Computational Connection Matrix Theory

…toward new tools in applied topology

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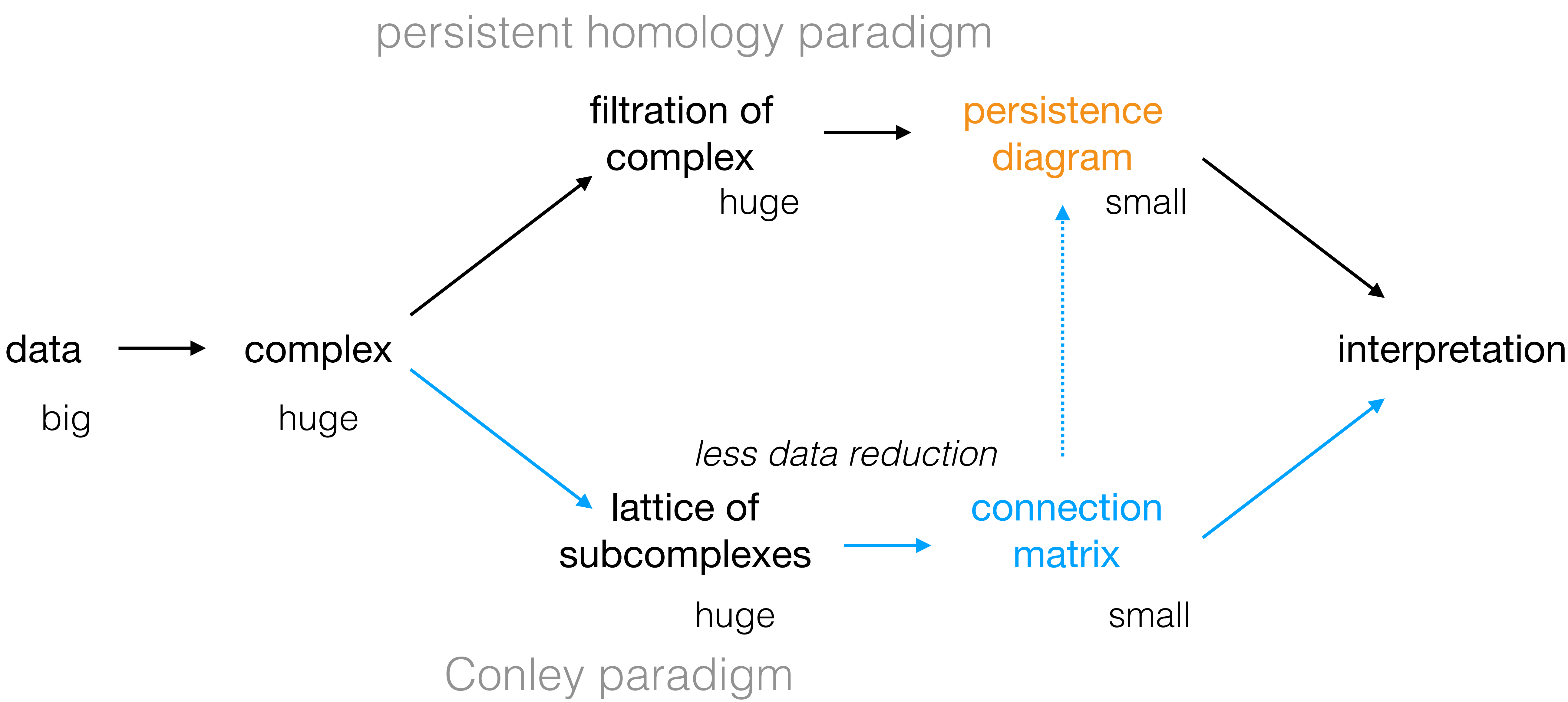
AMS Sectional Meeting, 2018

workflows


persistent homology paradigm
filtration of complex
huge
persistence diagram
small
data
big
complex
huge
interpretation
less data reduction
lattice of subcomplexes
huge
connection matrix
small
Conley paradigm

Data Structures

graded complexes

X cellular complex
(Lefschetz, CW)

(X, $\leq$) face poset

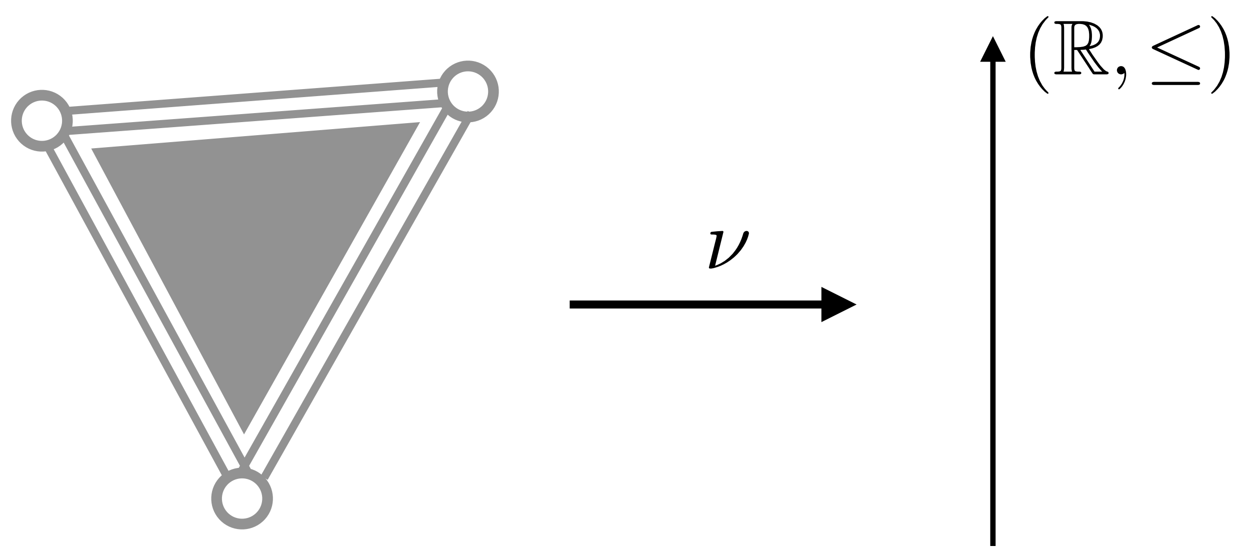
$\mathbb{R}$ poset

an order preserving map (X, $\leq$) $\xrightarrow{\nu}$ $(\mathbb{R}, \leq)$

filters X via pre-images of downsets

$\nu^{-1}(-\infty, a]$ is a subcomplex of X

the collection $\{\nu^{-1}(-\infty, a]\}_{a \in \mathbb{R}}$ is a filtration



$U \subseteq \mathbb{R}$ is a down-set if the following holds: $x \in U$ and $y \leq x$ implies $y \in U$

X cellular complex
(Lefschetz, CW)

