一. 向量组线性相关(无关)及线性表出的基本概念

1. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_m$线性相关

$\Leftrightarrow$存在不全为零的数$k_1,k_2,\cdots,k_m$,使得$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m=0$

2. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关

$\Leftrightarrow$"$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m=0\Rightarrow k_1=k_2=\cdots=k_m=0$"

3. $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)\Rightarrow R(A)=R\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$

(1)$\alpha_1,\alpha_2,\cdots,\alpha_n(\in R^m)$线性相关$\Leftrightarrow R(A)<n\Leftrightarrow Ax=0$有非零解

(2)$\alpha_1,\alpha_2,\cdots,\alpha_n(\in R^m)$线性无关$\Leftrightarrow R(A)=n\Leftrightarrow Ax=0$只有零解

4. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关(相关)$\Leftrightarrow R\{\alpha_1,\alpha_2,\cdots,\alpha_n\}=n(<n)$

5. β可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出

$\Leftrightarrow$存在$k_1,k_2,\cdots,k_n$,使得$\beta=k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n$

6. 向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性相关

$\Leftrightarrow$ 存在一个向量可由其余的线性表出

7. b可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出$\Leftrightarrow Ax=b$有解$\Leftrightarrow R(A)=R(A|b)$

8. 向量组(I)$:\alpha_1,\alpha_2,\cdots,\alpha_s$可由向量组(II)$:\beta_1,\beta_2,\cdots,\beta_t$线性表出

$\Rightarrow R(\mathrm{I})\le R(\mathrm{II})$.

9. 向量组(I)$:\alpha_1,\alpha_2,\cdots,\alpha_s$与向量组(II)$:\beta_1,\beta_2,\cdots,\beta_t$等价

$\Rightarrow R(\mathrm{I})=R(\mathrm{II})$.

1.设$\alpha_1,\alpha_2,\cdots,\alpha_r$可由$\beta_1,\beta_2,\cdots,\beta_s$线性表示,则$r$与$s$的关系是(　)

(A) $r \le s$　　(B) $r \ge s$　　(C) $r < s$　　(D) r与s无关.

答案:(D).

$\alpha_1,\alpha_2,\cdots,\alpha_r$线性相关性与$\beta_1,\beta_2,\cdots,\beta_s$的线性相关性不清楚

注:

$\alpha_1,\alpha_2,\cdots,\alpha_r$可由$\beta_1,\beta_2,\cdots,\beta_s$线性表出$\begin{cases} (1)\ \alpha_1,\alpha_2,\cdots,\alpha_r\text{线性无关} \Rightarrow r \le s \\ (2)\ r > s \Rightarrow \alpha_1,\alpha_2,\cdots,\alpha_r\text{线性相关} \end{cases}$

例:

(1): (I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$.

(2): (I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$.

(3): (I) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}$; (II) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$.

2.设$\alpha_1,\alpha_2,\alpha_3$线性相关,$\alpha_2,\alpha_3,\alpha_4$线性无关,**证明:**

(1).α_1能由α_2,α_3线性表出;

(2).α_4不能由$\alpha_1,\alpha_2,\alpha_3$线性表出.

证明:

(1)$\alpha_2,\alpha_3,\alpha_4$线性无关$\Rightarrow\alpha_2,\alpha_3$线性无关 $\Rightarrow\alpha_1$可由α_2,α_3线性表出

又$\alpha_1,\alpha_2,\alpha_3$线性相关 $\Rightarrow\alpha_1=l_2\alpha_2+l_3\alpha_3$

(2)若α_4可由$\alpha_1,\alpha_2,\alpha_3$线性表出$\Leftrightarrow\alpha_4=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3\Rightarrow$

$\alpha_4=k_1(l_2\alpha_2+l_3\alpha_3)+k_2\alpha_2+k_3\alpha_3=(k_1l_2+k_2)\alpha_2+(k_1l_3+k_3)\alpha_3$,矛盾!

3. 已知$\alpha_1,\alpha_2,\alpha_3$线性无关,**证明:** $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$线性无关.

证(法1): $\boxed{\text{令}\ k_1(\alpha_1+\alpha_2)+k_2(\alpha_2+\alpha_3)+k_3(\alpha_3+\alpha_1)=0}$

$$\Rightarrow (k_1+k_3)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3=0$$

由向量$\alpha_1,\alpha_2,\alpha_3$线性无关可得：$(k_1+k_3)=(k_1+k_2)=(k_2+k_3)=0,$

即$\begin{cases} k_1+\qquad k_3=0 \\ k_1+k_2\qquad =0 \\ \qquad k_2+k_3=0 \end{cases}$，其系数行列式为$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}=2\neq 0\Rightarrow$

$\boxed{k_1=k_2=k_3=0}\Rightarrow$向量$(\alpha_1+\alpha_2),(\alpha_2+\alpha_3),(\alpha_3+\alpha_1)$线性无关.

(课本P_{136},P_{140})

3. 已知$\alpha_1,\alpha_2,\alpha_3$线性无关, **证明:** $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$线性无关.

证(法2): 令$\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\beta_3=\alpha_3+\alpha_1$

$$\Rightarrow B=(\beta_1,\beta_2,\beta_3)=(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$$

$$=AP$$

又$|P|=\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=2\neq 0\Leftrightarrow P$可逆 $\Rightarrow R(B)=R(AP)=R(A)$

又由$\alpha_1,\alpha_2,\alpha_3$线性无关$\Rightarrow R(A)=3$ $\Rightarrow R(B)=3$ $\Rightarrow \beta_1,\beta_2,\beta_3$线性无关

注: 若β_1,β_2,β_3线性无关,则$R(A)$=$R(B)$=3 $\Rightarrow \alpha_1,\alpha_2,\alpha_3$线性无关.

$$\boxed{\alpha_1,\alpha_2,\alpha_3\text{线性无关}\Leftrightarrow \beta_1,\beta_2,\beta_3\text{线性无关}}$$

4. 已知$\alpha_1,\alpha_2,\cdots,\alpha_m;\beta_1=\alpha_1+\alpha_2,\beta_2=\alpha_2+\alpha_3,\cdots,\beta_m=\alpha_m+\alpha_1$.

证明:(1)当m为偶数时,向量组$\beta_1,\beta_2,\cdots,\beta_m$线性相关;

(2)当m为奇数时,

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关$\Leftrightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关.

证(法2): $B=(\beta_1,\beta_2,\cdots,\beta_m)=(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\cdots,\alpha_m+\alpha_1)$

$$=(\alpha_1,\alpha_2,\cdots,\alpha_m)\begin{pmatrix}1&0&\cdots&0&1\\1&1&\ddots&&0\\0&1&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&1&0\\0&\cdots&0&1&1\end{pmatrix}=AP$$

$$|P|=\begin{vmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \ddots & & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{vmatrix}=1+(-1)^{m+1}=\begin{cases} 0, & m=2k \\ 2, & m=2k+1 \end{cases}$$

$$\Rightarrow \begin{cases} m=2k\text{时}, P\text{不可逆} \Leftrightarrow R(P)<m \Rightarrow R(B)=R(AP)\le R(P)<m \\ m=2k+1\text{时}, P\text{可逆} \Leftrightarrow R(P)=m \Rightarrow R(B)=R(AP)=R(A) \end{cases}$$

$$\Rightarrow \begin{cases} m=2k\text{时}, \beta_1,\beta_2,\cdots,\beta_m\text{线性相关} \\ m=2k+1\text{时}, \beta_1,\beta_2,\cdots,\beta_m\text{与}\alpha_1,\alpha_2,\cdots,\alpha_m\text{有相同的相关与无关性} \end{cases}$$

注: $m=2k+1$时,

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关 $\Leftrightarrow R(A)=m \Leftrightarrow R(B)=m \Leftrightarrow$ $\beta_1,\beta_2,\cdots,\beta_m$线性无关

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性相关 $\Leftrightarrow R(A)<m \Leftrightarrow R(B)<m \Leftrightarrow$ $\beta_1,\beta_2,\cdots,\beta_m$线性相关

5. 已知$\alpha_1,\alpha_2,\cdots,\alpha_m;\beta_1=\alpha_1,\beta_2=\alpha_1+\alpha_2,\cdots,\beta_m=\alpha_1+\alpha_2+\cdots+\alpha_m$.

证明: $\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关$\Leftrightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关.

