OLS ; Ordinary Least Squares ; 최소제곱법

$$\min \boxed{\sum_{i=1}^{n} (y_i - \hat{y}_i)} \overset{let}{=} Q(\hat{\beta}_0, \hat{\beta}_1) \quad \text{for} \quad \hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

where $\hat{\beta}_0$, $\hat{\beta}_1$ are chosen to minimize.

$$\Longleftrightarrow \boxed{\frac{\partial Q(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_0} = 0 \quad \& \quad \frac{\partial Q(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_1} = 0}$$

$$\frac{\partial Q}{\partial \hat{\beta}_0} = \frac{\partial \left[\sum (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2 \right]}{\partial \hat{\beta}_0} = \sum 2(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))(-1) = 0$$

(1)

$$\Leftrightarrow \boxed{\sum x_i y_i = \sum (\hat{\beta}_1 x_i^2 + \hat{\beta}_0 x_i)}$$

$$\frac{\partial Q}{\partial \hat{\beta}_1} = \frac{\partial \left[\sum (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2 \right]}{\partial \hat{\beta}_1} = \sum 2(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))(-x_i) = 0$$

(2)

$$\Leftrightarrow \boxed{\sum y_i = n\hat{\beta}_0 - \sum \hat{\beta}_1 x_i}$$

식(1), (2)를 연립하면

[$\hat{\beta}_0$를 제거했을때]

$$n\sum x_i y_i - \sum x_i \sum y_i = \hat{\beta}_1 n \left(n\sum x_i^2 - (\sum x_i)^2 \right)$$

$$\hat{\beta}_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{n\sum x_i y_i - n^2 \bar{X}\bar{Y}}{n\sum x_i^2 - n^2(\bar{X})^2} = \frac{\sum x_i y_i - n\bar{X}\bar{Y}}{\sum x_i^2 - n(\bar{x})^2}$$

$\bar{X} = \frac{1}{n}\sum X_i$

[$\hat{\beta}_1$을 제거했을때]

$$\hat{\beta}_0 = \frac{\sum x_i \sum x_i y_i - \sum x_i^2 \sum y_i}{(\sum X_i)^2 - n\sum x_i^2} = \frac{(n\bar{X}\sum x_i \sum y_i - \bar{X}\sum x_i \sum y_i) + (\bar{Y}\sum x_i \sum y_i - n\bar{X}\sum x_i y_i)}{n\sum x_i^2 - (\sum x_i)^2}$$

$$= \bar{Y} - \bar{X}\hat{\beta}_1$$

$$\therefore \hat{\beta}_1 = \frac{\Sigma x_i y_i - n\bar{X}\bar{Y}}{\Sigma x_i^2 - n(\bar{X})^2}, \quad \hat{\beta}_0 = \bar{Y} - \bar{X}\hat{\beta}_1$$

$$\hat{\beta}_1 = \frac{\Sigma x_i y_i - n\bar{X}\bar{Y}}{\Sigma x_i^2 - n(\bar{X})^2} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$$

($\hat{\beta}_1$ 의 분모) 증명

$$\boxed{\Sigma(x_i - \bar{x})^2} = \Sigma(x_i^2 - 2x_i\bar{x} + (\bar{x})^2)$$

$= SST_x$

x의 총제곱합

$$= \Sigma x_i^2 - 2\bar{x}\Sigma x_i + n(\bar{x})^2$$

$$= \Sigma x_i^2 - 2n(\bar{x})^2 + n(\bar{x})^2$$

$$= \Sigma x_i^2 - n(\bar{x})^2$$

($\hat{\beta}_1$ 의 분자) 증명

$$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma(x_i y_i - \bar{x}y_i - \bar{y}x_i + \bar{x}\bar{y})$$

$$= \Sigma x_i y_i \boxed{- \bar{x}\Sigma y_i - \bar{y}\Sigma x_i} + n\bar{x}\bar{y}$$

$$= \Sigma x_i y_i \boxed{- n\bar{x}\bar{y} - n\bar{x}\bar{y}} + n\bar{x}\bar{y}$$

