Topology – Homework 1

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1) Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Proof: It suffices to show that A is the union of open sets. To that end, let

$$\mathscr{U} := \{U \in \mathscr{T} \mid \exists x \in A : x \in U \subseteq A\}$$

This is a collection of open sets. Now consider $\bigcup \mathscr{U}$, a candidate for A. To show this, let $x \in A$. Then there is an open U for which $x \in U$, hence, $A \subseteq \bigcup \mathscr{U}$. For the other direction, note that by definition, each U is a subset of A. Hence $\bigcup \mathscr{U} \subseteq A$.

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3) Show that the collection $\mathscr{T}_c := \{U \subseteq X | X \setminus U \hookrightarrow \mathbb{N} \vee X \setminus U = X\}$ is a topology on the set X. Is the collection $\mathscr{T}_\infty := \{U \mid X \setminus U$ is infinite or empty or all of $X\}$ a topology on X?

Proof: We check the three parts of the definition.

1) Let $U = X$. Then $X \setminus U = \emptyset$ which is countable, so $X \in \mathscr{T}_c$. Now let $U = \emptyset$. Then $X \setminus U = X$, so $\emptyset \in \mathscr{T}_c$.

2) Let $\mathscr{U}_{j \in J}$ be a subset of $\mathscr{T}_c$. We wish to show that $\bigcup \mathscr{U}_{j \in J} \in \mathscr{T}_c$. For each $\mathscr{U}_j$ there are two possibilities: either $X \setminus \mathscr{U}_j$ is countable or X itself. If any $\mathscr{U}_j = X$ then the whole union is as well. On the other hand, if no $\mathscr{U}_j$ is the whole set, each one satisfies "$X \setminus \mathscr{U}_j$ is countable". By De Morgan, $X \setminus \bigcup \mathscr{U}_{j \in J} = \bigcap (X \setminus \mathscr{U}_{j \in J})$ which is also countable.

3) If suffices to show $\mathscr{U}_j \cap \mathscr{U}_k$ is open for any $j, k \in J$. If they are disjoint then their intersection is empty, so open. If they are not disjoint, they have a nonempty intersection (in particular, each one is nonempty). Thus the complement of each in X is countable. Then we apply De Morgan to the complement (in X) of the intersection: $X \setminus (\mathscr{U}_j \cap \mathscr{U}_k) = (X \setminus \mathscr{U}_j) \cup (X \setminus \mathscr{U}_k)$, which is also countable since it is the union of two countable sets.

Then $\mathcal{T}_c$ is a topology on X.

But $\mathcal{T}_\infty$ is not a topology: Let $X = \mathbb{N}, U_1 = \{n \in X \mid \text{Composite}(n)\}, U_2 = \{n \in X \mid \text{Odd}(n)\}$. Then by De Morgan, $X \setminus (U_1 \cup U_2) = (X \setminus U_1) \cap (X \setminus U_2) = \{2\}$, which is not an "open" set.

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5) Show that if $\mathscr{A}$ is a basis for a topology on X, then the topology $\mathcal{T}$ generated by $\mathscr{A}$ equals the intersection of all topologies on X that contain $\mathscr{A}$. Prove the same if $\mathscr{A}$ is a subbasis.

Proof (basis):

($\subseteq$) Take $U \in \mathcal{T}$. Then $U = \bigcup_{j \in J} B_j$ for some collection of basis elements. Note that each B_j is also in any topology that contains $\mathscr{A}$.

($\supseteq$) Let U be in the intersection of all topologies that contain $\mathscr{A}$. Note that $\mathcal{T}$ contains $\mathscr{A}$, so U is open in $\mathcal{T}$.

Proof (subbasis):

($\subseteq$) Take $U \in \mathcal{T}$. Then $U = \bigcup \left\{B_j \mid B_j = \bigcap_{<\omega} B_i\right\}$ for a collection of subbasis elements B_i. Since any topology that contains $\mathscr{A}$ contains each B_i, it also contains each B_j (since B_j is a finite intersection of open sets). Then it also contains U, because U is a union of open sets.

($\supseteq$) Same as in the basis case: If U is in each topology that contains $\mathscr{A}$, it is also in $\mathcal{T}$ since it, too, contains $\mathscr{A}$.

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7) Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology
$\mathcal{T}_2 = \mathbb{R}_K$
$\mathcal{T}_3$ = the finite complement topology
$\mathcal{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis
$\mathcal{T}_5$ = the topology having all sets $(-\infty, a)$ as basis

Determine, for each of these, which of the others it contains.

Standard topology:

1. Does not refine $\mathcal{T}_2$ by the argument presented in lemma 13.4.
2. Does refine $\mathcal{T}_3$: Let U be a nontrivial open set of $\mathcal{T}_3$. Then it's missing only finitely many real numbers. Let x_0 be the least such real number. Then we can construct an open set $U' = U$ of $\mathcal{T}_1$ as follows: The leftmost part, U'_l is defined to be the union of all intervals of the form
$$(j-1, j), j \in \{r \mid r < x_0\}$$

If x_1 is the least missing element not equal to x_0, then we merely take the interval (x_0, x_1). We follow this construction until all missing elements x_i have been taken care of, and finally we define the rightmost part U'_r similarly as above. Then $U' = U'_l \cup U'_r \cup \bigcup(x_i, x_{i+1})$. Then by construction U' has every real number except the x_i, so is equal to U.

