SYNCHRONIZATION

— CONVERSATIONS ON NONLINEAR SCIENCES —

ABSTRACT. If self sustained, independent oscillators are coupled under a certain range of detuning, either with other oscillators or with an external periodic force, their frequencies may become entrained. We set up the main mechanism of synchronization, then briefly discuss various cases such as entrainment between force and an oscillator, in the presence of noise, between two or more oscillators such as in oscillatory media, and with chaotic oscillators.

1 PHASE DYNAMICS

1.1 Formulation

Let us consider an ordinary autonomous system (a system of ordinary differential equations without the explicit dependence of the independent variable) of M-dimesions, $M \geq 2$,

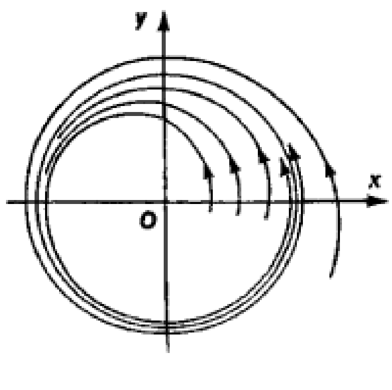
$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

with $\mathbf{x} = x_1, x_2, ..., x_M$, such that

$$\frac{dx_1}{dt} = f_1(x_1, x_2, ..., x_M)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, ..., x_M)$$
$$...$$

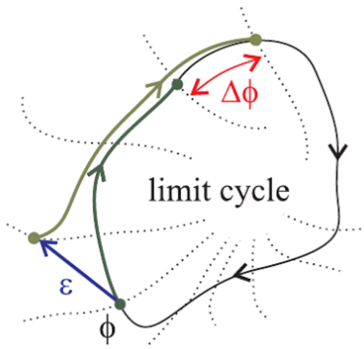
Then, if this system has a stable periodic solution, an isolated closed attractive trajectory forms (as in, all trajectories near this cycle converge toward it), called the *limit cycle*, in the *phase space*, the space of all the variables $\mathbf{x}$ (think of phase as a particular state of the variables of the dynamical system — in a phase space, then, any coordinate on it corresponds to a certain state of the variables, and thus we consider the particular point as a phase).

Intuitively, this can be easily seen as the oscillations are periodic, meaning the events repeat themselves in a cycle temporally. As an example, the diagram below illustrates the limit cycle in a two dimensional system with two variables, $x_1 \equiv x$, $x_2 \equiv y$.



While in a higher dimensional phase space, the limit cycle may take up various complex forms such as knots, *different trajectories cannot intersect with themselves*, even within chaotic systems (systems wherein even a slight deviation from an initial point leads to a completely diverging path from the trajectory that arose from the initial point, i.e. sensitive to small perturbations of initial conditions). This is due to the system being *deterministic*: only one trajectory can evolve from a given point in the phase space (after all, if two paths branch from a single initial point, what's to tell where the point should land up?) In fact, the following properties properties of chaotic systems, that they are sensitive to initial conditions, and that chaotic events do not repeat themselves (and, as such, do not exhibit periodic behavior), are exactly why a chaotic system must at least have 3 dimensions.

Another point we clarify from the start is that *systems completely driven by an external force cannot synchronize.* Upon a perturbation of a point on the limit cycle, the amplitude's (radial) perturbation decays such that the point returns back to the limit cycle — this corresponds to a negative *Lyapunov exponent* (the perturbation's temporal eigenvalue). However, the perturbation of the phase in a self-sustained oscillatory system does not diminish (as in, the point does not "catch up angularly" back to where the point would have been unperturbed after the perturbed point returns back to the limit cycle) — any stalling of the phase due to the perturbed trajectory from the amplitude perturbation remains such. Below, we provide an example of a perturbed point on the limit cycle by a force with amplitude ϵ — after being perturbed, the point returns back to the limit cycle, such that $\Delta A \to 0$, but $\Delta\phi$ does not diminish.



As the phase does not have a preferred value it wants to take on (such as the phase wanting to catch up or fall back to the unperturbed phase), we may consider the phase to be free. However, the phase of the forced oscillator is unambiguously related to the phase of the external force. Self-sustained oscillators are time-independent, in that all moments of time are equivalent for the oscillator; driven oscillators, on the other hand, do not have this property — moments of time differ since the periodic force varies with time, meaning there exists a preference phase. Thus, because the phase is not free, one cannot speak of synchronization through frequency entrainment. Of course, it would make sense for us to consider any other force other than periodic, since our goal is to have the oscillator mimic the periodicity of the external force.

