Modeling Popcorn Popping

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Abstract

Newton's Law of Cooling is the fundamental ingredient for a basic model of popcorn popping. Depending on the heat of a given stove or the time the popcorn is cooked, popcorn will either not pop at all or burn completely. Our main purpose is two-fold: to find both an optimal temperature and and optimal cooking time to minimize burned kernels and maximize popped kernels. Modeling this phenomena for a single kernel requires and understanding of the Ideal Gas Law ($PV = nrT$). For a whole batch of popcorn, a sample of kernels are drawn at random from a distribution based on the pressure required to rupture them. We find that our model favors cooking the popcorn at higher temperatures with lower cook times to avoid burning individual kernels.

1 Background

It's Friday night and you're at home watching a movie. Naturally, you venture to the pantry for some microwavable popcorn. Much to your dismay, you learn that someone has put an empty box of microwavable popcorn back in the pantry instead of recycling it. Before you get too upset, you remember that you have some popcorn kernels in your food storage that you will attempt to cook on the stove. Being the mathematician you are, instead of popping the kernels and using your instincts to decide when its done, you want to know, given the temperature of your stove top, what is the optimal time to pop the popcorn and what is a reasonable stovetop temperature? You want the best bowl of popcorn possible, meaning you want to minimize burned and unpopped kernels and maximize popped kernels.

There have been a few studies about the distribution of temperatures (or pressures) that popcorn kernels pop at [1,2]. From these studies, we learn that it is normally distributed with a mean around $180°C$ and variance around $4°C$. The basic idea is to draw kernels, meaning kernel temperatures, and with a set stove temperature, use differential equations to model the time the kernel will pop. However, the popcorn does not remain edible for all time; it eventually burns. There isn't any research on this, so coming up with this part of the model takes a little more creativity (see 2.1). We then take finite time steps and for each time t, and we record if that kernel is unpopped, popped, or burned. We do this for each kernel, meaning at each timestep t, we will have an ordered list of length k of unpopped, popped, or burned, where k is the number of kernels. We then use a suitable loss function (see 2.3) to determine the optimal time that gives the best ratio of unpopped, popped, and burned popcorn.

2 Equations and Modeling

Our initial idea is to model individual kernels for the time at which they pop. The process is to draw from a distribution of popping temperatures (or pressures) and then use that critical temperature and the temperature of the stove to determine the time at which the popcorn pops. Some background on the properties of popcorn kernels is necessary to develop an appropriate model for the distributions of these critical temperatures. There are many variables that potentially account for the differences in popping times, including variables that don't change the critical temperature but rather change the way the kernel absorbs heat. However, for this basic model, we will assume that the kernels heat uniformly. Variations in the critical temperatures, however, are created by differences in kernel size, pericarp strength (the strength of the outer shell), and water content. If we could somehow limit all these variations, then the popcorn kernels would all theoretically pop at the same time. However, the compounding of all these factors creates very noticeable differences in the popping times that we hope we can capture in our model.

Rather than mathematically accounting for all of these factors, we assume that the pericarp strength of a kernel will have a Gaussian distribution, like most other naturally occurring features. Hopefully, this will take in the variation in water content and other factors as well as shape and size. According to [2], we have that the typical hull strength of popcorn pericarp (σ_c) is around 10 mPa. Furthermore, the critical pressure is equivalent to

$$p_c = \frac{2t}{R_k} \times \sigma_c$$

where t is hull thickness and R_k is the kernel radius. Using the mean hull thickness and kernel radius provided in the resource, we get that $p_c \approx 10$ bar. We have to guess at how p_c is distributed because we can't find resources that describe how this varies over a group of kernels. Assuming that we have ideal gas conditions in the kernel, the Clausius-Clapeyron relation (see below) gives an expression for critical temperature based on the critical pressure.

$$T_c = \frac{T_0}{1 - (\frac{RT_0}{ML_v})\ln(\frac{p_c}{p_o})}$$

In this case, $T_0 = 373.15K$ and $p_0 = 1$ bar relate to the standard boiling conditions of pure water. $R = 8.3\ \frac{J}{\text{mol}K}$ is the ideal gas constant, $L_v = 2257\frac{J}{\text{g}}$ is the heat of vaporization of water and $M = 18\ \frac{g}{\text{mol}}$ is the molar mass of water. Plugging these numbers into this relation, using the mean critical pressure 10 bar, we get a critical temperature of $179.76°C$ which matches the critical temperature of our references [2,3].

