# Simple Random Sampling: Not So Simple

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# Simple Random Sampling

**Simple random sampling:** drawing $k$ objects from a group of $n$ in such a way that all $\binom{n}{k}$ possible subsets are equally likely.

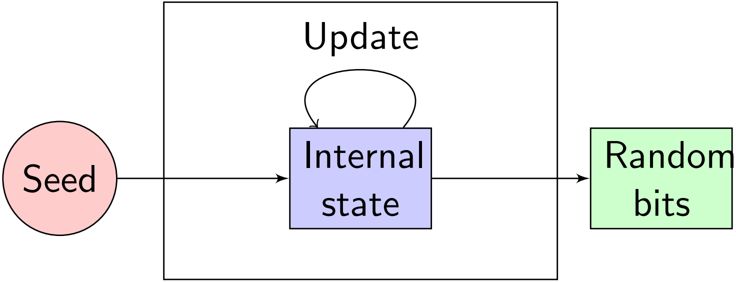
In practice, it is difficult to draw truly random samples.

Instead, people tend to draw samples using

1. A **pseudorandom number generator** (PRNG) that produces sequences of bits, plus
2. A sampling algorithm that maps a sequence of pseudorandom numbers into a subset of the population

# PRNGs

**Pseudorandom number generator:** a deterministic algorithm that produces sequences that are computationally indistinguishable from the uniform distribution



# Pigeons and Pigeonholes

## Theorem (Pigeonhole Principle)

*If there are $n$ pigeonholes and $m > n$ pigeons, then there exists at least one pigeonhole containing more than one pigeon.*

(Wikipedia)

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## Corollary (Too few pigeons)

*If $\binom{n}{k}$ is greater than the size of a PRNG's state space, then the PRNG cannot possibly generate all samples of size $k$ from a population of $n$.*

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Period of Mersenne Twister (standard PRNG in Statistics):
$2^{32\times 624} \approx 2 \times 10^{6010}$
Permutations of $2084$ objects: $2084! \approx 3 \times 10^{6013}$
**Less than $0.01\%$ of permutations can be generated**

# The good, the bad, and the ugly

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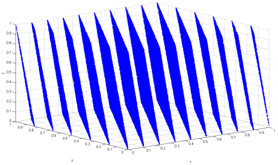


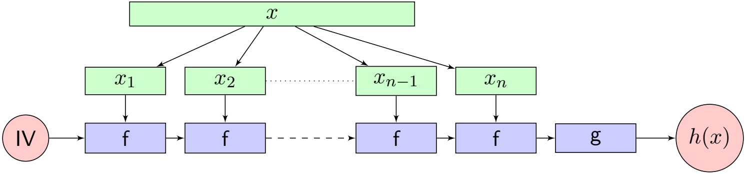
Triples of RANDU lie on 15 planes in 3D space

$$x_{n+1} = (65539 x_n) \mod 2^{31}$$

(Wikipedia)

# A better alternative

**One solution:** Find a class of PRNGs with infinite state space




Cryptographic hash functions:

- computationally infeasible to invert
- difficult to find two inputs that map to the same output
- small input changes produce large, unpredictable changes to output
- resulting bits are uniformly distributed

# SHA256 in practice

- Preliminary results: SHA256 PRNG produces samples with equal probabilities while other common PRNGs don't
- Replace the default PRNGs in Python
  `https://www.github.com/statlab/cryptorandom`
- Results apply more broadly to computer simulations: permutation tests, bootstrapping, MCMC, etc.

# Thanks!

`https://github.com/kellieotto/prng-slides`