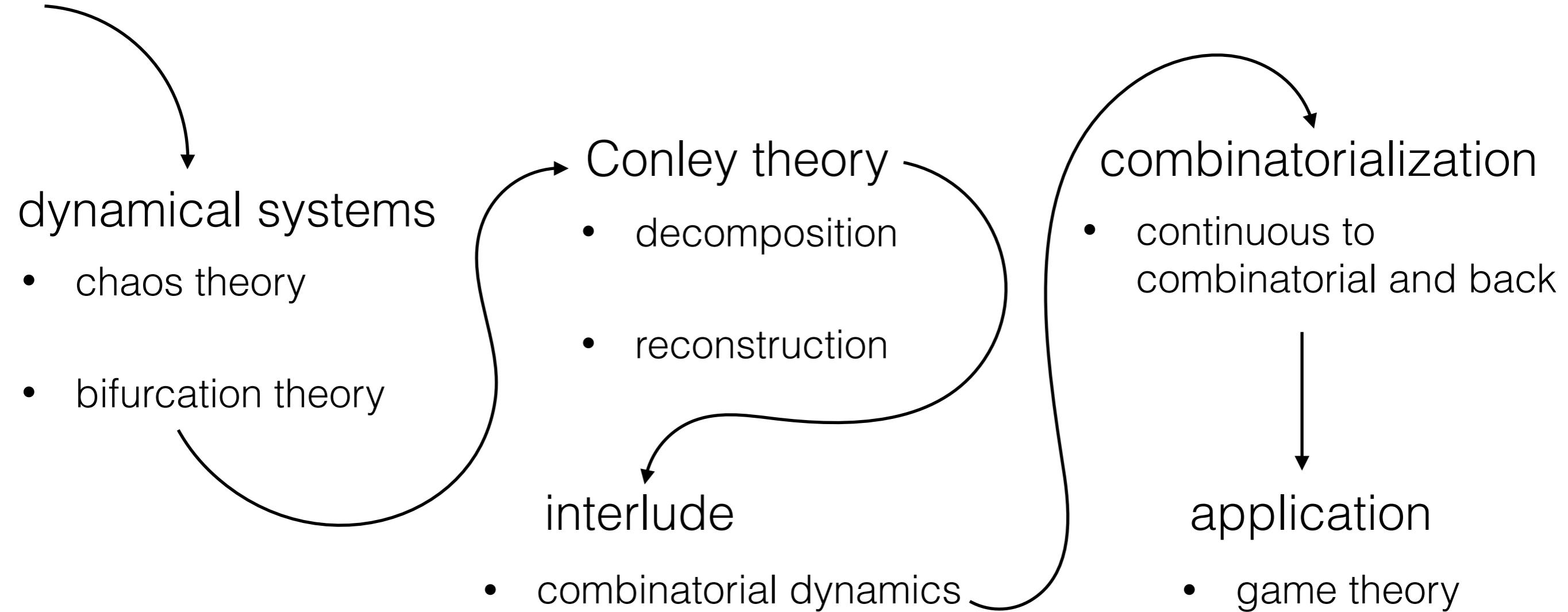


from dynamics to combinatorics and back again

*...an introduction to global dynamics via
computational Conley-Morse theory*

is there a map of the talk?



1. Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka, and P. Pilarczyk. A database schema for the analysis of global dynamics of multiparameter systems. *SIAM Journal on Applied Dynamical Systems*, 8(3):757–789, 2009.
 2. J. Bush, M. Gameiro, S. Harker, H. Kokubu, K. Mischaikow, I. Obayashi, and P. Pilarczyk. Combinatorial-topological framework for the analysis of global dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):047508, 2012.
 3. C. C. Conley. *Isolated invariant sets and the Morse index*. Number 38. American Mathematical Soc., 1978.
 4. J. Franks and D. Richeson. Shift equivalence and the conley index. *Transactions of the American Mathematical Society*, 352(7):3305–3322, 2000.
 5. W. D. Kalies, K. Mischaikow, and R. Vandervorst. An algorithmic approach to chain recurrence. *Foundations of Computational Mathematics*, 5(4):409–449, 2005.
 6. K. Mischaikow. Conley index theory. In *Dynamical systems*, pages 119–207. Springer, 1995.
 7. K. Mischaikow. Topological techniques for efficient rigorous computation in dynamics. *Acta Numerica*, 11:435–477, 2002.
 8. K. Mischaikow and M. Mrozek. Chaos in the lorenz equations: a computer-assisted proof. *Bulletin of the American Mathematical Society*, 32(1):66–72, 1995.
- ⋮ ⋮ ⋮

dynamical systems

on butterflies in your simulation,
can proverbial straws be more catastrophic than plastic ones?
why it's impossible to predict the weather.

given a metric space X ‘phase space’

(for the purposes of this talk) a **dynamical system** is a continuous map

$$f : X \rightarrow X$$

dynamical systems often include **parameters**:

$$f : X \times \Lambda \rightarrow X$$

typical **questions** one may ask of a dynamical system include:

for a particular initial condition, what happens in the long run?
how does the long run behavior depend upon the initial conditions?

...on the parameters within the model?

are there fixed points? i.e. solutions to $f(x) = x$

are there periodic orbits? i.e. solutions to $f^n(x) = x$

are there multiple attractors?

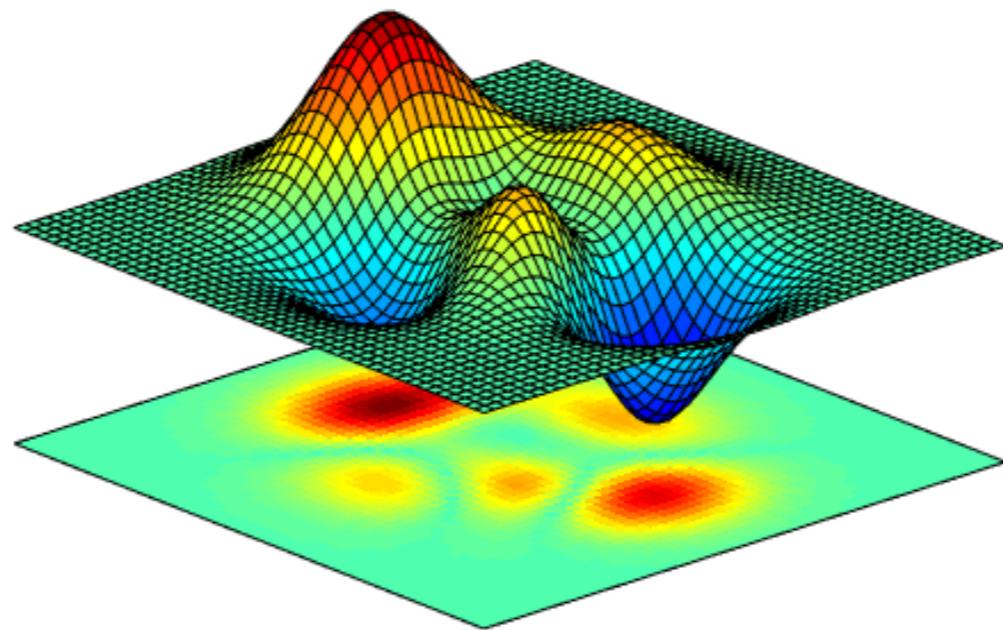
:

dynamics can be simple (gradient-like)

$$f : X \rightarrow \mathbb{R}$$

$$\dot{x} = -\nabla f(x)$$

$$x_{n+1} = x_n - \gamma \nabla f(x_n)$$



dynamics: equilibria + heteroclinic orbits

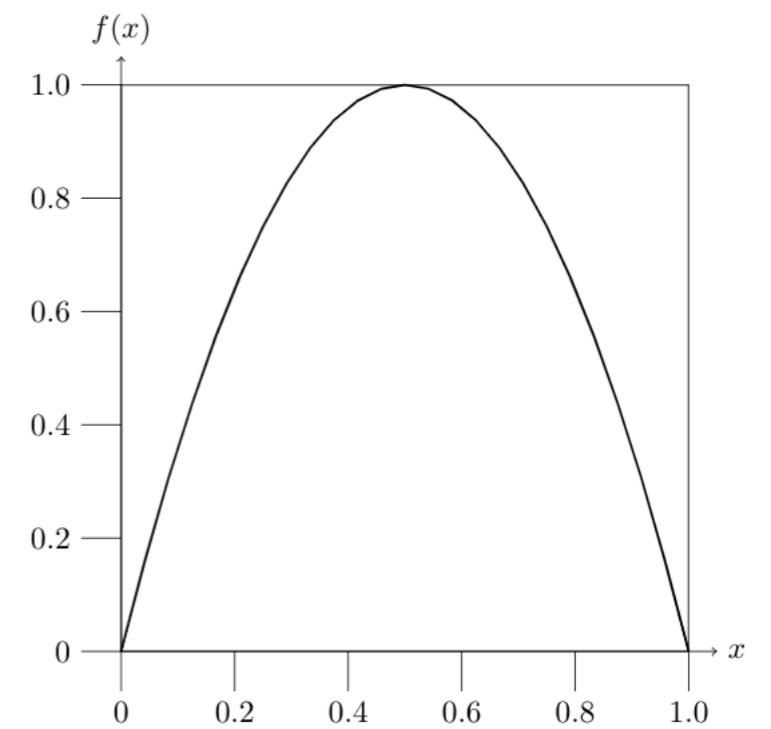
dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

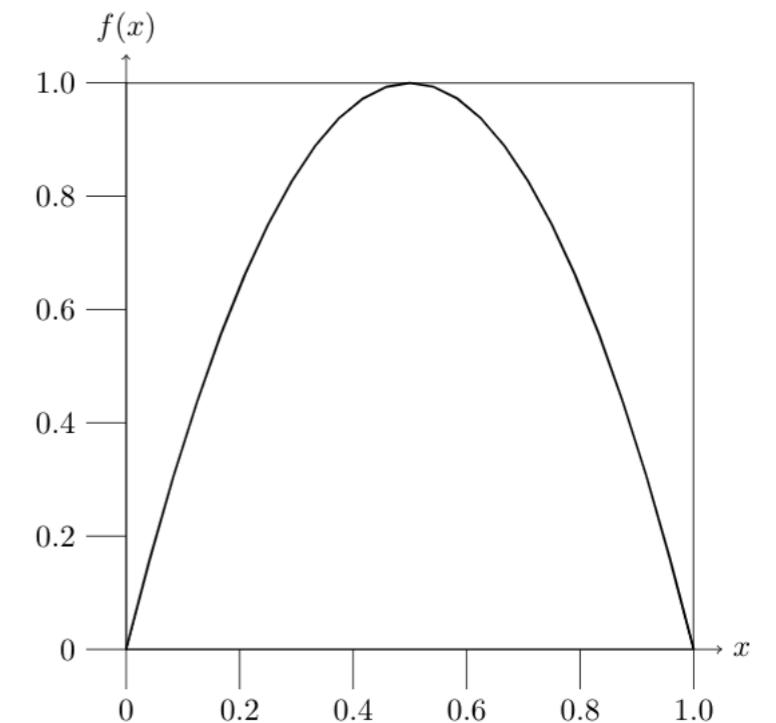
$$x_{n+1} = f(x_n)$$

deviations in
the 15th decimal

$$x_0 = 0.3333333333333333$$

a flap of a butterfly's wings in Brazil...

$$x_0 = 0.3333333333333330$$



dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

$$x_{n+1} = f(x_n)$$

deviations in
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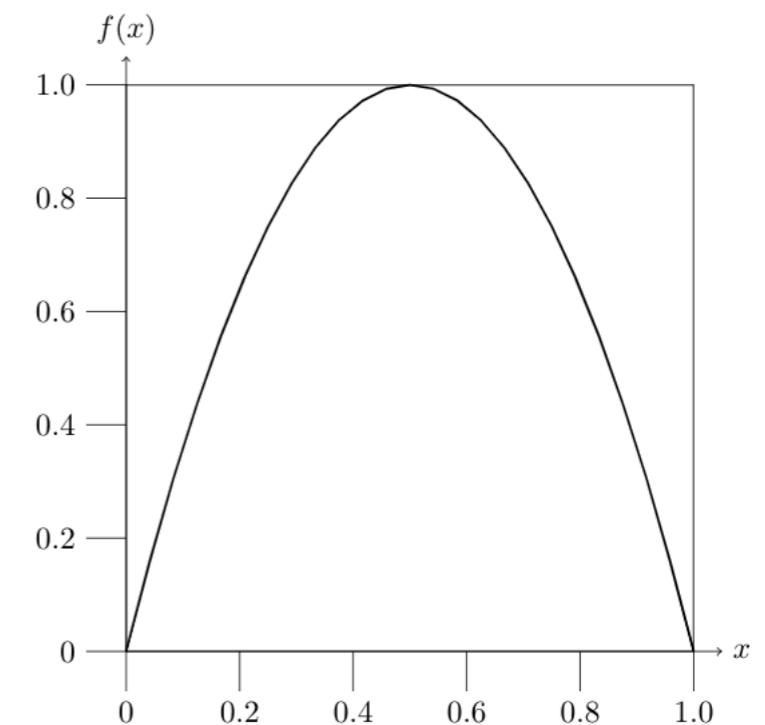
$$x_0 = 0.333333333333333$$

$$x_1 = 0.888888888888889$$

$$x_0 = 0.333333333333330$$

$$x_1 = 0.888888888888885$$

a flap of a butterfly's wings in Brazil...



dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

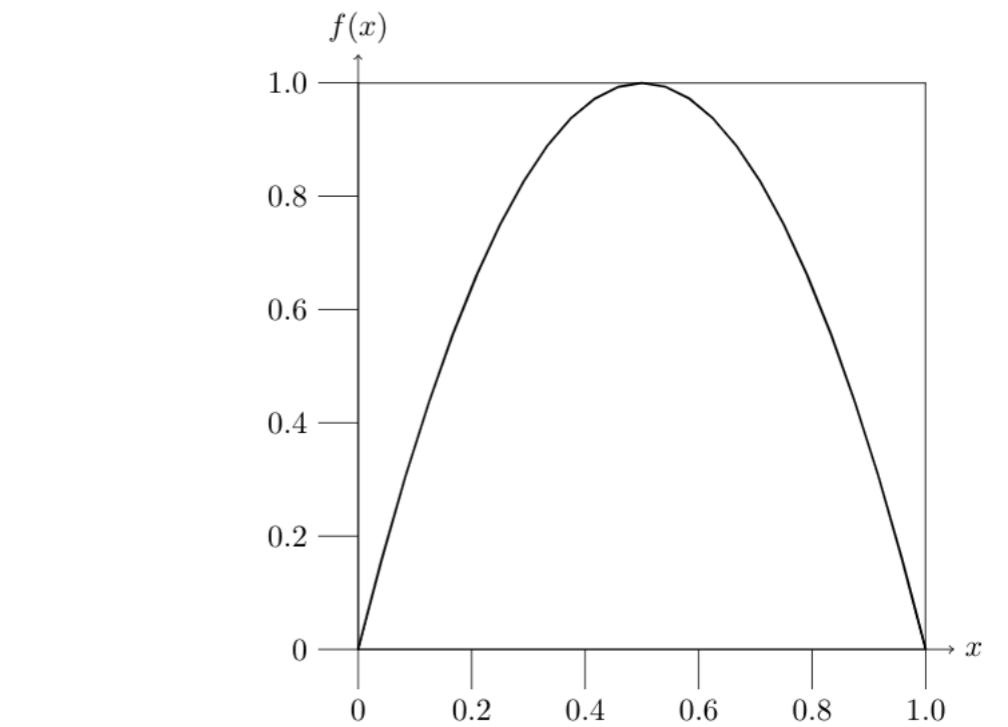
$$x_{n+1} = f(x_n)$$

deviations in
the 15th decimal

$$x_0 = 0.333333333333333$$

$$x_1 = 0.888888888888889$$

$$x_2 = 0.395061728395062$$



a flap of a butterfly's wings in Brazil...

