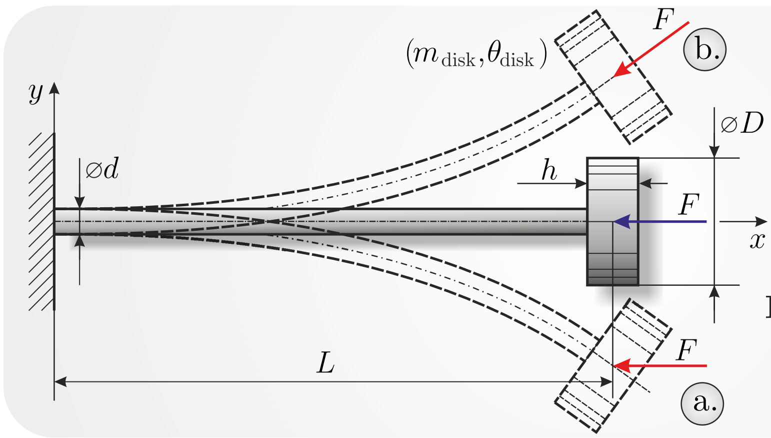
BME Faculty of Mechanical Engineering Department of Applied Mechanics	FINITE ELEMENT ANALYSIS 2^{nd} **Homework**	Name: Gergő Kelle
MSc Program SOLID MECHANICS specialization		Neptun ID: GBBNUL

Total: 20 **p.**

Divergence and flutter instability of Beck's column with rigid disk

Code 1	**Code 2**	**Code 4**
1	2	1

We investigate the dynamic stability of the built-in beam with rigid disk shown below. The initial harmonic motion takes place in the x-y plane. The beam is subjected to a concentrated force at the end of the beam. The cross section is circular and the beam material is linear elastic, isotropic and homogeneous. The structural mass matrix of the disk (which should be added to the structural mass matrix of the beam) is given on the right-hand side of the figure.



$$m_{\text{disk}} = \frac{D^2\pi}{4} h \rho_{\text{disk}}$$

$$\theta_{\text{disk}} = \frac{1}{4} m_{\text{disk}} \left(\frac{D}{2}\right)^2 + \frac{1}{12} m_{\text{disk}} h^2$$

$$\mathbf{M}_{\text{disk}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & 0 & m_{\text{disk}} & 0 \\ 0 & 0 & 0 & 0 & \theta_{\text{disk}} \end{pmatrix}$$

Data table

Code 1	L [m]	d [mm]	D [mm]	h [mm]	**Code 2**	ρ_{beam} [kg/m^3]	ρ_{disk} [kg/m^3]	**Code 3**	E_{beam} [GPa]
1	0.50	10	50	10	**1**	7810	7480	**1**	205
2	0.65	12.5	65	12	**2**	7730	7410	**2**	210
3	0.8	13	70	14	**3**	7900	7940	**3**	220
4	0.75	11.5	55	13	**4**	7850	7670	**4**	215
5	0.72	12.1	63	11	**5**	7650	7250	**5**	203

Tasks:

1. *Apply the equation of motion of multi-DOF systems! Use four BEAM1D elements to create the structural matrices for the finite element analysis! Apply the consistent mass matrix and consider the cases of conservative (a.) and nonconservative (b.) force!* 5 **p.**

2. *Expand the characteristic equation of the system!* 3 **p.**

3. *Create the root amplitude plots considering the first four roots of eigenequation of the system for cases (a.) and (b.)!* 4 **p.**

4. *Determine the critical loads (or load control parameters) for divergence (a.) and flutter (b.) instability of the system! Plot the time response of the system!* 4 **p.**

5. *Calculate the mode shapes at flutter instability using the first two critical loads (or load control parameters) when* ***S***$\Rightarrow$***U*** *and display the shapes! Create the phase plane portraits!* 4 **p.**

6. *Apply the freeware MAXIMA code for the calculations, refer to* ***LM.No.08****! Create a detailed report on the results in PDF! Submit Your work as "****name_firstname_HW2.pdf****"!*

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Finite Element Analysis
II. Homework

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Contents

1 Equation of motion on a multi-DOF system

Apply the equation of motion of multi-DoF systems! Use four *BEAM1D* elements to create the structural matrices for the finite element analysis! Apply the consistent mass matrix and consider the cases of conservative a. and nonconservative b. force!

