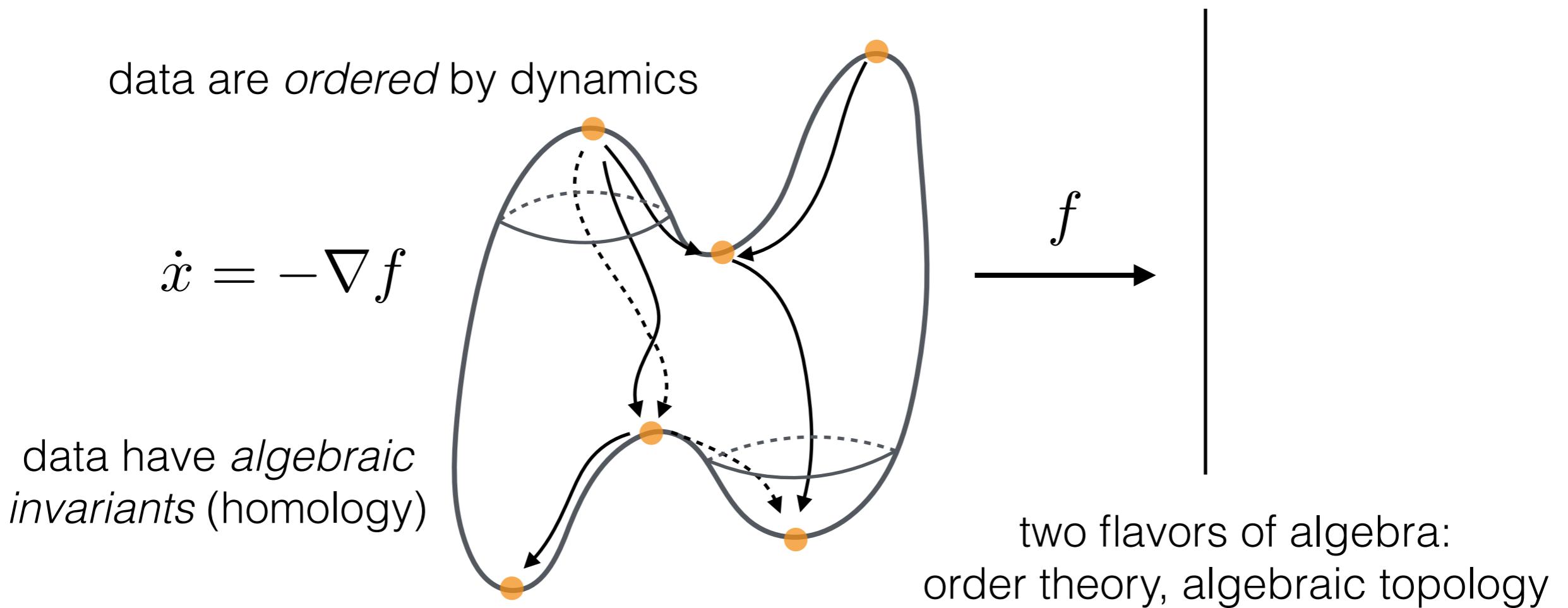


# Morse, Conley, and Computation

*...toward a computational  
homological theory of dynamics*

# dynamical musings

- a dynamical system engenders topological data
- local data (e.g. equilibria) and global data (attractors)
- topological data are ordered and measured with algebra



# Conley-Morse Theory

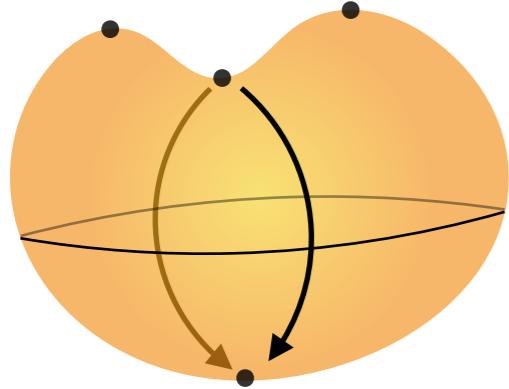
*'...if such rough equations are to be of use it is necessary to study them in rough terms.'*  
C. Conley, CBMS Monograph (1978)

first, the model of a (Morse-type) gradient system

## Morse theory

### list of ingredients

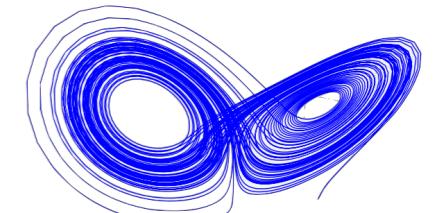
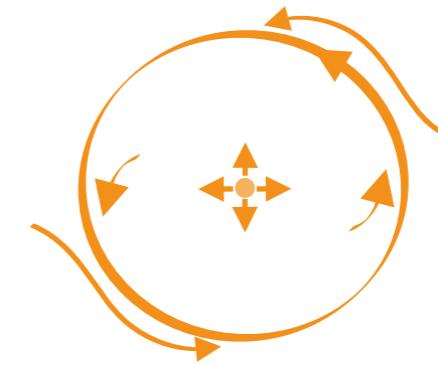
- Morse index
- gradient structure (height function)
- Morse homology



## Conley theory

### list of ingredients

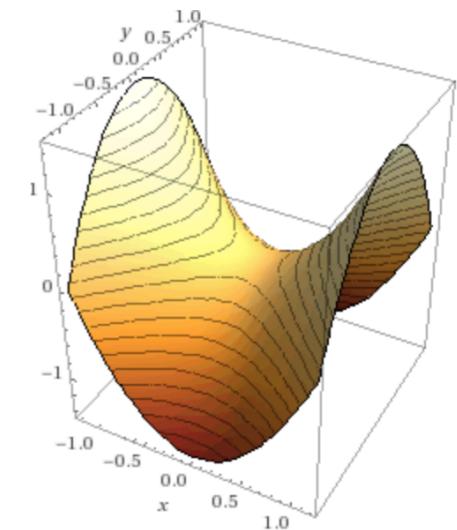
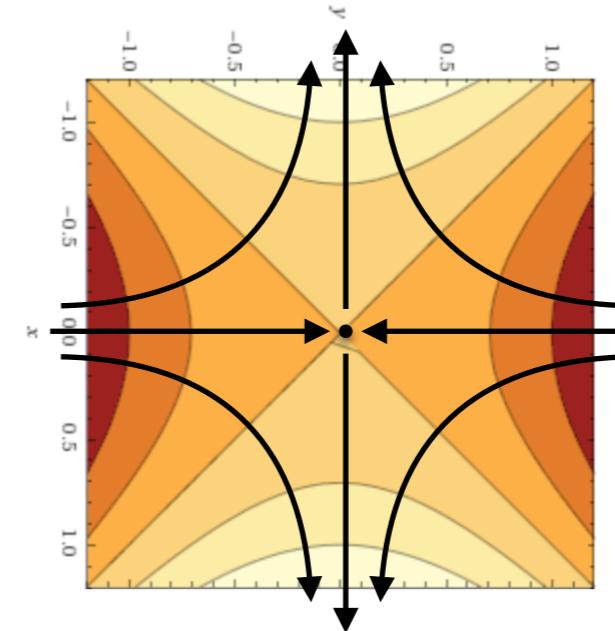
- Conley index
- lattice (of attractors)
- connection matrix



gradients are model systems: every dynamical system has gradient structure

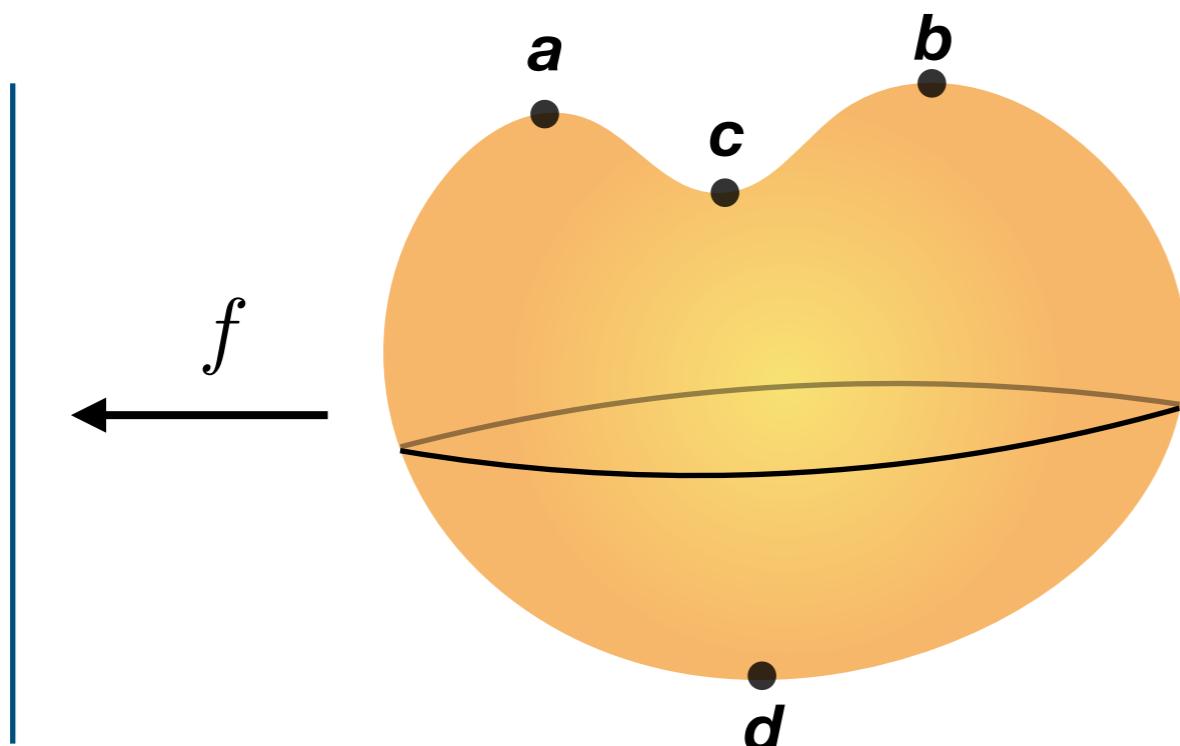
Morse indices **measure** fixed points

Morse index **quantifies** instability  
 $\dim W^u(p)$



*Morse indices as cyclic chain complex (zero differentials)*

$$M_n(p) = \begin{cases} \mathbb{Z}_2\langle p \rangle, & n = \dim W^u(p) \\ 0, & \text{else} \end{cases}$$



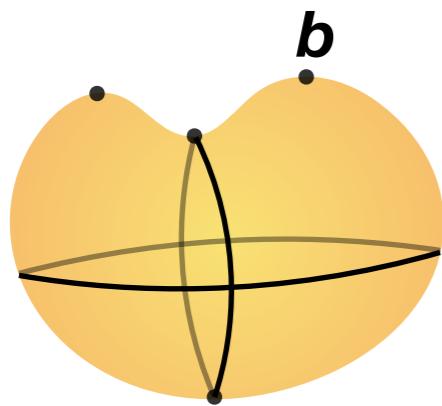
$$\dot{x} = -\nabla f(x)$$

Morse indices **assemble**

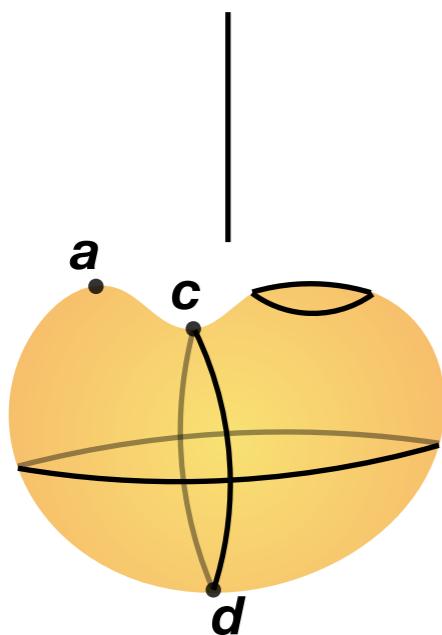
$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{Z}_2\langle a \rangle \oplus \mathbb{Z}_2\langle b \rangle \\ \downarrow (1 \quad 1) \\ \mathbb{Z}_2\langle c \rangle \\ \downarrow 0 \\ \mathbb{Z}_2\langle d \rangle \\ \downarrow \\ 0 \end{array} \qquad M_\bullet(f) = \bigoplus_{p \in crit(f)} M_\bullet(p)$$

*boundary operator counts connecting orbits mod 2*

a height function filters  
 (via *lattice* of sublevel sets)



$$0 \leftarrow \mathbb{Z}_2\langle d \rangle \xleftarrow{0} \mathbb{Z}_2\langle c \rangle \xleftarrow{(1 \quad 1)} \mathbb{Z}_2\langle a \rangle \oplus \mathbb{Z}_2\langle b \rangle \leftarrow 0$$



$$0 \leftarrow \mathbb{Z}_2\langle d \rangle \xleftarrow{0} \mathbb{Z}_2\langle c \rangle \xleftarrow{1} \mathbb{Z}_2\langle a \rangle \leftarrow 0$$

simple dynamics:  
 non-degenerate equilibria  
 heteroclinic orbits

$$f^{-1}(-\infty, x] \rightsquigarrow \{\mathbb{Z}_2\langle a \rangle : f(a) \leq f(x)\}$$

sublevel set

(Morse) subcomplex

Morse index of **b** recovered  
 as a subquotient  
 $0 \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{Z}_2\langle b \rangle \leftarrow 0 = M_\bullet(b)$

$$M_\bullet(b) = \frac{0 \leftarrow \mathbb{Z}_2\langle d \rangle \xleftarrow{0} \mathbb{Z}_2\langle c \rangle \xleftarrow{(1 \quad 1)} \mathbb{Z}_2\langle a \rangle \oplus \mathbb{Z}_2\langle b \rangle \leftarrow 0}{0 \leftarrow \mathbb{Z}_2\langle d \rangle \xleftarrow{0} \mathbb{Z}_2\langle c \rangle \xleftarrow{1} \mathbb{Z}_2\langle a \rangle \leftarrow 0}$$

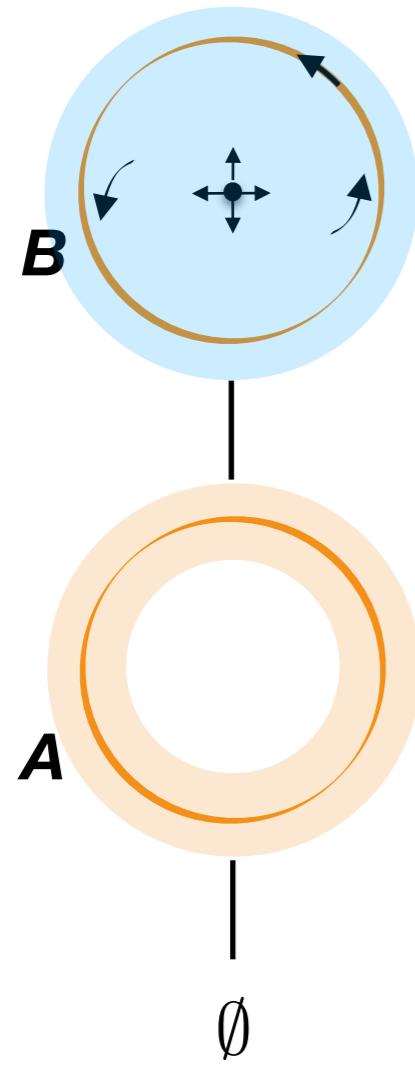
# Conley's focus: attractors, attracting blocks

*Conley theory is a purely topological generalization of Morse theory  
for general dynamical systems*

