x}| = \sqrt{\frac{2}{m}}\sqrt{h-U(x)} \Rightarrow \text{trajectories}$$

around center

for $\dot{x} \geq 0$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{h-U(x)} \to$$ separable 1^{st} order ODE

$$\sqrt{\frac{m}{2}}\int_A^B \frac{1}{\sqrt{h-U(x)}}\,dx = \int_0^{T/2} 1\,dt$$

$$T = \sqrt{2m}\int_A^B \frac{1}{\sqrt{h-U(x)}}\,dx$$ $\sim$ no closed form solution

↳ let's use approximation (linearize)

Approximate linearization

$$\ddot{x} + \underbrace{\frac{1}{m}F_s(x)}_{:= f(x)} = 0$$

$\sim$ For sake of simplicity let us have a symmetric nonlinearity: $f(x) = -f(-x)$

Linearization at equilibrium by power series

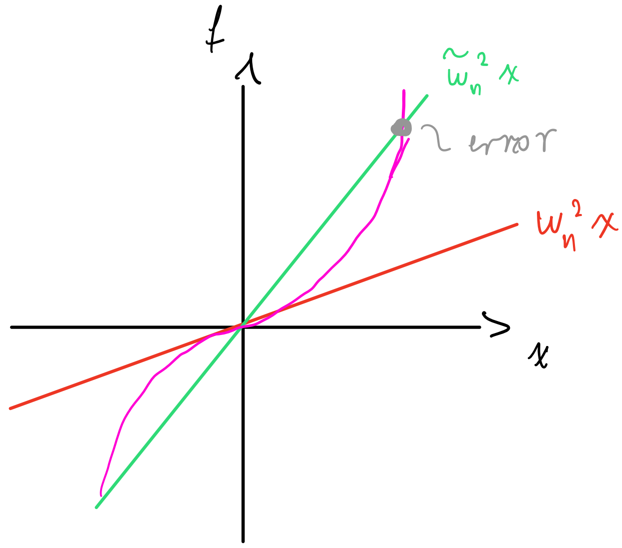
$$f(x) = \frac{1}{0!}\underbrace{f(0)}_{0} + \frac{1}{1!}\underbrace{f'(0)}_{:=\omega_n^2} x + \frac{1}{2!}\underbrace{f''(0)}_{\substack{0 \\ \text{sym eq.}}} x^2 + \frac{1}{3!} f'''(0)\, x^3 \ldots$$

1) lin. by tangent line

$$T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{f'(0)}}$$

It does not depend on the amplitude A!

which is not time



General example

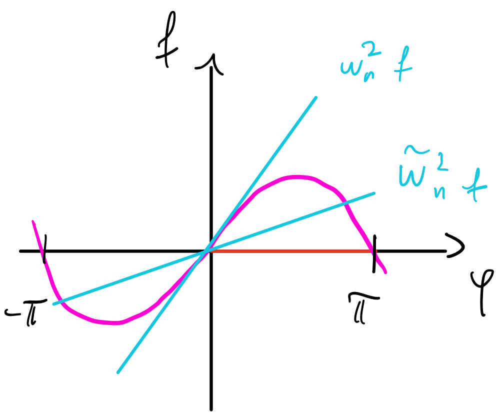
2) linearization by chord

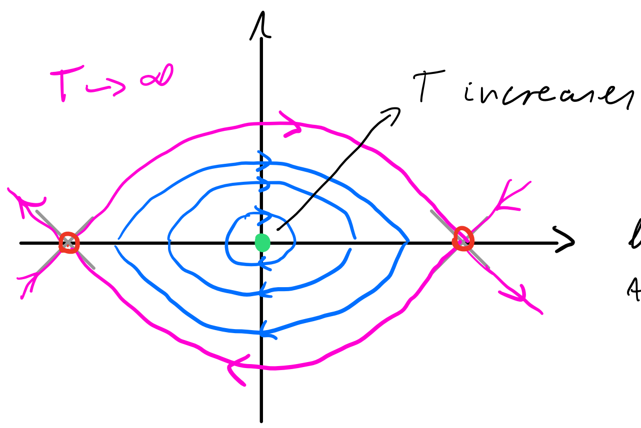
$$\tilde{\omega}_n^2 = \frac{f(A)}{A} \Rightarrow \tilde{T} \cong 2\pi\sqrt{\frac{A}{f(A)}} = T(A)$$

progressive spring $\rightarrow$ $\tilde{\omega}_n > \omega_n$ thus

$\tilde{T} < T$ decreases

Pendulum $\quad \ddot{\varphi} + \frac{g}{\ell}\sin\varphi = 0$





$\lim_{A \to \pi} T = +\infty$

VIII. Lecture

We have some better approximation, created with so called weight function $\varrho(x)$:

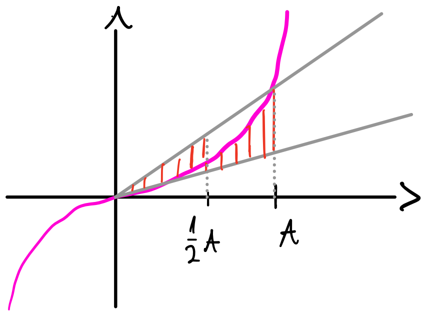
$$w_n^2 = \frac{\int_0^A \rho(x)\, f'(x)\, dx}{\int_0^A \varrho(x)\, dx}$$

We can choose ϱ arbitarly, for example:

$$\varrho(x) = 1 \Rightarrow w_n^2 = \frac{f(A) - \overbrace{f(0)}^{0}}{A - 0} = \frac{f(A)}{A} \quad \text{(chord)}$$

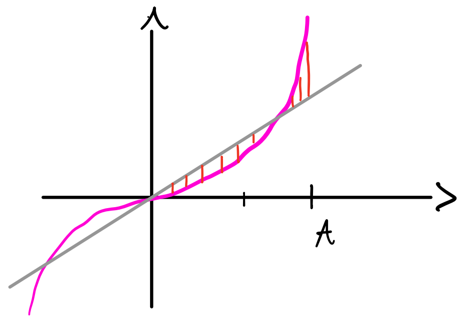
A better choice:

$$\varrho(x) = \begin{cases} 1, & x \in [0, A/2) \\ 1/2, & x \in [A/2, A] \end{cases}$$