证: $B=(\beta_1,\beta_2,\cdots,\beta_m)=(\alpha_1,\alpha_1+\alpha_2,\cdots,\alpha_1+\alpha_2+\cdots+\alpha_m)$

$$=(\alpha_1,\alpha_2,\cdots,\alpha_m)\begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & 1 & \cdots & 1\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}=AP$$

P可逆$\Rightarrow R(B)=R(AP)=R(A)$

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性无关$\Leftrightarrow R(A)=m\Leftrightarrow R(B)=m\Leftrightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关

$\alpha_1,\alpha_2,\cdots,\alpha_m$线性相关$\Leftrightarrow R(A)<m\Leftrightarrow R(B)<m\Leftrightarrow\beta_1,\beta_2,\cdots,\beta_m$线性相关

6. 设向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$中$\alpha_1\neq 0$,且每个$\alpha_i(i=1,2,\cdots,n)$都不能由它前面的$i-1$个向量$\alpha_1,\alpha_2,\cdots,\alpha_{i-1}$线性表出,求证:向量组

$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关.

证:设$k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n=0$

如果$k_n\neq 0 \Rightarrow \alpha_n=-\frac{k_1}{k_n}\alpha_1-\frac{k_2}{k_n}\alpha_2-\cdots-\frac{k_{n-1}}{k_n}\alpha_{n-1}$

α_n可由$\alpha_1,\alpha_2,\cdots,\alpha_{n-1}$线性表出,与假设矛盾.

因此$k_n=0$

$k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n=0$变为$k_1\alpha_1+k_2\alpha_2+\cdots+k_{n-1}\alpha_{n-1}=0$

同理可证$k_{n-1}=0$, 以此类推$k_{n-2},k_{n-3},\cdots,k_2,k_1$都等于零.

所以向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$线性无关

7.设A是n阶方阵,若存在正整数k,使$A^k x=0$有解向量α,且$A^{k-1}\alpha\neq 0$,

证明: $\alpha, A\alpha, A^2\alpha, \cdots, A^{k-1}\alpha$线性无关.

证: (1) $A^k\alpha=0 \Rightarrow A^m\alpha=0(m\geq k)$

(2) $A^{k-1}\alpha\neq 0 \Rightarrow$ "$lA^{k-1}\alpha=0 \Leftrightarrow l=0$"

令 $l_0\alpha+l_1A\alpha+l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}\alpha=0$

$\Rightarrow A^{k-1}(l_0\alpha+\underline{l_1A\alpha+l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}}\alpha)=0 \Rightarrow l_0=0$

$\Rightarrow A^{k-2}(l_1A\alpha+\underline{l_2A^2\alpha+\cdots+l_{k-1}A^{k-1}}\alpha)=0 \Rightarrow l_1=0$

$\Rightarrow l_i=0(i=0,1,\ldots,k-1)$

$\Rightarrow \alpha, A\alpha, A^2\alpha, \cdots, A^{k-1}\alpha$线性无关.

$7'$. 设A是n阶方阵,$\alpha_1,\alpha_2,\alpha_3$为$n$维向量组,其中$\alpha_1\neq 0$,且满足:

$A\alpha_1=2\alpha_1, A\alpha_2=\alpha_1+2\alpha_2, A\alpha_3=\alpha_2+2\alpha_3$,证明:$\alpha_1,\alpha_2,\alpha_3$线性无关.

证: $A\alpha_1=2\alpha_1 \Leftrightarrow (A-2I)\alpha_1=0;$

$A\alpha_2=\alpha_1+2\alpha_2 \Leftrightarrow (A-2I)\alpha_2=\alpha_1;$

$A\alpha_3=\alpha_2+2\alpha_3 \Leftrightarrow (A-2I)\alpha_3=\alpha_2.$

$$\boxed{\Rightarrow \begin{cases}(A-2I)^2\alpha_1=0\\ (A-2I)^2\alpha_2=0\\ (A-2I)^2\alpha_3=\alpha_1\end{cases}}$$

令 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=0$(1),则

$(A-2I)(k_1\alpha_1+k_2\alpha_2+k_3\alpha_3)=0 \Rightarrow k_2\alpha_1+k_3\alpha_2=0$(2)

$(A-2I)(k_2\alpha_1+k_3\alpha_2)=0 \Rightarrow k_3\alpha_1=0$(3)

又$\alpha_1\neq 0$ $\Rightarrow k_3=0$

将$k_3=0$代入(2)、(1),得 $k_1=k_2=k_3=0$

$\Rightarrow \alpha_1,\alpha_2,\alpha_3$线性无关.

注:$A\alpha_1,A\alpha_2,A\alpha_3$线性无关.

8. 设η^*是非齐次线性方程组$Ax=b$的一个解，$\xi_1,\xi_2,\cdots,\xi_{n-r}$是对应齐次线性方程组的一个基础解系，

证明：(1) $\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关；

(2) $\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关.

证：(1)(法1) $\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关

若 $\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性相关

$$\left.\begin{array}{l}\xi_1,\xi_2,\cdots,\xi_{n-r}\text{线性无关}\\ \text{若 }\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}\text{线性相关}\end{array}\right\}\Rightarrow \eta^*=k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}$$

即η^*为$Ax=0$的解. 矛盾！

$\Rightarrow$ $\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关.

证：(1)(法2) $k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} = 0$

若$k_0 \neq 0$，则η^*可由$\xi_1, \xi_2, \cdots, \xi_{n-r}$线性表出.

即η^*为$Ax = 0$的解. 矛盾！

故$k_0 = 0 \Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0$

$\Rightarrow \eta^*, \xi_1, \xi_2, \cdots, \xi_{n-r}$线性无关.

证:(1)(法3) $k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} = 0$

注: $\boxed{A\eta^* = b, A\xi_i = 0(i = 1, 2, \cdots, n-r)}$

$\Rightarrow A(k_0\eta^* + k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r}) = 0$

$\Rightarrow k_0 b = 0(b \neq 0) \Rightarrow k_0 = 0$

$\Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0$

$\Rightarrow \eta^*, \xi_1, \xi_2, \cdots, \xi_{n-r}$线性无关.

证:(2)(法1) $k_0\eta^*+k_1(\eta^*+\xi_1)+k_2(\eta^*+\xi_2)+\cdots+k_{n-r}(\eta^*+\xi_{n-r})=0$

$$\Rightarrow (k_0+k_1+\cdots+k_{n-r})\eta^*+k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}=0$$

$$\Rightarrow k_0+k_1+\cdots+k_{n-r}=0,\ k_1=k_2=\cdots=k_{n-r}=0$$

$$\Rightarrow k_0=k_1=k_2=\cdots=k_{n-r}=0$$

$\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关

(2)(法2)

$$B=(\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,,\cdots,\eta^*+\xi_{n-r})=(\eta^*,\xi_1,\xi_2,,\cdots,\xi_{n-r})\begin{pmatrix}1&1&1&\cdots&1\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\end{pmatrix}$$

$$=AP$$

P可逆 $\Rightarrow R(B)=R(AP)=R(A)=n-r+1\ \Rightarrow \eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关.

注:$\eta^*,\eta^*+\xi_1,\eta^*+\xi_2,\cdots,\eta^*+\xi_{n-r}$线性无关$\Leftrightarrow\eta^*,\xi_1,\xi_2,\cdots,\xi_{n-r}$线性无关

9. 设$\alpha_1,\alpha_2,\cdots,\alpha_n \in R^n$, **证明:**

$$\alpha_1,\alpha_2,\cdots,\alpha_n\text{线性无关} \Leftrightarrow D=\begin{vmatrix} \alpha_1^{\mathrm{T}}\alpha_1 & \alpha_1^{\mathrm{T}}\alpha_2 & \cdots & \alpha_1^{\mathrm{T}}\alpha_n \\ \alpha_2^{\mathrm{T}}\alpha_1 & \alpha_2^{\mathrm{T}}\alpha_2 & \cdots & \alpha_2^{\mathrm{T}}\alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_n^{\mathrm{T}}\alpha_1 & \alpha_n^{\mathrm{T}}\alpha_2 & \cdots & \alpha_n^{\mathrm{T}}\alpha_n \end{vmatrix} \neq 0.$$