1.1 Parameters and discretization

Table 1: Parameters for Homework ID 121

Parameter	Value	Dimension
L	0.5	m
d	10	mm
D	50	mm
h	10	mm
ρ_{beam}	7730	$\mathrm{kg/m^3}$
ρ_{disk}	7410	$\mathrm{kg/m^3}$
E_{beam}	205	GPa

Later calculations require the definition of some basic parameters which are the following:

$$m_{disk} = \frac{D^2\pi}{4} \cdot \rho_{disk}\ h = 0.1455\ \ [\mathrm{kg}]\,, \tag{1}$$

$$A = \frac{d^2\pi}{4} = 1.9635 \cdot 10^{-3}\ \ \left[\mathrm{m}^2\right], \tag{2}$$

$$I_z = \frac{d^4\pi}{64} = 4.9087 \cdot 10^{-10}\ \ \left[\mathrm{m}^4\right], \tag{3}$$

$$\theta_{disk} = \frac{1}{4} \cdot m_{disk} \left(\frac{D}{2}\right)^2 + \frac{1}{12} \cdot m_{disk}\ h^2 = 2.3946 \cdot 10^{-5}\ \ \left[\mathrm{kg\ m}^2\right]. \tag{4}$$

Due to the circular shape of the beam's cross section remaining constant along the X-axis, the area and second moment of inertia remains consistent throughout the beam. Before further calculations discretization must be done where in regard to the task *BEAM1D* models should be used. The information about discretization showed it Table **??**.

Table 2: Discretization

Node	x [mm]	y [mm]
1	0	0
2	125	0
3	250	0
4	375	0
5	500	0

(a) Node coordinates

Element	Local node 1	Local node 2
1	1	2
2	2	3
3	3	4
4	4	5

(b) Element connectivity

1.2 Equation of motion

a. conservative

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{K} + \mathbf{K}_G)\,\mathbf{u} = \mathbf{0} \tag{5}$$

b. nonconservative

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{K} + \mathbf{K}_G + \mathbf{K}_L)\,\mathbf{u} = \mathbf{0} \tag{6}$$

Where

$\mathbf{M}_\mathrm{e}$	:	Mass matrix
$\mathbf{K}_\mathrm{e}$	:	Stiffness matrix
$\mathbf{K}_\mathrm{G}$	:	Geometric stiffness matrix
$\mathbf{K}_\mathrm{L}$	:	Structural load stiffness matrix
$\mathbf{u}$	:	Vector of nodal displacements

Equations for these parameters showed in Table 3.

Table 3: Main equations

Parameter	**Equation**
Consistent mass matrix	$\mathbf{M}_e = \frac{Al\rho}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$ (7)
Material stiffness matrix	$\mathbf{K}_e = \frac{I_1 E l^3}{12}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$ (8)
Geometric stiffness matrix	$\mathbf{K}_{Ge} = \frac{N}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$ (9)
Structural load stiffness matrix	$\mathbf{K}_L = -\frac{\partial \mathbf{F}_{nc}}{\partial \mathbf{u}}$ (10)

The stiffness and mass matrices of all four discretized elements are identical due to the beam's uniform cross-section and isotropic material properties. Thus the global matrices are obtained by summing the identical mass or stiffness matrices with regard to it's DoF. This way we get 10 $\times$10 matrices that has to be condensed based on the boundary conditions of $v_1 = 0$ and $\varphi = 0$ resulting the ellimination of the 1st and 2nd rows and columns of the structural matrices. Condensed

matrices will be 8 × 8 matrices.
Another thing to point out is the mass matrix for the whole structure, because we have to consider the disk. Thus before the structural mass condensation, we should substitute the beam's parameters into M_e then calculate the global structural mass matrix of the beam M_{beam} and add the disk's mass matrix M_{disk} leaving us with the following global mass matrix:

$$\mathbf{M} = \mathbf{M}_{beam} + \mathbf{M}_{disk} \tag{11}$$

Where $\quad M_{disk} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & 0 & m_{disk} & 0 \\ 0 & 0 & 0 & 0 & \theta_{disk} \end{bmatrix}$