We make an educated approximation for the variance by noting that most kernels have a similar pericarp strength/size and that most kernel's critical temperature are within $\approx 10°C$ of the $180°C$. A pericarp pressure of $p_c = 9$ maps to a critical temperature of about $175.38°C$. A critical pressure of $p_c = 11$ maps to a critical pressure of $183.79°C$. Empirically, from the datasets we've seen in our research, this range should contain almost all of the critical temperatures. For this reason, we choose $\sigma = .5$, so that 97% of the critical pressures fall between 9 and 11. Note that while it's justifiable to assume that pericarp strength would be normally distributed because its a physical feature, we weren't sure that critical temperatures would be similarly distributed, especially because the relation depends logarithmically on p_c. Drawing from this critical pressure distribution, and mapping to the associated temperature, we see that this does in fact yield a normal distribution with $\mu = 179.5$ and $\sigma = 2$. This is our critical temperature distribution for individual kernels.

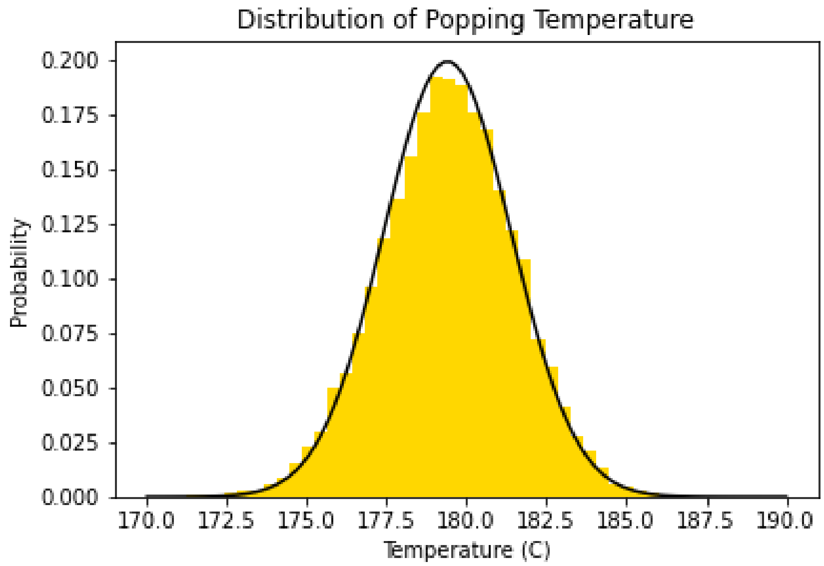


Figure 1: Histogram approximation of our temperature distribution with the corresponding PDF superimposed.

Now that we've drawn from this distribution, meaning we set a temperature for that specific kernel, we want finding a relation between the temperature at which a kernel will pop and the temperature of the stove. Newton's Law of Cooling gives us this relation, which is

$$\frac{dT}{dt} = r(T_s - T)$$

where r is our heat constant and T_s is our external temperature.
Thankfully this ODE is separable, giving

$$\frac{dT}{r(T_s - T)} = rdt$$
$$-\log(T_s - T) = rt + c$$
$$\log(T_s - T) = -(rt + c)$$
$$T_s - T = Ce^{-rt}$$
$$T = T_s - Ce^{-rt}$$

We use the initial condition that $T(0) = 20°C$ (approximately room temperature) giving us