证: $A=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, 则

$$A^{\mathrm{T}}A=\begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \alpha_2^{\mathrm{T}} \\ \vdots \\ \alpha_n^{\mathrm{T}} \end{pmatrix}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\begin{pmatrix} \alpha_1^{\mathrm{T}}\alpha_1 & \alpha_1^{\mathrm{T}}\alpha_2 & \cdots & \alpha_1^{\mathrm{T}}\alpha_n \\ \alpha_2^{\mathrm{T}}\alpha_1 & \alpha_2^{\mathrm{T}}\alpha_2 & \cdots & \alpha_2^{\mathrm{T}}\alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_n^{\mathrm{T}}\alpha_1 & \alpha_n^{\mathrm{T}}\alpha_2 & \cdots & \alpha_n^{\mathrm{T}}\alpha_n \end{pmatrix}$$

$$\alpha_1,\alpha_2,\cdots,\alpha_n\text{线性无关} \Leftrightarrow \det A \neq 0 \Leftrightarrow D=(\det A)^2 \neq 0.$$

二. 关于向量组的秩及矩阵秩的结论

1. 设A, B均为m行的矩阵，证明：

$$\max\{R(A), R(B)\} \le R(A, B) \le R(A) + R(B).$$

2. 设A, B为同型矩阵，证明：$R(A+B) \le R(A) + R(B).$

3. $R(AB) \le \min\{R(A), R(B)\}.$(待续)

4. 设$A_{m\times n}, B_{n\times p}$满足$AB = 0$，证明：$R(A)+R(B) \le n.$

5.导出结论：

(1) $R(A^*)=\begin{cases} n, & R(A)=n \\ 0, & R(A)<n-1 \\ 1, & R(A)=n-1 \end{cases}$

(2) A为n阶矩阵,且$A^2=A \Rightarrow R(A)+R(A-I)=n$.

(3) A为n阶矩阵,且$A^2+A=0 \Rightarrow R(A)+R(A+I)=n$.

(4) A,B为n阶矩阵,且$ABA=B^{-1} \Rightarrow R(AB+I)+R(AB-I)=n$.

1. proof：设A,B均为m行的矩阵，证明：

$$\max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B).$$

证(法1)：令$A=(\alpha_1,\alpha_2,\cdots,\alpha_n),B=(\beta_1,\beta_2,\cdots,\beta_p)$，则

$$(A,B)=(\alpha_1\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_p)$$

设A与B的列向量组的最大无关组分别为$\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s}$与$\beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t}$，

记
$$C=(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s},\beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t})$$

则(A,B)的列向量组可由C的列向量组线性表出.

$$\Rightarrow \left\{\begin{matrix} R(A) \\ R(B) \end{matrix}\right. \le R(A,B)\le R(C)\le s+t=R(A)+R(B)$$

$$\Rightarrow \max\{R(A),R(B)\}\le R(A,B)\le R(A)+R(B).$$

1. proof： 设A, B均为m行的矩阵, **证明：**

$$\max\{R(A), R(B)\} \le R(A, B) \le R(A) + R(B).$$

证(法2)： $(A, B) = (I_m, I_m)\begin{pmatrix} A & \\ & B \end{pmatrix} \Rightarrow R(A, B) \le R\begin{pmatrix} A & \\ & B \end{pmatrix} = R(A) + R(B)$

$$\left.\begin{cases} A = (A, B)\begin{pmatrix} I_n \\ 0 \end{pmatrix} \Rightarrow R(A) \le R(A, B) \\ B = (A, B)\begin{pmatrix} 0 \\ I_p \end{pmatrix} \Rightarrow R(B) \le R(A, B) \end{cases}\right\} \Rightarrow \max\{R(A), R(B)\} \le R(A, B)$$

$$\Rightarrow \max\{R(A), R(B)\} \le R(A, B) \le R(A) + R(B).$$

注： $\boxed{A, B\text{均为}n\text{列的矩阵} \Rightarrow \max\{R(A), R(B)\} \le R\begin{pmatrix} A \\ B \end{pmatrix} \le R(A) + R(B).}$

2. proof： 设A,B为同型矩阵, 证明： $R(A+B)\le R(A)+R(B)$.

证(法1): 令$A=(\alpha_1,\alpha_2,\cdots,\alpha_n),B=(\beta_1,\beta_2,\cdots,\beta_n)$,则

$$A+B=(\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n)$$

$$(A,B)=(\alpha_1,\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_n)$$

$\underline{\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots,\alpha_n+\beta_n}$可由$\underline{\alpha_1\alpha_2,\cdots,\alpha_n},\underline{\beta_1,\beta_2,\cdots,\beta_n}$线性表出

设A与B的列向量组的最大无关组分别为$\underline{\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s}}$与$\underline{\beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t}}$,

$\underline{\{\alpha_1\alpha_2,\cdots,\alpha_n,\beta_1,\beta_2,\cdots,\beta_n\}}$与$\underline{\{\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s},\ \beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t}}\}$等价

故(A,B)的列向量组可由$C=(\alpha_{i_1},\alpha_{i_2},\cdots,\alpha_{i_s},\beta_{j_1},\beta_{j_2},\cdots,\beta_{j_t})$的列向量组线性表出

$$\Rightarrow R(A+B)\le R(A,B)=R(C)\le s+t=R(A)+R(B)$$

2. proof: $R(A+B)\le R(A)+R(B)$

证(法2): $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \xrightarrow{r_2+r_1} \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \xrightarrow{c_2+c_1} \begin{pmatrix} A+B & B \\ B & B \end{pmatrix}$

$$\Rightarrow R(A+B)\le R\begin{pmatrix} A+B & B \\ B & B \end{pmatrix} = R\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = R(A)+R(B)$$

证(法3): $A+B=(A,B)\begin{pmatrix} I_n \\ I_n \end{pmatrix} \Rightarrow R(A+B)\le R(A,B)\ \ (\le R(A)+R(B))$

$$(A,B)=(I_m,I_m)\begin{pmatrix} A & \\ & B \end{pmatrix} \Rightarrow R(A,B)\le R\begin{pmatrix} A & \\ & B \end{pmatrix} = R(A)+R(B)$$

$$\Rightarrow R(A+B)\le R(A,B)\le R(A)+R(B).$$

4. 设$A_{m\times n}, B_{n\times p}$满足$AB=0$, **证明：** $R(A)+R(B)\le n.$

(设n阶矩阵A,B满足$AB=0$, **证明：** $R(A)+R(B)\le n.$ 课本P_{153})

分析： $R(A)+R(B)\le n \Leftrightarrow R(B)\le n-R(A)$

$$n-R(A) = \dim W \Leftarrow W=\{x \mid Ax=0, \forall x\in R^n\}$$

证： (1). 若$B=0$, 则$R(A)+R(B)=R(A)+0=R(A)\le n.$

(2). 设$B=(b_1,b_2,\cdots,b_p)\ne 0$, 则

$$(Ab_1,Ab_2,\cdots,Ab_p)=A(b_1,b_2,\cdots,b_p)=AB=0$$

$$Ab_i=0(i=1,2,\cdots,p)\text{且}Ab_l=0(b_l\ne 0)$$

$b_i(i=1,2,\cdots,p)$为$Ax=0$的解，故可由基础解系$\xi_1,\xi_2,\cdots,\xi_{n-r}$ $(r=R(A))$线性表出. 所以

$$R(B)=R\{b_1,b_2,\cdots,b_p\}\le R\{\xi_1,\xi_2,\cdots,\xi_{n-r}\}=n-r=n-R(A)$$

即 $R(A)+R(B)\le n.$

(Sylvester公式): 设$A_{m\times n}, B_{n\times p}$, 证明: $R(A)+R(B)-n\le R(AB)$

证: $R(A)=r \Rightarrow$ 存在P,Q可逆,使得$PAQ=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$$PAB = PAQ\underbrace{Q^{-1}B}_{C} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ (1)$R(AB)=R(PAB)=R\begin{pmatrix} C_1 \\ 0 \end{pmatrix}=R(C_1)$, (2)$R(B)=R(C)$

$C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow R(C_1)+R(C_2)\ge R\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=R(C) \Rightarrow$ (3)$R(C_1)\ge R(C)-R(C_2)$

$\Leftrightarrow R(AB)\ge R(B)-R(C_2)$

(4)$R(C_2)\le n-r \Rightarrow R(AB)\ge R(B)-(n-r)=R(B)+R(A)-n$

$\Rightarrow R(AB)\ge R(B)+R(A)-n$

Sylvester推论:$A_{m\times n}, B_{n\times p}$且$AB=0 \Rightarrow R(A)+R(B)\le n$

问题1. 设A,B为$AB=0$的任意两个非零矩阵，则必有( )

(A) A的列向量组线性相关，B的行向量组线性相关.