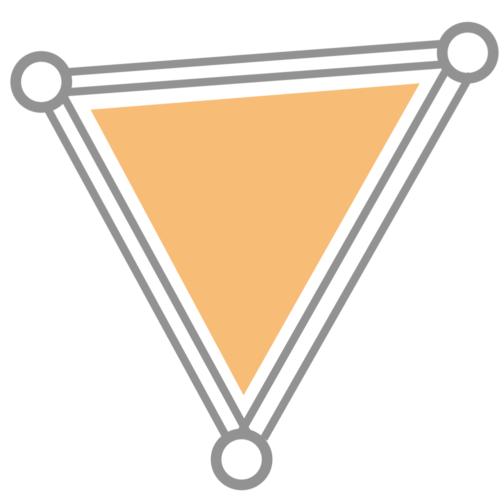
$(\mathsf{X}, \leq)$ face poset

P poset

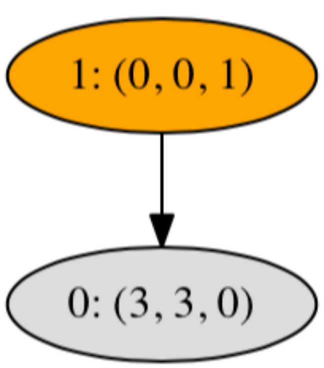
an order preserving map $(\mathsf{X}, \leq) \xrightarrow{\nu} (\mathsf{P}, \leq)$

filters X via pre-images of downsets

$\{\nu^{-1}(U)\}_{U \in \mathsf{O}(\mathsf{P})}$ is a lattice of subcomplexes



$\xrightarrow{\nu}$



count of cells in the fiber for each dim.

the lattice of down sets O(P) is

$\mathsf{O}(\mathsf{P}) := \{U \subset \mathsf{P} : U \text{ is a downset}\}$ $\quad \wedge := \cap \quad \vee := \cup$

Birkhoff

X cellular complex
(Lefschetz, CW)

(X, $\leq$) face poset

P poset

Definition (P-graded cell complex)

X, P, and a poset morphism ν from X to P

$$(\mathsf{X}, \leq) \xrightarrow{\nu} (\mathsf{P}, \leq)$$

a graded cell complex determines

a P-graded chain complex $(C(\mathsf{X}), \partial)$

$$C(\mathsf{X}) = \bigoplus_{p \in \mathsf{P}} C(\nu^{-1}(p))$$

boundary map is P-graded

$$\partial_{pq} \neq 0 \implies p \leq q$$

upper triangular wrt P

for a graded chain complex $C(\mathsf{X}) = \bigoplus_{p \in \mathsf{P}} C(\nu^{-1}(p))$

structure is determined by the fibers of ν

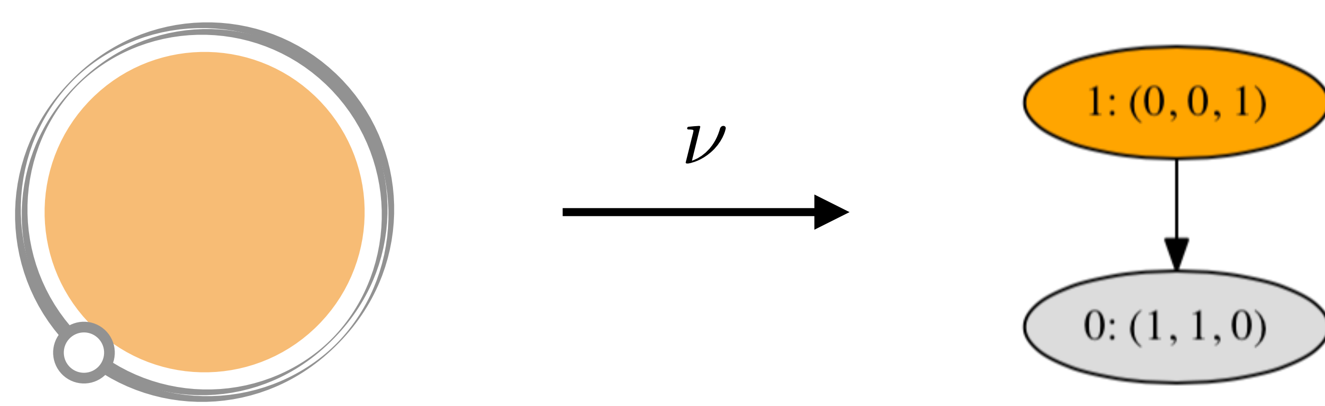
Definition (cyclic **P**-graded complex)

P-graded complex with cyclic fibers

$\partial_{pp} = 0$ for ***p*** in **P** ‘small’ objects

i.e. ∂ is strictly upper triangular wrt **P**

Betti numbers of fiber
(Conley index)



goal: replace graded complex with equivalent cyclic graded complex

Categories

goal: homologically-faithful data compression category **GCh(P)** of **P**-graded chain complexes

morphisms: **P**-graded chain maps

homotopy category **GK(P)** of **P**-graded chain complexes

localize about graded chain equivalences

interpretation of connection matrix for data analysis:

a *Conley complex* is a cyclic representative of isomorphism class in **GK(P)**

the boundary operator of a Conley complex is a *connection matrix*

moral: homotopy categories for chain-level data compression without loss of homological information subcategory **GK_0(P)** of cyclic **P**-graded complexes

$$\mathrm{GK_0(P)} \xrightarrow{\mathfrak{I}} \mathrm{GK(P)}$$

Theorem: over fields, the inclusion functor $\mathfrak{I}$ is full, faithful and essentially surjective (categorical equivalence)

thus there exists an inverse functor $\mathfrak{C}$ called a Conley functor

$$\mathrm{GK_0(P)} \underset{\mathfrak{C}}{\overset{\mathfrak{I}}{\rightleftarrows}} \mathrm{GK(P)}$$

taking a graded chain complex to a Conley complex

analogous to a homology functor

under the hood: discrete morse theory

Theorem: For any graded complex the persistent homology groups of $C_\bullet(\mathrm{P})$ and $\mathfrak{C}(C_\bullet(\mathrm{P}))$ are isomorphic

Harker + Mischaikow + S.