1.2 Equation

In the vicinity of the limit cycle,

$$\frac{d\phi(\mathbf{x})}{dt} = \omega_0$$

Then, let us now consider the phase perturbed by an external force, a coupling force. As the phase is a smooth function of $\mathbf{x}$, we can rewrite the left as

$$\frac{d\phi(\mathbf{x})}{dt} = \sum_k \frac{\partial\phi}{\partial x_k}\frac{dx_k}{dt}$$

such that clearly, using our initial definition $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$,

$$\sum_k \frac{\partial\phi}{\partial x_k} f_k(\mathbf{x}) = \omega_0$$

With a perturbation with amplitude ϵ, then

$$\frac{d\phi(\mathbf{x})}{dt} = \sum_k \frac{\partial\phi}{\partial x_k}(f_k(\mathbf{x}) + \epsilon p_k(\mathbf{x}, t)) = \omega_0 + \epsilon \sum_k \frac{\partial\phi}{\partial x_k} p_k(\mathbf{x}, t)$$

Because we consider a small perturbation from the attractive limit cycle, we may approximate the second term on the right hand side as $\epsilon \sum_k \frac{\partial \phi(\mathbf{x}_0)}{\partial x_k} p_k(\mathbf{x}_0, t)$, and rewrite our phase equation as

$$\frac{d\phi(\mathbf{x})}{dt} = \omega_0 + \epsilon Q(\phi, t)$$

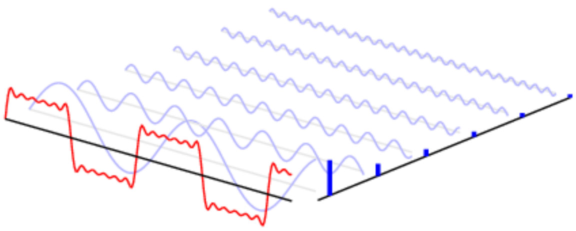
with our coupling $Q(\phi, t) = \sum_k \frac{\partial \phi(\mathbf{x}_0)}{\partial x_k} p_k(\mathbf{x}_0, t)$. Recall that this external force/perturbation is exactly what we are trying to entrain our oscillators to, and as such, the perturbation should be periodic.

2 LOCKING

Within a certain detuning range (frequency difference or mismatch) between the oscillator and the entrainer, even an infinitessimal coupling may bring two oscillations into synchronicity.

2.1 Phase dynamics

As Q is 2π periodic in ϕ and τ in time, we may express it as a double *Fourier series*, the decomposition of a temporal signal into the summation of frequencies that make it up. Let's slightly digress to summarize the purpose of a Fourier transform for those unfamiliar with it.



One can see the decomposition of the square-like wave into the summation of simple sinusodials with varying frequencies in the diagram above. How "much" one needs of a particular frequency is given by the transform's Fourier coefficients which can be interpreted as the amplitude of the individual sinusoid. We establish the sinusodials in the form $sin(n\cdot)$ (although many write it in exponential form as $e^{in\cdot}$), so that with increasing n (what I will call the counter), we are able to change the frequency of the sinusodial.

Thus, we may be able to express a transform of a function, $f(x)$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx$$

The above coefficients simply describe the mathematical definition of the weighted average of the original function— they provide the best approximation in gauging how many of each frequency one needs given the current counter count. They only range from $-\pi$ to π since a sine and cosine are completely defined within this range — they only repeat periodically from then on.

In our case, since both the phase and time are periodic, we may establish two counters l, k, in transforming Q into two Fourier series, yielding

$$Q(\phi, t) = \sum_{l,k} a_{l,k} e^{ik\phi + il\omega t}$$

with $\omega = 2\pi/\tau$ being the frequency of the entrainer (for now, we consider an external periodic force).

As the amplitude of the force ϵ is small, we may approximate the solution of ϕ by completely neglecting the external force (ϵ=0), such that

$$\phi = \omega_0 t + \phi_0$$

Plugging this back into Q, we get

$$Q(\phi, t) = \sum_{l,k} a_{l,k} e^{ik\phi_0} e^{i(k\omega_0 + l\omega)t}$$

We are interested in the essential dynamics of the system, so we look to average the force and only keep the harmonics that contribute to wide fluctuations in the phase. These are clearly then the terms that have the largest Fourier coefficients. We may invoke the Reimann-Lebesgue lemma, that states that if f is integrable over an interval, its Fourier coefficients tend to 0 as $n \to 0$. Since we saw before that an increase in n corresponds to an increase in frequency, the terms with the lowest frequencies (i.e. changing the slowest) will have the widest fluctuations in the phase.

Clearly, the terms satisfying this *resonance condition*

$$k\omega_0 + l\omega \approx 0$$

yield the slowly varying phase terms. For the simplest case where the detuning is small, such that $\omega \approx \omega_0$, then terms with $k = -l$ are resonant. Then, our new averaged forcing will yield

$$\sum_{l=-k} a_{l,k} e^{ik\phi + il\omega t} = \sum_{k} a_{-k,k} e^{ik(\phi - \omega t)} = q(\phi - \omega t)$$