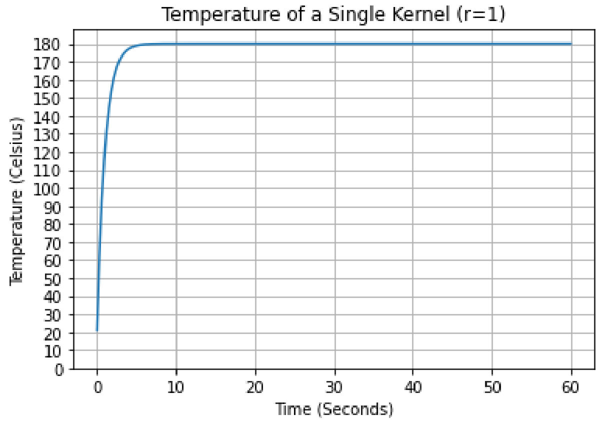
$$20 = T_s - C$$
$$C = T_s - 20$$

Thus our heating equation is

$$T = T_s(1 - e^{-rt}) + 20e^{-rt} \tag{1}$$

2.1 Basic Model of One Kernel

We first applied the equation solved above (1) to find the temperature of a kernel at a given time. Assuming our initial temperature to be room temperature ($T_0 = 20$°C) and the stove to be $T_s = 180$°C, we have the following relationship:



According to this model, the single kernel will reach a temperature close to the stove top in less than 5 seconds. This first attempt is clearly off because it takes much longer experimentally for a kernel to pop when it is placed on the stove. We realized that our problem was assume our conductive heat coefficient was $r = 1$. We investigated different values of r; below are the results for $r = .1$ and $r = .05$.

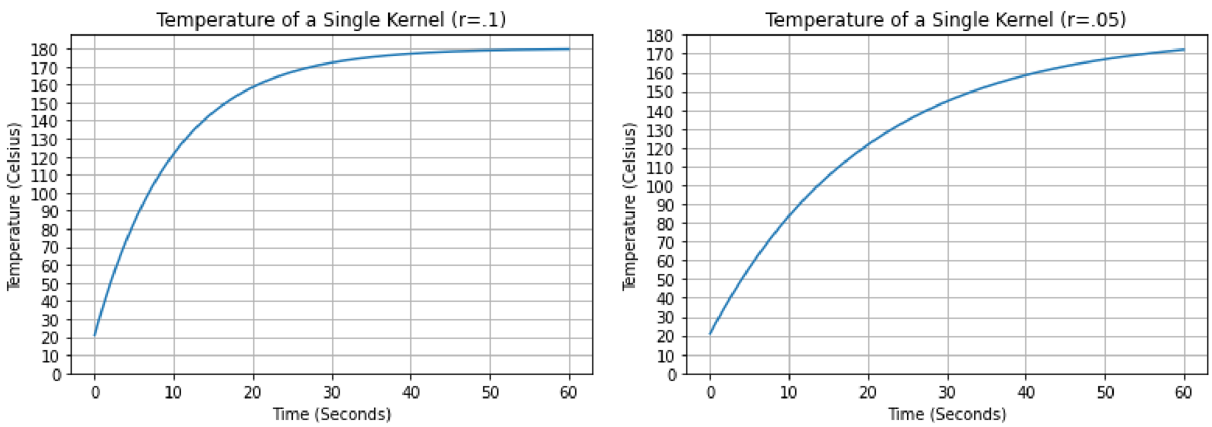


Figure 2: The figure on the left shows the relationship between time and temperature of a single kernel when the conductive heat coefficient is $r = .1$. On the right, the heat coefficient was set to $r = .05$.

This coefficient r would be easily calculated by having temperature values for two different time stamps. However, we don't have any experimental data and finding an explicit heat coefficient online for our specific situation was fruitless. To continue with this generic model, we assumed the general time that a kernel pops close to 1 minute. Based on this assumption, we chose $r = .05$ as our ideal conduction heat coefficient.

