Stats Reference

Kelly Sovacool

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Preface

I started compiling this reference for concepts & equations in statistics while taking an introductory probability theory course at U-M.

You can view the bookdown site or download the PDF file.

The source code is hosted on GitHub.

Chapter 1

Calculus Review

1.1 Logarithms

$\log_b a = x \leftrightarrow b^x = a$

$e^{c \ln x} = x^c$

1.2 Derivative & Integration rules

Derivative	Integral
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x dx = \frac{n^x}{\ln n} + c$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln \lvert ax+b \rvert + c$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx} \sin x = cosx$	$\int \sin x dx = -\cos x + c$
$\frac{d}{dx} \cos x = -\sin x$	$\int \cos x dx = \sin x + c$
$\frac{d}{dx} \tan x = \sec^2 x$	$\int \tan x = \ln \lvert \sec x \rvert + c$

$\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$

$\int x f(x) = xF(x) + f(x)$

1.2.1 Quotient Rule

$\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

1.2.2 Integration by substitution

$u = g(x)$

$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

1.2.3 Integration by parts

Assign u and dv, differentiate u to find du, integrate dv to find v, then solve:
$\int_a^b udv = [uv]_a^b - \int_a^b vdu$

1.3 Trigonometry

1.3.1 SOH CAH TOA

1.3.2 Basic Identities

1.3.3 Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Chapter 2

Probability

For the following section, A and B represent events in the sample space S.

2.1 Axioms

1. $\mathbb{P}(A) \geq 0 \quad \forall A \subset S$
2. $\mathbb{P}(S) = 1$
3. If $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

2.2 Union Rule

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$

2.3 De Morgan's Laws

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

2.4 Conditional Probability

$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)$

2.5 Bayes' Theorem

$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{i=1}^{k} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}$

2.6 Independence

If events A and B are independent: $\mathbb{P}(A|B) = \mathbb{P}(A)$

2.7 Counting Examples

- There are $n!$ ways to arrange n distinct elements in an ordered list.
- There are 6^n outcomes for n tosses of a 6-sided die.

Chapter 3

Distributions of Random Variables

3.1 Discrete

CDF: $F(k) = \sum_{k=0}^{i} p(k)$

3.1.1 Bernoulli

$X \sim \text{Bern}(p)$

$\mathbb{E}[X] = p$

$\text{Var}[X] = p(1-p)$

$$p(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & else \end{cases} \tag{3.1}$$

3.1.2 Binomial

$X \sim \text{Binom}(n, p)$

$\mathbb{E}[X] = np$

$\text{Var}[X] = np(1-p)$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

3.1.3 Poisson

$X \sim \text{Poisson}(\lambda \sim np)$

$\mathbb{E}[X] = \lambda$

$\text{Var}[X] = \lambda$

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

- Approximation to binomial when $n \to \infty$ and $p \to 0$.
- E.g. number of misprints per page of a book.

3.1.4 Geometric

$X \sim \text{Geom}(p)$

$\mathbb{E}[X] = \frac{1}{p}$

$\text{Var}[X] = \frac{1-p}{p^2}$

$$\begin{aligned} p(k) &= (1-p)^{k-1} \\ F(k) &= 1-(1-p)^k \end{aligned} \tag{3.2}$$

- Experiment with infinite trials; stop at first success.
- Memoryless.
- E.g. flip a coin until heads comes up.

3.2 Continuous

3.2.1 Uniform

$X \sim \text{Unif}(a, b)$

$\mathbb{E}[X] = \frac{b+a}{2}$

$\text{Var}[X] = \frac{(b-a)^2}{12}$

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases} \\ F(x) &= \begin{cases} \frac{x-a}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases} \end{aligned} \tag{3.3}$$

- Simplest continuous distribution.
- All outcomes equally likely.
- E.g. uniformly pick random point on disk of radius r. x is distance to center (not Uniform). $f(x) = \frac{2x}{r^2}$ when $0 \leq x \leq r$.

