Multivariate anisotropic wavelets on the torus

Ronny Bergmann

AG Bildverarbeitung
TU Kaiserslautern

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Anweiler am Trifels

Introduction

Periodic wavelets were first defined for the univariate case [PT95]

- based on shifts by $2\pi/N$, $N \in \mathbb{N}$
- fast wavelet transform [Se98]
- (localised) de la Vallée Poussin mean based wavelets [Se98]

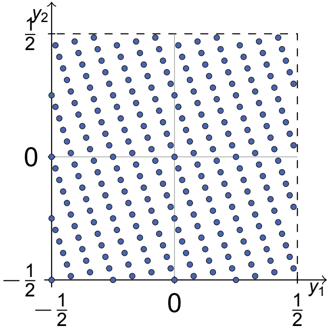
For the multivariate generalization

- a function may be shifted by $\mathbf{y} \in \mathbb{T}^d := [-\pi, \pi)^d$
- scaling factor j replaced by a matrix $\mathbf{J}$ [MS03]
- for fixed $|\det \mathbf{J}| = 2$: several matrices $\mathbf{J}$ available [LP10]

$\Rightarrow$ preference of direction

- handle “curse of dimension”

The pattern and the generating set



The pattern $\mathcal{P}(\mathbf{M})$,

$$\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix}$$

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be regular.

- pattern
  $\mathcal{P}(\mathbf{M}) := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$
- generating set
  $\mathcal{G}(\mathbf{M}) := \mathbf{M}\mathcal{P}(\mathbf{M}) = \mathbf{M}\left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbb{Z}^d$

We have

- $m := |\mathcal{P}(\mathbf{M})| = |\mathcal{G}(\mathbf{M})| = |\det \mathbf{M}|$
- the group $\big(\mathcal{P}(\mathbf{M}), + \bmod 1\big)$
- subpatterns $\mathcal{P}(\mathbf{N})$, für
  $\mathbf{M} = \mathbf{J}\mathbf{N}, \quad \mathbf{J}, \mathbf{N} \in \mathbb{Z}^{d\times d}$

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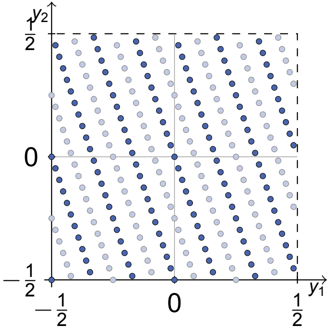

subpattern $\mathcal{P}(\mathbf{N}) \subset \mathcal{P}(\mathbf{M})$,

$$\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix} = \mathbf{J}_Y \begin{pmatrix} 28 & -12 \\ 6 & 2 \end{pmatrix}$$

$$\mathbf{J}_Y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be regular.

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  $\mathcal{P}(\mathbf{M}) := \left[-\frac{1}{2}, \frac{1}{2}\right)^d \cap \mathbf{M}^{-1}\mathbb{Z}^d$
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The Fourier transform

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be regular.
For fixed orderings of $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and $\mathcal{P}(\mathbf{M})$:

- Fourier matrix

$$\mathcal{F}(\mathbf{M}) := \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}}),\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^{m\times m}$$

- discrete Fourier transform for $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^m$

$$\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} := \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a} \in \mathbb{C}^m$$

$\Rightarrow$ **fast Fourier transform (B., 2013)**

We further need for $f \in L_2(\mathbb{T}^d)$ its Fourier coefficients

$$c_{\mathbf{k}}(f) := \langle f, \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ}\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}}\,\mathrm{d}\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d.$$

Translation invariant space

The space $V_{\mathbf{M}}^{\xi} := \operatorname{span}\{\mathrm{T}_{\mathbf{y}}\xi,\ \mathbf{y} \in \mathcal{P}(\mathbf{M})\} \subset L_2(\mathbb{T}^d)$ is **M**-invariant, i.e. for all $\mathbf{y} \in \mathcal{P}(\mathbf{M})$, $\varphi \in V_{\mathbf{M}}^{\xi}$ we have

$$\mathrm{T}_{\mathbf{y}}\varphi := \varphi(\circ - 2\pi\mathbf{y}) \in V_{\mathbf{M}}^{\xi}.$$

Further

$$\varphi = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\varphi,\mathbf{y}}\mathrm{T}_{\mathbf{y}}\xi, \quad a_{\mathbf{y}} \in \mathbb{C},$$

Translation invariant space

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Further

$$\varphi = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\varphi,\mathbf{y}} \mathrm{T}_{\mathbf{y}}\xi, \quad a_{\mathbf{y}} \in \mathbb{C},$$

in Fourier coefficients: For all $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$, $\mathbf{z} \in \mathbb{Z}^d$

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\varphi) = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\varphi,\mathbf{y}} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\xi) = \hat{a}_{\varphi,\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(\xi),$$

where $\hat{\mathbf{a}} = \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a}$.

