Topology – The Order, Product and Subspace Topologies

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Definition (Order topology). Generated by the basis

$$\begin{aligned}\mathscr{B} &= \{x \in X \mid a < x < b\} \\ &\cup \{x \in X \mid a < x \leq \max(X)\} \\ &\cup \{x \in X \mid \min(X) \leq x < b\}\end{aligned}$$

if $\max(X), \min(X)$ are well-defined.

Note: Open rays form a subbasis for the order topology.

Definition (Product topology). If X, Y are topologies, the order topology on $X \times Y$ is defined to be generated by the basis

$$\mathscr{B} = \{U \times V \mid U \in \mathscr{T}_X, V \in \mathscr{T}_Y\}$$

Theorem: If $\mathscr{B}$ is a basis for X and $\mathscr{C}$ for Y, then the following is a basis for $X \times Y$:

$$\{B \times C \mid B \in \mathscr{B}, C \in \mathscr{C}\}$$

Theorem: The following is a subbasis for the product topology on $X \times$Y:

$$S = \left\{\pi_1^{-1}(U) \mid U \in \mathscr{T}_X\right\} \cup \left\{\pi_2^{-1}(V) \mid V \in \mathscr{T}_Y\right\}$$

Definition (Subspace topology)**.** If X is a topology and $Y \subseteq X$ then the following is the subspace topology on Y:

$$\{Y \cap U \mid U \in \mathscr{T}_X\}$$

Lemma: If $\mathscr{B}$ is a basis for the topology on X then the following is a basis for the subspace topology on Y:

$$\{B \cap Y \mid B \in \mathscr{B}\}$$

Lemma: Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.

Theorem: If A is a subspace of X and B is a subspace of Y then the product topology on $A \times B$ is the same as the subspace topology $A \times B$ inherits from $X \times Y$.

Definition (Convex set)**.** Let Y be a subset of a totally ordered X. We say Y is **convex** if for each $a, b \in Y$ the set $\{x \in X \mid a < x < b\}$ is a subset of Y.

Theorem: Let X have an order topology and Y a convex subset of X. Then the order topology of Y is the same as the topology it inherits from X as a subspace.