MATH 147

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Final Notes

- Intro to Proofs.

- Well-Ordering principle:

  If $S \subseteq \mathbb{N}$ st: $S \neq \emptyset$, then S has a smallest element.

- What is $\mathbb{R}$?

  - The set of infinite decimal expansions:

    $$x = a.x_1x_2x_3\ldots, \qquad a \in \mathbb{Z}, \quad x_i \in \{0,1,2,\ldots,9\}$$

  - $\mathbb{R}$ is a field.
  - $\mathbb{R}$ is totally ordered.
    - Either $x=y$, $x<y$ or $x>y$.
    - If $x<y$ and $y<z$, then $x<z$.
    - If $0<x$, and $0<y$, then $0<xy$.

Def:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0, \end{cases}$$

Triangle Inequality: $|x+y| \leq |x|+|y|$.

- $\mathbb{Q}$ is also a totally ordered field.

  What differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Sequences

<u>Def</u>: A sequence of real numbers is a function, $f: \mathbb{N} \to \mathbb{R}$, (ordered list).

Let $a_n = f(n)$ for $n = 1, 2, \ldots$ and we denote the sequence $\{a_n\}$.

Limit:

<u>Def</u>: Let $\{a_n\}$ be a sequence, $L \in \mathbb{R}$ is called the <u>limit of the sequence $\{a_n\}$</u> if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st: $|a_n - L| < \varepsilon$. $\forall n \geq N$.

- If such L exists, a_n is <u>convergent</u> and $\lim_{n\to\infty} a_n = L$. Otherwise, $\{a_n\}$ is <u>divergent</u>.

<u>Prop^n</u>: If $\{a_n\}$ has a limit, then it's unique.

Infinite Limits:

<u>Def</u>: $\{a_n\}$ <u>diverges</u> to $+\infty$ (or $-\infty$) if $\forall M > 0$, $\exists N \in \mathbb{N}$ st: $a_n > M$, $\forall n \geq N$.

We write: $\lim_{n\to\infty} a_n = \pm\infty$.

<u>Prop^n</u>: If $\lim_{n\to\infty} a_n = L$, then $\{a_n\}$ is bounded, $\exists B > 0$ st: $|a_n| \leq B$, $\forall n \in \mathbb{N}$.

- Contrapositive: If $\{a_n\}$ not bounded, then it's not convergent.

Prop^n: If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, then

1) $\lim\limits_{n\to\infty} c = c, \quad \forall c \in \mathbb{R}$
2) $\lim\limits_{n\to\infty} (a_n + b_n) = L + M$
3) $\lim\limits_{n\to\infty} (\alpha \cdot a_n) = \alpha \cdot L \quad \forall \alpha \in \mathbb{R}$
4) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot M$
5) If $M \neq 0$, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$.
6) If $a_n \geq 0 \ \forall n$, $\lim\limits_{n\to\infty} a_n^m = L^m$, for $m > 0$.
7) If $a_n \geq b_n \ \forall n$, $L \geq M$.

The Squeeze Theorem:

Let $a_n \leq b_n \leq c_n$ for $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

Then $\lim\limits_{n\to\infty} b_n = L$.

Least Upper Bound Principle:

Let $S \subseteq \mathbb{R}$;

- S has an upper bound $\alpha \in \mathbb{R}$, $x \leq \alpha \ \forall x \in S$
- S has a lower bound $\alpha \in \mathbb{R}$, $x \geq \alpha \ \forall x \in S$.
- S is bounded from above and below.

Def: Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$. $\alpha \in \mathbb{R}$ is the <u>least upper bound</u> or <u>supremum</u> for S, if:

1) α is an upper bound for S.
2) if β is an upper bound for S then $\alpha \leq \beta$.

We write: $\alpha = \sup S$.

Similarly, we define the <u>greatest lower bound</u> or <u>infimum</u>, $\alpha = \inf S$.

- If S has no upper bound, we use the convention $\sup S = +\infty$

Equivalently:
α is an upper bound iff $\forall \varepsilon > 0, \exists y \in S$ st: $\alpha - \varepsilon < y$.

Axiom: <u>Least Upper Bound Principle</u>
Let $S \subseteq \mathbb{R}$ st: $S \neq \emptyset$ and S is bounded above
Then $\sup S \in \mathbb{R}$ exists.

<u>Corollary</u>: Same applies for infimum.

<u>Def</u>: $\{a_n\}$ is

- <u>increasing</u> if $a_n \leq a_{n+1}$
- <u>strictly increasing</u> if $a_n < a_{n+1}$
- <u>decreasing</u> if $a_n \geq a_{n+1}$
- <u>strictly decreasing</u> if $a_n > a_{n+1}$

$\Bigg\}\ \forall n \in \mathbb{N}$

<u>**Monotone Convergence Theorem (MCT):**</u>
Let $\{a_n\}$ be monotone (increasing or decreasing)
Then $\{a_n\}$ converges $\iff$ $\{a_n\}$ is bounded.

Subsequences

Def: Let $\{a_n\}$ be a sequence of reals.
Let $n_1 < n_2 < \cdots < n_k < \cdots$, a sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}$.

Def: (tail of a sequence)
Let $\{a_n\}$ be a sequence in $\mathbb{R}$, and let $N \in \mathbb{N}$
The subsequence $\{a_N, a_{N+1}, a_{N+2}, \dots\}$ is the tail of a sequence.

Prop^n: If $\lim_{n\to\infty} a_n = L$, then $\lim_{k\to\infty} a_{n_k} = L$, for any subsequence

Bolzano-Weierstrass Theorem (BW)

Every bounded sequence has a convergent subsequence

- Uses idea of Nested Interval Property.

Cauchy Sequences

Def: $\{a_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$
st: $\forall n \geq m \geq N$, $|a_n - a_m| < \varepsilon$.

Prop^n: Sequence is Cauchy $\Longleftrightarrow$ is convergent.
We say $\mathbb{R}$ is complete

Bernoulli's Inequality:

For $x \geq -1$, $(1+x)^n \geq 1+nx$, $n \geq 1$. (By induction).