Especially: $V_{\mathbf{N}}^{\varphi} \subseteq V_{\mathbf{M}}^{\xi}$, $\mathbf{M} = \mathbf{J}\mathbf{N}$

The wavelet transform

Given $\mathbf{M} = \mathbf{J}\mathbf{N}$, $|\det \mathbf{J}| = 2$, and $\xi, \varphi \in L_2(\mathbb{T}^d)$ such that

- $\mathrm{T}_{\mathbf{y}}\xi,\ \mathbf{y} \in \mathcal{P}(\mathbf{M})$, linear independent
- $\mathrm{T}_{\mathbf{x}}\varphi,\ \mathbf{x} \in \mathcal{P}(\mathbf{N})$, linear independent, i.e. $\dim V_{\mathbf{N}}^{\varphi} = \frac{1}{2} \dim V_{\mathbf{M}}^{\xi} = \frac{|\det \mathbf{M}|}{2}$

$\Rightarrow \exists$ Wavelet $\psi \in L_2(\mathbb{T}^d)$ s.t. $V_{\mathbf{M}}^{\xi} = V_{\mathbf{N}}^{\varphi} \oplus V_{\mathbf{N}}^{\psi}$

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Decompose $f = \displaystyle\sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{f,\mathbf{y}} \mathrm{T}_{\mathbf{y}}\xi = g + h = \sum_{\mathbf{x}\in\mathcal{P}(\mathbf{N})} a_{g,\mathbf{x}} \mathrm{T}_{\mathbf{x}}\varphi + \sum_{\mathbf{x}\in\mathcal{P}(\mathbf{N})} a_{h,\mathbf{x}} \mathrm{T}_{\mathbf{x}}\psi$

by computing

$$\hat{a}_{g,\mathbf{k}} = \frac{1}{\sqrt{|\det \mathbf{J}|}} \sum_{\mathbf{l}\in\mathcal{G}(\mathbf{J}^T)} \overline{\hat{a}_{\varphi,\mathbf{k}+\mathbf{N}^T\mathbf{l}}}\, \hat{a}_{f,\mathbf{k}+\mathbf{N}^T\mathbf{l}}, \quad \mathbf{k} \in \mathcal{G}(\mathbf{N}^T),$$

Analogously for $\hat{a}_{h,\mathbf{k}}$ using the coefficients $a_{\psi,\mathbf{y}}$ of $\psi \in V_{\mathbf{M}}^{\xi}$

$\Rightarrow$ **Fast wavelet transform in $\mathcal{O}(m)$ (B., 2013)**

Constructing wavelets

Let's take

- a nonnegative function $g : \mathbb{R}^d \to \mathbb{R}_+$ with
  - $\sum\limits_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{z}) = 1$
  - $g(\mathbf{x}) > 0$, $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2})^d$
- matrices $\mathbf{J}_1, \ldots, \mathbf{J}_n, \mathbf{M}_0 \in \mathbb{Z}^{d\times d}$, $\quad \mathcal{J}_{l,k} = (\mathbf{J}_l, \ldots, \mathbf{J}_k)$, $|\det \mathbf{J}_l| = 2$

$\Rightarrow$ $\mathbf{M}_l := \mathbf{J}_l \cdot \ldots \mathbf{J}_1\mathbf{M}_0$, $m_l := 2^l|\det \mathbf{M}_0|$

