Homework 1

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In [2]:

```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.patches as patches
```

In [4]:

```
import warnings
warnings.simplefilter('ignore')
```

1. Matrix Rank

For parts (a) and (b), we will assume that we have a matrix $A \in \mathbb{R}^{m \times n}$ such that the row rank is equal to r and a column rank equal to k.

(a)

We assume that A has k linearly independent columns, thus A can be written as

$$A = CW = [\, c_1 \quad \dots \quad c_k \,]\, W$$

where $A_i = C * W_i$, $C \in \mathbb{R}^{m \times k}$ and $W \in \mathbb{R}^{k \times n}$. We define c such that it spans the range of A. We can write c in row form such that

$$A = \begin{bmatrix} \bar{c}_1^T \\ \dots \\ \bar{c}_m^T \end{bmatrix} W$$

$$A = \begin{bmatrix} \bar{c}_1^T W \\ \dots \\ \bar{c}_m^T W \end{bmatrix}$$

We can see that each row in the resulting A is a linear combination of the matrix W. Because every row is a linear combination of W and there are only k rows in W, we can see that the row rank of A , r, must be less than or equal to k - i.e. $r \leq k$.

(b)

We assume that A has r linearly independent rows, thus A can be written as

$$A = VR = \begin{bmatrix} v_1 \\ \dots \\ v_r \end{bmatrix} R$$

where $\bar{A}_i^T = v_i^T R$, $v \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$. We define R such that it is a basis for A. We can write v in column form such that

$$A = \begin{bmatrix} \bar{v}_1^T & \dots & \bar{v}_m^T \end{bmatrix} R$$

$$A = \begin{bmatrix} \bar{v}_1^T R & \dots & \bar{v}_m^T R \end{bmatrix}$$

We can see that each row in the resulting A is a linear combination of the matrix R. Because every column is a linear combination of R and there are only r columns in R, we can see that the col rank of A, k, must be less than or equal to r - i.e. $k \geq r$.

2. Grammian Rank

We can first see that $\text{rank}(A) = \text{rank}(A^T)$ by inspection. We know that the $\text{rank}(A) = \text{row rank}(A)$ from question 1. We know the transpose of A results in the relationship that $\text{row rank}(A) = \text{col rank}(A^T) = \text{rank}(A^T)$. Thus, we can see that the $\text{rank}(A) = \text{rank}(A^T)$.

Using the rank-nullity theorem, we can see the rest of the equality.

Second, we will prove $\text{rank}(A) = \text{rank}(A^T A)$. From the rank nullity theorem, in order for $\text{rank}(A) = \text{rank}(A^T A)$, $\mathcal{N}(A) = \mathcal{N}(A^T A)$ must also be true.

Given $Ax = 0$, we can see that $A^T Ax = 0$ as one of our terms (Ax) zeros the left side. Given $A^T Ax = 0$, we can perform the following operations to show that $Ax = 0$.

$$A^T Ax = x^T A^T Ax = 0$$

$$|Ax|^2 = 0$$

$$|Ax| = 0$$

Since we know that for a given v, $|v| = 0$ implies $v = 0$, we can see that $Ax = 0$.

Thus we can see that the nullspaces of A and $A^T A$ are equivalent.

Third, we will prove $\text{rank}(A) = \text{rank}(AA^T)$. From the rank nullity theorem, in order for $\text{rank}(A) = \text{rank}(A^T A)$, $\mathcal{N}(A) = \mathcal{N}(AA^T)$ must also be true.

Given $Ax = 0$, we can see that $AA^T x = 0$.

3. Basis for Domain from Nullspace of A and Range of A^T

(a)

To symbolically compute $\begin{bmatrix} A^T & N \end{bmatrix}^{-1}$, we can check to see if $\begin{bmatrix} A^T & N \end{bmatrix}^{-1} = \begin{bmatrix} A^T & N \end{bmatrix}^T$ (as we know there is a relationship between the the inverse and transpose).

We know that,

$$[\,A^T \quad N\,]^{-1}[\,A^T \quad N\,] = I$$

Substituting the inverse for the transpose results inverse

$$\begin{bmatrix} A^T \\ N \end{bmatrix}[\,A^T N\,] = \begin{bmatrix} AA^T & AN \\ N^TA^T & N^TN \end{bmatrix} = \begin{bmatrix} AA^T & 0 \\ 0 & N^TN \end{bmatrix}$$

We can see that the resulting matrix is not quite the identity matrix, I. However, if we multiply the resulting matrix by its inverse it would result in the identity matrix.