computational Conley homology

applications + implementation

application i:

discrete flows

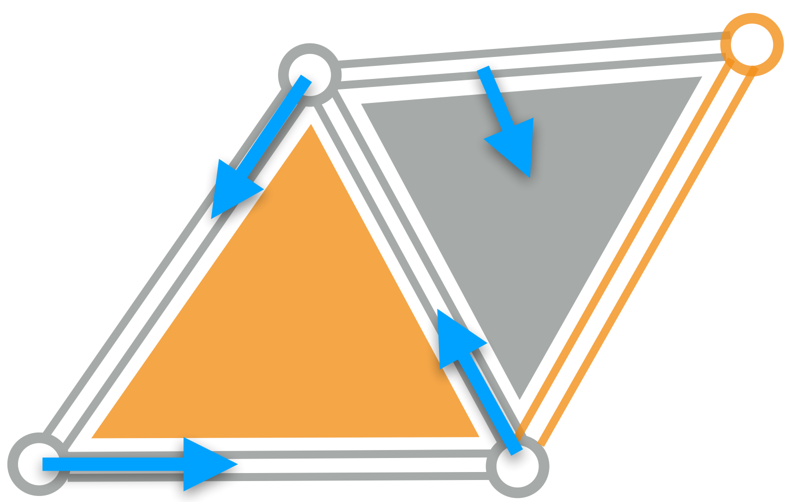
discrete flows

the simplest discrete flow is a *combinatorial vector field* on simplicial complex (Forman)

a combinatorial vector field is a partial matching $\{c_i\} \sqcup \{y_i < x_i\}$

two cells are matched only if one is a facet of another

(no acyclicity requirement)



arrows give the matching

cells are critical if they are not matched

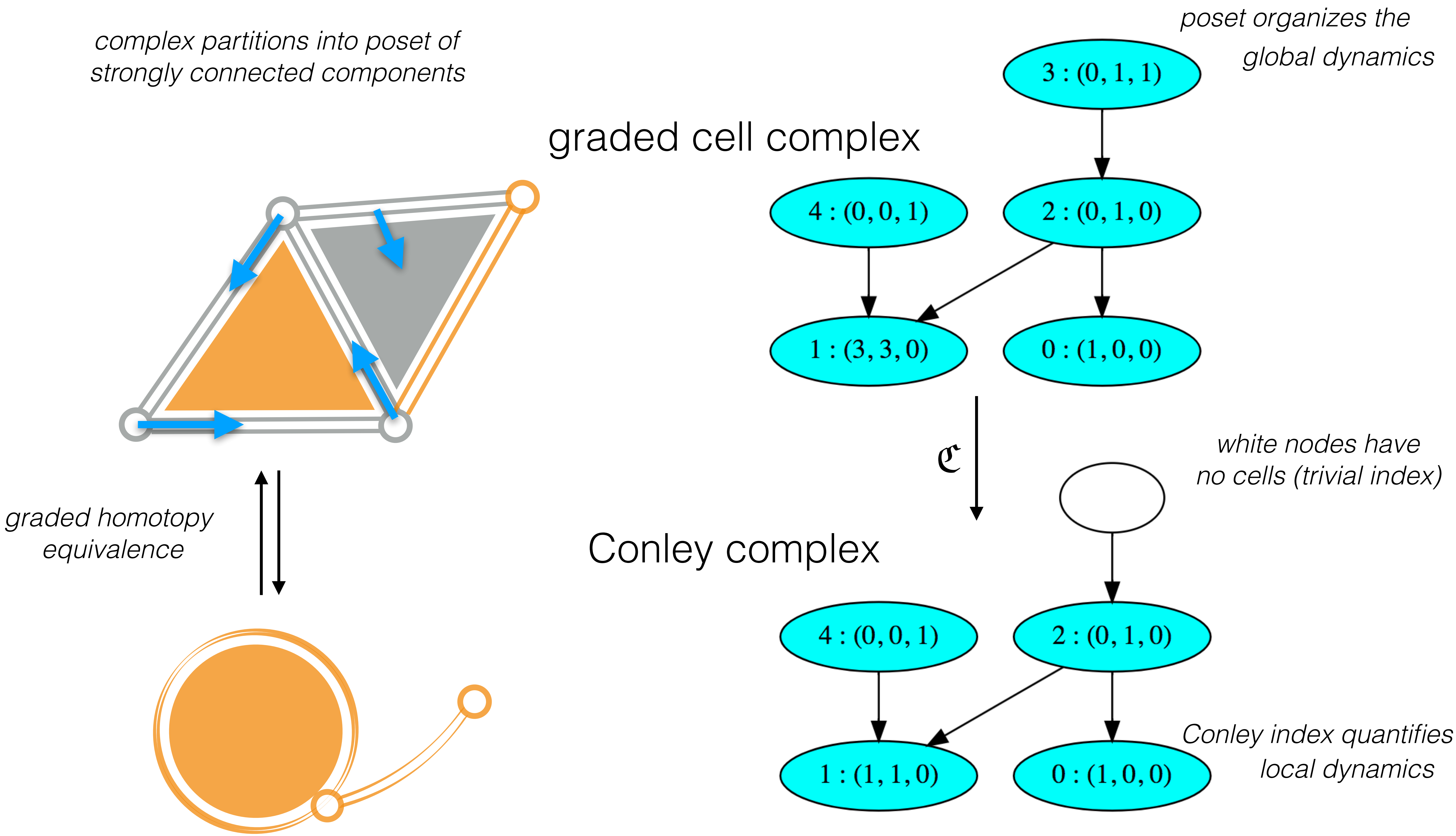
face poset and matching give a directed graph on complex

