Exercise sheet 3

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1 problem 1 - Weiss-Curie Ferromagnetism

Consider a System with the following Hamiltonian:

$$H(s) = -\frac{J}{2N}\sum_i \sum_j s_i s_j - h\sum_i s_i \tag{1}$$

1.1 (a, 2P)

show: That its Partition function can be written as:

$$Z(\beta, J, h) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-N\beta J x^2/2}\, C(x) \tag{2}$$

With:

$$C(x) = \prod_i \sum_{\{s_i\}} e^{\beta s_i (Jx+h)} \tag{3}$$

Let's consider first the general case of the summation done in eq. 3:

$$\Omega = \prod_i \sum_{s_{i[,j]}} f(s_{i[,j]}) \tag{4}$$

Where the index $j = 1, \dots, M$ runs over the members of the configuration s_i, $i = 1, \dots, N$. We may expand the expression to:

$$\Omega = \prod_i \left(f(s_{i[,1]}) + \dots + f(s_{i[,M]}) \right) \tag{5}$$

$$= \left(f(s_{1[,1]}) + \dots + f(s_{1[,M]}) \right) \times \dots \times \left(f(s_{N[,1]}) + \dots + f(s_{N[,M]}) \right) \tag{6}$$

Which can be reordered to:

$$\Omega = \left(f(s_{1[,1]}) \dots f(s_{N[,1]}) \right) + \dots + \left(f(s_{1[,M]}) \dots f(s_{N[,M]}) \right) \tag{7}$$

$$= \prod_i f(s_{i[,1]}) + \dots + \prod_i f(s_{i[,M]}) \tag{8}$$

Where the last expression corresponds to a sum over all possible configurations. Hence we may write:

$$\Omega = \prod_i \sum_{s_{i[,j]}} f(s_{i[,j]}) = \sum_{\{\vec{s}\}} \prod_i f(s_{i[,j]}) \tag{9}$$

Using this, we may write eq. 2 as:

$$Z(\beta, J, h) = \sum_{\{\vec{s}\}} \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-N\beta J x^2/2} \prod_i e^{\beta s_i (Jx+h)} \tag{10}$$

$$= \sum_{\{\vec{s}\}} \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-N\beta J x^2/2}\, e^{\beta(\sum_i s_i Jx + \sum_i s_i h)} \tag{11}$$

$$= \sum_{\{\vec{s}\}} \sqrt{\frac{N\beta J}{2\pi}}\, e^{\beta \sum_i s_i h} \int_{-\infty}^{+\infty} dx\, e^{-N\beta J x^2/2}\, e^{\beta J \sum_i s_i x} \tag{12}$$

We perform a variable ransform $y = \sqrt{N\beta J} x \implies dx = dy/\sqrt{N\beta J}$. Now we may write:

$$Z(\beta, J, h) = \sum_{\{\vec{s}\}} \sqrt{\frac{1}{2\pi}}\, e^{\beta \sum_i s_i h} \int_{-\infty}^{+\infty} dx\, e^{-y^2/2 - by} \tag{13}$$

with:

$$b = -\frac{\beta J}{\sqrt{N\beta J}} \sum_i s_i \tag{14}$$

$$b^2 = \frac{\beta J}{N} \sum_i \sum_j s_i s_j \tag{15}$$

This is a general form of a gaussian integral with solution:

$$\int_{-\infty}^{+\infty} dx\, e^{-y^2/2 - by} = \sqrt{2\pi}\, e^{b^2/2} \tag{16}$$

This leads to:

$$Z(\beta, J, h) = \sum_{\{\vec{s}\}} e^{\beta \sum_i s_i h + \frac{\beta J}{N} \sum_i \sum_j s_i s_j} \tag{17}$$

$$= \sum_{\{\vec{s}\}} e^{-\beta H(s)} \tag{18}$$

which is precisely the definition of the partition function.

1.2 (a, 2P)

Compute: $C(x)$ explicitely.

We have only 2 posible values for each s_i. We hence have:

$$C(x) = \prod_i \left(e^{\beta(Jx+h)} + e^{-\beta(Jx+h)} \right) \tag{19}$$

$$= \prod_i 2 \sin\left(\beta(Jx+h)\right) \tag{20}$$

$$= \sinh\left(\beta(Jx+h)\right)^N \tag{21}$$