$$x_0 = 0.333333333333330$$

$$x_1 = 0.888888888888885$$

$$x_2 = 0.395061728395075$$

dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

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Logistic map

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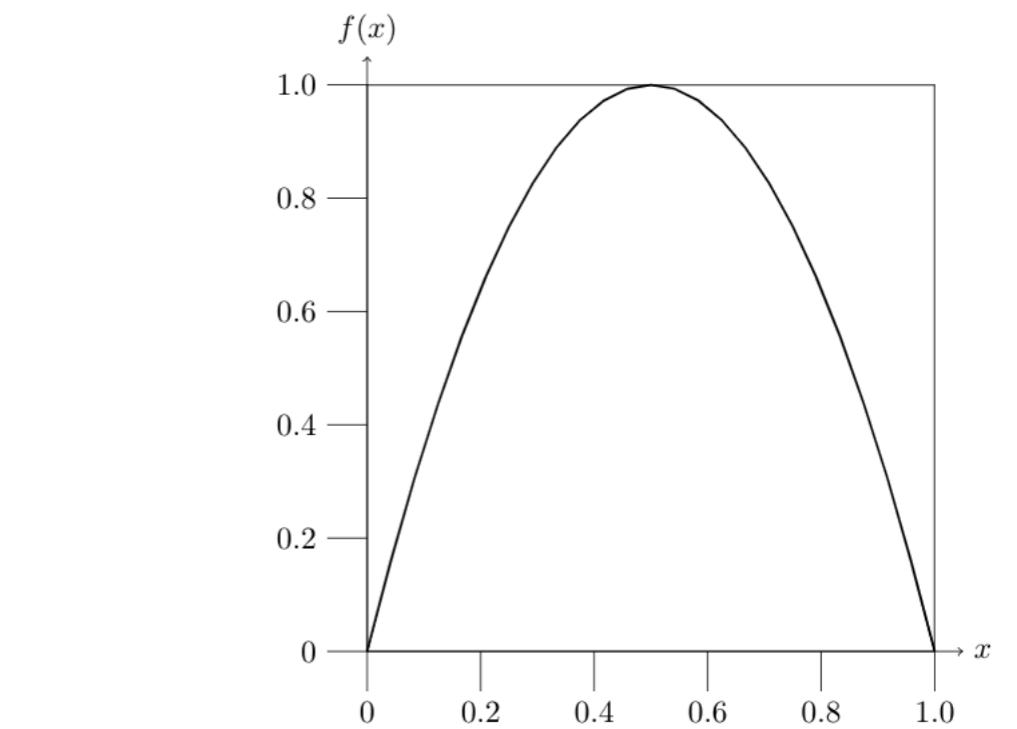
deviations in
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$$x_0 = 0.333333333333333$$

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$$x_2 = 0.395061728395062$$

$$x_3 = 0.955951836610273$$



a flap of a butterfly's wings in Brazil...

$$x_0 = 0.333333333333330$$

$$x_1 = 0.888888888888885$$

$$x_2 = 0.395061728395075$$

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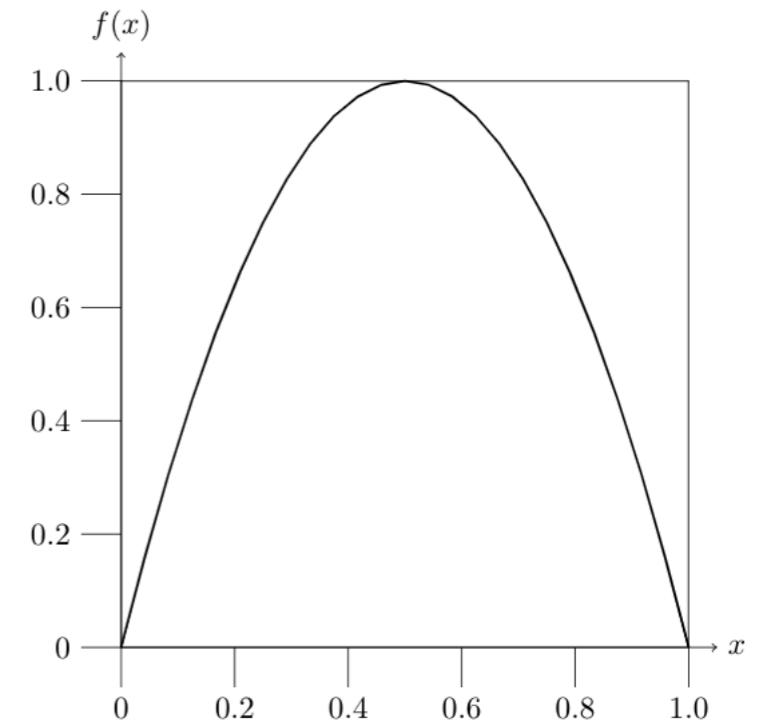
dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



deviations in
the 15th decimal

a flap of a butterfly's wings in Brazil...

$$x_0 = 0.3333333333333333$$

$$x_0 = 0.3333333333333330$$

$$x_1 = 0.8888888888888889$$

$$x_1 = 0.8888888888888885$$

$$x_2 = 0.395061728395062$$

$$x_2 = 0.395061728395075$$

$$x_3 = 0.955951836610273$$

$$x_3 = 0.955951836610284$$

⋮

⋮

$$x_{15} = 0.695026128241317$$

$$x_{15} = 0.695026128347429$$

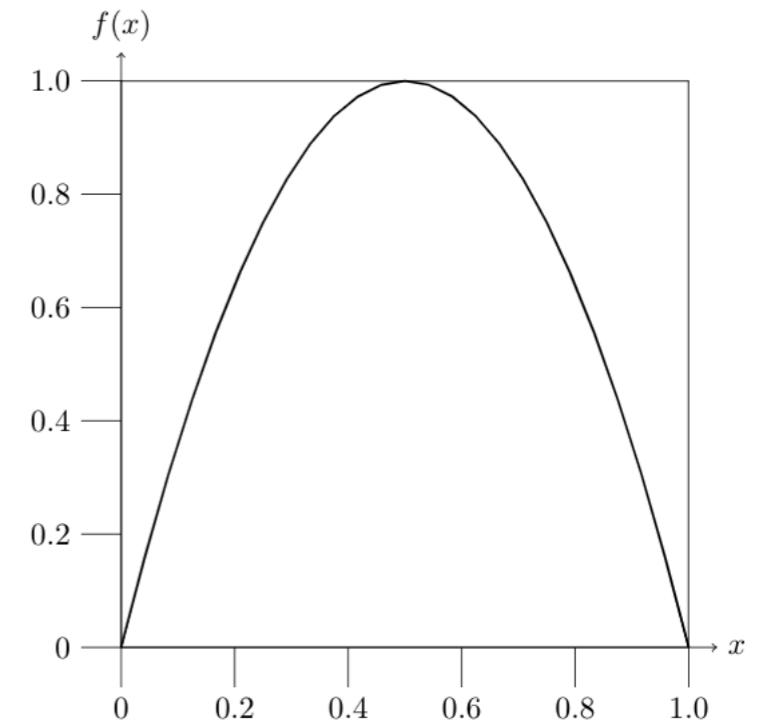
dynamics can be complex

$$f: [0, 1] \rightarrow [0, 1]$$

$$f(x) = 4x(1 - x)$$

Logistic map

$$x_{n+1} = f(x_n)$$



deviations in
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a flap of a butterfly's wings in Brazil...

$$x_0 = 0.3333333333333333$$

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$$x_2 = 0.395061728395062$$

$$x_2 = 0.395061728395075$$

$$x_3 = 0.955951836610273$$

$$x_3 = 0.955951836610284$$

⋮

⋮

$$x_{15} = 0.695026128241317$$

$$x_{15} = 0.695026128347429$$

⋮

⋮

$$x_{49} = 0.071160322456580$$

$$x_{49} = 0.906436654059206$$

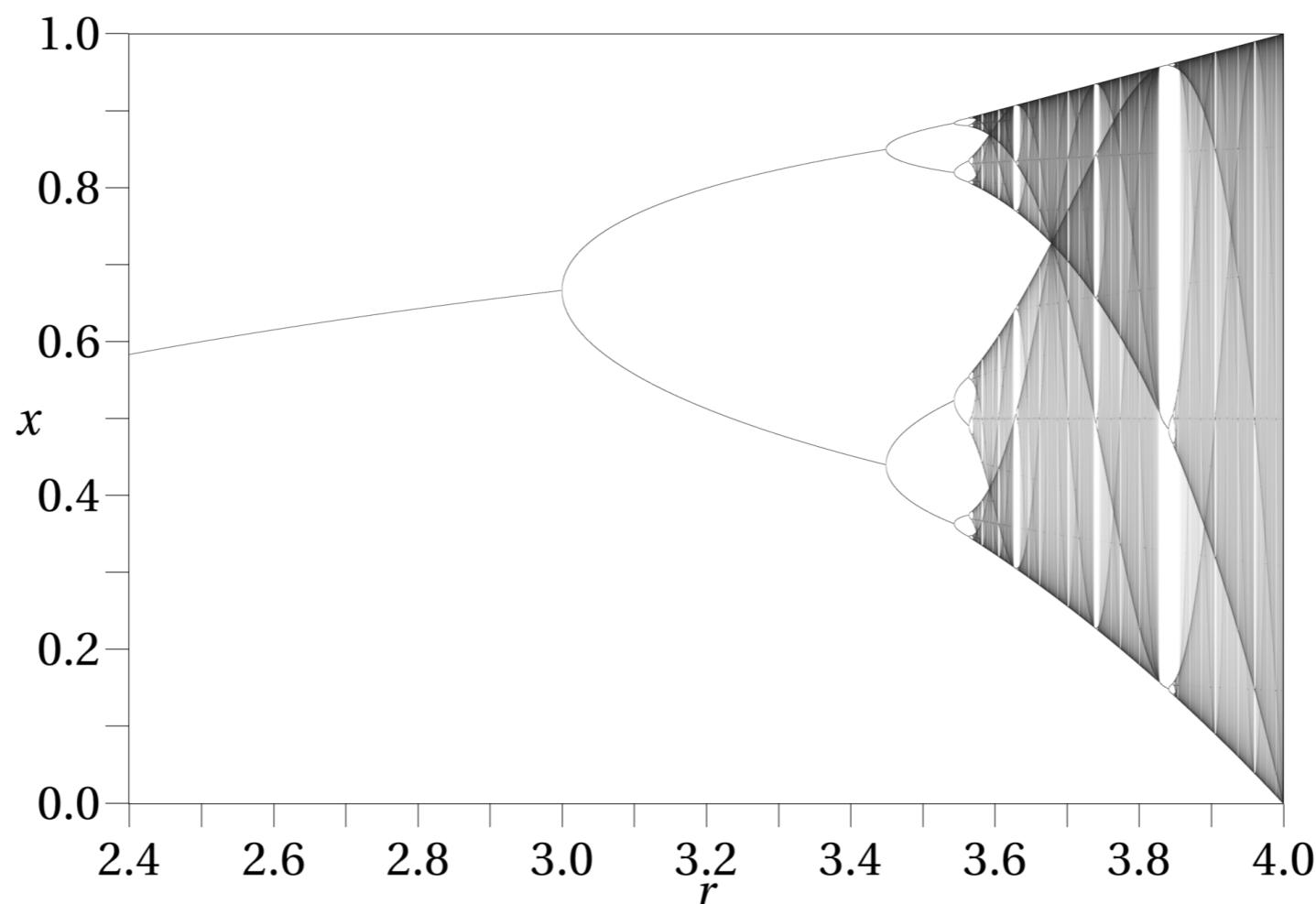
deviation is now on the
order of the entire phase space

...sets off a tornado in Texas.

dynamics can be complex II

$$x_{n+1} = f_r(x) = rx_n(1 - x_n)$$

r is a model parameter



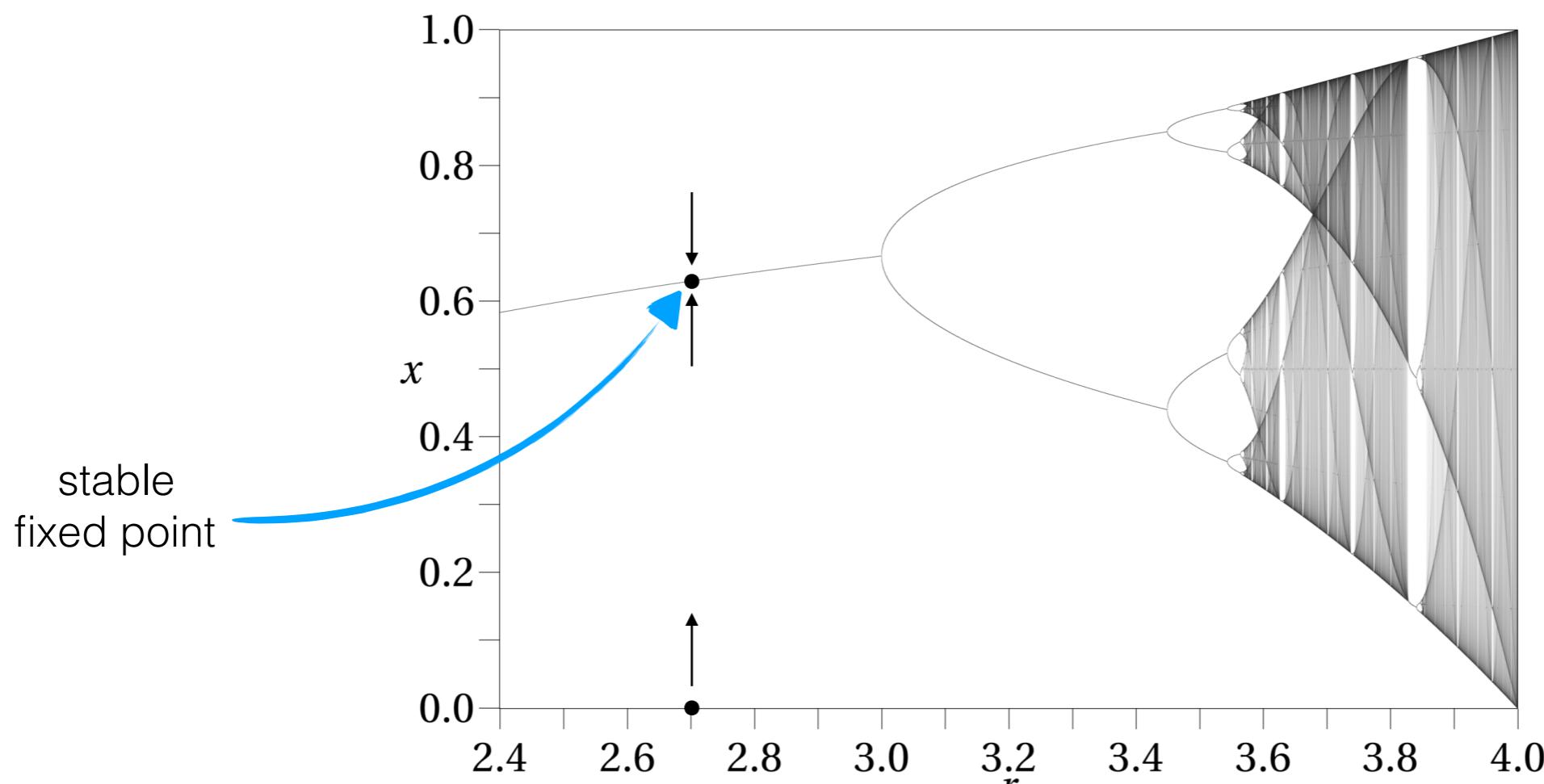
'bifurcation diagram'