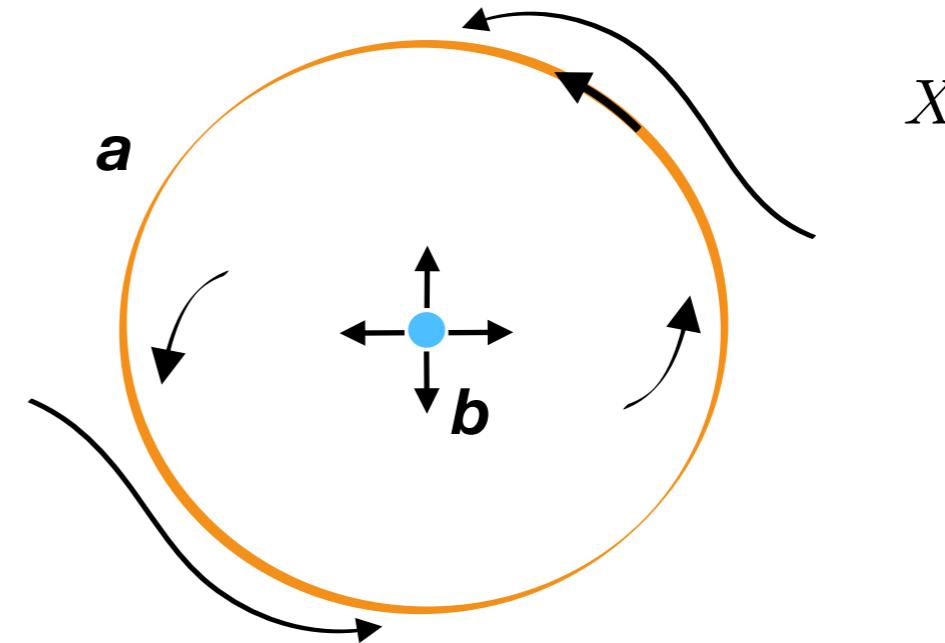
$X$  compact metric space

dynamics given by continuous flow  $\varphi : \mathbb{R} \times X \rightarrow X$

a compact set  $N$  is an **attracting block** if  $\varphi(N, t) \subset \text{int}(N)$  for all  $t > 0$



$\mathbf{L}$  sublattice of attracting blocks



Fact: the set of attracting blocks **ABlock**  
is bounded distributive lattice

$\wedge := \cap$     $\vee := \cup$

# Birkhoff's theorem

$L$  finite distributive lattice

the poset of *join irreducible* elements of  $L$  is

$$J(L) := \{x \in L \setminus \{0_L\} : \text{if } x = a \vee b, \text{ then } a = x \text{ or } b = x\}$$

*a join-irreducible has a unique predecessor*

$$Pred : J(L) \rightarrow L$$

$(P, \leq)$  poset

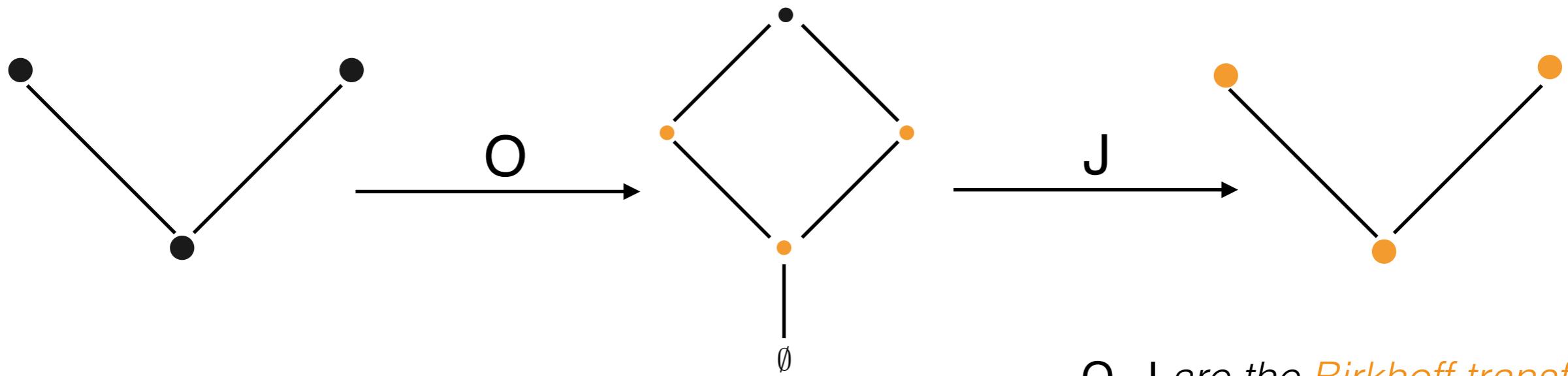
the lattice of lower sets is

$$O(P) := \{U \subseteq P : \text{if } x \in U \text{ and } y \leq x \text{ then } y \in U\}$$

$$\wedge := \cap \quad \vee := \cup$$

Fact:  $O, J$  are contravariant functors

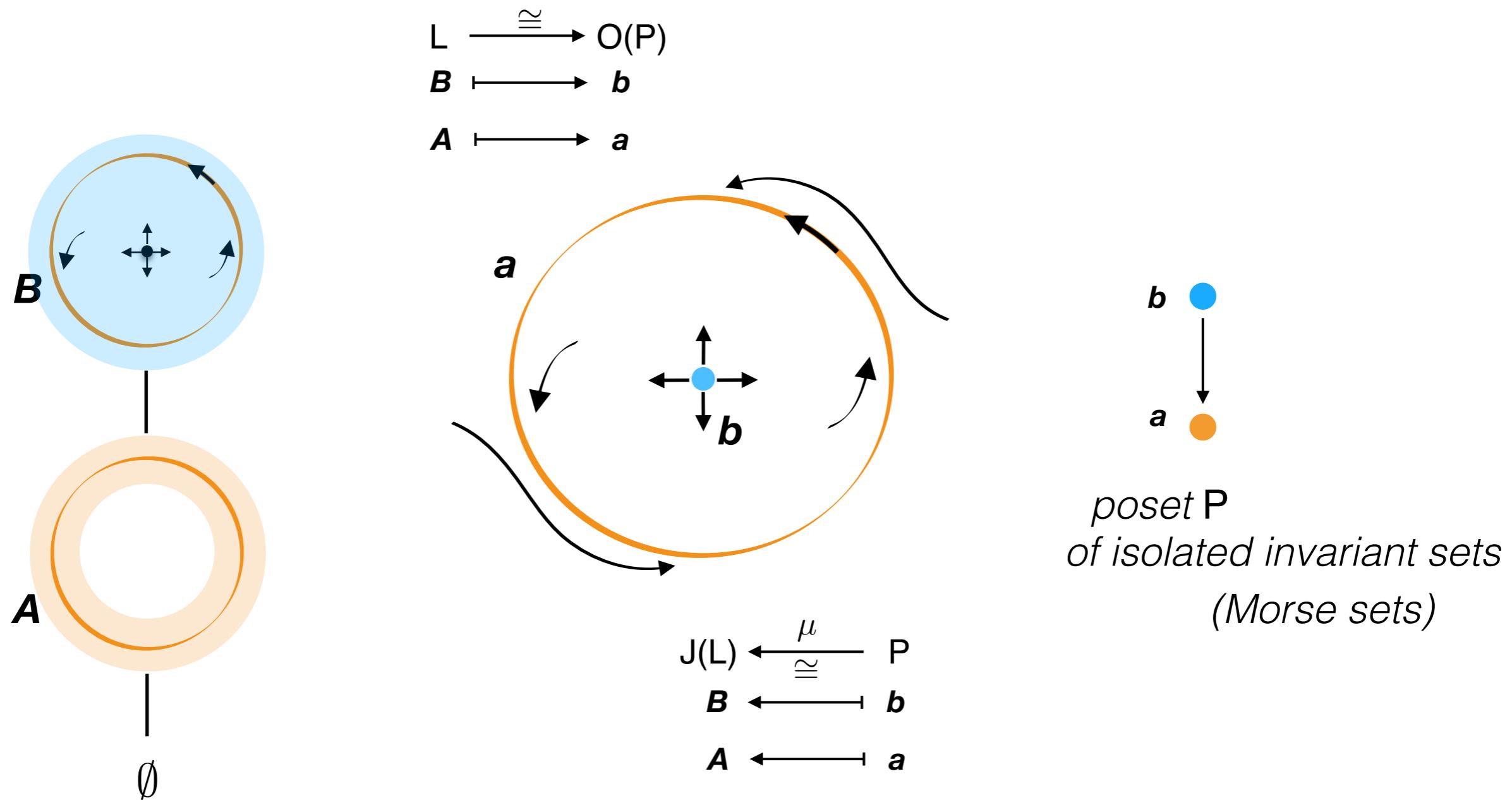
$$\text{Birkhoff: } O(J(L)) \cong L \quad J(O(P)) \cong P$$



$O, J$  are the *Birkhoff transforms*

# Conley-Morse Homology i

the Birkhoff transforms give dual perspective to dynamics



$L$  sublattice of attracting blocks

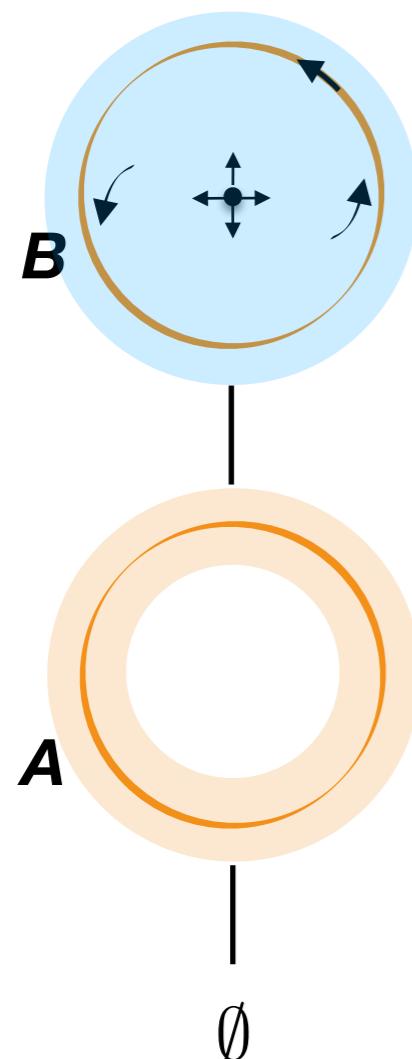
# Conley-Morse Homology ii

to generalize Morse homology

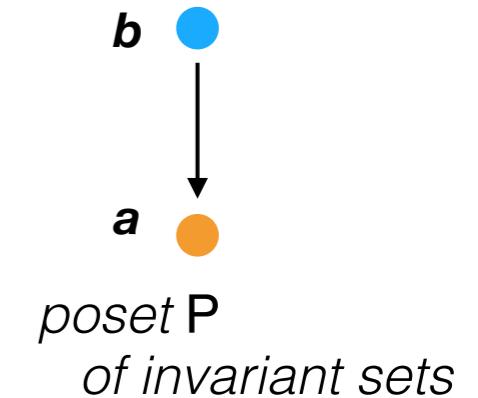
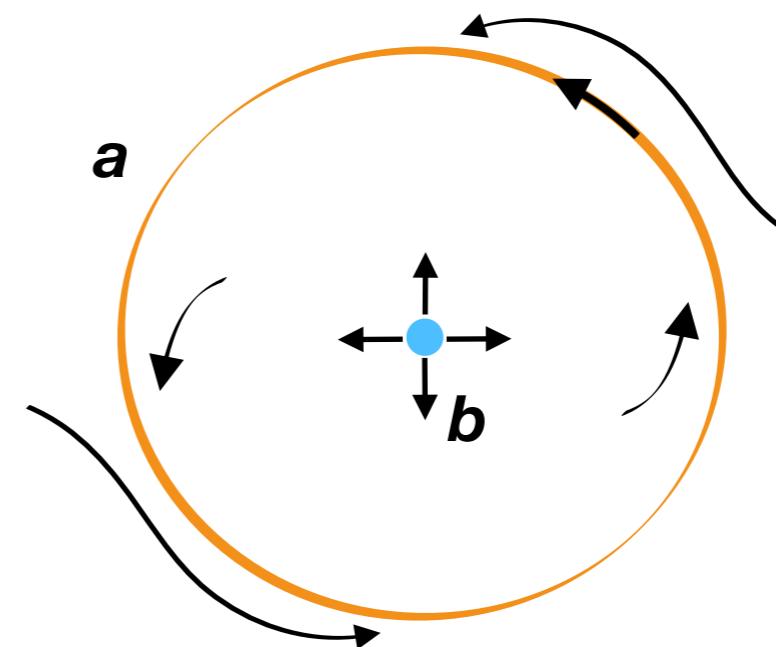
associate cyclic complex to isolated invariant sets (**Conley index**)

characterized by dynamics at the boundary (local instability)

$$CH_{\bullet}(b) = H_{\bullet}(B, \text{Pred}(B)) \quad B = \mu(b)$$



$$\begin{array}{ccc} J(L) & \xleftarrow[\cong]{\mu} & P \\ B & \longleftarrow b & \\ A & \longleftarrow a & \end{array}$$

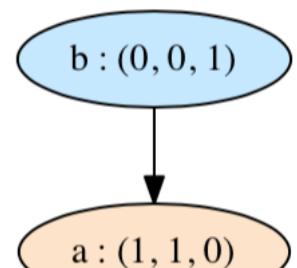


chain complex of Conley indices

$$0 \leftarrow \mathbb{Z}_2\langle a \rangle \xleftarrow{0} \mathbb{Z}_2\langle a \rangle \xleftarrow{1} \mathbb{Z}_2\langle b \rangle \leftarrow 0$$

boundary operator is called the **connection matrix**

$L$  sublattice of attracting blocks



# Conley-Morse Homology iii

to generalize Morse homology

Conley indices as input to chain complex

*what is the boundary operator?*

for  $L$  lattice of attracting blocks and  $J(L)$  join-irreducibles

**Theorem (Franzosa, Robbin & Salamon):** There exists a strictly upper triangular - wrt  $(J(L), \leq)$  - boundary operator

$$\Delta : \bigoplus_{p \in J(L)} CH_\bullet(p) \rightarrow \bigoplus_{p \in J(L)} CH_\bullet(p)$$

so that for any attracting block  $A$  in  $L$  the homology of

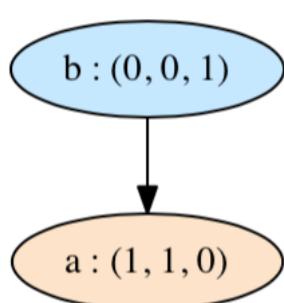
local to global

$$\Delta : \bigoplus_{p \in A} CH_\bullet(p) \rightarrow \bigoplus_{p \in A} CH_\bullet(p)$$

is isomorphic to  $H_\bullet(A)$

algebraic representation  
of dynamics

$\Delta$  is called a **connection matrix**



$$\Delta = \begin{array}{c|ccc} & \text{a} & \text{a} & \text{b} & \text{valuation} \\ & 0 & 1 & 2 & \text{cell dim.} \\ \hline \text{a} & 0 & 0 & 0 & \\ \text{a} & 1 & 0 & 0 & 1 \\ \text{b} & 2 & 0 & 0 & 0 \end{array}$$

Connection Matrix Data

=====

Boundaries of 0-cells:

Cell 0 (valuation a) : {}

Boundaries of 1-cells:

Cell 1 (valuation a) : {}

Boundaries of 2-cells:

Cell 2 (valuation b) : {1}

caveat: chain complex braids,  
graded module braids

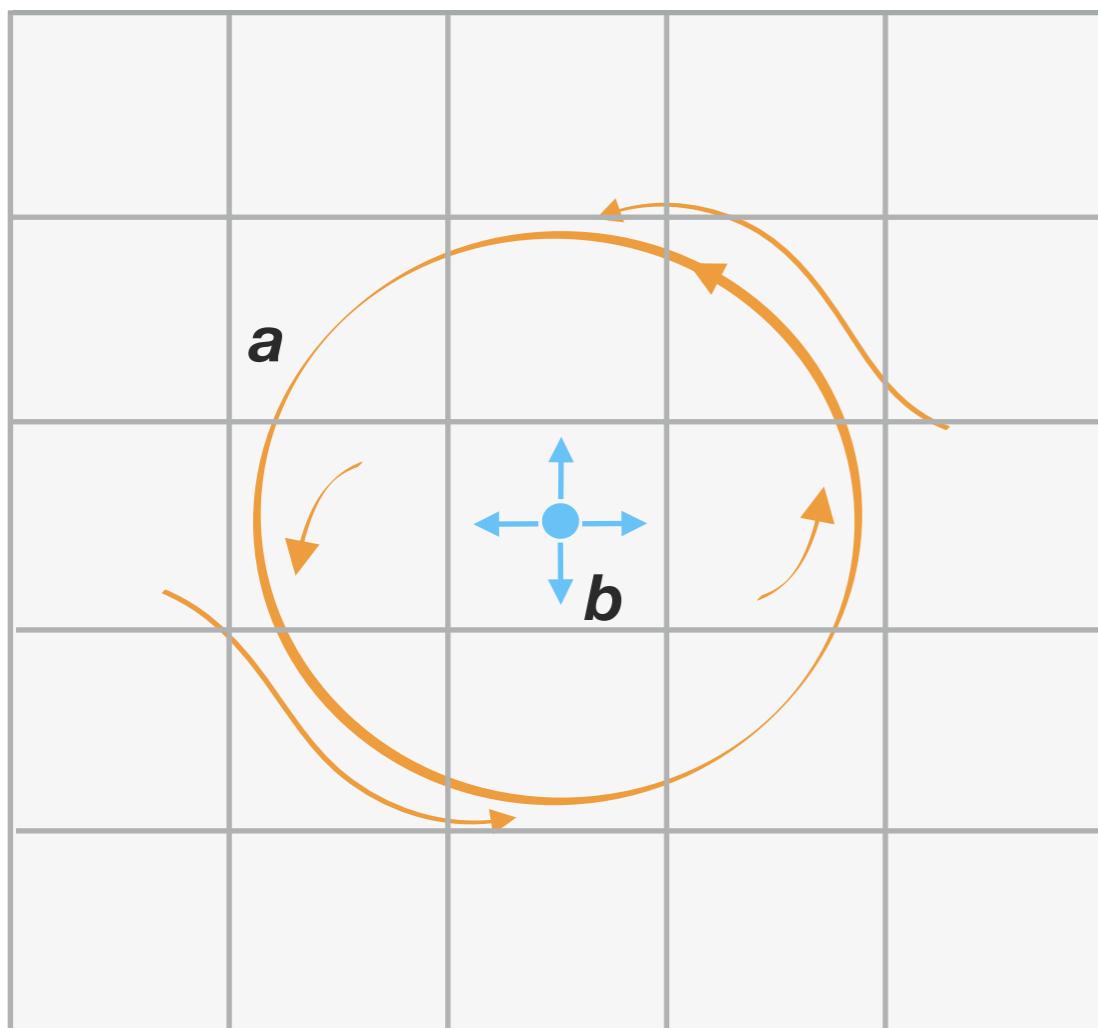
computational Conley theory

approximation + data structures

topological spaces are approximated  
with cell complexes (cubical, simplicial, polygonal)

$$X \subset \mathbb{R}^2$$

*approximation by cubical complex*



**Definition (Cell complex)**

$\mathcal{X} = (\mathcal{X}, \leq, \kappa, \dim)$  consists of poset  $(\mathcal{X}, \leq)$   
with two functions  $\dim : \mathcal{X} \rightarrow \mathbb{N}$  and  $\kappa : \mathcal{X} \times \mathcal{X} \rightarrow k$   
satisfying:

1.  $\dim$  is a poset morphism;
2. for all  $\xi, \xi' \in \mathcal{X}$   
 $\kappa(\xi, \xi') \neq 0 \implies \xi' \leq \xi$  and  $\dim(\xi) = \dim(\xi') + 1$
3. for all  $\xi, \xi' \in \mathcal{X}$

$$\sum_{\xi' \in \mathcal{X}} \kappa(\xi, \xi') \cdot \kappa(\xi', \xi'') = 0$$

a cell complex  $\mathcal{X}$  generates  
a chain complex  $(C_\bullet(\mathcal{X}), \partial)$

if the attracting blocks are representable by subcomplexes  
then we may compute algebraic invariants, e.g. homology

$X$  cell complex

$\text{Sub}(X)$  lattice of subcomplexes of  $X$

representation of attracting blocks

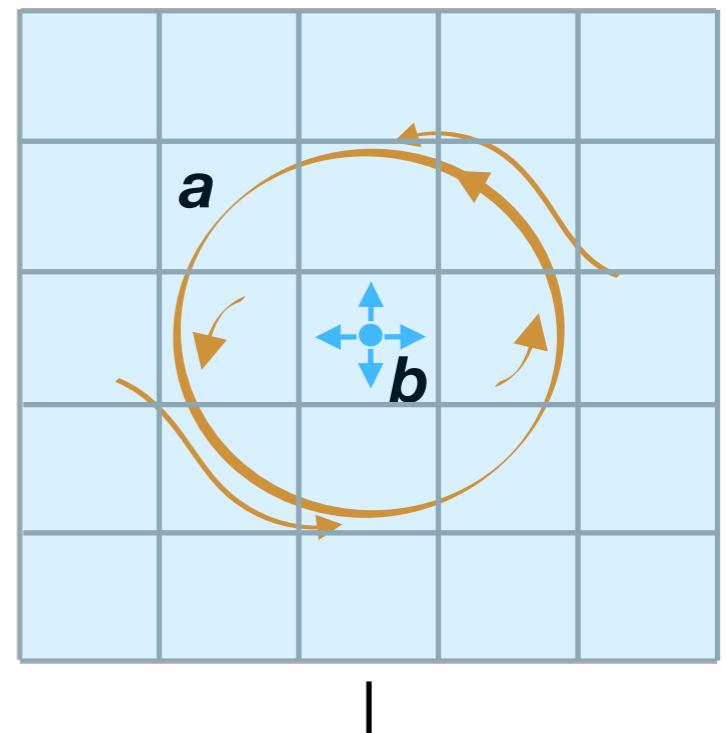
inclusion is a lattice homomorphism

$$L \xrightarrow{i} \text{Sub}(X)$$

$L$  lattice of  
attracting blocks

$\text{Sub}(X)$  lattice of  
subcomplexes

**B**



the Conley indices can be computed for both attractors...

*Betti numbers*

$$CH_{\bullet}(A) = H_{\bullet}(A) = (1, 1, 0)$$

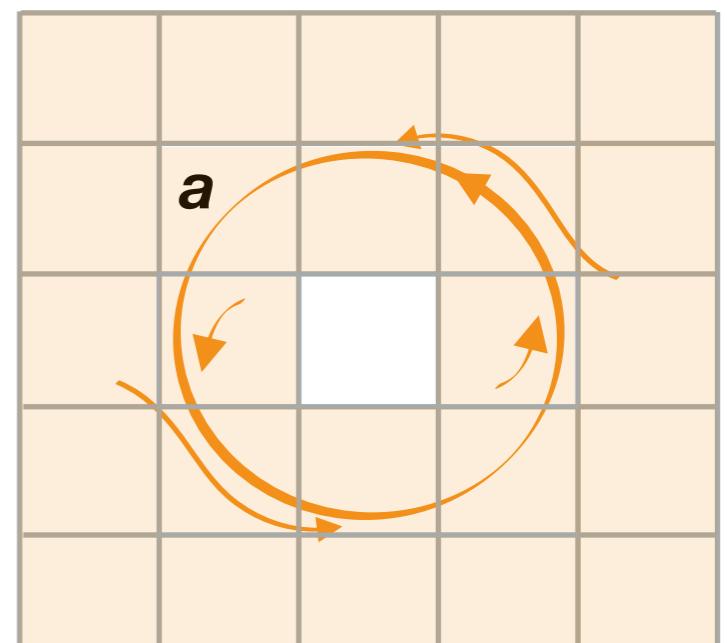
$$CH_{\bullet}(B) = H_{\bullet}(B) = (1, 0, 0)$$

... and the invariant sets (Morse sets)

$$CH_{\bullet}(a) = H_{\bullet}(A, \emptyset) = (1, 1, 0)$$

$$CH_{\bullet}(b) = H_{\bullet}(B, A) = (0, 0, 1)$$

**A**



$\emptyset$

# What is the data structure for Conley theory?

$L$  lattice of attracting blocks

$\text{Sub}(X)$  lattice of subcomplexes of  $X$

*inclusion is lattice morphism*

$$L \xrightarrow{i} \text{Sub}(X)$$

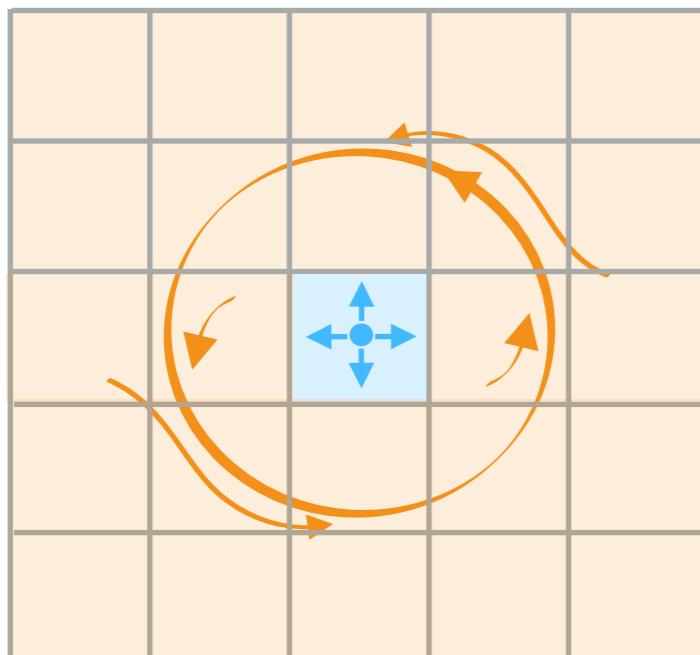
...applying the Birkhoff transform (contravariant)

$(X, \leq)$  face poset

$J(L)$  poset of join-irreducibles

$$J(\text{Sub}(X)) = (X, \leq)$$

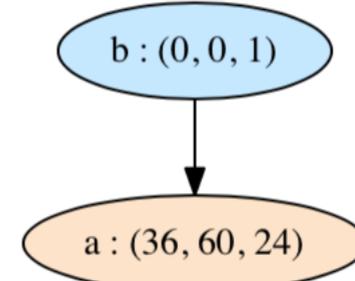
$$(X, \leq) \xrightarrow{J(i)} (J(L), \leq)$$



$$\xrightarrow{J(i)}$$

*poset morphism*

*count of cells in the fiber  $J(i)^{-1}(b)$  for each dim.*



# Axiomization (Conley theoretic data structure)

$X$  cell complex

$(X, \leq)$  face poset

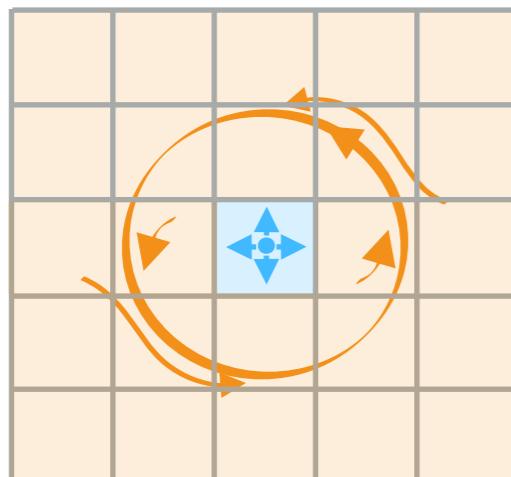
$P$  poset

## Definition ( $P$ -graded cell complex)

$X$ ,  $P$ , and a poset morphism  $\nu$  from  $X$  to  $P$

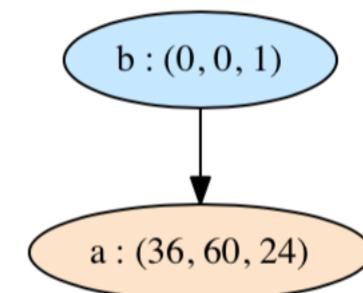
$$(X, \leq) \xrightarrow{\nu} (P, \leq)$$

*topological approximation*



$$\xrightarrow{\nu}$$

*dynamical information*



a graded cell complex determines

a  $P$ -graded chain complex  $(C(X), \partial)$

$$C(X) = \bigoplus_{p \in P} C(\nu^{-1}(p))$$

boundary map is  $P$ -graded

$$\partial_{pq} \neq 0 \implies p \leq q$$

upper triangular wrt  $P$

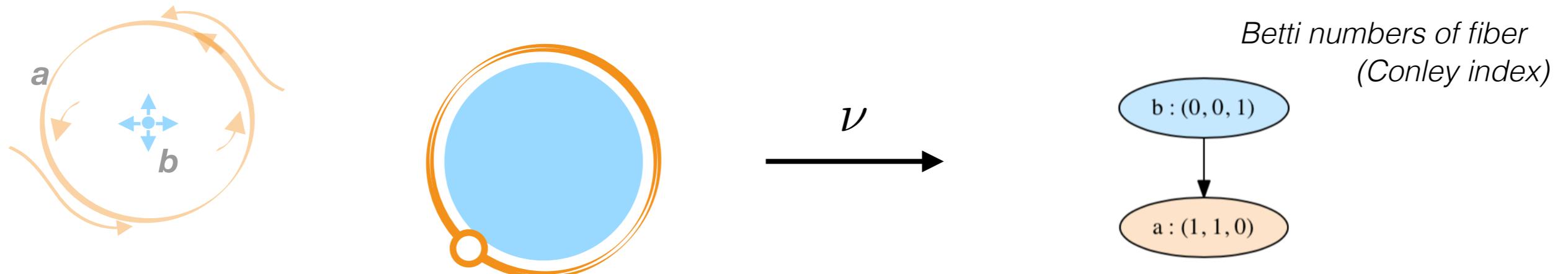
# Axiomization (Connection matrix)

## Definition (cyclic $\mathbf{P}$ -graded complex)

$\mathbf{P}$ -graded complex with cyclic fibers

$$\partial_{pp} = 0 \quad \text{for } p \text{ in } \mathbf{P} \quad \text{‘small’ objects}$$

i.e.  $\partial$  is strictly upper triangular wrt  $\mathbf{P}$

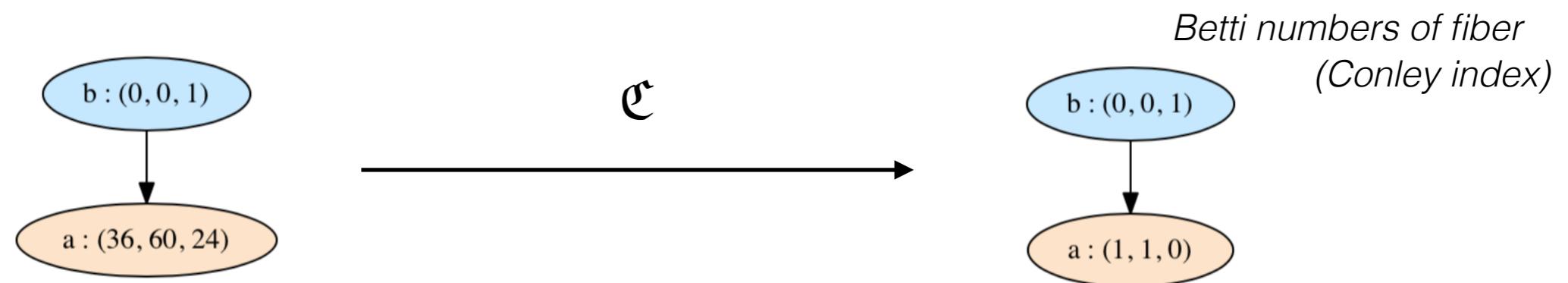


goal: replace graded complex with equivalent cyclic graded complex

## interpretation of connection matrix for computation:

a *Conley complex* is a cyclic graded complex whose fibers are Conley indices  
the boundary operator of a Conley complex is a *connection matrix*

**Theorem:** there is a functor  $\mathfrak{C}$  taking a graded complex to a Conley complex



*under the hood: discrete morse theory  
homotopy categories*

*computational perspective: chain-level data reduction  
without loss of homological information*