2.2 A First Attempt to Model the Kernel Burn Rate

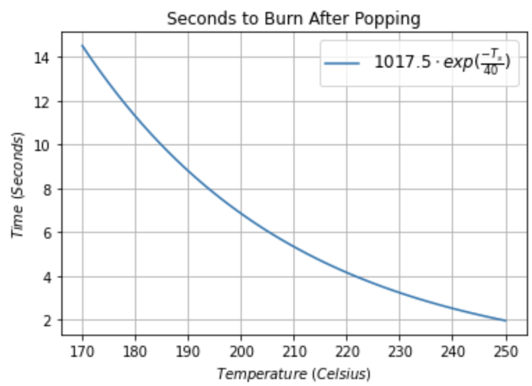
To make our model more realistic, we decided to add the possibility of the popcorn burning after having popped. After extensive online research, we were unable to find proven equations or other research into this problem with regards to popcorn. Thus, we created our own equation that uses temperature in Celsius to find the amount of time it takes for a kernel to burn after popping for a given stove temperature. Based on our own experiences and research into how long it takes for food to cook, we decided the most accurate representation of this relationship was in the form of

$$B(T_s) = ae^{-bT_s}$$

where a and b are scalars and T_s is our stove temperature. We next decided around how long is reasonable for a kernel to burn after popping. At 185° Celsius, we decided this should be around ten seconds and lower down to five seconds between 210° and 210° Celsius. This led us to use $a = 1017.5$ and negative stove temp divided by two times room temp (20° Celsius) as our exponent, hence $b = 1/40$. This gave us our final equation:

$$B(T_s) = 1017.5e^{-\frac{T_s}{40}}.$$

This model in shown in the figure below.



2.3 Finding the Optimal Time with Multiple Kernels

Using our models for popping and burning, we will now find the optimal time to pop a batch of kernels given various stove temperatures. To do this, we draw multiple kernels from our temperature distribution and for each time in our discrete time step, we keep track of the state the kernel is in (popped, unpopped, or burned). To determine the optimal time, we will minimize a simple loss function over our times. The loss function is

$$\mathcal{L}(\tilde{u}, \tilde{p}, \tilde{b}) = -\frac{log(1-\tilde{u})}{4} - \log(\tilde{p}) - log(1-\tilde{b})$$

where $\tilde{u}, \tilde{p}, \tilde{b}$ represent the proportion of unpopped, popped, and burned kernels respectively. The idea behind this loss function is we want to penalize unpopped and burned kernels, with a worse penalty on burned, and reward popped kernels.

- If the proportion of burned kernels is close to 1, then $-\log(1-\tilde{b}) \to \infty$ and if the proportion is close to 0 then $-\log(1-\tilde{b}) \to 0$. This is desirable since we don't want a lot of burned kernels.

- If the proportion of unpopped kernels is close to 1, then $-\log(1-\tilde{u}) \to \infty$ and if the proportion is close to 0 then $-\log(1-\tilde{u}) \to 0$. This is desirable since we don't want a lot of unpopped kernels. Moreover, dividing by a factor of 4 penalizes unpopped less than penalizing burned.

- If the proportion of popped kernels is close to 0, then $-\log(\tilde{p}) \to \infty$ and if the proportion is close to 1 then $-\log(\tilde{p}) \to 0$. This is desirable since we want a lot of popped kernels.

To verify that this loss function has the desired behavior, we did some toy calculations on 6 kernels and got the following results:

Test	Loss
all kernels popped	0
all unpopped	28.782
all burned	46.052
half unpopped, half popped	0.866
half popped, half burned	1.386
one unpopped, rest popped	0.228
two unpopped, rest popped	0.507

In this case, the minimum loss is when all kernels are popped. Having all burnt kernels is a greater loss than having all unpopped and having half burned and half popped is a greater loss than half unpopped and half popped, demonstrating that burned kernels are penalized more than unpopped kernels. Finally, decreasing the number of unpopped kernels from 2 to 1 decreases our loss function, showing that popped kernels are favored over unpopped (and thus burned since unpopped is favored over burned).

