Exercise sheet 1

Kai Keller

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1 problem 1 - Gaussian Integrals

1.1 (a)

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \tag{1}$$

The trick is to compute I^2 instead of I. We get:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, e^{-\frac{1}{2}(x^2+y^2)} \tag{2}$$

this we may express in polar choordinates:

$$I^2 = \int_{0}^{\infty} \int_{-\pi}^{\pi} dr\, d\theta\, r e^{-\frac{1}{2}r^2} = \dots \tag{3}$$

performing a variable transform from $r^2 \to u$ and perfoming the integration for θ, we may reduce to:

$$I^2 = \pi \int_{0}^{\infty} dr\, e^{-\frac{1}{2}u} = \dots \tag{4}$$

the convolution of e^{au} evaluates to $1/ae^{au}$ and we get:

$$I^2 = -2\pi e^{-\frac{1}{2}u}\Big|_0^{\infty} = 2\pi \tag{5}$$

and finally:

$$\boxed{I = \sqrt{2\pi}} \tag{6}$$

1.2 (b)

The steps from above also apply here. We just have to apply the rule used for eq. 5. We will achieve:

$$I^2 = -(2/a)\pi e^{-\frac{1}{2}u}\Big|_0^\infty = (2/a)\pi \tag{7}$$

and finally:

$$\boxed{I = \sqrt{(2/a)\pi}} \tag{8}$$

1.3 (c)

c.1

$$I = \int_{-\infty}^{\infty} dx\, xe^{-\frac{x^2}{2}} \tag{9}$$

performing a variable transform from $(x^2/2) \to u$ we get:

$$\boxed{I = \int_{u(-\infty)}^{u(\infty)} du\, e^{-u} = -e^{-u}\big|_\infty^\infty = 0} \tag{10}$$

c.2

$$I = \int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{x^2}{2}} \tag{11}$$

We may write I as:

$$I = \int_{-\infty}^{\infty} dx\, x \cdot xe^{-\frac{x^2}{2}} \tag{12}$$

we may use partial integration:

$$I = x\left(\int dx\, xe^{-\frac{x^2}{2}}\right)\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \tag{13}$$

$$\stackrel{(10)}{=} \cancelto{0}{xe^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \tag{14}$$

Which simply is the Gaussian integral from eq. 1. Hence

$$\boxed{I = \sqrt{2\pi}} \tag{15}$$

1.4 (d)

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2} - bx} \tag{16}$$

we will start by completing the square in the exponential function argument:

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x^2+2bx+b^2)+\frac{1}{2}b^2} = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x+b)^2} \times e^{\frac{1}{2}b^2} \tag{17}$$

Now we do a variable transform $x + b \to u$ and we get:

$$\boxed{I = e^{\frac{1}{2}b^2} \times \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}u^2} = e^{\frac{1}{2}b^2} \times \sqrt{2\pi}} \tag{18}$$

2 Problem 2

Assumption: The Grand potential and the internal energy are homogeneous of degree 1.

Theorem: let $f(x_1, \ldots, x_k)$ be differantiable and homogeneous of degree m:

$$f(\lambda x_1, \ldots, \lambda x_k) = \lambda^m f(x_1, \ldots, x_k) \tag{19}$$

Then it follows:

$$\frac{\partial f}{\partial x_1} x_1 + \cdots + \frac{\partial f}{\partial x_k} x_k = mf \tag{20}$$

2.1 (a)

show: $U = TS - PV + \mu N$ The state of a system, described by the internal energy, is in an equilibrium state, which means, that the state variables are constant. We first apply the Theorem from above to $U(V, S, N)$ and get:

$$\frac{\partial U}{\partial V}_{S,N} V + \frac{\partial U}{\partial S}_{V,N} S + \frac{\partial U}{\partial N}_{V,S} N \tag{21}$$

Now we may replace the partial derivatives and achieve:

$$\boxed{U = -PV + TS + \mu N} \tag{22}$$

show: $\Omega = -PV$ We start with the definition of the Grand potential, which is:

$$\Omega = U - TS - \mu N \tag{23}$$

Now we compare the total derivative of the potential to the one of the internal energy:

$$dU = -PdV - VdP + TdS + SdT + \mu dN + Nd\mu \tag{24}$$

$$d\Omega = dU - TdS - SdT - \mu dN - Nd\mu = -PdV - VdP \tag{25}$$

From this we can follow, that the potential can only depend on the volume and the pressure.

DUNNO FURTHER, THINK MORE!!!