plotted are values visited asymptotically from almost all initial conditions

dynamics can be complex II

$$x_{n+1} = f_r(x) = rx_n(1 - x_n)$$

r is a model parameter



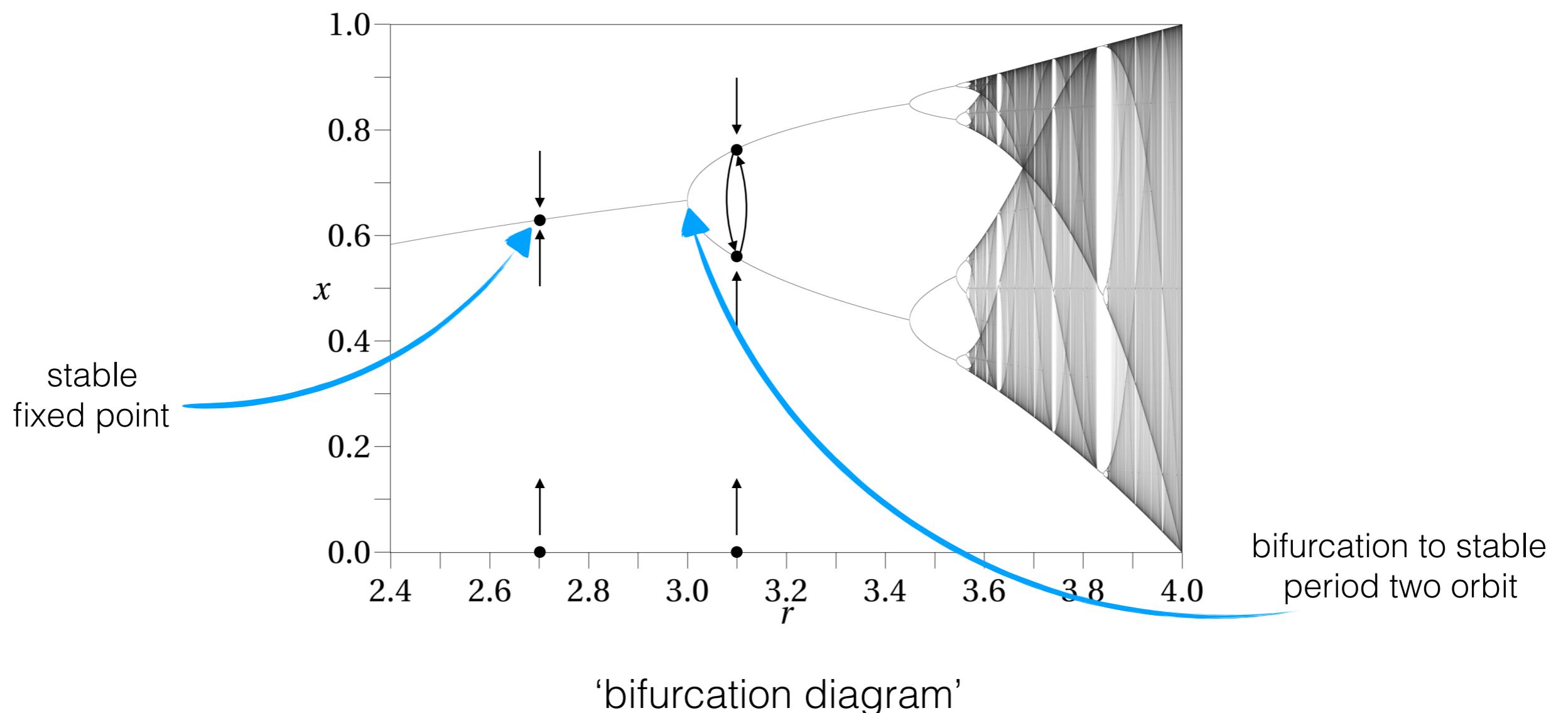
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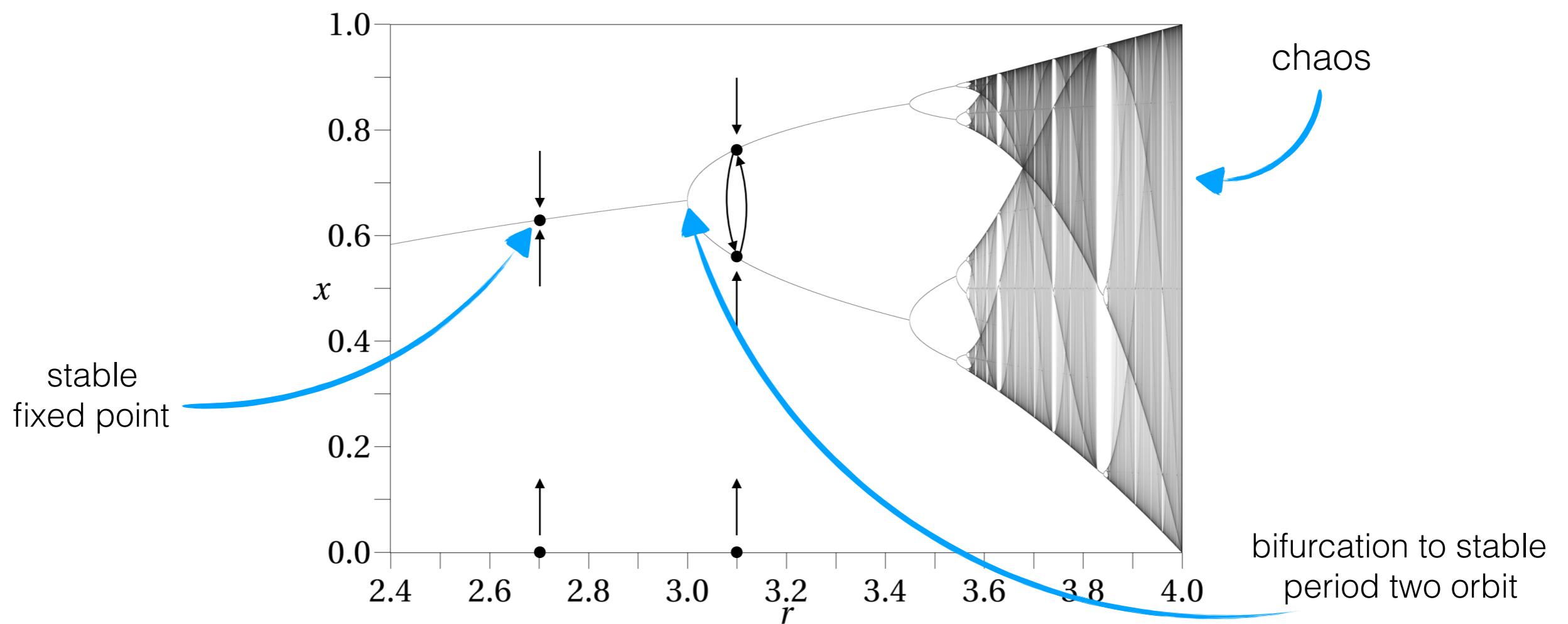


plotted are values visited asymptotically from almost all initial conditions

dynamics can be complex II

$$x_{n+1} = f_r(x) = rx_n(1 - x_n)$$

r is a model parameter



bifurcations: a single straw can break the camel's back
that is, a small perturbation in parameters can drastically change dynamics

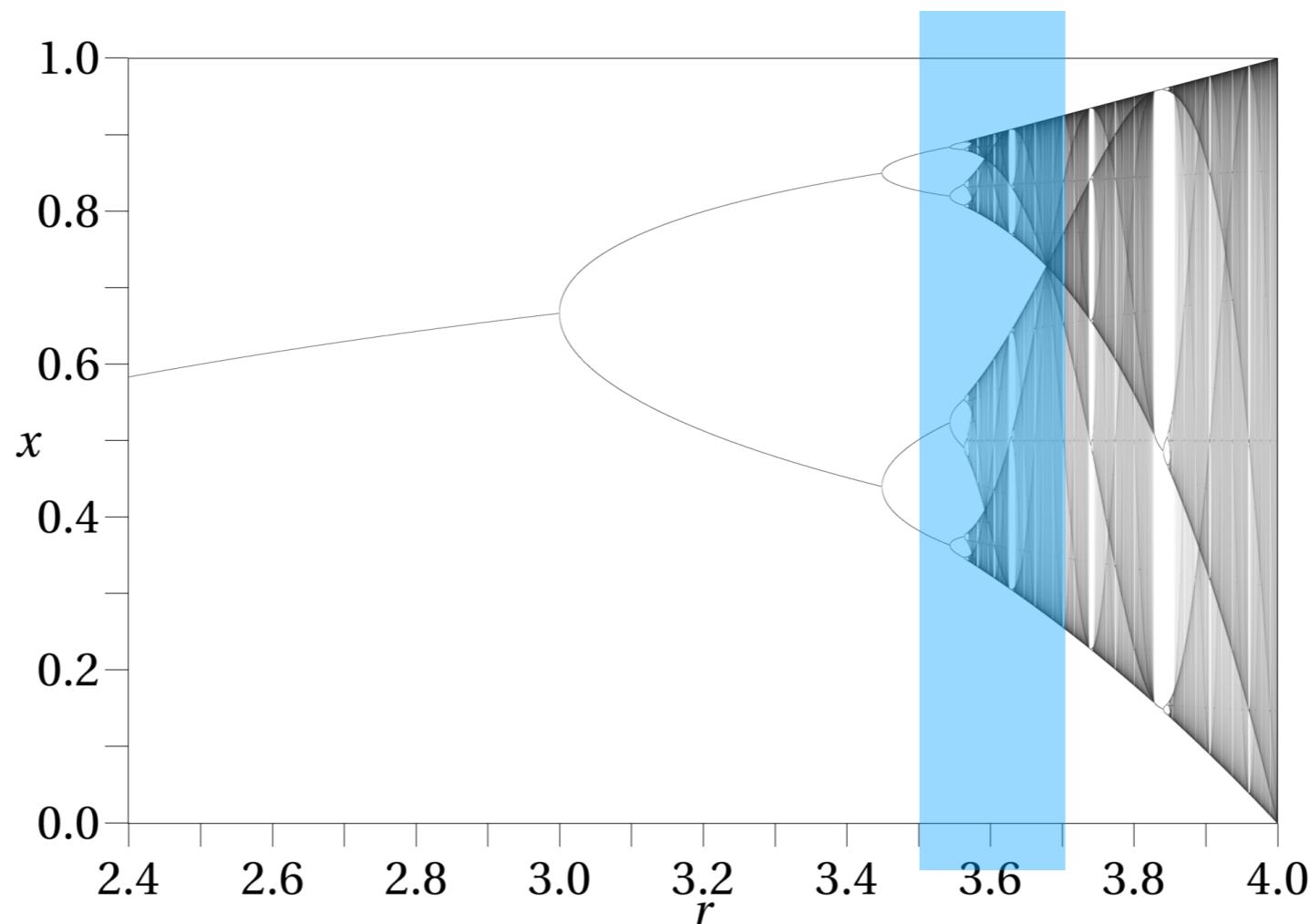
a thought experiment

suppose... that the logistic map is the perfect model
(say, for the onset of turbulence), and

suppose... that one can perform a perfect numerical
simulation...

*... it is still possible that the conclusions can be
wrong with high probability.*

suppose an experimentalist can measure r to within one decimal place



any computation probably will suggest the wrong dynamics

Conley's theory

decomposition + reconstruction

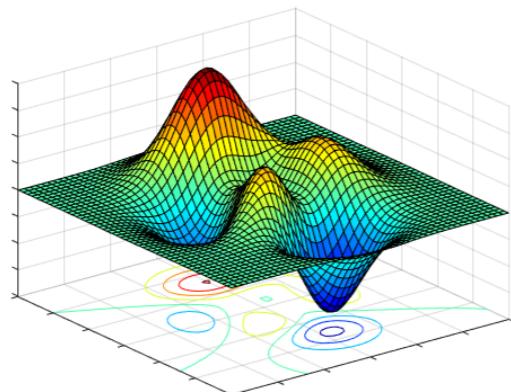
'...if such rough equations are to be of use it is necessary to study them in rough terms.'

C. Conley

decomposition

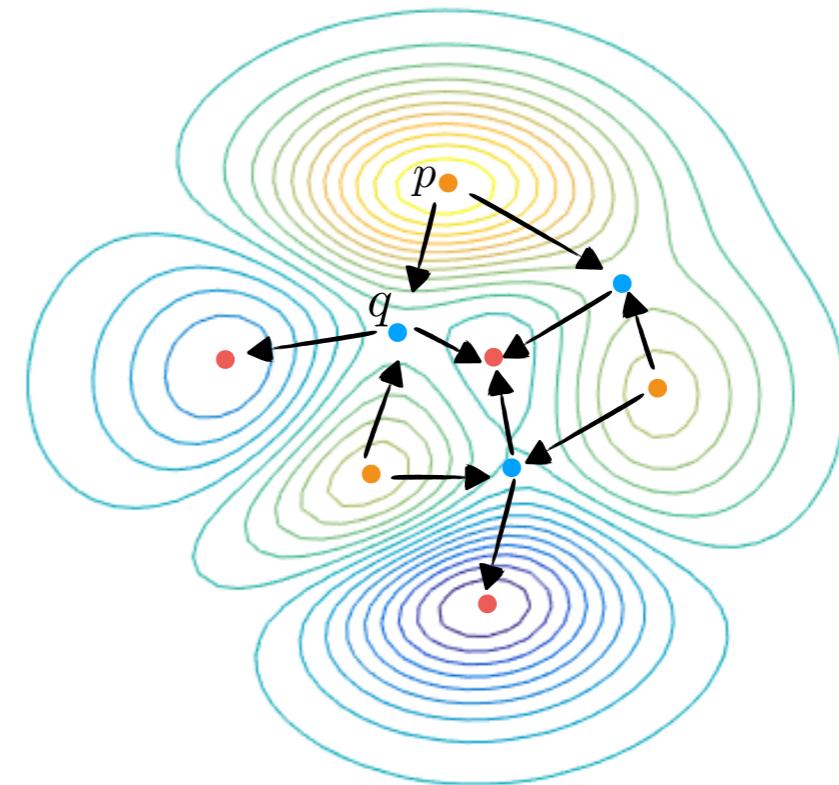
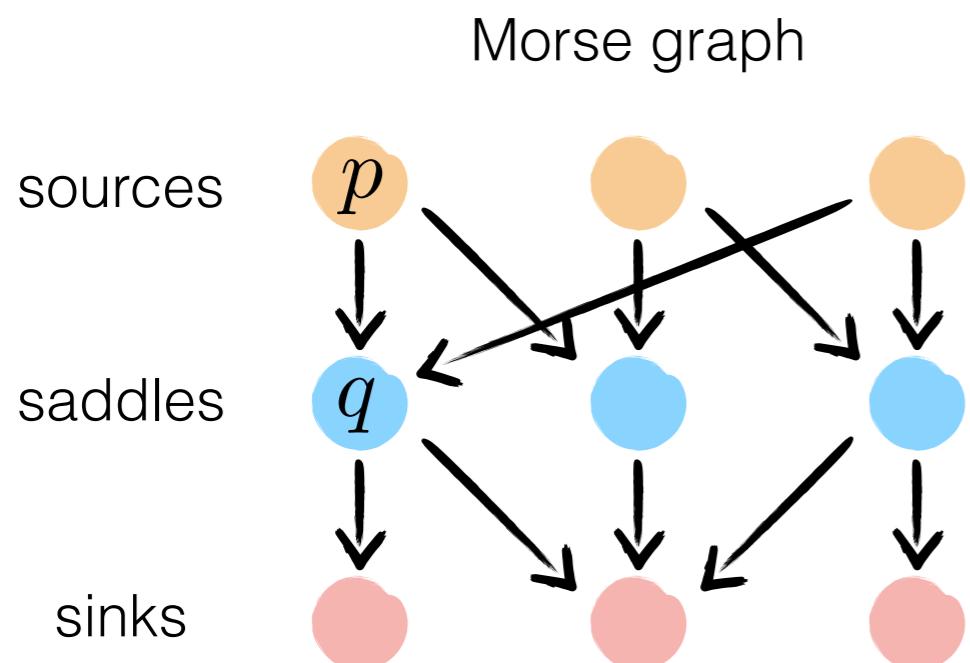
Morse theory as a rapid prototype,
Morse graphs for your dynamical system on a napkin,
how to stop worrying and learn to be robust to perturbations.