AM-GM inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

for $x_1, \dots, x_n \geq 0$.

Cauchy-Schwarz Inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Pf: Use Quadratic Formula or Linear Algebra

Functions:

Sum $(f+g)(x) = f(x) + g(x)$

difference $(f-g)(x) = f(x) - g(x)$

product $(fg)(x) = f(x) \cdot g(x)$

quotient $(f/g)(x) = \dfrac{f(x)}{g(x)}$

composition $(f \circ g)(x) = f(g(x))$

Def: Injective: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Surjective: $\text{range}(f) = Y$ for $f: X \to Y$.

Bijective: both injective and surjective.

Functional Limits

Def: Let f be a real-valued function defined everywhere on an open interval I except possibly at $a \in I$.

Then: $\lim_{x \to a} f(x) = L$, if:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \text{st: } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Sequential Characterization

<u>Prop</u>

Prop^n: Let f be a real valued function defined on an open interval I st: $a \in I$ (except possibly at $x=a$). Let $L \in \mathbb{R}$:

$$\lim_{x \to a} f(x) = L \iff \text{if } \forall \text{ sequence } \{x_n\} \text{ st: } x_n \in I, x_n \neq a \ \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n = a, \text{ then } \lim_{n \to \infty} f(x_n) = L.$$

Corollary: Assume $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} f(x) = M$, then $L = M$.

Prop^n: If $f(x) = g(x)$ $\forall x \in I \setminus \{a\}$, and $\lim_{x \to a} f(x) = L$

then $\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$.

Prop^n: Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

Then:

1) $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$
2) $\lim_{x \to a} (\alpha \cdot f(x)) = \alpha \cdot L$ $\quad \forall \alpha \in \mathbb{R}$
3) $\lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M$
4) If $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

Squeeze Theorem for functions

If $f(x) \leq g(x) \leq h(x)$ $\quad \forall x \in I \setminus \{a\}$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$.

One-sided limit:

Def: $\lim_{x\to a^+} f(x) = L$, if $\forall \varepsilon > 0$, $\exists \delta > 0$ st:

$$a < x < a+\delta \text{ and } x \in I \Rightarrow |f(x) - L| < \varepsilon$$

Similar for $\lim_{x\to a^-} f(x)$.

Trig Limit: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \Rightarrow \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$

If $f(x)$ is even, $\lim_{x\to 0} f(x)$ exists $\Leftrightarrow$ $\lim_{x\to 0} f(x)$ exists

Infinite limits:

- $\lim_{x\to a} f(x) = \infty$, if $\forall M \in \mathbb{R}$, $\exists \delta > 0$ st: $\forall x \in I$ with $0 < |x-a| < \delta \Rightarrow f(x) > M$
- Similar for $\lim_{x\to a} f(x) = -\infty$

- $\lim_{x\to \infty} f(x) = L$, if $\forall \varepsilon > 0$, $\exists N > 0$ st: $|f(x) - L| < \varepsilon, \forall x > N$.
- Similar for $\lim_{x\to -\infty} f(x) = L$.

$\Rightarrow$ The lines $y = L$ are <u>horizontal asymptotes</u>.

- Combine to get $\lim_{x\to \infty} f(x) = \infty$.

- We can get Squeeze Theorem for limits at infinity.

$\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$ and $g(x) \le f(x) \le h(x)$, $\forall x \ge N$.

then $\lim_{x\to\infty} f(x) = L$.

$\lim_{x \to \infty} \frac{\ln x}{x} = 0.$ $\lim_{h \to 0} e^h = 1.$

Continuity:

Def: Let $f: [b,c] \to \mathbb{R}$ and $a \in [b,c]$

f is continous at a if $\lim_{x \to a} f(x) = f(a)$.

- Sequential Characterization:
  - $\forall \{x_n\}$ st: $\{x_n\} \to a$, $\lim_{n \to \infty} f(x_n) = f(a)$.
- If not continuous, then discontinuous.

continuous $\Leftrightarrow \lim_{h \to 0} f(a+h) = f(a)$

- Thomae's function: f continuous on $\mathbb{R} \setminus \mathbb{Q}$

$$f(x) = \begin{cases} 1, & x = 0. \\ 1/q, & x = \frac{p}{q} \in \mathbb{Q},\ p, q \text{ coprime} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

- If f, g are continuous at a, then

$$\left.\begin{array}{l} 1)\ f \pm g, \\ 2)\ f \cdot g \\ 3)\ f/g \quad (g(a) \neq 0) \\ 4)\ \alpha \cdot f,\ \forall \alpha \in \mathbb{R} \end{array}\right\} \text{All continuous}$$

Compositions: If f is continuous at $x=a$ and g is continuous at $b=f(a)$, then $(g \circ f)$ is continuous at $x=a$.

Bounded:

Def: $f: E \to \mathbb{R}$ is bounded on E if $\exists M > 0$ st: $|f(x)| \leq M, \forall x \in E$

Intermediate Value Theorem (IVT)

Proof by Binary Search:

Thm: $f: [a,b] \to \mathbb{R}$ continuous and $f(a) < y < f(b)$, then $\exists c \in (a,b)$ st. $f(c) = y$.

Extreme Value Theorem:

Def: $f: E \to \mathbb{R}$ is bounded on E if $\exists M > 0$ st $|f(x)| \leq M, \forall x \in E$

Thm: $f: [a,b] \to \mathbb{R}$, then:

1) f is bounded on $[a,b]$
2) $\exists \alpha, \beta \in [a,b]$ st:

$$f(\alpha) = \sup_{a \leq x \leq b} f(x) := M$$

max

absolute

min

$$f(\beta) = \inf_{a \leq x \leq b} f(x) := m$$

Pf: • f is bounded by contradiction and sequential definition

• Define max as $\sup_{a \leq x \leq b} f(x)$

Use BW, and sequential

↳ $\exists x_n$ st: $M > f(x_n) > M - \frac{1}{n}$

then (x_n) converges.