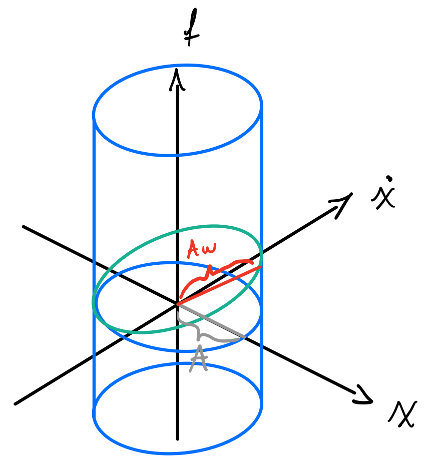


We can approximate by minimalization of the difference

$$\min_{w_n^2} \int_0^A (f(x) - w_n^2)^2\, dx$$



Linearization above the $(x, \dot{x})$ phase plane:



These were all direct linearizations! The first analytical (power series) approximation in Poincaré's small parameter method

$$\ddot{x} + f(x) = 0 \qquad \& \qquad f(x) = -f(-x)$$

↳ symmetrical

The power series of f looks like:

$$f(x) = \frac{1}{0!}\underbrace{f(0)}_{0} + \frac{1}{1!}\underbrace{f'(0)}_{\omega_n^2}x + \frac{1}{2!}\underbrace{f''(0)}_{0}x^2 + \frac{1}{3!}\underbrace{f'''(0)}_{\mu}x^3 + \dots$$

The simplest nonlinear case:

$$\ddot{x} + \omega_n^2 x + \mu x^3 = 0$$

Let's assume x as:

$$x(t) = \sum_{i=0}^{\infty} \varphi_i(t)\mu^i$$

Where the unknown functions $\varphi_\xi(t)$ satisfy:

$$\varphi_\xi(t) = \varphi_\xi(t + T_n), \quad \xi \in \mathbb{N} \qquad \text{s.t. } T_n = \frac{2\pi}{\beta}$$

Where β: is the non linear system ang. nat. freq.

$$\beta^2 = \underbrace{\alpha^2}_{h_0} + \mu h_1 + \mu^2 h_2 + \dots \Rightarrow w_n^2 = \beta^2 - \mu h_1 - \mu^2 h_2 - \dots$$

Why does it converges?

The IC looks like:

$$\left.\begin{array}{l} x(0) = A \\ \dot{x}(0) = 0 \end{array}\right\} \quad \begin{array}{l} \varphi_0(0) = A \\ \dot{\varphi}_0(0) = 0 \end{array} \quad \& \quad \begin{array}{l} \varphi_\xi(0) = 0 \\ \dot{\varphi}_\xi(0) = 0 \end{array}, \quad \xi \in \mathbb{N}$$

Let's substitute the series

$$(\ddot{\varphi}_0 + \mu\ddot{\varphi}_1 + \mu^2\ddot{\varphi}_2 + \dots) + (\beta^2 - \mu h_1 - \mu^2 h_2 - \dots) + (\varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \dots) + \mu(\varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \dots)^3 = 0$$

Let's use the polinomial balance

$$\boxed{\mu^0} \quad \left\{\begin{array}{l} \ddot{\varphi}_0 + \beta^2\varphi_0 = 0 \\ \varphi_0(0) = A \\ \dot{\varphi}(0) = 0 \end{array}\right\} \rightarrow \varphi_0(t) = A\cos(\beta t)$$

$\boxed{\mu^1}$

$$\begin{cases} \ddot{\varphi}_1 + \beta^2 \varphi_1 - h_1 \varphi_0 + \varphi_0^3 \Rightarrow \ddot{\varphi}_1 + \beta^2 \varphi_1 = \underbrace{\left(h_1 A - \frac{3}{4}A^3\right)}_{\text{Secular term}} \cos(\beta t) - \frac{1}{4}A^3 \cos(3\beta t) \\ \varphi_1(0) = 0 \\ \dot{\varphi}_1(0) = 0 \end{cases}$$

$= 0$ to have periodic sol for $\varphi_1(t)$

$t \cdot \cos(t)$ ✗

$\Downarrow$

$$h_1 = \frac{3}{4} A^2$$

$\Downarrow$

$$\beta^2 = w_n^2 + \frac{3}{4}\mu A^2 + \dots$$

$\Downarrow$ if we neglect h_2

$$T_n \simeq \frac{2\pi}{\sqrt{w_n^2 + \frac{3}{4}\mu A^2}}$$

Example: Pendulum

$$\ddot{\varphi} + \underbrace{\frac{g}{l}\varphi - \frac{g}{6l}\varphi^3}_{} = 0$$

$$\varphi - \frac{1}{6}\varphi^3 \simeq \sin\varphi$$

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

$$\mu = -\frac{g}{6l} \; ; \; w_n^2 = \frac{g}{l}$$

$$\beta^2 = \frac{g}{l} + \frac{3}{4}\left(-\frac{9}{6l}\right)A^2 = \frac{g}{l}\cdot\left(1 - \frac{A^2}{8}\right)$$

$$T(A) = \frac{2\pi}{\sqrt{\frac{g}{l} - \frac{3}{4}\cdot\frac{1}{6}\frac{g}{l} A^2}}$$

