A Second Order Non-smooth Variational Model for Restoring Manifold-valued Images

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FELIX KLEIN
ZENTRUM FÜR
MATHEMATIK

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Restoring Images: Denoising and Inpainting

Introducing discrete total variation models for real valued data.

- given noisy image $f\colon \mathcal{V} \to \mathbb{R}$, $\mathcal{V} \subseteq \mathcal{G} = \{1,\dots,N\} \times \{1,\dots,M\}$
- reconstruct original image $u_0\colon \mathcal{G} \to \mathbb{R}$
- pixel from $\mathcal{G}\setminus\mathcal{V}$ have to be inpainted
- approach: Minimize variational model

$$\mathcal{E}(u) \coloneqq \underbrace{F(f,u)}_{\text{data term}} + \underbrace{\alpha\, R(u)}_{\text{regularization term}}, \qquad \alpha > 0$$

- first order models (total variation, TV)   [Rudin, Osher, Fatemi, 1992]
    - isotropic model with discrete gradient $|\nabla u|$: $R_{\text{iso}}(u) \coloneqq \sum_{i,j} |\nabla u|$
    - anisotropic model: $R_{\text{aniso}}(u) \coloneqq \sum_{i,j} \big(|u_{i+1,j} - u_{i,j}| + |u_{i,j+1} - u_{i,j}|\big)$
    - edge preserving
- several higher order variational models avoid stair caising-effect
    [Chambolle, Lions, 1997; Setzer, Steidl, 2008; Bredies, Kunisch, Pock, 2010; Papafitsoros, Schönlieb, 2014]

Riemannian Manifolds
...which we want to apply in different spaces

Here: Images with pixel values in a Riemannian manifold $\mathcal{M}$

Goal: Second Order Model for images $f\colon \mathcal{V} \to \mathcal{M}$

Applications with manifold-valued images

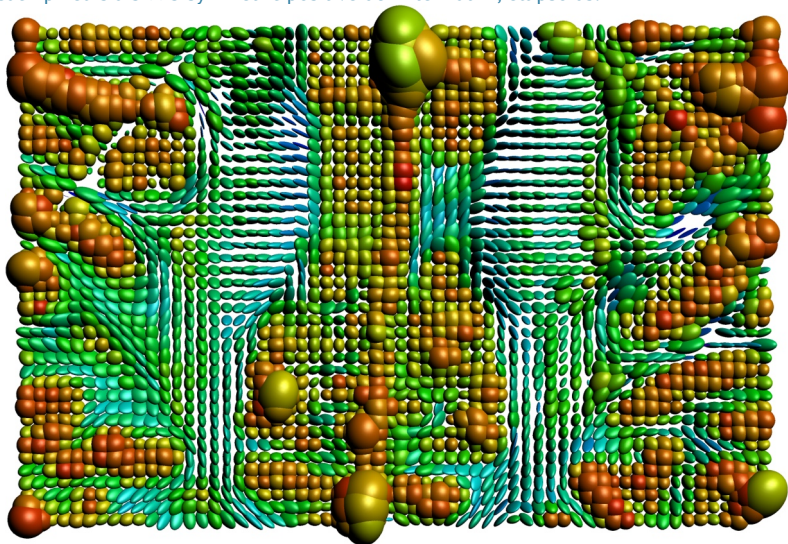$\mathbb{S}^1$ InSAR, HSI(HSV) color space, phase space

$\mathbb{S}^2$ directions, chromaticity-brightness colorspace

$SO(3)$ orientations, electron backscattered diffraction

$\mathcal{P}(s)$ DT-MRI, covariance matrices

Symmetric Positive Definite Matrices: $x^T A x > 0$

$\mathcal{P}(3)$: each pixel is a 3×3 symmetric positive definite matrix, ellipsoids: $x^T A x = 1$

First and Second Order Differences

Let's restrict ourselves to signals for now... $\mathcal{G} = \{1, \ldots, N\}, \mathcal{V} \subset \mathcal{G}$

On $\mathbb{R}^n$

- line $\gamma(t) = x + t(y - x)$
- distance $\|x - y\|_2$
- first order model

$$\sum_{i \in \mathcal{V}} \|f_i - u_i\|_2^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} \|u_i - u_{i+1}\|_2$$

Riemannian manifold $\mathcal{M}$

- geodesic path $\gamma_{\widehat{x,z}}(t)$
- geodesic distance $d \colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$
- first order model

$$\sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1})$$

First and Second Order Differences

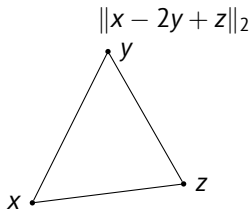Let's restrict ourselves to signals for now...