Thus, we now know that

$$\begin{bmatrix} AA^T & 0 \\ 0 & N^TN \end{bmatrix}^{-1} \begin{bmatrix} A^T \\ N \end{bmatrix}[\,A^T N\,] = I$$

$$\begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^TN)^{-1} \end{bmatrix} \begin{bmatrix} A^T \\ N \end{bmatrix}[\,A^T N\,] = I$$

We can now equate our equation from before to this resulting equation such that

$$[\,A^T \quad N\,]^{-1}[\,A^T \quad N\,] = \begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^TN)^{-1} \end{bmatrix} \begin{bmatrix} A^T \\ N \end{bmatrix}[\,A^T N\,]$$

$$[\,A^T \quad N\,]^{-1} = \begin{bmatrix} (AA^T)^{-1} & 0 \\ 0 & (N^TN)^{-1} \end{bmatrix} \begin{bmatrix} A^T \\ N \end{bmatrix}$$

$$[\,A^T \quad N\,]^{-1} = \begin{bmatrix} (AA^T)^{-1}A \\ (N^TN)^{-1}N^T \end{bmatrix}$$

(b)

We can now solve for x_1' and x_2' given A, N, and x.

$$x = [\,A^T \quad N\,] \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$[\,A^T \quad N\,]^{-1} x = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$\begin{bmatrix} (AA^T)^{-1}A \\ (N^TN)^{-1}N^T \end{bmatrix} x = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

Solving for x_1' and x_2',

$$x_1' = (AA^T)^{-1}Ax$$

$$x_2' = (N^TN)^{-1}N^TAx$$

4. Range and Nullspace

(a)

Given that $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$, we know that x and y are orthogonal to each other because we know that $\mathcal{R}(A)$ and $\mathcal{N}(A^T)$ are orthogonal subspaces of the co-domain. Therefore, if y and x are within these subspaces they must also be orthogonal to each other.

We can also prove this by decomposing A into rows as

$$A = \begin{bmatrix} - & \bar{a}_1^T & - \\ \dots & & \dots \\ - & \bar{a}_m^T & - \end{bmatrix}$$

Knowing that $\mathcal{R}(A^T)$ is the span of the rows of A, we can now calculate $Ax = 0$ (i.e. the nullspace) as

$$Ax = \begin{bmatrix} - & \bar{a}_1^T & - \\ \dots & & \dots \\ - & \bar{a}_m^T & - \end{bmatrix} x = \begin{bmatrix} \bar{a}_1^T x \\ \dots \\ \bar{a}_m^T x \end{bmatrix} = 0$$

Thus, we can see that if $x \in (N)(A)$ (i.e. $Ax = 0$) then x is orthogonal to each row of A. If $y \in \mathcal{R}(A)$, then x must be orthogonal to it as y is a function of the rows of A.

(b)

If $A \in \mathbb{R}^{5\times10}$ and there are 3 linearly independent columns, then we know that $rank(A) = 3$. Therefore, using the Fundamental Theorem of Linear Algebra we know that (given $m = 5$ and $n = 10$)

