



NTNU

The Riemannian Difference of Convex Algorithm in Manopt.jl

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joint work with

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Difference of Convex

We aim to solve

$$\arg \min_{p \in \mathcal{M}} f(p)$$

where

- ▶ \mathcal{M} is a Riemannian manifold
- ▶ $f: \mathcal{M} \rightarrow \mathbb{R}$ is a difference of convex function, i. e. of the form

$$f(p) = g(p) - h(p)$$

- ▶ $g, h: \mathcal{M} \rightarrow \overline{\mathbb{R}}$ are convex, lower semicontinuous, and proper

A Riemannian Manifold \mathcal{M}

A d -dimensional Riemannian manifold can be informally defined as a set \mathcal{M} covered with a “suitable” collection of charts, that identify subsets of \mathcal{M} with open subsets of \mathbb{R}^d and a continuously varying inner product on the tangent spaces.

[Absil, Mahony, and Sepulchre [2008](#)]

A Riemannian Manifold \mathcal{M}

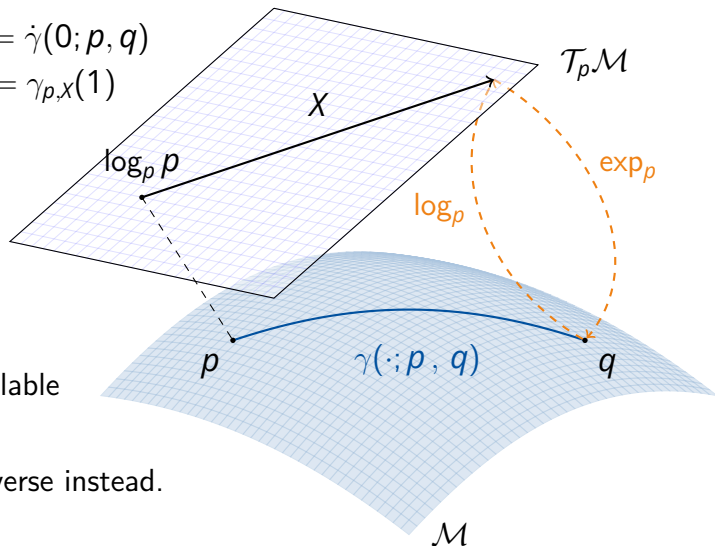
Notation.

- ▶ Logarithmic map $\log_p q = \dot{\gamma}(0; p, q)$
- ▶ Exponential map $\exp_p X = \gamma_{p,X}(1)$
- ▶ Geodesic $\gamma(\cdot; p, q)$
- ▶ Tangent space $\mathcal{T}_p \mathcal{M}$
- ▶ inner product $(\cdot, \cdot)_p$

Numerics.

\exp_p and \log_p maybe not available efficiently/ in closed form

\Rightarrow use a retraction and its inverse instead.



(Geodesic) Convexity

[Sakai 1996; Udriște 1994]

A set $\mathcal{C} \subset \mathcal{M}$ is called (strongly geodesically) **convex**
if for all $p, q \in \mathcal{C}$ the geodesic $\gamma(\cdot; p, q)$ is unique and lies in \mathcal{C} .

A function $f: \mathcal{C} \rightarrow \overline{\mathbb{R}}$ is called (geodesically) **convex**
if for all $p, q \in \mathcal{C}$ the composition $f(\gamma(t; p, q)), t \in [0, 1]$, is convex.

The Riemannian Subdifferential

Let \mathcal{C} be a convex set.

The **subdifferential** of f at $p \in \mathcal{C}$ is given by

[Lee 2003; Udriște 1994]

$$\partial_{\mathcal{M}} f(p) := \{ \xi \in \mathcal{T}_p^* \mathcal{M} \mid f(q) \geq f(p) + \langle \xi, \log_p q \rangle_p \text{ for } q \in \mathcal{C} \},$$

where

- ▶ $\mathcal{T}_p^* \mathcal{M}$ is the dual space of $\mathcal{T}_p \mathcal{M}$, also called **cotangent space**
- ▶ $\langle \cdot, \cdot \rangle_p$ denotes the duality pairing on $\mathcal{T}_p^* \mathcal{M} \times \mathcal{T}_p \mathcal{M}$