Approx formula: $T(A) \to \infty$, $A \to 2\sqrt{2} = 2.82$

The exact solution in $A = \pi$ when $T(A) \to \infty$ // big error //

$$\begin{cases} \ddot{\varphi}_1 + \beta^2 \varphi_1 = -\frac{1}{4}A^3 \cos(3\beta t) \\ \varphi_1(0) = A \\ \dot{\varphi}_1(0) = 0 \end{cases}$$

$$\varphi_1(t) = -\frac{A^3}{32\beta^2}\cos(\beta t) + \frac{A^3}{32\beta^2}\cos(3\beta t)$$

$$h_1 = \frac{3}{4}A^2$$

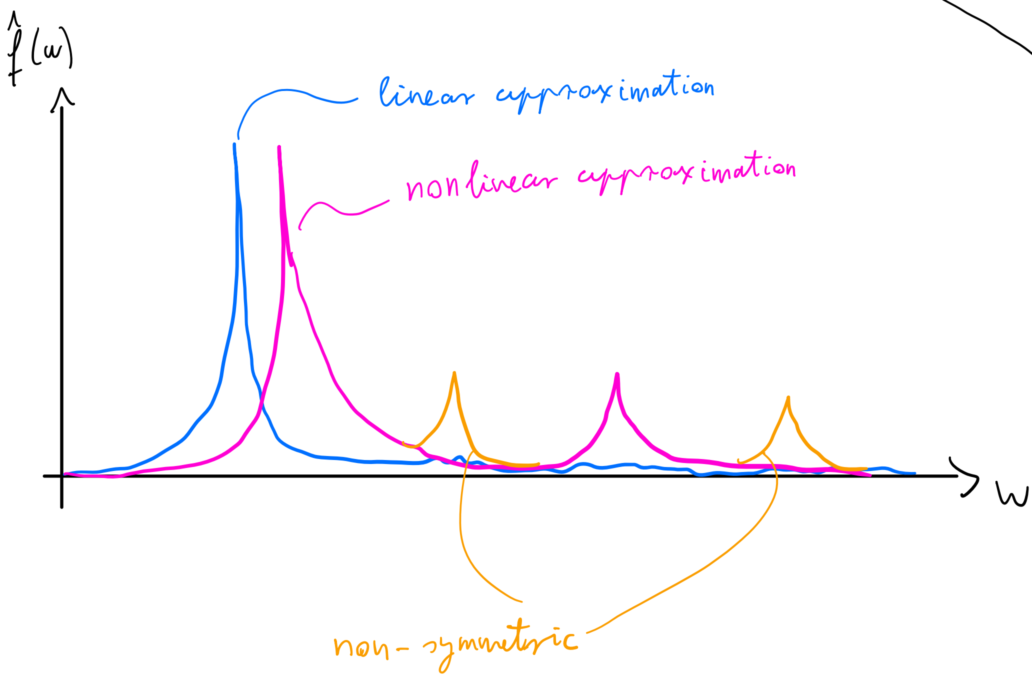
The first nonlinear approx solution

$$x(t) = \varphi_0(t) + \mu\,\varphi_1(t) + \dots = \left(A - \mu\frac{A^2}{32\beta}\right)\cos(\beta t) + \mu\frac{A^3}{32\beta^2}\cos(3\beta t) + \dots$$

Generally:

$$x(t) = \underbrace{\sum_{n=0}^{\infty} \left[\underbrace{\left(\sum_{s=0}^{\infty} a_s \mu^s \right)}_{\text{Taylor series}} \cos(2n+1)\beta t \right]}_{\text{Fourier series}}$$

In time series & phase plane the approximation error is not easy to recognise, but if we check the frequency domain



Linear → 1 peak

Non lin → ∞ peaks

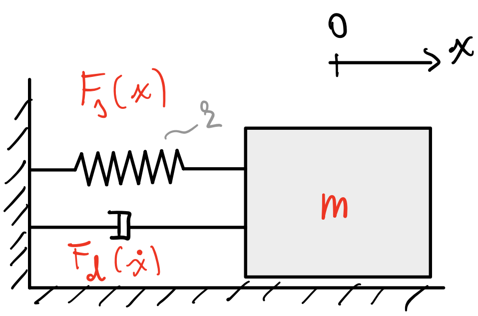
if odd freq. → symmetric

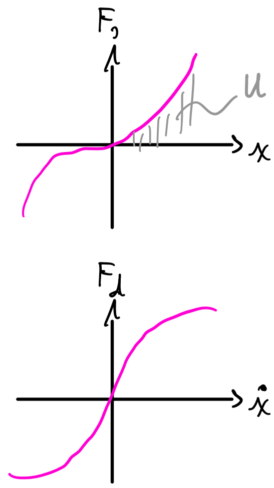
if asymmetric → all multipliers

IV. Lecture

Nonlinear - damping

General case





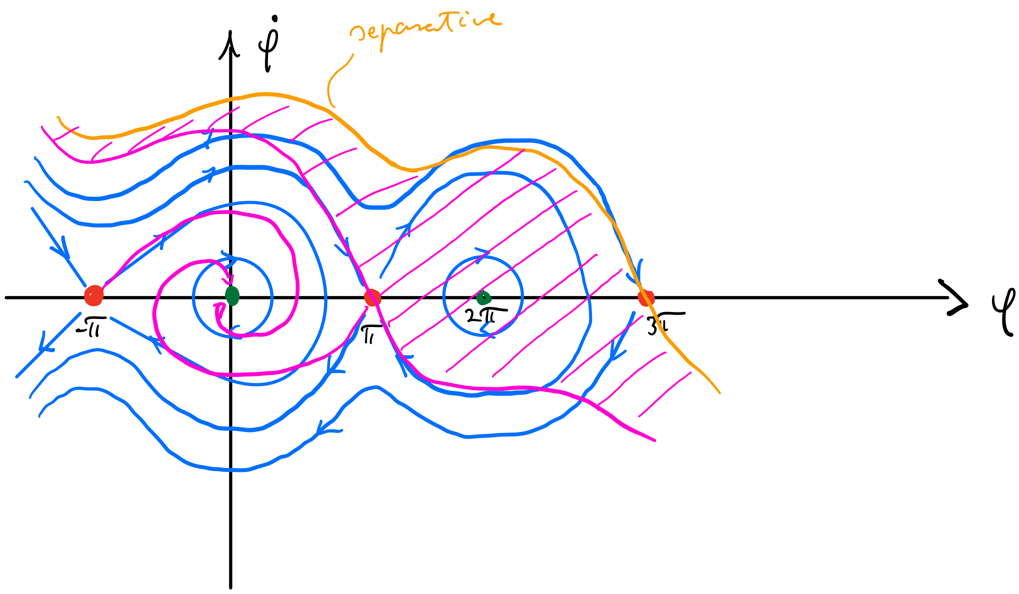
$$m\ddot{x} + F_d(\dot{x}) + F_g(x) = 0$$

$$\Downarrow$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = \int_{t_0}^{t} \underbrace{(-F_d(\dot{x})\,\dot{x})}_{P_d \leq 0}\,dt$$

We can see the positive damping gives us a negative power, which means the total mechanical energy decreases.