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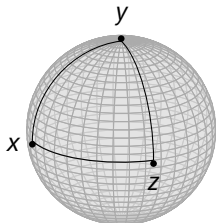
- second oder difference

$$\|x - 2y + z\|_2$$

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$$\sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1})$$

- How to model that on $\mathcal{M}$?

$\mathcal{M} = \mathbb{S}^2$

First and Second Order Differences

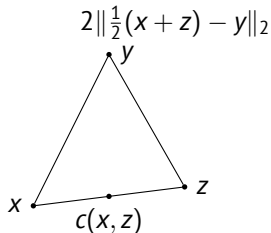Let's restrict ourselves to signals for now... $\mathcal{G} = \{1, \ldots, N\}, \mathcal{V} \subset \mathcal{G}$

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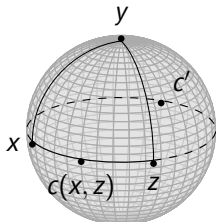
- second oder difference

$$2\|\tfrac{1}{2}(x + z) - y\|_2$$

Riemannian manifold $\mathcal{M}$

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$$\sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1})$$

- idea: mid point formulation

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First and Second Order Differences

Let's restrict ourselves to signals for now... $\mathcal{G} = \{1, \ldots, N\}, \mathcal{V} \subset \mathcal{G}$
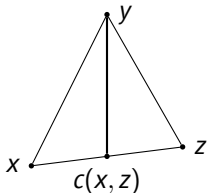
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- second oder difference

$$2\|c(x,z) - y\|_2, \ c(x,y) : \text{ mid point}$$
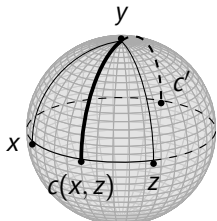
Riemannian manifold $\mathcal{M}$

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- idea: mid point formulation

$$\mathcal{M} = \mathbb{S}^2$$

A Second Order TV-type Model

Mid points between $x, z \in \mathcal{M}$:

$$\mathcal{C}_{x,z} := \left\{ c \in \mathcal{M} : c = \gamma_{\widehat{x,z}}\left(\tfrac{T}{2}\right) \text{ for any geodesic } \gamma_{\widehat{x,z}}, \ \ T := \mathcal{L}(\gamma_{\widehat{x,z}}) \right\}$$

The Absolute Second Order Difference:

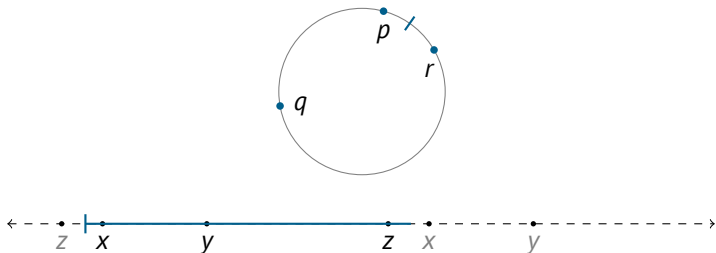$$d_2(x, y, z) := \min_{c \in \mathcal{C}_{x,z}} d(c, y), \qquad x, y, z \in \mathcal{M}.$$

$\Rightarrow$ Second Order TV-type Model for $\mathcal{M}$-valued signals f

$$\mathcal{E}(u) := \sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1}) + \beta \sum_{i \in \mathcal{G} \setminus \{1,N\}} d_2(u_{i-1}, u_i, u_{i+1})$$

Example: the Circle $\mathbb{S}^1$

Cyclic data: $p, q, r \in \mathbb{S}^1 := \left\{ s \in \mathbb{R}^2 \ : \ \|s\|_2 = 1 \right\} \iff x, y, z \in [-\pi, \pi)$
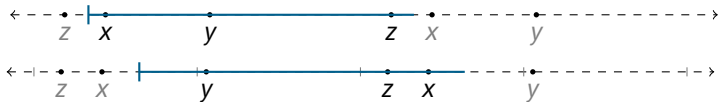
$d_2(x, y, z)$: Several unwrappings