$$\mathcal{R}(A) \in \mathbb{R}^3$$

$$\mathcal{N}(A^T) \in \mathbb{R}^2$$

$$\mathcal{N}(A) \in \mathbb{R}^7$$

$$\mathcal{R}(A^T) \in \mathbb{R}^3$$

5. Fundamental Theorem of Linear Algebra Pictures

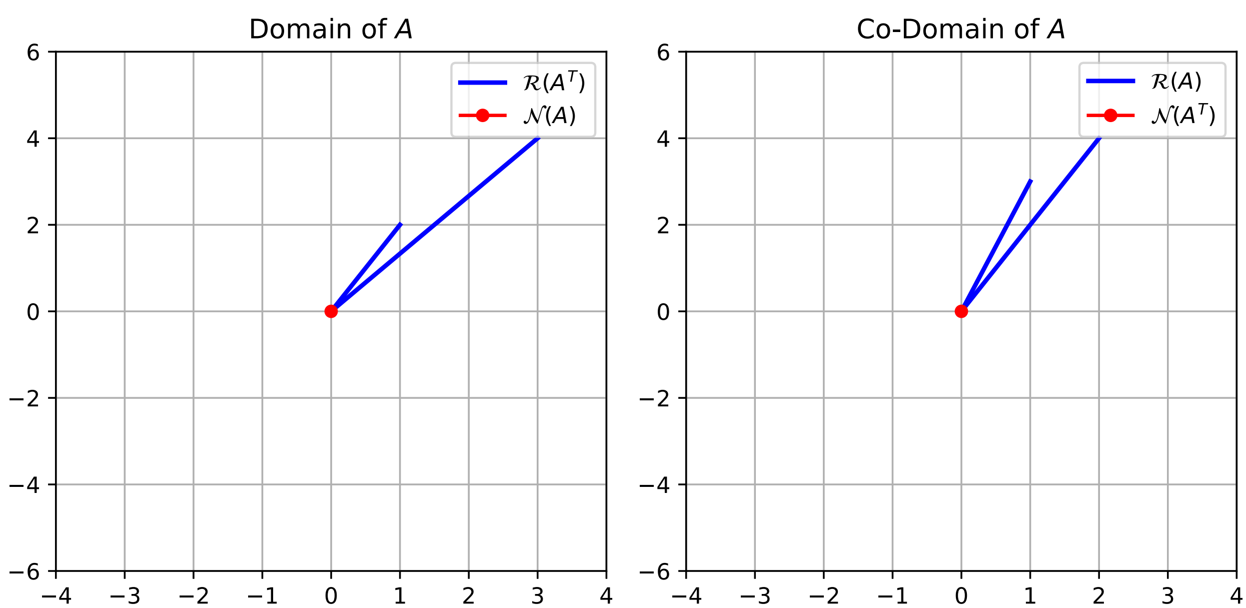
(a)

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

In [5]:

```
fig, ax = plt.subplots(1, 2, figsize=(8, 4))

# Plot range of A^T
ax[0].plot([0, 1], [0, 2], color='b', label='$\mathcal{R}(A^T)$', linewidth=2)
ax[0].plot([0, 3], [0, 4], color='b', linewidth=2)

# Plot nullspace of A
```

```
ax[0].plot([0], [0], color='r', label='$\mathcal{N}(A)$', marker='.', markersize=10)

ax[0].set_xlim([-4, 4])
ax[0].set_ylim([-6, 6])
ax[0].grid()
ax[0].legend()
ax[0].set_title('Domain of $A$')

# Plot the range of A
ax[1].plot([0, 1], [0, 3], color='b', label='$\mathcal{R}(A)$', linewidth=2)
ax[1].plot([0, 2], [0, 4], color='b', linewidth=2)

# Plot the nullspace of A^T
ax[1].plot([0], [0], color='r', label='$\mathcal{N}(A^T)$', marker='.', markersize=10)

ax[1].set_xlim([-4, 4])
ax[1].set_ylim([-6, 6])
ax[1].grid()
ax[1].legend()
ax[1].set_title('Co-Domain of $A$')

fig.tight_layout()

plt.show()
```