Using our new averaged forcing in for Q, we get

$$\frac{d\phi}{dt} = \omega_0 + \epsilon q(\phi - \omega t)$$

Furthermore, introducing the difference between the phase of the oscillator and that of the external force as as

$$\psi = \phi - \omega t$$

and the detuning as

$$\nu = \omega - \omega_0$$

we arrive at:

$$\frac{d\psi}{dt} = -\nu + \epsilon q(\psi)$$

The simplest 2π periodic function is sine, leading to what is often called the Adler equation:

$$\frac{d\psi}{dt} = -\nu + \epsilon sin(\psi)$$

One furthermore finds that in the general case

$$n\omega \approx m\omega_0$$

where m and n are integers without a common divisor, one can plug this new relation into Q and find that the coupling now takes the form

$$\hat{q}(m\phi - n\omega t)$$

such that naturally, introducing the phase difference as $\psi = m\phi - n\omega t$ with detuning $n\omega - m\omega_0$, we arrive at

$$\frac{d\psi}{dt} = -\nu + \epsilon m\hat{q}(\psi)$$

Synchronization in these regimes is called *synchronization of order* $n : m$ (n force pulses within m oscillations).

2.2 Entrainment — phase lock

For the phases to be locked, we must establish that their differences do not grow or shrink over time through their frequency difference. We do not look for solutions where $\psi = 0$ because the force and oscillation may have started out with different initial phases, in which case their phases will be inevitably different.

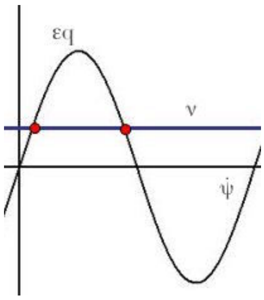
Thus, we look for cases where

$$\dot{\psi} = 0 = -\nu + \epsilon q(\psi)$$
$$\nu = \epsilon q(\psi)$$

As q is phase-periodic in 2π, we need only look for solutions within the domain $[0, 2\pi)$. Clearly, if

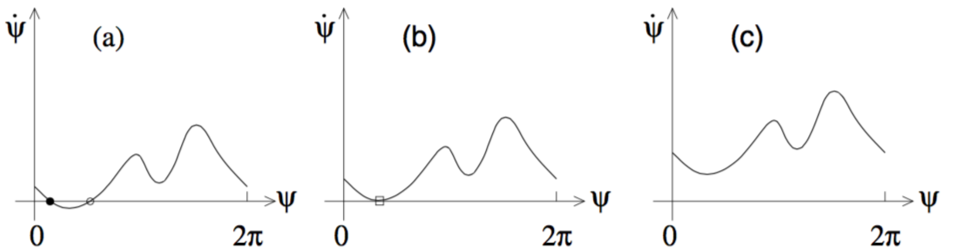
$$\epsilon q_{min} < \nu < \epsilon q_{max}$$

then there lies at least a pair of stable and unstable values (which one can determine through the sign of $\ddot{\psi}$ at the solutions) of ψ that satisfy the above relation.



Only the simplest sinusoid is shown above — of course, in the case of more extrema, there can be several pairs of points.

Below are plots of $\dot{\psi}$ against ψ for three different scenarios:



(a) shows the instance inside the synchronization region, (b) shows the state at the border, and (c) shows the unsynchronized state. Over time, the phase converge toward the stable point, and the oscillator is locked in phase with the external force. As the system approaches the border of the synchronization region, the stable and unstable phases collide to form a semistable phase.

Let us take a look when the detuning lies outside the synchronization region. One can formally express the solution to $\frac{d\psi}{dt} = -\nu + \epsilon q(\psi)$ as

$$\int^{\psi} \frac{d\psi}{\epsilon q(\psi) - \nu} = t$$

such that the period of this phase difference, now a function of time $\psi(t)$ (as the frequency difference prevents the t term from disappearing),

$$\tau_\psi = \frac{2\pi}{\Omega_\psi} = \left| \int_0^{2\pi} \frac{d\psi}{\epsilon q(\psi) - \nu} \right|$$

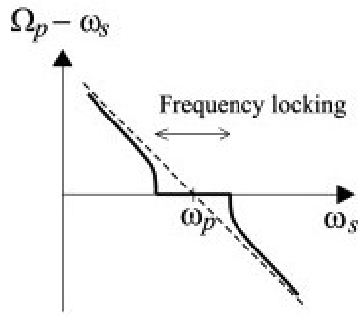
where Ω_ψ is what we call the *beat frequency*, such that the mean (time average) velocity of the phase, called the *observed frequency*, Ω, follows

$$< \dot{\phi} > \equiv \Omega = \omega + \Omega_\psi$$

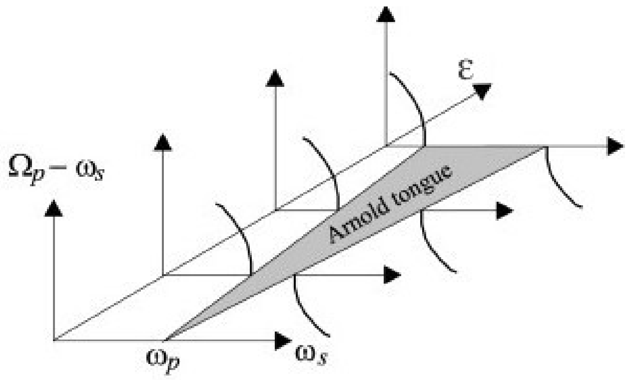
One can reduce the coupling function $q(\psi)$ through a Taylor reduction of the denominator of the integral (valid near the border of the synchronization region), and find that