$$|x - 2y + z|$$

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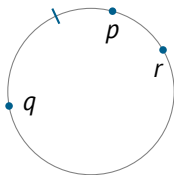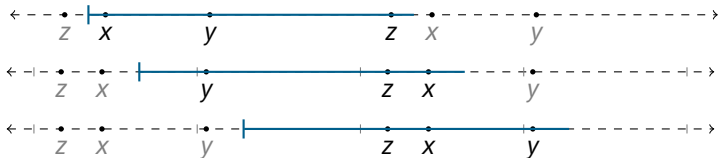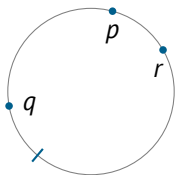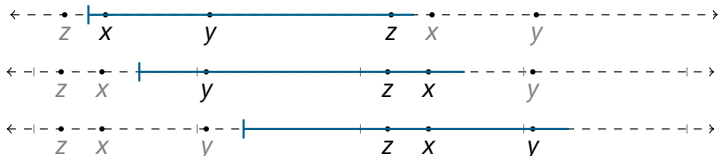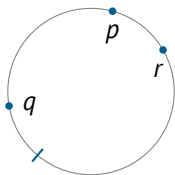
$d_2(x, y, z)$: Several unwrappings

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Example: the Circle $\mathbb{S}^1$

Cyclic data: $p, q, r \in \mathbb{S}^1 := \left\{ s \in \mathbb{R}^2 : \|s\|_2 = 1 \right\} \Longleftrightarrow x, y, z \in [-\pi, \pi)$

$d_2(x, y, z)$: Minimize over all unwrappings

$$d_2(x, y, z) := \min_{\mu \in \mathbb{R}} \left| (x + \mu) - 2(y + \mu) + (z + \mu) \right| = \left| (x - 2y + z)_{2\pi} \right|$$

Proximal Mappings

Proximal algorithms are standard tool for solving non-smooth, constrained minimization problems.

To compute a minimizer of $\mathcal{E}(u)$ we need proximal mappings

For a proper, closed, convex function $\varphi\colon \mathbb{R}^n \to (-\infty, +\infty]$ and $\lambda > 0$
the proximal mapping is defined by   [Moreau, 1965; Rockafellar, 1976]

$$\operatorname{prox}_{\lambda\varphi}(g) := \underset{u \in \mathbb{R}^n}{\arg\min}\, \frac{1}{2}\|u - g\|_2^2 + \lambda\varphi(u).$$

For a function $\varphi\colon \mathcal{M}^n \to (-\infty, +\infty]$ and $\lambda > 0$
the proximal mapping is defined by   [Ferreira, Oliveira, 2002]

$$\operatorname{prox}_{\lambda\varphi}(g) := \underset{u \in \mathcal{M}^n}{\arg\min}\, \frac{1}{2} \sum_{i=1}^{n} d(u_i, g_i)^2 + \lambda\varphi(u).$$

Note: For a minimizer u^* of φ it holds $\operatorname{prox}_{\lambda\varphi}(u^*) = u^*$.

The Cyclic Proximal Point Algorithm
Splitting large φ into smaller problems.

To find $\arg\min_x \varphi(x)$ use Picard iteration: Proximal Point Algorithm

$$x^{(k)} = \text{prox}_{\lambda\varphi}(x^{(k-1)}), \quad k > 0$$

$\Rightarrow$ fast evaluation of $\text{prox}_{\lambda\varphi}$ needed

For $\varphi = \sum_{l=1}^{c} \varphi_l$ use Cyclic Proximal Point Algorithm (CPPA)   [Bertsekas, 2011]

$$x^{(k+\frac{l+1}{c})} = \text{prox}_{\lambda_k\varphi_l}(x^{(k+\frac{l}{c})}), \quad i = 0, \ldots, c-1, \ k > 0$$

For real-valued data:
Converges to a minimizer if $\{\lambda_k\} \in \ell_2(\mathbb{Z})\backslash\ell_1(\mathbb{Z})$.

Minimizing the Second Order Model
Applying the CPPA to our model on manifolds.

