Math 104C: Midterm

Kelly Wang 5351010

Question 1

(a) In one-step form, let us substitute in K_1, K_2.

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right]$$

First to show it's consistent, from definition 14.4, a numerical method is consistent (with the ODE of the initial value problem) if the method is at least of order 1.
Define $\Phi = \frac{1}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right]$. Then for our one-step method we have:

$$\begin{aligned}
\tau_{n+1} &= y(t_{n+1}) - [y(t_n) + \Delta t \Phi(t_n, y(t_n), \Delta t)] \\
&= \Delta t y'(t_n) - \Delta t \Phi(t_n, y(t_n), \Delta t) + O(\Delta t)^2 \\
&= \Delta t\left[f(t_n, y(t_n)) - \Phi(t_n, y(t_n), \Delta t)\right] + O(\Delta t)^2
\end{aligned}$$

By assuming the increment function Φ is continuous, a one-step method is consistent with ODE $y' = f(t, y)$ if and only if

$$\Phi(t, y, 0) = f(t, y)$$

Let $\Delta t = 0$, we can see the following:

$$\begin{aligned}
&\frac{1}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right] \\
&= \frac{1}{2}\left[f(t_n, y_n) + f(t_n, y_n)\right] \\
&= \frac{1}{2}\left[2f(t_n, y_n)\right] \\
&= f(t_n, y_n)
\end{aligned}$$

Thus this method is consistent.

Next to show it's stable, we want to show that Φ is Lipschitz in y. Let us first simplify our previous definition on Φ.

$$\begin{aligned}
\Phi &= \frac{1}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right] \\
&= f\left(\frac{t_n}{2}, \frac{y_n}{2}\right) + f\left(\frac{t_n}{2} + \frac{\Delta t}{2}, \frac{y_n}{2} + \frac{\Delta t}{2} f(t_n, y_n)\right) \\
&= f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t_n, y_n)\right) \\
\Phi(t, y, \Delta t) &= f\left(t + \frac{\Delta t}{2}, y + \frac{\Delta t}{2} f(t, y)\right)
\end{aligned}$$

Now using our simplified version of Φ (increment function) we find the following:

$$\begin{aligned}
|\Phi(t, y_1, \Delta t) - \Phi(t, y_2, \Delta t)| &= \left| f\left(t + \frac{\Delta t}{2}, y_1 + \frac{\Delta t}{2} f(t, y_1)\right) - f\left(t + \frac{\Delta t}{2}, y_2 + \frac{\Delta t}{2} f(t, y_2)\right)\right| \\
&\leq L \left| y_1 + \frac{\Delta t}{2} f(t, y_1) - y_2 - \frac{\Delta t}{2} f(t, y_2)\right| \\
&\leq L|y_1 - y_2| + \frac{\Delta t}{2} L |f(t, y_1) - f(t, y_2)| \\
&\leq \left(1 + \frac{\Delta t}{2} L\right) L |y_1 - y_2| \leq \tilde{L} |y_1 - y_2|
\end{aligned}$$

where $\tilde{L} = \left(1 + \frac{\Delta t_0}{2} L\right) L$ and $\Delta t \leq \Delta t_0$, i.e for sufficiently small Δt. This shows Φ is Lipschitz in y.

Lastly, since we were able to show this one-step method is consistent with a Lipschitz in y increment function $\Phi(t, y, \Delta t)$, it is convergent. (Theorem 14.2)

(b) In our Explicit Trapezoidal method we have

$$\begin{aligned}
y_{n+1} &= y_n + \frac{\Delta t}{2}[\lambda y_n + \lambda(y_n + \Delta \lambda y_n)] \\
&= \left[1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2\right] y_n
\end{aligned}$$

This gives us the stability function $R(z) = 1 + z + \frac{z^2}{2}$ and the set of linear stability consists of all the complex numbers such that $|R(z)| \leq 1$. So we have $R(z) = e^z + O(z^3)$. This shows that R approximates $e^{\Delta t \lambda}$ to the third order in Δt. This makes sense because the local truncation error should be $O(\Delta t)^3$ making the method to the second order.

(c) Since 1b gave us the stability function $R(z) = 1 + z + \frac{z^2}{2}$ and the linear stability consists of all complex numbers such that $|R(z)| \leq 1$, we want $|1 + z + \frac{z^2}{2}| \leq 1$ where $Im\{z\} = 0$ and $0 \leq Re\{1 + z + \frac{z^2}{2}\} \leq 1$.

If we look at possible values of z, we can see that when $Im\{z\} = 0$ the previous requirement holds true when $Re\{z\} \in [-2, 0]$. Therefore, our stability interval defined by intersection of the stability region with the real axis is $Re\{z\} \in [-2, 0]$ for $Im\{z\} = 0$.