and helping functions

- $$B_{\mathcal{J}_{l,n}}(\mathbf{x}) := \begin{cases} \Big(\sum\limits_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{J}_l^{\mathrm{T}}\mathbf{z})\Big) B_{\mathcal{J}_{l+1,n}}(\mathbf{J}_l^{-\mathrm{T}}\mathbf{x}) & l \leq n \\ g(\mathbf{x}) & \text{else.} \end{cases}$$
- $$\tilde{B}_{\mathcal{J}_{l,n}}(\mathbf{x}) := \mathrm{e}^{-2\pi\mathrm{i}\mathbf{x}^{\mathrm{T}}\mathbf{w}_l}\Big(\sum\limits_{\mathbf{z}\in\mathbb{Z}^d} g(\mathbf{x}+\mathbf{J}_l^{\mathrm{T}}\mathbf{z}-\mathbf{v}_l)\Big) B_{\mathcal{J}_{l+1,n}}(\mathbf{J}_l^{-\mathrm{T}}\mathbf{x}), \quad l \leq n,$$

  where $\mathbf{v}_l \in \mathcal{P}(\mathbf{J}_l^{\mathrm{T}})\setminus\{\mathbf{0}\}$ and $\mathbf{w}_l \in \mathcal{P}(\mathbf{J}_l)\setminus\{\mathbf{0}\}$ (both unique.)

Constructing wavelets II

Definition

Define scaling functions $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}$ and wavelets $\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}$ in Fourier coefficients

$$c_{\mathbf{k}}\left(\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right) := \frac{1}{\sqrt{m_l}} B_{\mathcal{J}_{l+1,n}}(\mathbf{M}_l^{-\mathrm{T}}\mathbf{k}),\ \mathbf{k} \in \mathbb{Z}^d, \quad l = 0, \ldots, n$$

$$c_{\mathbf{k}}\left(\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}}\right) = \frac{1}{\sqrt{m_l}} \tilde{B}_{\mathcal{J}_{l+1,n}}(\mathbf{M}_l^{-\mathrm{T}}\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^{d\times d}, \quad l = 0, \ldots, n-1.$$

If g arbitrary smooth $\Rightarrow$ $B_{\mathcal{J}_{l,n}}$ smooth $\Rightarrow$ localization

Theorem (B., 2013)

For $l = 0, \ldots, n-1$

a) $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} \in \operatorname{span}\left\{ T_{\mathbf{y}}\varphi_{\mathbf{M}_{l+1}}^{\mathcal{J}_{l+2,n}} ;\ \mathbf{y} \in \mathcal{P}(\mathbf{M}_{l+1})\right\} =: V_{l+1}$

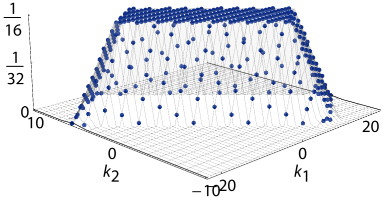
b) $V_{l+1} = V_l \oplus \operatorname{span}\left\{ T_{\mathbf{y}}\psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} ;\ \mathbf{y} \in \mathcal{P}(\mathbf{M}_l)\right\}.$

With slight restriction on $g \Rightarrow \psi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} = \psi_{\mathbf{M}_l}^{(\mathbf{J}_{l+1})}$ and $\varphi_{\mathbf{M}_l}^{\mathcal{J}_{l+1,n}} = \varphi_{\mathbf{M}_l}^{(\mathbf{J}_{l+1})}$.

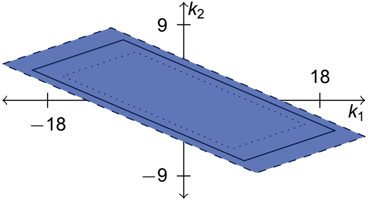
Example of a wavelet

De la Vallée Poussin-type scaling functions and wavelets

- based on $g = B_\Xi$, $\Xi = \begin{pmatrix} 1 & 0 & \frac{1}{10} & 0 \\ 0 & 1 & 0 & \frac{1}{10} \end{pmatrix}$
- $\mathbf{M} = \begin{pmatrix} 28 & -12 \\ 12 & 4 \end{pmatrix} = \mathbf{J}_X \begin{pmatrix} 14 & -6 \\ 12 & 4 \end{pmatrix}$
- $c_{\mathbf{k}}(\varphi_{\mathbf{M}}) = g(\mathbf{M}^{-\mathrm{T}}\mathbf{k})$