To minimize

$$\mathcal{E}(u) := \sum_{i \in \mathcal{V}} d(f_i, u_i)^2 + \alpha \sum_{i \in \mathcal{G} \setminus \{N\}} d(u_i, u_{i+1}) + \beta \sum_{i \in \mathcal{G} \setminus \{1, N\}} d_2(u_{i-1}, u_i, u_{i+1})$$

take each summand as one φ_l in the CPPA.
For the involved proximal maps we have

- $\varphi(u_i) = d(f_i, u_i)^2$, analytical solution   [Ferreira, Oliveira, 2002]
- $\varphi(u_i, u_{i+1}) = d(u_i, u_{i+1})$, analytical solution   [Weinmann, Storath, Demaret, 2014]
- $\varphi(u_{i-1}, u_i, u_{i+1}) = d_2(u_{i-1}, u_i, u_{i+1})$
  - analytical solution for $\mathbb{S}^1$ [B., Laus, Steidl, Weinmann, 2014]
  - numerical solution otherwise   [Bačák, B., Steidl, Weinmann, 2015]

Proximal Mapping of a Second Order Difference on $\mathcal{M}$

To compute

$$\text{prox}_{\lambda d_2}(g) = \underset{u \in \mathcal{M}^3}{\arg\min} \Big\{ \frac{1}{2} \sum_{i=1}^{3} d(u_i, g_i)^2 + \lambda d_2(u_1, u_2, u_3) \Big\}$$

(sub)gradient descent method requires gradient of $d_2(x, y, z) = d(c(x, z), y)$

$$\nabla_{\mathcal{M}^3} d_2 = (\nabla_{\mathcal{M}} d_2(\cdot, y, z), \nabla_{\mathcal{M}} d_2(x, \cdot, z), \nabla_{\mathcal{M}} d_2(x, y, \cdot))^{\mathrm{T}}.$$

For $y \neq c(x, z)$ and an ONB $\{\xi_1, \ldots, \xi_n\}$ of $T_x\mathcal{M}$

- $\nabla_{\mathcal{M}} d_2(x, \cdot, z)(y) = \dfrac{\log_y c(x, z)}{\|\log_y c(x, z)\|_y} \in T_y\mathcal{M}$

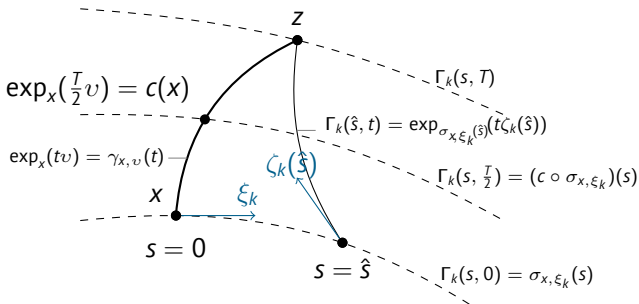- $\nabla_{\mathcal{M}} d_2(\cdot, y, z)$ by chain rule

$$\nabla_{\mathcal{M}} d_2(\cdot, y, z)(x) = \sum_{k=1}^{n} \left\langle \frac{\log_{c(x)} y}{\|\log_{c(x)} y\|_{c(x)}}, D_x c[\xi_k] \right\rangle_{c(x)}, \quad \xi_k \in T_x\mathcal{M}.$$

Proximal Mapping of a Second Order Difference on $\mathcal{M}$

To compute

$$\text{prox}_{\lambda d_2}(g) = \arg\min_{u \in \mathcal{M}^3}\Big\{\frac{1}{2}\sum_{i=1}^{3}d(u_i, g_i)^2 + \lambda d_2(u_1, u_2, u_3)\Big\}$$

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Geodesic Variation
How do small changes in x affect the geodesic?

- $\sigma_{x,\xi_k}(s)$, $k = 1, \ldots, n$, $s \in (-\varepsilon, \varepsilon)$:
    curve starting in $\sigma_{x,\xi_k}(0) = x$ with direction $\sigma'_{x,\xi_k}(0) = \xi_k$
- $\zeta_k(s)$: direction in $\sigma_{x,\xi_k}(s)$ towards z.
- $\Rightarrow$ geodesic variation of the geodesic $\gamma_{\widehat{x,z}}$

$$\Gamma_k(s,t) \coloneqq \exp_{\sigma_{x,\xi_k}(s)}\left(t\zeta_k(s)\right), \quad s \in (-\varepsilon, \varepsilon),\ t \in [0, T],\ T = d(x,z)$$

Jacobi Fields

Indicate directions of change along the geodesic and rephrase computation of the gradient to...