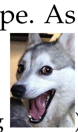
$$|\Omega_\psi| \sim \sqrt{\nu - \nu_{max}}$$

such that the synchronization region can be portrayed as



and, as we vary the parameter ϵ,



One can infer the qualitative reason on why the Arnold tongue takes its shape. As the amplitude of the force decreases, the force can not sway the phase of the oscillator as forcibly (omg ), and as such, the detuning range for synchronization decreases.

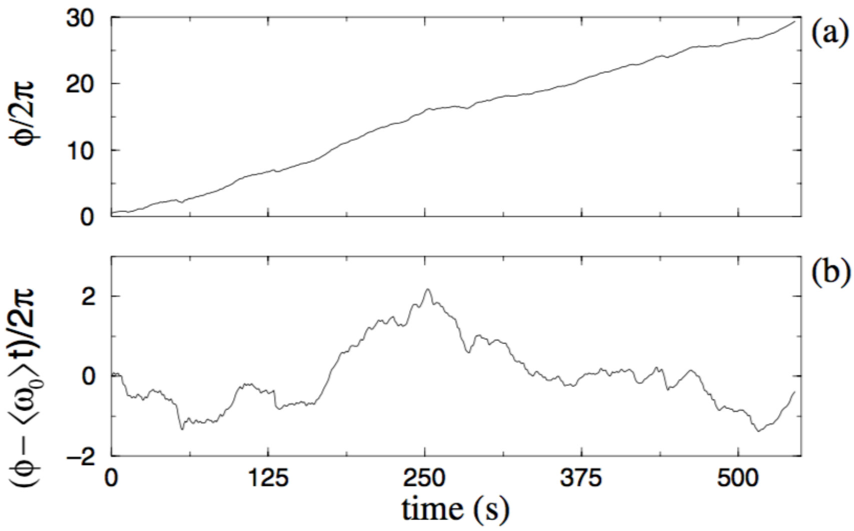
Toward the border of the synchronization region, the phase difference evolves irregularly — it temporarily synchronizes with the external force in short epochs (so that the difference remains constant), then slips in phase by 2π to reach the next most stable equilibria — these jumps are called *phase slips*. As we move further and further away from the synchronization region, the phase difference shapes itself to evolve regularly linearly due to the constant frequency mismatch.

2.3 Noise

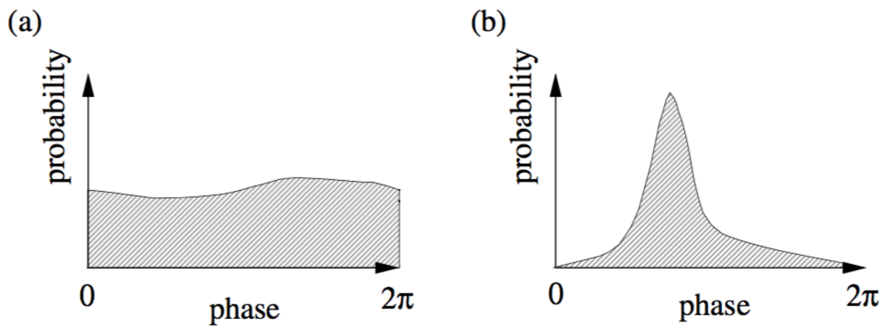
The presence of external noise is expressed in the phase dynamics through what is known as Langevin dynamics:

$$\frac{d\psi}{dt} = -\nu + \epsilon q(\psi) + \xi(x, t)$$

Recall that the phase of a self-sustained oscillator is free — thus, any perturbation of the phase remains there permanently. Meaning, even small perturbations coming from weak noise wil build up, which may lead to large deviations from the linear growth of the phase ($\phi = \omega_0 + \phi_0$). This is sometimes regarded as *phase diffusion*, with the phase point's motion regarded as a *random walk*. Two diagrams, without and with the presence of noise, are shown below in (a) and (b), respectively.

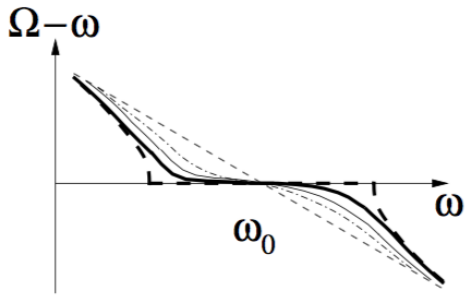


Analysis is often done by utilizing the Fokker-Planck equation, a probability continuity equation. The purpose of doing so is to watch how the phase probability evolves over time. Due to phase diffusion, the phase has almost equal probability to attain any value. However, the phase does have a tendency to remain in stable equilibrium, so the more the force outweighs the noise, the more the phase probability will evolve into its noise-free state, probablity peaking at the stable equilibrium. The evolution of the phase dynamics over time can be characterized through the perturbation's Lyapunov exponent.