After we have the condensated matrices for the conservative force, we examine the divergence instability, and for that purpose the corresponding equation of motion is

$$\mathbf{M}_c\ddot{\mathbf{u}}_c + (\mathbf{K}_c + \mathbf{K}_{Gc})\,\mathbf{u}_c = \mathbf{0} \tag{12}$$

For case b., the nonconservative part of the force vector depends on the **u** nodal displacement. The nonconservative part of the force is

$$F_{nc}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -F\sin\varphi_5 & 0 \end{bmatrix} \tag{13}$$

As we are dealing with small values of φ, we can simplify and linearize the expression by expanding it in a Taylor series:

$$\sin\varphi = \varphi - \cancel{\frac{\varphi^3}{3!} + \dots}$$

thus

$$\sin\varphi \approx \varphi$$

leaving us with a linearized nonconservative part of the force which is the following:

$$F_{nc}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -F\varphi_5 & 0 \end{bmatrix} \tag{14}$$

From (14) equation (10) can be determined, therefore the nonconservative force case of the equation of motion can be calculated:

$$\mathbf{M}_c\ddot{\mathbf{u}}_c + (\mathbf{K}_c + \mathbf{K}_{Gc} + \mathbf{K}_{Lc})\,\mathbf{u}_c = \mathbf{0} \tag{15}$$

2 Characteristic equation

Expand the characteristic equation of the system!

To analyze both cases a. and b., we need to start by expanding the characteristic equations of the system, and then create the root amplitude plots by considering the first four roots of the eigenequation. The test function for both cases can be expressed using

$$\mathbf{u}_c(t) = C\boldsymbol{\phi}_c \mathrm{e}^{pt}, \tag{16}$$

$$\ddot{\mathbf{u}}(t) = p^2 C\boldsymbol{\phi}_c \mathrm{e}^{pt}. \tag{17}$$

Where
- C : A constant
- $\boldsymbol{\phi}_c$: The mode shape vector
- p : The characteristic exponent

Using the test function in the equation of motion for both cases and simplifying it, we obtain the following equation in regards of a. case

$$\left(p^2\mathbf{M}_c + \mathbf{K}_c + \mathbf{K}_{Gc}\right) C\boldsymbol{\phi}_c \mathrm{e}^{pt} = \mathbf{0}. \tag{18}$$

By solving this equation, we can derive the nontrivial solution, which leads us to the eigenequation of

$$\det\left(p^2\mathbf{M}_c + \mathbf{K}_c + \mathbf{K}_{Gc}\right) = 0, \tag{19}$$

As for b. case the corresponding equation of motion leads to the following eigenequation:

$$\left(p^2\mathbf{M}_c + \mathbf{K}_c + \mathbf{K}_{Gc} + \mathbf{K}_{Lc}\right) C\boldsymbol{\phi}_c \mathrm{e}^{pt} = \mathbf{0}. \tag{20}$$

$$\det\left(p^2\mathbf{M}_c + \mathbf{K}_c + \mathbf{K}_{Gc} + \mathbf{K}_{Lc}\right) = 0. \tag{21}$$

In order to calculate the $\mathbf{K}_{Gc}$ and $\mathbf{K}_{Lc}$, we need to solve the eigenequations for different values of the F force at the right end. To simplify the calculations, we can use dimensionless parameters ν and λ instead of the original p and F parameters. The characteristic exponent can be expressed as a function of ν and λ.