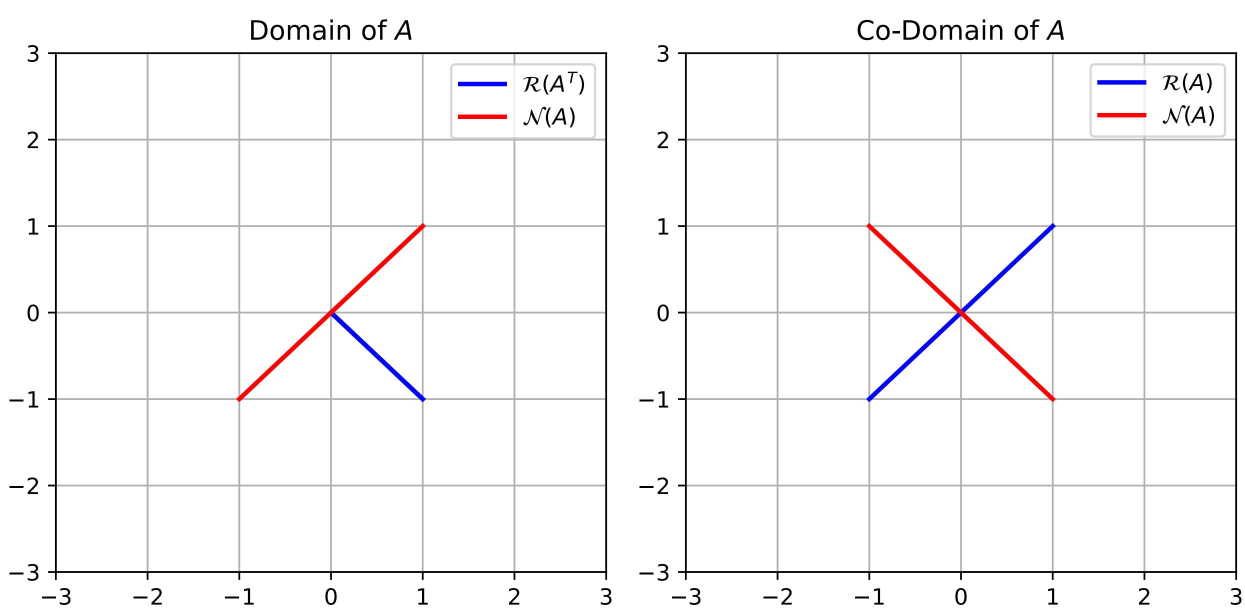
(b)

Given $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

In [9]:

```
fig, ax = plt.subplots(1, 2, figsize=(8, 4))

# Plot range of A^T
ax[0].plot([0, 1], [0, -1], color='b', label='$\mathcal{R}(A^T)$', linewidth=2)
ax[0].plot([0, 1], [0, -1], color='b', linewidth=2)

# Plot nullspace of A
ax[0].plot([-1, 1], [-1, 1], color='r', label='$\mathcal{N}(A)$', linewidth=2)
```

```python
ax[0].set_xlim([-3, 3])
ax[0].set_ylim([-3, 3])
ax[0].grid()
ax[0].legend()
ax[0].set_title('Domain of $A$')

# Plot the range of A
ax[1].plot([0, 1], [0, 1], color='b', label='$\mathcal{R}(A)$', linewidth=2)
ax[1].plot([0, -1], [0, -1], color='b', linewidth=2)

# Plot the nullspace of A^T
ax[1].plot([-1, 1], [1, -1], color='r', label='$\mathcal{N}(A)$', linewidth=2)

ax[1].set_xlim([-3, 3])
ax[1].set_ylim([-3, 3])
ax[1].grid()
ax[1].legend()
ax[1].set_title('Co-Domain of $A$')

fig.tight_layout()

plt.show()
```



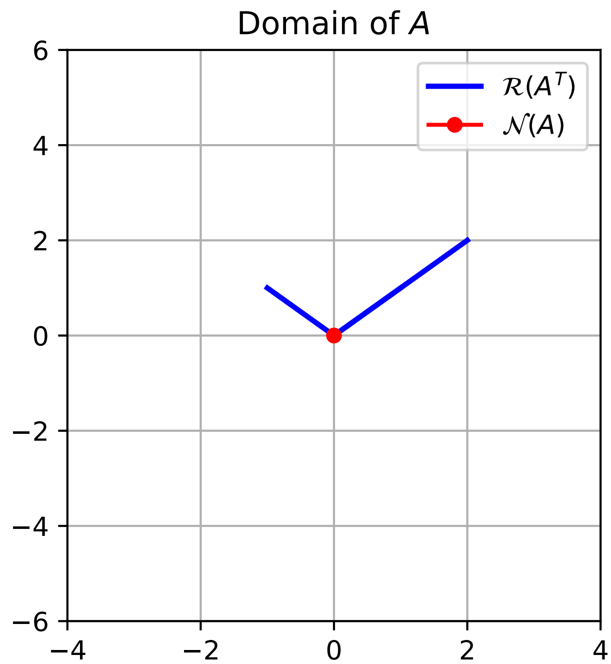
(c)