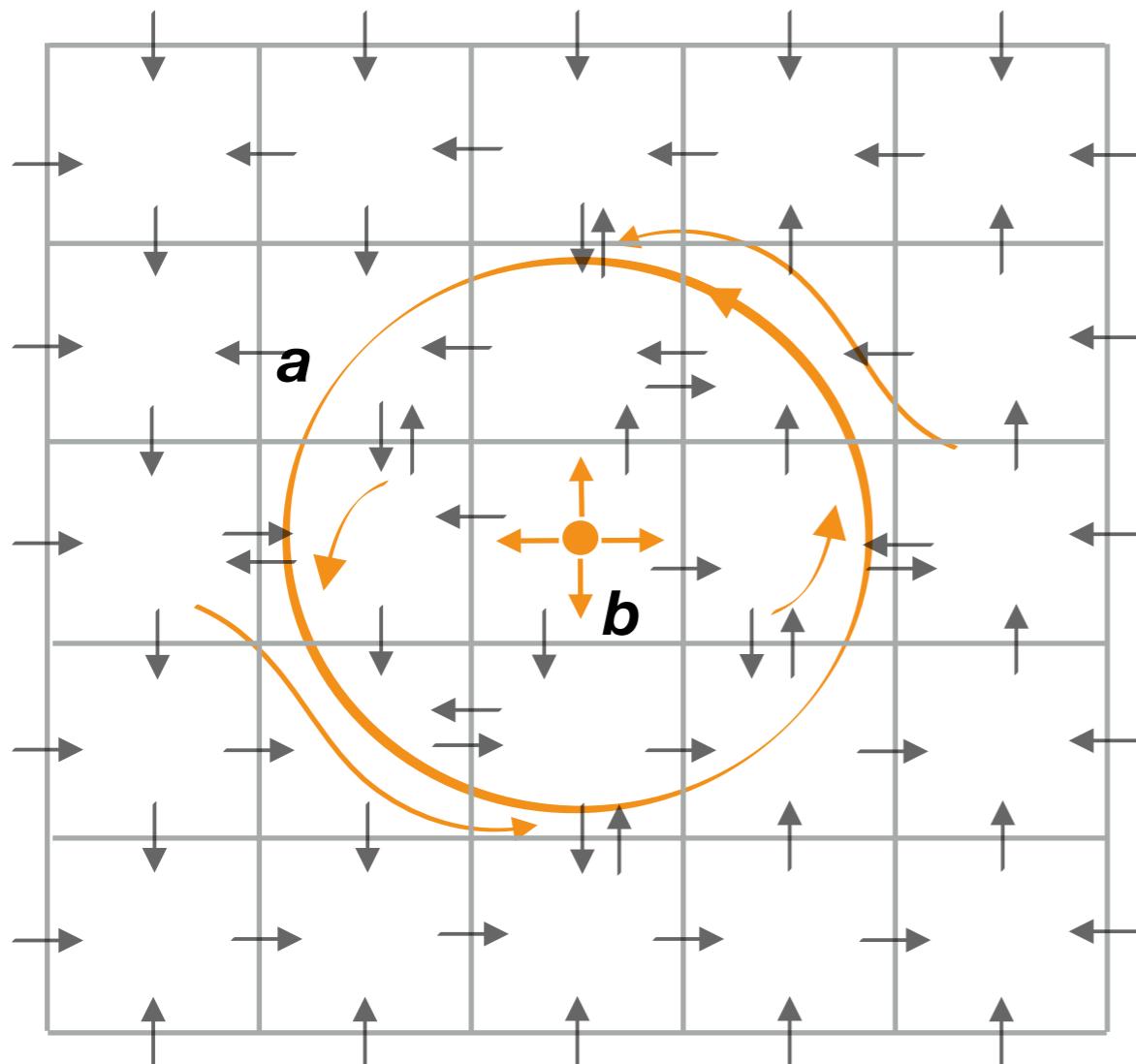
# computational Conley homology

applications + implementation

application i:

transversality

topological spaces are approximated  
with cell complexes



continuous dynamics are approximated  
with directed graph  $\mathcal{F}$  on top cells  $\mathcal{X}_n$

lattice of forward invariant sets:

$$\text{Invset}^+(\mathcal{F}) := \{U \subseteq \mathcal{X}_n : \mathcal{F}(U) \subset U\}$$

poset of strongly connected components:

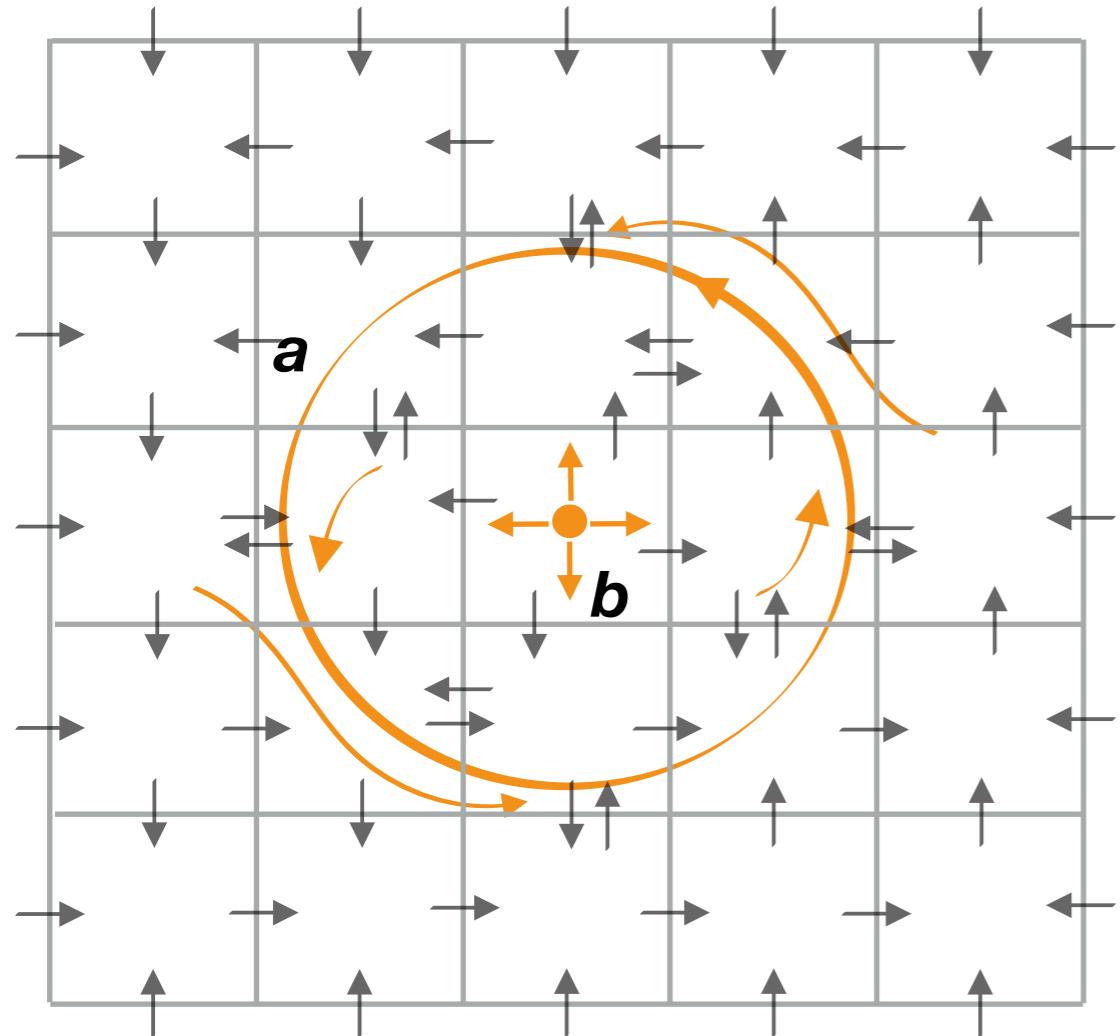
$$\text{SC}(\mathcal{F}) := \text{J}(\text{Invset}^+(\mathcal{F}))$$

*maximal recurrent sets of graph*

poset  $\text{SC}(\mathcal{F})$  strongly connected components of  $\mathcal{F}$

$$(\mathcal{X}_n, \leq) \longrightarrow \text{SC}(\mathcal{F})$$

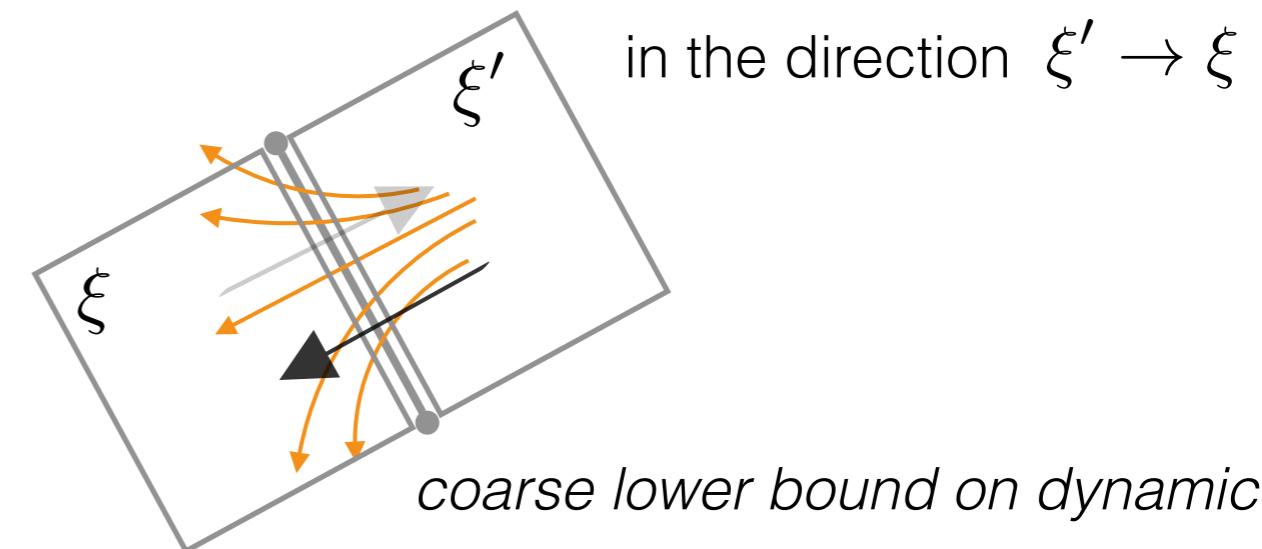
# transversality models



*transversality model*

there is an edge  $\xi \rightarrow \xi'$  between adjacent top cells

unless flow is transverse to  $\xi \cap \xi'$



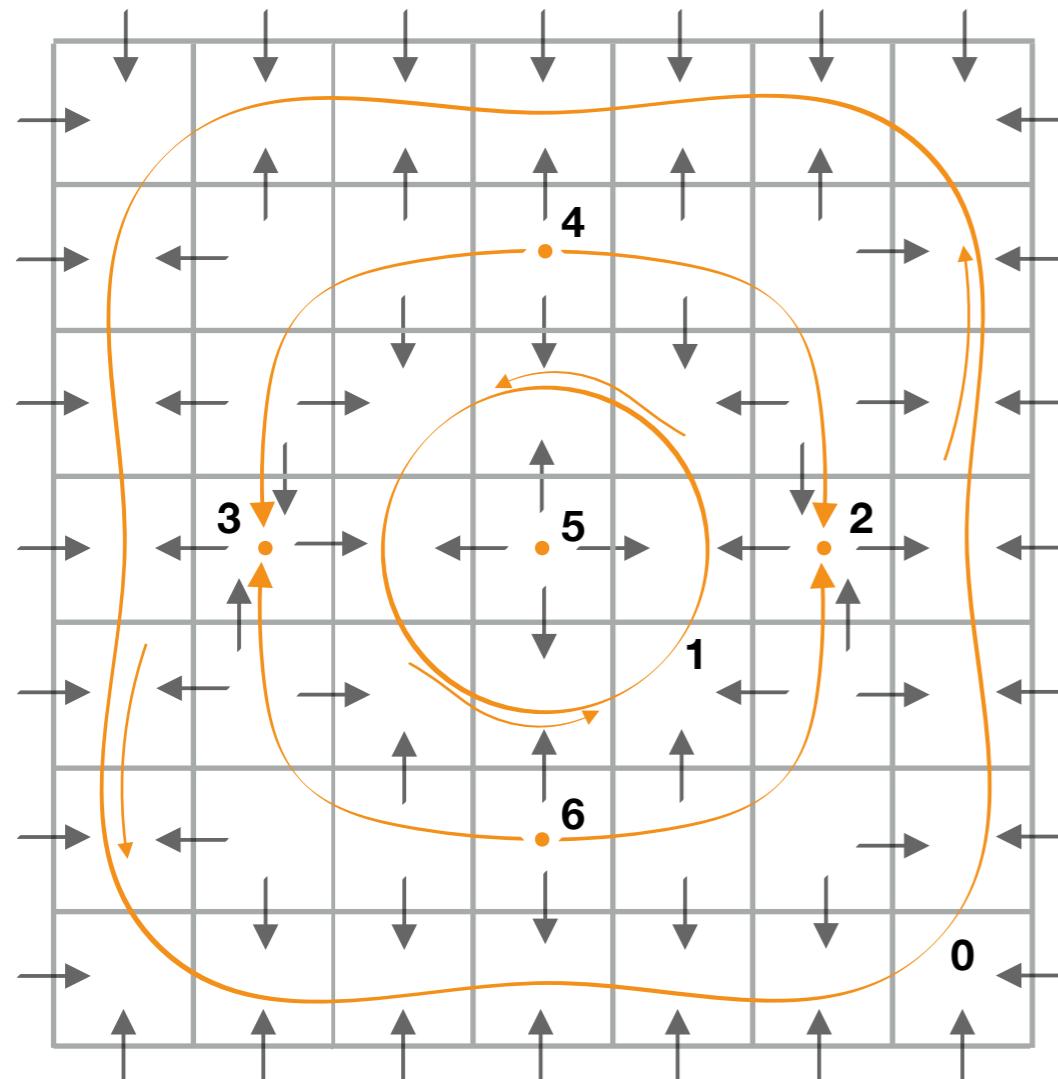
**Theorem:** if the graph is a transversality model then there is an extension  $\nu$

$$\begin{array}{ccc}
 (\mathcal{X}_n, \leq) & \xrightarrow{\quad} & (\mathcal{X}, \leq) \\
 & \downarrow \nu & \\
 SC(\mathcal{F}) & \xleftarrow{\quad \text{graded cell complex} \quad} &
 \end{array}$$

1.  $A = \{\nu^{-1}(a)\}_{a \in O(SC(\mathcal{F}))}$  is a lattice of attracting blocks for  $\varphi$
2.  $\mathfrak{C}(\mathcal{X}, \nu)$  is a Conley complex for  $\varphi$

**Remark:** Computations + theorems are valid for any differential equation which is transverse to top cell boundaries in direction indicated

# transversality models ii

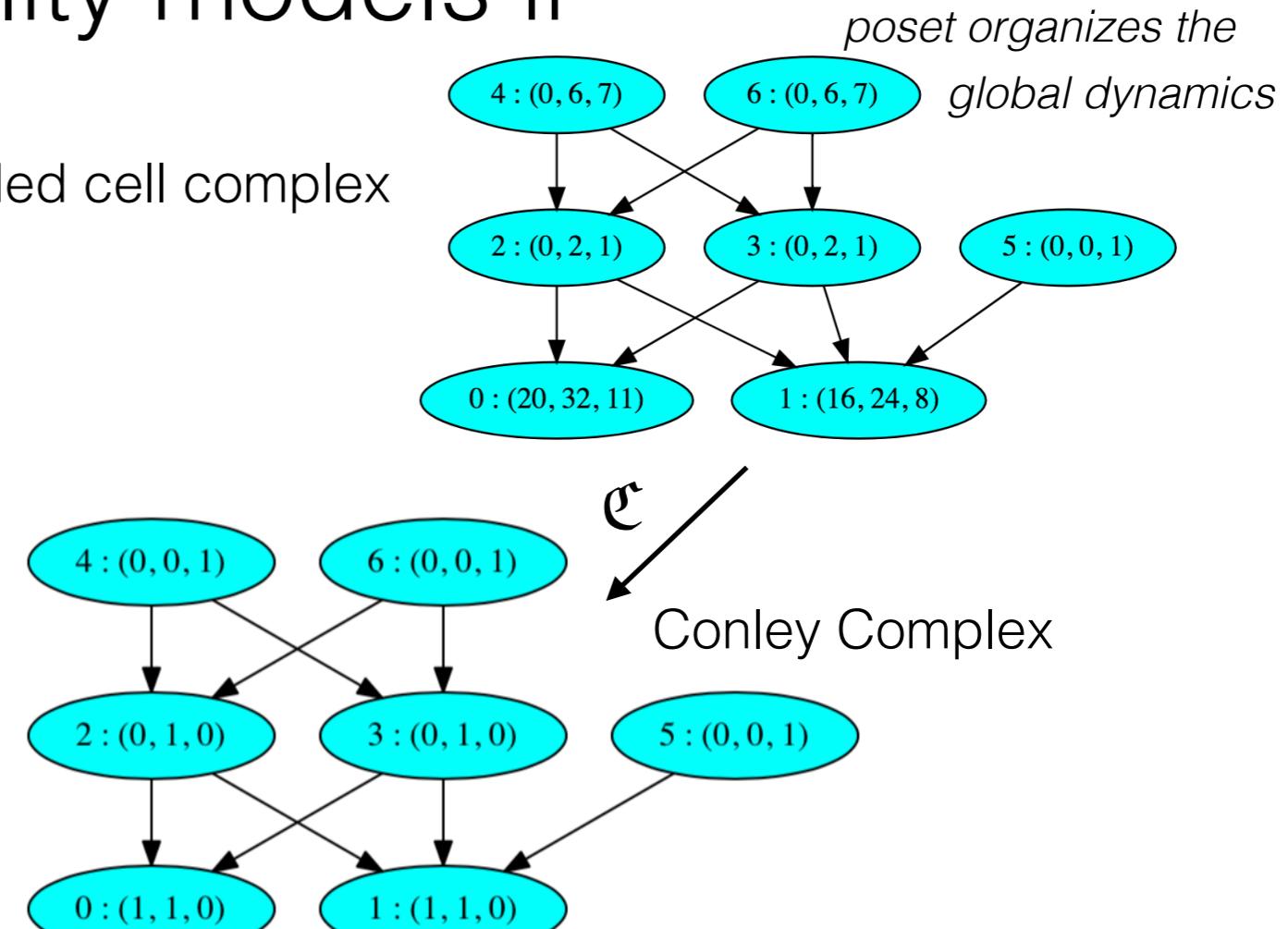


graded cell complex

connection matrix is represented  
with respect to a basis

different bases give different  
qualitative descriptions of dynamics

*in this example: four different bases*



Connection Matrix Data

=====

Boundaries of 0-cells (by cell index):