Using this loss function, we ran our model, comparing 4 different stove temperature.

The algorithm is as follows:

1. Draw k number of kernels from our temperature distribution.
2. Using the solved ODE (1), find the time that each kernel pops.
3. Using the burn model, find the times that the kernels will burn, given the time that they popped.
4. Using part 2 and 3, for each discrete time step, label each kernel as unpopped, popped, or burned. Now each time step has a list of unpopped, popped, or burned kernels associated with it.
5. calculate the loss at each time step, and choose the time that gives the minimal loss.

We ran this algorithm with 4 different stove temperatures, $185°C, 195°C, 205°C, 215°C$, and plotted the results at the optimal time (see Figure 3). Note that the position of the kernel on the figure is random and has no bearing on the results of the algorithm.

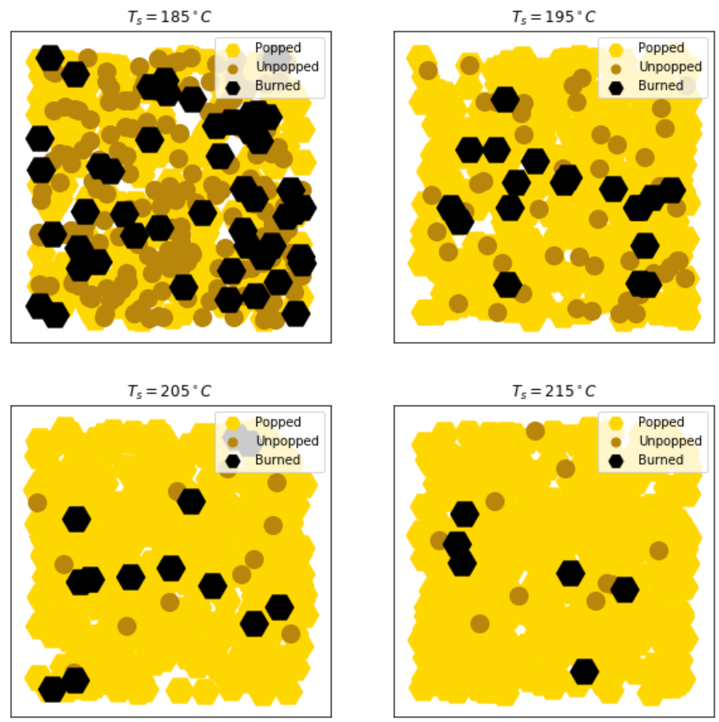


Figure 3: A visualization of the amount of unpopped, popped, and burned kernels at the optimal time for each stove temperature. To create this visualization, each kernel was assigned a random point in the square. Note: the location of the kernel has no bearing on the algorithm.

The optimal time, numbered of unpopped, popped, and burned, and the loss for each stove temperature are summarized in the table below.

Stove Temp ($^\circ C$)	Optimal Time (s)	Unpopped	Popped	Burn	Loss
185	69.924	190	261	49	0.873
195	51.904	49	432	19	0.211
205	42.645	12	475	13	0.084
215	39.218	10	484	6	0.05

2.4 Advanced Model Kernel Burn Rate

While the results from 2.2 seen in figure 2 gave a decent batch of popcorn (at least at 215° Celsius), they are not realistic. A main weak point of the model is the assumption that popped kernels burn at a rate that is independent of all other kernels in the pan. One observation we have about the burning rate of popcorn is the first kernel does not burn quickly after popping, but after the last kernel pops several popped kernels begin to burn within seconds. This points toward the burning rate of popped kernels in the pan being dependent on the popping rate of kernels in the pan. This may be due to the moisture that is released when a kernel pops. To approximate the effect the popping rate has on our burn rate, we added a constant multiplied by the percentage of kernels that remained unpopped. This will approximate the popping rate well since the kernel popping temperatures are drawn from a normal distribution (Figure 1) and generally the more kernels that remain unpopped, the higher the popping rate is (once kernels have begun to pop). Once the popping rate is zero, we are left with the burn rate that is essentially independent of the other kernels in the pan, and an equation similar to 2.2. Thus our advanced kernel burn rate function has the form