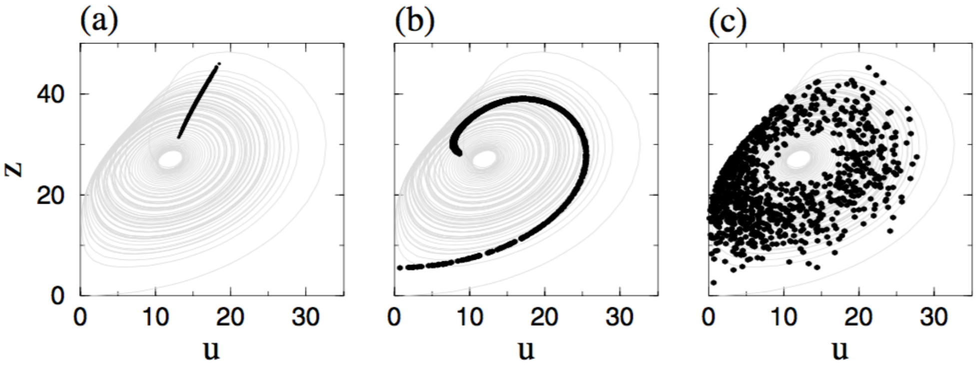
We can also infer that if the intensity of the noise is unbounded or very strong, the noise may even facilitate synchronization. If the noise is weak, the phase will simply fluctuate around its current state that it resides in, but if the noise is strong enough, it may push the phase over onto the next stable equilibria, causing phase slips. Since, however, the phase has a tendency to grow over time due to its t term, the noise required to hurtle the phase into a phase slip of $+2\pi$ is often lower than -2π. As such, the phase slips appear irregularly.

Since noise is, in essence, a form of perturbation, we expect the synchronization region to be distorted slightly, most prominently towards the edge of the region where the forcing barely reconciles the detuning, and least in the center, where the force can easily entrain the oscillator.

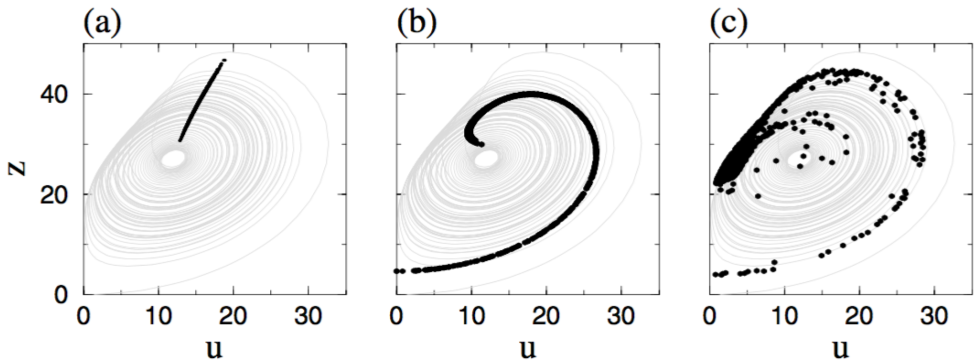


2.4 Chaos

In a chaotic system, even infinintessimally close trajectories may end up exponentially diverging. This means that, while the phase, in average, does circulate around a chaotic system's version of a limit cycle called a *strange attractor*, each orbit has a different return time. Thus, the phase does not rotate uniformly as it experiences irregular accelerations, and it inevitably diffuses across the strange attractor — like the case of a noise-driven periodic oscillator.

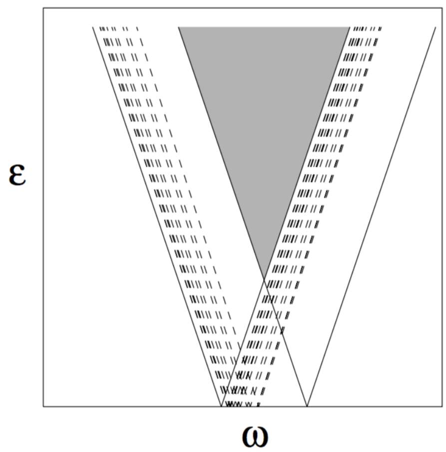


Thus, if an external force where to synchronize with the oscillator, it must be strong enough to overcome phase diffusion.



Notice, however, that while most of the phases become entrained with the force, some are inevitably left behind. This is due to the high instability of chaotic systems, and the resulting phase diffusions as well, so even "full" synchronization will cause some to not be entrained.

This can be seen as well through the Arnold tongue of these diffusive oscillators. Since each orbit has a different frequency, the Arnold tongues developed from those individual frequencies are offset from each other, and "full" synchronization takes place in their mean fields, the intersection region of all the tongues.