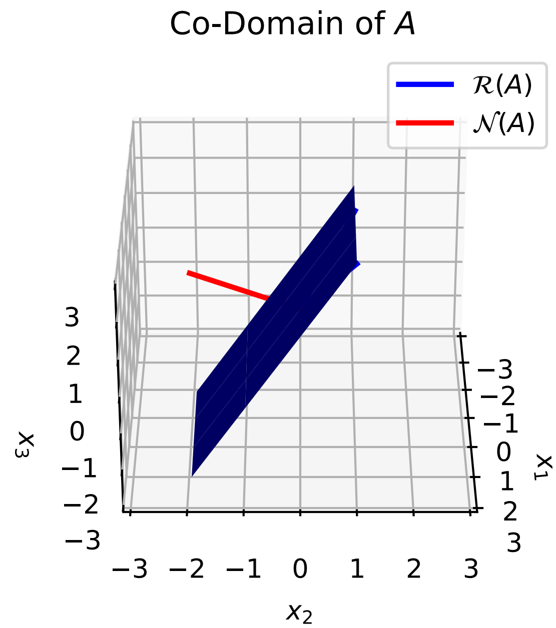
Given $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

In [45]:

```python
fig = plt.figure(figsize=(8, 4))

#2D
# Plot range of A^T
ax = fig.add_subplot(1, 2, 1)
ax.plot([0, -1], [0, 1], color='b', label='$\mathcal{R}(A^T)$', linewidth=2)
ax.plot([0, 1], [0, 1], color='b', linewidth=2)
ax.plot([0, 2], [0, 2], color='b', linewidth=2)
```

```python
# Plot nullspace of A
ax.plot([0], [0], color='r', label='$\mathcal{N}(A)$', marker='.', markersize=10)

ax.set_xlim([-4, 4])
ax.set_ylim([-6, 6])
ax.grid()
ax.legend()
ax.set_title('Domain of $A$')

#3D
ax = fig.add_subplot(1, 2, 2, projection='3d')

# Plot the range of A
x_1 = np.arange(-2, 2, 1)
x_2 = np.arange(-2, 2, 1)
x_2, x_1 = np.meshgrid(x_1, x_2)
x_3 = 2 * x_2

surf = ax.plot_surface(x_1, x_2, x_3, color='blue')

ax.plot([0, -1], [0, 1], [0, 2], color='b', label='$\mathcal{R}(A)$', linewidth=2)
ax.plot([0, 1], [0, 1], [0, 2], color='b', linewidth=2)

# Plot the nullspace of A^T
ax.plot([0, 0], [0, -2], [0, 1], color='r', label='$\mathcal{N}(A)$', linewidth=2)

ax.set_xlim3d([-3, 3])
ax.set_ylim3d([-3, 3])
ax.set_zlim3d([-3, 3])
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$x_3$')
ax.legend()
ax.set_title('Co-Domain of $A$')
ax.view_init(30, 0)

#fig.tight_layout()

plt.show()
```





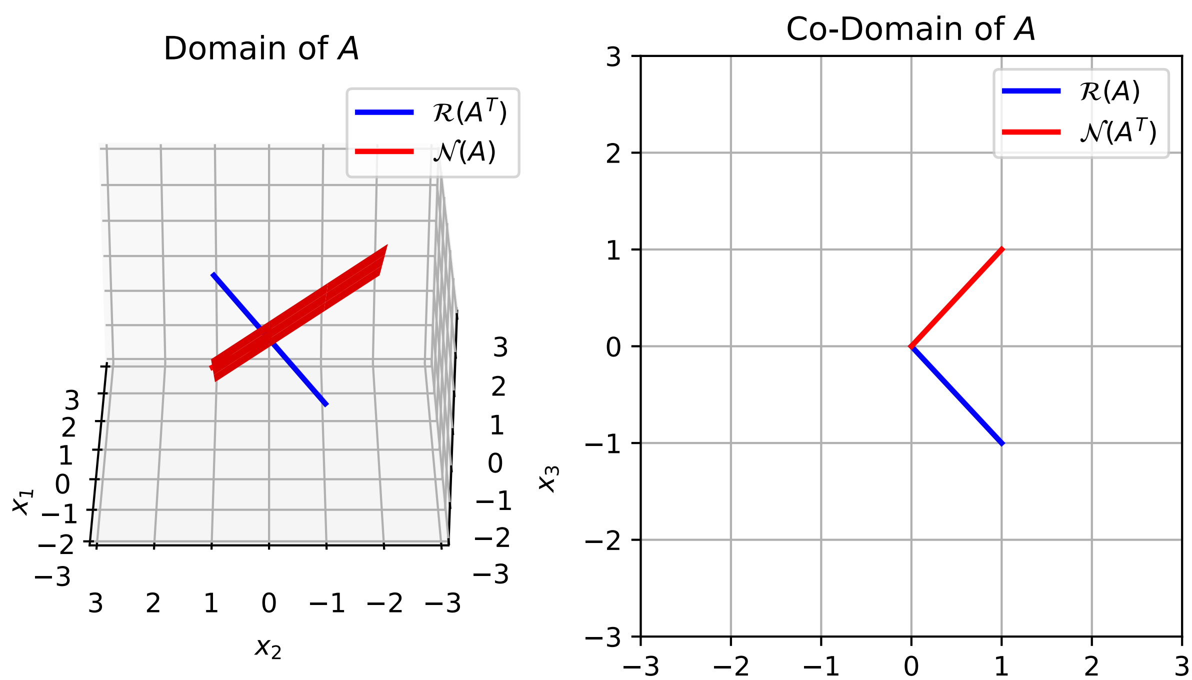
(d)