The Jacobi field J_k, $k = 1, \ldots, n$, along $\gamma_{\widehat{x,z}}$ is defined by

$$J_k(t) := \frac{\partial}{\partial s}\Gamma_k(s,t)\big|_{s=0}, \quad t \in [0, T]$$

Lemma

$$D_x c[\xi_k] = J_k\left(\tfrac{T}{2}\right)$$

Lemma

A Jacobi field J_k fulfills the System of ODEs

$$\frac{D^2}{\mathrm{d}t^2}J_k + R(J_k, \dot{\gamma}_{\widehat{x,z}})\dot{\gamma}_{\widehat{x,z}} = 0, \qquad J_k(0) = \xi_k, \, J_k(T) = 0$$

with the Riemannian curvature tensor $R\colon T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M}$.

Riemannian Symmetric Spaces
At least in beautiful Spaces we can solve the system of ODEs

- For every $x \in \mathcal{M}$ exists an isometry s_x on $\mathcal{M}$, i.e.

$$s_x(x) = x, D_x s[\xi] = -\xi, \qquad \text{for all } \xi \in T_x\mathcal{M}$$

- covariant derivative

$$\nabla R = 0$$

$\Rightarrow$ the system of ODEs simplifies to a a linear system of ODEs with constant coefficients

- Examples of symmetric spaces
  - Spheres $\mathbb{S}^n$
  - Hyperbolic Spaces
  - positive definite matrices $\mathcal{P}(s)$
  - Grasmannians

  Riemannian symmetric spaces are the most beautiful and most important Riemannian manifolds.

  (J.-H. Eschenburg, 1997)

The Derivative $D_x c[\xi_k]$ in Symmetric Spaces
Finally we are able to evaluate the Jacobi field at the mid point $T/2$

With parallel transport of an orthonormal frame $\{\Theta_1(t), \ldots, \Theta_n(t)\}$, i.e.

$$J_k(t) = \sum_{i=1}^{n} a_{k,i}(t)\Theta_i(t).$$

and a specific ONB $\{\xi_1, \ldots, \xi_n\}$ at $t = 0$ and κ_i "characterize curvature"

Corollary

With $T = d(x, z)$ we have for $k = 1, \ldots, n$

$$D_x c[\xi_k] = J_k\left(\tfrac{T}{2}\right) = \begin{cases} \dfrac{\sinh\left(\sqrt{-\kappa_k}\frac{T}{2}\right)}{\sinh(\sqrt{-\kappa_k}T)}\Theta_k(\tfrac{T}{2}), & \text{if } \kappa_k < 0, \\[2ex] \dfrac{\sin\left(\sqrt{\kappa_k}\frac{T}{2}\right)}{\sin(\sqrt{\kappa_k}T)}\Theta_k(\tfrac{T}{2}), & \text{if } \kappa_k > 0, \\[2ex] \tfrac{1}{2}\Theta_k(\tfrac{T}{2}), & \text{if } \kappa_k = 0. \end{cases}$$

$\Rightarrow$ Gradient descent to approximate the Proximal Mapping of the second order difference on a manifold.

A Signal of Cyclic Data

Denoising a phase valued signal.

- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_{\mathrm{o}} = (f_{\mathrm{o},i})_{i=1}^{500}$
- f could be a wrapped version of the gray plot, hence
- jumps $> \pi$ at $\frac{5}{16}$ and $\frac{11}{16}$ are due to the representation system
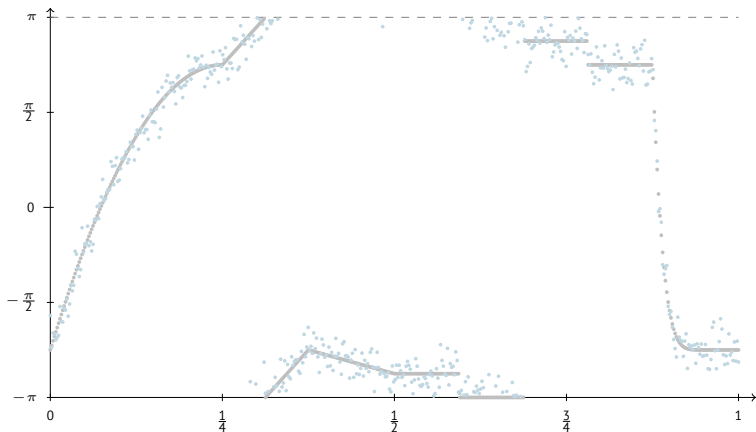
A Signal of Cyclic Data

Denoising a phase valued signal.

- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_o = (f_{o,i})_{i=1}^{500}$
- adding wrapped Gaussian noise, $\sigma = 0.2$
- noisy data $f_n = (f_0 + \eta)_{2\pi}$

A Signal of Cyclic Data

Denoising a phase valued signal.