(B) A的列向量组线性相关，B的列向量组线性相关.

(C) A的行向量组线性相关，B的行向量组线性相关.

(D) A的行向量组线性相关，B的列向量组线性相关.

分析： A为$m\times n$矩阵，B为$n\times s$矩阵

$$\left.\begin{array}{l}(1)\ AB=0\Rightarrow R(A)+R(B)\le n\\(2)\ A,B\text{非零}\Rightarrow R(A)\ge 1,R(B)\ge 1\end{array}\right\}\Rightarrow 1\le R(A)<n,1\le R(B)<n.$$

(1) A的列秩$=R(A)<n\Rightarrow A$的列向量组线性相关.

(2) B的行秩$=R(B)<n\Rightarrow B$的行向量组线性相关.

问题2. 设$m\times n$矩阵A的秩$R(A)=m<n$,则( )

(A) A的任意m个列向量所成向量组线性无关.

(B) A的任意一个m阶子式不为零

(C) 若$BA=0$,则$B=0$

(D) 通过行初等变换,必可化为$(E_m,0)$

解:(法1) $$BA=0\Rightarrow \left.\begin{array}{r} R(B)+R(A)\le m \\ R(A)=m \end{array}\right\}\Rightarrow R(B)\le 0\Rightarrow R(B)=0\Leftrightarrow B=0$$

(法2) $$BA=0\Leftrightarrow \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots \\ b_{r1} & \cdots & b_{rm} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}=0\Leftrightarrow \sum_{j=1}^{m} b_{ij}\alpha_j=0(\forall i)$$

$$R(A)=m\Leftrightarrow \alpha_1,\alpha_2,\cdots,\alpha_m\text{线性无关}\Rightarrow b_{ij}=0(\forall i,j)\Leftrightarrow B=0$$

问题3. 设$A=\begin{pmatrix} 1 & 2 & -2 \\ 4 & t & 3 \\ 3 & -1 & 1 \end{pmatrix}$, B为3阶非零矩阵, 且$AB=0$, 求t的值

解： $\left.\begin{array}{l} (1)\ AB=0 \Rightarrow R(A)+R(B)\le 3 \Rightarrow R(A)\le 3-R(B) \\ (2)\ B\text{为3阶非零矩阵} \Rightarrow R(B)\ge 1 \end{array}\right\} \Rightarrow R(A)\le 2<3$

$\Rightarrow |A|=0 \quad \Rightarrow t=-3$

导出结论(1).proof： $R(A^*)=\begin{cases} n, & R(A)=n \\ 0, & R(A)<n-1 \\ 1, & R(A)=n-1 \end{cases}$

证：$R(A)=n-1 \Rightarrow \left\{ \begin{array}{l} (1)A\text{中有}n-1\text{阶子式不为零} \Rightarrow A^* \neq 0 \Rightarrow R(A^*)\geq 1 \\ (2)\det(A)=0 \Rightarrow AA^* = \det(A)I = 0 \\ \qquad \Rightarrow R(A)+R(A^*)\leq n \\ \qquad \Rightarrow R(A^*)\leq n-R(A)=n-(n-1)=1 \end{array} \right\}$

$\Rightarrow R(A^*)=1.$

导出结论(2).proof：A为n阶矩阵,$A^2=A \Rightarrow R(A)+R(A-I)=n.$

证:$A^2=A \Leftrightarrow A(A-I)=0 \Rightarrow (1)R(A)+R(A-I)\leq n.$

$(2)R(A)+R(A-I) = R(A)+R(I-A) \geq R(A+I-A)=R(I)=n.$

$\Rightarrow R(A)+R(A-I)=n.$

三. 如何判断向量组线性表出关系?

(1) $Ax=b$有解 $\Leftrightarrow R(A)=R(A|b) \Leftrightarrow b$可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表出

$\Leftrightarrow \alpha_1,\alpha_2,\cdots,\alpha_n$与$\alpha_1,\alpha_2,\cdots,\alpha_n,b$等价

(2) $\begin{cases}(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_n;(\mathrm{II}):b_1,b_2,\cdots,b_l \\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n);B=(b_1,b_2,\cdots,b_l)\end{cases}$, (II)可由(I)表出 $\Leftrightarrow R(A)=R(A|B)$

证: (II)可由(I)表出 $\Leftrightarrow \forall b_j(j=1,2,\cdots,l), b_j$可由$\alpha_1,\alpha_2,\cdots,\alpha_n$表出 $\Leftrightarrow R(A)=R(A|b_j)$

$\Leftrightarrow R(A)=R(A|b_1)=R(A|b_2)=\cdots=R(A|b_l) \Leftrightarrow R(A)=R(A|b_1,b_2,\cdots,b_l)=R(A|B)$

(3) $\begin{cases}(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_n;(\mathrm{II}):b_1,b_2,\cdots,b_l \\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n);B=(b_1,b_2,\cdots,b_l)\end{cases}$, (II)与(I)等价 $\Leftrightarrow R(A)=R(B)=R(A|B)$

证: (II)与(I)等价 $\Leftrightarrow \begin{cases}(\mathrm{II})\text{可由}(\mathrm{I})\text{表出} \Leftrightarrow R(A)=R(A|B) \\ (\mathrm{I})\text{可由}(\mathrm{II})\text{表出} \Leftrightarrow R(B)=R(B|A)\end{cases} \Leftrightarrow R(A)=R(B)=R(A|B)$

1.设向量组: $a_1=(1,-1,1,-1)^T$、$a_2=(3,1,1,3)^T$;

向量组: $b_1=(2,0,1,1)^T$、$b_2=(1,1,0,2)^T$、$b_3=(3,-1,2,0)^T$.

证明: 向量组 a_1、a_2 与 b_1、b_2、b_3 等价.

证: 设 $A=(a_1,a_2), B=(b_1,b_2,b_3)$

两向量组等价 $\Leftrightarrow R(A)=R(B)=R(A|B)$

$$(A|B)=\left(\begin{array}{cc:ccc}1&3&2&1&3\\-1&1&0&1&-1\\1&1&1&0&2\\-1&3&1&2&0\end{array}\right)\xrightarrow[r_1+r_2]{\substack{r_3+r_4\\r_2+r_3}}\left(\begin{array}{cc:ccc}1&3&2&1&3\\0&4&2&2&2\\0&2&1&1&1\\0&4&2&2&2\end{array}\right)\xrightarrow[-\frac{1}{2}r_2+r_3]{-r_2+r_4}\left(\begin{array}{cc:ccc}1&3&2&1&3\\0&2&1&1&1\\0&0&0&0&0\\0&0&0&0&0\end{array}\right),$$

$$\Rightarrow\left\{\begin{array}{l}(1)R(A)=R(A|B)=2\\(2)\begin{vmatrix}2&1\\1&1\end{vmatrix}=1\neq0\Rightarrow R(B)=2\end{array}\right\}\Rightarrow R(A)=R(B)=R(A|B)$$

$\Leftrightarrow$ 两向量组等价

四. 如何从向量组线性表出的观点认识两矩阵的乘积?

1. $A=(a_{ij})_{m\times n}, B=(b_{ij})_{n\times p}, C=(c_{ij})_{m\times p}$, 且$C=AB$, 则

(1)矩阵$C=AB$的列向量组能由A的列向量组线性表出.

(2)矩阵$C=AB$的行向量组能由B的行向量组线性表出.

证: (1) $A=(\alpha_1,\alpha_2,\cdots,\alpha_n), C=(c_1,c_2,\cdots,c_p)$, 有

$$\boxed{C=AB} \Leftrightarrow (c_1,c_2,\cdots,c_p)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$\Leftrightarrow c_j=\sum_{i=1}^{n} b_{ij}\alpha_i=b_{1j}\alpha_1+\cdots+b_{nj}\alpha_n \ (j=1,2,\cdots,p)$$

$\Leftrightarrow$ 矩阵$C=AB$的列向量组能由A的列向量组线性表出.

$\Rightarrow$ 矩阵$C^T==B^T A^T$的列向量组能由B^T的列向量组线性表出.

矩阵$C=AB$的行向量组能由B的行向量组线性表出.

