Massachusetts Institute of Technology
6.046J/18.410J: Introduction to Algorithms
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Handout 9
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$T(n) = aT(\frac{n}{b}) + f(n)$

$c_{crit} = \log_b a$

Master Theorem Worksheet

This is a worksheet to help you master solving recurrence relations using the Master Theorem. For each recurrence, either give the asympotic solution using the Master Theorem (state which case), or else state that the Master Theorem doesn't apply. You should be able to go through these **25** recurrences in **10** minutes.

Problem 1-1. $T(n) = 3T(n/2) + n^2$

$c_{crit} = \log_2 3 \approx 1.585$
$c = 2$

~~1~~ $c > c_{crit}$

~~2~~ $f(n) = \Theta(n^{c_{crit}} \log^k n)$ for a $k \geq 0$
$n^2 = \Theta(n^{\log_2 3} \log^k n)$

3 $f(n) = \Omega(n^2)$ ✓

$\boxed{T(n) = \Theta(n^2),\ \text{case } 3}$

Problem 1-2. $T(n) = 7T(n/2) + n^2$

$c_{crit} = \log_2 7 \approx 2.8$

1 $c < c_{crit}$
$f(n) = O(n^c)$
$n^2 = O(n^2)$

$\boxed{T(n) = \Theta(n^{\log_2 7})\ \text{case } 1}$

Problem 1-3. $T(n) = 4T(n/2) + n^2$

$c_{crit} = \log_2 4 \approx 2$ ~~1~~

2 $f(n) = \Theta(n^{c_{crit}} \log^k n)$ for $k \geq 0$
$n^2 = \Theta(n^2 \log^k n)$
true when $k = 0$

$\boxed{T(n) = \Theta(n^2 \log n),\ \text{case } 2}$

Problem 1-4. $T(n) = 3T(n/4) + n \lg n$

$c_{crit} = \log_4 3 \approx 0.8$

~~1~~ $n\log n = O(n^1)$
~~2~~ $n\log n = \Theta(n^{\log_4 3} \log^k n)$ for $k \geq 0$
3 $n\log n = \Omega(n^1)$
$1 > \log_4 3$

$3\left(\frac{n\log n}{4}\right) \leq k\, n\log n$ for $k < 1$
true when $k \geq 3/4$

$\boxed{T(n) = \Theta(n\log n),\ \text{case } 3}$

Problem 1-5. $T(n) = 4T(n/2) + \lg n$

$\log_2 4 = 2$
$c = 0$?