Phase plane (damped pendulum)



Back to the diff. eq.:

$$\ddot{x} + \frac{1}{m}F_d(\dot{x}) + \underbrace{\frac{1}{m}F_g(x)}_{\approx x} = 0 \qquad \text{// dimensionless time}$$

$$w_n^2 x'' + \frac{1}{m} F_d(w_n \dot{x}) + \frac{k}{m} x = 0$$

$$\Downarrow$$

$$w_n^2 x'' + \frac{1}{m} F_d(w_n \dot{x}) + \frac{k}{m} x = 0$$

$$\Downarrow$$

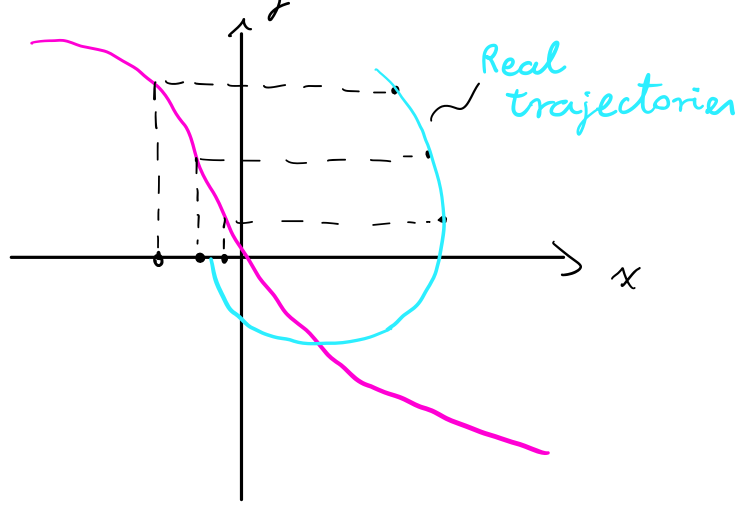
$$x'' + \frac{1}{m w^2} F_d(w_n \dot{x}) + \underbrace{\frac{k}{m w_n^2}}_{1} x = 0$$ // $\varphi(\dot{x})$ - dimensionless damping

Using the Cauchy transformation $\begin{cases} x' = y \\ y' = -x - \varphi(y) \end{cases}$

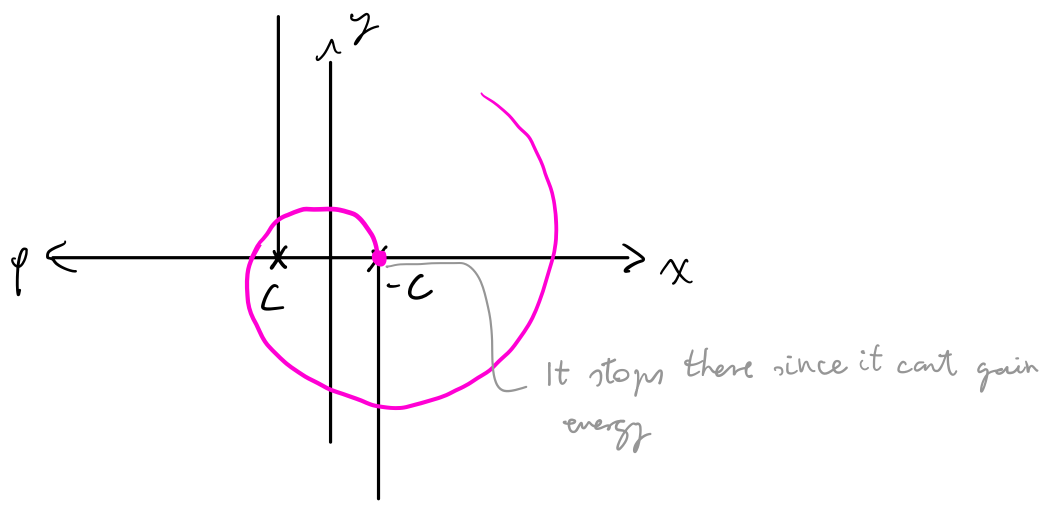
The trajectories in the phase plane:

$$\frac{dy}{dx} = -\frac{x + \varphi(y)}{y}$$

$$\frac{dx}{dy} = -\frac{1}{\frac{x + \varphi(y)}{y}}$$

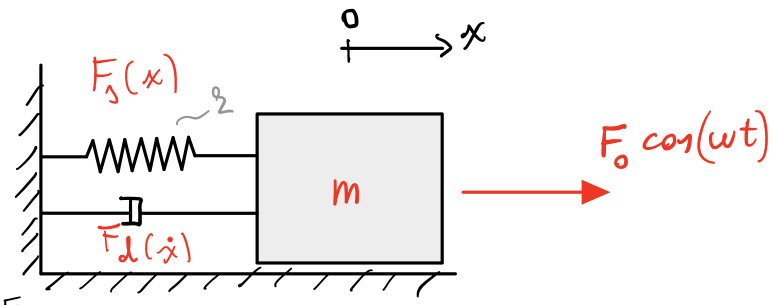


This method accurate in case of Coulomb friction



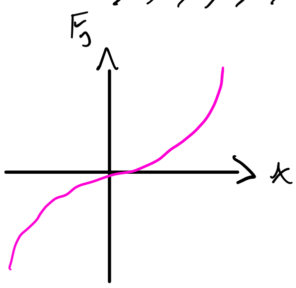
Harmonically excited 1DoF nonlinear oscillator

~> Nonlinear forced system



The eq. of motion is for a nonlin. harmonically excited oscillation of 1DoF with mass m, nonlinear spring $F_s(x)$ and linear damping $F_d(\dot{x})$:

$$m\ddot{x} + c\dot{x} + F_s(x) = F_0 \cos(\omega t)$$



The general solution for the eq. of motion can't be determined by calculating the homogeneous and inhomogeneous part of the function due to nonlinear spring:

$$x(t) \neq x_h(t) + x_p(t)$$

To get the characteristic of the system, the inhomogeneous part of the solution have to be obtained, because

$$\lim_{t \to \infty} x(t) = x_p(t)$$

Which means that $x_p(t)$ is the stationary solution after the $x_h(t)$ transient solution dies out.