The Fenchel Conjugate

The **Fenchel conjugate** of a function $f: \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is given by

$$f^*(\xi) := \sup_{x \in \mathbb{R}^n} \langle \xi, x \rangle - f(x) = \sup_{x \in \mathbb{R}^n} \begin{pmatrix} \xi \\ -1 \end{pmatrix}^T \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

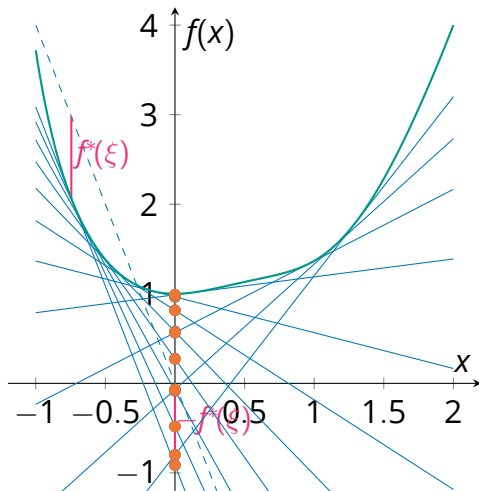
- ▶ given $\xi \in \mathbb{R}^n$: maximize the distance between $\xi^T \cdot$ and f
- ▶ **can also** be written in the epigraph

The **Fenchel biconjugate** reads

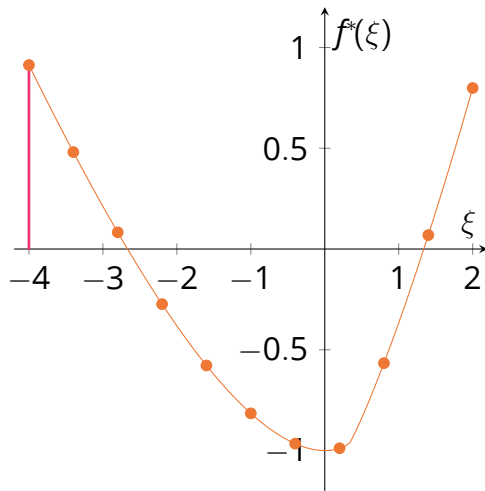
$$f^{**}(x) = (f^*)^*(x) = \sup_{\xi \in \mathbb{R}^n} \langle \xi, x \rangle - f^*(\xi).$$

Illustration of the Fenchel Conjugate

The function f



The Fenchel conjugate f^*



The Riemannian Difference of Convex Algorithm

The Euclidean DCA

Idea 1. At x_k , approximate $h(x)$ by its affine minorization

$$h_k(x) := h(x^{(k)}) + \langle x - x^{(k)}, y^{(k)} \rangle \text{ for some } y^{(k)} \in \partial h(x^{(k)})$$

\Rightarrow iteratively minimize $g(x) - h_k(x) = g(x) - h(x^{(k)}) - \langle x - x^{(k)}, y^{(k)} \rangle$

Idea 2. Using duality theory finding a new $y^{(k)} \in \partial h(x^{(k)})$ is equivalent to

$$y^{(k)} \in \arg \min_{y \in \mathbb{R}^n} \left\{ h^*(y) - g^*(y^{(k-1)}) - \langle y - y^{(k-1)}, x^{(k)} \rangle \right\}$$

Idea 3. Reformulate 2 using a proximal map \Rightarrow DCP

on manifolds this was done in

[Almeida, Neto, Oliveira, and Souza 2020; Souza and Oliveira 2015]

In the Euclidean case, all three models are equivalent.

A Fenchel Duality on a Hadamard Manifold

Let

- ▶ $T\mathcal{M} = \dot{\bigcup}_p T_p\mathcal{M}$ denote the **tangent bundle**
- ▶ analogously $T^*\mathcal{M}$ denotes the **cotangent bundle**
- ▶ \mathcal{M} be a Hadamard manifold (non-positive sectional curvature).

Definition

[Silva Louzeiro, RB, and Herzog 2022]

Let $f: \mathcal{M} \rightarrow \overline{\mathbb{R}}$.

