A review of "On the Failure of the Bootstrap for Matching Estimators" (Abadie and Imbens; 2008)

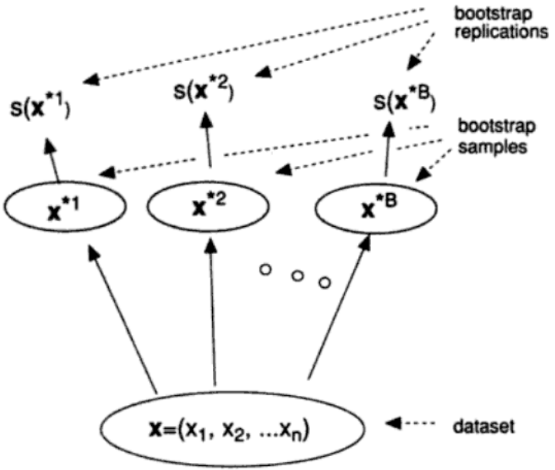
Andrew Do, Kellie Ottoboni, Simon Walter

April 8, 2016

Problem

- Matching is sometimes done to control for pretreatment covariates in observational studies
- Matching estimators are nonlinear functions of the data and do not follow any nice known distribution
- **Problem:** how do you find standard errors for matching estimators?
- Two common ways: asymptotic approximations and resampling methods

The bootstrap



Sample with replacement from the observed data, pretending it is the population, to approximate the distribution of the statistic TO DO: Anyone know of a better graphic? TO DO: cite

On the failure of the bootstrap

The bootstrap estimate of the variance of the matching estimator $\hat{\tau}$ is given by

$$\hat{V}^B = \frac{1}{B} \sum_{b=1}^{B} (\hat{\tau}_b - \hat{\tau})^2$$

Abadie and Imbens show that $\hat{V}^B$ is not generally valid for matching estimators.

They focus on the case of one-to-one matching on a single continuous covariate.

Notation and Assumptions

- Suppose we have a random sample of N_0 units from the control population and a random sample of N_1 units from the treated population, with $N = N_0 + N_1$
- Each unit has a pair of potential outcomes, $Y_i(0)$ and $Y_i(1)$, under the control and active treatments
- Let W_i indicate treatment: we observe $Y_i = W_i Y_i(1) + (1 - W_i) Y_i(0)$
- In addition to the outcome, we observe a (scalar) covariate X_i for each individual

We're interested in the **average treatment effect for the treated** (ATT):

$$\tau = \mathbb{E}(Y_i(1) - Y_i(0) \mid W_i = 1)$$

Notation and Assumptions

We make the usual assumptions for matching:

- Unconfoundedness: For almost all x,

$$(Y_i(0), Y_i(1)) \perp\!\!\!\perp W_i \mid X_i = x \text{ almost surely}$$

- Overlap: For some $0 < c < 1$ and almost all x,

$$c \leq \mathbb{P}(W_i = 1 \mid X_i = x) \leq 1 - c$$

Notation and Assumptions

D_i is the distance between the covariate values for observation i and the closest control group match:

$$D_i = \min_{j=1,\ldots,N:W_j=0} \|X_i - X_j\|$$

$\mathcal{J}(i)$ is the set of closest matches for treated unit i.

$$\mathcal{J}(i) = \begin{cases} \{j \in \{1,\ldots,N\} : W_j = 0, \|X_i - X_j\| = D_i\} & \text{if } W_i = 1 \\ \emptyset & \text{if } W_i = 0 \end{cases}$$

If X is continuous, this set will consist of one unit with probability 1. In bootstrap samples, units may appear more than once.