returning to simple (Morse-theoretic) dynamics



dynamics: equilibria + heteroclinic orbits

global dynamics are organized by a graph



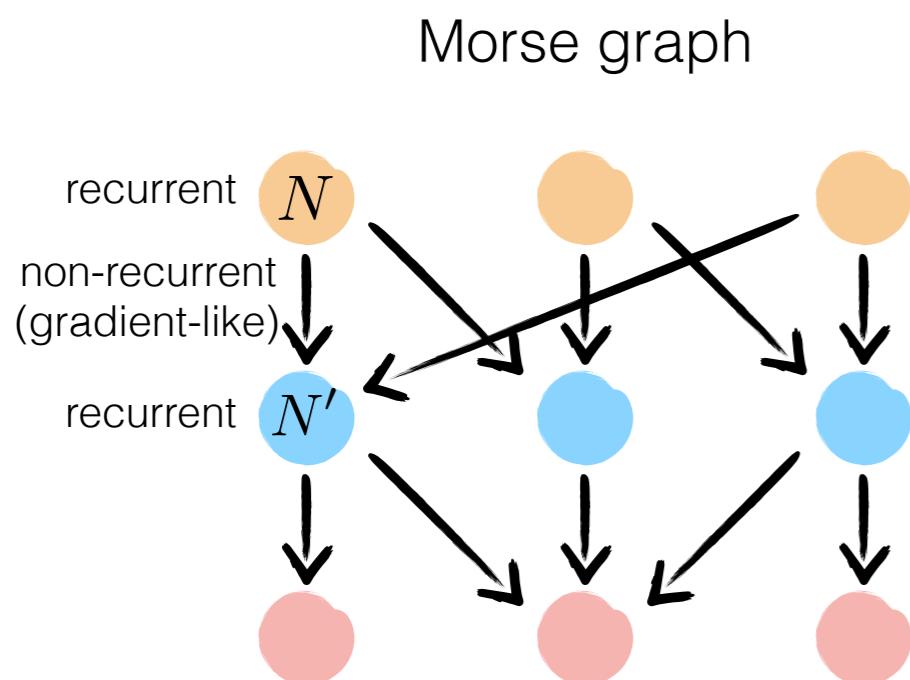
fixed points become vertices

edge set satisfies the property:

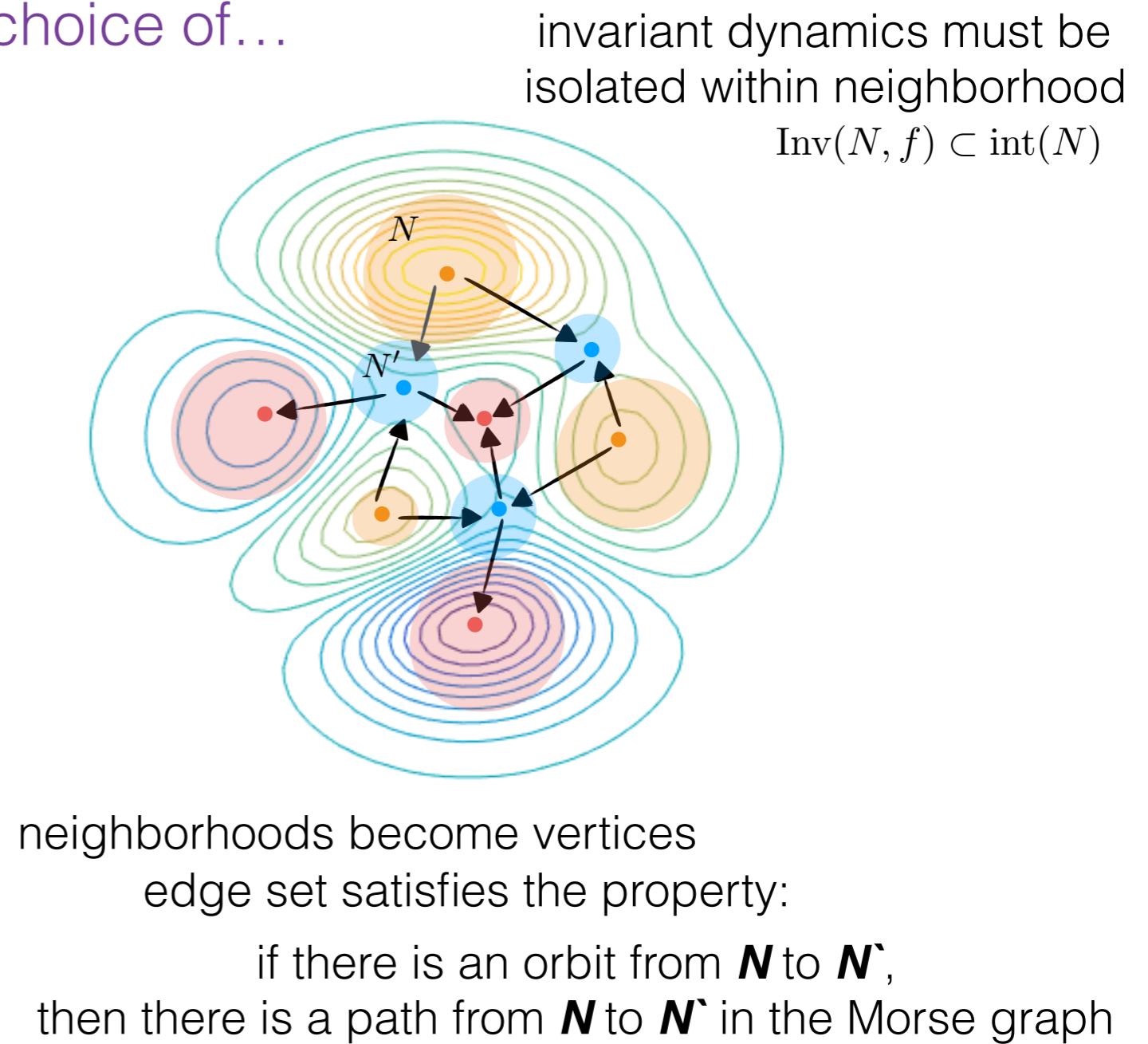
if there is an orbit from p to q ,
then there is a path from p to q in the Morse graph

(isolating) neighborhoods instead of fixed points

a choice of neighborhoods is a choice of...
resolution

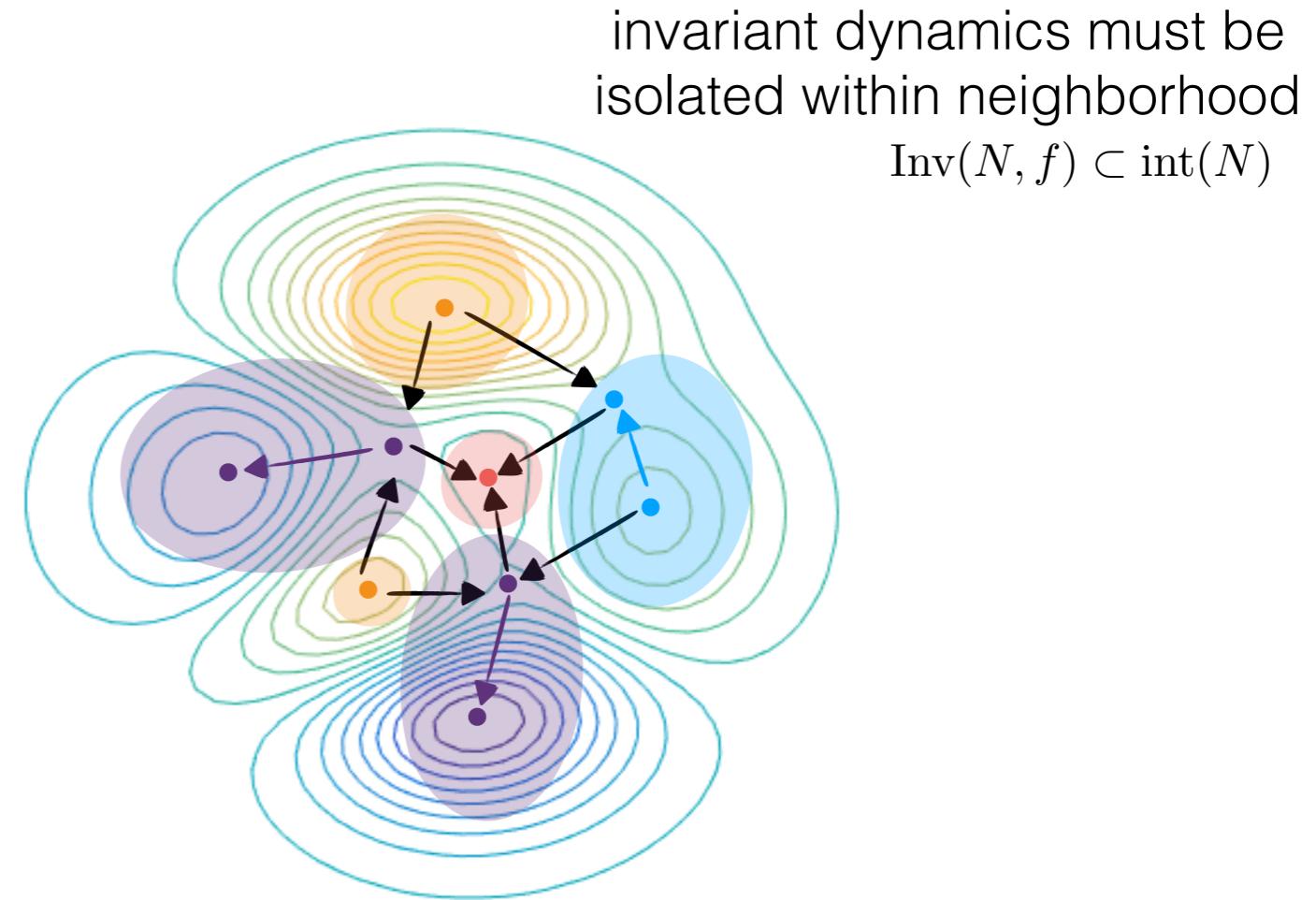
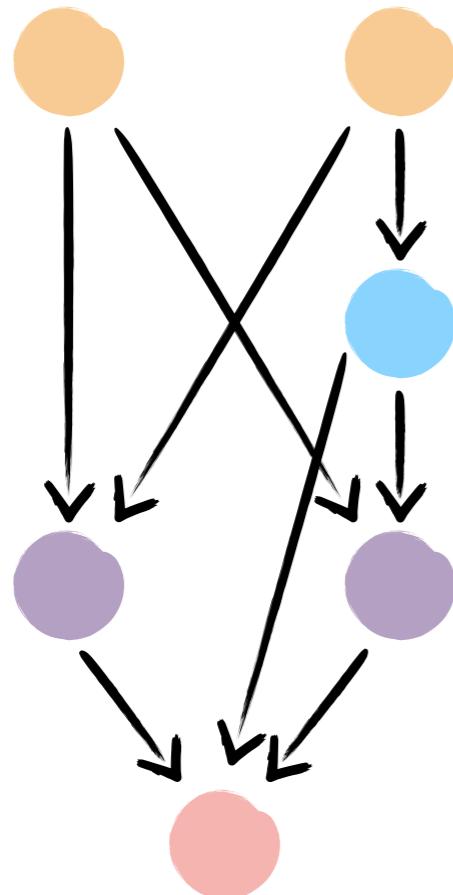


at this resolution the
decomposition remains the same



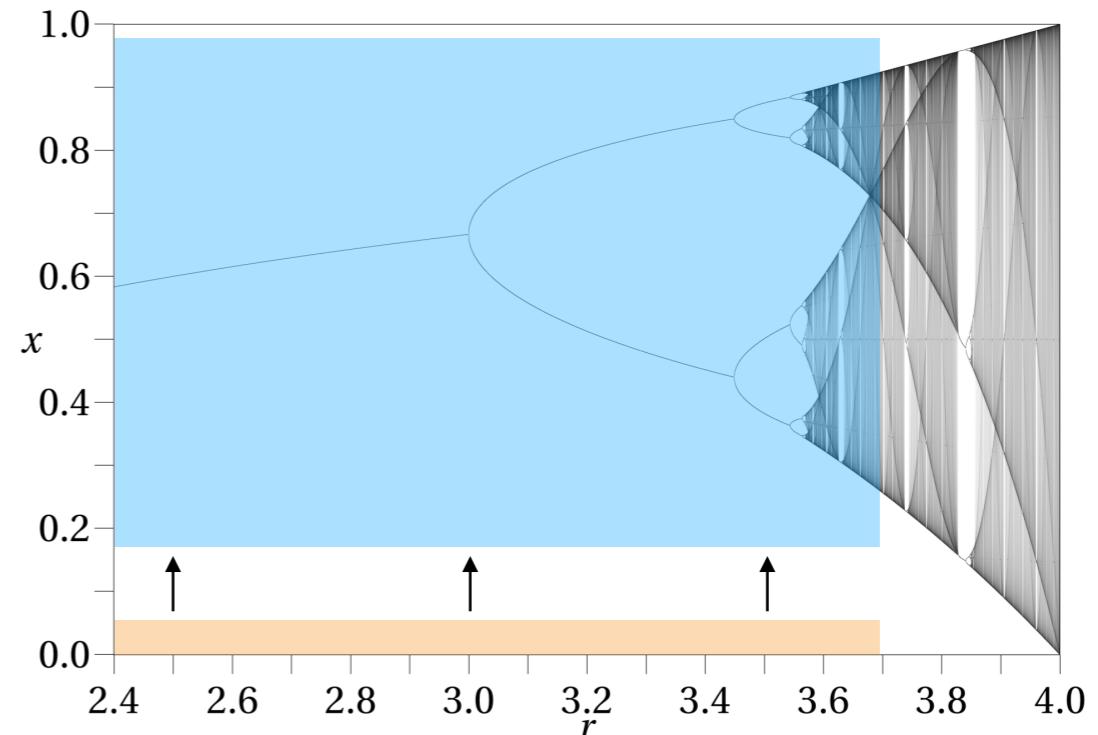
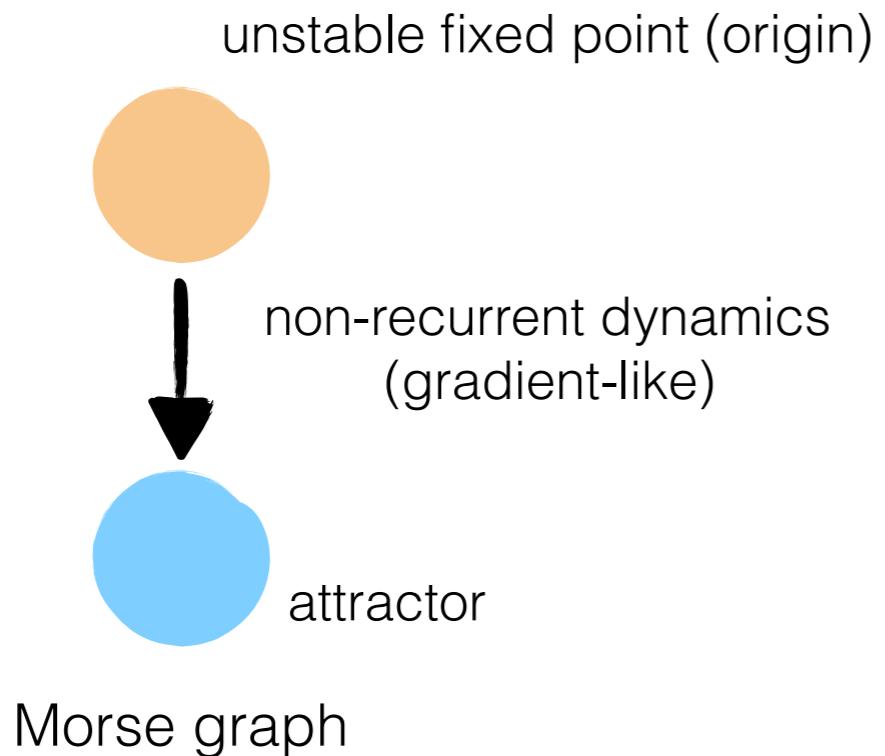
a choice of neighborhoods is a choice of...
resolution

choosing
different neighborhoods



the graph is robust with respect to perturbation of the system

decompose dynamics over a set of parameters



a Morse graph ...