(2) $B=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}, C=\begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_m\end{pmatrix}$,有

$$\boxed{C=AB} \Leftrightarrow \begin{pmatrix}\boxed{\gamma_1}\\ \gamma_2\\ \vdots\\ \gamma_m\end{pmatrix}=\begin{pmatrix}\boxed{a_{11} & a_{12} & \cdots & a_{1n}}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$$

$$\Leftrightarrow \gamma_i=\sum_{j=1}^{n}a_{ij}\beta_j \quad (i=1,2,\cdots,m)$$

$\Leftrightarrow$ 矩阵$C=AB$的行向量组能由B的行向量组线性表出.

1. 设$A_{m\times n}$, $B_{n\times p}$, **证明：** $R(AB)\le \min\{R(A),R(B)\}$.

证(法1): 设$C_{m\times p}=AB$,则

$$(c_1,\cdots,c_p)=(\alpha_1,\cdots,\alpha_n)\begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}$$

$$\Rightarrow c_k=b_{1k}\alpha_1+b_{2k}\alpha_2+\cdots+b_{nk}\alpha_n,\ (k=1,\ldots,p)$$

$\Rightarrow$ 矩阵$C=AB$的列向量组能由A的列向量组线性表出.

$$R(AB)=R(C)=R\{c_1,\cdots,c_p\}\le R\{\alpha_1,\cdots,\alpha_n\}=R(A)$$

$$R(AB)=R(C)=R(C^{\mathrm{T}})=R(B^{\mathrm{T}}A^{\mathrm{T}})\le R(B^{\mathrm{T}})=R(B)$$

$$\Rightarrow R(AB)\le \min\{R(A),R(B)\}.$$

1.设$A_{m\times n}, B_{n\times p}$,证明: $R(AB)\le \min\{R(A),R(B)\}$

证(法2): 设$R(A)=r$,则存在m阶可逆矩阵P,n阶可逆矩阵Q,使得

$$PAQ=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow PAB=PAQ\underbrace{Q^{-1}B}_{C}=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}C=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=\begin{pmatrix} C_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (1)R(AB)=R(PAB)=R\begin{pmatrix} C_1 \\ 0 \end{pmatrix}\le r=R(A) \\ (2)R(AB)=R((AB)^{\mathrm{T}})=R(B^{\mathrm{T}}A^{\mathrm{T}})\le R(B^{\mathrm{T}})=R(B) \end{cases}$$

$$\Rightarrow R(AB)\le \min\{R(A),R(B)\}$$

2.设A是$n\times m$矩阵,B是$m\times n$矩阵(其中$n<m$),I是n阶单位矩阵.

若$AB=I$,证明:B的列向量组线性无关.

证(法1):设$B=(\beta_1,\beta_2,\cdots,\beta_n)$,

令$x_1\beta_1+x_2\beta_2+\cdots+x_n\beta_n=0$,即$(\beta_1,\beta_2,\cdots,\beta_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=0$

$\Leftrightarrow Bx=0 \Rightarrow 0=ABx=Ix=x \Rightarrow \beta_1,\beta_2,\cdots,\beta_n$线性无关.

证(法2):$\left.\begin{cases} (1)R(B)\le n \\ (2)R(B)\ge R(AB)=R(I)=n \end{cases}\right\} \Rightarrow R(B)=n$

$\Leftrightarrow \beta_1,\beta_2,\cdots,\beta_n$线性无关.

五."矩阵A与B等价"与"向量组$\alpha_1,\alpha_2,\cdots,\alpha_n$与$\beta_1,\beta_2,\cdots,\beta_n$等价"的关系?

(1)**矩阵A与B等价** $\Leftrightarrow A_{m\times n} \xrightarrow{\text{有限次初等变换}} B_{m\times n}$

$\Leftrightarrow$ 存在初等矩阵$E_1,\cdots,E_s,F_1,\cdots,F_t$,使得$E_s\cdots E_1AF_1\cdots F_t=B$

$\Leftrightarrow$ 存在可逆矩阵P,Q,使得$PAQ=B \Leftrightarrow R(A)=R(B)$

(2)**矩阵A与B行等价** $\Leftrightarrow A \xrightarrow{\text{有限次行初等变换}} B$

$\Leftrightarrow$ 存在可逆矩阵P,使得$PA=B$.

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}=\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} \Leftrightarrow \sum_{j=1}^{m} p_{ij}\alpha_j=\beta_i\,(i=1,2,\cdots,m)$$

$$PA=B \Leftrightarrow \sum_{j=1}^{m} p_{ij}\alpha_j=\beta_i(i=1,2,\cdots,m)$$

$\Rightarrow$ B的行向量组可由A的行向量组线性表出

$$PA=B \Leftrightarrow A=P^{-1}B=QB \Leftrightarrow \sum_{j=1}^{m} q_{ij}\beta_j=\alpha_i(i=1,2,\cdots,m)$$

$\Rightarrow$ A的行向量组可由B的行向量组线性表出

即A与B的行向量组等价

(3)矩阵A与B列等价 $\Leftrightarrow A \xrightarrow{\text{有限次列初等变换}} B$

$\Leftrightarrow$ 存在可逆矩阵Q,使得$AQ=B.$

$$\Leftrightarrow \sum_{i=1}^{n} q_{ij}\alpha_i=\beta_j$$

$$(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}=(\beta_1,\beta_2,\cdots,\beta_n)$$

$$AQ = B \Leftrightarrow \sum_{i=1}^{n} q_{ij}\alpha_i = \beta_j (j = 1, 2, \cdots, n)$$

$\Rightarrow$ B的列向量组可由A的列向量组线性表出

$$AQ = B \Leftrightarrow A = BQ^{-1} = BP \Leftrightarrow \sum_{i=1}^{n} p_{ij}\beta_i = \alpha_j (j = 1, 2, \cdots, n)$$

$\Rightarrow$ A的列向量组可由B的列向量组线性表出

即A与B的列向量组等价

注: 1^0. 矩阵A与B等价 $\Leftrightarrow A_{m\times n} \xrightarrow{\text{有限次初等变换}} B_{m\times n}$

$\Rightarrow$ A与B的行向量组未必等价,A与B的列向量组未必等价.

2^0. 设两列向量组等价(m维),若它们所含向量个数不相同,则它们对应的两个矩阵是不同型的,因而不等价;若它们所含向量个数相同(如都为n个向量),那么它们对应的两个$m\times n$矩阵列等价,从而一定等价,但不一定行等价.

3^0. 举例

$$(1)A=\begin{pmatrix}1&2\\0&0\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&3\end{pmatrix}=\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}$$

$\Rightarrow$ A与B等价，但α_1,α_2与β_1,β_2不等价.

$$(2)A=\begin{pmatrix}1&2\\0&0\end{pmatrix}=(\alpha_1,\alpha_2),\ B=\begin{pmatrix}0&0\\1&3\end{pmatrix}=(\beta_1,\beta_2)$$

$\Rightarrow$ A与B等价，但α_1,α_2与β_1,β_2不等价.

六. 矩阵的行初等变换对列向量组和行向量组各有什么作用?

$A_{m\times n} \xrightarrow{\text{行初等变换}} B_{m\times n}$ ($\Leftrightarrow$ 矩阵A与B的行等价)

$\Rightarrow \begin{cases} \text{(1)矩阵}A\text{与}B\text{的行向量组等价.} \\ \text{(2)矩阵}A\text{与}B\text{的列向量组有相同的线性(相关与无关)关系.} \end{cases}$

注: $A \xrightarrow{\text{行初等变换}} B$(行阶梯形 或 简化行阶梯形)

(求最大无关组及用最大无关组表出其它向量的理论基础)

(3) $R(A) = A$的行秩$=A$的列秩.

1. 求向量组:

$$\alpha_1=(1,2,3,4)^T,\alpha_2=(2,3,4,5)^T,\alpha_3=(3,4,5,6)^T,\alpha_4=(4,5,6,7)^T$$

的秩与一个最大无关组,并用所求最大无关组表示其余向量.

解:$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\xrightarrow[(-1)r_1+r_2]{\substack{(-1)r_3+r_4\\(-1)r_2+r_3}}\begin{pmatrix}1&2&3&4\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

$$\xrightarrow[(-1)r_1+r_2]{\substack{(-1)r_3+r_4\\(-1)r_2+r_3}}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{pmatrix}\xrightarrow[(-1)r_2]{2r_2+r_1}\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{pmatrix}\Rightarrow R(A)=2$$

α_1,α_2为一个最大无关组;

$$\begin{cases}\alpha_3=-\alpha_1+2\alpha_2\\\alpha_4=-2\alpha_1+3\alpha_2\end{cases}$$

七. 相关与无关两个对偶结论

1. $(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_s;$ $(\mathrm{II})\alpha_1,\alpha_2,\cdots,\alpha_s,\alpha_{s+1},\cdots,\alpha_m;$

(I)相关 $\xrightarrow{\text{个数}\uparrow}$ (II)相关; (II)无关 $\xrightarrow{\text{个数}\downarrow}$ (I)无关.