Fixed Point:

Def: Let $f : [a,b] \to R$ and $p \in [a,b]$.
p is a fixed point if $f(p) = p$.

Prop: $f : [a,b] \to [a,b]$ continuous has a fixed point.

very close kinda

Proof: Consider $g(x) = f(x) - x$ on $[a,b]$.
We have $g(a) \geq 0 \geq g(b)$
$\therefore$ by IVT, $\exists p \in [a,b]$ st $g(p) = 0$
$\Rightarrow f(p) = p$.

Range:

Def: Consider $f : [a,b] \to \mathbb{R}$
then define $\text{Rng}(f) = \{ f(x) : x \in [a,b] \}$

Theorem: The range of a continuous function on $[a,b]$ is closed and bounded interval.

Pf: Apply EVT, Apply IVT.

Monotone functions

- Either increasing or decreasing.

Theorem: If f is continuous and injective on an interval then it's strictly monotone.

Pf: (by contradiction)

then not monotone

$\Rightarrow \exists x, y, z: \ f(x) < f(y) > f(z)$

OR $f(x) > f(y) < f(z)$.

Inverse Functions

Thm: If f is continuous and injective:

1) f^{-1} is strictly increasing if f is strictly increasing

2) f^{-1} is strictly decreasing if f is strictly decreasing.

Pf: Let $g = f^{-1}$, given $y_1 < y_2$

$\exists \ x_1, x_2$ s.t. $f(x_1) = y_1 < f(x_2) = y_2$

and $x_1 < x_2$, cause otherwise $x_1 \geq x_2$, $f(x_1) \geq f(x_2)$

- g is continuous. (by contradiction)

Uniform Continuity

Def: $f : [a,b] \to \mathbb{R}$ is uniformly continuous
if $\forall \varepsilon > 0$, $\exists \delta > 0$ st
$\forall x, y \in [a,b]$
$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

OR: $\{x_n\}, \{y_n\} \subseteq I$ st
$\lim_{n\to\infty} |x_n - y_n| = 0$, then $\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0$

- Uniformly Continuous $\Longrightarrow$ continuous

closed, bounded interval.

Theorem: $f : [a,b] \to \mathbb{R}$ be continuous
then f is uniformly continuous on $[a,b]$.

Pf: - Contradiction, sequential, BW, back sequential

<u>Differentiation:</u>

<u>Def:</u> Let $f: I \to \mathbb{R}$ and let $a \in I$. Then f is <u>differentiable at $x=a$</u> if

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = L$$

exists and <u>finite</u>. $(L \in \mathbb{R})$

Derivative of f at a is:

$$\frac{df}{dx}(a) = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

f is <u>differentiable on I</u> if f is differentiable at every point of I.

<u>Note:</u>

When $I=[a,b]$, we consider one-sided limits.

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}, \text{ or } f'(b) = \lim_{h \to 0^-} \frac{f(b+h)-f(b)}{h}.$$

<u>Def:</u> If f is differentiable at $x=a$, then the tangent line to f at a is:

$$T(x) = f(a) + f'(a)\cdot(x-a).$$

Instantaneous velocity: $v(a) = s'(a)$.

<u>Theorem:</u> if $f: I \to \mathbb{R}$ is differentiable at a, then f is continuous at a.

<u>Pf:</u> $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists

then $\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$

$= f'(a) \cdot 0 = 0$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$ $\square$

- Converse not necessarily true, (think $f(x) = |x|$)

<u>Leibniz Notation:</u>

$\Delta x = x - a$

$\Delta f = f(a + \Delta x) - f(a)$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$

$= \frac{df}{dx}(x)$

$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{a}$

① $f(x) = x^n$ $\qquad\qquad$ $f'(x) = nx^{n-1}$

② $f(x) = \sin(x)$ $\qquad\qquad$ $f'(x) = \cos(x)$

$f(x) = \cos(x)$ $\qquad\qquad$ $f'(x) = -\sin(x)$

③: $\exists f: \mathbb{R} \to \mathbb{R}$ st. f is continuous on $\mathbb{R}$ but differentiable nowhere.

— 1872, K. Weierstrass

Rules:

(i) Differentiation is linear:

$$(\alpha f + \beta g)'(x) = \alpha f'(x) + \beta \cdot g'(x). \qquad (\alpha, \beta \in \mathbb{R})$$

(ii) Product Rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(iii) Quotient Rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$$

Pf: (i) Exercise.

(ii)
$$\lim_{x\to a} \frac{f(x)\cdot g(x) - f(a)\cdot g(a)}{x-a}$$
$$= \lim_{x\to a} \frac{f(x)\cdot g(x) - f(a)\cdot g(x) + f(a)g(x) - f(a)\cdot g(a)}{x-a}.$$
$$= \lim_{x\to a} \frac{g(x)\cdot[f(x)-f(a)] + f(a)\cdot[g(x)-g(a)]}{x-a}$$
$$= \lim_{x\to a} g(x)\cdot \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + f(a)\cdot \lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$
$$= g(a)\cdot f'(a) + f(a)\cdot g'(a)$$

(iii) Similar to Product Rule.

Chain Rule:

Thm A: Let $f:(a,b)\longrightarrow \mathbb{R}$ s.t. $c\in(a,b)$

f is differentiable at $c \Longleftrightarrow \exists\, \varphi$ on (a,b) that is continuous at c and s.t. $f(x)-f(c) = \varphi(x)\cdot(x-c)$ for $x\in(a,b)$.

Characterization of derivative in terms of secant slopes

Pf: ($\Rightarrow$) Let $\varphi(x) = \begin{cases} f'(c) & x = c \\ \dfrac{f(x)-f(c)}{x-c}, & x \neq c. \end{cases}$

$\varphi(x)$ is continuous at $x = c$, because $\lim\limits_{x \to c} \varphi(c)$ is the same

as $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$. (consider points around and close to c, but not equal)

$= f'(c) = \varphi(c)$.