Thus, the force amplitude has a minimum threshold for it to be able to synchronize the chaotic or noisy oscillator with itself. Of course, as the individual frequencies' differences get larger, so does the threshold of the amplitude. In regions without complete overlap, partial synchronization takes place: the phase follows the force for a short epoch, then instability take over and causes it to diverge onto a different path.

3 A LOOK: Media synchronization

We have talked about how a self-sustained oscillator may become entrained to an external force with one way coupling: from the force to the oscillator, but not the other way around. However, oscillators can be entrained with other oscillators that share local or even global coupling. Building from our previous models, we can observe some interesting particularities that arise from these coupling variations.

In this section, we will focus on an oscillatory lattice where the oscillators are locally coupled with each other. We use this example to interactively highlight the main interactions in mutual couplings, as the mathematical analysis is similar to that in the previous section. Furthermore, we will observe how a synchronized oscillatory media collapses into chaos through turbulence.

3.1 Lattice

With weak coupling, we may model the phase dynamics with an extension of the approximation we had used before:

$$\frac{d\phi_k}{dt} = \omega_k + \epsilon q(\phi_{k-1} - \phi_k) + \epsilon q(\phi_{k+1} - \phi_k)$$

with $k = [1, ..., N]$.

Clearly, with vanishing coupling ($\epsilon = 0$), the oscillators all oscillate independently with individual frequencies. On the opposite limit, for a very stronge coupling, the difference in the natural frequencies of each oscillator becomes negligibile, and the oscillators synchronize together. Then, between these two regimes, we expect partial synchronization, where clusters of individually synchronized states form about the lattice.

3.2 Medium

In many cases, oscillatory lattices cannot be considered discretely, and are considered as oscillatory media.

In many cases, the coupling constant is directly related to the spacing between the coupled sites. We exploit this in our treatment of the continuous limit, by rescaling $\epsilon = \frac{\tilde{\epsilon}}{(\Delta x)^2}$. We also Taylor expand q and rearrange the terms to conclude to

$$\frac{d\phi_k}{dt} = \omega_k + \tilde{\epsilon} q'(0) \frac{\phi_{k-1} - 2\phi_k + \phi_{k+1}}{(\Delta x)^2} + \tilde{\epsilon} q''(0) \frac{(\phi_{k+1} - \phi_k)^2 + (\phi_k - \phi_{k-1})^2}{2(\Delta x)^2} + ...$$

Furthermore by imposing the limit $\phi_{k+1} - \phi_k = O(\Delta x)$, $\Delta x \to 0$, we see that the second term on the right hand side converge towards the second derivative of the phase, and that the third term converge towards the square of the first derivative. Thus, we conclude to

$$\frac{\partial \phi(x,t)}{\partial t} = \omega(x) + \alpha \nabla^2 \phi(x,t) + \beta (\nabla \phi(x,t))^2$$

We note some logical inferences of the phase dynamics that assure us our approximation is sound. We keep in mind that the symmetry $x \to -x$ requires that the total power of the derivative (whether it be higher order or higher nonlinearities) should be even (e.g. $\nabla^4 \phi$, $(\nabla \phi)^2 \nabla^2 \phi$, etc). Furthermore, the phase should not directly appear in the above equation, for the phase dynamics should remain impartial to any phase shifts.

3.3 The Complex Ginzburg-Landau equation (CGLE)

Let us extend our averaged amplitude equation for a weakly perturbed oscillator to include local couplings. We will see confirm that this amplitude equation can, through further reduction, provide us the phase dynamics of a medium.

$$\frac{dA_k}{dt} = \mu A_k - (\gamma + i\alpha)|A_k|^2 A_k + (\beta + i\delta)(A_{k+1} - 2A_k + A_{k-1})$$

A discussion on the derivation of such a form is provided at the end of the paper.

Let us once again exploit the spacing between interaction constants in setting $\beta = \frac{\tilde{\beta}}{(\Delta x)^2}$ and $\delta = \frac{\tilde{\delta}}{(\Delta x)^2}$,

$$\frac{dA}{dt} = \mu A_k - (\gamma + i\alpha)|A|^2 A + (\tilde{\beta} + i\tilde{\delta})(\nabla A)^2$$

and then normalize (rescale) the amplitude by $\sqrt{\gamma/\mu}$ to conclude to what is known as the *complex Ginzburg-Landau equation*:

$$\frac{\partial a(x,t)}{\partial t} = a - (1 + ic_3)|a|^2 a + (1 + ic_1)\nabla^2 a$$

wherein, on the right hand side, we may see the following physical manifestations: the first term describes the linear growth of the oscillations, the real part of the second term the nonlinear saturation of frequency, the imaginary part of the second term the nonlinear shift in frequency, the real part of the third term the spactial interaction (diffusion) of dissipative types, and the imaginary part of the third the diffusion of reactive types. We would like to remind the necessity of the second term on the right side being of cubic form, as spatial translational symmetry necessitates that a be invariant under an arbitrary phase translation $e^{i\phi}$.