The **Fenchel conjugate** of f is the function $f^*: T^*\mathcal{M} \rightarrow \overline{\mathbb{R}}$ defined by

$$f^*(p, \xi) := \sup_{q \in \mathcal{M}} \left\{ \langle \xi, \log_p q \rangle - f(q) \right\}, \quad (p, \xi) \in T^*\mathcal{M}.$$

The Dual Difference of Convex Problem

Given the Difference of Convex problem

$$\arg \min_{p \in \mathcal{M}} g(p) - h(p)$$

and the Fenchel duals g^* and h^* we can state the dual difference of convex problem as

[RB, Ferreira, Santos, and Souza 2024]

$$\arg \min_{(p, \xi) \in T^* \mathcal{M}} h^*(p, \xi) - g^*(p, \xi).$$

On $\mathcal{M} = \mathbb{R}^n$ this indeed simplifies to the classical dual problem.

Theorem.

[RB, Ferreira, Santos, and Souza 2024]

$$\inf_{(q, X) \in T^* \mathcal{M}} \left\{ h^*(q, X) - g^*(q, X) \right\} = \inf_{p \in \mathcal{M}} \{ g(p) - h(p) \}.$$

The Dual Difference of Convex Problem

The primal and dual Difference of Convex problem

$$\arg \min_{p \in \mathcal{M}} g(p) - h(p) \quad \text{and} \quad \arg \min_{(p, \xi) \in T^* \mathcal{M}} h^*(p, \xi) - g^*(p, \xi)$$

are equivalent in the following sense.

Theorem.

[RB, Ferreira, Santos, and Souza 2024]

If p^* is a solution of the primal problem, then $(p^*, \xi^*) \in T^* \mathcal{M}$ is a solution for the dual problem for all $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$.

If $(p^*, \xi^*) \in T^* \mathcal{M}$ is a solution of the dual problem for some $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$, then p^* is a solution of the primal problem.

Derivation of the Riemannian DCA

We consider the first order Taylor approximation of h at some point $p^{(k)}$:
 With $\xi \in \partial h(p^{(k)})$ we set

$$h_k(p) := h(p^{(k)}) + \langle \xi, \log_{p^{(k)}} p \rangle_{p^{(k)}}$$

Using [musical isomorphisms](#) we identify $X = \xi^\# \in T_p \mathcal{M}$,
 where we call X a subgradient. [Locally](#) h_k [minorizes](#) h , i. e.

$$h_k(q) \leq h(q) \quad \text{locally around } p^{(k)}$$

\Rightarrow Use $-h_k(p)$ as [upper bound](#) for $-h(p)$ in $f = g - h$.

Note. On \mathbb{R}^n the function h_k is linear.

On a manifold h_k is nonlinear and not even necessarily [convex](#), even on a Hadamard manifold.

The Riemannian DC Algorithm

[RB, Ferreira, Santos, and Souza 2024]

Input: An initial point $p^{(0)} \in \text{dom}(g)$, g and $\partial_{\mathcal{M}}h$

1: Set $k = 0$.

2: **while** not converged **do**

3: Take $X^{(k)} \in \partial_{\mathcal{M}}h(p^{(k)})$

4: Compute the next iterate $p^{(k+1)}$ as

$$p^{(k+1)} \in \arg \min_{p \in \mathcal{M}} g(p) - (X^{(k)}, \log_{p^{(k)}} p)_{p^{(k)}}. \quad (*)$$

5: Set $k \leftarrow k + 1$

6: **end while**

Note. In general the subproblem $(*)$ can not be solved in closed form. But an approximate solution yields a good candidate.

For example: Given g , $p^{(k)}$, and $X^{(k)}$ and $\text{grad } g \Rightarrow$ Gradient descent.

Convergence of the Riemannian DCA

Let $\{p^{(k)}\}_{k \in \mathbb{N}}$ and $\{X^{(k)}\}_{k \in \mathbb{N}}$ be the iterates and subgradients of the RDCA.

Theorem.

[RB, Ferreira, Santos, and Souza 2024]

If \bar{p} is a cluster point of $\{p^{(k)}\}_{k \in \mathbb{N}}$, then $\bar{p} \in \text{dom}(g)$ and there exists a cluster point \bar{X} of $\{X^{(k)}\}_{k \in \mathbb{N}}$ s. t. $\bar{X} \in \partial g(\bar{p}) \cap \partial h(\bar{p})$.