($\Leftarrow$)

if $x \neq c$, $\varphi(x) = \dfrac{f(x)-f(c)}{x-c}$.

Since we're given φ continuous on (a,b) it's continuous at $x = c$.

$\therefore \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} = \varphi(c) \in \mathbb{R}$

$\hookrightarrow$ limit exists and is finite.

$\Rightarrow$ $f(x)$ is differentiable at c.

$\square$

Thm (Chain Rule):

Let $g:(c,d)\to\mathbb{R}$ and $f:(a,b)\to(c,d)$
and $x_0\in(a,b)$.

If f is differentiable at x_0
and g is differentiable at $f(x_0)$ then:

$$[g(f(x_0))]' = g'(f(x_0))\cdot f'(x_0).$$

__Pf:__ Since $f'(x_0)$ exists, Thm A implies
$\exists\, \varphi$ on (a,b) continuous at x_0
st: $f(x)-f(x_0)=\varphi(x)\cdot(x-x_0)$ $(**)$
and $f'(x_0)=\varphi(x_0)$

and since $g'(f(x_0))$ exists by Thm A,
$\exists\, \psi$ on (c,d), continuous at $f(x_0)$
st: $g(f(x_0))-g(y)=\psi(y)\cdot(y-f(x_0))$
and $g'(f(x_0))=\psi(f(x_0))$. $(*)$

Plug $y:=f(x)$ in $(*)$: $g(f(x))-g(f(x_0))=\psi(f(x))\cdot(f(x)-f(x_0))$

Use $(**)$: $=\underbrace{\psi(f(x))\cdot\varphi(x)}_{h(x)}\cdot(x-x_0)$

Note: $h(x_0) = \psi(f(x_0))\varphi(x_0)$

$= g'(f(x_0)) \cdot f'(x_0)$

and $h(x)$ is continuous at x_0.

By thm A: (using reverse direction)

$g(f(x))$ is differentiable at x_0 and

$(g(f(x_0)))' = h(x_0) = g'(f(x_0)) \cdot f'(x_0)$. $\square$

Inverse Functions:

$f(f^{-1}(x)) = x$ (Differentiate both sides)

$f'(f^{-1}(x))[f^{-1}(x)]' = 1$

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$$

Thm (Inverse function):

Let $f: (a,b) \to (c,d)$ be strictly monotone and continuous

Let $f^{-1}: (c,d) \to (a,b)$ be inverse of f. (strictly monotone and cont.)

If f is differentiable at x_0 and $f'(x_0) \neq 0$ then $(f^{-1})'(f(x_0)) = \psi(f(x_0)) = \dfrac{1}{f'(x_0)}$.

<u>Pf</u>: By Thm A:

$\exists\ \varphi$ on (a,b), continuous at x_0

st: $f(x)-f(x_0)=\varphi(x)\cdot(x-x_0) \quad \forall x\in(a,b)$

and $\varphi(x_0)=f'(x_0)\neq 0$.

$\Rightarrow\ f(x)=f(x_0)+\varphi(x)(x-x_0) \quad \maltese$

We know:

$$f(f^{-1}(y))=y, \qquad \forall y\in(c,d)$$

take $x:=f^{-1}(y)$ and use $\maltese$

$$y=f(x)=f(x_0)+\varphi(f^{-1}(y))(f^{-1}(y)-x_0)$$

Use $y_0=f(x_0)$

$$\Rightarrow\ =f(x_0)+\underbrace{\varphi(f^{-1}(y))}_{\neq 0}(f^{-1}(y)-f^{-1}(y_0))$$

$\therefore$

$$f^{-1}(y)-f^{-1}(y_0)=\left(\frac{1}{\varphi(f^{-1}(y))}\right)(y-y_0)$$

$\psi(y)$, for $y\neq y_0$

$$\Leftrightarrow\ f^{-1}(y)=f^{-1}(y_0)-\psi(y)\cdot(y-y_0)$$

$$\psi(y)=\begin{cases}\dfrac{1}{\varphi(f^{-1}(y))}, & y\neq y_0\\ \dfrac{1}{f'(x_0)} & y=y_0\end{cases}$$

Then $\psi(y)$ is continuous at $y_0 = f(x_0)$:

$$\lim_{y \to y_0} \psi(y) = \lim_{y \to y_0} \frac{1}{\varphi(f^{-1}(y))} = \lim_{x \to x_0} \frac{1}{\varphi(x)}$$

$$= \frac{1}{\varphi(x_0)} = \frac{1}{f'(x_0)} = \psi(y_0)$$

By Thm A: f^{-1} is differentiable at $y_0 = f(x_0)$

$\square$

Use Inverse Func. Thm:

① $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$(\tan x)' = \sec^2 x$, $(\arctan x)' = \frac{1}{1+x^2}$

$(\cot x)' = -\csc^2 x$, $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$

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for $x<1$, $1+x \le e^x \le \frac{1}{1-x}$

$\Rightarrow$ $x \le e^x - 1 \le \frac{x}{1-x}$

$\Rightarrow$ $1 \le \frac{e^x-1}{x} \le \frac{1}{1-x} \xrightarrow{x \to 0} 1$

$\therefore$ by squeeze thm: $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Since, $(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$

$$(e^x)' = e^x$$

For $a > 0$: $(a^x)' = \ln(a) \cdot a^x$

$$\Rightarrow (\ln x)' = \frac{1}{x}$$

Implicit Diff: on $x^2y + y^2x = 6$ (Hard to solve for $y = f(x)$)

Define an implicit function: $y = f(x)$, find $f'(1)$

Sol: Differentiate both sides

$$2xy + x^2y' + 2yy'x + y^2 = 0$$

Solve for y':

$$y' = \frac{-2xy - y^2}{x^2 + 2yx},$$

Plug $x = 1, \Rightarrow y = 2$

$$y' = \frac{-4 - 4}{1 + 4} = \frac{-8}{5}$$

Similar trick for when variables are part of the exponent.