The solution to the CGLE has stable plane wave solutions

$$a(x,t) = (1-\kappa^2)^{1/2} exp[i\kappa x - i(c_3 + (c_1 - c_3)\kappa^2)t]$$

Let us perform linear stability analysis in order to determine the criteria for stability of these plane wave solutions.

Letting $a_0 \equiv (1-\kappa^2)^{1/2}$, let us add a perturbative term $u(x,t)$ to a_0 and analyse its extended qualitative behaviour. We then have

$$\tilde{a}(x,t) = (a_0 + u(x,t))exp[i(\kappa x - \omega t)]$$

where $\omega \equiv c_3 + (c_1 - c_3)\kappa^2$.

Plugging this back into our CGLE, linearizing with respect to $u(x,t)$ (as we consider small perturbations) and recognizing that, as we are dealing with the long wavelength limit, $\frac{\partial a}{\partial \zeta} \approx 0$ for all temporal and spatial variables ζ, we get

$$\frac{\partial u}{\partial t} = [-(1+ic_3)(1-\kappa^2) - 2(c_1 - i)\kappa\frac{\partial}{\partial x} + (1+ic_1)\frac{\partial^2}{\partial x^2}]u - (1+ic_3)(1-\kappa^2)\bar{u}$$

where $\bar{u}$ signifies u's complex conjugate.

One can easily see that, after considering u in its Fourier amplitudes such that $u = \int_{-\infty}^{\infty} u_q(t)e^{iqx}dq$ we may reformulate the above into a 2×2 operator, which we denote $\hat{L}$, such that

$$\frac{\partial}{\partial t}\begin{bmatrix} u_q \\ \bar{u_q} \end{bmatrix} = \hat{L}\begin{bmatrix} u_q \\ \bar{u_q} \end{bmatrix}$$

with eigenvalues λ, which contains the Lyapunov exponent of the perturbation.

For nontrivial eigenvalues, finding $|\hat{L} - \lambda I| = 0$, and expanding the solution in the long wavelength limit looking for purely imaginary eigenvalues (i.e. $\Re[\lambda] = 0$) yield the following stability criterion:

$$1 + c_1 c_3 - \frac{2\kappa^2(1+c_3^2)}{a_0^2} > 0$$

$$\kappa^2 < \frac{1 + c_1 c_3}{2c_3^2 + c_1 c_3 + 3}$$

Thus, for $\kappa = 0$ *(band centre state)*, the phase diffusion coefficient concludes to the Benjamin-Feir-Newell criterion wherein the plane solutions are stable if

$$1 + c_1 c_3 > 0$$

Furthermore, if $c_1 = c_3$, we see that as we approach the above threshold, we conclude to the Eckhaus instability condition

$$\kappa^2 < \frac{1}{3}$$

Toward the stability threshold, the amplitude remains near 1, but the phase evolves irregularly. This state is called *phase turbulence*.

We realize that the stability criterion that we have determined above may also be realized by rewriting the CGLE into the Kuramoto-Sivashinksy approximation in the long wavelength limit. In the approximation of the unnormalized CGLE, the Landau Stuart equation, the model reads

$$\frac{dA}{dt} = (1 + i\eta)A - (1 + i\alpha)|A|^2 A$$

which by decomposing in polar coordinates, $A = Re^{i\theta}$, to rewrite the above as

$$\frac{dR}{dt} = R(1 - R^2)$$
$$\frac{d\theta}{dt} = \eta - \alpha R^2$$

and solving the above to yield

$$R(t) = [1 + \frac{1 - R^2}{R_0^2} e^{-2t}]^{-1/2}$$
$$\theta(t) = \theta_0 + (\eta - \alpha)t - \frac{\alpha}{2} ln(R_0^2 + (1 - R_0^2)e^{-2t})$$

we readily see that on the limit cycle, θ rotates with constant velocity $\omega_0 = \eta - \alpha$ and with a slight deviation of initial amplitude from unity, the additional phase shift is then approximately equal to $-\alpha ln R_0$, such that clearly,

$$\phi = \theta - \alpha ln R_0$$

Then, rewriting the phase dependence in the form

$$\phi(X, Y) = tan^{-1} \frac{Y}{X} - \frac{c_3}{2} ln(X^2 + Y^2)$$

,where $a = X + iY = cos\phi + isin\phi$, and perturbation in the form

$$p_X = \nabla^2 X(\phi) - c_1 \nabla^2 Y(\phi)$$
$$p_Y = \nabla^2 Y(\phi) + c_1 \nabla^2 X(\phi)$$

,we conclude to the *Kuramoto-Sivashinksy equation*

$$\frac{\partial \phi}{\partial t} = -c_3 + (1 + c_1 c_3)\nabla^2 \phi + (c_3 - c_1)(\nabla \phi)^2 - \frac{1}{2} c_1^2 (1 + c_3^2) \nabla^4 \phi$$

Clearly we can observe the Benjamin-Newell-Feir criterion in the Laplacian term. The stabilitizng fourth spacial derivative term is also included near the *Hopf bifurcation* (where a system's stability switches and a periodic (oscillatory) solution emerges).