~ Amplitude of nonlinear forced system ~

To determine $x_p(t)$ the following assumptions have to be made:

-> The damping is negligible: $c = 0$

-> The nonlinear spring is symmetric: $F_s(x) = -F_s(-x)$

-> The nonlinearity is estimated with Taylor series:

$$F_s(x) = \underset{>0}{\underbrace{s}}\, x + s_3 x^3$$

a) $s_3 > 0$: progressive, hardening

b) $s_3 < 0$: degressive, softening

The resulting eq. of motion:

$$m\ddot{x} + s x + s_3 x^3 = F_0 \cos(\omega t)$$

$$\ddot{x} + \omega_n^2 x + \mu x^3 = f_0 \omega_n^2 \cos(\omega t)$$

Where $\omega_n^2 = \frac{s}{m}$, $\mu = \frac{s_3}{m}$, $f_0 = \frac{F_0}{m\omega_n^2}$

~ Linearization by Poincaré's small parameter method ~

1. Introducing the small parameter

$$\mu\phi = f_0 \omega_n^2$$

$$\mu\Psi = \omega^2 - \omega_n^2$$

↳ Subs into the eq. of motion:

$$\ddot{x} + \omega^2 x \overbrace{- \omega^2 x + \omega_n^2 x}^{-\mu\Psi x} + \mu x^3 = \mu\phi\cos(\omega t)$$

$$\ddot{x} + \omega^2 x = \mu\left[\Psi x - x^3 + \phi\cos(\omega t)\right]$$

2. Form the solution.

The solution is searched as a power series of the μ small parameter:

$$x_r(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) \dots$$

We know that the solution has to be periodic function, since the excitation is also periodic.

$$x_z(t) = x_z(t+T) \quad \text{with } T = \frac{2\pi}{\omega}$$

3. Substitution

Substitute the power series back to the eq. of motion

$$(\ddot{x}_0 + \mu_1 \ddot{x}_1 + \ldots) + w^2 (x_0 + \mu x_1 + \ldots) = \mu \left[\psi (x_0 + \mu x_1 + \ldots) - (x_0 + \mu x_1 + \ldots)^3 + \phi \cos(wt) \right]$$

The μ terms have to be balanced by degree

$$\mu^0: \quad \ddot{x}_0 + w^2 x_0 = 0$$

$$\mu^1: \quad \ddot{x}_1 + w^2 x_1 = \psi x_0 - x_0^3 + \phi \cos(wt)$$

The solution for unknown functions x_0 comes from solving the second order diff. eq.

$$x_0(t) = A\cos(wt) + B\sin(wt)$$

The diff eq. for x_1 can be written in other form using the function $x_0(t)$

$$\ddot{x}_1 + w^2 x_1 = \underbrace{\left(\psi A - \frac{3}{4} A^3 - \frac{3}{4} A B^2 + \phi\right)}_{\text{Resonance!}} \cos(wt) +$$

$$+ \underbrace{\left(\psi B - \frac{3}{4} A^2 B + \frac{3}{4} B^3\right)}_{\text{Resonance!}} \sin(wt) \ldots$$

The resonating terms have to vanish to fullfill the periodicity requirement, therefore:

$$\psi B - \frac{3}{4} A^2 B + \frac{3}{4} B^3 = 0 \qquad // \ B \Rightarrow 0$$

$$\psi A - \frac{3}{4} A^3 - \frac{3}{4} A B^2 + \phi = 0 \qquad // \ A$$

~ The amplitude equation ~

The A amplitude of the $x_p(t)$ stationary solution of nonlinear forced systems can be determined by the following equation

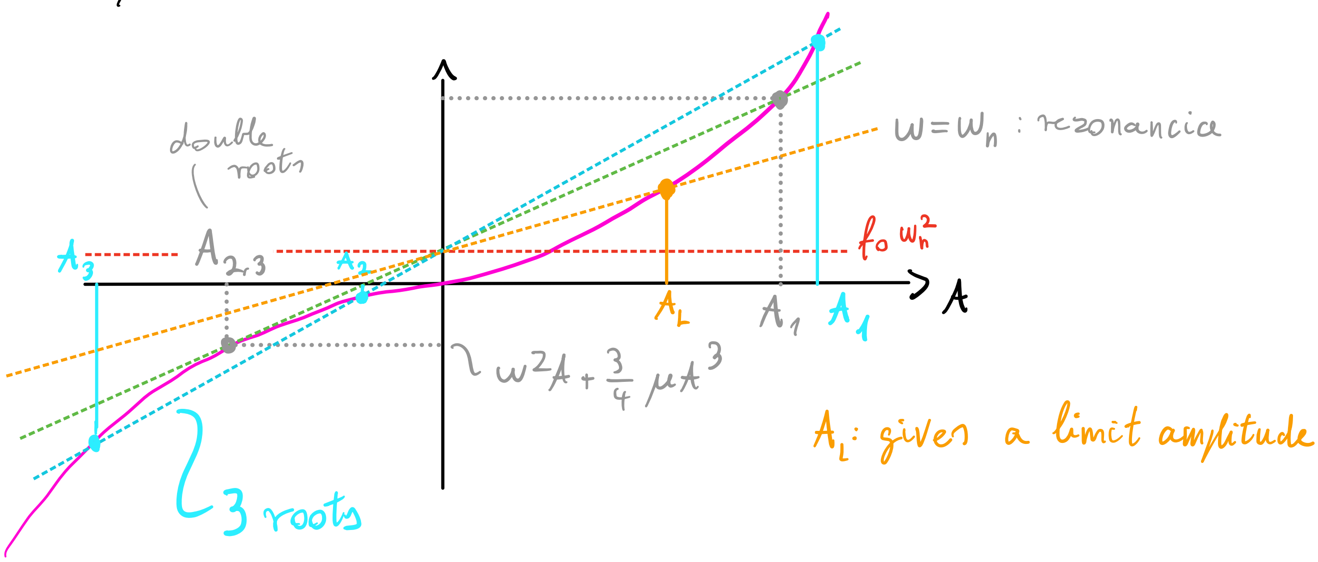
$$f_0 w_n^2 + w^2 A = w_n^2 A + \frac{3}{4}\mu A^3$$

The characteristic of the nonlinearity depends on the sign of the μ parameter

$\mu > 0$: Hyperbolic (hardening) char.