Question 2

(a) Let $y' = u, y'' = u', u(0) = 0$ then we can rewrite our second order ODE as $u' - 10u + y = \cos t$ for $y(0) = 0$ and $0 \leq t \leq \pi$. This gives us the following first order system of ODEs.

$$\begin{aligned}
y' &= u \\
u' &= 10u - y + \cos t
\end{aligned}$$

(b) Using our first order system of ODEs from 2a we can express $f(t, y)$ as a vector.

$$
\begin{aligned}
f(t, v) &= (u, 10u - y + \cos t), v(t) = (u, y) \\
K_1 &= (u, 10u - y + \cos t) \\
K_2 &= (u + \Delta t, 10u - y + \cos t) + \Delta t K_1 \\
&= (u + \Delta t, 10u - y + \cos t) + \Delta t(u, 10u - y + \cos t)) \\
&= (u + \Delta t, 10u - y + \cos t) + (u\Delta t, 10u\Delta t - y\Delta t + \cos t\Delta t)) \\
&= (u + u\Delta t, 10u + 10u\Delta t - y - y\Delta t + \cos t + \cos t\Delta t) \\
y_{n+1} &= y_n + \Delta t\left[\frac{1}{2}K_1 + \frac{1}{2}K_2\right] \\
&= y_n + \Delta t\left[\frac{1}{2}(u, 10u - y + \cos t) + \frac{1}{2}((u + u\Delta t, 10u + 10u\Delta t - y - y\Delta t + \cos t + \cos t\Delta t))\right]
\end{aligned}
$$

To find the first step, let $n = 0$:

$$
\begin{aligned}
y_1 &= 0 + \Delta t\left[\frac{1}{2}(0, \cos t) + \frac{1}{2}(0, \cos t + \cos t\Delta t)\right] \\
&= \frac{\Delta t}{2}\,(0, 2\cos t + \cos t\Delta t)
\end{aligned}
$$

(c) No, this problem is not numerically stiff because we are using the Explicit Trapezoidal RK method. Even though it would remain bounded, from 2b we can see that the $\cos t$ terms would create an oscillating pattern which violates stiffness properties.

Question 3

(a) Given the multi-step method

$$y_{n+1} = 5y_{n-1} - 4y_n + \Delta t\,[4f_n + 2f_{n-1}]$$

Let us move all y to the LHS.

$$y_{n+1} + 4y_n - 5y_{n-1} = \Delta t\,[4f_n + 2f_{n-1}]$$

Then

$$
\begin{aligned}
\rho(\xi) &= \xi^2 + 4\xi - 5 \\
&= (\xi - 1)(\xi + 5) \\
\sigma(\xi) &= 4\xi + 2
\end{aligned}
$$

Thus, $\rho(1) = 0, \rho(1) = \sigma(1)$ so the method is at least of order 1 and therefore consistent with $y' = f(t, y)$.

(b) No, it's not zero-stable because a multistep method is zero-stable if the zeros of ρ satisfy the root condition and in our case it fails the first root condition where $|\xi| \leq 1 \forall i = 1, 2, \ldots, m$ since our roots of ρ are $1, -5$

(c) No, it's not convergent because a consistent multistep method is convergent if and only if it is zero-stable. Here our multistep method is consistent but it is not zero-stable

Question 4

(a) Suppose the forward Euler method and apply to the model for linear stability.

$$\begin{aligned} y_{n+1} &= y_n + \Delta t \lambda y_n = (1 + \Delta t \lambda) y_n \\ &= (1 + \Delta t \lambda)(1 + \Delta t \lambda) y_{n-1} = (1 + \Delta t \lambda)^2 y_{n-1} \\ &= \cdots = (1 + \Delta t \lambda)^{n+1} y_0 \end{aligned}$$

We find that our forward Euler solution is $y_n = (1 + \Delta t \lambda)^n y_0$. For this numerical approximation to remain bounded for long term behavior (as $n \to \infty$), we need

$$|1 + \Delta t \lambda| \leq 1$$

Denoting $z = \Delta t \lambda$, the set

$$\mathcal{S} = \{z \in \mathbb{C} : |1 + z| \leq 1\}$$

So the region of A-stability of the forward Euler method is where the unit disk is centered at -1.

(b) Suppose the backward Euler method and similar to 4a, apply the method to the model for linear stability.

$$y_{n+1} = y_n + \Delta t \lambda y_{n+1}$$

Then solving for y_{n+1} we have

$$y_{n+1} = \left[\frac{1}{1 - \Delta t \lambda} \right] y_n$$

so its stability function is $R(z) = \frac{1}{(1-z)}$ and its A-stability region is the set of complex numbers z such that $|1 - z| \geq 1$. In other words, the region of A-stability of the backward Euler method is where the exterior of the unit disk is centered at 1.

(c) A method is L-stable if it is A-stable and $(z) \to 0$ as $z \to \infty$. For backward Euler method we have $R(z) = \frac{1}{(1-z)}$, we can clearly see that as $z \to \infty$, $R(z)$ goes to 0.