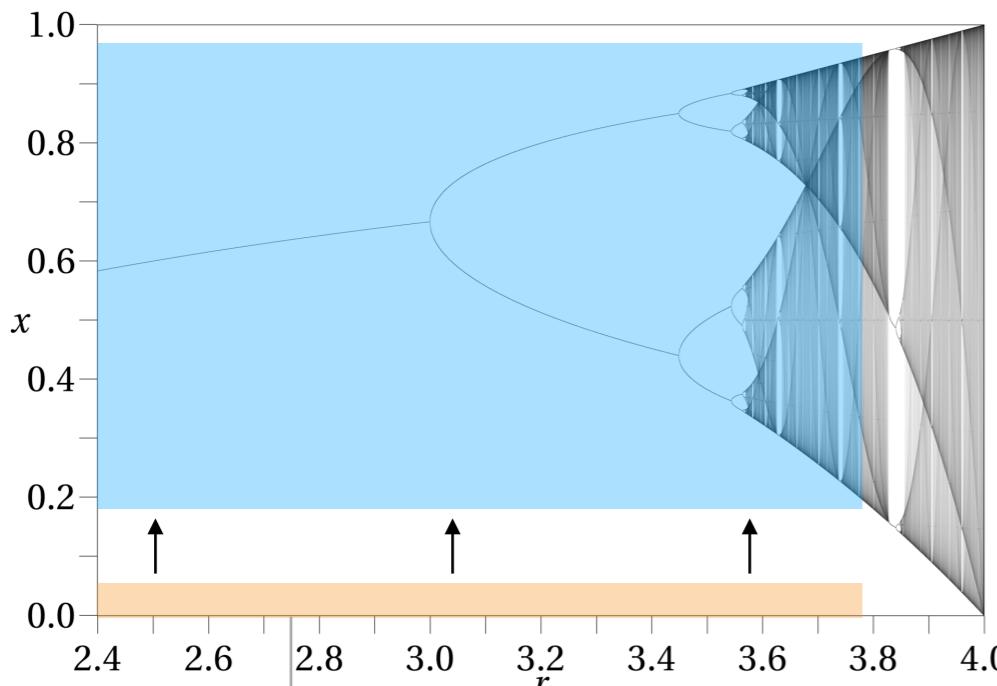
- is robust (valid) over the set of parameters
- compact representation of global dynamics
- capable of capturing complex dynamics

reconstruction

invariant sets leave algebraic topological footprints,
Conley-Morse graphs = Morse graphs + Conley indices,
a one-dimensional zoo.

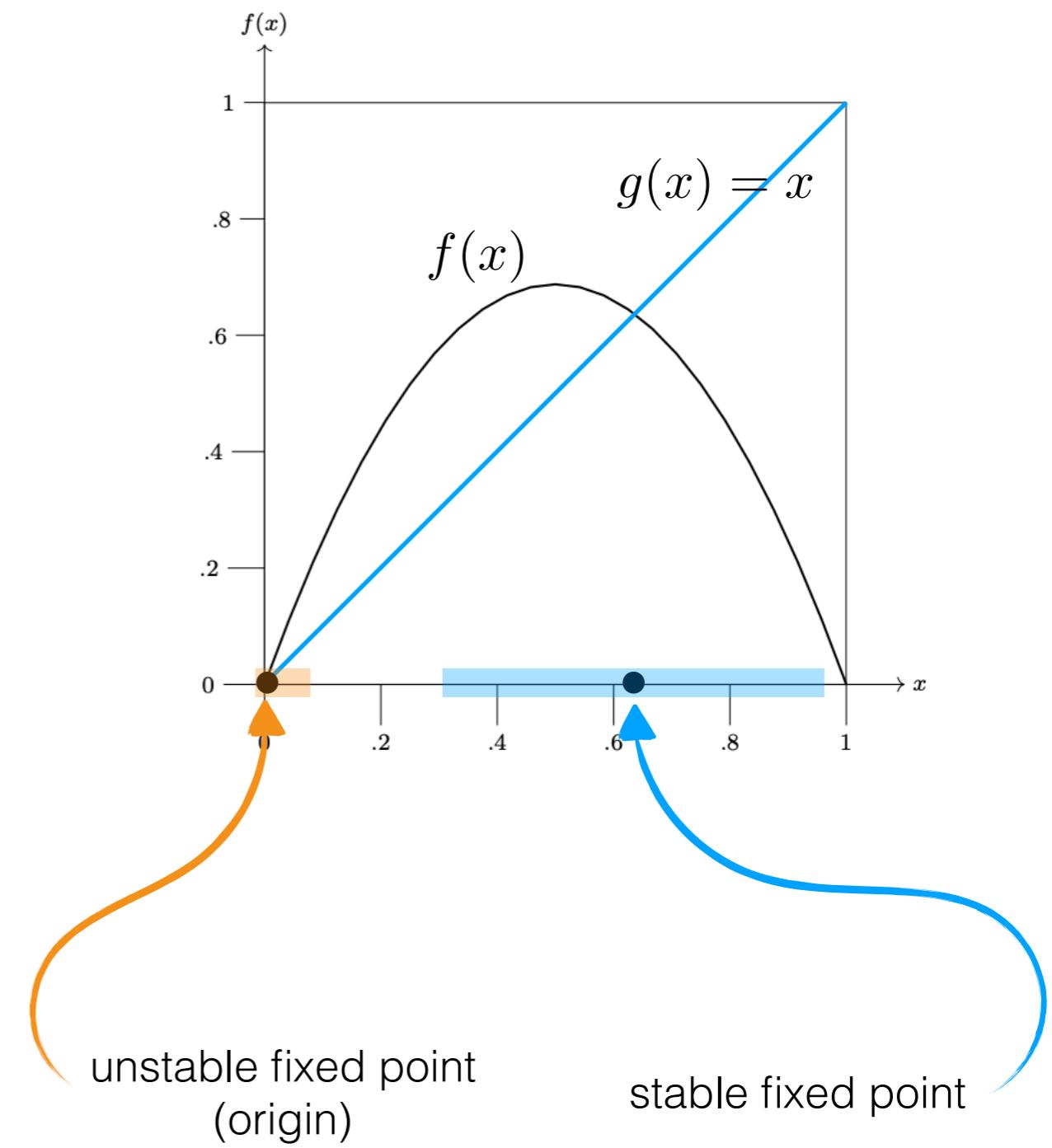
logistic redux

$$r = 2.75$$



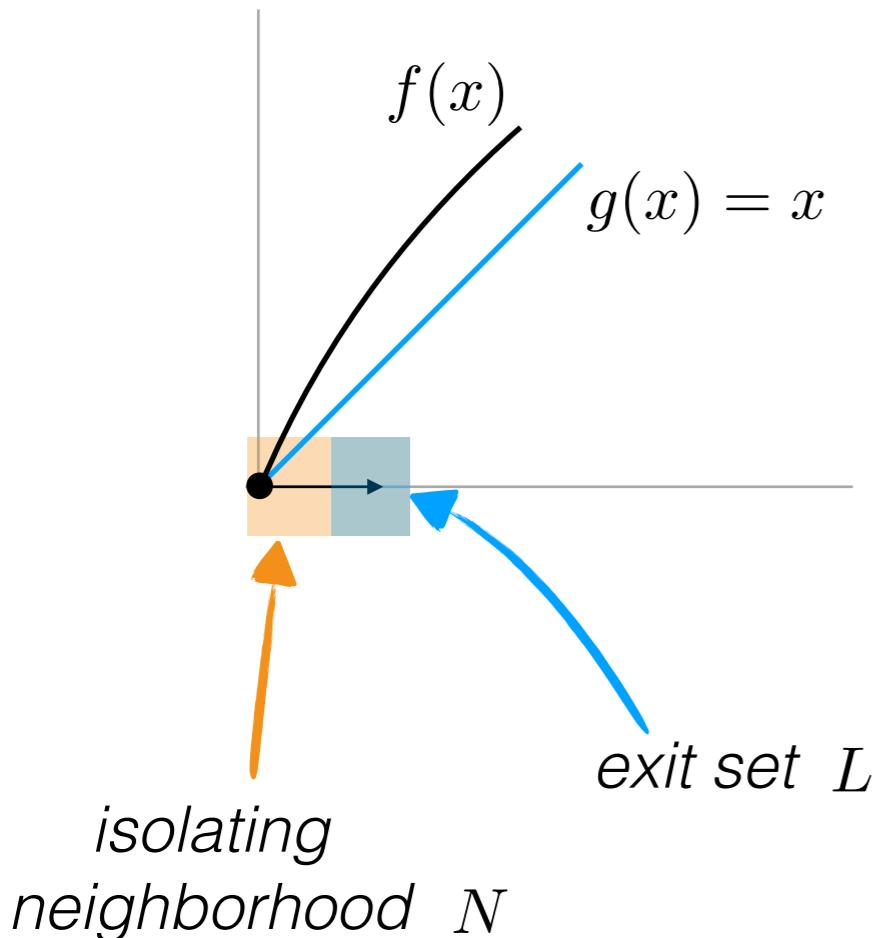
we are here

$$f(x) = 2.75x(1 - x)$$



reconstruction via Conley index

local picture near $x = 0$



in this case...

$$\begin{aligned} H_\bullet(N/L, [L]) &= (H_0(N/L, [L]), H_1(N/L, [L])) \\ &= (0, 0) \end{aligned}$$

$$(f_{N,L})_\bullet = (0 \rightarrow 0, 0 \rightarrow 0)$$

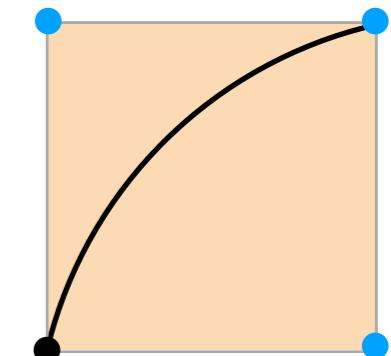
(pointed) quotient space

$$(N/L, [L])$$



induced self-map

$$f_{N,L}: (N/L, [L]) \rightarrow (N/L, [L])$$



Conley index is the associated linear map

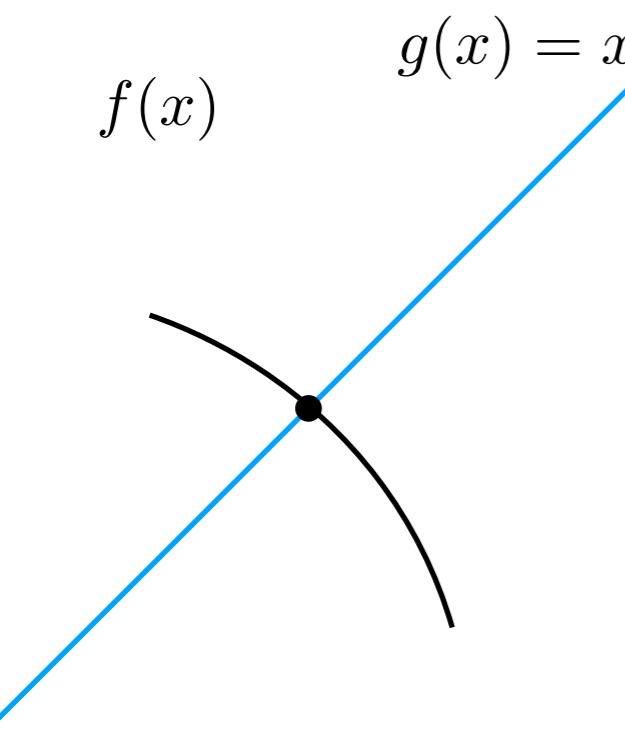
$$(f_{N,L})_\bullet: H_\bullet(N/L, [L]) \rightarrow H_\bullet(N/L, [L])$$

we write the Conley index as $(0, 0)$

(caveat: shift equivalence class)

reconstruction via Conley index II

*local picture
near attracting fixed point*



isolating neighborhood N

no exit set

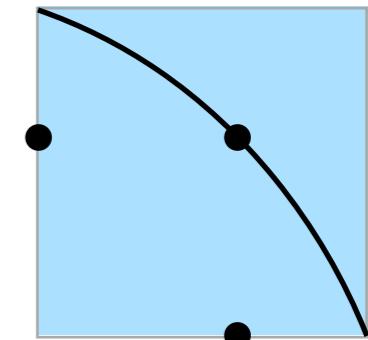
(pointed) quotient space

$$(N/L, [L])$$



induced self-map

$$f_{N,L}: (N/L, [L]) \rightarrow (N/L, [L])$$



Conley index is the associated linear map

$$(f_{N,L})_*: H_*(N/L, [L]) \rightarrow H_*(N/L, [L])$$

$$H_*(N/L, [L]) = (\mathbb{F}, 0)$$

$$(f_{N,L})_* = (\mathbb{F} \xrightarrow{\text{id}} \mathbb{F}, 0 \rightarrow 0)$$

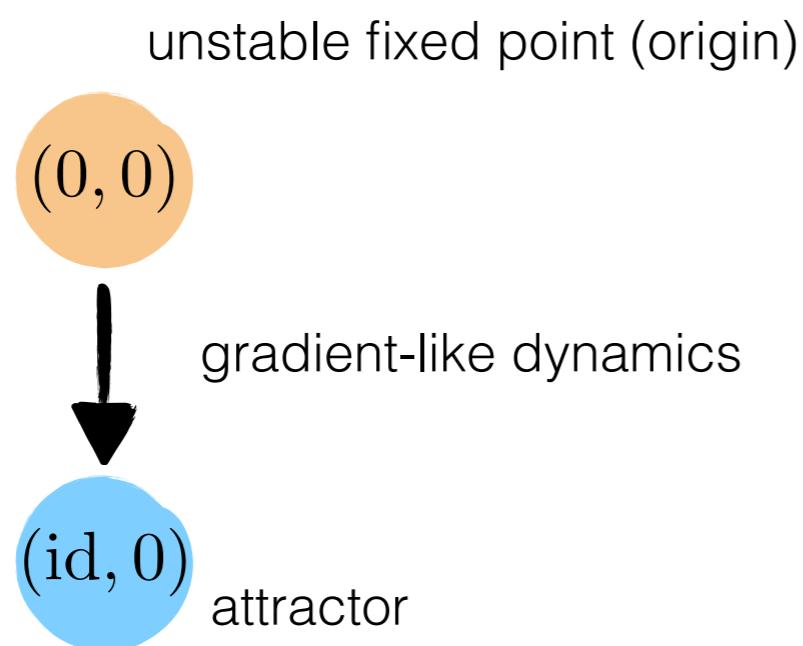
we write the Conley index as $(\text{id}, 0)$

characterize recurrent dynamics over a set of parameters

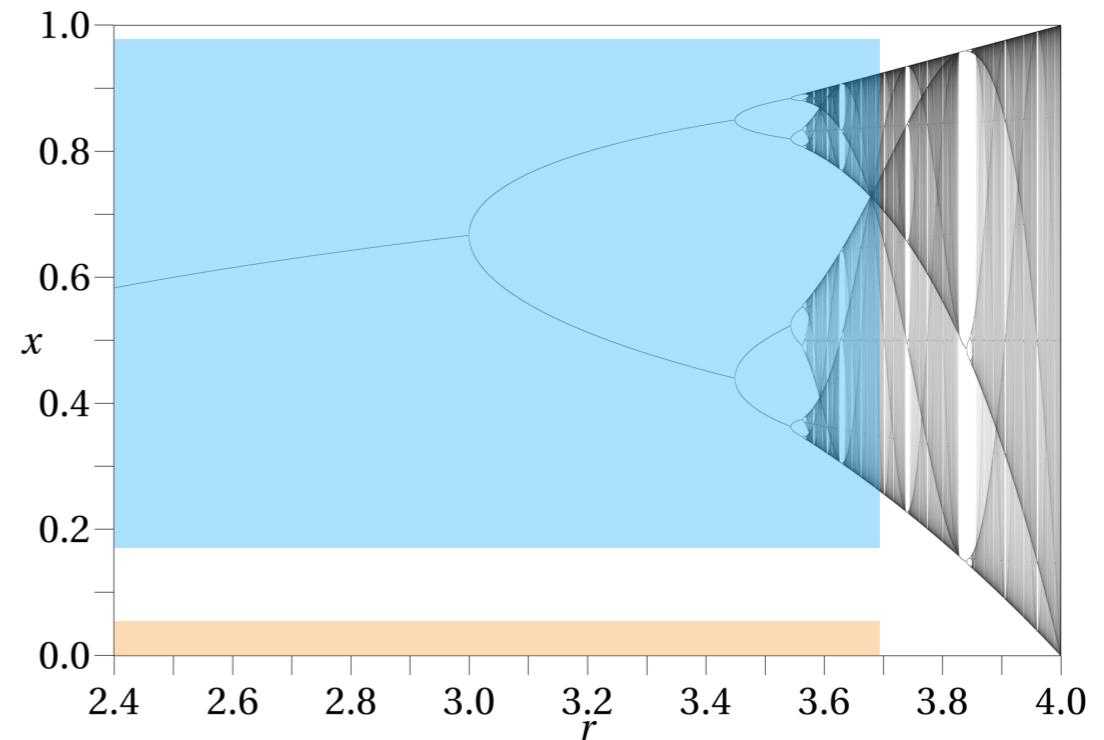
Theorem: If \mathbf{N} is an isolating neighborhood for each r in a path connected set, then the Conley index associated to \mathbf{N} is the same for all f_r

i.e., robust with respect to parameters

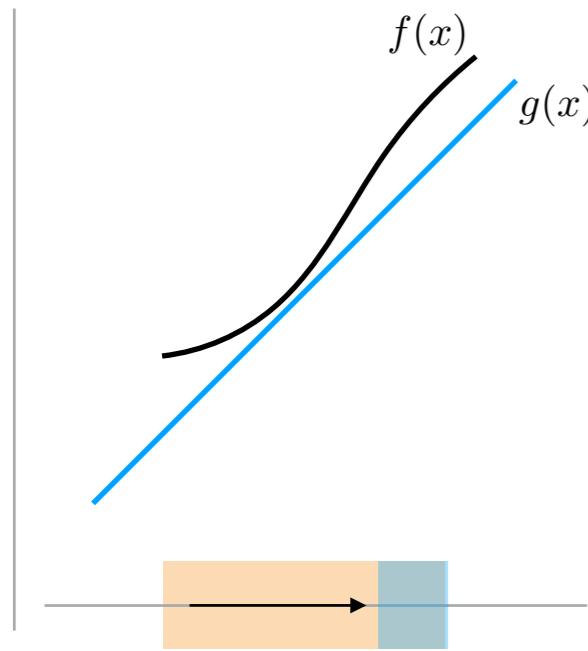
Conley-Morse graph = Morse graph + Conley indices



