Homework 2

Subject code: BMEGEMMNWEP, *Subject Name:* Elasticity & Plasticity

The stress history at a particular material point is given as follows:

$$\mathbf{0} \longrightarrow \boldsymbol{\sigma}_A \longrightarrow \boldsymbol{\sigma}_B \longrightarrow \mathbf{0}$$

Step 1: From the initial stress- and strain-free configuration, the stress is increased to stress state $\boldsymbol{\sigma}_A$. The variation between states 0 and A is linear for all stress components.

Step 2: In the second step, the stress state is modified from $\boldsymbol{\sigma}_A$ to $\boldsymbol{\sigma}_B$. The variation between states A and B is linear for all stress components.

Step 3: In the last step, the stress components are reduced to zero values. Therefore, this step is an unloading step. The final configuration is labelled with state C. The variation between states B and C is linear for all stress components.

The material behavior is linear elastic – linear isotropic hardening governed by the Mises yield criterion. Material constants:

E: Young's modulus.

ν: Poisson's ratio.

σ_{Y0}: Initial yield stress.

H: Plastic hardening modulus.

The stress components are given using the Voigt notation! Use the Voigt notation in your calculations!

<u>Tasks:</u>

a) Let σ and τ to be the non-zero normal and shear stress components in $\boldsymbol{\sigma}_B$. Illustrate the stress loading in the $\{\sigma, \tau\}$ coordinate system. Plot the corresponding yield loci at state A, B and at the particular state (labelled with Y), where the initial yield stress is reached.

a) Determine the total strain components at states Y, A, B and C!

b) Determine the plastic strain components at states Y, A, B and C!

Use analytical calculation and do not use numerical methods!

Your data can be downloaded from the following link:

`http://www.mm.bme.hu/~kossa/NEPTUN-HW2.pdf`

The string **NEPTUN** has to be replaced with the Student's NEPTUN code. The link above is case-sensitive! Use uppercase letters for the file-name!

The homework assignment must include comprehensive and detailed derivations for all calculations, with each step of the calculation explained thoroughly in written form. The primary focus should be on the solution process rather than the numerical values obtained. The use of computer software is permitted for numerical calculations, with the accompanying code to be included as an appendix to the document.

Budapest University of Technology and Economics
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Elasticity and Plasticity
II. Homework

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Contents

Initial data

The initial data given based on my NEPTUN code can be seen on Figure 2.

NEPTUN code	E [GPa]	$\nu\,[-]$	σ_{Y0} [MPa]	H [GPa]	σ_A [MPa]	σ_B [MPa]
GBBNUL	160	0.24	110	14	$\begin{pmatrix}150\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}200\\0\\0\\100\\0\\0\end{pmatrix}$

Figure 1: Initial values based on NEPTUN code

Loading Process and Yield Locus

As the task description describe the loading process can be devided into 4 different step as Table 1 shows.

	IC	**Step 1**	**Step 2**	**Step 3**	**Step 4**
Name	0	Y	A	B	C (or 0)
Time step [s]	0	t_Y	$t_A = 1$	$t_B = 2$	$t_C = 3$
Stress	$\mathbf{0}$	$\boldsymbol{\sigma}_Y$	$\boldsymbol{\sigma}_A$	$\boldsymbol{\sigma}_B$	$\mathbf{0}$
Strain	$\mathbf{0}$	$\boldsymbol{\varepsilon}_Y^e \neq \mathbf{0}$ $\boldsymbol{\varepsilon}_Y^p = \mathbf{0}$	$\boldsymbol{\varepsilon}_A$	$\boldsymbol{\varepsilon}_B$	$\boldsymbol{\varepsilon}_C^e = \mathbf{0}$ $\boldsymbol{\varepsilon}_C^p \neq \mathbf{0}$
Type of deformation	-	Elastic	Elastic-Plastic	Elastic-Plastic	Unloading

Table 1: Loading steps and notations

From Step 3 to Step 4 there is an unloading process thus the structure comes back to a stress- and strain free state regarding only the elastic part of the strain while the plastic part is non-zero. Separating 0 to A step into only elastic and elastic-plastic deformation helps us to solve the problem easier.