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$.

In [53]:

```
fig = plt.figure(figsize=(8, 4))

#3D
ax = fig.add_subplot(1, 2, 1, projection='3d')

# Plot range of A^T
ax.plot([0, 1], [0, 1], [0, 1], color='b', label='$\mathcal{R}(A^T)$', linewidth=2)
ax.plot([0, -1], [0, -1], [0, -1], color='b', linewidth=2)

# Plot nullspace of A
x_1 = np.arange(-2, 2, 1)
x_2 = np.arange(-2, 2, 1)
x_2, x_1 = np.meshgrid(x_1, x_2)
x_3 = -x_2 - x_1

surf = ax.plot_surface(x_1, x_2, x_3, color='r')

ax.plot([0, -1], [0, 1], [0, 0], color='r', label='$\mathcal{N}(A)$', linewidth=2)
ax.plot([0, -1], [0, 0], [0, 1], color='r', linewidth=2)

ax.set_xlim3d([-3, 3])
ax.set_ylim3d([-3, 3])
ax.set_zlim3d([-3, 3])
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$x_3$')
ax.legend()
ax.set_title('Domain of $A$')
ax.view_init(30, 180)
```

```
#2D
ax = fig.add_subplot(1, 2, 2)

# Plot the range of A
ax.plot([0, 1], [0, -1], color='b', label='$\mathcal{R}(A)$', linewidth=2)

# Plot the nullspace of A^T
ax.plot([0, 1], [0, 1], color='r', label='$\mathcal{N}(A^T)$', linewidth=2)

ax.set_xlim([-3, 3])
ax.set_ylim([-3, 3])
ax.grid()
ax.legend()
ax.set_title('Co-Domain of $A$')

#fig.tight_layout()

plt.show()
```



6. Representations of Affine Sets

(a)

(i)

Given $A = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$ and $b = 1$, we know that $\mathcal{N}(A) = \mathcal{R}(N)$. So if we can define $\mathcal{N}(A)$, we can solve can find an N such that the equality holds true.

$$\mathcal{N}(A) = \begin{bmatrix} -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \mathcal{R}(N)$$

We know that N is defined by a scalar of the columns of $\mathcal{R}(N)$, thus we can define N as

$$N = \begin{bmatrix} -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We can solve for d by substituting $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} 1 & -2 & 0 \end{bmatrix} d = 1$$

$$d_1 - 2d_2 = 1$$

$$d_1 = 1 + 2d_2$$

We select $d_3 = 0$ and $d_2 = 1$; thus we can solve for $d_1 = 3$.

Thus, we find an $N = \begin{bmatrix} -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $d = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ for our given A and b.

(ii)

Given $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we know that $\mathcal{N}(A) = \mathcal{R}(N)$. So if we can define $\mathcal{N}(A)$, we can solve can find an N such that the equality holds true.

$$\mathcal{N}(A) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \mathcal{R}(N)$$

We know that N is defined by a scalar of the columns of $\mathcal{R}(N)$, thus we can define N as

$$N = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

We can solve for d by substituting $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This results in two equations:

$$d_1 - 2d_2 = 1$$

$$d_2 + d_3 = 1$$

We select $d_3 = 0$; thus we can solve for $d_2 = 1$ and $d_1 = 3$.