"部分相关⇒整体相关" "整体无关⇒部分无关"

2. r维向量组的每个向量添加$n-r$个分量,成为n维向量组,则:

(1)若r维向量组线性无关,则n维向量组也线性无关;

(2)若n维向量组线性相关,则r维向量组也线性相关.

即:$(\mathrm{I}):\alpha_1,\alpha_2,\cdots,\alpha_s\in R^r;\beta_1,\beta_2,\cdots,\beta_s\in R^{n-r};$令$\gamma_i=\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}(i=1,2,\cdots,s),$

且$(\mathrm{II}):\gamma_1,\gamma_2,\cdots,\gamma_s\in R^n,$则

(1)$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关$\Rightarrow\gamma_1,\gamma_2,\cdots,\gamma_s$线性无关;

(2)$\gamma_1,\gamma_2,\cdots,\gamma_s$线性相关$\Rightarrow\alpha_1,\alpha_2,\cdots,\alpha_s$线性相关.

证：(I)：$(\alpha_1,\alpha_2,\cdots,\alpha_s)\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0$，(II)：$(\gamma_1,\gamma_2,\cdots,\gamma_s)\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0 \Leftrightarrow \begin{pmatrix}\alpha_1,\alpha_2,\cdots,\alpha_s\\\beta_1,\beta_2,\cdots,\beta_s\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_s\end{pmatrix}=0$

令$A=(\alpha_1,\alpha_2,\cdots,\alpha_s)$，$B=(\beta_1,\beta_2,\cdots,\beta_s)$，则 (I)：$Ax=0$，(II)：$\begin{pmatrix}A\\B\end{pmatrix}x=0 \Leftrightarrow \begin{cases}Ax=0\\Bx=0\end{cases}$

则(II)的前r个方程就是(I)的方程$\Rightarrow$(II)的解必是(I)的解，即

$$\boxed{\{0\}\subset\{\text{(II)的解集}\}\subset\{\text{(I)的解集}\}}$$

(1)$\alpha_1,\alpha_2,\cdots,\alpha_s$线性无关$\Leftrightarrow$(I)只有零解$\Rightarrow$(II)只有零解

$\Leftrightarrow\gamma_1,\gamma_2,\cdots,\gamma_s$线性无关.

(2)$\gamma_1,\gamma_2,\cdots,\gamma_s$线性相关$\Leftrightarrow$(II)有非零解$\Rightarrow$(I)有非零解

$\Leftrightarrow\alpha_1,\alpha_2,\cdots,\alpha_s$线性相关.

1.设有向量组$\gamma_i=(a_i,a_i^2,\cdots,a_i^n)^{\mathrm{T}},(i=1,2,\cdots,m)(m\le n)$,试证:向量组$\gamma_1,\gamma_2,\cdots,\gamma_m$线性无关,(其中:$a_1,a_2,\cdots,a_m$为$m$个互不相等且不为零的常数).

证:$A=(\gamma_1,\cdots,\gamma_m)=\begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_m^n \end{pmatrix}(m\le n)$, 由$A$的前$m$行与$m$列构成的子式

$$D=\begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ \vdots & \vdots & & \vdots \\ a_1^m & a_2^m & \cdots & a_m^m \end{vmatrix}=a_1a_2\cdots a_m\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \end{vmatrix}$$

$$=(\prod_{i=1}^{m}a_i)\{\prod_{1\le i<j\le m}(a_j-a_i)\}\neq 0,$$

令$\beta_i=(a_i,a_i^2,\cdots,a_i^m)^{\mathrm{T}}\Rightarrow\beta_1,\beta_2,\cdots,\beta_m$线性无关$\Rightarrow\gamma_1,\gamma_2,\cdots,\gamma_m$线性无关.

八. 线性方程组解的结构

1.设$R(A)=r<n$,则$Ax=0$有基础解系且所含向量个数为$n-r$,

即$\dim W=n-r$,(其中n为方程组未知量的个数,$W=\{x\mid Ax=0\}$).

2.若$R(A)$=n,则$Ax=0$只有零解,无基础解系.

3.$Ax=0$的通解:设$\xi_1,\xi_2,\cdots,\xi_{n-r}$为$Ax=0$一个基础解系,则

$$\forall\alpha(Ax=0\text{的解}),$$

$$\alpha=k_1\xi_1+k_2\xi_2+\cdots+k_{n-1}\xi_{n-r},\forall k_1,k_2,\cdots,k_{n-1}\in R.$$

4.$Ax=b$的通解:设η_0为$Ax=b$一个特解,$\xi_1,\xi_2,\cdots,\xi_{n-r}$为其导出组的一个基础解系,则

$$\forall\alpha(Ax=b\text{的解}),$$

$$\alpha=\eta_0+k_1\xi_1+k_2\xi_2+\cdots+k_{n-1}\xi_{n-r},\forall k_1,k_2,\cdots,k_{n-1}\in R.$$

1.解方程组 $\begin{cases} x_1+2x_2+3x_3=0 \\ 3x_1+6x_2+10x_3=0 \\ 2x_1+5x_2+7x_3=0 \\ x_1+2x_2+4x_3=0 \end{cases}$

解：$A=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 10 \\ 2 & 5 & 7 \\ 1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow R(A)=3=n \Rightarrow$ 只有零解$x=0.$

2.解方程组 $\begin{cases} 3x_1+x_2+x_3=5 \\ 3x_1+2x_2+3x_3=3 \\ x_2+2x_3=2 \end{cases}$

解: $\overline{A}=\left(\begin{array}{ccc|c} 3 & 1 & 1 & 5 \\ 3 & 2 & 3 & 3 \\ 0 & 1 & 2 & 2 \end{array}\right)\to\left(\begin{array}{ccc|c} 3 & 1 & 1 & 5 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right)\to\left(\begin{array}{ccc|c} 3 & 1 & 1 & 5 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right)$

$\Rightarrow R(A)=2\neq R(\overline{A})=3\Rightarrow$ 无解.

3. 解方程组 $\begin{cases} x_1 - 5x_2 + 3x_3 - x_4 + 2x_5 = 1 \\ 2x_1 - 9x_2 + 6x_3 + 5x_4 + 3x_5 = 3 \\ x_1 - 4x_2 + 3x_3 + 6x_4 + x_5 = 2 \end{cases}$

解：$\overline{A} = \left(\begin{array}{ccccc|c} 1 & -5 & 3 & -1 & 2 & 1 \\ 2 & -9 & 6 & 5 & 3 & 3 \\ 1 & -4 & 3 & 6 & 1 & 2 \end{array}\right) \xrightarrow[r_2 \leftrightarrow r_3]{\substack{(-1)r_1+r_2 \\ (-1)r_3+r_2}} \left(\begin{array}{ccccc|c} 1 & -5 & 3 & -1 & 2 & 1 \\ 1 & -4 & 3 & 6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\xrightarrow[5r_2+r_1]{(-1)r_1+r_2} \left(\begin{array}{ccccc|c} 1 & 0 & 3 & 34 & -3 & 6 \\ 0 & 1 & 0 & 7 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow R(A) = 2 = R(\overline{A}) < 5 \Rightarrow$ 有无穷多解.

$$\begin{pmatrix} 1 & 0 & 3 & 34 & -3 & \vert & 6 \\ 0 & 1 & 0 & 7 & -1 & \vert & 1 \\ 0 & 0 & 0 & 0 & 0 & \vert & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 6 - 3x_3 - 34x_4 + 3x_5 \\ x_2 = 1 - 7x_4 + x_5 \end{cases}$$

$\Rightarrow$ 原方程组有特解: $\eta=(6,1,0,0,0)^{T}$

$\Rightarrow$ 导出组的基础解系:

$$\xi_1 = (-3,0,1,0,0)^{T}, \xi_2 = (-34,-7,0,1,0)^{T}, \xi_3 = (3,1,0,0,1)^{T}$$

原方程组的通解为:

$$x = \eta + k_1\xi_1 + k_2\xi_2 + k_3\xi_3 (k_1, k_2, k_3 \in R)$$

4.设A是$m\times 3$矩阵，且$R(A)=1$.如果非齐次线性方程组$Ax=b$的三个解向量η_1,η_2,η_3满足：

$$\eta_1+\eta_2=\begin{pmatrix}1\\2\\3\end{pmatrix},\quad \eta_2+\eta_3=\begin{pmatrix}0\\-1\\1\end{pmatrix},\quad \eta_3+\eta_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

求$Ax=b$的通解.