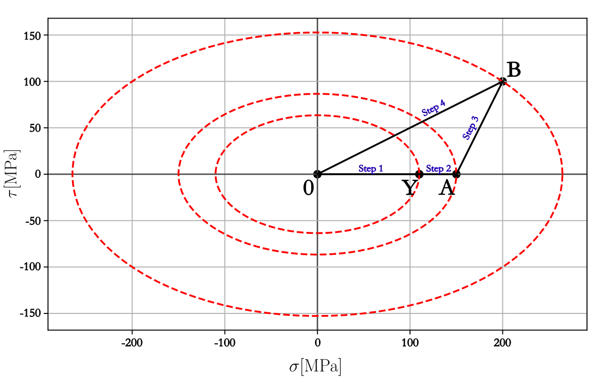

Figure 2: Yield locus illustration

The material behavior is governed by the Mises yield criterion (as the task description says). The distance of point B from 0 is calculated as:

$$F = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} - \sigma_Y = 0 \tag{1}$$

$$\sqrt{\sigma_{B_x}^2 + 3\tau_{B_{xy}}^2} = 264.575\,[\mathrm{MPa}] \tag{2}$$

1 Step 1 - 0 to Y

Step 1 involves transitioning from point 0 to point Y under purely elastic deformation. Here, the material obeys Hooke's Law, allowing for a direct calculation of strain given the applied stress through the elastic modulus and Poisson's ratio as

$$\boldsymbol{\varepsilon}_Y = \boldsymbol{C}^e\ \boldsymbol{\sigma}_Y, \tag{3}$$

$$\boldsymbol{\varepsilon}_Y = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{A_x}\ t_Y \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

In 1 second we want to get from point 0 to A with a linearly increasing load. Due to the linearity t_Y can be calculated as

$$t_Y = \frac{\sigma_{Y0}}{\sigma_{A_x}} = 0.733\ [\mathrm{s}]. \tag{5}$$

Since we determined every unknown parameter $\boldsymbol{\varepsilon}_Y$ can be determined.

$$\boldsymbol{\varepsilon}_Y = \begin{bmatrix} 0.0006875 \\ -0.000165 \\ -0.000165 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Keep in mind that during this step there is no plastic deformation $\boldsymbol{\varepsilon}^p_Y = \mathbf{0}$.

2 Step 2 - Y to A

In Step 2, transitioning from point Y to point A involves the onset of plastic deformation. Due to plastic deformation we have to use the elastic-plastic compliance matrix and the time derivative of stress and strain as:

$$\dot{\boldsymbol{\varepsilon}}_A = \boldsymbol{C}^{ep}_A\ \dot{\boldsymbol{\sigma}}_A, \tag{7}$$

$$\boldsymbol{C}^{ep}_A = \boldsymbol{C}^e + \frac{9}{4H} \frac{\tilde{\boldsymbol{s}}_A \otimes \tilde{\boldsymbol{s}}_A}{\sigma^2_{Y_A}}. \tag{8}$$

In order to determine the elastic-plastic compliance matrix first, we have to calculate the deviatoric stress tensor $\tilde{\boldsymbol{s}}_A$ for point A, that can be done as

$$\tilde{\boldsymbol{s}}_A = \boldsymbol{\sigma}_A - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}_A)\ \mathrm{I} = \begin{bmatrix} \frac{2}{3}\sigma_{A_x} & -\frac{1}{3}\sigma_{A_x} & -\frac{1}{3}\sigma_{A_x} & 2\tau_{xy} & 2\tau_{xz} & 2\tau_{yz} \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

The increasing load over time, can be represented as $\boldsymbol{\sigma}_A(t)$. Due to the linear increase this can be written as simple as

$$\boldsymbol{\sigma}_A(t) = \begin{bmatrix} \sigma_{A_x}\ t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \quad \rightarrow \quad \dot{\boldsymbol{\sigma}}_A(t) = \begin{bmatrix} \sigma_{A_x} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10}$$

From that the deviatoric stress tensor is

$$\tilde{\boldsymbol{s}}_A = \begin{bmatrix} 100t & -50t & -50t & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{11}$$

The yield stress function of time can ve calculated as

$$\sigma_{Y_A}^2 = \sigma_{A_x}^2 + 3\tau_{xy}^2 = (150t)^2 \tag{12}$$

After the simplification of Equation (8) we get the following equation:

$$\boldsymbol{C}_A^{ep} = \boldsymbol{C}^e + \frac{1}{H} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

To obtain the total strain at point A, we have to combine the already present strain with the incremental strain resulting from elastic-plastic deformation, computed through an integration (applied stress over the time interval from t_Y to t_A).