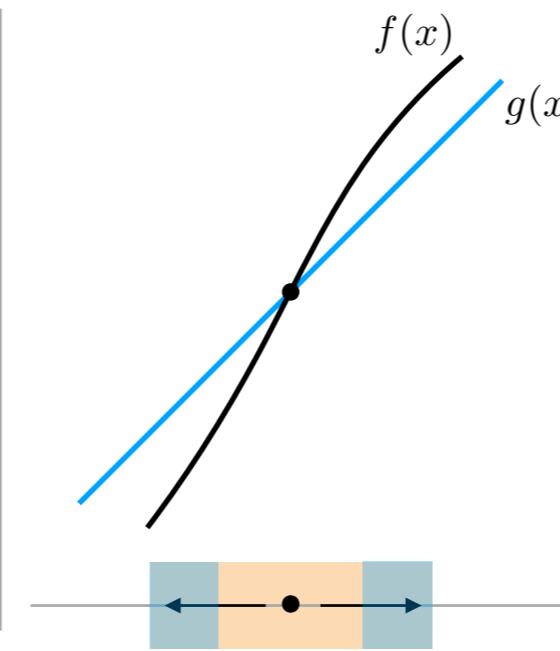
Conley-Morse graph



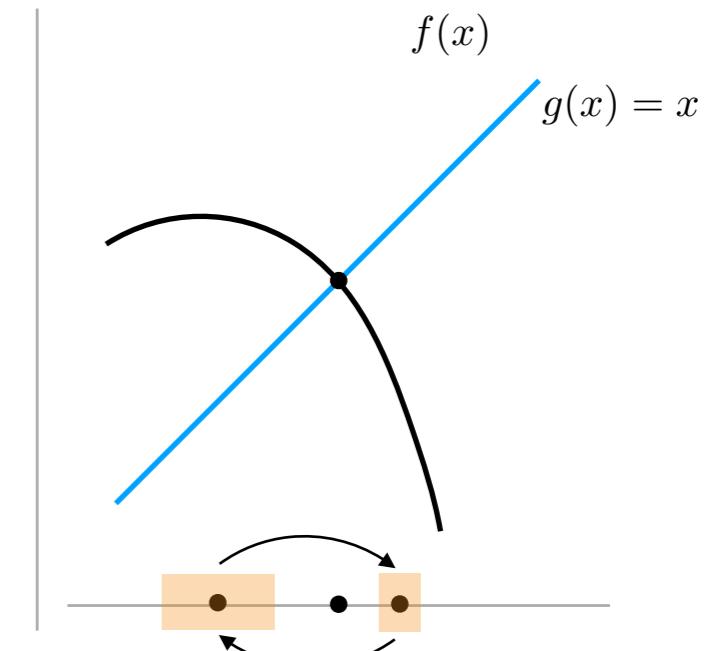
a zoo of Conley indices



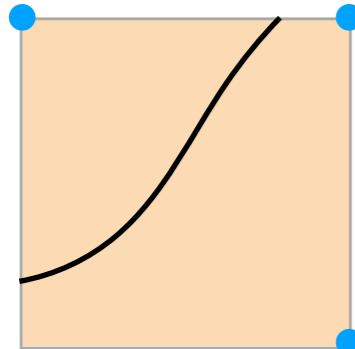
empty invariant set



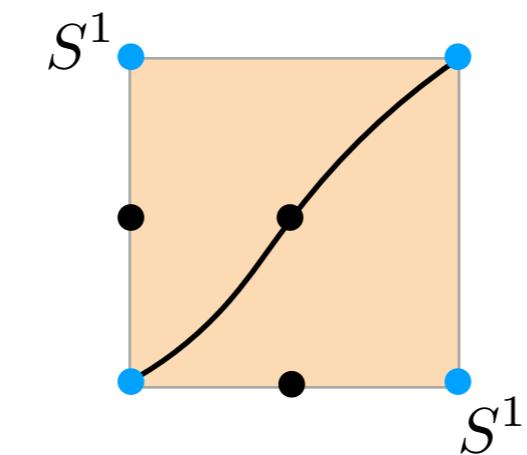
unstable fixed point



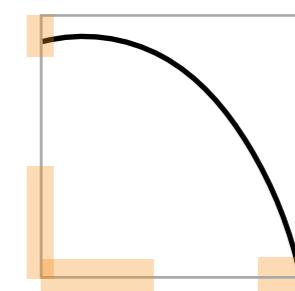
stable period 2 orbit



$$H_{\bullet}(N/L, [L]) = (0, 0)$$



$$H_{\bullet}(N/L, [L]) = (0, \mathbb{F})$$



$$H_{\bullet}(N/L, [L]) = (\mathbb{F} \oplus \mathbb{F}, 0)$$

Conley
index:

$$(0, 0)$$

$$(0, \text{id})$$

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

interlude

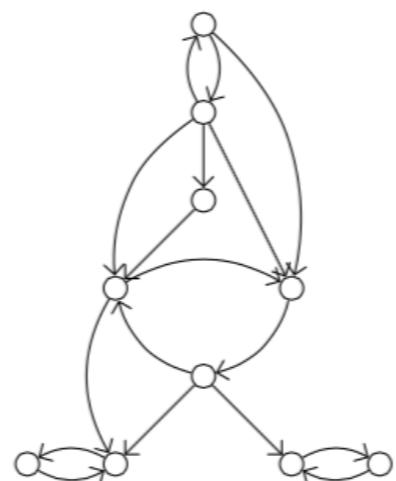
combinatorial dynamics

combinatorial dynamics

vertices = states of the system

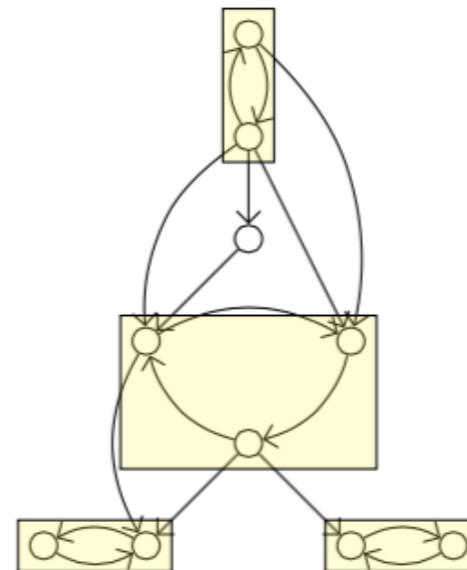
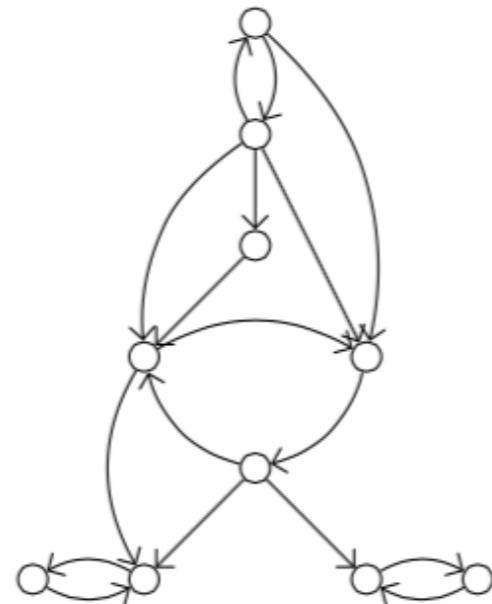
directed edges = possible dynamics

graphs can be manipulated by a computer



decomposition

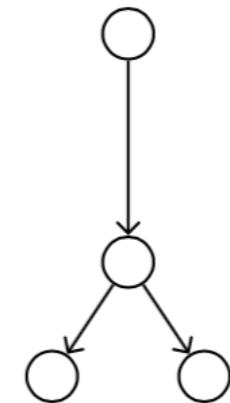
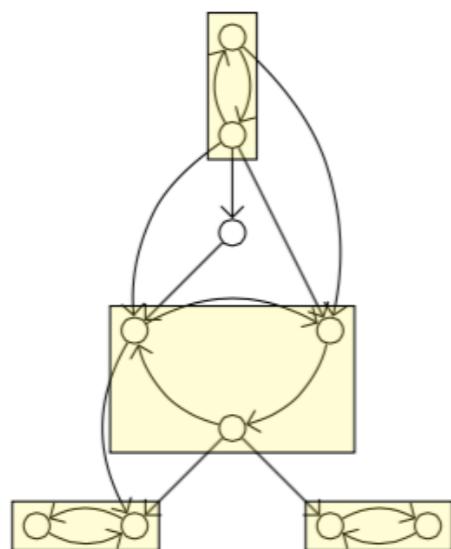
recurrent vs. non-recurrent



cyclic strongly connected component:

maximal sets of vertices in which any two vertices are connected with a directed path, and contains at least one cycle

decomposition II



(combinatorial) Morse graph

cyclic
strongly connected path components

poset of cyclic
strongly connected components

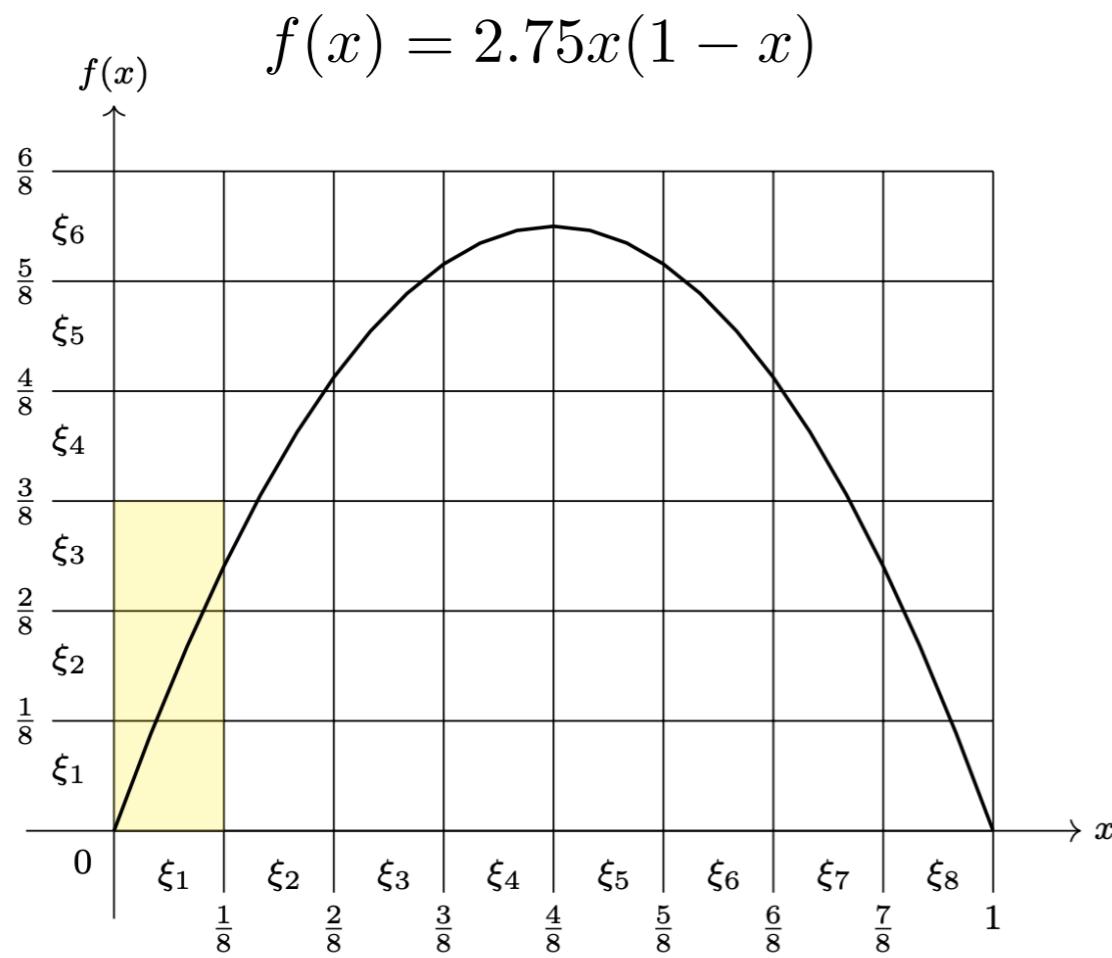
combinatorialization

how to go from wiggles to bits and back again

'Our physical world is wiggly... but in nature wiggles don't come 'pre-bitted'. If you want to eat a chicken you have to cut it up – it doesn't come bitten.'

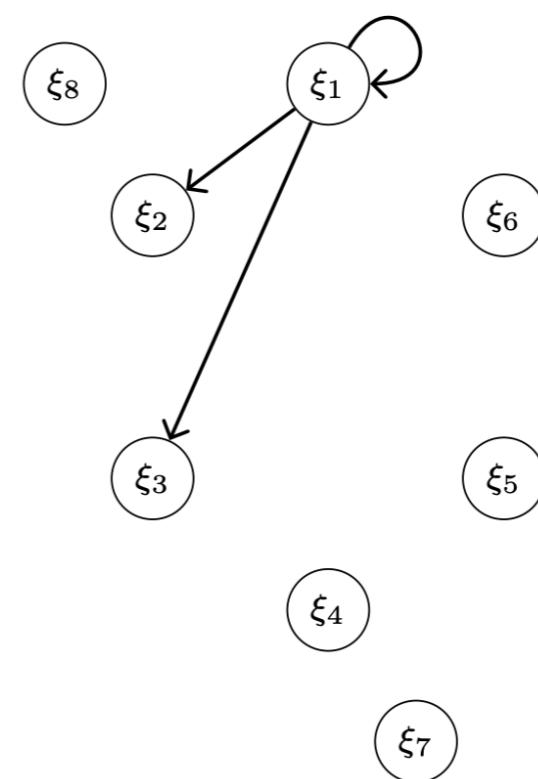
A. Watts

step 1: discretize phase space
(grid on phase space)



