Problem 1: Take the symmetric matrix,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

find a way to diagonalize this matrix orthogonally, i.e. factor,

$$A = \mathcal{O} D \mathcal{O}^t$$

where D is diagonal and $\mathcal{O}$ is orthogonal.
Solve this problem completely "*by-hand*".

Problem 2: Locate the <u>maximum point</u> (x, y) for the following linear form,

$$f(x, y) = 0.5x + 1.5y$$

subject to the constraints that,

(i) $x + y \leq 6$
(ii) $3x + y \leq 15$
(iii) $x + 3y \leq 15$
(iv) $x \geq 0$
(v) $y \geq -1$

Solve this problem by *drawing the convex polygon region* and graphically determining where the maximum occurs. Be careful, there is more than one answer.

Problem 3: There is an important linear in Linear Algebra, the "*Cayley-Hamilton Theorem*", which says that if $A \in \mathbb{R}^{n\times n}$ and if $p(\lambda)$ is the characteristic polynomial of A, i.e.

$$p(\lambda) \overset{\text{(def)}}{=} \det(\lambda I_n - A) = \underbrace{c_n\lambda^n + c_{n-1}\lambda^{n-1} + ... + c_2\lambda^2 + c_1\lambda^1 + c_0\lambda^0}_{\text{this is the determinant expanded out as a polynomial}}$$

then if you "substitute" A in place of λ you will get,

$$c_nA^n + c_{n-1}A^{n-1} + ... + c_2A^2 + c_1A^1 + c_0A^0 = \underbrace{\mathbf{0}}_{\text{zero matrix}}$$

(here $A^0 = I_n$ is the identity matrix).

What makes this theorem really funny is that at first it seems obvious. If you replace λ by A you will get, $p(A) = \det(AI_n - A) = \det(A - A) = \det(\mathbf{0}) = 0$. However, this is "*symbolic non-sense*". In the polynomial λ is a real number, not a matrix. Furthermore, $p(A)$ is supposed to be equal to the zero matrix, not the number 0. The Cayley-Hamilton Theorem basically says the symbolic non-sense is justified.

Find the characteristic polynomial for the matrix,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

and verify by computation that the matrix satisfies the polynomial equation as stated by the Cayley-Hamilton Theorem.

Problem 4: Prove the Cayley-Hamilton Theorem with the assumption that A is a *symmetric matrix* (use the fact that symmetric matrices have an orthogonal decomposition). This theorem is more general and works for all square matrices, but it is much harder to prove!