$$\boldsymbol{\varepsilon}_A = \boldsymbol{\varepsilon}_Y + \int_{t_Y}^{t_A} \boldsymbol{C}_A^{ep} \dot{\boldsymbol{\sigma}}_A \ \mathrm{d}t \tag{14}$$

$$\boldsymbol{\varepsilon}_A = \boldsymbol{\varepsilon}_Y + \sigma_{A_x} \left(t_A - t_Y\right) \begin{bmatrix} \frac{1}{H} + \frac{1}{E} \\ -\frac{1}{2H} - \frac{\nu}{E} \\ -\frac{1}{2H} - \frac{\nu}{E} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

The resulting total, elastic and plastic strain is determined as:

$$\boldsymbol{\varepsilon}_A = \begin{bmatrix} 0.0038 \\ -0.00165 \\ -0.00165 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{16}$$

$$\boldsymbol{\varepsilon}_A^e = \boldsymbol{C}^e \ \boldsymbol{\sigma}_A = \begin{bmatrix} 0.000938 \\ -0.000225 \\ -0.000225 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon}_A^p = \boldsymbol{\varepsilon}_A - \boldsymbol{\varepsilon}_A^e = \begin{bmatrix} 0.00286 \\ -0.00143 \\ -0.00143 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

3 Step 3 - A to B

In Step 3, the calculation process is similar to Step 2. The two key differences is that the shear stress component $\tau_{B_{xy}}$ is non-zero meaning a different loading condition, and the time interval extends from 1 to 2 seconds.

$$\boldsymbol{\sigma}(t) = (\boldsymbol{\sigma}_B - \boldsymbol{\sigma}_A) \ (t-1) + \boldsymbol{\sigma}_A = \begin{bmatrix} (\sigma_{B_x} - \sigma_{A_x}) \ (t-1) + \sigma_{A_x} \\ 0 \\ 0 \\ \tau_{B_{xy}} (t-1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 50\,(t-1) + 150 \\ 0 \\ 0 \\ 100\,(t-1) \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

In order to use the Equation (8) for point B we need to determine $\tilde{\boldsymbol{s}}_B$ with the help of Equation (9).

$$\tilde{\boldsymbol{s}}_B = \begin{bmatrix} \frac{2}{3}\left((\sigma_{B_x} - \sigma_{A_x}) \ (t-1) + \sigma_{A_x}\right) \\ -\frac{1}{3}\left((\sigma_{B_x} - \sigma_{A_x}) \ (t-1) + \sigma_{A_x}\right) \\ -\frac{1}{3}\left((\sigma_{B_x} - \sigma_{A_x}) \ (t-1) + \sigma_{A_x}\right) \\ 2\tau_{B_{xy}} (t-1) \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

$$\sigma_{Y_B}^2 = \left((\sigma_{B_x} - \sigma_{A_x}) \ (t-1)\right)^2 + 3\left(\tau_{B_{xy}} (t-1)\right)^2 \tag{20}$$

In order to calculate the strain at point B we need to take the time derivative of Equation (18) as

$$\dot{\boldsymbol{\sigma}}_B = \frac{\mathrm{d}\boldsymbol{\sigma}(t)}{\mathrm{d}t} \tag{21}$$

$$\boldsymbol{\varepsilon}_B = \boldsymbol{\varepsilon}_A + \int_{t_A}^{t_B} \boldsymbol{C}_B^{ep} \dot{\boldsymbol{\sigma}}_B \ \mathrm{d}t \tag{22}$$

$$\boldsymbol{\varepsilon}_B = \begin{bmatrix} 0.0113 \\ -0.0053 \\ -0.0053 \\ 0.0079 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

$$\boldsymbol{\varepsilon}^e_B = \boldsymbol{C}^e\ \boldsymbol{\sigma}_B = \begin{bmatrix} 0.0013 \\ -0.0003 \\ -0.0003 \\ 0.0016 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon}^p_B = \boldsymbol{\varepsilon}_B - \boldsymbol{\varepsilon}^e_B = \begin{bmatrix} 0.0100 \\ -0.0050 \\ -0.0050 \\ 0.0064 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

4 Step 4 - B to C

During the unloading step the elastic strain comes back to zero (since the stress components are reduced to zero) while the plastic strain remains the same. Thus calculating with the same algorithm we get $\boldsymbol{\varepsilon}^e_C = \mathbf{0}$ and $\boldsymbol{\varepsilon}^p_C = \boldsymbol{\varepsilon}^p_B$.