$\mu < 0$: Eliptic (softening) char.

For $w > w^*$ there is 3 distinct solution for the amplitude equation: A_1, $A_{2,3}$. The eq. can be solved graphically



The w^* angular frequency can be calculated if a $A_{2,3}$ double zero root exits, which means that $f(A) = 0$ and $f'(A) = 0$

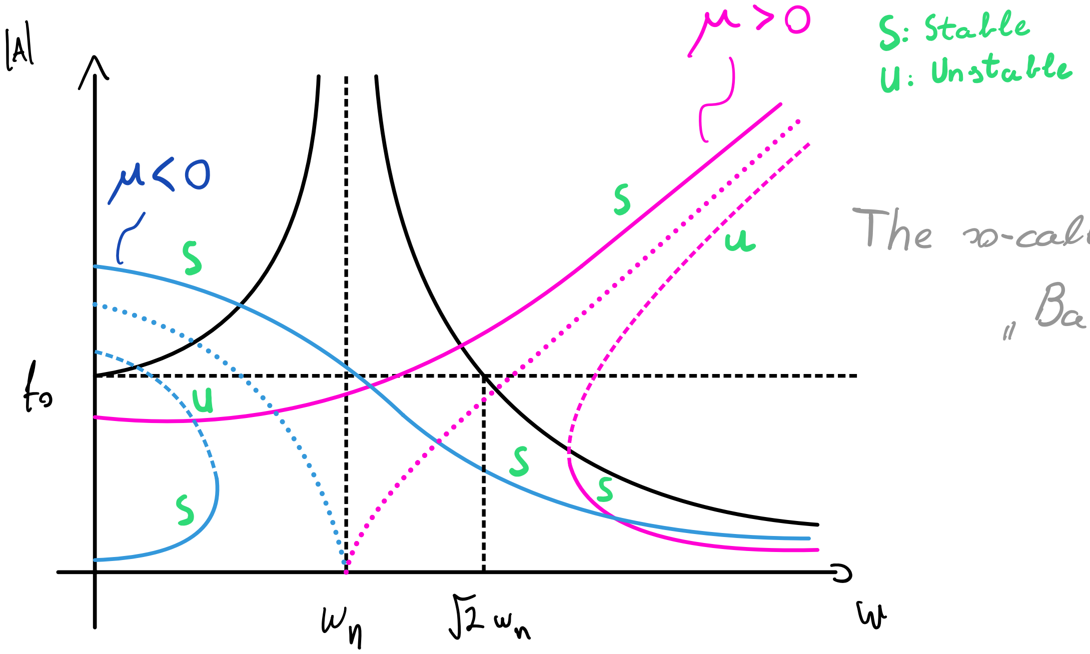
$$A^*_{2,3} = -\sqrt[3]{\frac{2 f_0 w_n^2}{3\mu}}$$

$$w^* = \sqrt{w_n^2 + \frac{3}{2}\sqrt[3]{\frac{3}{2} f_0^2 w_n^4 \mu}}$$

The amplification plot shows the amplitude of the oscillation in function of the angular frequency of the external force

$\mu > 0$: Hyperbolic (hardening) char.

$\mu < 0$: Eliptic (softening) char.



The so-called: „Backbone curve"

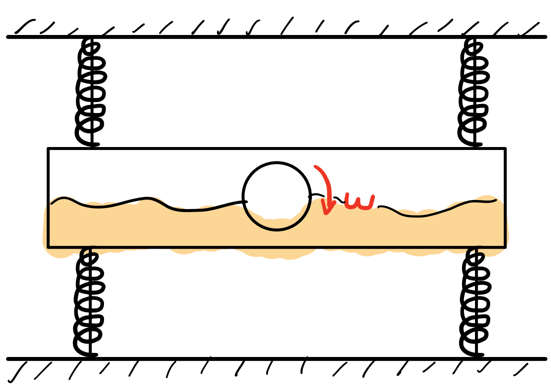
Backbone curve:

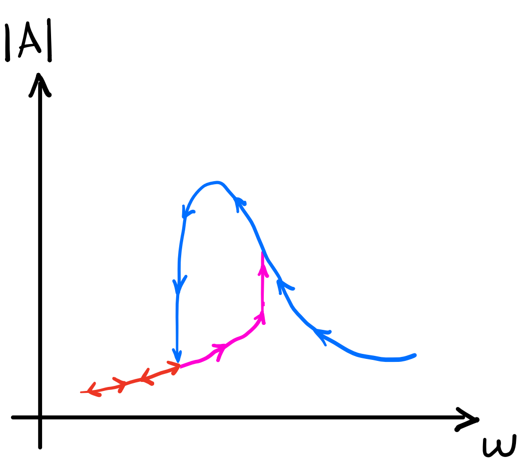
The backbone curve of the mechanical system is the curve of maximal amplitude of the periodic response, where $f_0 \to 0$

$$\frac{\omega^2}{\omega_n^2} - \frac{A^2}{\frac{4}{3\mu}\,\omega_n^2} = 1$$

Example:

Wet sand drier machine with a shaker



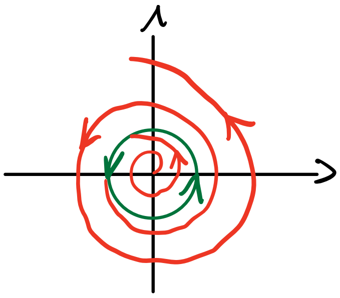


Nonlinear 1DoF self-excited vibrations

Self-excited system:

The nonlinear self-excited vibration is mathematically a limit cycle on the phase plane of the nonlinear system.