■ comparison of f_o & f_n with

A Signal of Cyclic Data

Denoising a phase valued signal.

- comparison of f_o & f_n with f_1
- denoising TV_1 ($\alpha = \frac{3}{4}$, $\beta = 0$)
- but: stair casing

A Signal of Cyclic Data

Denoising a phase valued signal.

- comparison of f_0 & f_n with f_2
- denoising TV_2 ($\alpha = 0$, $\beta = \frac{3}{2}$)
- but: no plateaus

A Signal of Cyclic Data

Denoising a phase valued signal.

- comparison of f_o & f_n with f_3
- denoising TV_1 & TV_2 ($\alpha = \frac{1}{2}$, $\beta = 1$)
- smallest mean squared error

Bernoulli's Lemniscate on the Sphere $\mathbb{S}^2$

$$\gamma(t) := \frac{a\sqrt{2}}{\sin^2(t) + 1}\left(\cos(t), \cos 1 \sin(t)\right)^{\mathrm{T}}, \quad t \in [0, 2\pi], a = \frac{\pi}{2\sqrt{2}}.$$

Generate a sphere-valued signal by putting it into the tangential plane of the north pole

$$\gamma_S(t) = \log_p(\gamma(t)), p = (0, 0, 1)^{\mathrm{T}}$$
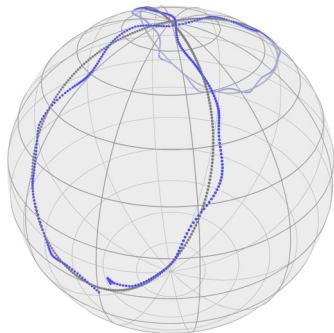

noisy lemniscate of Bernoulli on $\mathbb{S}^2$, Gaussian noise, $\sigma = \frac{\pi}{30}$, on $T_p\mathbb{S}^2$.

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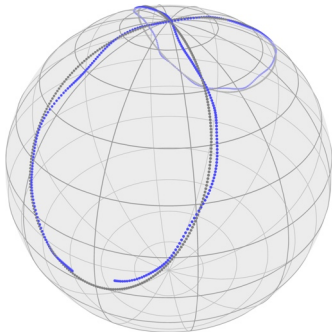

reconstruction with TV_1, $\alpha = 0.21$, MAE $= 4.08 \times 10^{-2}$.

Bernoulli's Lemniscate on the Sphere $\mathbb{S}^2$

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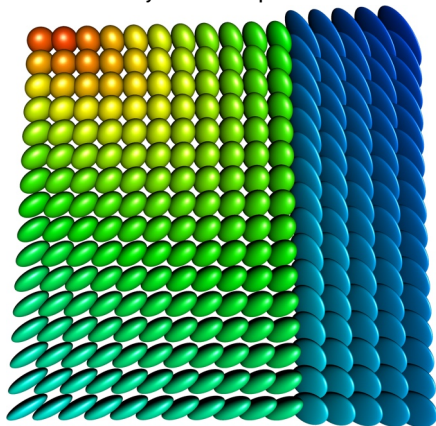Generate a sphere-valued signal by putting it into the tangential plane of the north pole

$$\gamma_S(t) = \log_p(\gamma(t)), p = (0,0,1)^{\mathrm{T}}$$

reconstruction with TV_2, $\alpha = 0$, $\beta = 10$, MAE $= 3.66 \times 10^{-2}$.

Bernoulli's Lemniscate on the Sphere $\mathbb{S}^2$

$$\gamma(t) := \frac{a\sqrt{2}}{\sin^2(t) + 1}\big(\cos(t), \cos 1 \sin(t)\big)^{\mathrm{T}}, \quad t \in [0, 2\pi], a = \frac{\pi}{2\sqrt{2}}.$$

Generate a sphere-valued signal by putting it into the tangential plane of the north pole

$$\gamma_S(t) = \log_p(\gamma(t)), p = (0, 0, 1)^{\mathrm{T}}$$

reconstruction with TV_1 & TV_2, $\alpha = 0.16$, $\beta = 12.4$, MAE $= 3.27 \times 10^{-2}$.