Logarithmic Differentiation.

Problem: $f(x) = (\tan^2 x)^{\cos x}$, [Note: $f(x) \geq 0$]

Find $f'(x)$.

Sol:

$\ln(f(x)) = (2\cos x)(\ln|\tan x|)$

We know $(\ln|x|)' = \frac{1}{x}$, for $x \neq 0$.

Then: differentiate both sides.

$$\frac{f'(x)}{f(x)} = -2\sin x \cdot \ln|\tan x| + \frac{2\cos x}{\tan x} \cdot \sec^2 x.$$

Then we solve for f'!

Maxima and Minima: let $f: [a,b] \to \mathbb{R}$.

$c \in [a,b]$ is called absolute maximum point if $f(x) \leq f(c)$ $\forall x \in [a,b]$

- - - - - - - - - minimum - - - -

$f(x) \geq f(c)$ - - -

Recall from EVT, if f is continuous then both the absolute max/min points exist

Def: Let $f: [a,b] \to \mathbb{R}$.

A point $c \in [a,b]$ is a local maximum if $\exists\, \delta > 0$ st: $f(x) \leq f(c)$ $\forall x \in (c-\delta, c+\delta)$.

. . . . local minimum if $\exists\, \delta > 0$ st: $f(x) \geq f(c)$ $\forall x \in (c-\delta, c+\delta)$.

If $c \in (a,b)$ is an absolute max then it's also a local max.

Thm: (Fermat's Theorem):
Let $f: [a,b] \to \mathbb{R}$ and let $c \in (a,b)$ st: c is a local extremum. Then either $f'(c) = 0$ or $f'(c)$ does not exist. (We call those critical points)

Pf: (For local max).
Assume $f'(c)$ exists and for a contradiction $f'(c) \neq 0$. Then WLOG, let $f'(c) > 0$.
So $\exists\, \delta > 0$ st $\dfrac{f(x) - f(c)}{x - c} > 0$ and if $(\forall x \in (c-\delta, c+\delta))$ $x > c$, then $f(x) > f(c)$, contradiction. $\square$

A direct proof can also be given.

(Converse of Fermat's Thm is not true)

- Use in optimization problems $\to$ must check endpoints.

Thm (Rolle's Theorem):

Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous, f is differentiable on (a,b) and $f(a) = f(b)$.

Then $\exists\ c \in (a,b)$ st: $f'(c) = 0$.

Pf: If f is constant on $[a,b]$, then any c value works.

Otherwise $\exists x \in (a,b)$ st: $f(x) \neq f(a)$ $\left(f(x) \neq f(b)\right)$

By EVT, f attains a max and min and they can't both be at the endpoints. So there's a max or min point, $c \in (a,b)$.

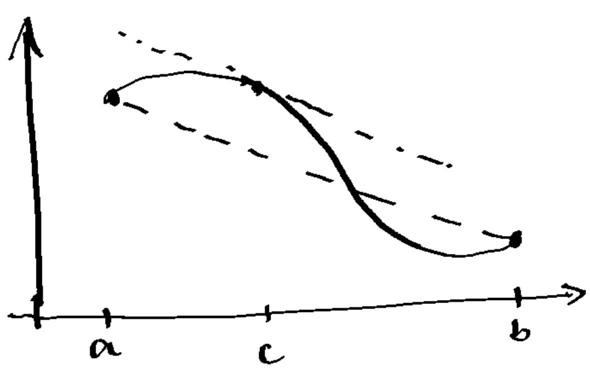
Since f is differentiable on (a,b) by Fermat's Theorem, $f'(c) = 0$. $\square$

Mean Value Theorem (MVT):

Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and differentiable on (a,b). Then there's a point $c \in (a,b)$ st:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically:



- There's a point in (a,b) st. slope at that point is equal to average slope.

Pf: Let $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$

g is continuous on $[a,b]$ and differentiable on (a,b).

Note: $g(a) = 0 = g(b)$

By Rolle's Thm: $\exists\, c \in (a,b)$ st: $g'(c) = 0$.

$$0 = g'(c) = f'(\cancel{x}) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

Antiderivatives:

Def: Given $f: (a,b) \to \mathbb{R}$, an antiderivative is a function $F: (a,b) \to \mathbb{R}$ st: $F'(x) = f(x)$, $\forall x \in (a,b)$.

- If F is an antiderivative, then $F + c$, $c \in \mathbb{R}$ also is.

Thm: If $f'(x) = 0$, $\forall x \in (a,b)$, then f is constant on (a,b).

Pf: Take $x \in [a,b]$, $x > a$ and apply MVT on $[a,x]$.

$\exists\, c \in (a,x)$ st:

$$0 = f'(c) = \frac{f(x)-f(a)}{x-a}$$

$$\Rightarrow f(x) = f(a) \quad \forall x \in [a,b]. \qquad \square$$

Cor: If $f'(x) = g'(x)$, $\forall x \in (a,b)$, then $\exists C \in \mathbb{R}$ st:

$$f(x) = g(x) + C.$$

Pf: Consider $h(x) = f(x) - g(x)$.

- Only solutions to $f'(x) = f(x)$ $\forall x \in \mathbb{R}$, is $f(x) = c \cdot e^x$.

↳ Strategy: consider $g(x) = f(x) \cdot e^{-x}$

then
$$g'(x) = f' e^{-x} - f \cdot e^{-x}$$
$$= f e^{-x} - f \cdot e^{-x} = 0$$
$$\Rightarrow g(x) = C \Rightarrow f(x) = c \cdot e^x.$$

Increasing Function Theorem:

(* Doesn't work for strictly increasing.)

Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

Then: (a) f is increasing on $[a,b] \Leftrightarrow f'(x) \geq 0$, $\forall x \in (a,b)$

(b) .... decreasing .... $f'(x) \leq 0$ ......