a discrete flow line is a sequence

$$y_0 < x_0 > y_1 < x_1 > \ldots > y_k < x_k$$

simplicial complex partitions into poset of strongly connected components

discrete flows ii

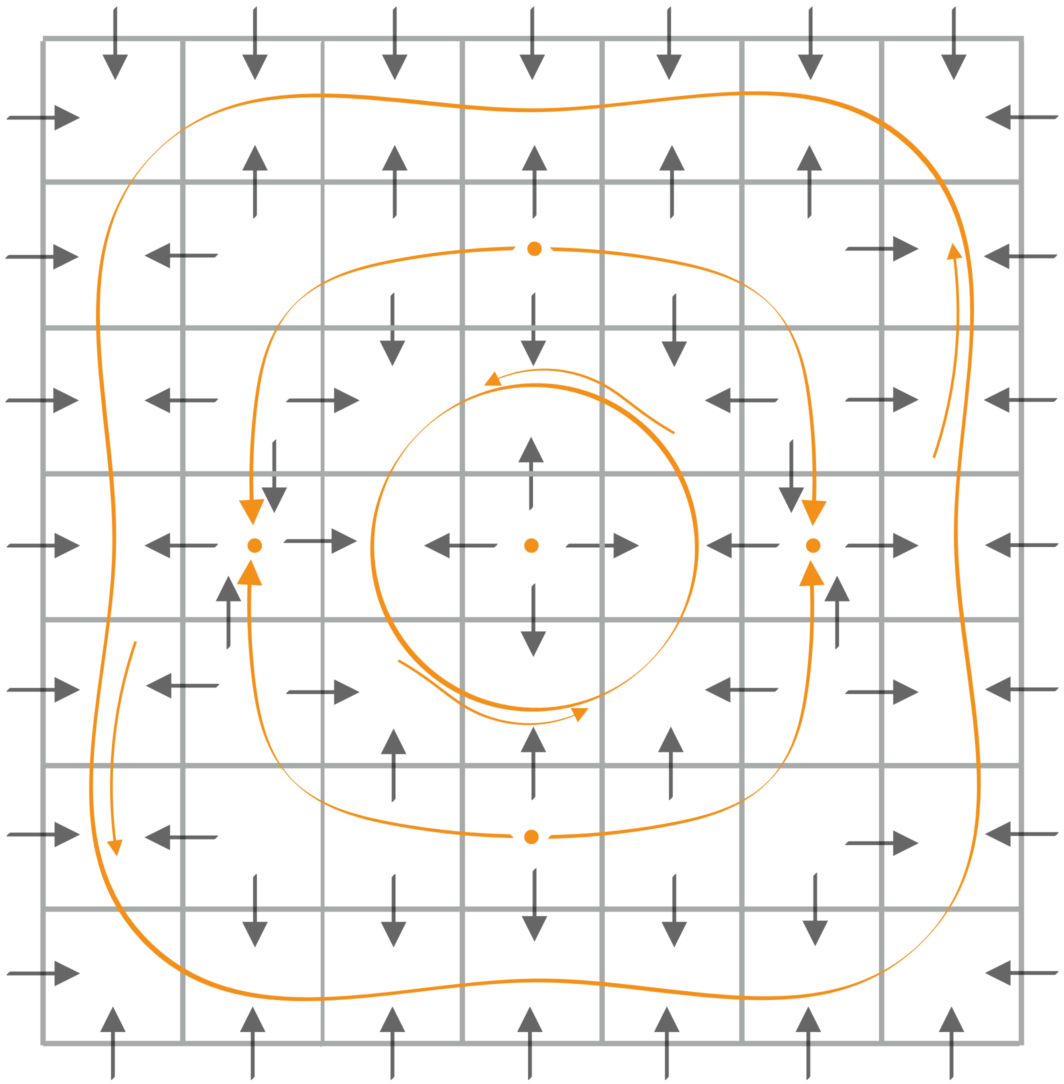


Remark: computational connection matrix theory generalizes to multi-vectors (Mrozek)

application ii:

transversality

transversality models



topological spaces are approximated with cell complexes

continuous dynamics are approximated with graph on top cells X_n

poset **P** of strongly connected components

$$(X_n, \leq) \longrightarrow (P, \leq)$$

transversality model

there is an edge $\xi \to \xi'$ between adjacent top cells unless flow is transverse to $\xi \cap \xi'$ in the direction $\xi' \to \xi$

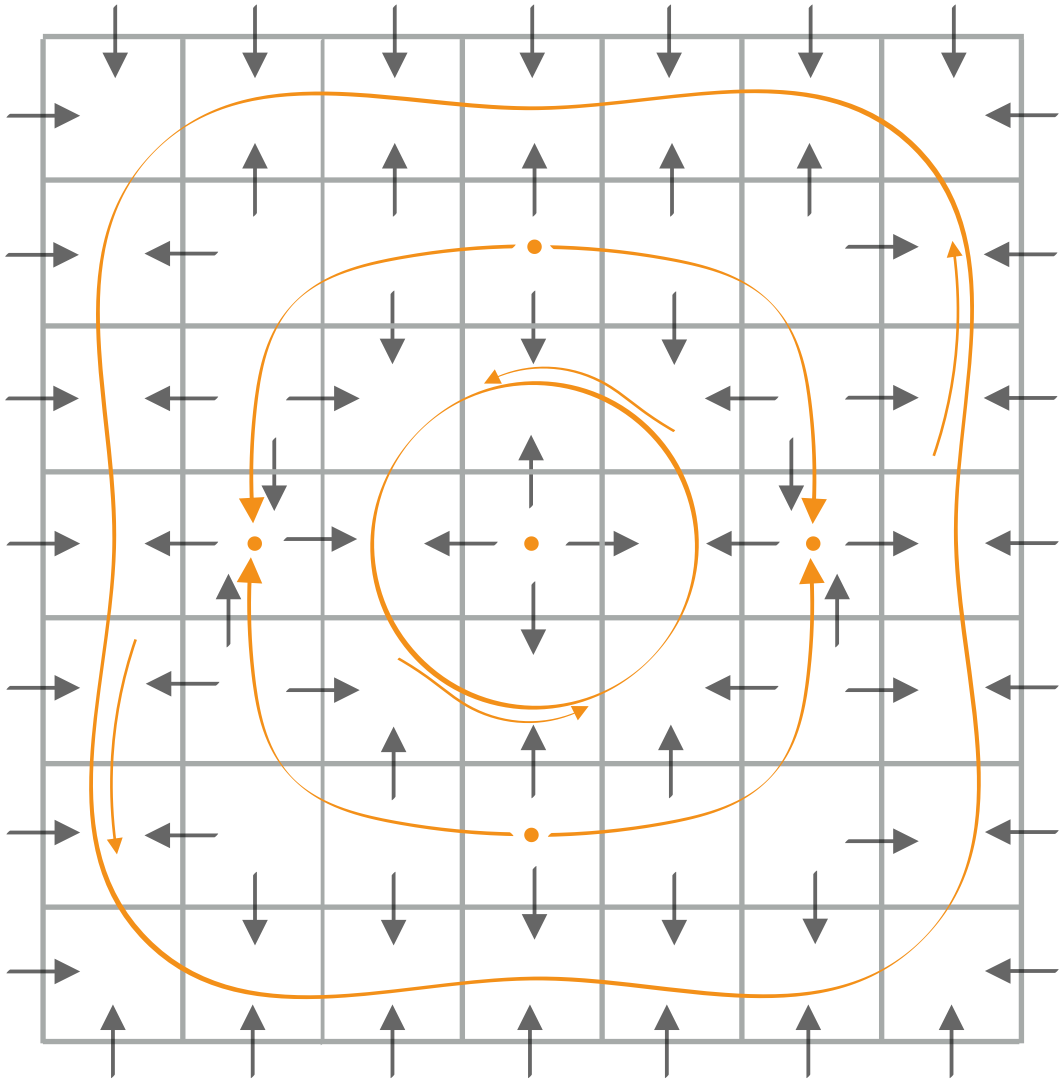
Theorem: if the graph is a transversality model then there is an extension

$$\begin{tikzcd} (X_n, \leq) \arrow[r] \arrow[d] & (X, \leq) \arrow[dl, dashed, "\text{graded cell complex}"] \\ (P, \leq) & \end{tikzcd}$$

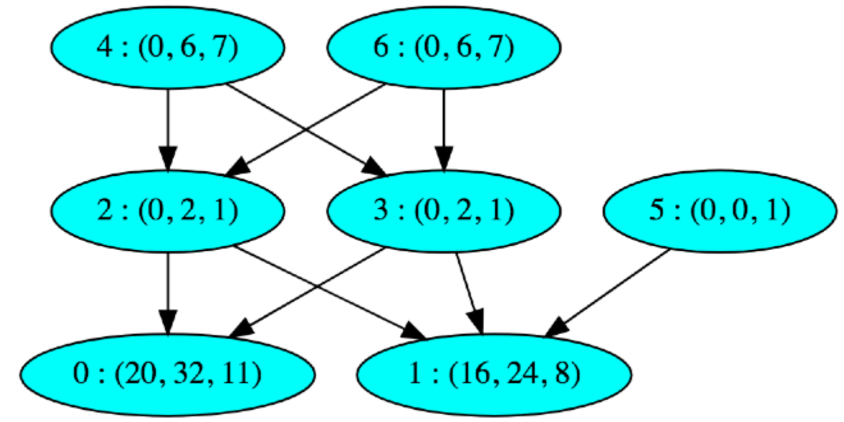
Remark: Computations + theorems are valid for any differential equation $\dot{x} = f(x)$ which is transverse to top cell boundaries in direction indicated