解: A是$m\times 3$矩阵, $R(A)=1$

$\Rightarrow$ $Ax=0$的基础解系中含有$3-1=2$个线性无关的解向量.

(法1): 令$\eta_1+\eta_2=a, \eta_2+\eta_3=b, \eta_3+\eta_1=c$, 则

$$\eta_1=\frac{1}{2}(a+c-b)=\begin{pmatrix}1\\3/2\\1/2\end{pmatrix},\quad \eta_2=\frac{1}{2}(a+b-c)=\begin{pmatrix}0\\1/2\\5/2\end{pmatrix},$$

$$\eta_3=\frac{1}{2}(b+c-a)=\begin{pmatrix}0\\-3/2\\-3/2\end{pmatrix}$$ 为$Ax=b$的特解.

$$\Rightarrow \eta_1-\eta_2=\begin{pmatrix}1\\1\\-2\end{pmatrix},\ \eta_1-\eta_3=\begin{pmatrix}1\\3\\2\end{pmatrix}$$ 为$Ax=0$的基础解系.

故$Ax=b$的通解:

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\3/2\\1/2\end{pmatrix}+k_1\begin{pmatrix}1\\1\\-2\end{pmatrix}+k_2\begin{pmatrix}1\\3\\2\end{pmatrix},\ \forall k_1,k_2\in R.$$

(法2): $(\eta_1+\eta_2)-(\eta_2+\eta_3)=\begin{pmatrix}1\\3\\2\end{pmatrix}$, $(\eta_1+\eta_2)-(\eta_3+\eta_1)=\begin{pmatrix}0\\2\\4\end{pmatrix}$ 线性无关.

$\Rightarrow$ 为$Ax=0$的基础解系.

$A(\frac{1}{2}(\eta_2+\eta_3))=\frac{1}{2}(b+b)=b \Rightarrow \frac{1}{2}(\eta_2+\eta_3)=\begin{pmatrix}0\\-1/2\\1/2\end{pmatrix}$为$Ax=b$的特解.

故$Ax=b$的通解:

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\-1/2\\1/2\end{pmatrix}+k_1\begin{pmatrix}1\\3\\2\end{pmatrix}+k_2\begin{pmatrix}0\\2\\4\end{pmatrix},\ \forall k_1,k_2\in R.$$

5.设矩阵$A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$,其中$\alpha_2,\alpha_3,\alpha_4$线性无关,$\alpha_1=2\alpha_2-\alpha_3$.

如果$\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4$,求非齐次线性方程组$Ax=\beta$的通解.

解:(1)$\alpha_1=2\alpha_2-\alpha_3 \Rightarrow \alpha_1,\alpha_2,\alpha_3$线性相关$\Rightarrow \alpha_1,\alpha_2,\alpha_3,\alpha_4$线性相关

$\Rightarrow Ax=0$有非零解$\Leftrightarrow Ax=0$有基础解系.

(2)$\alpha_2,\alpha_3,\alpha_4$线性无关,$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性相关$\Rightarrow R\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}=3$

$\Leftrightarrow R(A)=3 \Rightarrow$基础解系含向量的个数为$4-R(A)=4-3=1$.

(3)$$\alpha_1=2\alpha_2-\alpha_3 \Rightarrow \alpha_1-2\alpha_2+\alpha_3+0\alpha_4=0 \Leftrightarrow (\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}=0$$

$\Rightarrow Ax=0$的基础解系为$(1,-2,1,0)^{\mathrm{T}}$.

(4)$$\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\Rightarrow (1,1,1,1)^{\mathrm{T}}\text{为}Ax=\beta\text{的特解.}$$

$\Rightarrow Ax=\beta$的通解为:$(1,1,1,1)^T+k(1,-2,1,0)^T,\forall\in R$.

$5'.A_{3\times 3}\neq 0$且$A^2=0$，证明：$\begin{cases}(1)Ax=0\text{的线性无关解向量最大个数}=2\\(2)Ax=b\text{的线性无关解向量最大个数}=3\end{cases}$

证：(1)

$<1>.A^2=AA=0\Rightarrow 2\leq 2R(A)=R(A)+R(A)\leq 3\Rightarrow 1\leq R(A)\leq \frac{3}{2}\Leftrightarrow R(A)=1$

$<2>.W=\{x\mid Ax=0\}\Rightarrow \dim W=n-R(A)=3-1=2$

$\Leftrightarrow Ax=0$的线性无关解向量最大个数=2

(2)

$<1>\begin{cases}\xi_1,\xi_2\text{为}Ax=0\text{的两个线性无关的解}\\ \eta^*\text{为}Ax=b\text{的解}\end{cases}\Rightarrow \eta^*,\eta^*+\xi_1,\eta^*+\xi_2$为$Ax=b$的3个线性无关的解.

$<2>$.设$Ax=b$的线性无关解的个数>3，不妨设$A\eta_i=b(i=1,2,3,4)$.

$\Rightarrow\begin{cases}\xi_1=\eta_1-\eta_4,\xi_2=\eta_2-\eta_4,\xi_3=\eta_3-\eta_4\text{为}Ax=0\text{的解.}\\ \xi_1,\xi_2,\xi_3\text{线性无关.}\end{cases}$

与$\dim W=2$矛盾！

6.设矩阵$A=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$,线性方程组$Ax=\beta$的通解:$\begin{pmatrix}2\\1\\0\\3\end{pmatrix}+k\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}(\forall k)$.

试问: α_4能否由$\alpha_1,\alpha_2,\alpha_3$线性表出?为什么?

解:(1)$\xi_1=(1,-1,2,0)^{\mathrm{T}}$为$Ax=0$的基础解系且$R(A)=3$.(基础解系含有一个向量)

(2)若α_4能由$\alpha_1,\alpha_2,\alpha_3$线性表出,则 $\alpha_4=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3$

$$\Leftrightarrow k_1\alpha_1+k_2\alpha_2+k_3\alpha_3-\alpha_4=0\Leftrightarrow(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\begin{pmatrix}k_1\\k_2\\k_3\\-1\end{pmatrix}=0\Leftrightarrow A\begin{pmatrix}k_1\\k_2\\k_3\\-1\end{pmatrix}=0$$

$\Rightarrow\xi_2=(k_1,k_2,k_3,-1)^{\mathrm{T}}$为$Ax=0$的解.

(3)ξ_1与ξ_2线性无关 $\Rightarrow Ax=0$的基础解系含解向量的个数≥ 2. 矛盾!

故α_4不能由$\alpha_1,\alpha_2,\alpha_3$线性表出.

7.证明: $R(A^{\mathrm{T}}A)=R(A)$.

证: 设A为$m\times n$矩阵, x为n维列向量, 则$A^{\mathrm{T}}A$为$n\times n$矩阵

$$令 W_1=\{x \mid Ax=0\},\ W_2=\{x \mid (A^{\mathrm{T}}A)x=0\}$$

(1) $\forall x\in W_1, Ax=0 \Rightarrow A^{\mathrm{T}}(Ax)=A^{\mathrm{T}}0=0 \Rightarrow x\in W_2 \Rightarrow W_1\subset W_2$.

(2) $\forall x\in W_2, A^{\mathrm{T}}Ax=0 \Rightarrow x^{\mathrm{T}}A^{\mathrm{T}}Ax=x^{\mathrm{T}}0=0 \Rightarrow Ax=0 \Rightarrow x\in W_1 \Rightarrow W_2\subset W_1$.

$$\Rightarrow W_1=W_2$$

$$\Rightarrow\ n-R(A)=\dim W_1=\dim W_2=n-R(A^{\mathrm{T}}A)$$

$$\Rightarrow\ R(A^{\mathrm{T}}A)=R(A).$$

注: $\boxed{R(A^{\mathrm{T}}A)=R(A)=R(AA^{\mathrm{T}})}$

$7'$.设$A_{m\times p}, B_{p\times n}$,证明: $R(AB)\le \min\{R(A),R(B)\}$.