Pf: $(\Leftarrow)$ Let $x_2 > x_1 \in I$, then apply MVT on $[x_1, x_2]$ to get $\exists c \in (x_1, x_2)$ st: $0 \leq f'(c) = \dfrac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}}$

$$\Rightarrow f(x_2) \geq f(x_1).$$

$(\Rightarrow)$ $\dfrac{f(x) - f(c)}{x - c} \geq 0$ $\quad \forall x \neq c,\ x \in [a,b]$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0. \quad \square$$

First Derivative Test for Extrema:

Let f be continuous on $I = [a,b]$ and differentiable on $I \setminus \{c\}$. $(c \in (a,b))$.

If $\exists\ \delta > 0$ with $(c-\delta, c+\delta) \subseteq I$ st:

$f'(x) \geq 0$ resp. $f'(x) \leq 0$ for $c - \delta < x < c$

and $f'(x) \leq 0$ resp. $f'(x) \geq 0$ for $c < x < c+\delta$.

then f has a local max resp. local min at $x = c$.

<u>Pf:</u> Take $x \in (c-\delta, c)$ then by MVT:

$\exists\, d \in (x, c)$ st: $f'(d) = \dfrac{f(c) - f(x)}{c - x} \geq 0.$

$\Rightarrow f(x) \leq f(c). \quad \forall x \in (c-\delta, c)$

Similarly, $f(x) \leq f(c) \quad \forall x \in (c, c+\delta)$.

$\Rightarrow f(x) \leq f(c), \quad \forall x \in (c-\delta, c+\delta)$

so c is a local max.

$\square$

Darboux's Theorem:

If f ~~is~~ differentiable on $[a,b]$ and if $\exists\, \alpha \in \mathbb{R}$ st: $f'(a) < \alpha < f'(b)$, then $\exists\, c \in (a,b)$ st: $f'(c) = \alpha$.

☆ A type of IVT for derivatives (which aren't necessarily continuous).

Pf: Let $g(x) = f(x) - \alpha x$, on $[a,b]$ (differentiable)

$g(x)$ attains an absolute min on $[a,b]$ by EVT.

Since, $g'(a) = f'(a) - \alpha < 0$, can show a and b aren't the absolute min of $g(x)$.

$\therefore$ g attains min at some $c \in (a,b)$ and apply Fermat's Thm, $g'(c) = 0 \Rightarrow f'(c) = \alpha$. $\square$

Thm: Let $f, g : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Assume:

- $\exists c \in (a,b)$ st: $f(c) = g(c)$
- $f'(x) \le g'(x)$ $\forall x \in (a,b)$.

Then:
$$f(x) \le g(x) \quad \forall x > c$$
and
$$f(x) \ge g(x) \quad \forall x < c.$$

Higher Derivatives

Let $f : (a,b) \to \mathbb{R}$ be a differentiable function.

If $f' : (a,b) \to \mathbb{R}$ is also differentiable, we say
$$f''((x)) = (f')'(x) = \frac{d}{dx} f'(x) = \frac{d^2 f}{dx^2}(x)$$
is the second derivative of f.

Similarly, for $n = 0, 1, 2 \ldots$ we write

$$f^{(0)}(x) = f(x)$$
$$f^{(1)}(x) = f'(x)$$
$$f^{(2)}(x) = f''(x)$$
$$\vdots$$

$f^{(n)}(x)$ is the n^{th} derivative of f, if it exists.

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d^n f}{dx^n}(x)$$

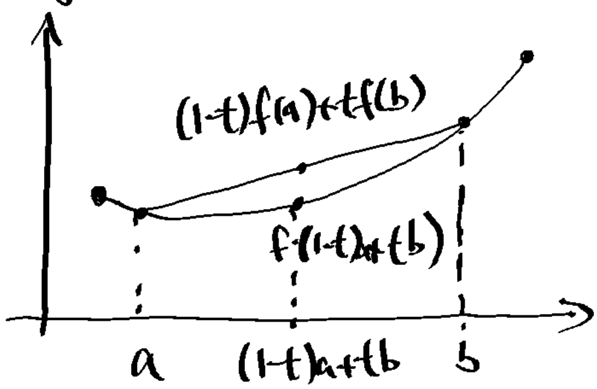
Convexity

Def: A function $f: I \to \mathbb{R}$ is convex on I if

$$f((1-t)x + ty) \leq (1-t)f(x) + t \cdot f(y)$$

for any $x, y \in I$ and any $t \in [0,1]$.

- $(1-t)x + ty$, parametrizes all points on $[x, y]$
- Geometrically, the chord joining any points $(a, f(a)) \longrightarrow (b, f(b))$ lies above the graph of f.



Def: A function f is concave if $-f$ is convex.

$\Rightarrow$ switches $\leq$ to $\geq$.

Prop: f is convex on I if and only if

$$\frac{f(z)-f(x)}{z-x} \le \frac{f(y)-f(z)}{y-z}$$

for any $x<z<y \in I$.

- Geometrically, it means the slope of secant lines are increasing.

Theorem: If $f: I \to \mathbb{R}$ is twice differentiable, then: f is convex on $I \iff f''(x) \ge 0, \ \forall x \in I$.

Pf: Show f' is increasing on I, then reconstruct prop. above

2^{nd} Derivative Test:

Let $f: (a,b) \to \mathbb{R}$, suppose f' and f'' exist and are continuous. If $f'(c)=0$,

- $f''(c)<0$, then c is a local min
- $f''(c)>0$, then c is a local max.

Pf: $\exists \delta$, s.t. $x \in (c-\delta, c+\delta)$ we have $f''(x) \underset{>}{\overset{\text{or}}{<}} 0$ by continuity.

Thus f' is strictly increasing and we can find $f'<0$ and $f'>0$ on either side. Then use def of local extrema.

Curve Sketching:

1) Find Domain
2) Symmetries; even or odd
3) Roots; $f(x)=0$
4) Vertical Asymptotes
5) Horizontal Asymptotes
6) Find f', and critical points, $f'=0$ or DNE
7) Find local extrema and intervals of increase/decrease.
8) Find f'', and inflection points, $f''=0$ or DNE
9) Find convexity of f,
10) Plot.