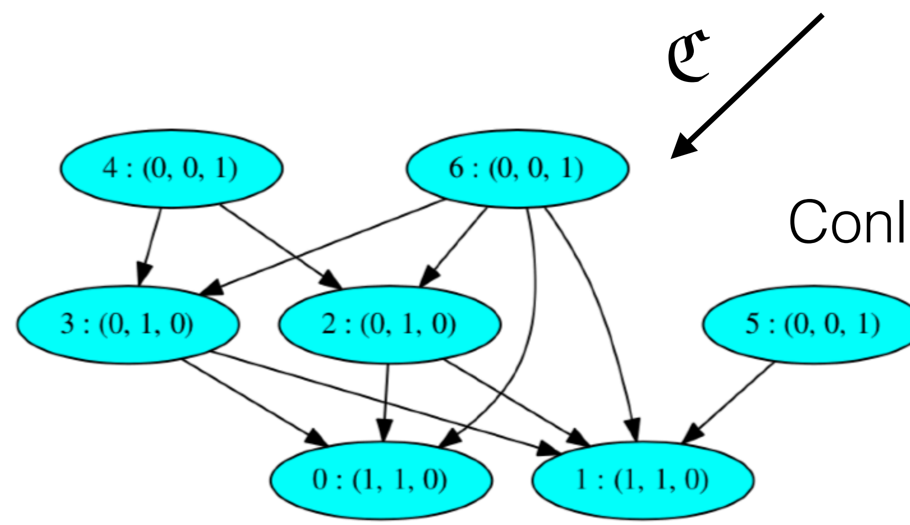
Harker + Mischaikow + S. + Vandervorst

transversality models ii



graded cell complex





Conley Complex

nontrivial boundaries are given edges

connection matrix is represented

with respect to a basis

different bases give different

qualitative descriptions of dynamics

in this example: four different bases

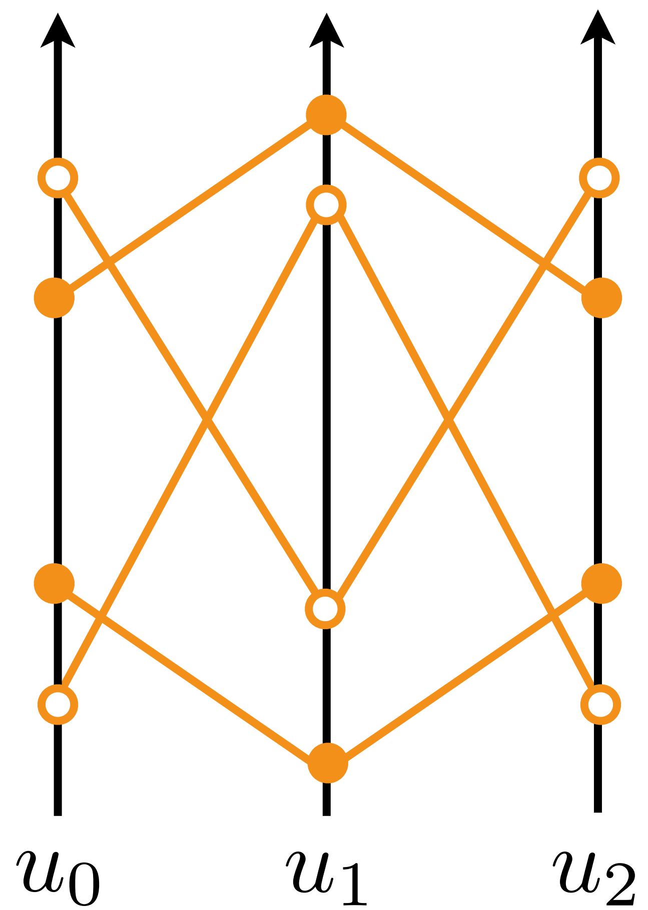
```
Connection Matrix Data
======================
  Boundaries of 0-cells (by cell index):
    Cell 0 (valuation 1) : set([])
    Cell 1 (valuation 0) : set([])
  Boundaries of 1-cells (by cell index):
    Cell 2 (valuation 2) : set([0L, 1])
    Cell 3 (valuation 3) : set([0L, 1])
    Cell 4 (valuation 0) : set([])
    Cell 5 (valuation 1) : set([])
  Boundaries of 2-cells (by cell index):
    Cell 6 (valuation 6) : set([2, 3, 4, 5])
    Cell 7 (valuation 5) : set([5])
    Cell 8 (valuation 4) : set([2, 3])
```

application iii:

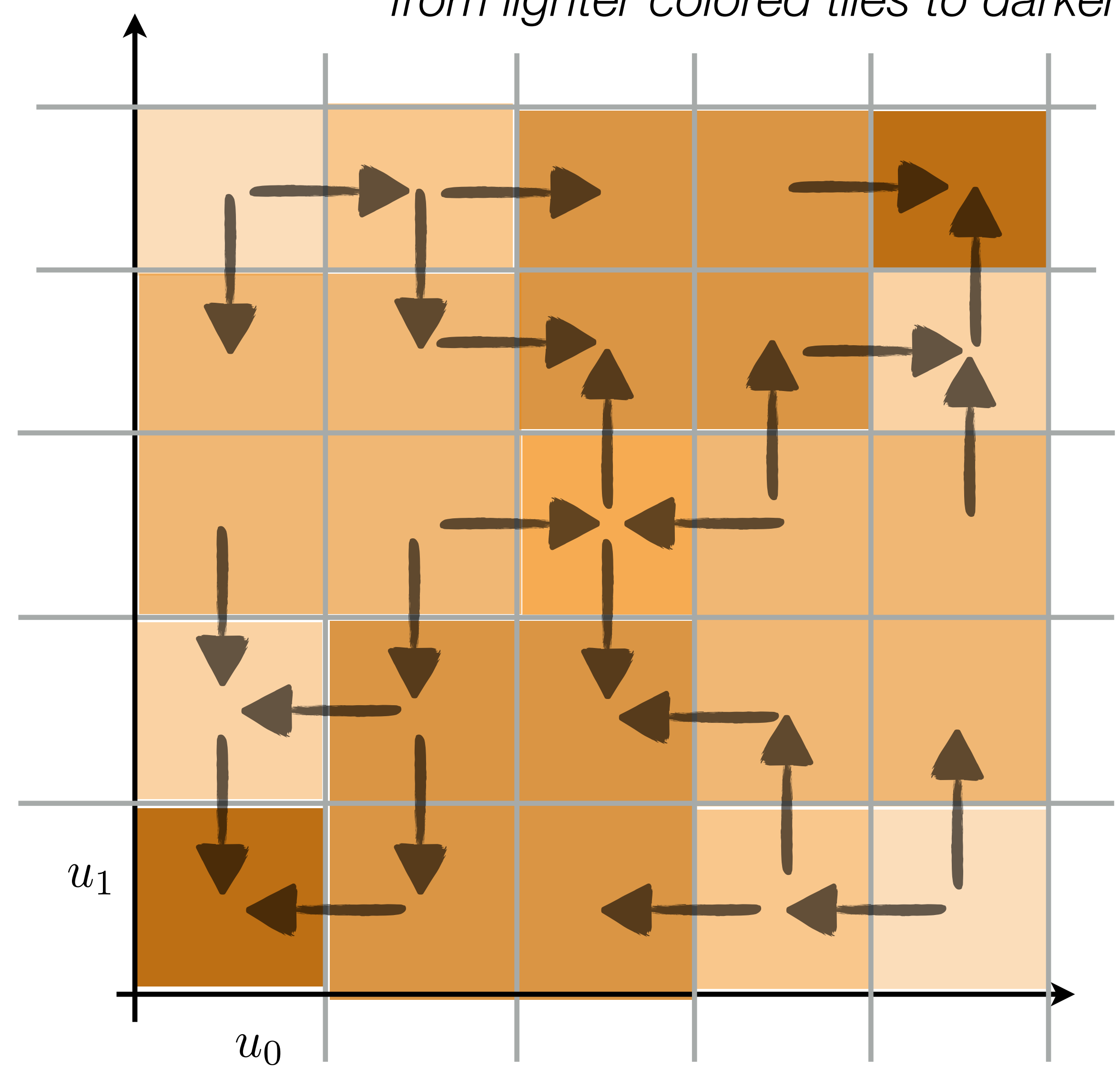
Morse theory on braids

Morse theory on braids

van den Berg, Ghrist, van der Vorst,
Inventiones Math. 2003



Solutions flow across boundary edges
from lighter colored tiles to darker



Braided equilibrium solutions to parabolic PDE
with periodic boundary conditions

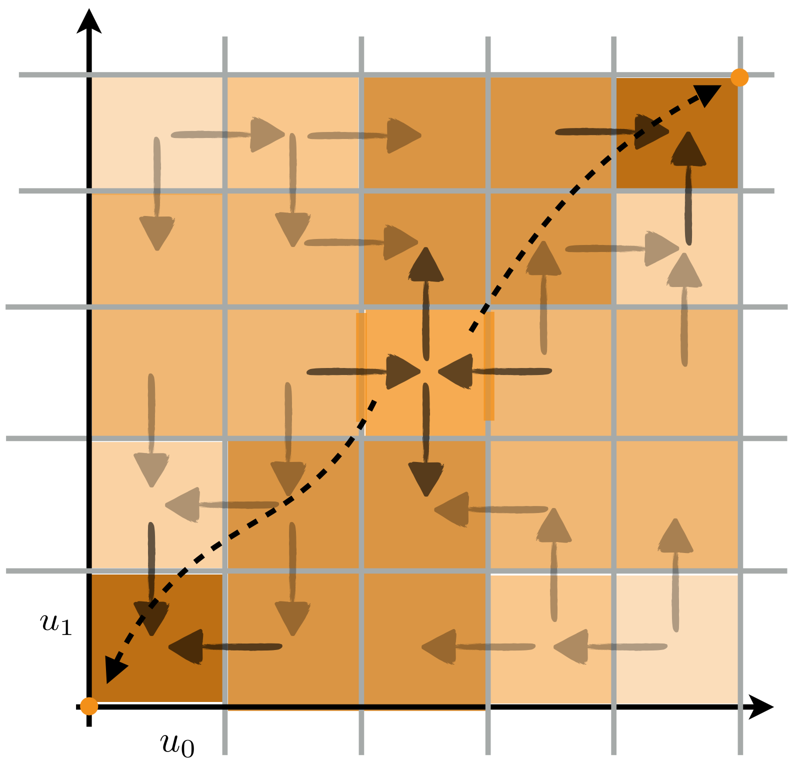
transversality model in $\mathbb{R}^2$
graded cubical complex in $\mathbb{R}^2$

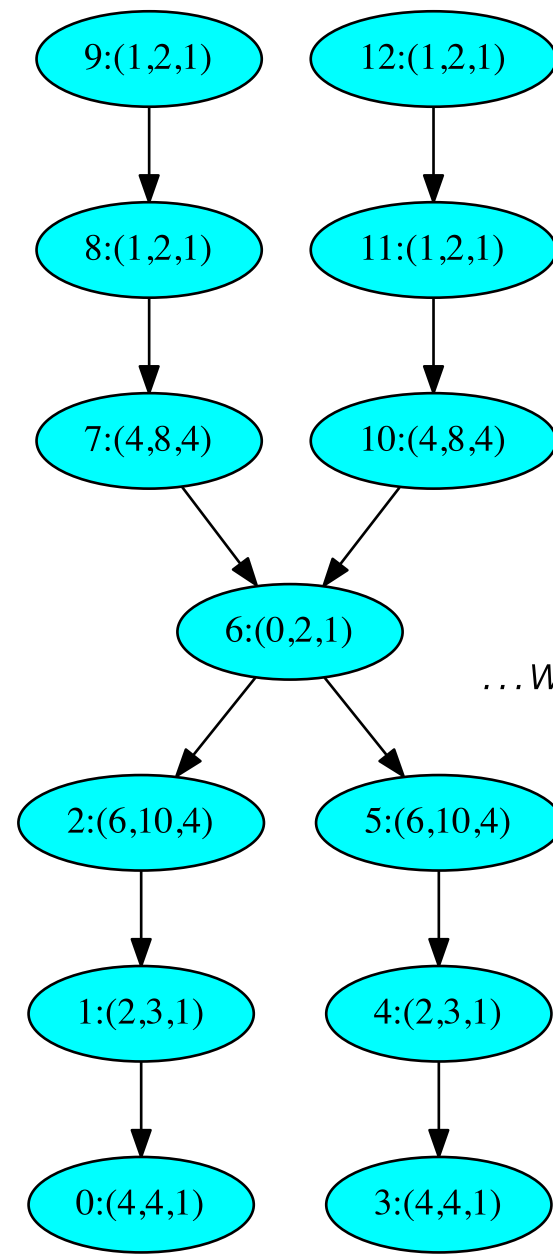
Fact: Nontrivial Conley indices imply existence of solutions to PDE