\Rightarrow Every cluster point of $\{p^{(k)}\}_{k \in \mathbb{N}}$, if any, is a critical point of f .

Proposition.

[RB, Ferreira, Santos, and Souza 2024]

Let g be σ -strongly (geodesically) convex. Then

$$f(p^{(k+1)}) \leq f(p^{(k)}) - \frac{\sigma}{2} d^2(p^{(k)}, p^{(k+1)})$$

and $\sum_{k=0}^{\infty} d^2(p^{(k)}, p^{(k+1)}) < \infty$, so in particular $\lim_{k \rightarrow \infty} d(p^{(k)}, p^{(k+1)}) = 0$.

Software



Manifolds.jl & Manopt.jl – Why Julia?

Goals.

- ▶ abstract definition of manifolds
 - ⇒ implement abstract solvers on a generic manifold
 - ▶ well-documented and well-tested
 - ▶ fast.
- ⇒ “Run your favourite solver on your favourite manifold”.

Why Julia?

julialang.org

- ▶ high-level language, properly typed
- ▶ multiple dispatch (cf. `f(x)`, `f(x::Number)`, `f(x::Int)`)
- ▶ just-in-time compilation, solves two-language problem
⇒ “nice to write” and as fast as C/C++
- ▶ I like the community



[Axen, Baran, RB, and Rzecki 2023]

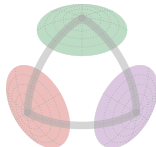
Goal. Provide an interface to implement and use Riemannian manifolds.

Interface `AbstractManifold` to model manifolds

Functions like `exp(M, p, X)`, `log(M, p, X)` or `retract(M, p, X, method)`.

Decorators for implicit or explicit specification of an embedding, a metric, or a group,

Efficiency by providing in-place variants like `exp!(M, q, p, X)`



Goal. Provide a library of Riemannian manifolds, that is efficiently implemented and well-documented

[Axen, Baran, RB, and Rzecki 2023]

Meta. generic implementations for $\mathcal{M}^{n \times m}$, $\mathcal{M}_1 \times \mathcal{M}_2$, vector- and tangent-bundles, esp. $T_p\mathcal{M}$, or Lie groups

Library. Implemented functions for

- ▶ Circle, Sphere, Torus, Hyperbolic, Projective Spaces, Hamiltonian
- ▶ (generalized, symplectic) Stiefel, (generalized) Grassmann, Rotations
- ▶ Symmetric Positive Definite matrices, with fixed determinant
- ▶ (several) Multinomial matrices, Symmetric, Symplectic matrices
- ▶ Tucker & Oblique manifold, Kendall's Shape space
- ▶ ...

Concrete Manifold Examples.

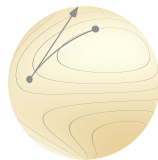
Before first run] `add Manifolds` to install the package.

Load packages with `using Manifolds` and

- ▶ Euclidean space `M1 = \mathbb{R}^3` and 2-sphere `M2 = Sphere(2)`
- ▶ their product manifold `M3 = M1 \times M2`
- ▶ A signal of rotations `M4 = SpecialOrthogonal(3)^10`
- ▶ SPDs `M5 = SymmetricPositiveDefinite(3)` (affine invariant metric)
- ▶ a different metric `M6 = MetricManifold(M5, LogCholeskyMetric())`

Then for `any` of these

- ▶ Generate a point `p=rand(M)` and a vector `X = rand(M; vector_at=p)`
- ▶ and for example `exp(M, p, X)`, or in-place `exp!(M, q, p, X)`



Manopt.jl

Goal. Provide optimization algorithms on Riemannian manifolds.

Features. Given a `Problem p` and a `SolverState s`,
implement `initialize_solver!(p, s)` and `step_solver!(p, s, i)`
⇒ an algorithm in the `Manopt.jl` interface

Highlevel interfaces like `gradient_descent(M, f, grad_f)`
on any manifold `M` from `Manifolds.jl`.

All provide `debug` output, `recording`, `cache` & `counting` capabilities,
as well as a library of `step sizes` and `stopping criteria`.

