习题2.1

求行列式
$$\begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix}$$

$$= x\begin{vmatrix} x & y & & & \\ & x & y & & \\ & & \cdots & & \\ & & & x & y \\ & & & & x \end{vmatrix} + (-1)^{n+1} y \begin{vmatrix} y & & & & \\ x & y & & & \\ & x & \cdots & & \\ & & \ddots & y & \\ & & & x & y \end{vmatrix}$$

$$= x^n + (-1)^{n+1} y^n$$

习题2.2

$$1(5)\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 3 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & n-1 & 2 \\ 2 & 2 & 2 & \cdots & 2 & n \end{vmatrix}$$

$$\underline{\underline{-r_1 + r_i}}\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & n-3 & 0 \\ 2 & 0 & 0 & \cdots & 0 & n-2 \end{vmatrix} = (-1)^{2+1}\begin{vmatrix} 2 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n-2 \end{vmatrix}$$

$$= 2\left(n-2\right)!$$

$$3(2)\begin{vmatrix} 1 & 2 & 3 & & (n-1) & n \\ 1 & -1 & 0 & & 0 & 0 \\ & 2 & -2 & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & 0 & -(n-2) & 0 \\ 0 & 0 & 0 & 0 & (n-1) & -(n-1) \end{vmatrix}$$

$$\underline{\underline{c_n + c_{n-1}}}\begin{vmatrix} 1 & 2 & 3 & & (n-1)+n & n \\ 1 & -1 & 0 & & 0 & 0 \\ & 2 & -2 & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & 0 & -(n-2) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(n-1) \end{vmatrix}$$

$$
\xlongequal{c_n + c_{n-1}}
\begin{vmatrix}
1 & 2 & 3 & & (n-1)+n & n \\
1 & -1 & 0 & & 0 & 0 \\
 & 2 & -2 & & 0 & 0 \\
 & & & \ddots & & \\
0 & 0 & 0 & 0 & -(n-2) & 0 \\
0 & 0 & 0 & 0 & 0 & -(n-1)
\end{vmatrix}
$$

$$
\xlongequal[\substack{\cdots \\ c_2 + c_1}]{c_{n-1} + c_{n-2}}
\begin{vmatrix}
\sum\limits_{k=1}^{n} k & \sum\limits_{k=2}^{n} k & \sum\limits_{k=3}^{n} k & & (n-1)+n & n \\
0 & -1 & 0 & & 0 & 0 \\
 & 0 & -2 & & 0 & 0 \\
 & & & \ddots & & \\
0 & 0 & 0 & 0 & -(n-2) & 0 \\
0 & 0 & 0 & 0 & 0 & -(n-1)
\end{vmatrix}
$$

$$
=(-1)^{n-1}(n-1)!\sum_{k=1}^{n} k=(-1)^{n-1}(n-1)!\frac{n(n-1)}{2}=(-1)^{n-1}\frac{(n+1)!}{2}
$$

$$2(3)\quad D_n=\begin{vmatrix} \cos\theta & 1 & & \cdots & & & \\ 1 & 2\cos\theta & 1 & \cdots & & & \\ & 1 & 2\cos\theta & \cdots & & & \\ & & \vdots & \ddots & \vdots & & \\ & & & \cdots & 2\cos\theta & 1 & \\ & & & \cdots & 1 & 2\cos\theta & 1 \\ & & & \cdots & & 1 & 2\cos\theta \end{vmatrix}=\cos n\theta$$

证明：（1）当$n=1$时，$D_1=\cos\theta$

（2）假设当$n\le k$时，$D_k=\cos k\theta$

$$D_{k+1}=2\cos\theta D_k+(-1)^{k+1+k}\begin{vmatrix} \cos\theta & 1 & & & & \\ 1 & 2\cos\theta & 1 & & & \\ & 1 & 2\cos\theta & & & \\ & & & \ddots & 1 & \\ & & & & 2\cos\theta & 0 \\ & & & & 1 & 1 \end{vmatrix}$$

$$D_{k+1}=2\cos\theta D_k+(-1)^{k+1+k}\begin{vmatrix} \cos\theta & 1 & & & & \\ 1 & 2\cos\theta & 1 & & & \\ & 1 & 2\cos\theta & & & \\ & & & \ddots & 1 & \\ & & & & 2\cos\theta & 0 \\ & & & & 1 & 1 \end{vmatrix}$$

$$=2\cos\theta D_k-D_{k-1}$$

$$=2\cos\theta\cos k\theta-\cos(k-1)\theta$$

$$=2\cos k\theta\cos\theta-\cos(k-1)\theta$$

$$=\cos(k+1)\theta$$

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$

$$2\cos\alpha\cos\beta-\cos(\alpha-\beta)=\cos(\alpha+\beta)$$

$\therefore$ 对$\forall n$,有$D_n=\cos n\theta$

$$3(1)\quad \begin{vmatrix} a_1+\lambda_1 & a_2 & & a_n \\ a_1 & a_2+\lambda_2 & & a_n \\ & & \ddots & \\ a_1 & a_2 & & a_n+\lambda_n \end{vmatrix} (\lambda_i \neq 0, i=1,2,\cdots,n)$$

$$= \begin{vmatrix} a_1+\lambda_1 & a_2 & & a_n \\ -\lambda_1 & \lambda_2 & & 0 \\ & & \ddots & \\ -\lambda_1 & 0 & & \lambda_n \end{vmatrix} = \left[a_1+\lambda_1+\frac{\lambda_1 a_2}{\lambda_2}+\cdots+\frac{\lambda_1 a_n}{\lambda_n} \right] \lambda_2 \cdots \lambda_n$$