Jensen's Inequality:

Let $f: (a,b) \to \mathbb{R}$ be convex.
Let $x_1, \dots, x_n \in (a,b)$ and $t_1, \dots, t_n \geq 0$, st: $\sum_{i=1}^{n} t_i = 1$.

Then:

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i).$$

If f is strictlly convex and $t_i > 0$, then equality holds only when $x_1 = x_2 = \dots = x_n$.

Pf: Induct on n, the base case $(n=2)$ is the definition.

* If f is concave, inequality is reversed.

L'Hôpital's Rule:

- Indeterminate forms:

$\frac{0}{0}$: If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} \text{ is indeterminate.}$$

$\frac{\infty}{\infty}$: If $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} \text{ is also indeterminate.}$$

Thm: Let $f,g: [a,b]\to\mathbb{R}$, $f(a)=g(a)=0$

and $g(x)\neq 0$ for $x\in(a,b)$.

If f and g are differentiable at $x=a$, and $g'(a)\neq 0$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Pf: Write

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \quad \text{for } x\in(a,b).$$

Apply Ratio rule for limits.

Thm (Cauchy's Mean Value Theorem):

f,g continuous on $[a,b]$ and differentiable on (a,b) and $g'(x)\neq 0$ $\forall x\in(a,b)$, then there's $c\in(a,b)$ st:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Pf: Define: $h(x) = \dfrac{f(b)-f(a)}{g(b)-g(a)} \cdot (g(x)-g(a)) - (f(x)-f(a))$

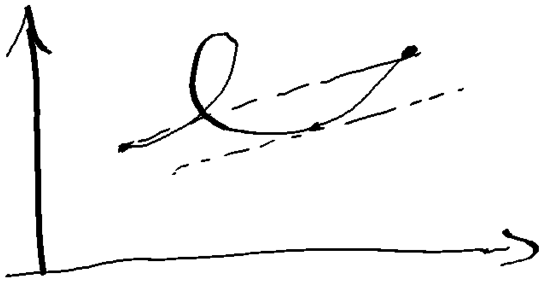
$\forall x \in [a,b]$.

Note, $h(a)=h(b)=0$ and apply Rolle's Theorem

Note: if $g(x)=x$, we get regular MVT.

- Geometric intuition: Consider the parametric curve

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $a \le t \le b$, Then Cauchy MVT says, $\exists c$ st: the slope of the curve at $(f(c), g(c))$ is equal to slope of the line joining endpoints.



L'Hôpital's Rule I:

- We treat $\lim x \to a^+$, $x \to a^-$ and $x \to a$ are similar.
- Let $-\infty \le a < b \le \infty$, allowing for unbounded intervals.
- Let f, g be differentiable on (a,b) st: $g'(x) \neq 0$, $\forall x \in (a,b)$.

Suppose: $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$.

a) If $\lim_{x \to a^+} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a^+} \dfrac{f(x)}{g(x)} = L$.

b) If $\lim_{x \to a^+} \dfrac{f'(x)}{g'(x)} = \pm\infty$, then $\lim_{x \to a^+} \dfrac{f(x)}{g(x)} = \pm\infty$.

Pf: Let $a < \alpha < \beta < b$, then $g(\alpha) \neq g(\beta)$.

By Cauchy MVT, $\exists c \in (\alpha, \beta)$ st:

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(c)}{g'(c)}.$$

a) since $\lim\limits_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, then $\forall \varepsilon > 0$, $\exists c \in (a,b)$ st.

$$L-\varepsilon < \frac{f'(x)}{g'(x)} < L+\varepsilon. \qquad x \in (a, c)$$

$$\Leftrightarrow \quad L-\varepsilon < \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} < L+\varepsilon \qquad \text{for } a < \alpha < \beta \leq c$$

Take $\alpha \to a^+$, $L-\varepsilon < \frac{f(\beta)}{g(\beta)} < L+\varepsilon \Rightarrow \lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = L$.

b) Similar to a).

L'Hôpital's Rule II

Let $-\infty \leq a < b \leq \infty$, f, g differentiable on (a,b) and $g'(x) \neq 0$ $\forall x \in (a,b)$.

Suppose $\lim\limits_{x\to a^+} f(x) = \pm\infty = \lim\limits_{x\to a^+} g(x)$.

a) if $\lim\limits_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = L$.

b) if $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \pm\infty$, then $\lim\limits_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty$.

Pf: (For (a) when $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = +\infty$)

- Let $\varepsilon > 0$ be given, then $\exists\, c \in (a,b)$ st:

$$L-\varepsilon < \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} < L+\varepsilon \qquad \text{for } a<\alpha<\beta\le c$$

$$\Leftrightarrow \quad L-\varepsilon < \frac{\frac{f(\beta)}{g(\alpha)} - \frac{f(\alpha)}{g(\alpha)}}{\frac{g(\beta)}{g(\alpha)} - 1} < L+\varepsilon, \text{ we know } g(\beta)\neq g(\alpha)$$

Let $\alpha \to a^+$, and fix β. Then: $\frac{f(\beta)}{g(\alpha)} \to 0$ and $\frac{g(\beta)}{g(\alpha)} \to 0$.

So, we can choose δ st: α close to a^+ and

$$L-2\varepsilon < \frac{f(\alpha)}{g(\alpha)} < L+2\varepsilon \Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L. \qquad \square$$

Other indeterminate forms:

$\infty - \infty,\ 0\cdot\infty,\ 1^\infty,\ \infty^0,\ 0^0 \rightarrow$ Algebraically Manipulate

Linear Approximations

Def: Let $f(x)$ be differentiable at $x=a$.

The linear approximation of f at $x=a$ is the function

$$L_a(x) = f(a) + f'(a)(x-a).$$

Properties:
1. $L_a(a) = f(a)$
2. $L_a'(a) = f'(a)$
3. L_a only linear polynomial that satisfies 1 and 2

- error$(x) = |f(x) - L_a(x)|$, depends on how far x is from a and how "curved" the graph is around $x=a$.