Inpainting of $\mathcal{P}(3)$ valued Images

Illustrate symmetric positive definite 3×3 matrices as ellipsoids
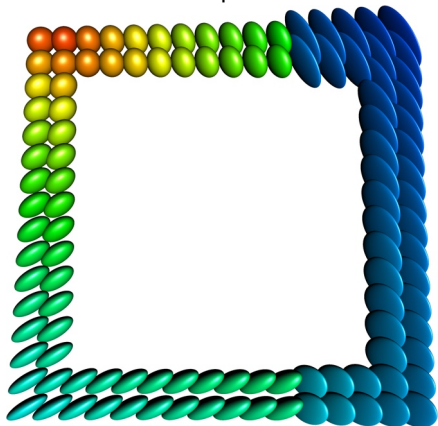

original data

Inpainting of $\mathcal{P}(3)$ valued Images

Illustrate symmetric positive definite 3×3 matrices as ellipsoids
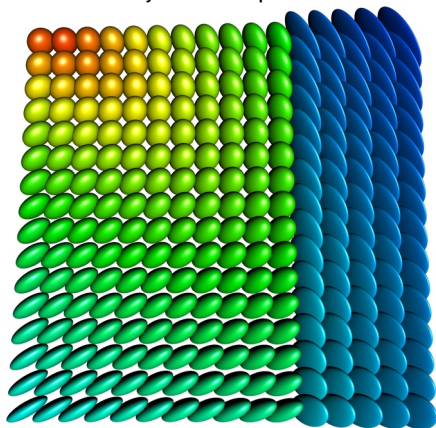

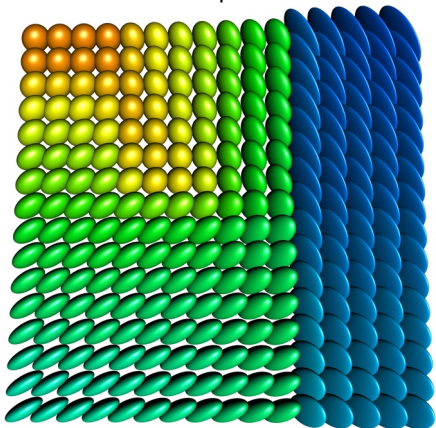original data   lost (a lot of) data

Inpainting of $\mathcal{P}(3)$ valued Images

Illustrate symmetric positive definite 3×3 matrices as ellipsoids
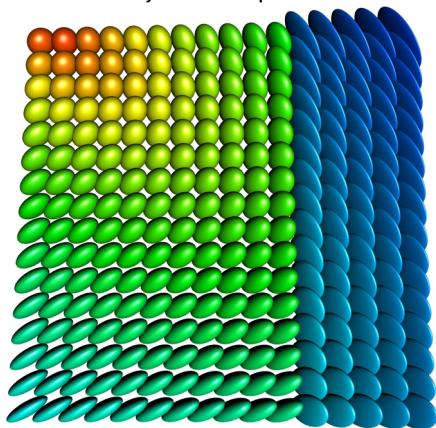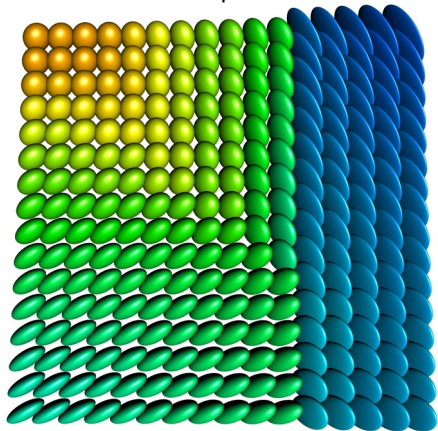

original data

inpainted with $\alpha = \beta = 0.05$,
MAE $= 0.0929$

Inpainting of $\mathcal{P}(3)$ valued Images

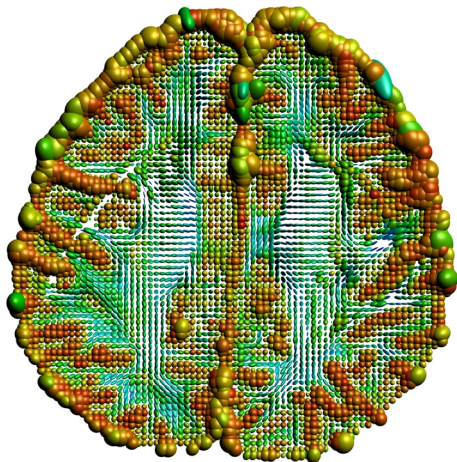Illustrate symmetric positive definite 3×3 matrices as ellipsoids
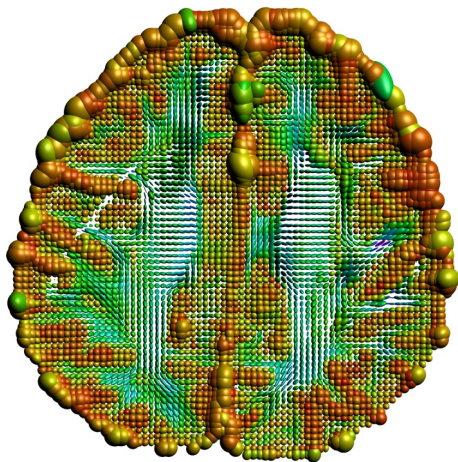