3.2.2 General Normal

$X \sim \text{N}(\mu, \sigma)$

$\mathbb{E}[X] = \mu$

$\text{Var}[X] = \sigma^2$

$$
\begin{gathered}
f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2} \\
F(x)=\phi(Z=\frac{x-\mu}{\sigma})
\end{gathered}
\tag{3.4}
$$

- To find CDF, convert to standard normal, then use Z table.
- E.g. biological variables.
- E.g. exam scores.

3.2.3 Standard Normal

$X \sim \mathrm{N}(0,1)$

$\mathbb{E}[X]=0$

$\mathrm{Var}[X]=1$

$$
\begin{gathered}
f(x)=\frac{1}{\sqrt{2\pi}}e^{-(x^2)/2} \\
F(x)=\phi(x)
\end{gathered}
\tag{3.5}
$$

- To find CDF, use Z table.
- Special case of the normal with no parameters.

3.2.3.1 Normal Approximation to the Binomial Distribution

When $X \sim \mathrm{Binom}(n,p)$, $n \to \infty$, & $p \to \frac{1}{2}$:

$\mathbb{E}[X]=np=\mu$, $\sigma=\sqrt{np(1-p)}$, $z=\frac{x-np}{\sqrt{np(1-p)}}$

$F_z(a) \to \phi(a)$

$\mathbb{P}(a \leq z \leq b) \approx \phi(b)-\phi(a)$

via De Maivre-Laplace Central Limit Theorem

3.2.4 Exponential

$X \sim \mathrm{Exp}(\lambda)$

$\mathbb{E}[X]=\frac{1}{\lambda}$

$\mathrm{Var}[X]=\frac{1}{\lambda^2}$

$$
\begin{gathered}
f(x)=\begin{cases}\lambda e^{-\lambda x} & x>0 \\ 0 & else\end{cases} \\
F(x)=\begin{cases}1-e^{-\lambda x} & x>0 \\ 0 & else\end{cases}
\end{gathered}
\tag{3.6}
$$

- Memoryless.

- λ = rate.
- Continuous version of Geom(p).

3.2.4.1 Hazard & Survival

Survival: $S_T(t) = \mathbb{P}(T > t) = 1 - \mathbb{P}(T \leq t) = 1 - F_T(t) = e^{-\int_{u=o}^{t} \lambda(u)du}$

Density: $f_T(t) = F'_T(t) = -S'_T(t)$

Hazard: $\lambda(t) = \frac{f_T(t)}{S_T(t)} = \frac{-S'_T(t)}{S_T(t)} = -\frac{d}{dt} \log S_T(t)$

3.2.5 Gamma

$X \sim \Gamma[\alpha, \lambda]$

$\mathbb{E}[X] = \frac{\alpha}{\lambda}$

$\mathrm{Var}[X] = \frac{\alpha}{\lambda^2}$

3.2.6 Chi Square

$X \sim \chi^2[n]$

$\mathbb{E}[X] = n$

$\mathrm{Var}[X] = 2n$

- Special case of Γ where $\alpha = \frac{n}{2}$ and $\lambda = \frac{1}{2}$.

3.3 Properties

3.3.1 Density Functions

PMF: $p(k)$ PDF: $f(x)$

- Derivative of the distribution function.
- Nonnegative everywhere.
- Integral over its domain is 1: $\int_a^b f(x) = 1$

3.3.2 Distribution Functions

CDF: $F(x)$

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to +\infty} F(x) = 1$
- Nondecreasing.

3.3.3 Parameters

$$\text{Law of total expectation:} \quad \mathbb{E}[X] = \sum_j \mathbb{E}(E|F_j)\mathbb{P}(F_j)$$

$$\text{Discrete:} \quad \mathbb{E}[X] = \mu = \sum_{i=1}^{k} x_i p_i$$

$$\text{Continuous:} \quad \mathbb{E}[X] = \mu = \int_{-\infty}^{\infty} x f(x) dx \tag{3.7}$$

$$\text{Var}[X] = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sigma^2$$

$$\sigma = \sqrt{\text{Var}[X]}$$

$$Z = \frac{x - \mu}{\sigma}, \quad Z \sim \text{N}(0, 1)$$

3.4 Distributions of Functions

X is a random variable. $Y = g(x)$ is a function of X.