$$p = \sqrt{\nu \frac{Iz\ E_{beam}}{A\ L^4\ \rho_{beam}}} \tag{22}$$

and the force can be written as

$$F = \lambda \frac{I_z\ E_{beam}}{L^2} \tag{23}$$

Where
- ν: The frequency parameter
- λ: The load control parameter

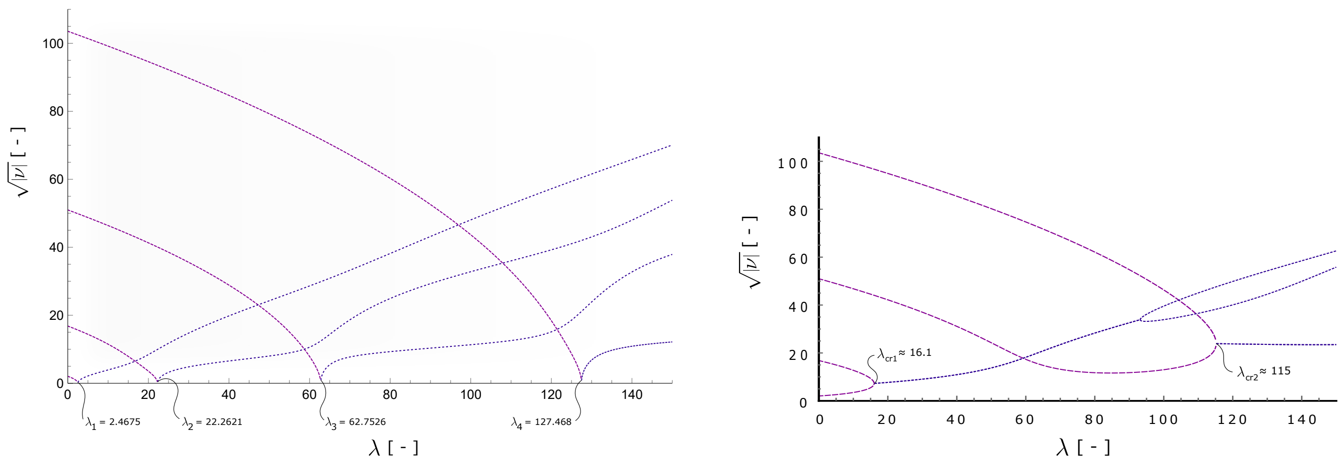
As in a more representative form for later visualization (22) and (23) can be written in the following form:

$$\sqrt{\nu} = \sqrt{\frac{\rho_{beam}\ A\ L^4\ p^2}{I_z\ E_{beam}}} \tag{24}$$

$$\lambda = \frac{F\ L^2}{I_z\ E_{beam}} \tag{25}$$

3 Root amplitude plots

Create the root amplitude plots considering the first four roots of eigenequation of the system for cases a. and b. !



(a) Root amplitude plot for divergence instability (b) Root amplitude plot for flutter instability

Figure 1: Root amplitude plots for divergence and flutter instability (step size: $\delta\lambda = 0.1$)

4 Task 4 & 5

Determine the critical loads (or load control parameters) for divergence a. and flutter b. instability of the system! Plot the time response of the system!
Calculate the mode shapes at flutter instability using the first two critical loads (or load control parameters) when S $\rightarrow$ U and display the shapes! Create the phase plane portraits!

This task requires to determine the load control parameters that lead to divergence and flutter instability of the system, and to create time response plots for each scenario. To calculate the critical load control parameters for a. divergence instability, we need to solve an eigenequation with a frequency parameter of zero, which (as Figure 1 shows) gives us four solution that Table 4 contains.