$$= \left[\frac{a_1}{\lambda_1}+1+\frac{a_2}{\lambda_2}+\cdots+\frac{a_n}{\lambda_n} \right] \lambda_1 \lambda_2 \cdots \lambda_n$$

$$= \left[1+\sum_{i=1}^{n} \frac{a_i}{\lambda_i} \right] \lambda_1 \lambda_2 \cdots \lambda_n$$

$$3(2)\quad \begin{vmatrix} a_1^n & a_1^{n-1}b_1 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & & a_2b_2^{n-1} & b_2^n \\ & & \ddots & & \\ a_n^n & a_n^{n-1}b_1 & & a_nb_n^{n-1} & b_n^n \\ a_{n+1}^n & a_{n+1}^{n-1}b_1 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix} (a_i \neq 0, i = 1,2,\cdots,n+1)$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & b_1/a_1 & \cdots & \left(b_1/a_1\right)^{n-1} & \left(b_1/a_1\right)^n \\ 1 & b_2/a_2 & & \left(b_2/a_2\right)^{n-1} & \left(b_2/a_2\right)^n \\ & & \ddots & & \\ 1 & b_n/a_n & & \left(b_n/a_n\right)^{n-1} & \left(b_n/a_n\right)^n \\ 1 & b_{n+1}/a_{n+1} & \cdots & \left(b_{n+1}/a_{n+1}\right)^{n-1} & \left(b_{n+1}/a_{n+1}\right)^n \end{vmatrix}$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \prod_{1\le j<i\le n+1} \left(b_i/a_i - b_j/a_j \right)$$

已知n阶矩阵A满足$A^2=A$, 证明: $A=I$或$|A|=0$

错误解法

$$A^2=A$$

$$A(A-I)=0$$

$$|A||A-I|=0$$

$$|A|=0 \ or \ |A-I|=0$$

$$\Downarrow$$

$$A=I$$

正确解法

A 可逆 $\Rightarrow$ $A^{-1}A^2=A^{-1}A$

A不可逆 $\Rightarrow$ $|A|=0$

习题2.3

1(2).
$$\begin{vmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = -35$$

1(4).
$$\begin{vmatrix} a & & & & & & & b \\ & a & & & & & b & \\ & & \ddots & & & \iddots & & \\ & & & a & b & & & \\ & & & b & a & & & \\ & & \iddots & & & \ddots & & \\ & b & & & & & a & \\ b & & & & & & & a \end{vmatrix}$$

$$D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix} D_{2(n-1)} = \left(a^2 - b^2\right) D_{2(n-1)}$$
$$= \left(a^2 - b^2\right)^2 D_{2(n-2)}$$
$$= \cdots$$
$$= \left(a^2 - b^2\right)^{n-1} D_2$$
$$= \left(a^2 - b^2\right)^n$$

习题2.4

2.⑵ $A_{n\times n}$，证明 $\left|A^*\right| = \left|A\right|^{n-1}$

$$AA^* = |A|I \Longrightarrow |A|\left|A^*\right| = |A|^n$$

A可逆

$\Downarrow$

$|A| \neq 0$

$\Downarrow$

$\left|A^*\right| = |A|^{n-1}$

A不可逆

$\Downarrow$

$|A| = 0$

$\Downarrow$

$\left|A^*\right| = 0?$

为什么？用反证！

反证

$\left|A^*\right| \neq 0$

$\Downarrow$

A^*可逆

$\Downarrow$

$AA^*(A^*)^{-1} = |A|(A^*)^{-1}$

$\Downarrow$

$A = |A|(A^*)^{-1} = 0$

$\Downarrow$

$A^* = 0$

$\left|A^*\right| \neq 0$

$A_{n\times n}\neq 0, A^{*}=A^{T}$，证明：$A$是可逆矩阵。

$$A_{n\times n}\neq 0, A^{*}=A^{T},$$

$$\Downarrow$$

$$A_{ij}=a_{ij}$$

$$\Downarrow$$

$$|A|=\sum_{j=1}^{n}a_{1j}A_{1j}=\sum_{j=1}^{n}a_{1j}a_{1j}=\sum_{j=1}^{n}a_{1j}^{2}\neq 0$$

$$\Downarrow$$

A可逆

复习题二

$2.A_{n\times n}\neq 0,\exists k\in Z^{+},s.t.A^{k}=0,$证明：$A$不可逆。

反证：A可逆 $\Rightarrow |A|\neq 0 \Rightarrow |A^{k}|=|A|^{k}\neq 0$

矛盾

A不可逆 $\Leftarrow A^{k}=0$

3. $A_{3\times3}, a_{ij}=A_{ij}$, 求$|A|$.

$A_{4\times 3}$,求$\left|AA^T\right|$.

解：$AA^Tx=0$和$A^Tx=0$同解。

系数矩阵的秩相等 $R(AA^T)=R(A^T)=R(A)<4$

$\left|AA^T\right|=0$

下证 $AA^Tx=0$和$A^Tx=0$同解。

$A^Tx=0\Rightarrow AA^Tx=0$。

$AA^Tx=0\Rightarrow x^TAA^Tx=0\Rightarrow (A^Tx)^T(A^Tx)=0\Rightarrow A^Tx=0$