Manopt family.



manoptjl.org

[RB 2022]



manopt.org

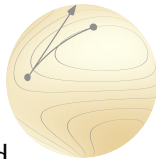
[Boumal, Mishra, Absil, and Sepulchre 2014]



pymanopt.org

[Townsend, Koep, and Weichwald 2016]

List of Algorithms in Manopt.jl



Derivative Free Nelder-Mead, Particle Swarm, CMA-ES

Subgradient-based Subgradient Method, Convex Bundle Method,
Proximal Bundle Method

Gradient-based Gradient Descent, Conjugate Gradient, Stochastic,
Momentum, Nesterov, Averaged, ...
Quasi-Newton with (L-)BFGS, DFP, Broyden, SR1,...
Levenberg-Marquard

Hessian-based Trust Regions, Adaptive Regularized Cubics (ARC)

nonsmooth Chambolle-Pock, Douglas-Rachford, Cyclic Proximal Point

constrained Augmented Lagrangian, Exact Penalty, Frank-Wolfe

nonconvex Difference of Convex Algorithm, DCPA



Illustrating a few Keyword Arguments

Given functions $f(M, p)$ and $\text{grad}_f(M, p)$, a manifold M and a start point p_0 .

- ▶ `q = gradient_descent(M, f, grad_f, p0)` to perform gradient descent
- ▶ Given the Euclidean gradient $\nabla f(E, p)$ use for conversion
`q = gradient_descent(M, f, ∇f , p0; objective_type=:Euclidean)`
- ▶ print iteration number, cost and change every 10th iterate

```
q = gradient_descent(M, f, grad_f, p0;
                    debug=[:Iteration, :Cost, :Change, 10, "\n"]
                    )
```
- ▶ record `reocord=[:Iterate, :Cost, :Change]`, `return_state=true`
 Access: `get_solver_result(q)` and `get_record(s)`
- ▶ modify stop: `stopping_criterion = StopAfterIteration(100)`
- ▶ cache calls `cache=(:LRU, [:Cost, :Gradient], 25)` (uses `LRUCache.jl`)
- ▶ count calls `count=[:Cost, :Gradient]` (prints with `return_state=true`)

The Difference of Convex Algorithm in Manopt.jl

The algorithm is implemented and released in Julia using `Manopt.jl`¹. It can be used with any manifold from `Manifolds.jl`

A solver call looks like

```
q = difference_of_convex_algorithm(M, f, g, ∂h, p0)
```

where one has to implement $f(M, p)$, $g(M, p)$, and $\partial h(M, p)$.

- ▶ a sub problem is generated if keyword `grad_g=` is set
- ▶ an efficient version of its cost and gradient is provided
- ▶ you can specify the sub-solver using `sub_state=` to also set up the specific parameters of your favourite algorithm

¹see https://manoptjl.org/stable/solvers/difference_of_convex/

A Numerical Example

Rosenbrock and First Order Methods

Problem. We consider the classical Rosenbrock example²

$$\arg \min_{x \in \mathbb{R}^2} a(x_1^2 - x_2)^2 + (x_1 - b)^2,$$

where $a, b > 0$, usually $b = 1$ and $a \gg b$, here: $a = 2 \cdot 10^5$.

Known Minimizer $x^* = \begin{pmatrix} b \\ b^2 \end{pmatrix}$ with cost $f(x^*) = 0$.

Goal. Compare first-order methods, e. g. using the (Euclidean) gradient

$$\nabla f(x) = \begin{pmatrix} 4a(x_1^2 - x_2) \\ -2a(x_1^2 - x_2) \end{pmatrix} + \begin{pmatrix} 2(x_1 - b) \\ 0 \end{pmatrix}$$

A “Rosenbrock-Metric” on \mathbb{R}^2

In our Riemannian framework, we can introduce a new metric on \mathbb{R}^2 as

$$G_p := \begin{pmatrix} 1 + 4p_1^2 & -2p_1 \\ -2p_1 & 1 \end{pmatrix}, \text{ with inverse } G_p^{-1} = \begin{pmatrix} 1 & 2p_1 \\ 2p_1 & 1 + 4p_1^2 \end{pmatrix}.$$

We obtain $(X, Y)_p = X^T G_p Y$

The exponential and logarithmic map are given as

$$\exp_p(X) = \begin{pmatrix} p_1 + X_1 \\ p_2 + X_2 + X_1^2 \end{pmatrix}, \quad \log_p(q) = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 - (q_1 - p_1)^2 \end{pmatrix}.$$

`Manifolds.jl`:

Implement these functions on `MetricManifold(\mathbb{R}^2 , RosenbrockMetric())`.