Table 4: Critical Load Control Parameters for Divergence Instability

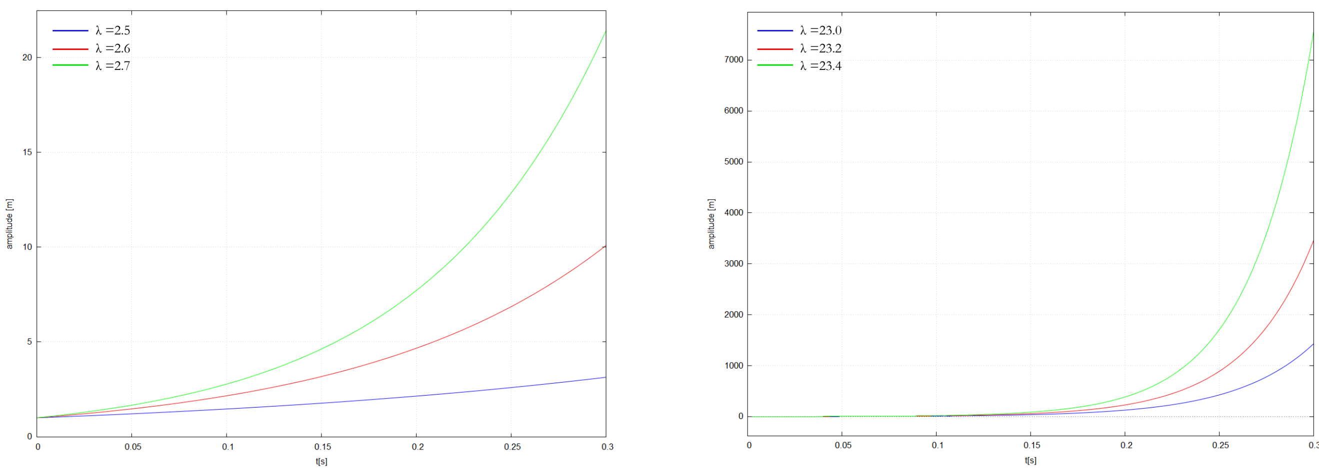
Critical Load Control Parameter	Value
λ_1	2.4675
λ_2	22.2621
λ_3	62.7526
λ_4	127.468

As for flutter instability, we need to use numerical analysis to estimate the critical load control parameters. When the system experiences flutter instability, two roots merge and form a complex conjugate pair. The estimated critical load control parameters for flutter instability b. are

$$\lambda_{cr1} \approx 16.1 \ ,$$

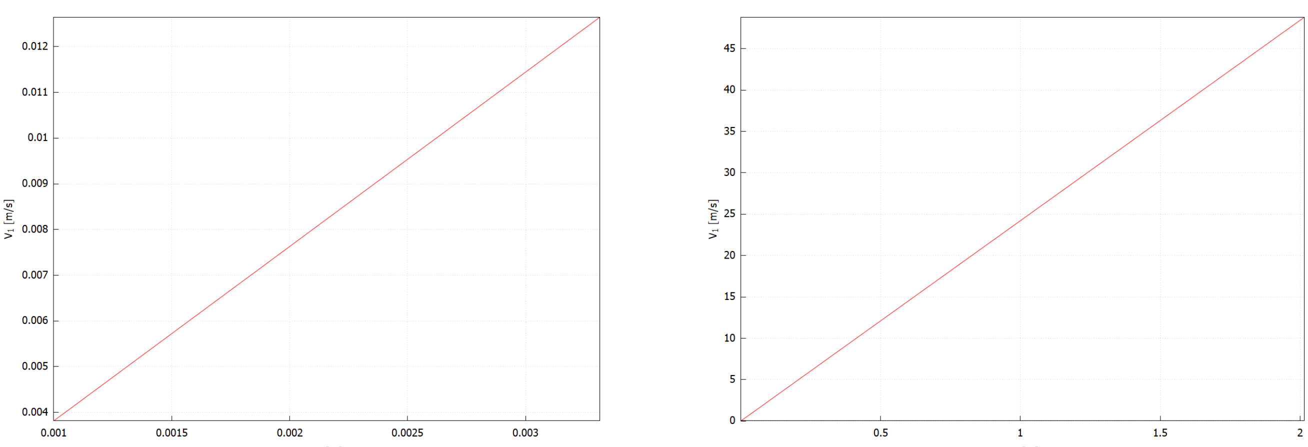
$$\lambda_{cr2} \approx 115$$

which are shown in Figure 4b. The corresponding force value can be calculated easily with Eq. (23). Finally, we need to create time response plots for both types of instability. At these points the system goes from a stable stance to an unstable stance S $\rightarrow$ U thus i plotted the results above those critical λ values. The resulting phase plane portraits are showed in Figure 2, 3, 4 and 5.