The limit cycle is a closed isolated orbit in the phase plane of autonomous system (no excitation)



~ Liénard's Theorem ~

The sufficient condition for existence of a limit cycle is stated by the Liénard's criterion

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

The conditions of the Liénard's criterion

① Nonlinearity of the spring. The $g(x)$ nonlinearity function of the spring have to be odd: $x\,g(x) > 0$

② Nonlinearity of the damping. The $f(x)$ nonlinearity function of the damping have to be even.

$$f(x) = f(-x)$$

The $F(x)$ function have to exist as the integral of $f(x)$ nonlinearity function of the damping.

$$F(x) = \int_0^x f(\xi)\, d\xi$$

$$\lim_{x \to \infty} F(x) = \infty$$

Condition for the $F(x)$ function:

→ The $F(x)$ has two roots: $F(a) = F(-a) = 0$

→ The only positive root of $F(x)$ is $F(a) = 0$

→ The $F(x)$ function increases monotonically $\lim_{x \to \infty} F(x) = \infty$

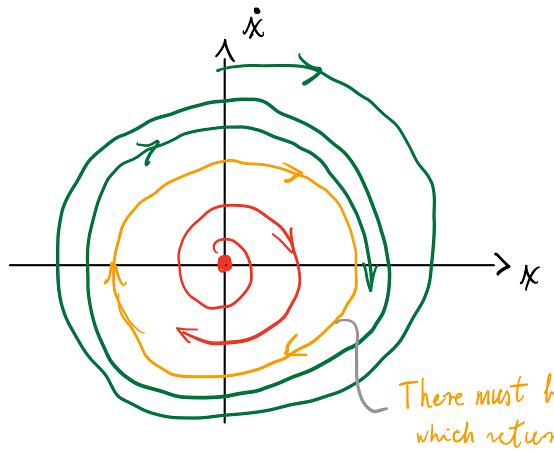
IF the Liénard's condition are fulfilled, then there is an orbitally asymptotically stable limit cycle in the system.

PROOF

The proof is proceeded by linearization of the nonlinear differential equation around 0.

$$f(x) = f(0) + \underbrace{x f'(0)}^{0} + \dots$$

$$g(x) = \underbrace{g(0)}_{0} + x g'(0) + \dots$$



The linearised differential equation

$$\ddot{x} + \underbrace{f(0)}_{<0} \dot{x} + \underbrace{g'(0)}_{>0} x = 0$$

For large x values there is positive damping, so the trajectories converge to an isolated periodic orbit (limit cycle)

Bendixon's theorem

The necessary condition for existence of a limit cycle is stated by the Bendixon's criterion

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\} \quad B(x,y) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \underline{\nabla} \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

IF there exists a simply connected set S in the phase plane (x,y) such that B either positive ⊕ or negative ⊖ in S, then there is no limit cycle in S

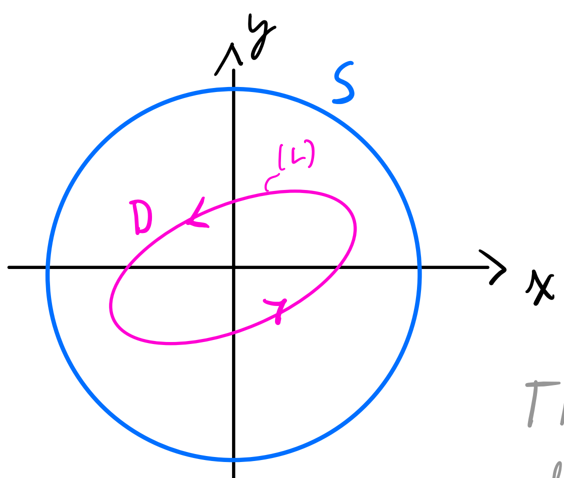
PROOF

Consider the vector field $\underline{v}$ which is perpendicular to the limit cycle L because it satisfie the diff. eq.

$$\underline{v}(x,y) = \begin{bmatrix} -g(x,y) \\ f(x,y) \\ 0 \end{bmatrix}$$

$$\oint_L \underline{v}(\underline{r})\, d\underline{r} = \int_0^T (-gf + fg)\, dt = 0$$

$$\oint_L \underline{v}(\underline{r})\, d\underline{r} = \underbrace{\int_A \text{curl}\, \underline{v}(\underline{r})\, d\underline{A} = \int_A B(\underline{r})\, dA}_{\text{Stokes-theorem}} \neq 0$$



There is a contradiction, because B have to be either positive or negative in S, thus limit cycle can't exist in S.

~ Hopf's bifurcation theorem

The Hopf's theorem states the neccesary and sufficient conditions for existence of limit cycles at once. The system:

$$\dot{x} = f(x, y, \mu)$$
$$\dot{y} = g(x, y, \mu)$$

Where $(x, y) \in \mathbb{R}^2$ are state variables, $\mu \in \mathbb{R}$ system parameter and f, g is a 4-times continuous differentiable functions.

The investigated system is linearized around its equlibrium point at the origin $f(0,0,\mu) = 0$, $g(0,0\mu) = 0$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} f'_x(0,0,\mu) & f'_y(0,0,\mu) \\ g'_x(0,0,\mu) & g'_y(0,0,\mu) \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x \\ y \end{bmatrix} + h.o.t$$

The characteristic roots (eigenvalues) can be calculated from the $\underline{\underline{A}}(\mu)$ coefficient matrix as follows

$$\det(\lambda \underline{\underline{I}} - \underline{\underline{A}}(\mu)) = 0$$

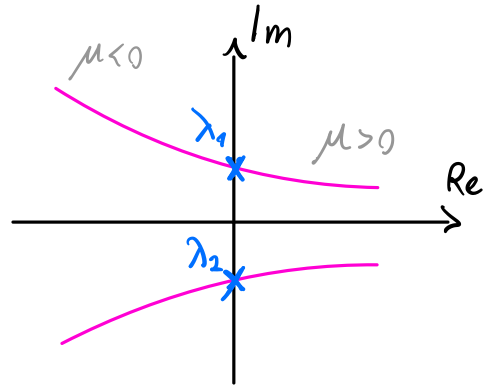
$$\lambda_{1,2}(\mu) = \sigma(\mu) \pm i\omega(\mu)$$