step 2: discretize dynamics
directed graph

grid elements are vertices

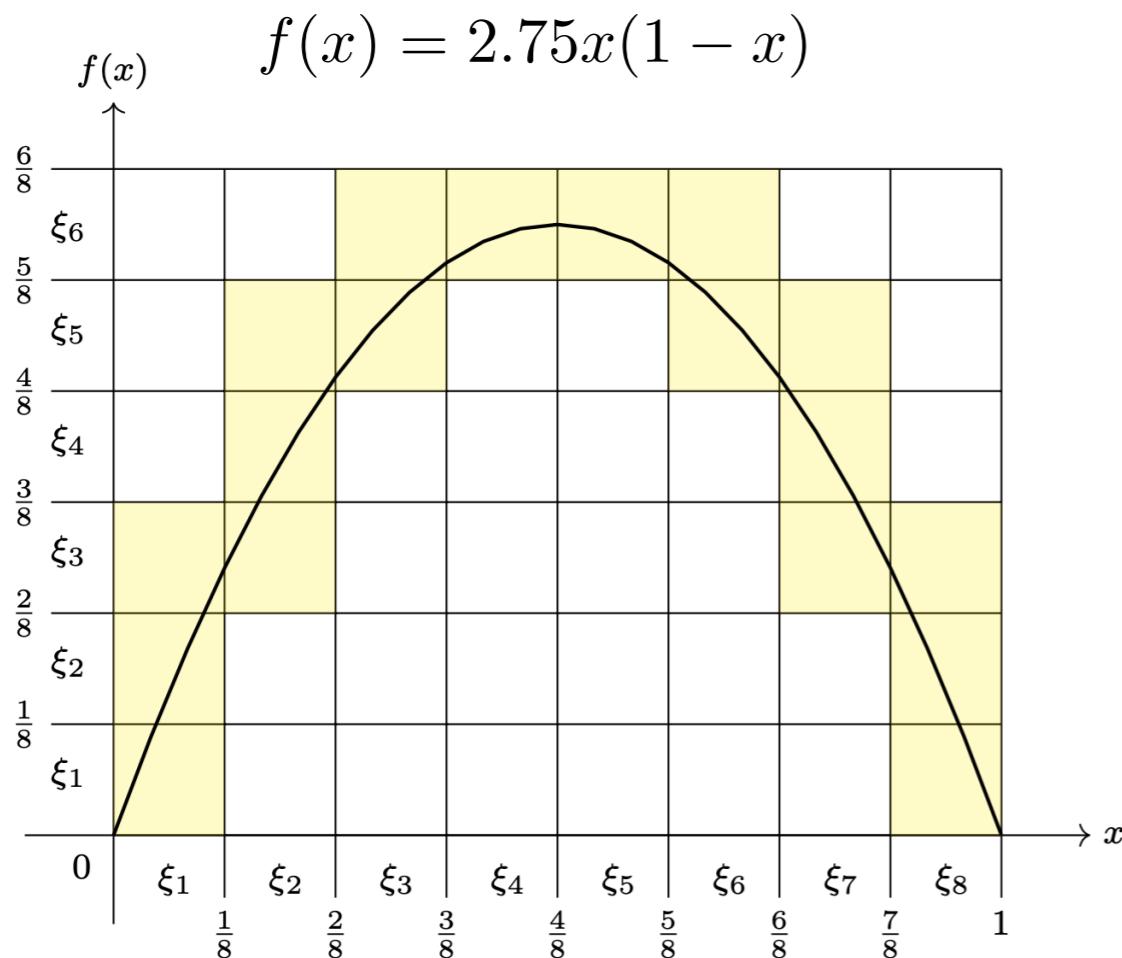


edge set obeys the property:

$$f(\xi_i) \cap \xi_j \neq \emptyset \implies \xi_i \rightarrow \xi_j$$

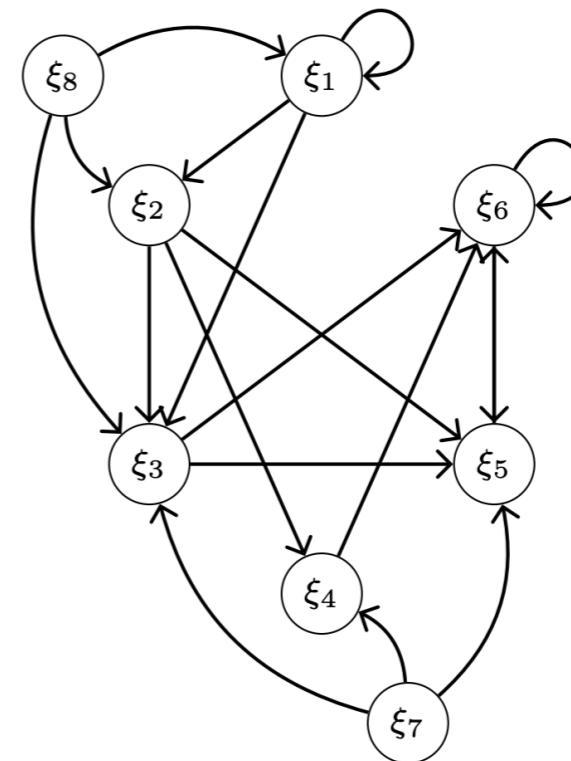
such a graph is called an **outer approximation** of f

step 1: discretize phase space
(grid on phase space)



step 2: discretize dynamics
directed graph

grid elements are vertices

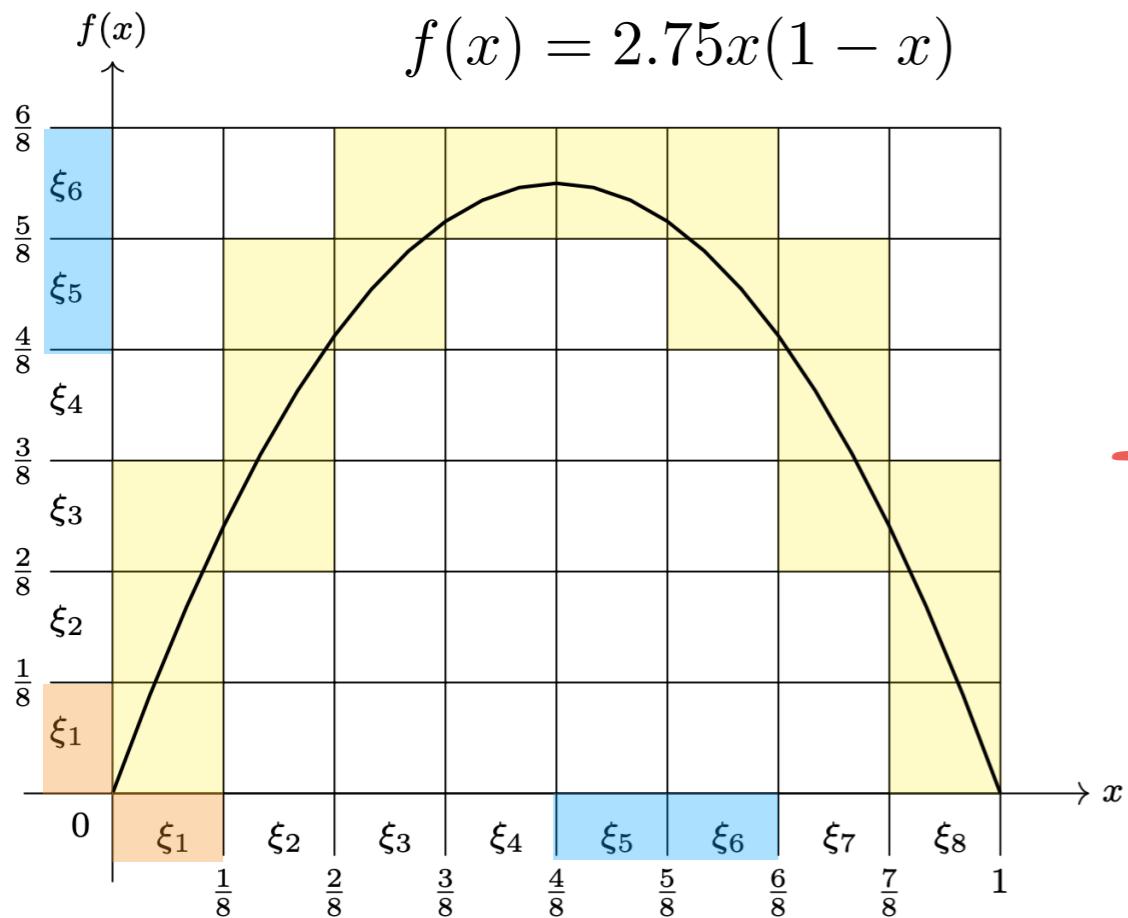


edge set obeys the property:

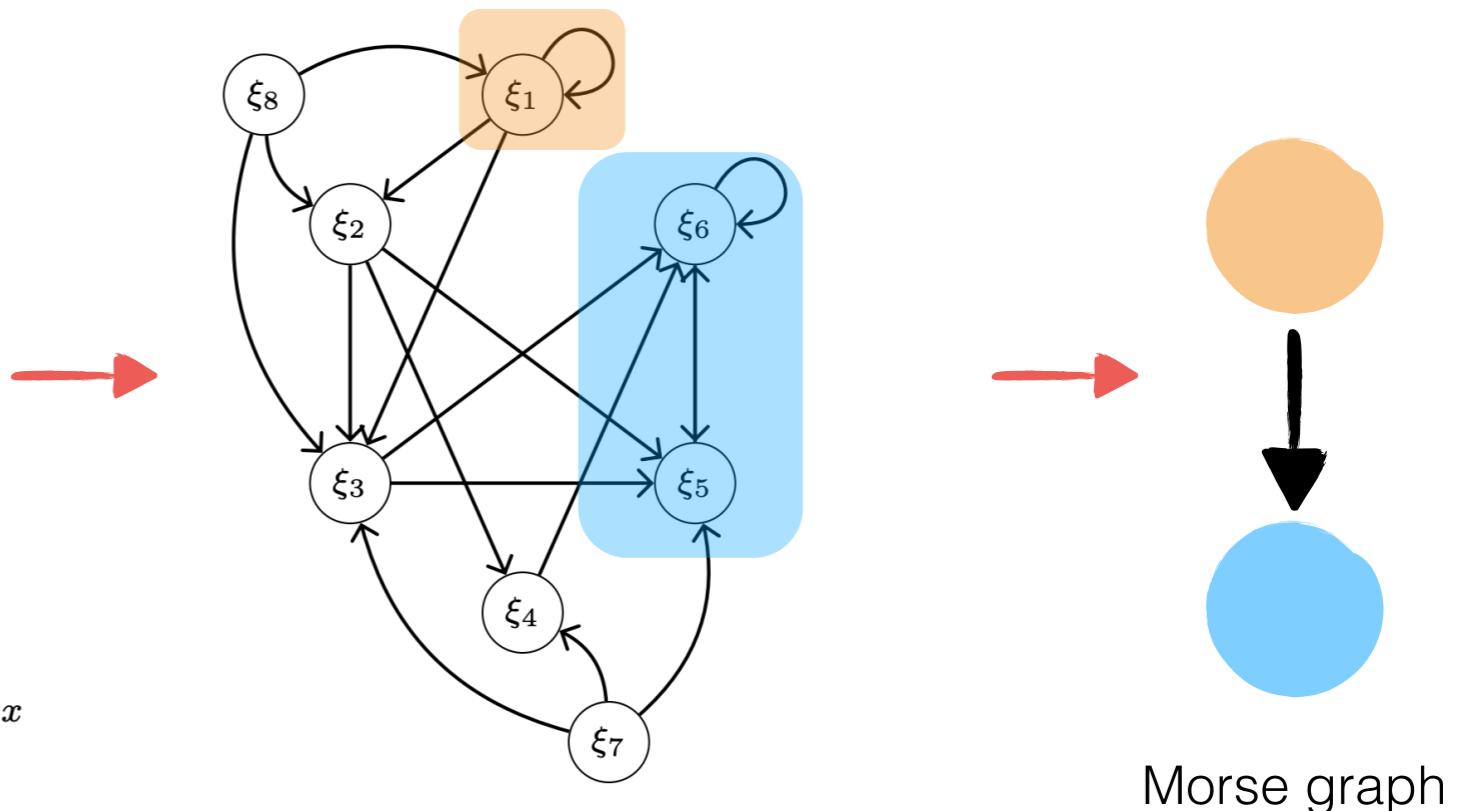
$$f(\xi_i) \cap \xi_j \neq \emptyset \implies \xi_i \rightarrow \xi_j$$

such a graph is called an **outer approximation** of f

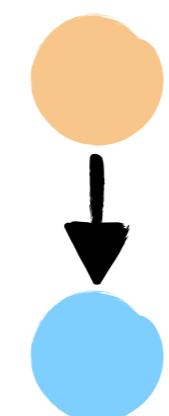
computational recipe



cyclic strongly connected components



Morse graph

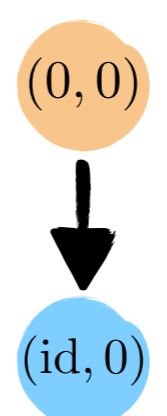


Morse graph

compute Conley indices



computational homology



Conley-Morse graph

Theorem: Conley-Morse graphs arising from outer approximations are valid for the underlying system f

application...
the global dynamics of games

is there more to a game than its Nash equilibria?
a *long* evening with a friend and two pennies,
how complex is learning to play a game?

'matching pennies'-style game:
 two players (**A** and **B**) simultaneously put down a penny, either heads or tails up;
 if same face then player **A** wins, if opposite faces player **B** wins

payoff matrices

$$A = \begin{matrix} & \begin{matrix} H & T \end{matrix} \\ \begin{matrix} H \\ T \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix}$$

$$B = \begin{matrix} & \begin{matrix} H & T \end{matrix} \\ \begin{matrix} H \\ T \end{matrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{matrix}$$

suppose **A** plays **H** and **B** plays **T**, then **A**'s payoff is $\mathbf{A}_{HT}=0$ and **B**'s is $\mathbf{B}_{TH}=1$

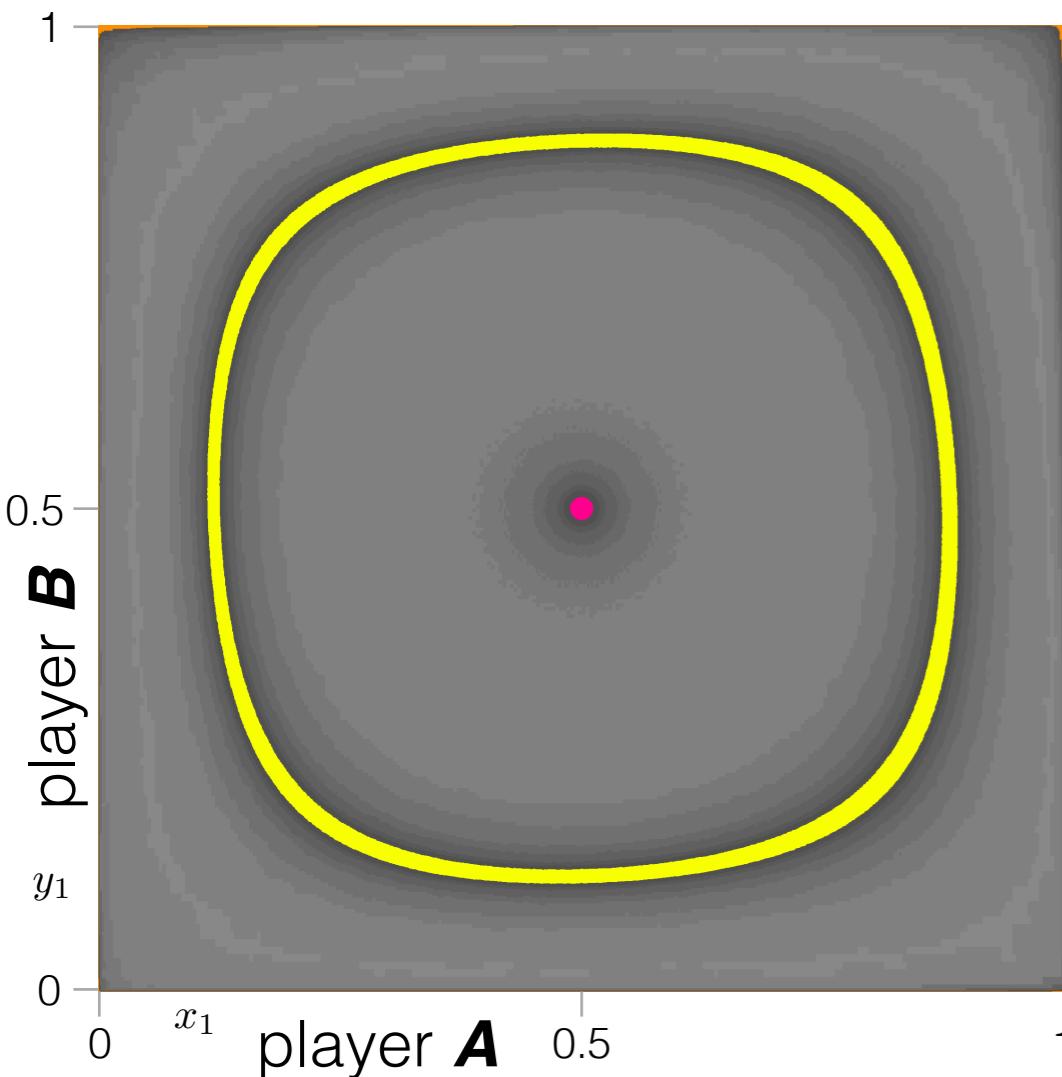
player **A** strategy: choose **H** with probability x_1 , choose **T** with probability $x_2 = 1 - x_1$
 player **B** strategy: choose **H** with probability y_1 , choose **T** with probability $y_2 = 1 - y_1$

a *learning dynamic* governs how player should update their strategy

$$x_i(t+1) = \frac{x_i(t)^{1-\alpha} e^{\beta(Ay)_i}}{\sum_j x_j(t)^{1-\alpha} e^{\beta(Ay)_j}}, \quad y_i(t+1) = \frac{y_i(t)^{1-\alpha} e^{\beta(Bx)_i}}{\sum_j y_j(t)^{1-\alpha} e^{\beta(Bx)_j}}.$$