4 DISCUSSION: Averaged amplitude for a weakly nonlinear oscillator

Here we will see how we arrive at the averaged amplitude equation for a weakly nonlinear oscillator.

4.1 The amplitude equation

Consider the following oscillator with natural frequency ω_0:

$$\ddot{x} + \omega_0^2 x = f(x, \dot{x}) + \epsilon p(t)$$

We consider the above oscillator, recognizable through the left side of the equation, to have two parts: a nonlinear function $f(x, \dot{x})$ determining the properties of our self sustained autonomous oscillator, and the force following a periodic function satisfying $p(t) = p(t + \tau)$ having a frequency $\omega = 2\pi/\tau$.

With the above differential equation, we expect a solution that will be entrained to the external force's frequency ω:

$$x(t) = \frac{1}{2}(A(t)e^{i\omega t} + c.c.)$$

To continute with our expectation, let us rewrite the above differential equation into an entrained one with frequency ω. Such a treatment is valid, of course, for small detuning.

$$\ddot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x + f(x, \dot{x}) + \epsilon p(t)$$

Then, rewriting the above system as $\dot{x} = y$ with y then clearly following

$$y(t) = \frac{1}{2}(i\omega A(t)e^{i\omega t} + c.c.)$$

Then, by resolving x and y,we find that we conclude to the equation for the complex amplitude:

$$\dot{A} = \frac{e^{-i\omega t}}{i\omega}[(\omega^2 - \omega_0^2)x + f(x, y) + \epsilon p(t)]$$

4.2 Averaging

While we have reexpressed our solutions in terms of a complex amplitude, it is not any easier to solve. Thus, we will average the equation for a solvable approximation. Our point in averaging is to neglect all the fast oscillating parts ($e^{\pm in\omega t}$ for $n = [1, 2, ...]$), preserving the core dynamics.

The complex equation above can clearly be decomposed into three parts. Since the analysis of the first term on the right side is trivial, let us analyze the other two term below.

◇ $\frac{e^{-i\omega t}}{i\omega}\epsilon p(t)$

In averaging the above term, we cannot simply drop the exponential part, since the perturbation is a function of time; we must average both the terms combined. In averaging, we only utilize the first Fourier harmonic of the perturbation. In general, this averaged harmonic $\frac{\epsilon}{i} < e^{-i\omega t}p(t) >$ is nonzero, and will provide a complex constant $-i\epsilon E$. Of course, it will not be a function of time since the point of averaging is to remove all temporal dependencies.

◇ $\frac{e^{-i\omega t}}{i\omega}f(x, y)$

Since f is a polynomial in x, y (e.g. $x^2, y^2, ...$), it will be a polynomial in $Ae^{i\omega t}$ and $A^*e^{i\omega t}$ as well, since $A = Re^{i\theta}$ with $R = \sqrt{x^2 + y^2}$. As such, consider the general case

$$\frac{e^{-i\omega t}}{i\omega}(Ae^{i\omega t})^n(A^*e^-i\omega t)^m$$

$$\frac{1}{i\omega}A^n e^{in\omega t}A^{*m}e^{-i(m+1)\omega t}$$

$$\frac{1}{i\omega} A^n A^{*m} e^{i[n-(m+1)]\omega t}$$

Thus, all the independent oscillatory terms will individually average out to 0, except when

$$n - (m+1) = 0$$
$$m = n - 1$$

for this term will not have an oscillatory dependency, and thus the core dynamics of this term is preserved. As such, the averaged term will exhibit a form

$$A^{n+*m} = R^n e^{in\theta} R^m e^{-im\theta} = R^{n+m} e^{i\theta} = R^{n+m-1} R e^{i\theta} = A \cdot g(|A|^k)$$

In considering only the first order, as our amplitude is small, we only need to care about the linear ($\propto A$) and the first nonlinear term ($\propto |A|^2 A$).

Thus, our final averaged amplitude equation takes the form

$$\dot{A} = -i \frac{\omega^2 - \omega_0^2}{2\omega} A + \mu A - (\gamma + i\kappa)|A|^2 A - i\epsilon E$$

with the parameters taking the following physical manifestations: μ and γ describe the linear and nonlinear growth/decay of the oscillation, respectively. For amplitudes to be stable, we need growth for amplitudes smaller than our stable, and decay for those larger. These clearly correspond to $\mu, \gamma > 0$. κ describes the nonlinear dependence of oscillation frequency on the amplitude (as $\dot{A} \propto i(\kappa|A|^2)A$). To clarify, the imaginary terms correspond to oscillatory terms in a differential function as it takes the form $e^{i\cdot}$ (and thus, gives a dependency on the frequency of the oscillation), while nonimaginary terms simply yield exponential growth or decay.