$$= \Sigma x_i y_i - n\bar{x}\bar{y}$$

$$\therefore \hat{\beta}_1 = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\Sigma(x_i-\bar{x})^2} = \frac{\Sigma(x_i-\bar{x})\left(\beta_1(x_i-\bar{x})+(\varepsilon_i-\bar{\varepsilon})\right)}{SST_x}$$

$$y_i = \beta_0+\beta_1 x_i+\varepsilon_i$$

$$\underbrace{E(y)}_{=\bar{Y}=\bar{y}} = \beta_0+\beta_1 \underbrace{E(x)}_{=\bar{X}=\bar{x}} + \underbrace{E(\varepsilon)}_{=\bar{\varepsilon}}$$

$$\Rightarrow y_i-\bar{y} = \beta_1(x_i-\bar{x})+(\varepsilon_i-\bar{\varepsilon})$$

$$E(\hat{\beta}_1) = E\left[\frac{1}{SST_x}\left(\Sigma\beta_1(x_i-\bar{x})^2+\Sigma\varepsilon_i(x_i-\bar{x})-\bar{\varepsilon}\,\Sigma(x_i-\bar{x})\right)\right]$$

$$=E\left[\beta_1\,\cancelto{1}{\frac{\Sigma(x_i-\bar{x})^2}{SST_x}}\right]+E\left[\frac{\Sigma\varepsilon_i(x_i-\bar{x})}{SST_x}\right]-\bar{\varepsilon}\,E\left[\cancelto{0}{\frac{\Sigma(x_i-\bar{x})}{SST_x}}\right]$$

$$\left(\because \Sigma(x_i-\bar{x}) = \Sigma\left(x_i-\frac{1}{n}\Sigma x_i\right) = \Sigma x_i - \Sigma\frac{1}{n}\Sigma x_i = \Sigma x_i - \Sigma x_i = 0\right)$$

$$=E[\beta_1]+E\left[\frac{\Sigma\varepsilon_i(x_i-\bar{x})}{SST_x}\right]$$

$$\varepsilon_i \sim N(0,\sigma^2)$$

$$=\beta_1+E\left(\frac{1}{SST_x}\right)E(\Sigma x_i-\bar{x})\,\Sigma\big(\underbrace{E(\varepsilon_i)}_{0}\big) = \beta_1$$

$$Var(\hat{\beta}_1) = E([\hat{\beta}_1 - E(\hat{\beta}_1)]^2)$$

$$= E((\hat{\beta}_1 - \beta_1)^2) = E([\frac{\Sigma \varepsilon_i (x_i - \bar{x})}{SST_x}]^2)$$

$E(\hat{\beta}_1)$는 증명하는 풀이 中

$$\hat{\beta}_1 = \beta_1 + \frac{\Sigma \varepsilon_i (x_i - \bar{x})}{SST_x} \Rightarrow \hat{\beta}_1 - \beta_1 = \frac{\Sigma \varepsilon_i (x_i - \bar{x})}{SST_x}$$

$$= \frac{1}{SST_x^2} E(\Sigma(\varepsilon_i (x_i - \bar{x})))^2 + E\left[2\sum_{i=1}^{n}\sum_{i>j} (x_i-\bar{x})(x_j-\bar{x}) \varepsilon_i \varepsilon_j\right]$$

$$= \frac{1}{SST_x^2} E(\varepsilon_i^2) \underbrace{\boxed{\Sigma(x_i - \bar{x})^2}}_{=SST_x} + 2\sum_{i=1}^{n}\sum_{i>j} (x_i-\bar{x})(x_j-\bar{x}) \underbrace{E(\varepsilon_i \varepsilon_j)}_{0}$$

by Gause-Markov thm,

$E(\varepsilon_i^2) = \sigma^2$

$E(\varepsilon_i \varepsilon_j) = 0$ for $\forall i \neq j$

$$= \frac{E(\varepsilon_i^2)}{SST_x} = \frac{Var(\varepsilon_i)}{SST_x} = \frac{\sigma^2}{SST_x}$$

$$= \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}$$