$c_{\mathbf{k}}(\varphi_{\mathbf{M}})$



schematic

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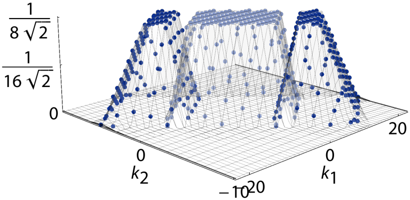
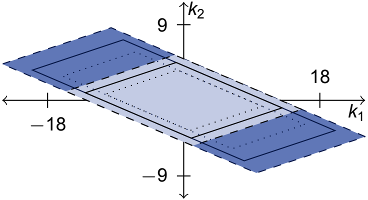


$c_{\mathbf{k}}(\varphi_{\mathbf{N}})$, $|c_{\mathbf{k}}(\psi_{\mathbf{N}})|$



schematic

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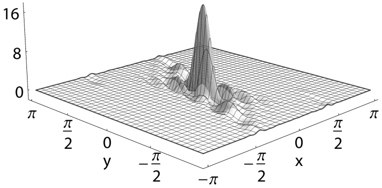

$\varphi_{\mathbf{M}}$

Example of a wavelet

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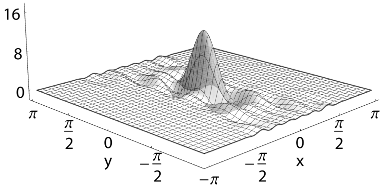
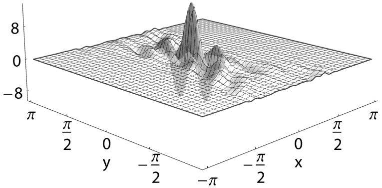


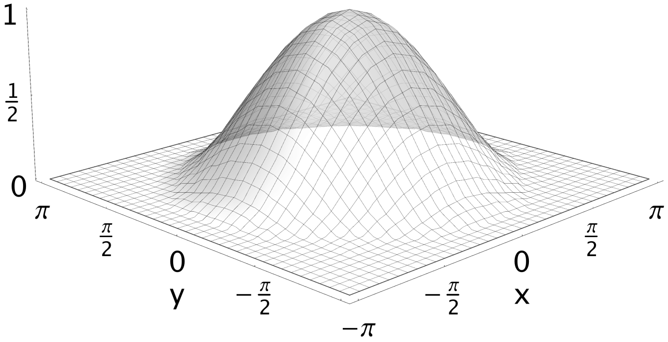
$\varphi_{\mathbf{N}}$



$\psi_{\mathbf{N}}^{(\mathbf{J}_X)}$

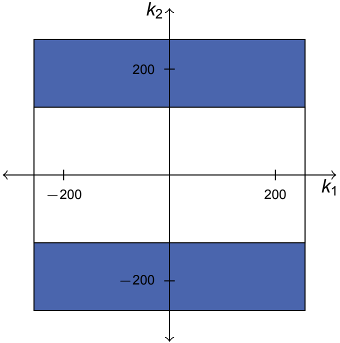
Example of a decomposition

- radial function based on a piecewise quadratic function
- jump in second directional derivative on a circle
- sampling with $\mathbf{M} = 512\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Rightarrow 512 \times 512$ pixel image.

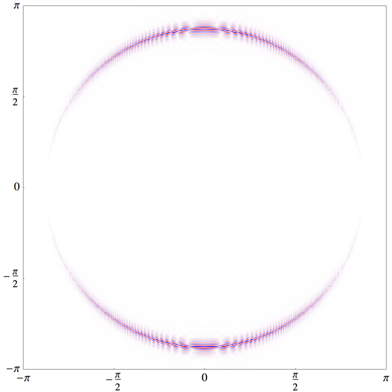


Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y \mathbf{N}_1$$



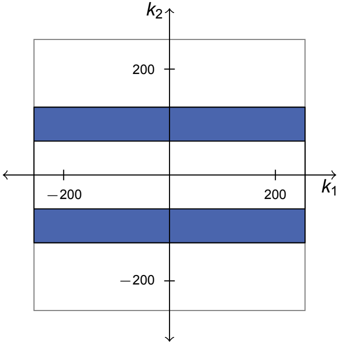
Wavelet $\psi^{(\mathbf{J}_Y)}_{\mathbf{N}_1}$, $\mathbf{N}_1 = \begin{pmatrix} 512 & 0 \\ 0 & 256 \end{pmatrix}$