Cell 0 (valuation 1) : {}

Cell 1 (valuation 0) : {}

Boundaries of 1-cells (by cell index):

Cell 2 (valuation 2) : {0, 1}

Cell 3 (valuation 3) : {0, 1}

Cell 4 (valuation 0) : {}

Cell 5 (valuation 1) : {}

Boundaries of 2-cells (by cell index):

Cell 6 (valuation 6) : {2, 3, 4, 5}

Cell 7 (valuation 5) : {5}

Cell 8 (valuation 4) : {2, 3}

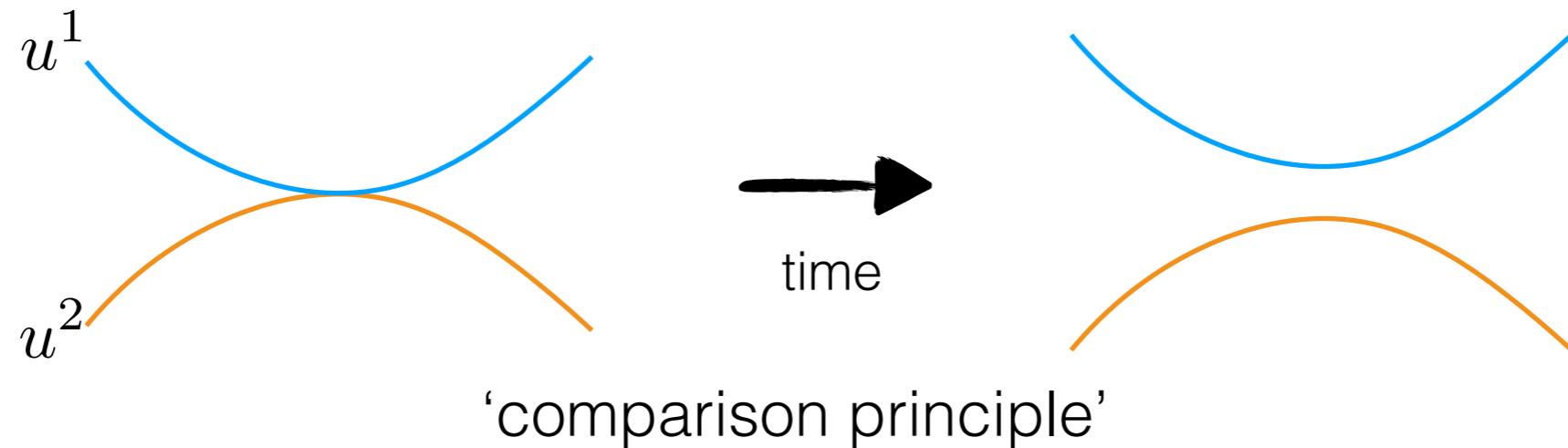
application ii:

Morse theory on braids

## instantiation: dynamics on braids

$$u_t = u_{xx} + f(x, u, u_x)$$

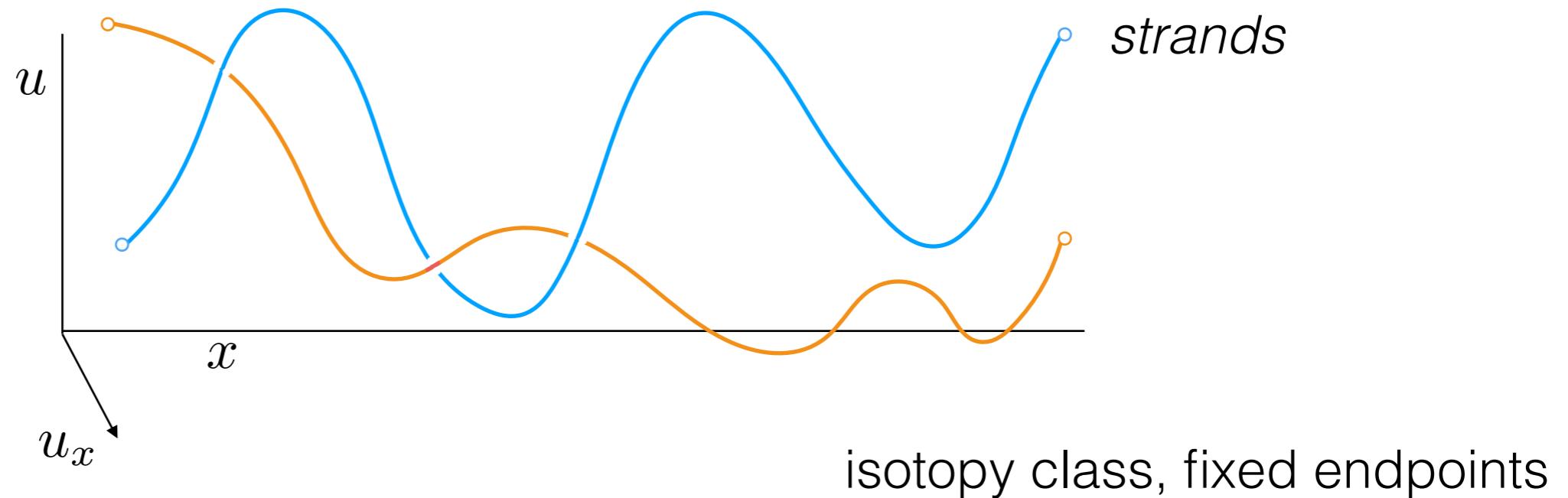
parabolic dynamics decreases intersections



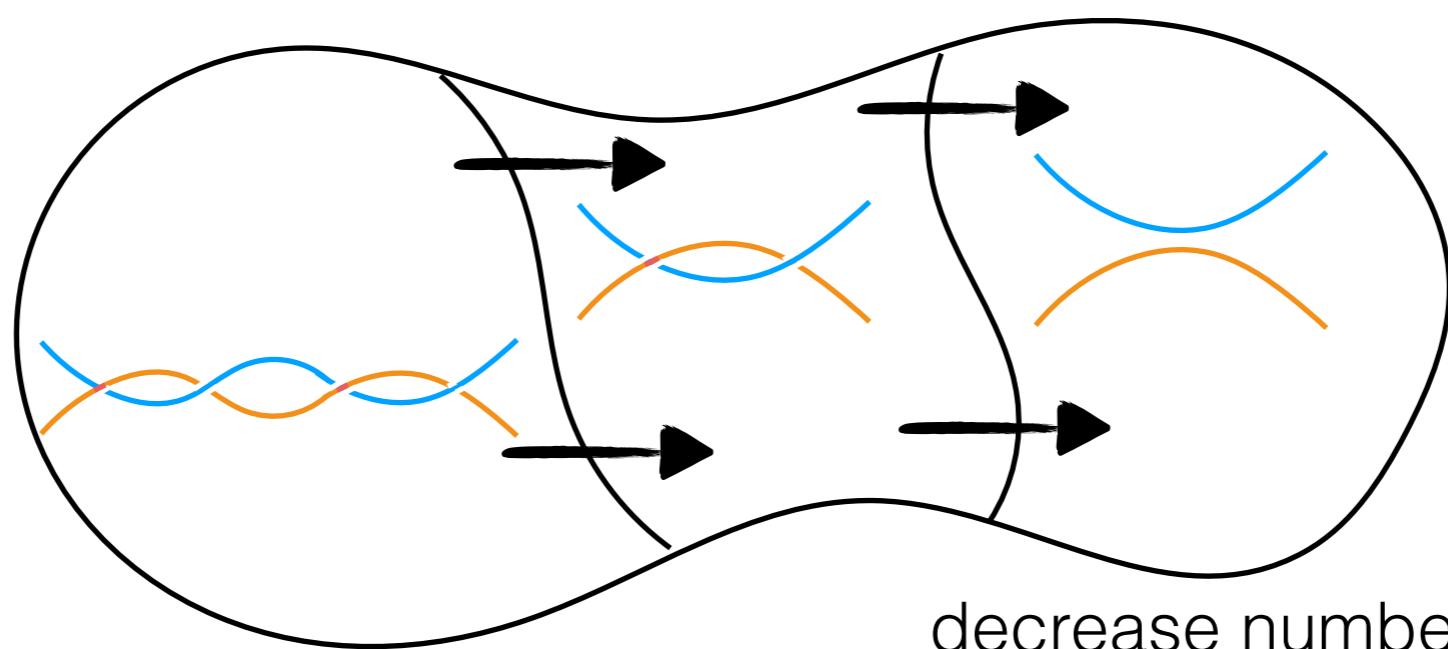
proof:

$$\begin{aligned}\frac{\partial}{\partial t}(u^1(x, t) - u^2(x, t)) &= u_{xx}^1 + f(x, u^1, 0) - u_{xx}^2 - f(x, u^2, 0) \\ &= u_{xx}^1 - u_{xx}^2 > 0\end{aligned}$$