$$\boldsymbol{\varepsilon}_C = \boldsymbol{\varepsilon}^P_C = \boldsymbol{\varepsilon}^P_B = \begin{bmatrix} 0.0100 \\ -0.0050 \\ -0.0050 \\ 0.0064 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

Appendix: MATLAB Code

```
%%
clear all
%%
% Define constants
E = 160e9;  % Pa (Pascals)
nu = 0.24;  % Dimensionless
sig_Y0 = 110e6;  % Pa
H = 14e9;  % Pa
sig_A_x = 150e6;  % Pa
sig_B_x = 200e6;  % Pa
tau_B_xy = 100e6;  % Pa

%% Preprocessing (Mises yield criterion)

F_A = sqrt(sig_A_x^2);
F_B = sqrt(sig_B_x^2 + 3*tau_B_xy^2);

t_Y = sig_Y0 / sig_A_x;
disp(['t_Y = ', num2str(t_Y), ' s']);

%% Define Compliance Matrix (Elastic)
C_e_matrix = (1/E) * [
    1, -nu, -nu, 0, 0, 0;
    -nu, 1, -nu, 0, 0, 0;
    -nu, -nu, 1, 0, 0, 0;
    0, 0, 0, 2*(1 + nu), 0, 0;
    0, 0, 0, 0, 2*(1 + nu), 0;
    0, 0, 0, 0, 0, 2*(1 + nu)
];

%% Compute Stress at Yield and Plastic Strain at Point A
sig_Y0_matrix = [
    sig_Y0;
    0;
    0;
    0;
    0;
    0
];

% 0 to Y point
eps_Y = C_e_matrix * sig_Y0_matrix;

% Y to A point
t_A = 1;
integrand_AY = [
    (1/H + 1/E);
    (-1/(2*H) - nu/E);
    (-1/(2*H) - nu/E);
    0;
    0;
    0
```

```
];
eps_A = eps_Y + sig_A_x * (t_A - t_Y) * integrand_AY;

sig_A_matrix = [
    sig_A_x;
    0;
    0;
    0;
    0;
    0
];
eps_A_e = C_e_matrix * sig_A_matrix;
eps_A_p = eps_A - eps_A_e;
%% Define Parameters for A to B Segment
syms sig_A_x_v2 sig_B_x_v2 tau_B_xy_v2 t_v2;

%% Compute Stress Vector at Point B
sig_B_t = [
    (sig_B_x_v2 - sig_A_x_v2) * (t_v2 - 1) + sig_A_x_v2;
    0;
    0;
    tau_B_xy_v2 * (t_v2 - 1);
    0;
    0
];

sig_B_t_num = subs(sig_B_t, {sig_A_x_v2, sig_B_x_v2, tau_B_xy_v2}, {sig_A_x,
   sig_B_x, tau_B_xy});

%% Compute the deviatoric stress tensor and dyadic product
s_B_v2 = [
    (2/3) * sig_B_t(1);
    (-1/3) * sig_B_t(1);
    (-1/3) * sig_B_t(1);
    2 * sig_B_t(4);
    0;
    0
];

dyadic_s_B_v2 = kron(s_B_v2, s_B_v2.');

sig_Y_B_2 = sig_B_t(1)^2 + 3 * sig_B_t(4)^2;

C_p_matrix_B = (9/(4*H)) * dyadic_s_B_v2 / sig_Y_B_2;
C_p_matrix_B_num = subs(C_p_matrix_B, {sig_A_x_v2, sig_B_x_v2, tau_B_xy_v2}, {
   sig_A_x, sig_B_x, tau_B_xy});

C_ep_matrix_B_num = C_e_matrix + C_p_matrix_B_num;

%% Integrate from A to B
t_A = 1;
t_B = 2;

dsig_B_t_num = diff(sig_B_t_num, t_v2);
```

```

% Define your integrand function
integrand_function = @(t_v2) C_ep_matrix_B_num * dsig_B_t_num;

integral_result = int(integrand_function, t_v2, t_A, t_B);
eps_B = eps_A + double(integral_result);

eps_B_e = C_e_matrix * [sig_B_x;
                        0;
                        0;
                        tau_B_xy;
                        0;
                        0];
eps_B_p = eps_B - eps_B_e;

%% From B to C
eps_C = eps_B_p;
eps_C_p = eps_C;
eps_C_e = [zeros(6,1)];
```

Listing 1: Elasticity and Plsticity - Homework 2 : MATLAB code