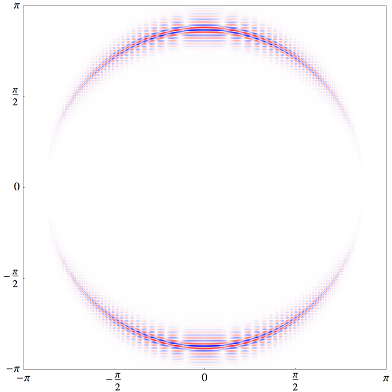
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y \mathbf{J}_Y \mathbf{N}_2$$



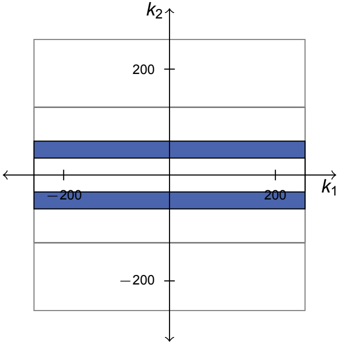
Wavelet $\psi_{\mathbf{N}_2}^{(\mathbf{J}_Y)}$, $\mathbf{N}_2 = \begin{pmatrix} 512 & 0 \\ 0 & 128 \end{pmatrix}$



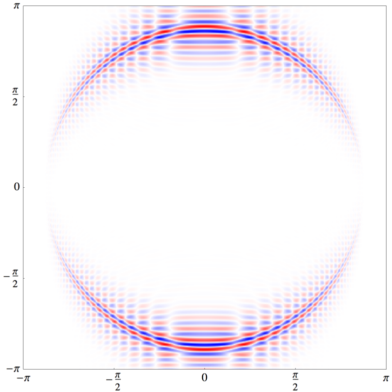
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_Y\mathbf{N}_3$$



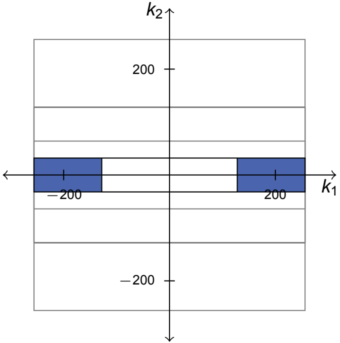
Wavelet $\psi^{(\mathbf{J}_Y)}_{\mathbf{N}_3}$, $\mathbf{N}_3 = \begin{pmatrix} 512 & 0 \\ 0 & 64 \end{pmatrix}$



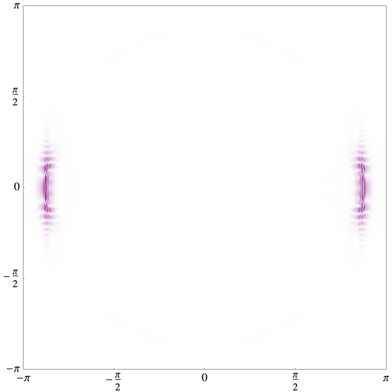
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_X\mathbf{N}_4$$



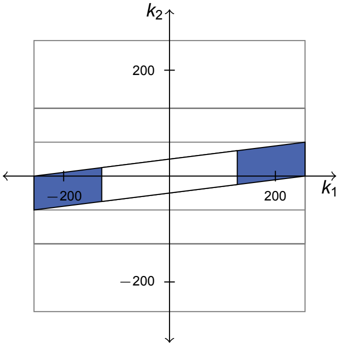
Wavelet $\psi^{(\mathbf{J}_X)}_{\mathbf{N}_4}$, $\quad \mathbf{N}_4 = \begin{pmatrix} 256 & 0 \\ 0 & 64 \end{pmatrix}$



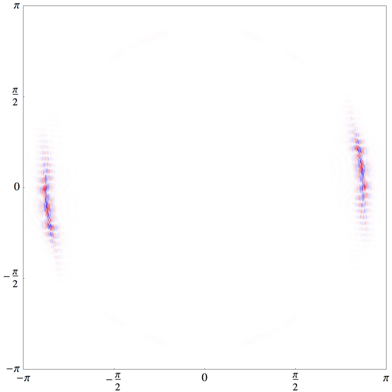
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_5$$



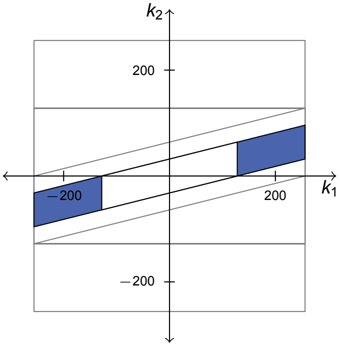
Wavelet $\psi^{(\mathbf{J}_X)}_{\mathbf{N}_5}$, $\quad \mathbf{N}_5 = \begin{pmatrix} 256 & 32 \\ 0 & 64 \end{pmatrix}$