original data

inpainted with $\alpha = 0.1$,
MAE $= 0.0712$

Diffusion Tensor Magnetic Resonance Imaging

Dataset $\mathcal{P}(3)^{112 \times 112 \times 50}$ of the human head, traversal plane $z = 28$

original data,

denoised, $\alpha = 0.01$, $\beta = 0.05$
(Data available from The Camino Project, cmic.cs.ucl.ac.uk/camino)

Diffusion Tensor Magnetic Resonance Imaging

Dataset $\mathcal{P}(3)^{112 \times 112 \times 50}$ of the human head, traversal plane $z = 28$
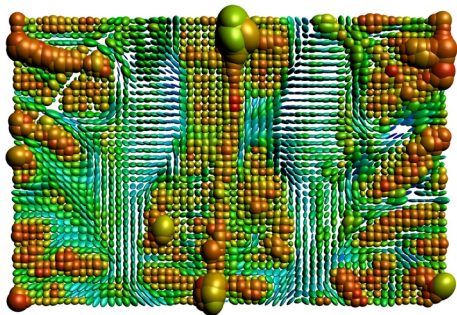

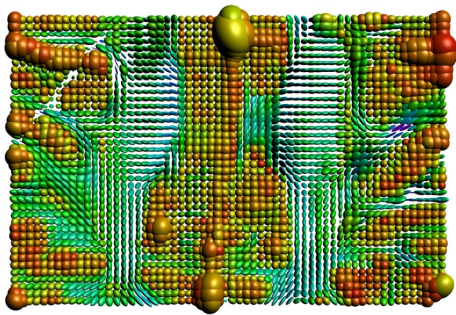original data,   denoised, $\alpha = 0.01$, $\beta = 0.05$
                                  (Data available from The Camino Project, cmic.cs.ucl.ac.uk/camino)

a detail

Conclusion

We have for manifold valued images $f\colon \mathcal{V} \to \mathcal{M}$

- a model for second order differences
- a first and second order TV-type functional $\mathcal{E}(u)$
- cyclic proximal point algorithm to minimize the functional $\mathcal{E}(u)$
- computed the proximal mappings using Jacobi fields
- proof of convergence
- examples of denoising and inpainting for manifold valued data

Future work

- an isotropic second order TV
- other couplings
- other algorithms

Literature

[1]   M. Bačák, R. Bergmann, G. Steidl, and A. Weinmann. A second order non-smooth variational model for restoring manifold-valued images. *Preprint, ArXiv #1506.02409*, 2015.

[2]   R. Bergmann, F. Laus, G. Steidl, and A. Weinmann. Second order differences of cyclic data and applications in variational denoising. *SIAM J. Imaging Sci.*, 2014.

[3]   R. Bergmann and A. Weinmann. A second order TV-type approach for inpainting and denoising higher dimensional combined cyclic and vector space data. *Preprint, ArXiv #1501.02684*, 2015.

[4]   J. Lellmann, E. Strekalovskiy, S. Koetter, and D. Cremers. Total variation regularization for functions with values in a manifold. *IEEE ICCV*, 2013.

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Literature

[1] M. Bačák, R. Bergmann, G. Steidl, and A. Weinmann. A second order non-smooth variational model for restoring manifold-valued images. *Preprint, ArXiv #1506.02409*, 2015.

[2] R. Bergmann, F. Laus, G. Steidl, and A. Weinmann. Second order differences of cyclic data and applications in variational denoising. *SIAM J. Imaging Sci.*, 2014.

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[4] J. Lellmann, E. Strekalovskiy, S. Koetter, and D. Cremers. Total variation regularization for functions with values in a manifold. *IEEE ICCV*, 2013.

[5] A. Weinmann, L. Demaret, and M. Storath. Total variation regularization for manifold-valued data. *SIAM J. Imaging Sci.*, 2014.

Thank you for your attention.