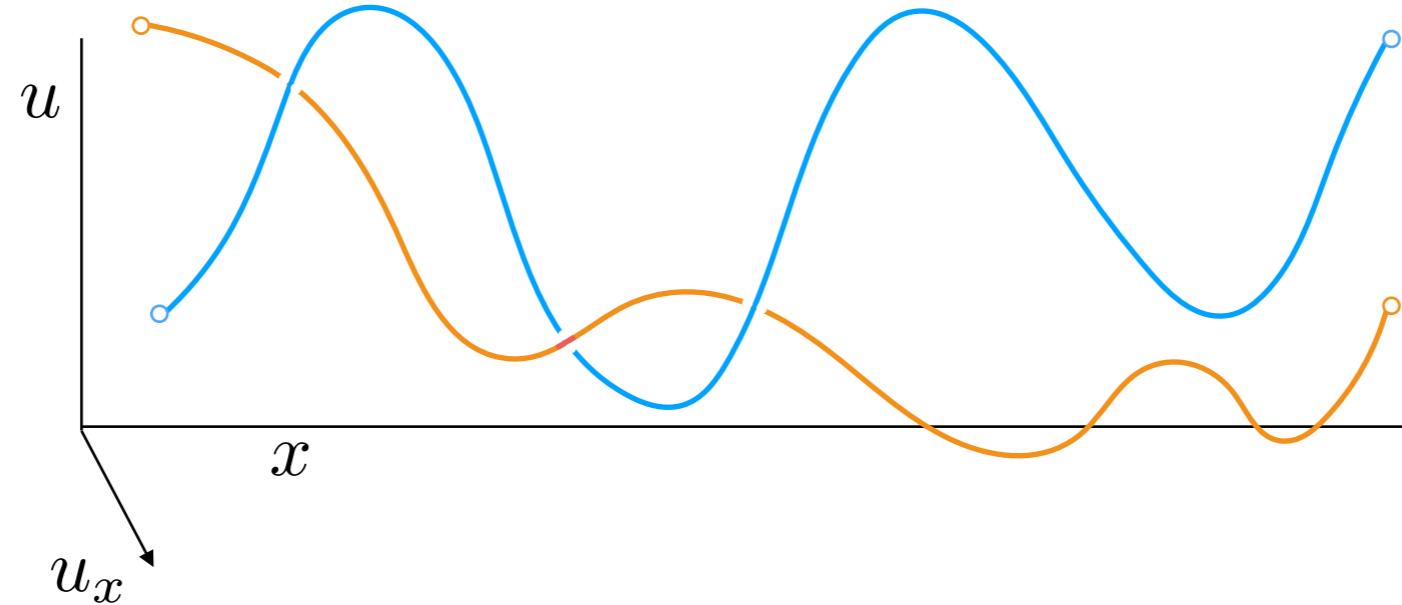
functions lift to braids



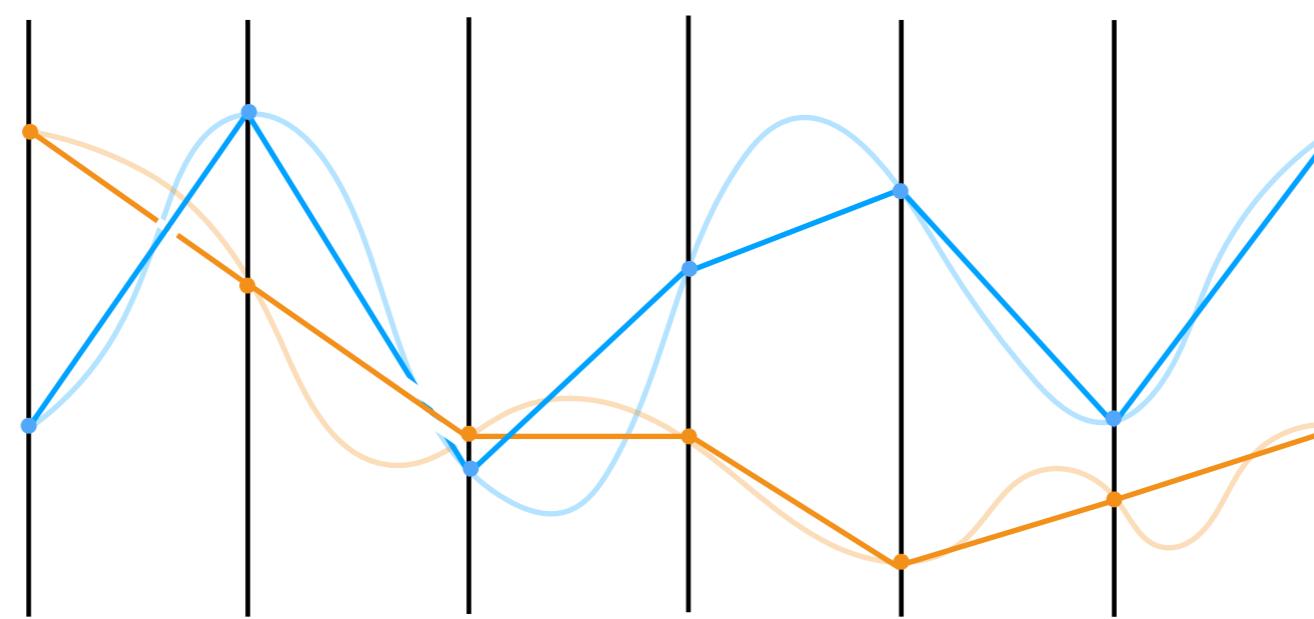
dynamics on braid classes



functions lift to braids

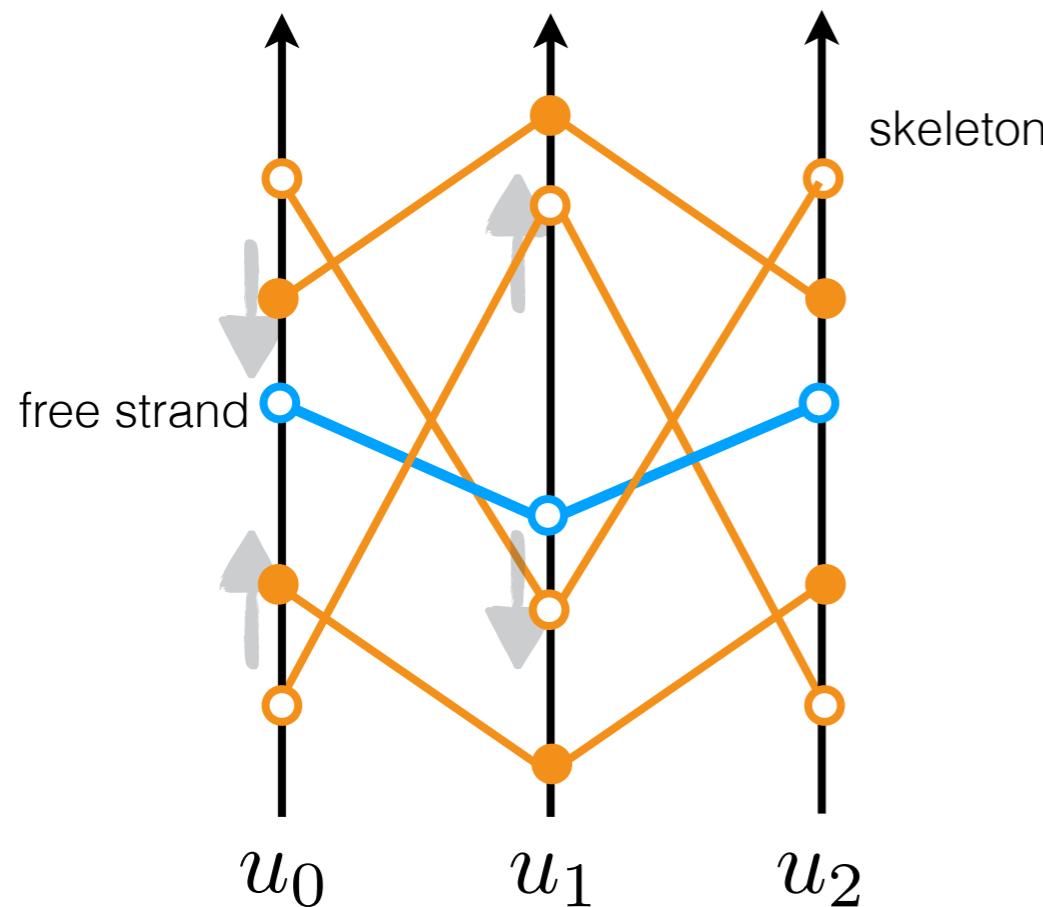


combinatorialization



# Morse theory on braids

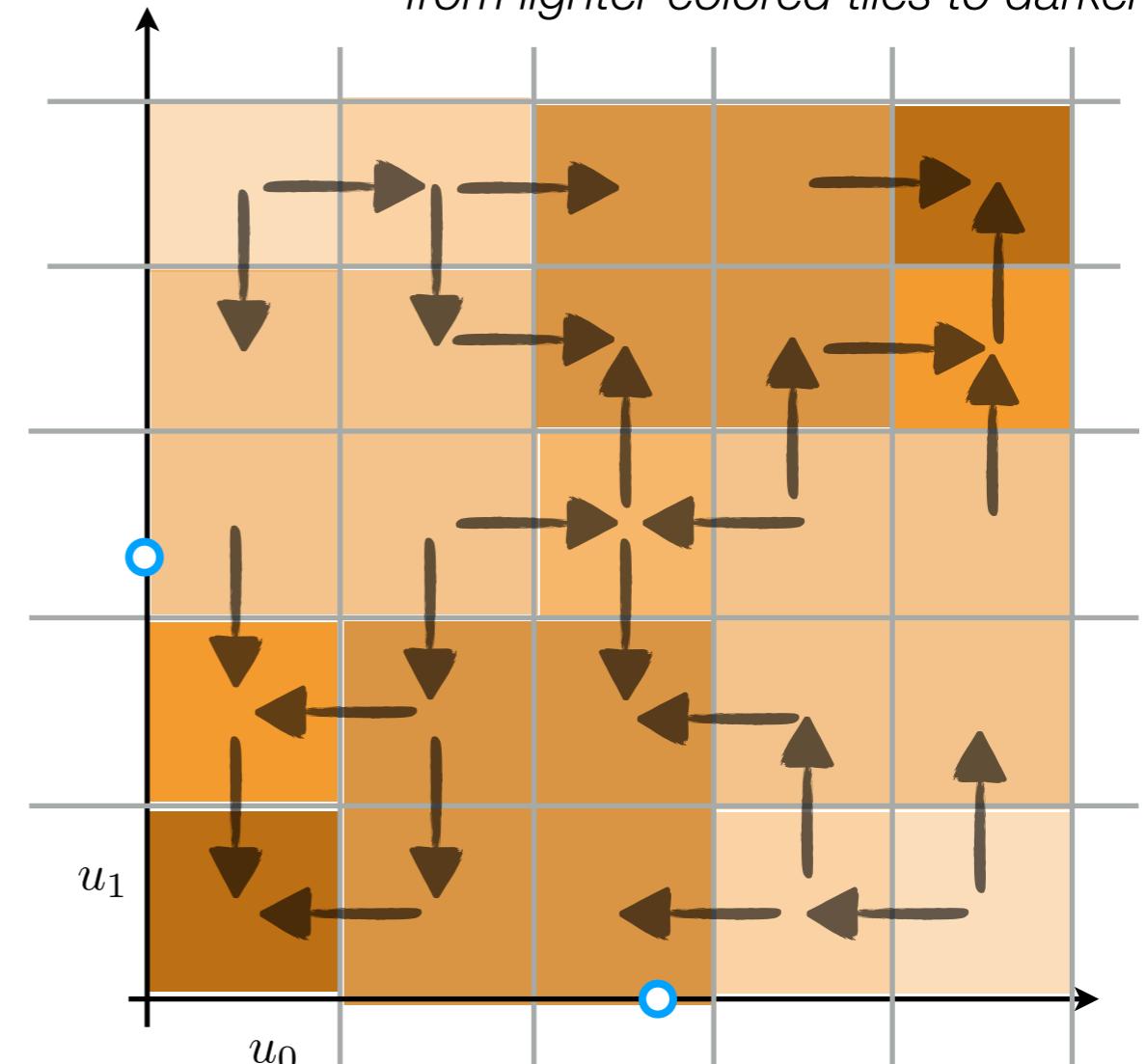
van den Berg, Ghrist, van der Vorst,  
Inventiones Math. 2003



Braided equilibrium solutions to parabolic PDE  
with periodic boundary conditions



*Solutions flow across boundary edges  
from lighter colored tiles to darker*



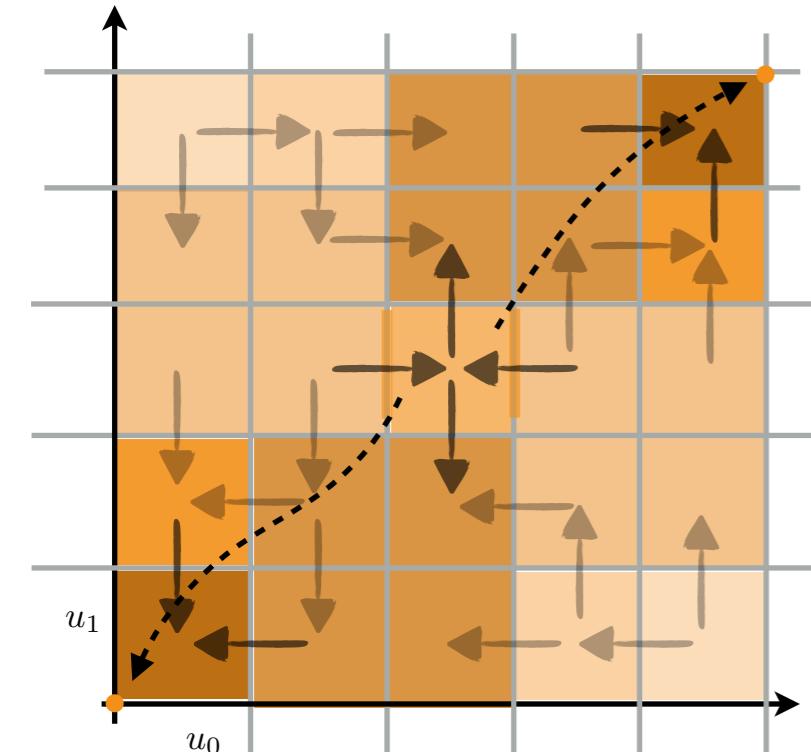
transversality model in  $\mathbb{R}^2$   
graded cubical complex in  $\mathbb{R}^2$

Fact: Nontrivial Conley indices imply existence of solutions to PDE

Fact: Nonzero entry in connection matrix between adjacent elements proves existence of connecting orbit

# braids i

$$(X, \leq) \xrightarrow{\nu} (P, \leq)$$

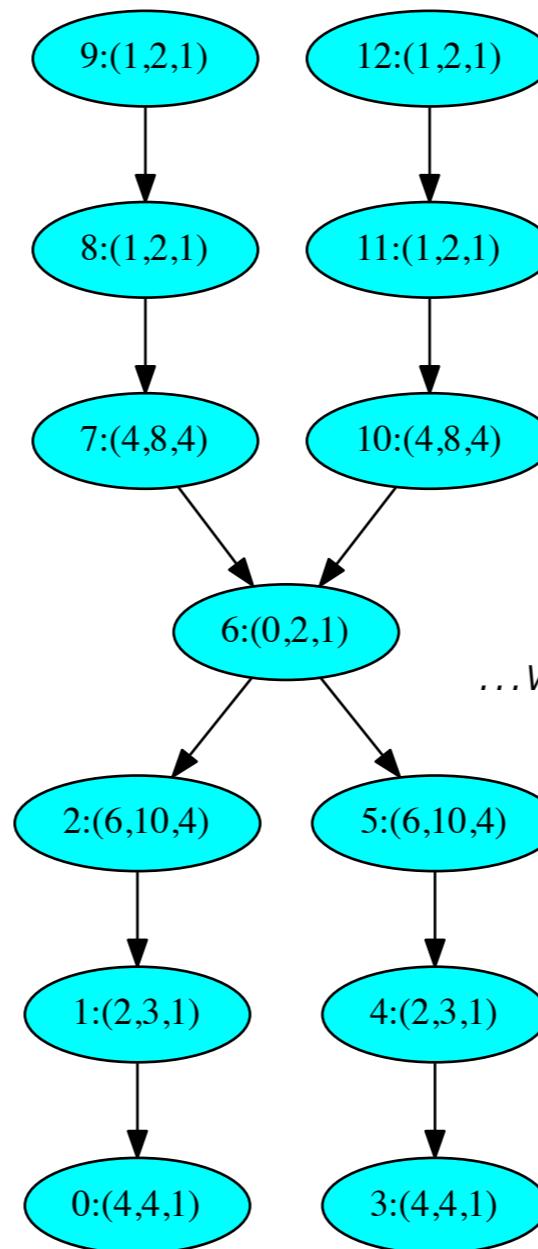


Conley complex



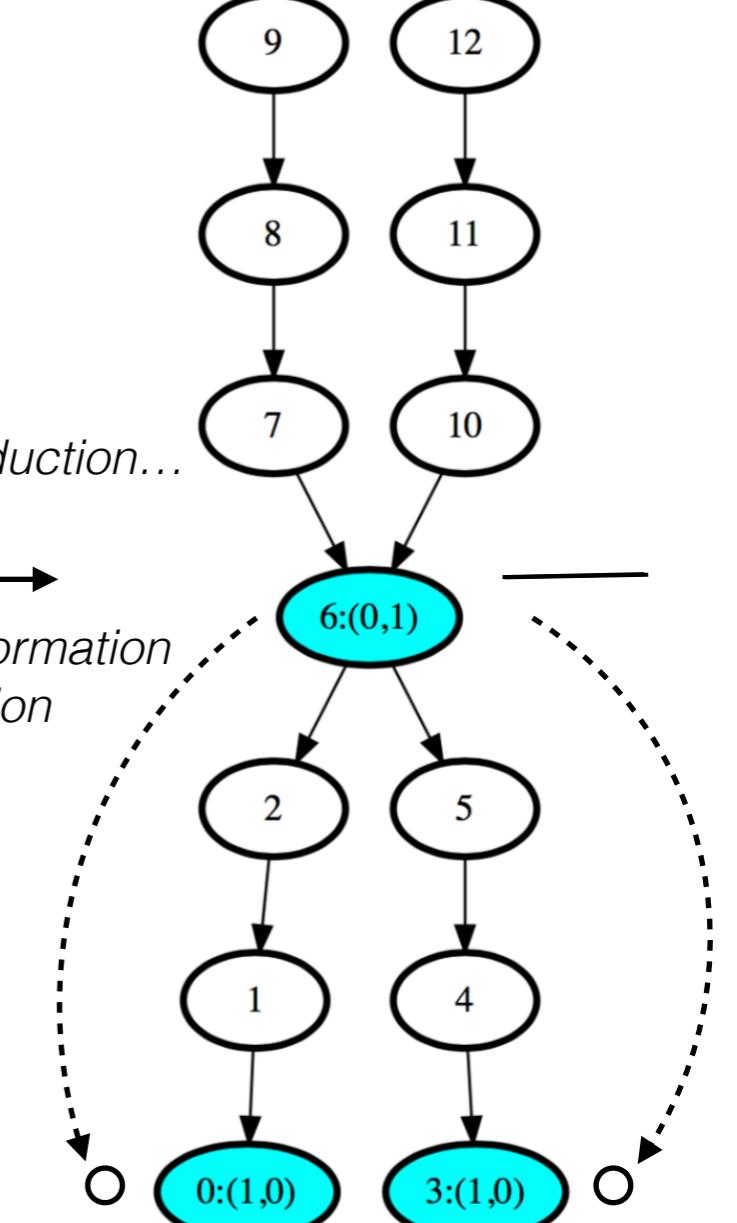
$$\Delta^M = \begin{matrix} & 0 & 3 & 6 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 & 0 \end{matrix}$$

node index  
cell dim.

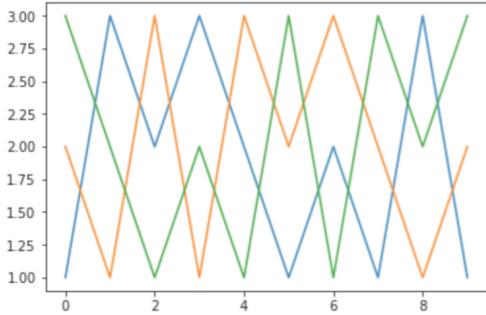


chain-level data compression  
144 cells  $\longleftrightarrow$  3 cells  
without loss of homological information

*data reduction...  
...without information reduction*



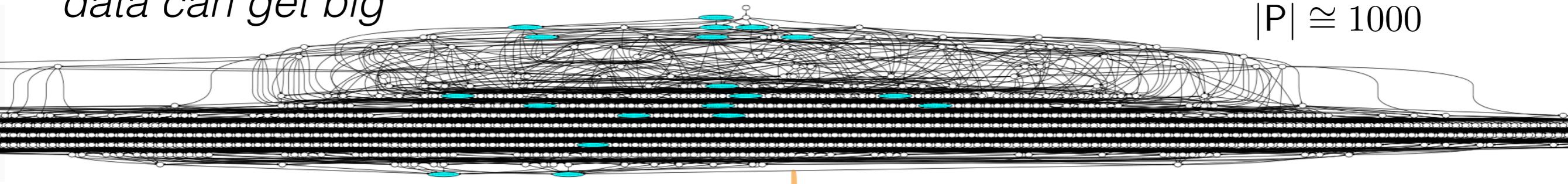
white nodes have no cells (trivial index)