'Experience Weighted Attraction'

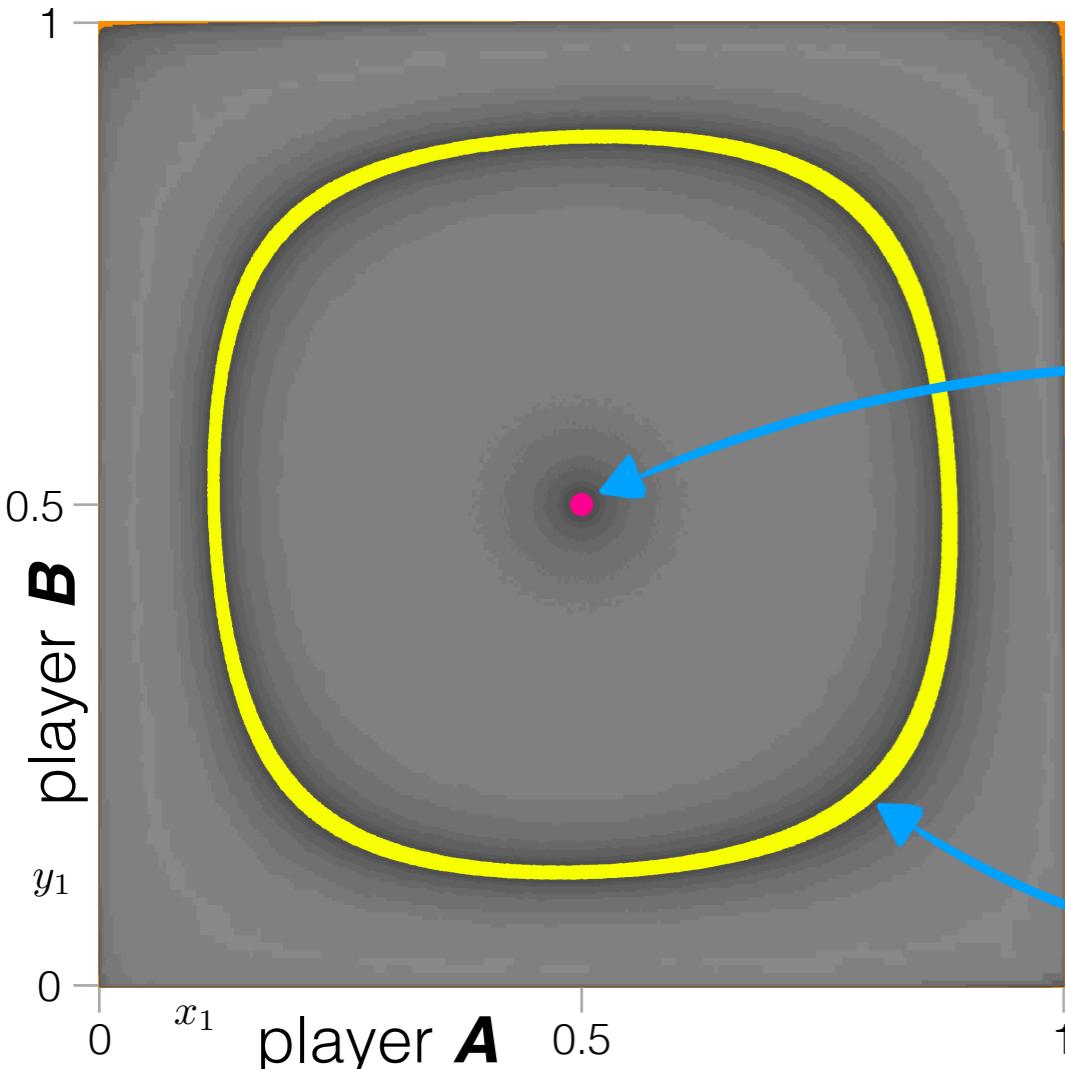
computational recipe for Conley-Morse theory



1. discretize phase space at chosen scale
2. build directed graph by *outer approximating* map
3. find cyclic strongly connected components (in color)
4. compute Conley-Morse graph

phase space $X = [0, 1] \times [0, 1]$

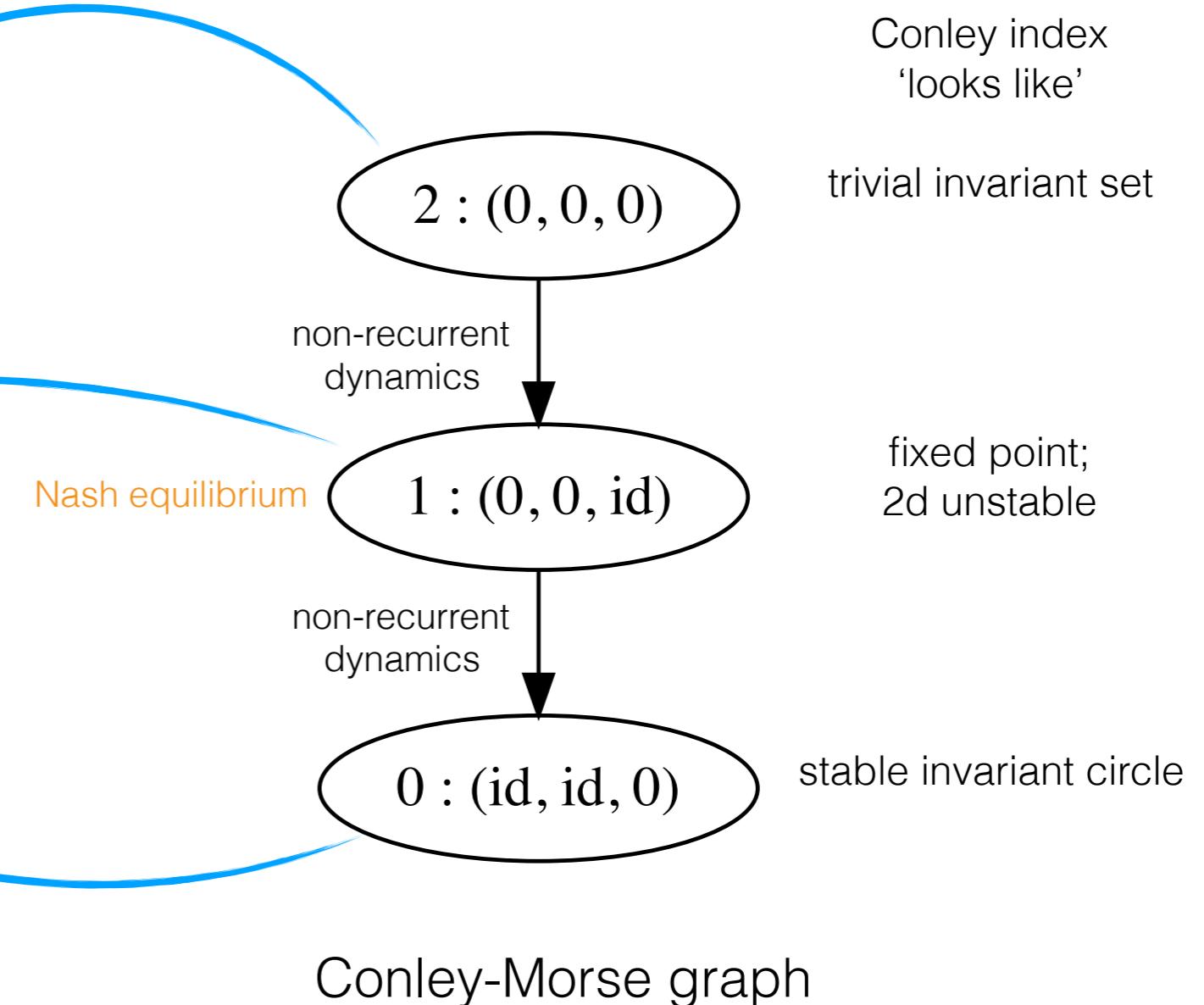
$$\alpha = 0.18, \beta = \sqrt{2}$$



phase space $X = [0, 1] \times [0, 1]$

$$\alpha = 0.18, \beta = \sqrt{2}$$

Given this method of learning the game,
our two players will not converge to the Nash equilibrium!
Instead, they cycle around the Nash equilibrium.



from zero to one

payoff matrices depend upon parameter: $A = \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}$ $B = \begin{pmatrix} -r & 1 \\ 1 & -r \end{pmatrix}$

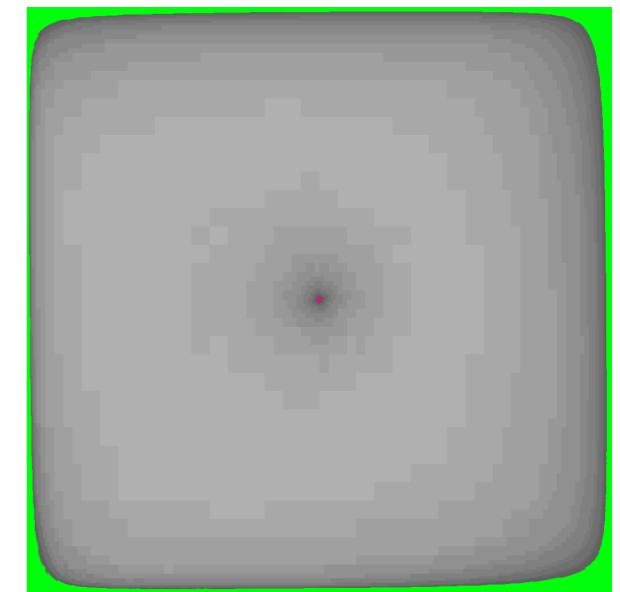
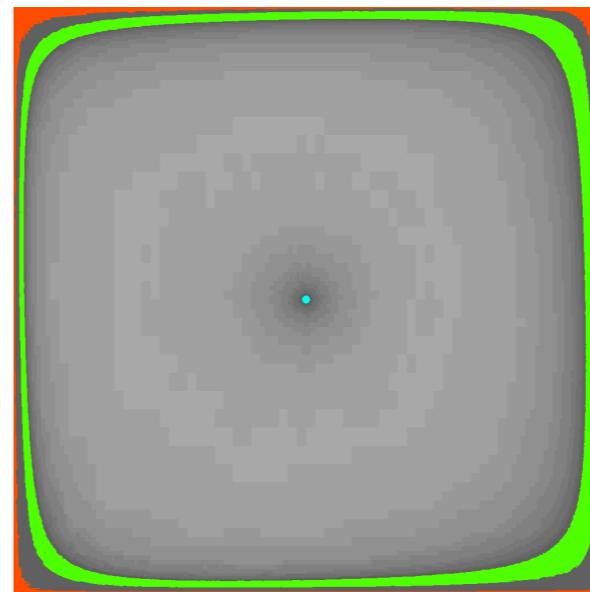
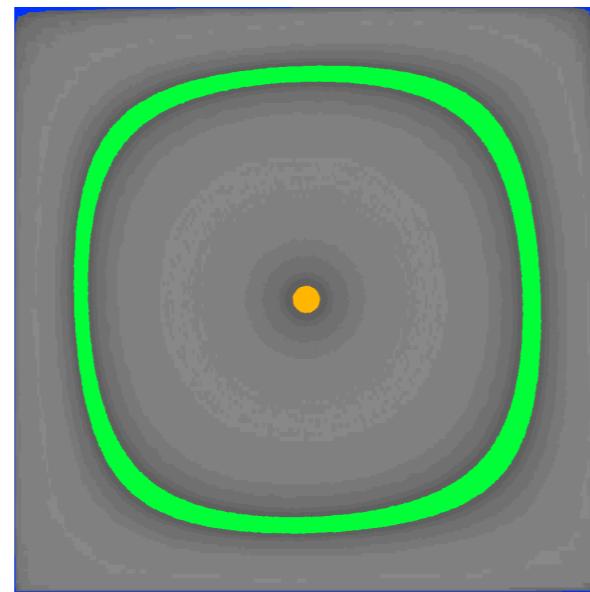
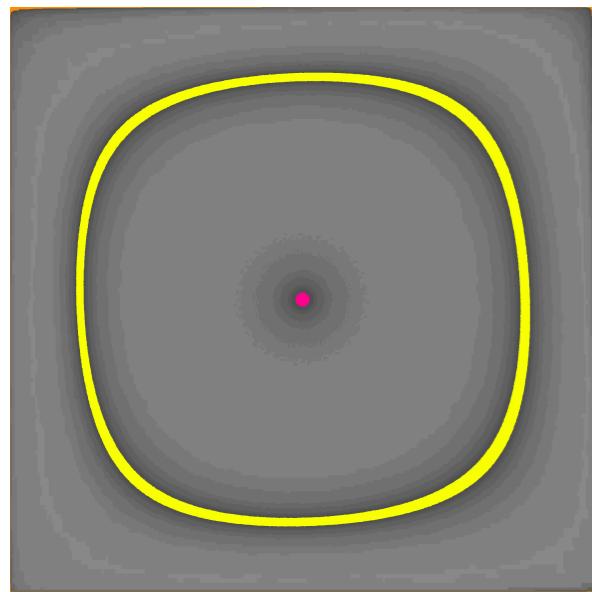
'zero-sum game'

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -.1 \\ -.1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -.1 & 1 \\ 1 & -.1 \end{pmatrix}$$

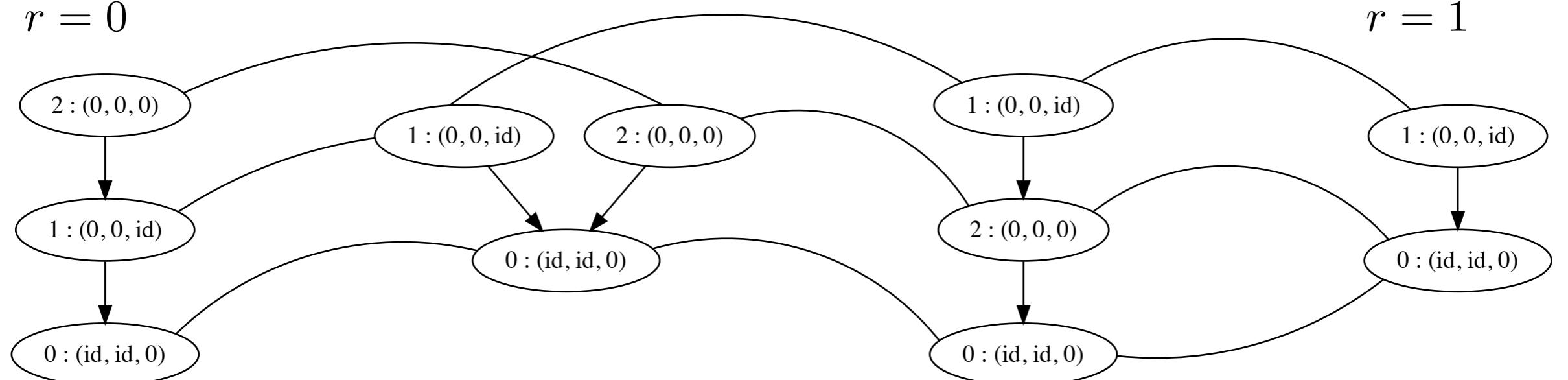
$$A = \begin{pmatrix} 1 & -.5 \\ -.5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -.5 & 1 \\ 1 & -.5 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$



$r = 0$

r



returning to our dynamical questions...

too hard

~~for a particular initial condition, what happens in the long run?~~
~~how does the long run behavior depend upon the initial conditions?~~

these may be (partially) answered
using Conley theory

is there a fixed point?
is there a periodic orbit?
are there multiple attractors?
how does behavior depend on the parameters within the model?

thank you for your attention

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