Thus, we find an $N = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $d = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ for our given A and b.

(iii)

Given $A = \begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we know that $\mathcal{N}(A) = \mathcal{R}(N)$. So if we can define $\mathcal{N}(A)$, we can solve can find an N such that the equality holds true.

$$\mathcal{N}(A) = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathcal{R}(N)$$

We know that N is defined by a scalar of the columns of $\mathcal{R}(N)$, thus we can define N as

$$N = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We can solve for d by substituing $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \end{bmatrix} d = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This results in two equations:

$$d_1 - 2d_2 + d_4 + d_5 = -1$$

$$d_2 + d_3 - d_4 + d_5 = 1$$

We select $d_3 = 0$, $d_4 = 0$, and $d_5 = 0$; thus we can solve for $d_2 = 1$ and $d_1 = 1$.

Thus, we find an $N = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $d = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ for our given A and b.

(b)

(i)

Given $N = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $A \in \mathbb{R}^{2\times 3}$, we know that $N^T A^T = 0$ based on the relationship that $\mathcal{N}(A) = \mathcal{R}(N)$. Thus we can write,

$$N^T A^T = \begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} = 0$$

Solving this results in the following relationships,

$$a_{11} - 2a_{12} - 2a_{13} = 0$$

$$a_{21} - 2a_{22} - 2a_{23} = 0$$

Thus, we can set $a_{12} = a_{13} = a_{22} = a_{23} = 1$; thus we can solve for $a_{11} = 4$ and $a_{21} = 4$. Therefore, we can define A as,

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

To solve for b, we can use the same principle as in part (a) by substituting our two known equations $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} 4 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = b$$

Solving for b

$$b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

Thus, we find an $A = \begin{bmatrix} 4 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ for our given N and d.

(ii)

Given $N = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$, $d = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $A \in \mathbb{R}^{1\times 3}$, we know that $N^T A^T = 0$ based on the relationship that $\mathcal{N}(A) = \mathcal{R}(N)$. Thus we can write,

$$N^T A^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = 0$$

Solving this results in the following relationships,

$$a_{11} + 2a_{12} + a_{13} = 0$$

$$a_{13} = 0$$

We can set $a_{12} = 1$; thus we can solve for $a_{11} = -2$ and $a_{13} = 0$. Therefore, we can define A as,

$$A = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$$

To solve for b, we can use the same principle as in part (a) by substituting our two known equations $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = b$$

Solving for b

$$b = -3$$

Thus, we find an $A = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$ and $b = -3$ for our given N and d.

(iii)

Given $N = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $d = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, and $A \in \mathbb{R}^{3\times5}$, we know that $N^T A^T = 0$ based on the relationship that $\mathcal{N}(A) = \mathcal{R}(N)$. Thus we can write,

$$N^T A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \\ a_{15} & a_{25} & a_{35} \end{bmatrix} = 0$$

Solving this results in the following relationships,

$$a_{11} + a_{13} + a_{15} = 0$$

$$a_{21} + a_{23} + a_{25} = 0$$

$$a_{31} + a_{33} + a_{35} = 0$$

$$-2a_{11} + a_{12} + a_{14} + a_{15} = 0$$

$$-2a_{21} + a_{22} + a_{24} + a_{25} = 0$$

$$-2a_{31} + a_{32} + a_{34} + a_{35} = 0$$

We can set $a_{i1} = a_{i4} = a_{i5} = 1$ (where $i = 1, 2, 3$); thus we can solve for $a_{i2} = 0$ and $a_{i3} = -2$. Therefore, we can define A as,

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \end{bmatrix}$$

To solve for b, we can use the same principle as in part (a) by substituting our two known equations $x = Nz + d$ into $Ax = b$ such that

$$A(Nz + d) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ad = b$.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = b$$

Solving for b

$$b = \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$$

Thus, we find an $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$ for our given N and d.