- We can generate a second degree polynomial
$$p(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2.$$
- Intuitively, if we use higher degree polynomials that match our function at higher derivatives then the error is reduced.

<u>Def</u>: Assume f has n derivatives at $x=a$. The n^{th} degree Taylor polynomial for f at $x=a$ is:
$$P_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Note: $P_{n,a}^{(m)}(a) = f^{(m)}(a), \quad \forall m = 0, \ldots, n.$
- We try to bound the error of $P_{n,a}(x)$ as an approximation of $f(x)$.

Taylor's Theorem:

Let $n \in \mathbb{N}$, $I = [\alpha, \beta]$, Consider $f: I \to \mathbb{R}$ st: $f, f', \dots, f^{(n)}$ are continuous on I and $f^{(n+1)}$ exists on (α, β).

If $a \in I$, then $\forall x \in I$, $\exists$ c between a and x

st: $$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

- Viewed as higher-order extension of MVT.

Def: The remainder is

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$
$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Pf: Let $x \neq a$ st $x \in I$.

Let $F(x) = f(x) - f(a) - \dots - \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

Note $F(a) = F'(a) = \dots = F^{(n)}(a) = 0$

and $G(x) = (x-a)^{n+1}$, $G(a) = G'(a) = \dots = G^{(n)}(a) = 0$

Apply Cauchy MVT $n+1$ times:

1) $\exists c_1 \in (a, x)$ st: $\dfrac{F(x)}{G(x)} = \dfrac{F(x) - \overbrace{F(a)}^{=0}}{G(x) - \underbrace{G(a)}_{=0}} = \dfrac{F'(c_1)}{G'(c_1)}$

2) Apply to F' and G' on (a, c_1)

$\exists c_2 \in (a, c_1)$ st:

$$\frac{F'(c_1)}{G'(c_1)} = \frac{F'(c_1) - F'(a)}{G'(c_1) - G'(a)} = \frac{F''(c_2)}{G''(c_2)} \quad \dots$$

Applying $n+1$ times

$\Rightarrow$ $a < c_{n+1} < \cdots < c_1 < x$. st:

$$\frac{F(x)}{G(x)} = \frac{F'(c_1)}{G'(c_1)} = \cdots = \frac{F^{(n)}(c_n)}{G^{(n)}(c_n)} = \frac{F^{(n+1)}(c_{n+1})}{G^{(n+1)}(c_{n+1})}$$

Let $c = c_{n+1}$, then:

$$F(x) = \frac{F^{(n+1)}(c)}{G^{(n+1)}(c)} \cdot G(x)$$

Note: $F^{(n+1)}(c) = f^{(n+1)}(c)$ and $G^{(n+1)}(c) = (n+1)!$

$$\Rightarrow \quad F(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

$\square$

<u>Corollary</u>: Let $M = \sup\{|f^{(n+1)}(x)| : x \in [\alpha, \beta]\}$

- is attained and finite by EVT since $f^{(n+1)}$ continuous on closed interval

Then: $|f(x) - P_{n,a}(x)| = |R_{n,a}(x)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$.

Newton's Method:

- Estimate solutions to $f(x) = 0$.
- Approximate $f(x)$ linearly and find closer estimates.

① Pick initial guess x_1

② Approximate $T_1(x) = f(x_1) + f'(x_1)(x - x_1)$

→ crosses x-axis at $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

③ Replace x_1 by x_2 and repeat.

Theorem:

Let $I = [a,b]$ and $f : I \to \mathbb{R}$ be twice differentiable

Suppose:

(1) $f(a) \cdot f(b) < 0 \Rightarrow \exists c \in (a,b)$ st: $f(c) = 0$ by IVT

(2) $\exists m, M$ st: $|f'(x)| \geq m > 0$ and $|f''(x)| < M, \forall x \in I$

and $K := \frac{M}{2m}$

Then $\exists\ I^* \subset I$ st: $c \in I^*$ and for any $x_1 \in I^*$

- $\{x_n\} \subseteq I^*$ where $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

and

- $\{x_n\} \to c$.

Moreover, $|x_{n+1} - c| \leq K |x_n - c|^2 \quad \forall n \in \mathbb{N}$.

Proof: Let $y_1 \in I$. By Taylor's Thm,

$\exists r_1$ between y_1 and c st:

$$0 = f(c) = f(y_1) + f'(y_1)(c - y_1) + \frac{1}{2} f''(r_1)(c - y_1)^2$$

$$\Rightarrow -f(y_1) = f'(y_1)(c - y_1) + \frac{1}{2} f''(r_1)(c - y_1)^2$$

If: $y_2 := y_1 - \frac{f(y_1)}{f'(y_1)}$ then

$$y_2 = y_1 + (c - y_1) + \frac{1}{2} \frac{f''(r_1)}{f'(y_1)} \cdot (c - y_1)^2$$

$$\Rightarrow y_2 - c = \frac{f''(r_1)}{2 f'(y_1)} (c - y_1)^2$$

Since $r_1 \in I$, then $|y_2 - c| \leq K |y_1 - c|^2$ $\quad (|f''| \leq M$ and $|f'| > m)$

Choose $\frac{1}{K} > \delta > 0$. and $I^* = [c - \delta, c + \delta] \subseteq I$.

If $x_n \in I^*$, then $|x_n - c| \leq \delta$ and $|x_{n+1} - c| \leq K |x_n - c|^2 \leq K\delta^2 < \delta$

$\Rightarrow x_{n+1} \in I^*$, Hence if $x_1 \in I^* \Rightarrow x_n \in I^* \quad \forall n \in \mathbb{N}$.

We can show inductively $|x_{n+1} - c| \leq (K\delta)^n |x_1 - c|$

but $K\delta < 1$

$$\Rightarrow \lim_{n \to \infty} x_n = c.$$

$\square$