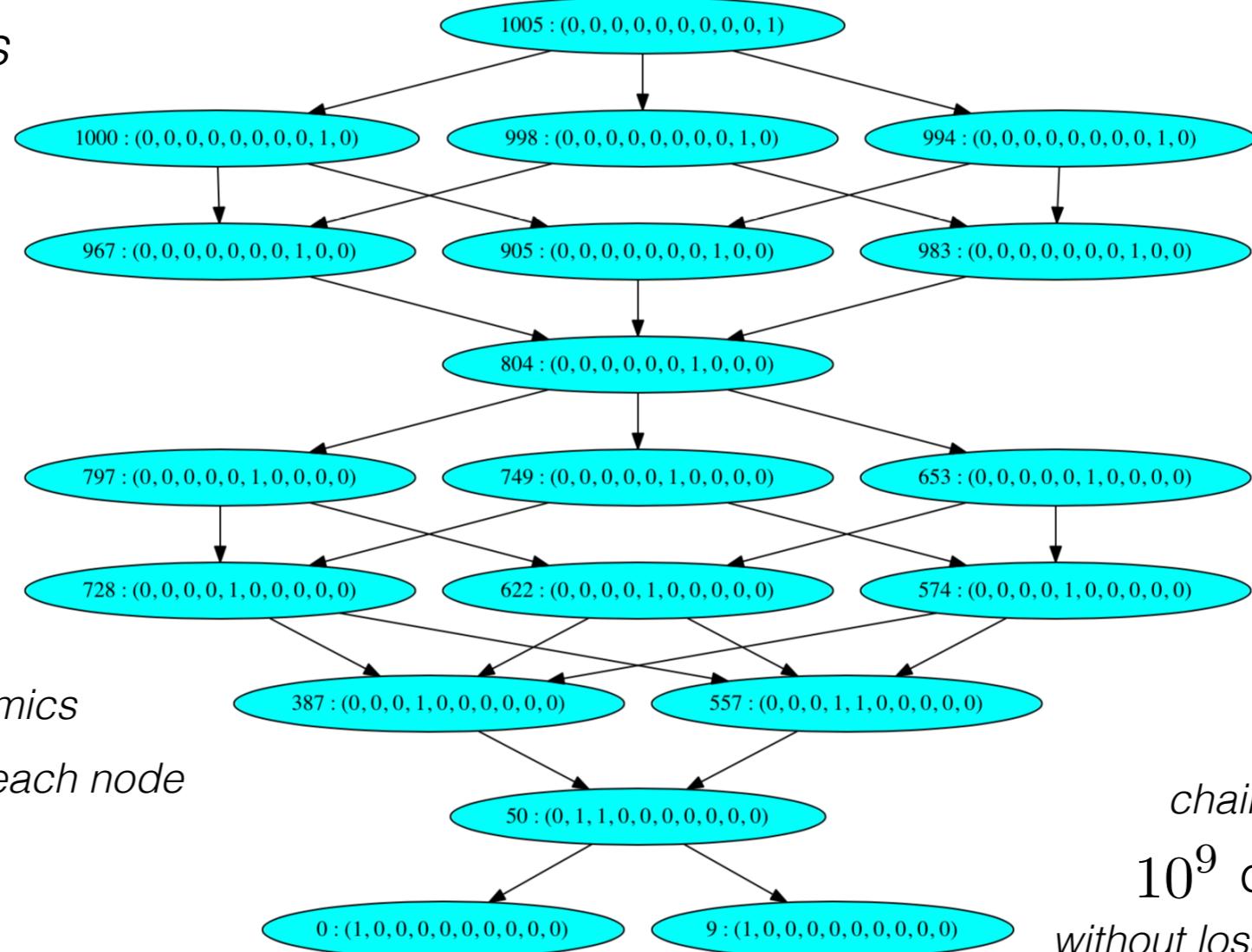
*data can get big*

## braids ii

*initial* graded cubical complex in  $\mathbb{R}^9$   
 $10^9$  cells  
 $|P| \cong 1000$



*restrict poset to nodes  
with nontrivial index*



*Conley-Morse Graph*

*organizes global dynamics*

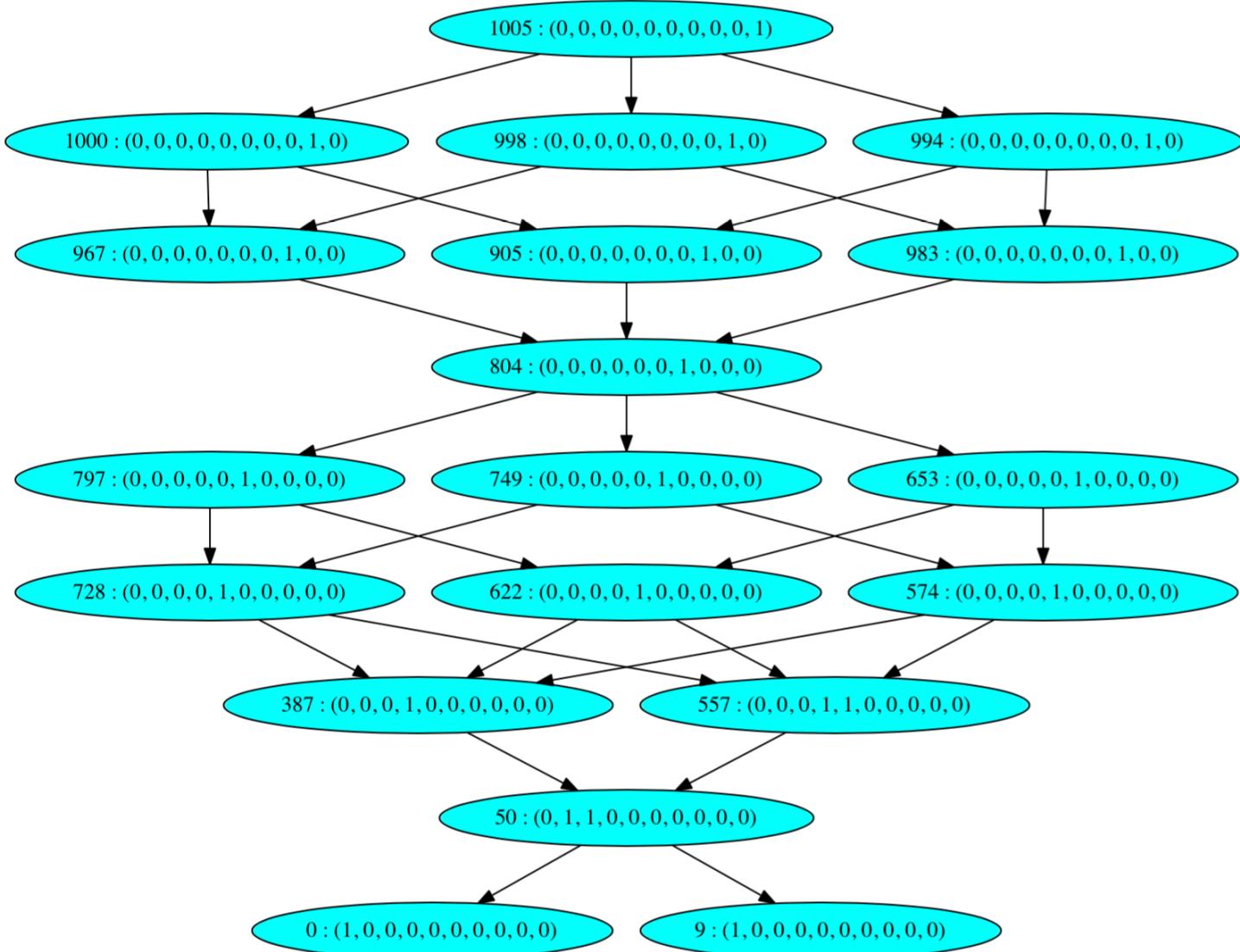
*Conley index for each node*

*Conley complex*  
21 cells  
19 nodes

*chain-level data compression*  
 $10^9$  cells  $\xleftrightarrow{} 21$  cells  
*without loss of homological information*

# braids iii

*order data*



*Conley-Morse Graph*

organizes global dynamics

Conley index for each node

*chain data*

**Connection Matrix Data**

=====

Boundaries of 0-cells in Conley complex:

```

0 : set()
1 : set()

```

Boundaries of 1-cells in Conley complex:

```

2 : {0, 1}

```

Boundaries of 2-cells in Conley complex:

```

3 : set()

```

Boundaries of 3-cells in Conley complex:

```

4 : {3}
5 : {3}

```

Boundaries of 4-cells in Conley complex:

```

6 : {4, 5}
7 : {4, 5}
8 : {4, 5}
9 : set()

```

Boundaries of 5-cells in Conley complex:

```

10 : {8, 9, 6}
11 : {8, 9, 7}
12 : {9, 6, 7}

```

Boundaries of 6-cells in Conley complex:

```

13 : set()

```

Boundaries of 7-cells in Conley complex:

```

14 : {13}
15 : {13}
16 : {13}

```

Boundaries of 8-cells in Conley complex:

```

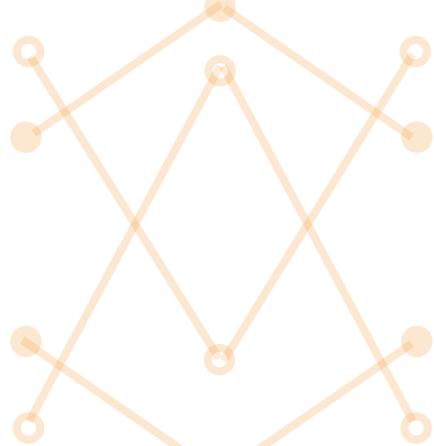
17 : {14, 15}
18 : {16, 14}
19 : {16, 15}

```

*Conley Complex*

connection matrix

boundaries can be queried from the data structure



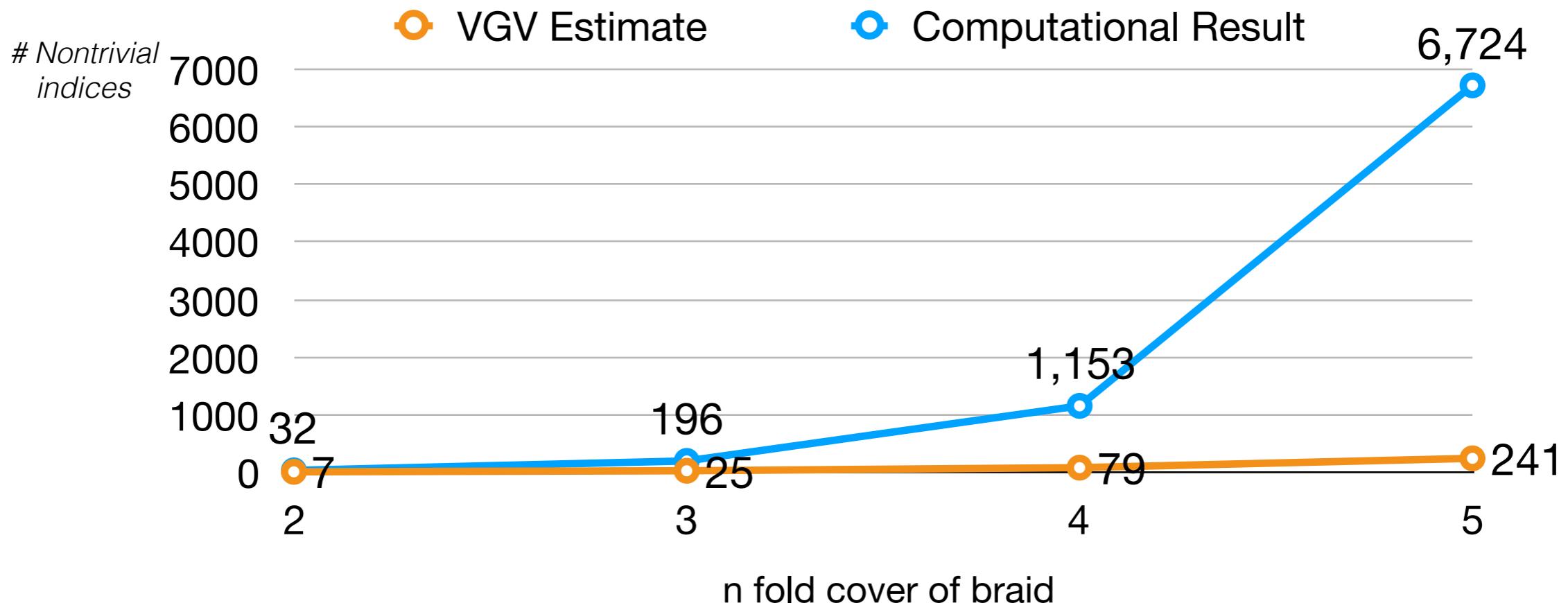
**Fig. 7.** The lifted skeleton of Example 1 with one free strand

from van den Berg, Ghrist, van der Vorst,  
Inventiones Math. 2003

### Theorem (van den Berg, Ghrist, Vandervorst)

For an  $n$ -fold cover of this braid there are at least  
 $3^n - 2$  nontrivial Conley indices

Compare this estimate to our computational result:



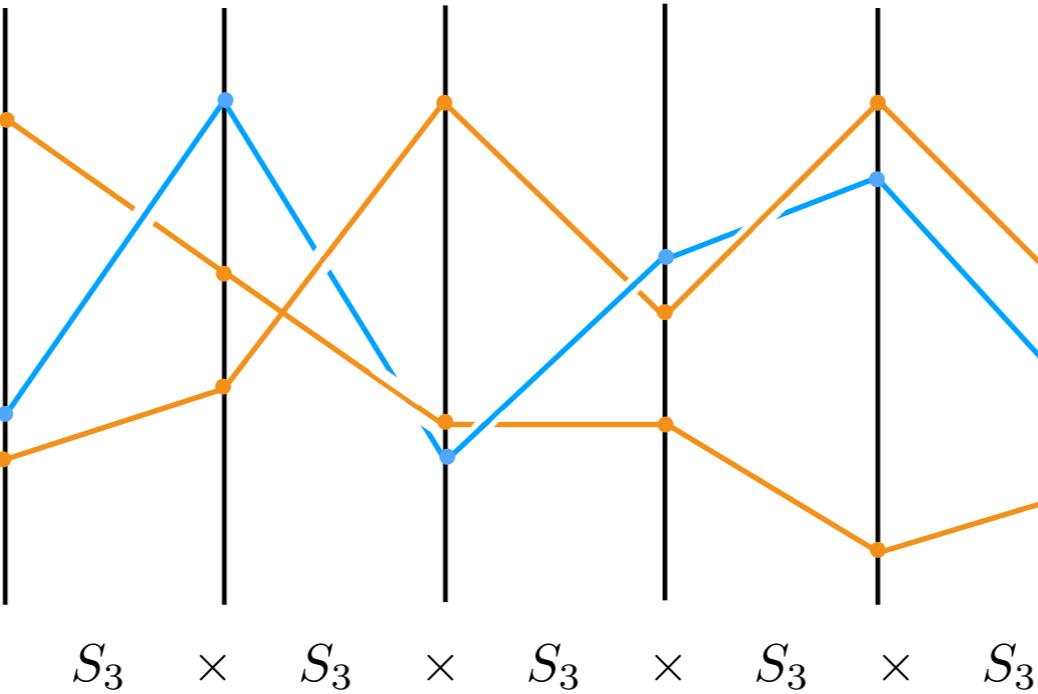
Remark: the 5-fold cover gives a 10-dimensional graded complex containing over 60 billion cells

application iii:

database approach to dynamics

look at all possible five dimensional braids on three strands

i.e. *all braids of this type*



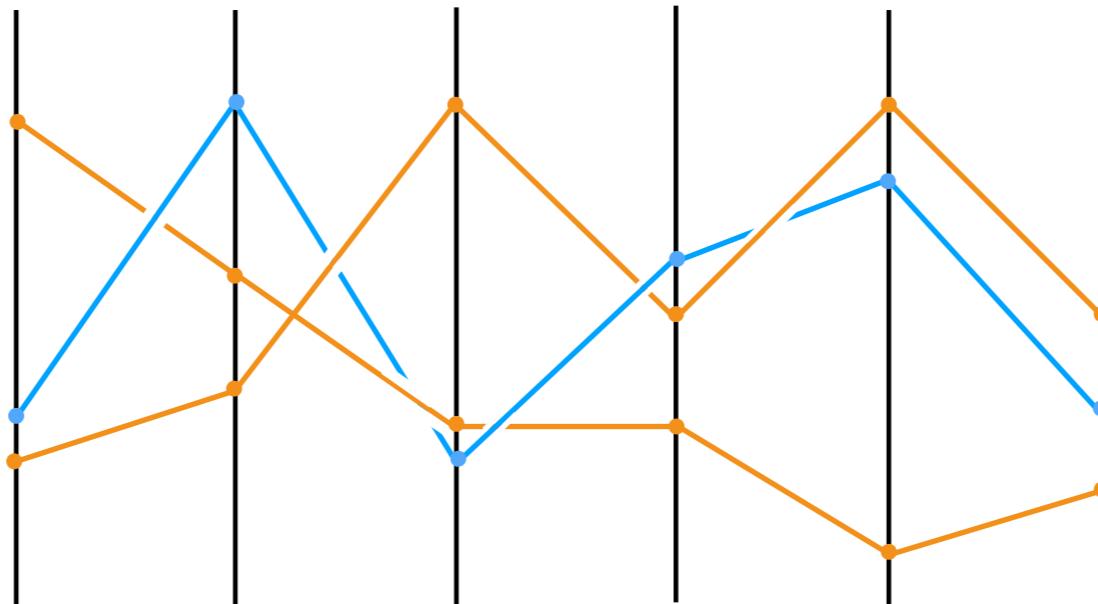
*compute database* of all Conley complexes