The Riemannian Gradient w.r.t. the new Metric

Let $f: \mathcal{M} \rightarrow \mathbb{R}$. Given the Euclidean gradient $\nabla f(p)$, its Riemannian gradient $\text{grad} f: \mathcal{M} \rightarrow T\mathcal{M}$ is given by

$$\text{grad} f(p) = G_p^{-1} \nabla f(p).$$

While we could implement this denoting $\nabla f(p) = (f'_1(p) \ f'_2(p))^T$ using

$$\left\langle \text{grad} f(q), \log_q p \right\rangle_q = (p_1 - q_1)f'_1(q) + (p_2 - q_2 - (p_1 - q_1)^2)f'_2(q),$$

but it is [automatically](#) done in [Manopt.jl](#).

The Experiment Setup

Algorithms. We now compare

1. The Euclidean gradient descent algorithm on \mathbb{R}^2 ,
2. The Riemannian gradient descent algorithm on \mathcal{M} ,
3. The Difference of Convex Algorithm on \mathbb{R}^2 ,
4. The Difference of Convex Algorithm on \mathcal{M} .

For DCA third we split f into $f(x) = g(x) - h(x)$ with

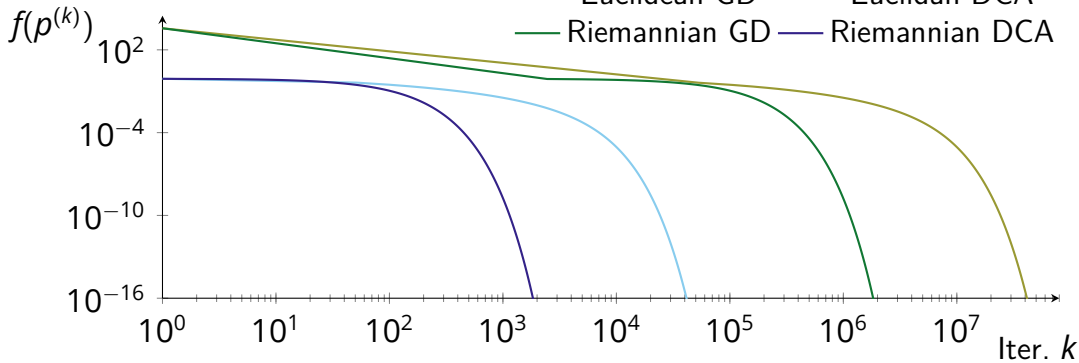
$$g(x) = a(x_1^2 - x_2)^2 + 2(x_1 - b)^2 \quad \text{and} \quad h(x) = (x_1 - b)^2.$$

Initial point. $p_0 = \frac{1}{10} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with cost $f(p_0) \approx 7220.81$.

Stopping Criterion.

$$d_{\mathcal{M}}(p^{(k)}, p^{(k-1)}) < 10^{-16} \text{ or } \|\text{grad} f(p^{(k)})\|_p < 10^{-16}.$$

Results



Algorithm	Runtime (sec.)	# Iterations
Euclidean GD	305.567	53 073 227
Euclidean DCA	58.268	50 588
Riemannian GD	18.894	2 454 017
Riemannian DCA	7.704	2 459

Summary

- ▶ Nonsmooth, nonconvex problems on manifold: [difference of convex](#)

$$\arg \min_{p \in \mathcal{M}} g(p) - h(p)$$

- ▶ The Difference of Convex Algorithm
- ➡ Relation to Fenchel Duality on Hadamard manifolds
- ➡ Convergence on Hadamard manifolds
- ▶ [Manifolds.jl](#) and [Manopt.jl](#)
- ➡ Numerically solve optimization problems on Riemannian manifolds

Outlook.

- ▶ couple [Manopt.jl](#) with (Euclidean) AD tools using [ManifoldDiff.jl](#)
- ▶ What [is](#) (Fenchel) duality on manifolds?

Selected References



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ronnybergmann.net/talks/2024-Aachen-Difference-of-Convex-Manopt.pdf