The conditions for the characteristic roots (eigenvalues) are

$\omega(0) > 0$ → really immaginary

$\sigma(0) > 0$ → if $\mu = 0$ pure characteristic

$$\sigma'(0) = \left.\frac{\partial \sigma}{\partial \mu}\right|_{\mu=0} > 0$$

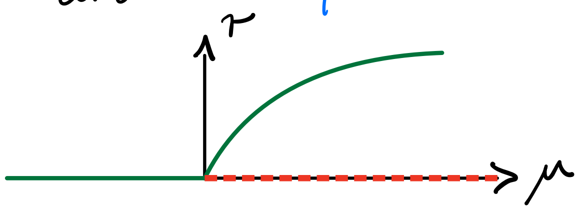


IF the conditions are fulfilled, then there exists a $\mu(x_0) \in C^2(-\varepsilon, \varepsilon)$, where the solution is for $x_0 \in (-\varepsilon, \varepsilon)$ starting at the initial condition $(x_0, 0)$ periodic with the approximate time period of $\frac{2\pi}{\omega_n(0)}$

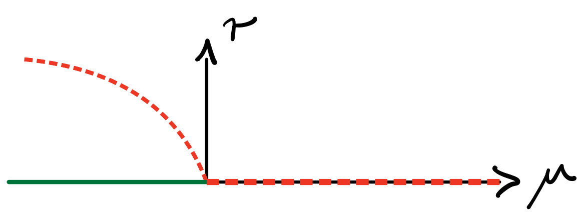
$$\mu(x_0) = c\, x_0^2 + \delta(x_0)$$

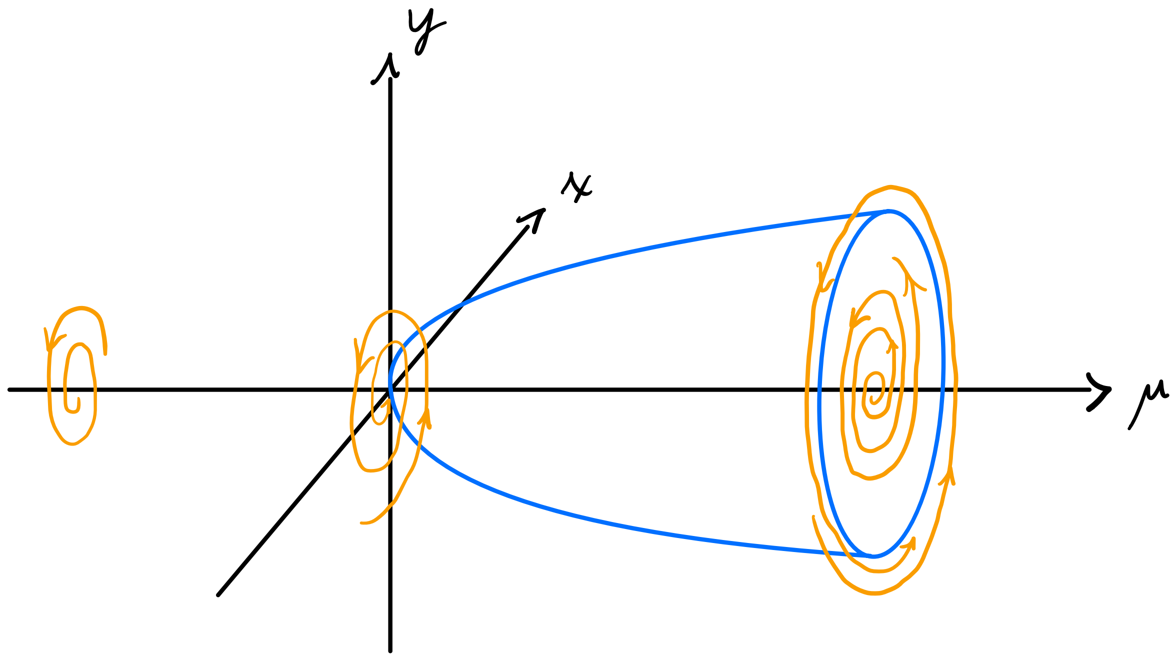
The stability and the bifurcation depends on the value of c:

→ IF $c > 0$ then the closed orbits are asymptotically stable and the bifurcation is supercritical



→ IF $c < 0$ then the closed orbits are unstable and the bifurcation is subcritical





The Hopf's bifurcation theorem does not give a method to calculate the value of c. The theory of normal forms searches for algorithms for calculating c

~ Theory of normal formes

The theory of normal forms searches for algorithms to calculate the value of c and so amplitude of limit cycles.

$$\dot{\underline{x}} = \underline{\underline{A}}(\mu)\,\underline{x}$$

~ Jordan - normal form

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \sigma(\mu) & \omega(\mu) \\ -\omega(\mu) & \sigma(\mu) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} \sum\limits_{j+k=2,3} a_{jk}(\mu)\,\xi^j \eta^k \\ \sum\limits_{j+k=2,3} b_{jk}(\mu)\,\xi^j \eta^k \end{bmatrix}$$

~ Poincaré normal form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \sigma(\mu) & \omega(\mu) \\ -\omega(\mu) & \sigma(\mu) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \delta u^3 + \beta u^2 v + \delta u v^2 + \beta v^3 \\ -\beta u^3 + \delta u^2 v - \beta u v^2 + \delta v^3 \end{bmatrix}$$

The Poincaré normal form can be written in polar coordinates as well. The polar transformation is:

$$u = r \sin \varphi$$

$$v = r \cos \varphi$$

The tranformation parameters can be calculated as:

$$\dot{r} = \sigma(\mu) r + \delta(\mu) r^3$$

$$\dot{\varphi} = \omega(\mu) + \text{h.o.t.}$$

The value of r can be given in closed form with the $\delta(0)$ Poincaré-Ljapunov constant

$$r = \sqrt{-\frac{\sigma'(0)}{\delta(0)}\mu}$$

The $\delta(0)$ Poincaré-Ljapunov constant has the role of parameter c, the sub- and supercritical bifurcation depends on it.