Fact: Nonzero entry in connection matrix between adjacent elements proves existence of connecting orbit

braids i

$$(\mathrm{X}, \leq) \xrightarrow{\quad \nu \quad} (\mathrm{P}, \leq)$$

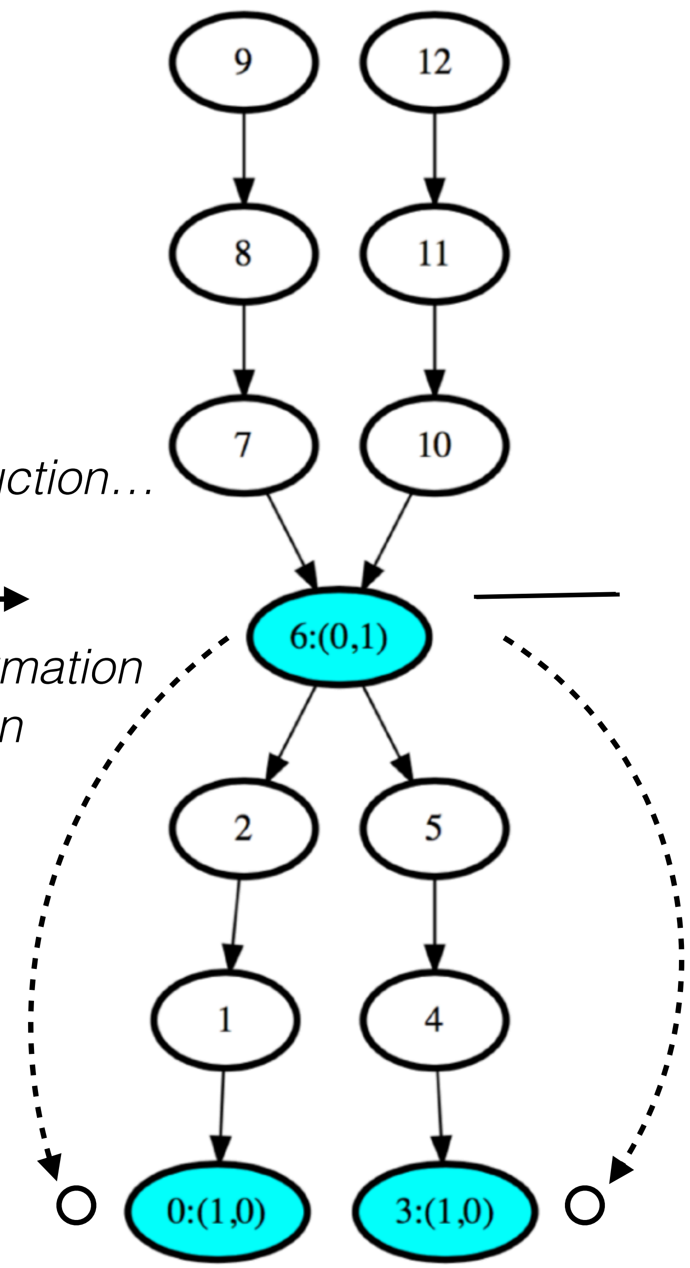




data reduction…

$\xrightarrow{\mathfrak{C}}$

…without information reduction



graded homotopy equivalence

$\uparrow\downarrow$

Conley complex

$$\Delta^M = \begin{array}{cc|c|c|c} & & 0 & 3 & 6 \\ & & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 & 1 \\ \hline 6 & 1 & 0 & 0 & 0 \end{array}$$

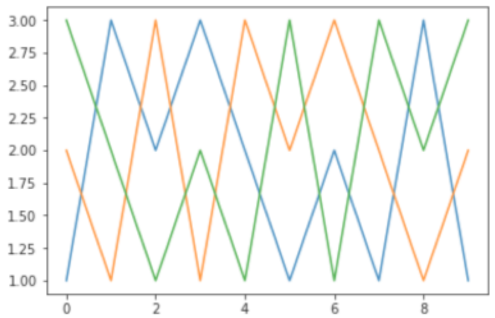
node index
cell dim.