Notation and Assumptions

Estimate the counterfactual for each treated unit as:

$$\hat{Y}_i(0) = \frac{1}{\#\mathcal{J}(i)} \sum_{j \in \mathcal{J}(i)} Y_i$$

The matching estimator of τ is then

$$\hat{\tau} = \frac{1}{N_1} \sum_{i:W_i=1} \left(Y_i - \hat{Y}_i(0) \right)$$

Notation and Assumptions

An alternative way of writing the estimator is

$$\hat{\tau} = \frac{1}{N_1} \sum_{i=1}^{N} (W_i - (1 - W_i) K_i) Y_i$$

where K_i is the weighted number of times that unit i is used as a match:

$$K_i = \begin{cases} 0 & \text{if } W_i = 1 \\ \sum_{j: W_j = 1} 1\{i \in \mathcal{J}(j)\} \frac{1}{\#\mathcal{J}(j)} & \text{if } W_i = 0 \end{cases}$$

Bootstrap

- We obtain a **bootstrap sample** Z_b by taking a random sample with replacement from $Z = (X, W, Y)$.
- Let $\hat{\tau}_b = t(Z_b)$ be the matching statistic computed on bootstrap sample b.
- The bootstrap variance of $\hat{\tau}$ is the variance of $\hat{\tau}_b$ conditional on the original data Z:

$$V^B = \mathbb{E}\left[(\hat{\tau}_b - \hat{\tau})^2 \mid Z\right]$$

- We estimate it by generating B bootstrap samples from Z and taking the following average:

$$\hat{V}^B = \frac{1}{B}\sum_{b=1}^{B}(\hat{\tau}_b - \hat{\tau})^2$$

Bootstrap

- The bootstrap works for linear statistics that are consistent and asymptotically normal
- For nonlinear statistics, the bootstrap requires additional smoothness conditions TO DO: cite?
- TO DO: Abadie and Imbens show that $\hat{\tau}$ is asymptotically normal with ugly variance
- Why does the bootstrap fail here?

Bootstrap

Issue: the bootstrap fails to replicate the distribution of K_i, even in large samples.

Example:

- Suppose the ratio N_1/N_0 is small (i.e. there are many more controls than treated)
- In the original sample, few controls are used as a match more than once
- In bootstrap samples, treated units may appear multiple times, creating situations where $\mathbb{P}(K_{b,i} > 1) > \mathbb{P}(K_i > 1)$ TO DO: IS THIS TECHNICALLY CORRECT? IS THERE A BETTER WAY TO PUT THIS?

Bootstrap

some theory?

Placing Abadie and Imbens in the literature

- Csörgő and Mason (1989) established that linear statistics are consistently estimated by the bootstrap if and only if they are asymptotically normal.
- X (198?) suggests rigorous results for non-linear statistics require in addition to asymptotic normality that the statistic is a smooth function of the data.
- Formalizing this requires a notion of smoothness of random quantities called Fréchet differentiability, but we will elide it.

Placing Abadie and Imbens in the literature

- The revision history of the manuscript suggests that at least initially Abadie and Imbens were not particularly familiar with this prior work and some vestiges of this position remain.
- The results of Abadie and Imbens are not surprising and not novel to specialists familiar with the theoretical work of the 1980s.
- But it is valuable because it informs practitioners one of the (many) limitations of the vanilla bootstrap.

Contribution to theoretical understanding of the bootstrap

- In a 2006 version of their paper Abadie and Imbens claim this is the first case for which the bootstrap is inconsistent for a statistic that is asymptotically normal and $\sqrt{n}$-consistent.
- But this is not true. Beran (1982) establishes that a Hodges-type estimator for the mean:

$$\theta(X_1, \ldots, X_n) = \begin{cases} b\bar{X}_n \text{ if } |\bar{X}_n| < n^{-1/4} \\ \bar{X}_n \text{ if } |\bar{X}_n| \geq n^{-1/4} \end{cases}$$

  is not consistently estimated by the bootstrap when the true mean is zero.
- The proof of this fact is not easy and requires some knowledge of random measures.

Contribution to theoretical understanding of the bootstrap

- In the final version of the paper, Abadie and Imbens emphasize the novelty of an example for which the bootstrap is inconsistent for a statistic that is asymptotically normal, $\sqrt{n}$-consistent and <u>asymptotically unbiased</u>.
- This is not the first example either because Beran's 1982 example is also asymptotically unbiased.
- It is also easy to construct simpler examples where the bootstrap fails that are unbiased in finite samples too, although they seem not to have previously appeared in the literature.