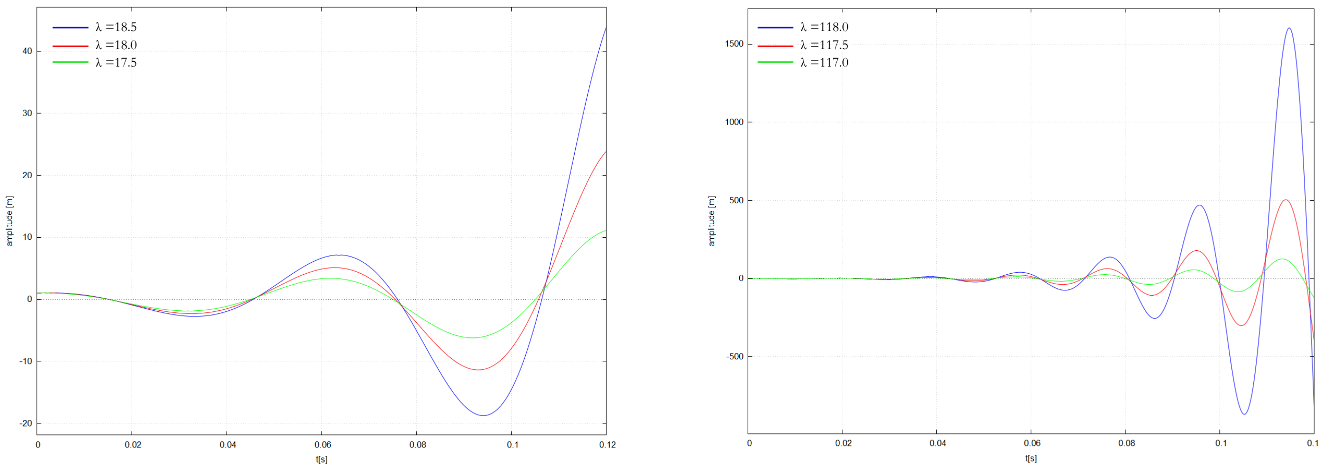
(a) Around λ_1 (b) Around λ_2

Figure 2: Time response of the system at the first and second divergence instability state



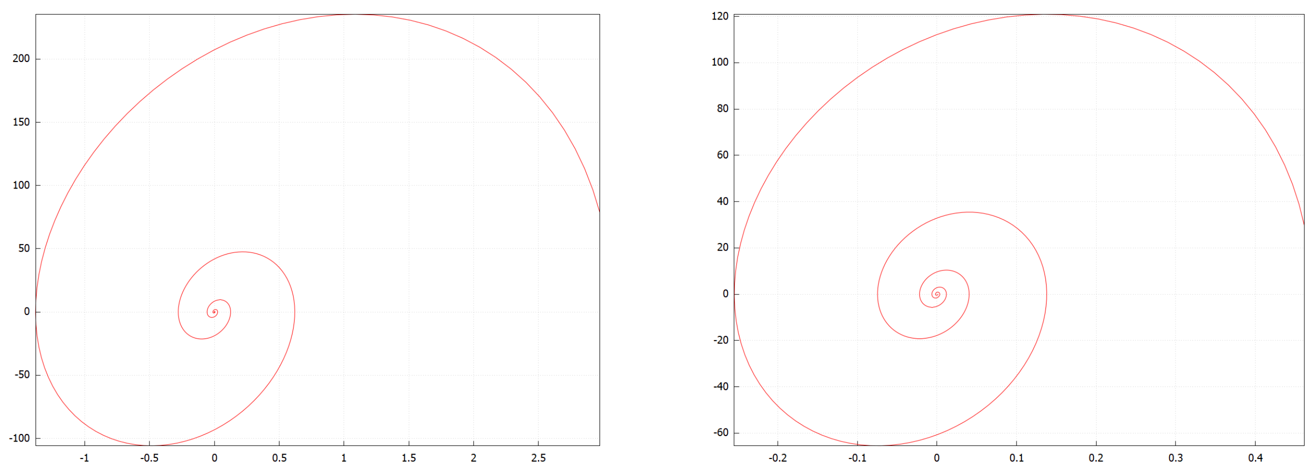
(a) At $\lambda = 2.5$ (b) At $\lambda = 23$

Figure 3: Phase plane portrait for divergence instability above the first two critical load parameter