chain-level data compression

144 cells $\rightleftarrows$ 3 cells

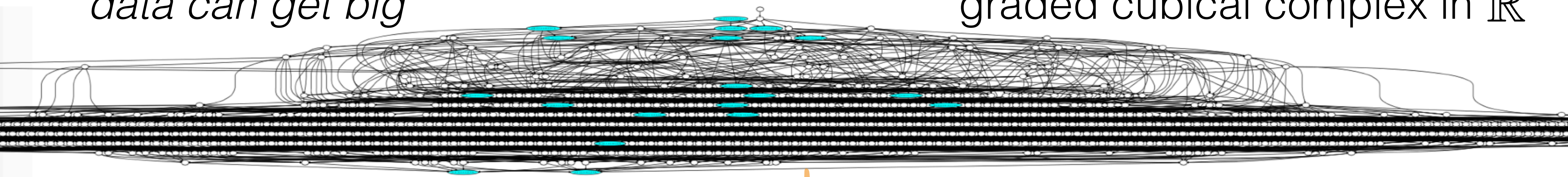
without loss of homological information

braids ii



data can get big

graded cubical complex in $\mathbb{R}^9$

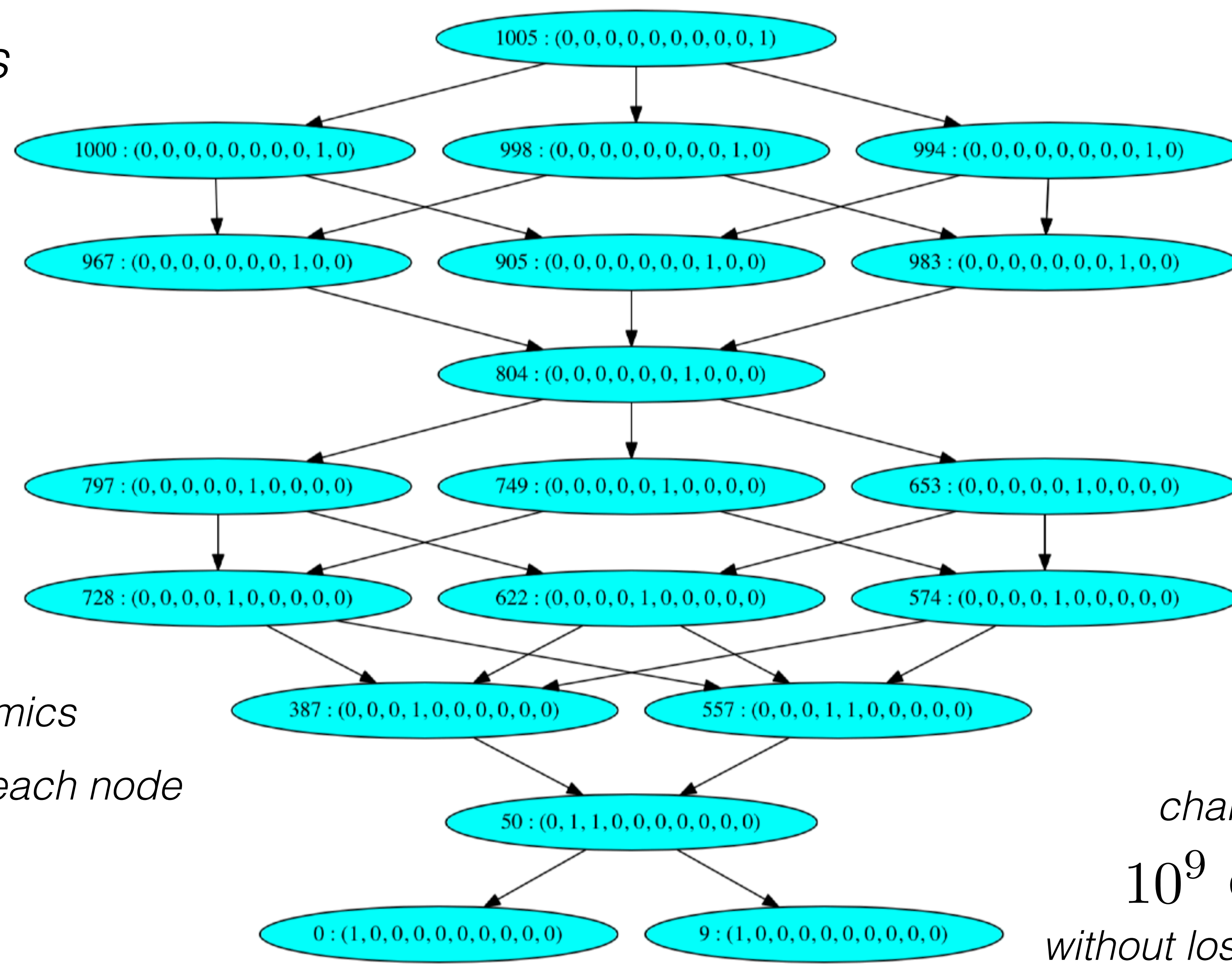


initial graded cubical complex

10^9 cells

$|P| \cong 1000$

restrict poset to nodes with nontrivial index



Conley-Morse Graph

organizes global dynamics

Conley index for each node

Conley complex

21 cells

19 nodes

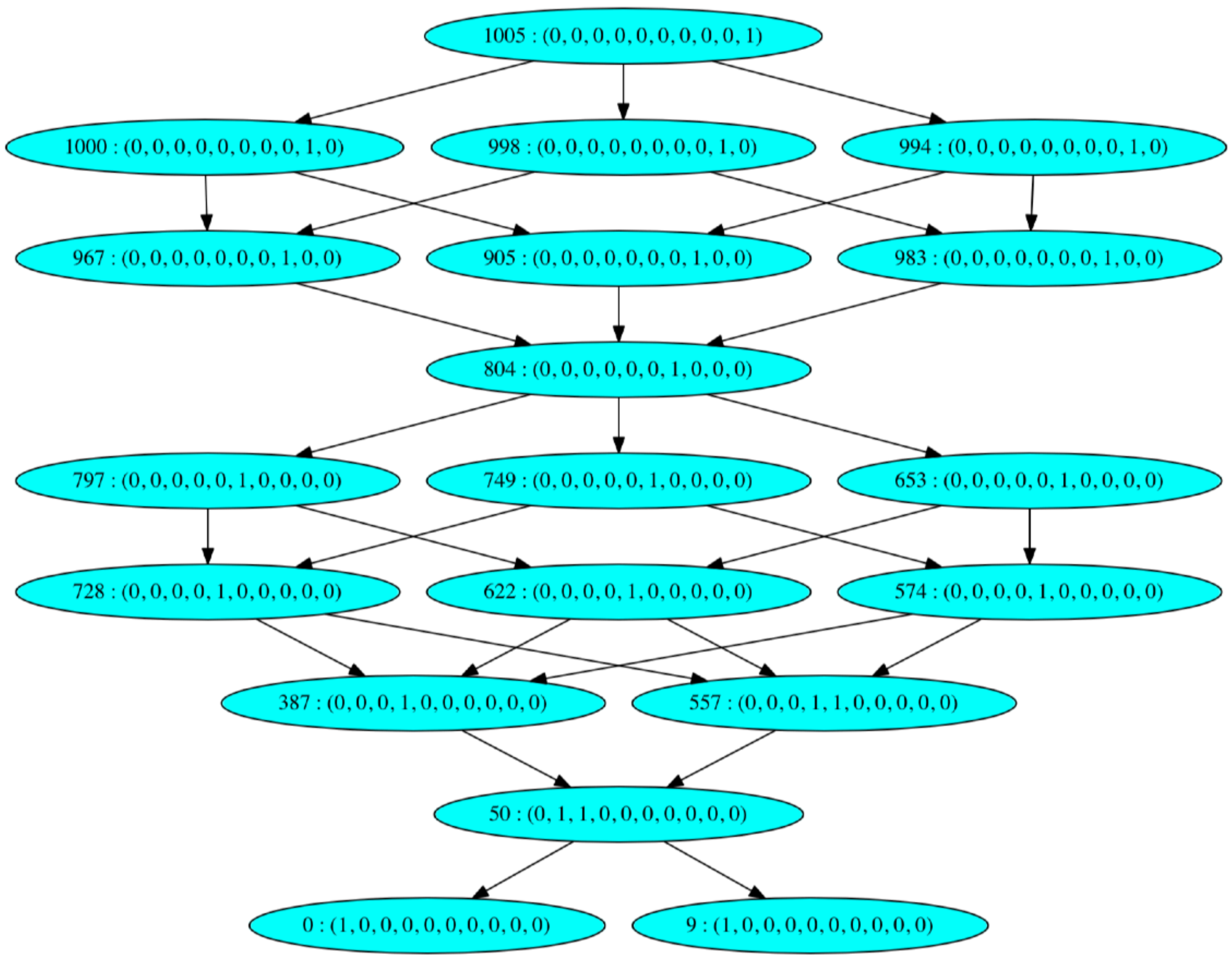
chain-level data compression

10^9 cells $\rightleftarrows$ 21 cells

without loss of homological information

braids iii

order data



Conley-Morse Graph

organizes global dynamics

Conley index for each node

chain data

```
Connection Matrix Data
======================
  Boundaries of 0-cells in Conley complex:
    0 : set()
    1 : set()
  Boundaries of 1-cells in Conley complex:
    2 : {0, 1}
  Boundaries of 2-cells in Conley complex:
    3 : set()
  Boundaries of 3-cells in Conley complex:
    4 : {3}
    5 : {3}
  Boundaries of 4-cells in Conley complex:
    6 : {4, 5}
    7 : {4, 5}
    8 : {4, 5}
    9 : set()
  Boundaries of 5-cells in Conley complex:
    10 : {8, 9, 6}
    11 : {8, 9, 7}
    12 : {9, 6, 7}
  Boundaries of 6-cells in Conley complex:
    13 : set()
  Boundaries of 7-cells in Conley complex:
    14 : {13}
    15 : {13}
    16 : {13}
  Boundaries of 8-cells in Conley complex:
    17 : {14, 15}
    18 : {16, 14}
    19 : {16, 15}
```

Conley Complex

connection matrix

boundaries can be queried from the data structure thank you for your attention

Collaborators:
S. Harker
K. Mischaikow
R. van der Vorst