An example of bootstrap inconsistency for an unbiased statistic

- We give one here: suppose X is drawn from the location family $\{N(\mu, 1)\}_{\mu \in \mathbb{R}}$.
- Our estimate for μ is

$$\theta(\hat{F}) = \theta(X_1, \ldots, X_n) = \bar{X} + \#\{(i,j) : X_i = X_j, i \neq j\}$$

- Under the true sampling distribution the second summand is almost surely zero.
- But under the bootstrap distribution the second summand is at least one with probability $1 - n!/n^n$

An example of bootstrap inconsistency for an unbiased statistic

- The previous example was bizarre and unnatural.
- But it does not seem hard to extend this to cases of practical interest involving ties.
- For example the critical value of Wilcoxon rank-sum and signed rank tests might be approximated using the bootstrap when ties are present in the data.

Should we follow Abadie and Imbens recommendation?

- Main conclusion is only their prior work based on asymptotic normality or subsampling have formal justification.
- This is not satisfying: this is only ‘first order’ correct but the bootstrap is used because it is often ‘second order’ correct.
- What does this mean ... ?

- For many statistics of interest we can form Edgeworth expansions of the distribution function:

$$\mathbb{P}(\sqrt{n}(\hat{\theta} - \theta_0)/\sigma \leq x) = \Phi(x) + n^{-1/2} p_1(x)\phi(x) + \dots$$

- The $p_j(x)$ are polynomials with coefficients determined by low order moments of the statistic.
- The bootstrap distribution has a similar expansion:

$$\mathbb{P}^*(\sqrt{n}(\hat{\theta}^* - \hat{\theta})/\hat{\sigma} \leq x) = \Phi(x) + n^{-1/2} \hat{p}_1(x)\phi(x) + \dots$$

- Here the $\hat{p}_j(x)$ are $p_j(x)$ with population moments substituted for their empirical counterparts.
- Bootstrap is second order correct if

$$n^{-1/2}\hat{p}_1(x)\phi(x) = n^{-1/2} p_1(x)\phi(x) + o(n^{-1/2})$$

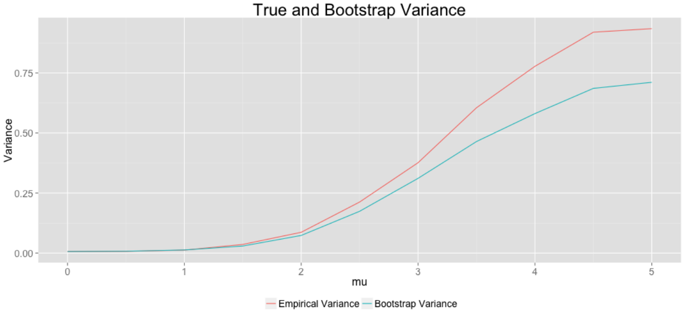
Can we rescue the bootstrap for matching estimators

- Yes! We think...
-

Effect of covariate distributions

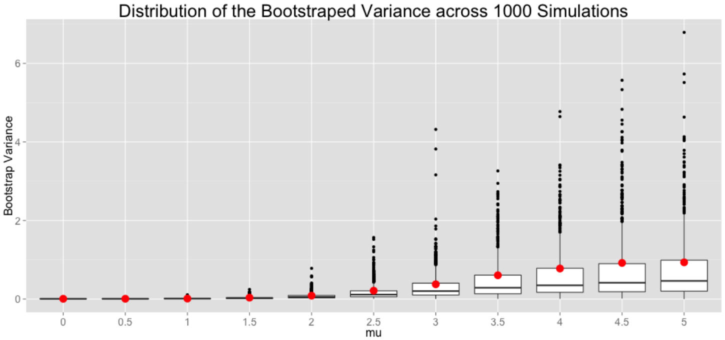
- Potential outcomes $Y(1)$ and $Y(0)$ defined as before
- Treatment assigned at random with $\frac{N_1}{N_0} = \alpha = 2$ fixed
- Change the covariate distributions: $X_i \sim N(0, 1)$ if $W_i = 1$ and $X_i \sim N(\mu, 1)$ if $W_i = 0$.
- We vary μ from 0 to 5
- TO DO: THIS SLIDE IS UGLY

Results



The bootstrap tends to underestimate the variance of $\hat{\tau}$. The bias increases as the distance μ between treatment and control groups increases.

Results



Red points indicate the observed variance of the test statistic. The distribution of bootstrap variances has a long right tail. The skew worsens as μ grows.