3. Does not refine $\mathscr{T}_4$ for the same reason it doesn't refine the lower limit topology: given $x \in (a, b]$ there is no open interval that contains x and is in $(a, x]$.

4. Does refine $\mathscr{T}_5$: Let $B_1 = (-\infty, a)$ be a basis element of $\mathscr{T}_5$ and $x \in B_1$. Then the open interval $(x-1, a)$ contains x and is a subset of B_1, therefore the standard topology refines $\mathscr{T}_5$ by lemma 13.3.

"Harmonic" topology

1. Refines $\mathscr{T}_1$ by lemma 13.4.

2. Refines $\mathscr{T}_3$ since fineness is transitive and $\mathscr{T}_1$ refines $\mathscr{T}_3$.

3. Does not refine $\mathscr{T}_4$, argument is the same as why $\mathscr{T}_1$ does not refine $\mathscr{T}_4$.

4. Refines $\mathscr{T}_5$ by transitivity.

Finite complement topology:

1. Does not refine $\mathscr{T}_1$: let $x \in \mathbb{R}$ and consider the interval $(x-1, x+1)$. If U is an open set of the finite complement topology, U will necessarily contain (uncountably many) reals $y > x+1$, so U is not a subset of $(x-1, x+1)$.

2. Does not refine $\mathscr{T}_2$ for the same reason.

3. Does not refine $\mathscr{T}_4$ for the same reason.

4. Does not refine $\mathscr{T}_5$ for the same reason.

Upper limit topology

1. Refines $\mathscr{T}_1$ by a similar argument in lemma 13.4.

2. Refines $\mathscr{T}_2$: Suppose $0 \neq x \in \mathbb{R}$. We're only interested in basis elements B that aren't in $\mathscr{T}_1$, so those that have at least one element of the form $\frac{1}{n}, n \in \mathbb{N}$. Let p be the greatest such number smaller than x. By stipulation, the closed interval $[p, x]$ is a subset of B. Thus the $\mathscr{T}_4$ basis element $(p, x]$ is a subset of B. On the other hand, if $x = 0$ then B is of the form $(a, b) \setminus K$ for some $a < 0, b > 0$. Then we take the basis element $(a, 0]$. So by lemma 13.3, the upper limit topology refines $\mathbb{R}_k$.

3. Refines $\mathscr{T}_3$ by transitivity.

4. Refines $\mathscr{T}_5$ by transitivity.

Infinite left topology First, note that only two open sets of the topology are not basis elements: $\mathbb{R}$ and $\emptyset$. This is because $\bigcap_n(-\infty, a_i) = (-\infty, \min\{a_i \mid i \leq n\})$ and $\bigcup_I(-\infty, a_i) = (-\infty, \limsup a_i)$, so the basis is closed under finite intersections and arbitrary unions.

1. Does not refine $\mathscr{T}_1$: Let $x \in (a,b)$. Then all open sets of $\mathscr{T}_5$ have every $y < a$ as an element, so by lemma 13.3 it can't refine $\mathscr{T}_1$.
2. Does not refine $\mathscr{T}_2$ for the same reason.
3. Does not refine $\mathscr{T}_3$: Let U be an open set in $\mathscr{T}_3$, then $U = \mathbb{R} \setminus F$ for some finite subset $F \subseteq \mathbb{R}$. Suppose F has at least one element in it. Let x be the least element in F. For any $y > x$ with $y \in U$, the only way for y to be included in an open set U' of $\mathscr{T}_5$ is for U' to include x as well. Thus open sets of $\mathscr{T}_3$ are not in general open sets of $\mathscr{T}_5$.
4. Does not refine $\mathscr{T}_4$: If $(a,b]$ is a basis element of $\mathscr{T}_4$ then any basis element of $\mathscr{T}_5$ that includes b will include a as well.

8a) Apply lemma 13.2 to show the countable collection

$$\mathscr{B} = \{(a,b) \mid a < b\} \qquad a,b \in \mathbb{Q}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Proof: Let U be an open set in the standard topology and x a real such that $x \in U$. Then we can write U as the union of open intervals of reals (a_i, b_i). For at least one such interval is $x \in (a_i, b_i)$. Now choose a rational p with $a_i < p < x$ and a rational q with $x < q < b_i$. Then $x \in (p,q) \subseteq (a_i, b_i)$, so by lemma 13.2 $\mathscr{B}$ is a basis for the standard topology.

8b) Show that the collection

$$\mathscr{C} = \{[a,b) \mid a < b\} \qquad a,b \in \mathbb{Q}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Proof: Let $x \in \mathbb{R} \setminus \mathbb{Q}$. For no rational a,b is the interval $[a,b)$ a subset of $[x,b)$ that also includes x. Therefore by lemma 13.2 $\mathscr{C}$ does not generate the same topology.