$$B(U_p) = ae^{-bT_s} + c(U_p)$$

Where U_p is the percentage of kernels than remain unpopped. After running this model for several values of a, b, c, we lowered a to 740 while leaving b unchanged from section 2.2 to account for us optimizing this part of the equation to one kernel instead of the whole pan. We then ran our modeling function for several values of c, and chose to use $c = 7$ for our final equation since values for c less than 7 resulted in batches with high numbers of burned and unpopped kernels, and increasing c to be greater than 7 resulted in all kernels popping for several temperatures. Thus our final advanced burn rate equation is given by

$$B(U_p) = 740e^{-\frac{T_s}{40}} + 7(U_p)$$

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2.5 Finding the Optimal Time with Multiple Kernels Revisited

With the new burning model the algorithm is modified as follows:

1. Draw k number of kernels from our temperature distribution.
2. Using the solved ODE (1), find the time that each kernel pops.
3. Using the new burn model, find the times that the kernels will burn, given the number of popped kernel at the current time. This differs from the last algorithm since the time it will take to burn changes with each timestep (i.e changes as the ratio of popped kernels changes).
4. Using part 2 and 3, for each discrete time step, label each kernel as unpopped, popped, or burned. Now each time step has a list of unpopped, popped, or burned kernels associated with it.
5. Calculate the loss at each time step, and choose the time that gives the minimal loss.

We ran this new algorithm with same 4 as stove temperatures, and plotted the results at the optimal time (see Figure 4).

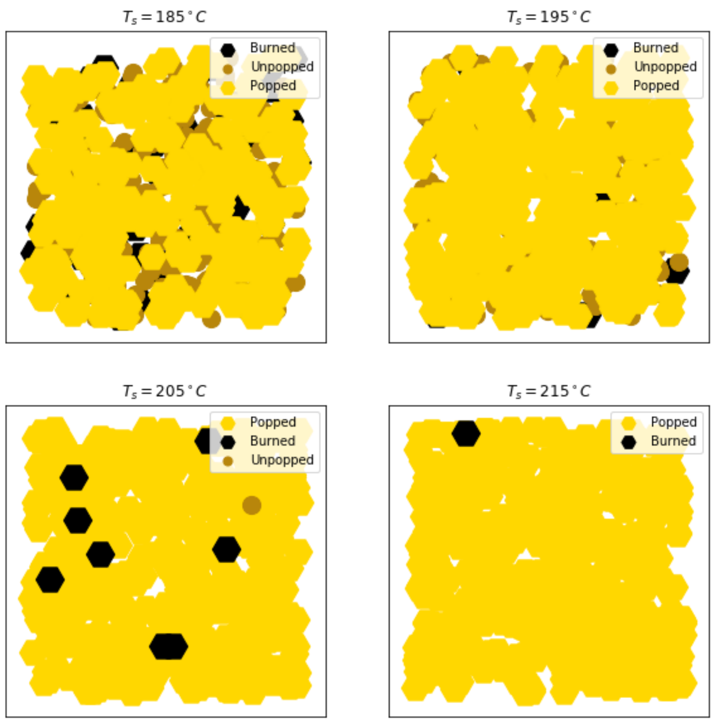

Figure 4: A visualization of the amount of unpopped, popped, and burned kernels at the optimal time for each stove temperature, using the burning model that is dependent on the number of popped kernels at time t. To create this visualization, each kernel was assigned a random point in the square. Note: the location of the kernel has no bearing on the algorithm.