(a) Around λ_{cr1} (b) Around λ_{cr2}

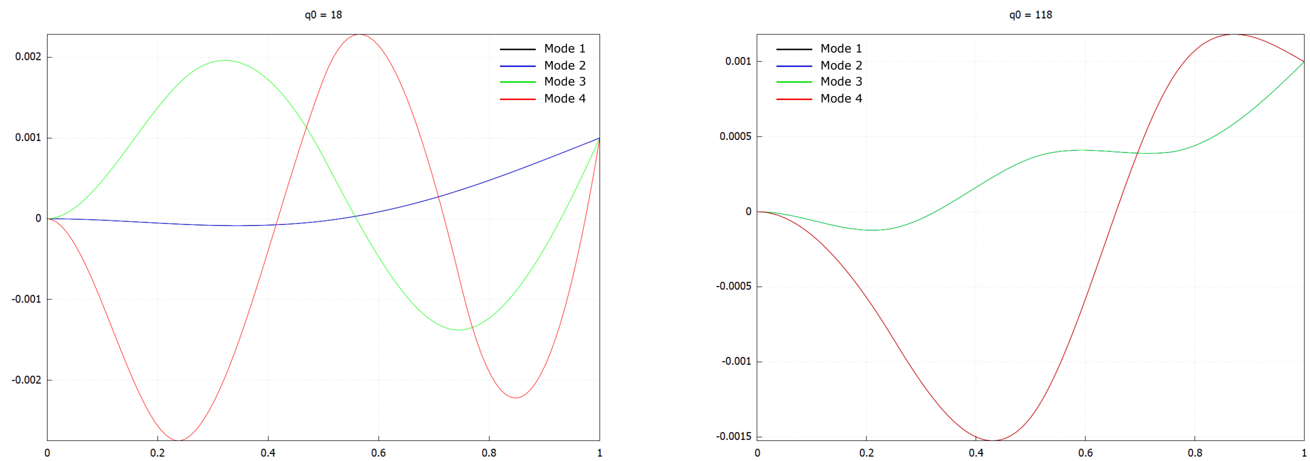
Figure 4: Time response of the system around the first and second flutter instability state



(a) $\lambda = 18$ (b) $\lambda = 118$

Figure 5: Phase plain portraits above the first and second flutter instability state

The mode shapes at flutter instability should be investigated around the load control parameter of flutter type of loss of stability, namely at λ_{cr1} and λ_{cr2} showed in Figure 6. From the mode shapes we can see that modes are merging together, at Figure 6a 2 modes merged, while Figure 6b shows that 2 pair of modes merged.



(a) $\lambda = 18$ (b) $\lambda = 118$

Figure 6: Phase plain portraits above the first and second flutter instability state

For comparison calculating with $\lambda = 0$ gives us the mode shapes with no prestress effect where modes does not merge together, this state is presented in Figure 7.

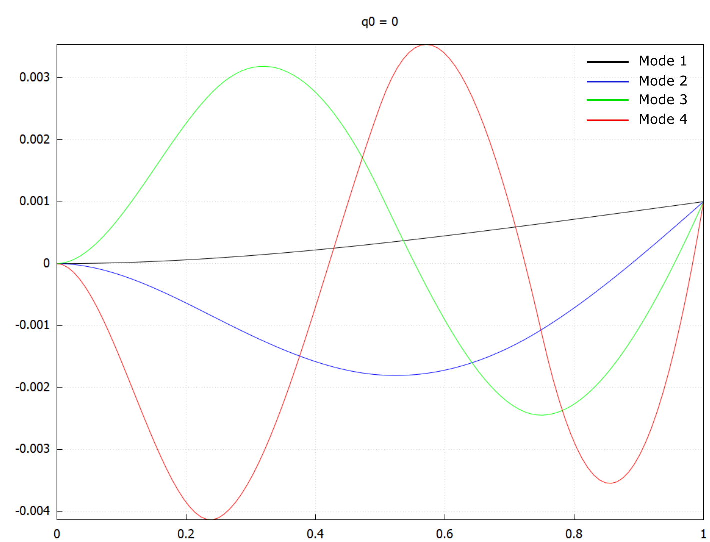


Figure 7: Mode shapes without prestress effect