Problem 1: Consider the following function of three variables,

$$f(x, y, z) = (3x^2 + 2y^2 + z^2)e^{-x^2-y^2-z^2}$$

(i) Determine the critical points of f "*by-hand*", where "*by-hand*" means that we allow the use of Mathematica! Of course, you should see if you can justify the result yourself with what Mathematica shows you. You will have 7 critical points for this function.

(ii) Check which of those critical points are max/min/saddle points by using the Hessian. It is absolutely important here to calculate the Hessian matrix *numerically.* For entertainment purposes you should calculate the Hessian matrix symbolically on Mathematica to see how complicated it gets! Here the Hessian matrix is a diagonal matrix so the eigenvalues will be presented for you on the diagonal.

Problem 2: Consider the following convex function,

$$f(x, y) = \log(x^2 + y^2 - 2x + 2) + \log(x^2 + y^2 - 2y + 2)$$

(i) Use Newton's method to find where the minimum is located. Start at the point $(0.9, 0.1)$ by running around 20 to 30 iterations and see which point you end up at. However, if you use other points, such as $(1, 2)$, you will find this algorithm blows up numerically. It is a good algorithm when it works but it is limited.

(ii) Solve the same problem using `optim`. You should experiment with both initial points and using different algorithm methods for it. Here the `optim` works really well and is stable.

Problem 3: Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^4$ be three-vectors (in four-dimensional space),

$$\mathbf{x}_1 = (1,1,0,0) \quad \mathbf{x}_2 = (1,0,0,1) \quad \mathbf{x}_3 = (1,1,1,1)$$

These vectors span a three-dimensional "*hyperplane*", i.e. simply consider the following set,

$$H = \{a\ \mathbf{x}_1 + b\ \mathbf{x}_2 + c\ \mathbf{x}_3 \mid a, b, c \in \mathbb{R}\}$$

(here a, b, c are just the "*free parameters*").

Let $\mathbf{y} \in \mathbb{R}^4$ denote the following vector,

$$\mathbf{y} = (1,2,3,4)$$

Calculate,

$$\mathrm{Proj}_H(\mathbf{y})$$

i.e. the *projection* from $\mathbf{y}$ onto the hyperplane H.

(i) Solve this problem by first finding an orthonormal basis for H by Gram-Schmidt and then projecting $\mathbf{y}$ onto that basis.

(ii) Solve this problem by using the least-squares minimum formula. Both of your answers need to be same.