7. Equivalent Representations of Spaces

(a)

For $A \in \mathbb{R}^{m \times n}$ and $U \in \mathbb{R}^{m \times m}$, we know that $x \in \mathcal{N}(A)$ if and only if x is a solution to the system $Ax = 0$. In addition, $x \in \mathcal{N}(UA)$ if and only if x is a solution to the system $UAx = 0$.

Since U is an invertable matrix, we can multiply our equation $UAx = 0$ by U^{-1} on both sides resulting in $Ax = 0$. Therefore, every x that is in the nullspace of A is also in the null space of UA, and every x not in the null space of A is not in the null space of UA. Therefore, we can state that $\mathcal{N}(A) = \mathcal{N}(UA)$.

(b)

For $A \in \mathbb{R}^{m \times n}$ and $V \in \mathbb{R}^{n \times n}$, if we suppose there exists a $y \in \mathcal{R}(AV)$ then $y = AVu$ for some u. Let $x = Vu$ for some x. Thus, we see that for some x, $y = Ax$. Therefore, $y \in \mathcal{R}(A)$ and thus $\mathcal{R}(AV)$ must be a subset of $\mathcal{R}(A)$.

If we suppose there exists a $y \in \mathcal{R}(A)$, then $y = Ax$ for some x. This can be written as $y = Ax = AIx = AVV^{-1}x$, such that $y = AVV^{-1}x$. Let $x = Vu$ for some u, we know that $u = V^{-1}x$ (given that V is invertible). Substituting this relationship into our equation for y, we see that $y = AVu$. Therefore, $y \in mathcalR(AV)$ and thus $\mathcal{R}(A)$ must be a subset of $\mathcal{R}(AV)$.

Given that $\mathcal{R}(AV)$ must be a subset of $\mathcal{R}(A)$ **and** $\mathcal{R}(A)$ must be a subset of $\mathcal{R}(AV)$, then $\mathcal{R}(A) = \mathcal{R}(AV)$.

8. Vector Derivatives

(a)

Let $f(x) = x_1^4 + 3x_1x_2^2 + e^{x_2} + \frac{1}{x_1x_2}$, solving for $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 4x_1^3 + 3x_2^2 - \frac{1}{x_1^2x_2} & 6x_1x_2 + e^{x_2} - \frac{1}{x_1x_2^2} \end{bmatrix}$$

(b)

Let $f(x) = \begin{bmatrix} \beta x_1 + \alpha x_2 \\ \beta(x_1 + x_2) \\ \alpha^2 x_1 + \beta x_2 \\ \beta x_1 + \frac{1}{\alpha} x_2 \end{bmatrix}$, solving for $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \beta & \alpha \\ \beta & \beta \\ \alpha^2 & \beta \\ \beta & \frac{1}{\alpha} \end{bmatrix}$$

(c)

Let $f(x) = \begin{bmatrix} e^{x^TQx} \\ (x^TQx)^{-1} \end{bmatrix}$, solving for $\frac{\partial f}{\partial x}$ we can use the product and chain rule.

Using the deriviation provided in the lecture notes, we know that $\frac{\partial}{\partial x}(x^TQx) = x^T(Q + Q^T)$. Thus, we need to use the chain rule to solve for our functions given in $f(x)$.

First, solving for e^{x^TQx} (set as $f_1(x)$),

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial g}\frac{\partial g}{\partial x}$$

where $g(x) = x^T Qx$. Knowing that the derivative of e^x is e^x, we know that

$$\frac{\partial f_1}{\partial x} = (e^{x^T Qx})(x^T(Q + Q^T))$$

Second, solving for $(x^T Qx)^{-1}$ We can solve for the derivative of an inverse using the identity matrix such that

$$(I)' = (KK^{-1})' = K'K^{-1} + K(K^{-1})'$$
$$(K^{-1})' = -K^{-1}K'K^{-1}$$

for any matrix K. Replacing K with our matrix $x^T Qx$ yields the following

$$\frac{\partial f_2}{\partial x} = -(x^T Qx)^{-1}(x^T(Q + Q^T))(x^T Qx)^{-1}$$

Thus, we find that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x^T(Q + Q^T)e^{x^T Qx} \\ -(x^T Qx)^{-1}(x^T(Q + Q^T))(x^T Qx)^{-1} \end{bmatrix}$$

In [ ]:

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