*each braid gives a graded cubical complex with 100,000 cells*

$$|S_3 \times S_3 \times S_3 \times S_3 \times S_3| = 7776$$

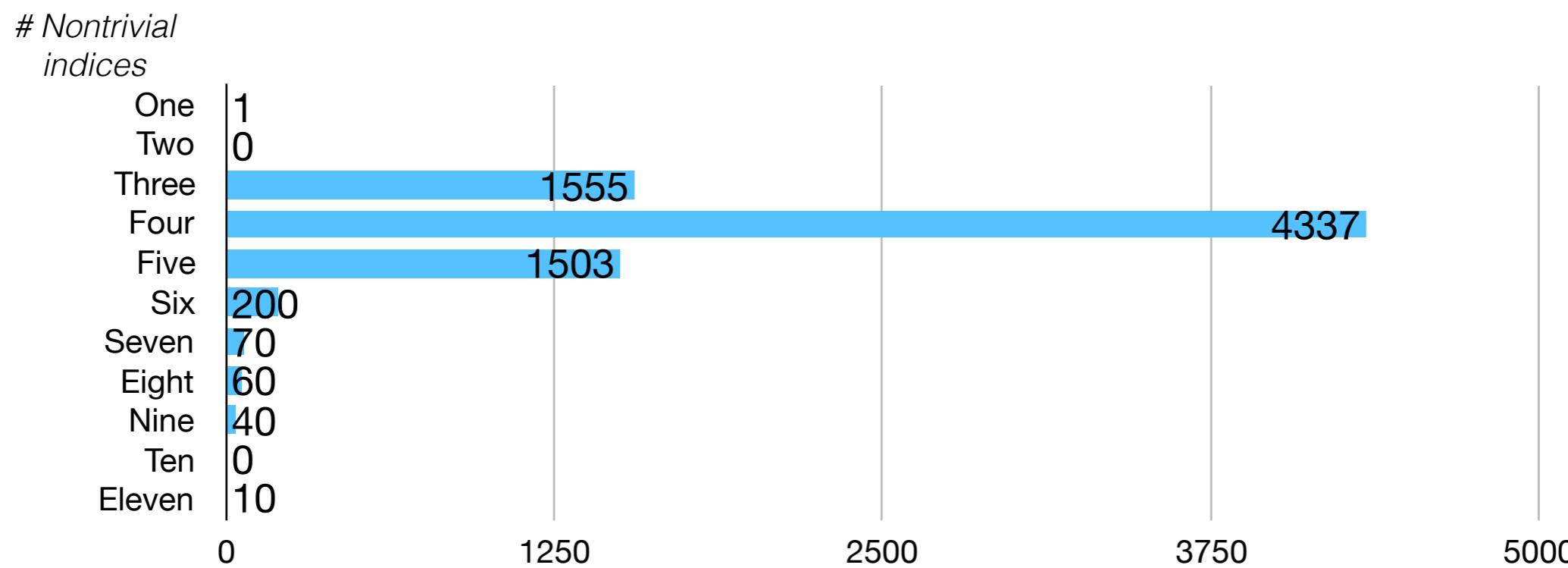
*database* of 7776 Conley complexes

database of all Conley complexes for 5-D braids on 3 strands



query the database:

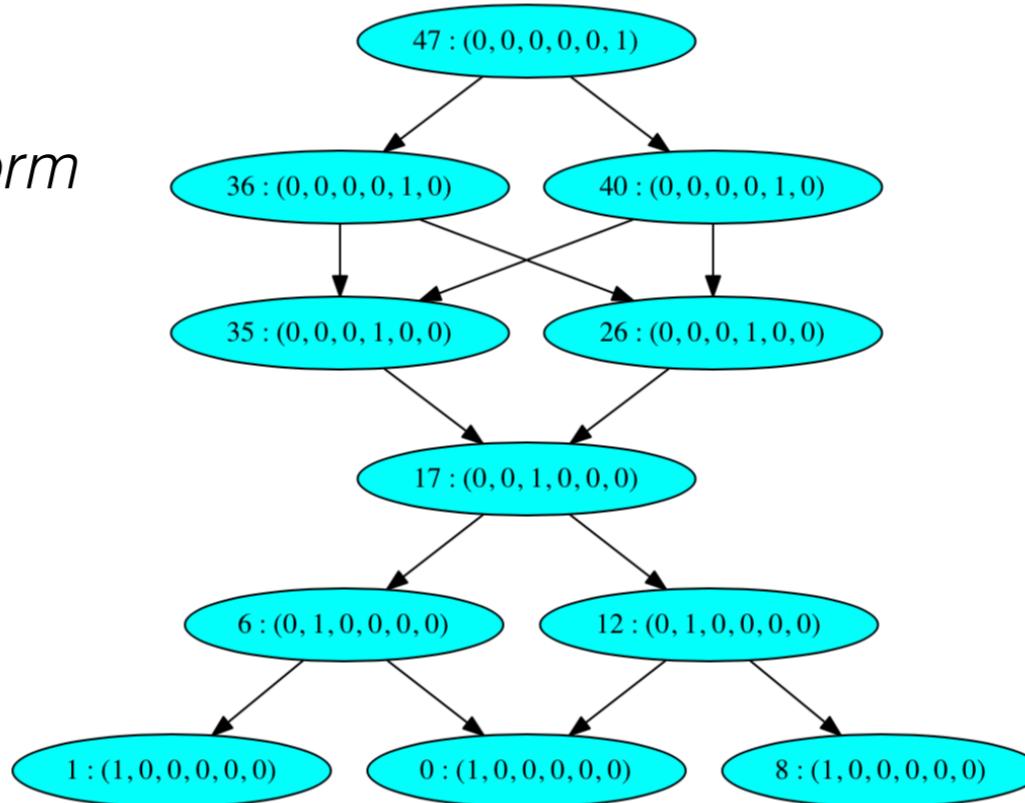
*'how many braids have precisely  $n$  nontrivial Conley indices?'*



query the database:

*'what are the Conley-Morse graphs that have 11 nontrivial indices?'*

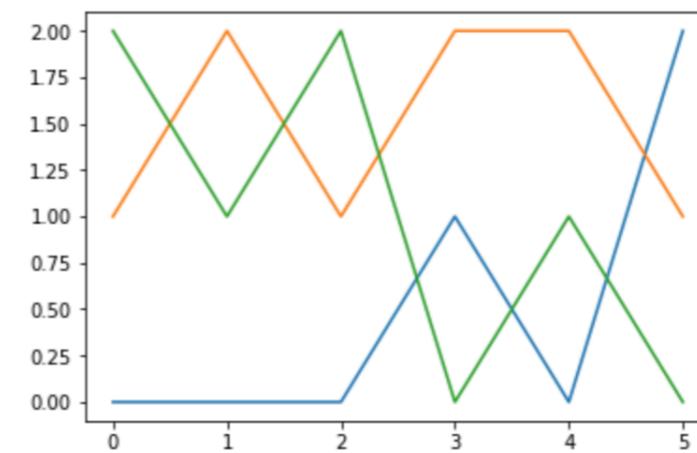
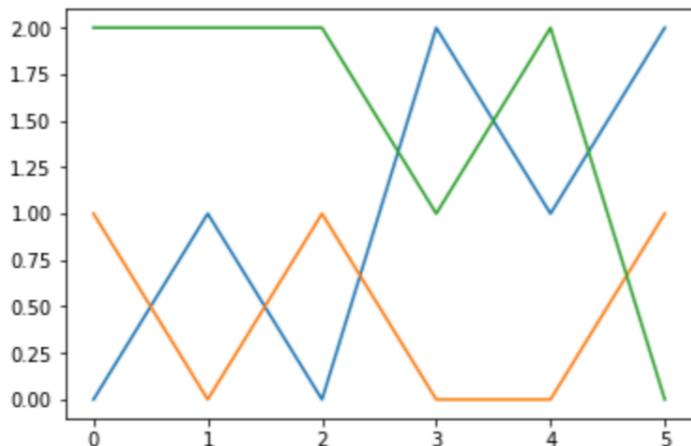
*all 10 are of this form*



query the database:

*'what are the braids that produce 11 nontrivial indices?'*

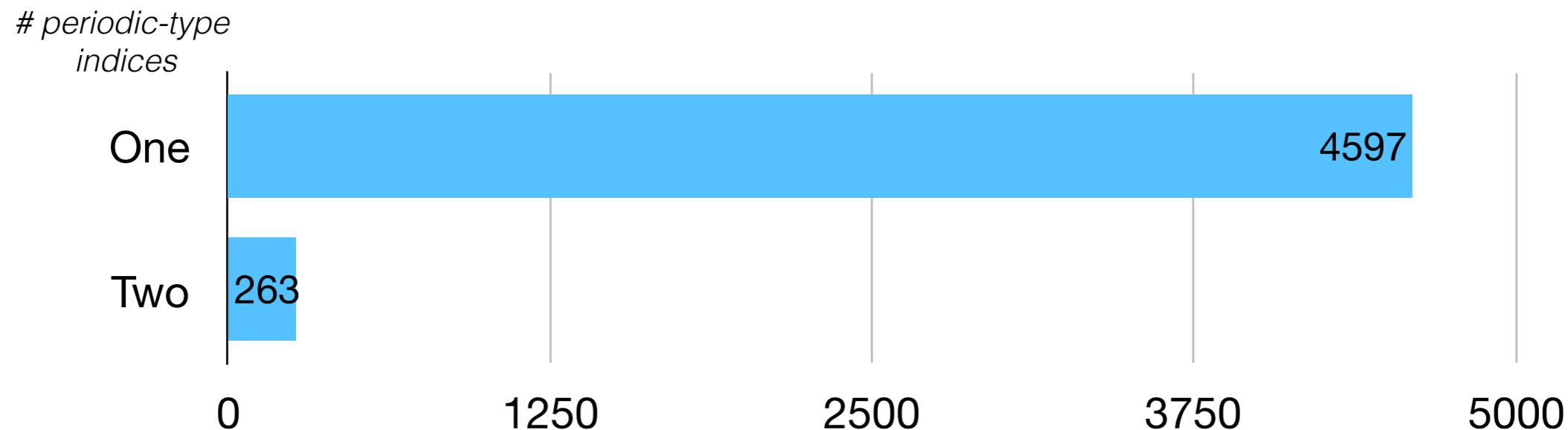
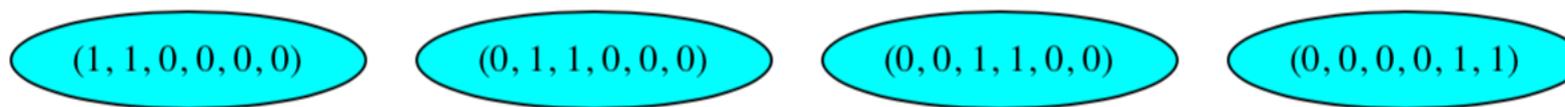
*the braids are translates of one of these two (dual) braids*



query the database:

*'how many braids produce a Conley index that looks like a periodic orbit?'*

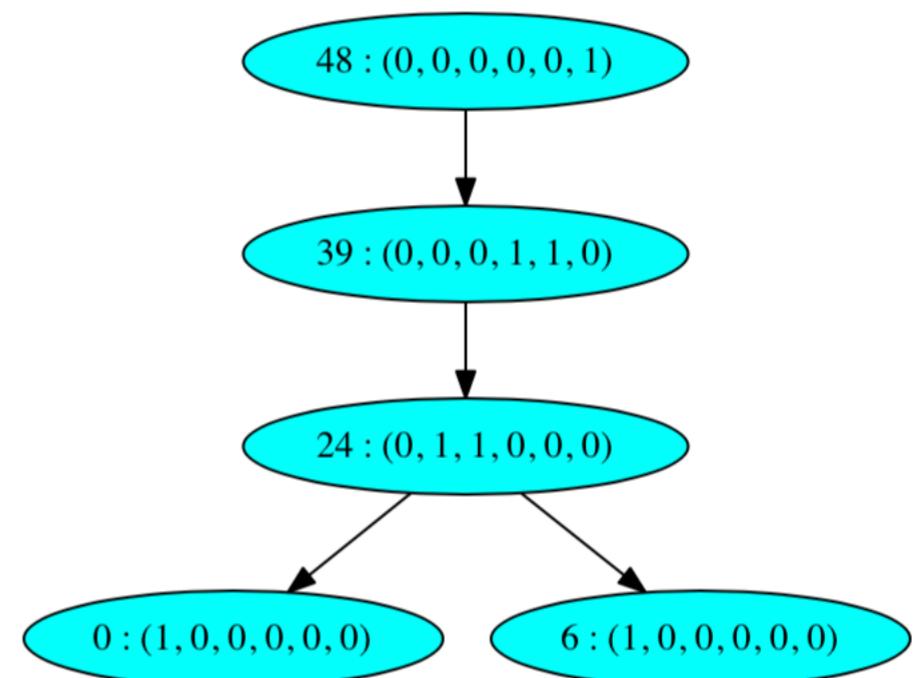
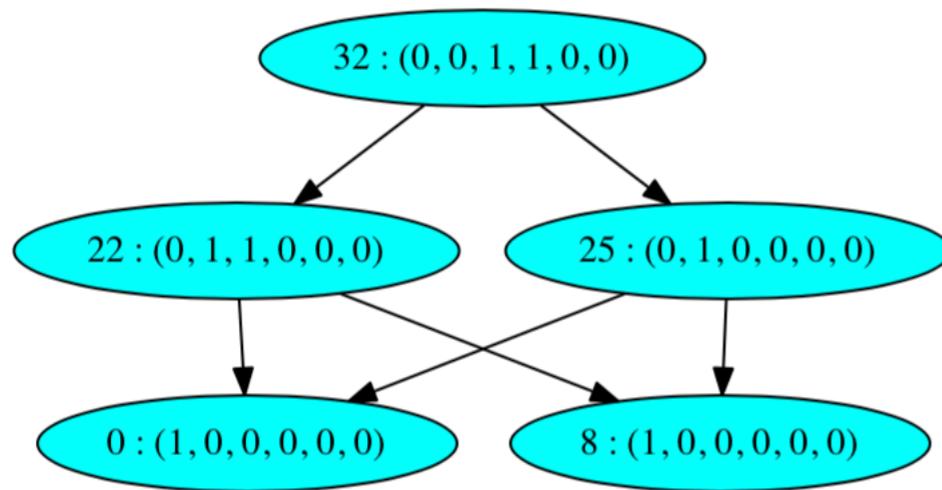
*i.e. contain one or more of the following indices*



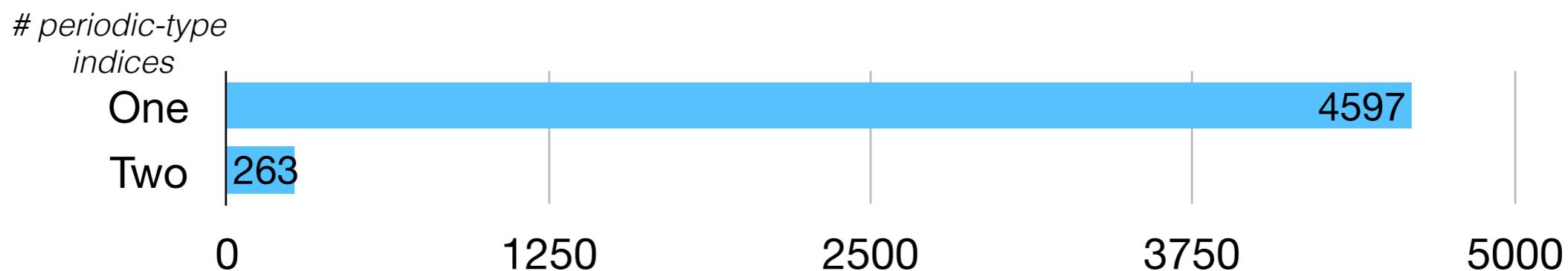
query the database:

'what are the Conley-Morse graphs that have two or more periodic-type indices?'

all of the 263 are of the following two forms:



upshot: we can examine not only high-dimensional braids,  
but also ask questions about dynamics in the space of braids



thank you for your attention

Collaborators:

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