The optimal time, numbered of unpopped, popped, and burned, and the loss for each stove temperature using the new burning model are summarized in the table below.

Stove Temp ($^\circ C$)	Optimal Time (s)	Unpopped	Popped	Burn	Loss
185	69.545	203	247	50	0.941
195	50.306	108	379	13	0.364
205	46.208	1	491	8	0.035
215	41.767	0	499	1	0.004

Interestingly, we can think of this new model as a modified SIR model. Moving from unpopped to popped is not determined by any interactions between the groups, but moving from popped to burned is dependent upon the amount of popped kernels present. We see this behavior in Figure 5 where the number of unpopped, popped, and burned kernels are plotted overtime for a model with a stove temperature of $185^\circ C$ and a burning model that is dependent on the number of popped kernels.

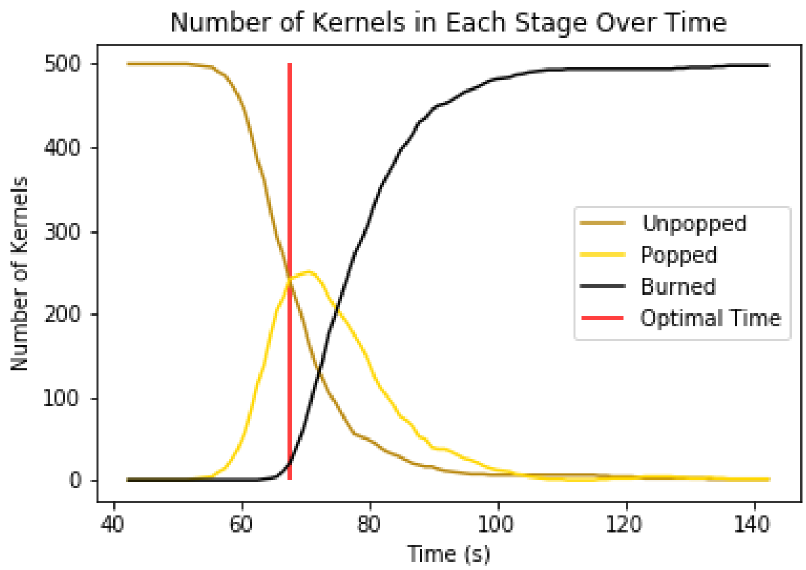

Figure 5: A visualization of the amount of unpopped, popped, and burned kernels over time for a stove temperature of 185°C using the second iteration of the burned function. The optimal time is represented by the vertical red line.

3 Conclusion

As demonstrated above, we implemented our model based on the heating equation derived from Newton's Law of Cooling with two different burn models. The first burn model indicated that with 185°C stove temperature, we got a lot of unpopped and burned popcorn kernels if we stop at the optimal cooking time of 69.924 seconds. Moreover, we ended up getting less unpopped and burned kernels as the temperature of the stove increased. As a result, we decided to construct a new burn model to mimic a more realistic result. This burn model took into account that the popped kernel burn rate is likely dependent on the current kernel popping rate. This second burn function showed that with 185°C stove temperature, we got around the same amount of unpopped and burned popcorn kernels than the first burn function if we stop at the optimal cooking time of 69.545 seconds. However, higher temperatures gave a more realistic model with the second burn model. Moreover, both of our burn functions favor higher stove temperatures, which did not line up with our expectation. This result might be due to the amount of time it takes for the kernels to heat up to their critical popping temperature. When the temperature of the stove is higher, it provides more energy and allows the kernels to all pop in a shorter amount of time before any burning happens.

While we modeled popcorn as close as we could to reality, our model has several weaknesses, largely due to the lack of available data on this process. This lack of data led us to experiment and use our best judgement for several coefficients in our equations. The first one of these was our choice for the heat coefficient, $r = .05$. The rest of these coefficients were our choices of a, b, c in our burning rate equations. While we were as rigorous as possible in choosing these equations, our choices for the coefficients were based on our own observations.

4 Works Cited

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