1 $c < c_{crit}$

$\boxed{T(n) = \Theta(n^2),\ \text{case } 1}$

Problem 1-6. $T(n) = T(n-1) + n$

Inadmissible, not divide-conquer algorithm

Problem 1-7. $T(n) = 4T(n/2) + n^2 \lg n$

$\log_2 4 = 2$

② $n^2 \log n = \theta(n^2 \log^k n)$ for $k \geq 0$

true when $k = 1$

$\boxed{T(n) = \theta(n^2 \log^2 n), \text{ case } 2}$

Problem 1-8. $T(n) = 5T(n/2) + n^2 \lg n$

$\log_2 5 \approx 2.3$

① $c < c_{crit}$

$2 < 2.3$

$\boxed{T(n) = \theta(n^{\log_2 5}), \text{ case } 1}$

Problem 1-9. $T(n) = 3T(n/3) + n/\lg n$

Inadmissible

Problem 1-10. $T(n) = 2T(n/4) + c$

$c_{crit} = \log_4 2 = \frac{1}{2}$

$c = O(n^c)$

$\boxed{T(n) = \theta(n^{\frac{1}{2}}), \text{ case } 1}$

Problem 1-11. $T(n) = T(n/4) + \lg n$

$c_{crit} = \log_4 1 \approx 0$

$\log n = \theta(n^0 \log^k n)$ when $k = 1$

$\boxed{T(n) = \theta(\log^2 n), \text{ case } 2}$

Problem 1-12. $T(n) = T(n/2) + T(n/4) + n^2$

Doesn't apply

Problem 1-13. $T(n) = 2T(n/4) + \lg n$

$c_{crit} = \log_4 2 = \frac{1}{2}$

$c = 0$

$0 < \frac{1}{2}$ $\log n = O(n)$

$\boxed{T(n) = \theta(n^{\frac{1}{2}}), \text{ case } 1}$

Problem 1-14. $T(n) = 3T(n/3) + n \lg n$

$c_{crit} = \log_3 3 = 1$

$c = 1$

$n \log n = \theta(n \log^k n)$ when $k = 1$

$\boxed{T(n) = \theta(n \log^2 n), \text{ case } 2}$

Problem 1-15. $T(n) = 8T((n - \sqrt{n})/4) + n^2$

Master theorem does not apply

Problem 1-16. $T(n) = 2T(n/4) + \sqrt{n}$

$c_{crit} = \log_4 2 = \frac{1}{2}$

$c = \frac{1}{2}$

$\sqrt{n} = \Theta(n^{\frac{1}{2}} \log^k n)$ when $k = 0$

$\boxed{T(n) = \Theta(n^{\frac{1}{2}} \log n), \text{ case } 2}$

Problem 1-17. $T(n) = 2T(n/4) + n^{0.51}$

$c_{crit} = \log_4 2 = \frac{1}{2}$

$c = 0.51$

$n^{0.51} = \Omega(n^{0.51})$ $\quad c > c_{crit}$

$2\left(\frac{n^{0.51}}{4}\right) \leq k n^{0.51}$ when $k = \frac{1}{2}$

$\boxed{T(n) = \Theta(n^{0.51}), \text{ case } 3}$

Problem 1-18. $T(n) = 16T(n/4) + n!$

$c_{crit} = \log_4 16 = 2$

$c > c_{crit}$ as $n \to \infty$

$\boxed{T(n) = \Theta(n!), \text{ case } 3}$

Problem 1-19. $T(n) = 3T(n/2) + n$

$c_{crit} = \log_2 3 \approx 1.6$

$c = 1$

$c < c_{crit}$

$1 < 1.6$

$n = O(n)$

$\boxed{T(n) = \Theta(n^{\log_2 3}), \text{ case } 1}$

Problem 1-20. $T(n) = 4T(n/2) + cn$

$c_{crit} = \log_2 4 = 2$

$c = 1$

$c < c_{crit}$ $\quad cn = O(n^1)$

$\boxed{T(n) = \Theta(n^2), \text{ case } 1}$

Problem 1-21. $T(n) = 3T(n/3) + n/2$

$c_{crit} = \log_3 3 = 1$

$c = 1$

$n/2 = \Theta(n \log^k n)$ for $k \geq 0$

true when $k = 0$

$\boxed{T(n) = \Theta(n \log n), \text{ case } 2}$

Problem 1-22. $T(n) = 4T(n/2) + n/\lg n$

$c_{crit} = \log_2 4 = 2$

$c = 1$

$\frac{n}{\log n} = O(n)$

$\boxed{T(n) = \Theta(n^2), \text{ case } 1}$

Problem 1-23. $T(n) = 7T(n/3) + n^2$

$c_{crit} = \log_3 7 \approx 1.8$

$c = 2$

$c > c_{crit}$

$n^2 = \Omega(n^2)$

$\boxed{T(n) = \Theta(n^2), \text{ case } 3}$

Problem 1-24. $T(n) = 8T(n/3) + 2^n$

$c_{crit} = \log_3 8 \approx 1.9$

$c > 2$ so

$c > c_{crit}$

$\boxed{T(n) = \Theta(2^n), \text{ case } 3}$

Problem 1-25. $T(n) = 16T(n/4) + n$

$c_{crit} = \log_4 16 = 4$

$c = 1$

$c < c_{crit}$

$n = O(n^1)$

$\boxed{T(n) = \Theta(n^4), \text{ case } 1}$