3.4.1 Transformation Method

If $Y = g(x)$ is monotonic:

$f_Y(y) = \frac{1}{|g'(x)|} f_X(x)$

Derive $g'(x)$ from $g(x)$. Integrate f_Y to find F_Y.

Note: monotonic = invertible = one-to-one.

3.4.2 CDF Method

Must use when $Y = g(x)$ is not monotonic:

$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(x) \leq y) \rightarrow$ solve for x and frame in terms of $F_X(y)$.

Differentiate F_Y to find f_Y.

Chapter 4

Joint Distributions

$\mathbb{P}(x \in A, y \in B) = \int_A \int_B f(x,y)dydx$

4.1 Marginals

$f_X = \int f(x,y)dy$

$f_Y = \int f(x,y)dx$

$\mathbb{P}(x \in A) = \mathbb{P}(x \in A, y \in (-\infty, \infty)) = \int_A \int_{-\infty}^{\infty} f(x,y)dydx$

$\mathbb{P}(y \in B) = \mathbb{P}(x \in (-\infty, \infty), y \in B) = \int_{-\infty}^{\infty} \int_B f(x,y)dydx$

4.2 Independence

$f(x,y) = f_X(x)f_Y(y) \quad \forall x, y$

$F(x,y) = F_X(x)F_Y(y) \quad \forall x, y$

4.2.1 Minimum & Maximum

Max: $F_{\text{Max}}(t) = \mathbb{P}(\text{Max} \leq t) = \mathbb{P}(x \leq t, y \leq t) \rightarrow$ use independence $\rightarrow = F_X(t)F_Y(t)$

Min: $1 - F_{\text{Max}}$

4.3 Sums of Independent Random Variables

4.3.1 Distributions

Convolution (CDF): $F_{X+Y}(a) = \mathbb{P}(X + Y \leq a) = \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy$

Density (PDF): $f_{X+Y} = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$

4.3.2 Uniform

4.3.3 Normal

The sum of n normal RVs $\sum_i^n X_i$ is normally distributed with parameters:

$$\mu = \sum_i^n \mu_i$$
$$\sigma^2 = \sum_i^n \sigma_i^2$$
$$\sigma = \sqrt{\sum_i^n \sigma_i^2} \neq \sum_i^n \sqrt{\sigma_i^2}$$

4.3.4 Gamma

4.3.5 Poisson

$X_1 \sim \text{Poisson}(\lambda_1)$

$X_2 \sim \text{Poisson}(\lambda_2)$

$Y = X_1 + Y_2$

$Y \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)$

$\mathbb{P}(X_1 + X_2 = n) = \frac{1}{n!} e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n$

4.3.6 Binomial

$X_1 \sim \text{Binom}(n, p)$

$X_2 \sim \text{Binom}(m, p)$

$Y = X_1 + Y_2$

$Y \sim \text{Binom}(n + m, p)$

$\mathbb{P}(X_1 + X_2 = k) = \binom{n+m}{k} = \sum_{i=0}^n \binom{n}{i}\binom{m}{k-i}$

4.3.7 Geometric

4.4 Conditional Joint Distributions

4.4.1 Discrete

$P_{X|Y} = \frac{P(x,y)}{P_Y(y)} = \mathbb{P}(X = x|Y = y)$

$\mathbb{E}[X|Y = y] = \sum_x x P_{X|Y}(x|y)$

4.4.2 Continuous

$f_{X|Y} = \frac{f(x,y)}{f_Y(y)}$

$\mathbb{E}[X|Y=y] = \displaystyle\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$

$F_{X|Y}(a,y) = \mathbb{P}(X \leq a|Y=y) = \displaystyle\int_{-\infty}^{a} f_{X|Y}(x|y)dx$

4.4.3 Bayes' Theorem (Continuous)

$f_{X|Y} = \frac{f_{Y|X}(y|x)f_x(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x)f_x(x)}{\int f_{Y|X}(y|x)f_x(x)dx}$

4.5 Order Statistics

4.6 Transformations of Joint Distributions

4.6.1 The Jacobian

$(u,v) = G(x,y)$

$\text{Jac}(x,y) = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

$f_{u,v}(u,v) = \frac{1}{|\text{Jac}(x,y)|} f_{x,y}(x,y)$

Chapter 5

Expectation