证: 设$W_1=\{x\mid Bx=0,\forall x\in R^n\}$, $W_2=\{x\mid ABx=0,\forall x\in R^n\}$,

(1) $\dim W_1=n-R(B)$, $\dim W_2=n-R(AB)$

(2) $\forall x\in W_1 \Longrightarrow Bx=0 \Longrightarrow ABx=A0=0 \Longrightarrow x\in W_2$

$$\Longrightarrow W_1\subset W_2 \Longrightarrow \dim W_1\le \dim W_2$$

$$\Longleftrightarrow n-R(B)\le n-R(AB)$$

$$\Longleftrightarrow R(AB)\le R(B)$$

$$\Longrightarrow R(AB)=R[(AB)^T]=R(B^TA^T)\le R(A^T)=R(A)$$

$$\Longrightarrow R(AB)\le \min\{R(A),R(B)\}.$$

7″.证明: $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$有解.

证: 设A为$m \times n$矩阵,b为m维列向量,则

$$(A^{\mathrm{T}}A, A^{\mathrm{T}}b) = A^{\mathrm{T}}(A,b) \Rightarrow R(A^{\mathrm{T}}A, A^{\mathrm{T}}b) = R[A^{\mathrm{T}}(A,b)]$$

$$\leq R(A^{\mathrm{T}}) = R(A) = R(A^{\mathrm{T}}A)$$

$$R(A^{\mathrm{T}}A) \leq R(A^{\mathrm{T}}A, A^{\mathrm{T}}b)$$

$$\Rightarrow R(A^{\mathrm{T}}A) = R(A^{\mathrm{T}}A, A^{\mathrm{T}}b)$$

$\Rightarrow$ $A^{\mathrm{T}}Ax = A^{\mathrm{T}}b$有解.

8.设有三维列向量$\alpha_1=\begin{pmatrix}1+\lambda\\1\\1\end{pmatrix},\alpha_2=\begin{pmatrix}1\\1+\lambda\\1\end{pmatrix},\alpha_3=\begin{pmatrix}1\\1\\1+\lambda\end{pmatrix},\beta=\begin{pmatrix}0\\\lambda\\\lambda^2\end{pmatrix}$

问λ为何值时，$\begin{cases}(1)\beta\text{可由}\alpha_1,\alpha_2,\alpha_3\text{线性表出,且表出唯一;}\\(2)\beta\text{可由}\alpha_1,\alpha_2,\alpha_3\text{线性表出,且表出不唯一;}\\(3)\beta\text{不能由}\alpha_1,\alpha_2,\alpha_3\text{线性表出.}\end{cases}$

解: 设$\beta=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3$,则$\begin{cases}(1+\lambda)x_1+x_2+x_3=0\\x_1+(1+\lambda)x_2+x_3=\lambda\\x_1+x_2+(1+\lambda)x_3=\lambda^2\end{cases}$

$$\bar{A}=\left(\begin{array}{ccc|c}1+\lambda&1&1&0\\1&1+\lambda&1&\lambda\\1&1&1+\lambda&\lambda^2\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1+\lambda&\lambda^2\\1&1+\lambda&1&\lambda\\1+\lambda&1&1&0\end{array}\right)$$

$$\rightarrow\left(\begin{array}{ccc|c} 1 & 1 & 1+\lambda & \lambda^2 \\ 0 & \lambda & -\lambda & \lambda-\lambda^2 \\ 0 & -\lambda & -\lambda^2-2\lambda & -\lambda^2(1+\lambda) \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & 1 & 1+\lambda & \lambda^2 \\ 0 & \lambda & -\lambda & \lambda-\lambda^2 \\ 0 & 0 & -\lambda(\lambda+3) & \lambda(1-2\lambda-\lambda^2) \end{array}\right)$$

$\Rightarrow$

(1)若$\lambda\neq 0$且$\lambda\neq -3$,方程组有唯一解,β可由$\alpha_1,\alpha_2,\alpha_3$唯一线性表出.

(2)若$\lambda=0$,则方程组有无穷多解,β可由$\alpha_1,\alpha_2,\alpha_3$线性表出,表出不唯一.

(3)若$\lambda=-3$,则$\bar{A}=\left(\begin{array}{ccc|c} 1 & 1 & -2 & 9 \\ 0 & -3 & 3 & -6 \\ 0 & 0 & 0 & 6 \end{array}\right)\Rightarrow R(A)=2<3=R(\bar{A})$

$\Rightarrow$ 方程组无解 $\Rightarrow$ β不能由$\alpha_1,\alpha_2,\alpha_3$线性表出.

9.问a,b为何值时,线性方程组$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_2+2x_3+2x_4=1 \\ -x_2+(a-3)x_3-2x_4=b \\ 3x_1+2x_2+x_3+ax_4=-1 \end{cases}$

有唯一解、无解、无穷多组解.并求出其唯一解和一般解.

解: $\bar{A}=\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$

(1)当$a\neq 1$时,$R(A)=R(\bar{A})=4$,原方程组有唯一解,且其唯一解为

$$x_1=\frac{b-a+2}{a-1},x_2=\frac{a-2b-3}{a-1},x_3=\frac{b+1}{a-1},x_4=0$$

(2)当$a=1$时, $R(A)=2$.

(I)当$b\neq -1$时, $R(\bar{A})=3>2=R(A)\Rightarrow$ 原方程组无解.

(II)当$b=-1$时, $R(\bar{A})=R(A)=2<4\Rightarrow$ 原方程组有无穷多组解.

将$a=1$及$b=-1$代入,易求的其通解为:

$$\bar{A}\to\left(\begin{array}{cccc|c}1&0&-1&-1&-1\\0&1&2&2&1\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)\Rightarrow\begin{cases}x_1=-1+x_3+x_4\\x_2=1-2x_3-2x_4\end{cases}\Rightarrow\eta_0=\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix}\Rightarrow x=\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+k_1\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+k_2\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}$$

$(\forall k_1,k_2)$.

九. 几何空间

1. 设$\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha=0$, **证明:** α,β,γ共面.

证: $\alpha\bullet(\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha)=\alpha\bullet 0=0$

$$\alpha\bullet(\alpha\times\beta)+\alpha\bullet(\beta\times\gamma)+\alpha\bullet(\gamma\times\alpha)=\alpha\bullet 0=0$$

即: $[\alpha\beta\gamma]=\alpha\bullet(\beta\times\gamma)=0\ \Rightarrow \alpha,\beta,\gamma$共面

2.设平面π与$\pi': 5x-y+3z-2=0$垂直,且与π'的交线落在xoy平面,求π的方程.

解: 平面π与$\pi': 5x-y+3z-2=0$的交线落在xoy平面内

$\Rightarrow$ 平面π与π'的交线方程与π'与平面xoy的交线一致.

即 平面π与π'的交线方程为$\begin{cases} 5x-y+3z-2=0 \\ z=0 \end{cases}$

$\Rightarrow$ 可设平面π的方程为: $(5x-y+3z-2)+\lambda z=0$

$\because$ 平面π与$5x-y+3z-2=0$垂直

$\Rightarrow$ π的法向量$(5,-1,3+\lambda)$与π'的法向量$(5,-1,3)$正交

$$\Rightarrow (5,-1,3+\lambda)\cdot(5,-1,3)=25+1+9+3\lambda=0$$

$$\Rightarrow \lambda=-\frac{35}{3}$$

$\Rightarrow$ 平面π的方程为：$(5x-y+3z-2)-\frac{35}{3}z=0$

化简为：$5x-y-\frac{26}{3}z-2=0$

3.求过直线$\begin{cases} x+3y-5=0 \\ x-y-2z+4=0 \end{cases}$且在$x$轴, y轴上截距相等的平面方程.

解: 设所求平面为$x+3y-5+\lambda(x-y-2z+4)=0$

即 $(1+\lambda)x+(3-\lambda)y-2\lambda z+(-5+4\lambda)=0$

令 $y=z=0$, 得x轴上截距为: $\dfrac{5-4\lambda}{1+\lambda}$

类似地, 得y轴上截距: $\dfrac{5-4\lambda}{3-\lambda}$

由题设得: $\dfrac{5-4\lambda}{1+\lambda}=\dfrac{5-4\lambda}{3-\lambda} \Rightarrow \lambda=1$

故所求平面: $2x+2y-2z-1=0.$