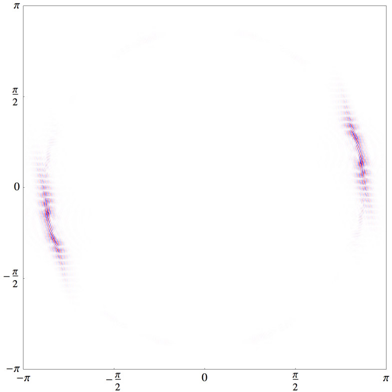
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y \mathbf{J}_{Y-} \mathbf{J}_Y \mathbf{J}_X \mathbf{N}_6$$



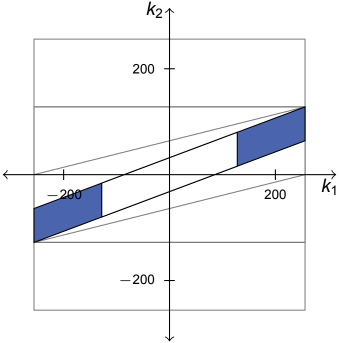
Wavelet $\psi^{(\mathbf{J}_X)}_{\mathbf{N}_6}$, $\mathbf{N}_6 = \begin{pmatrix} 256 & 64 \\ 0 & 64 \end{pmatrix}$



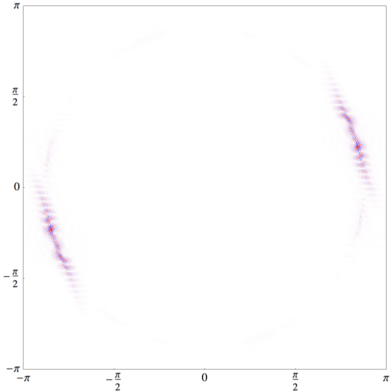
Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y \mathbf{J}_{Y-} \mathbf{J}_{Y-} \mathbf{J}_X \mathbf{N}_7$$



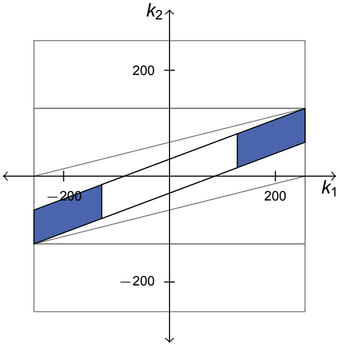
Wavelet $\psi^{(\mathbf{J}_X)}_{\mathbf{N}_7}$, $\quad \mathbf{N}_7 = \begin{pmatrix} 256 & 96 \\ 0 & 64 \end{pmatrix}$

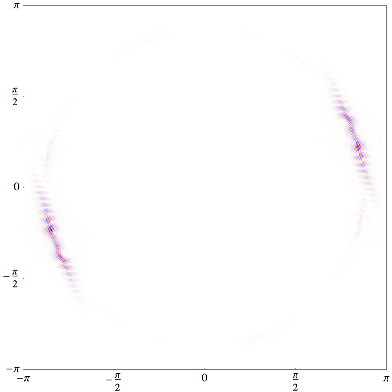


Corresp. wavelet part

Example of a decomposition

$$\mathbf{M} = \mathbf{J}_Y\mathbf{J}_{Y-}\mathbf{J}_{Y-}\mathbf{J}_X\mathbf{N}_7$$





Wavelet $\psi^{(\mathbf{J}_X)}_{\mathbf{N}_7}$, $\mathbf{N}_7 = \begin{pmatrix} 256 & 96 \\ 0 & 64 \end{pmatrix}$

Corresp. wavelet part

numbering: binary in matrices and directly in multiples of 32 in shear

Conclusion

Patterns $\mathcal{P}(\mathbf{M})$ and generating sets $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$

- generalize equally spaced points
- still resemble an FFT
- fast wavelet transform based with corresponding TI spaces

The constructed wavelets generalize the onedimensional
de la Vallée Poussin wavelets

- to arbitrary dyadic scaling matrices
- based on arbitrary smooth functions g

$\Rightarrow$ localization

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Thank your for your attention.