Homework 1

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In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
```

In [2]:
```
import warnings
warnings.simplefilter('ignore')
```

1. Projections

(a)

To compute the projection of $x = [1, 2, 3]^T$ onto $y = [1, 1, -2]^T$, we will use the following equation:

$$proj_y x = y(y^T y)^{-1} y^T x$$

We can perform this calculation using Numpy and built-in matrix multiplication, transpose, and inverse functions.

In [3]:
```
x = np.array([[1], [2], [3]])
y = np.array([[1], [1], [-2]])

#print(x)
#print(y)

proj_yx = y.dot(np.linalg.inv(np.transpose(y).dot(y)).dot(np.transpose(y).dot(x)))

print(proj_yx)
```

```
[[-0.5]
 [-0.5]
 [ 1. ]]
```

The result is $proj_y x = [-0.5, -0.5, 1]^T$.

(b)

To compute the projection of $x = [1, 2, 3]^T$ onto the range $Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ we will use the following equation:

$$proj_Y x = Y(Y^T Y)^{-1} Y^T x$$

We can perform this calculation using Numpy and built-in matrix multiplication, transpose, and inverse functions.

In [4]:

```
x = np.array([[1], [2], [3]])
y = np.array([[1, 1], [-1, 0], [0, 1]])

#print(x)
#print(y)

proj_yx = y.dot(np.linalg.inv(np.transpose(y).dot(y)).dot(np.transpose(y).dot(x)))

print(proj_yx)
```

```
[[1.]
 [2.]
 [3.]]
```

The result is $proj_y x = [1, 2, 3]^T$.

2. Block Matrix Computations

(a)

$$AB = \begin{bmatrix} A_{11}B_{11} + \ldots + A_{1N}B_{N1} & \ldots & A_{11}B_{1K} + \ldots + A_{1N}B_{NK} \\ \vdots & & \vdots \\ A_{M1}B_{11} + \ldots + A_{MN}B_{N1} & \ldots & A_{M1}B_{1K} + \ldots + A_{1N}B_{NK} \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_{11} \in \mathbb{R}^{n_1 \times k_1}$, $B_{1K} \in \mathbb{R}^{n_1 \times k_K}$, $B_{N1} \in \mathbb{R}^{n_N \times k_1}$, and $B_{NK} \in \mathbb{R}^{n_N \times k_K}$.

(b)

$$AB = \begin{bmatrix} A_1B_1 & \ldots & A_1B_k \\ \vdots & & \vdots \\ A_mB_1 & \ldots & A_mB_k \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_1 \in \mathbb{R}^{n \times 1}$ and $B_k \in \mathbb{R}^{n \times 1}$.

(c)

$$AB = \begin{bmatrix} | \\ A_1 \\ | \end{bmatrix} [- \quad B_1 \quad -] + \ldots + \begin{bmatrix} | \\ A_n \\ | \end{bmatrix} [- \quad B_n \quad -].$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_1 \in \mathbb{R}^{1 \times k}$ and $B_n \in \mathbb{R}^{1 \times k}$.

(d)

$$ADB = \begin{bmatrix} A_1DB_1 & \dots & A_1DB_k \\ \vdots & & \vdots \\ A_mDB_1 & \dots & A_mDB_k \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_1 \in \mathbb{R}^{n\times 1}$ and $B_k \in \mathbb{R}^{n\times 1}$.

(e)

$$ADB = \sum_{x=1}^{n}\sum_{y=1}^{n} \begin{bmatrix} | \\ A_x \\ | \end{bmatrix} D_{xy} \begin{bmatrix} - & B_y & - \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_1 \in \mathbb{R}^{1\times k}$ and $B_k \in \mathbb{R}^{1\times k}$.

(f)

$$AB = \begin{bmatrix} AB_1 & \dots & AB_k \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B_1 \in \mathbb{R}^{n\times 1}$ and $B_k \in \mathbb{R}^{n\times 1}$.

(g)

$$AB = \begin{bmatrix} A_1B \\ \vdots \\ A_mB \end{bmatrix}.$$

Given the resulting matrix, we know the required dimensions of the sub-blocks of B are $B \in \mathbb{R}^{n\times k}$ (since there are no sub-blocks of B).

3. Linear Transformations of Sets

(a) Affine Sets

Given $\mathcal{X}_1 = \{x | x_1 + x_2 = 1, x \in \mathbb{R}^2\}$ and $\mathcal{X}_2 = \{x | x_1 - x_2 = 1, x \in \mathbb{R}^2\}$, we can draw the set of points for Ax for $x \in \mathcal{X}_1$ and $x \in \mathcal{X}_2$.

For the condition where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can solve for Ax such that

$$Ax = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can now draw the set of points by finding two points and then drawing the line through these two points.

We can now define the set of points for when $x \in \mathcal{X}_1$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 1$$

thus our first point is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 0$$

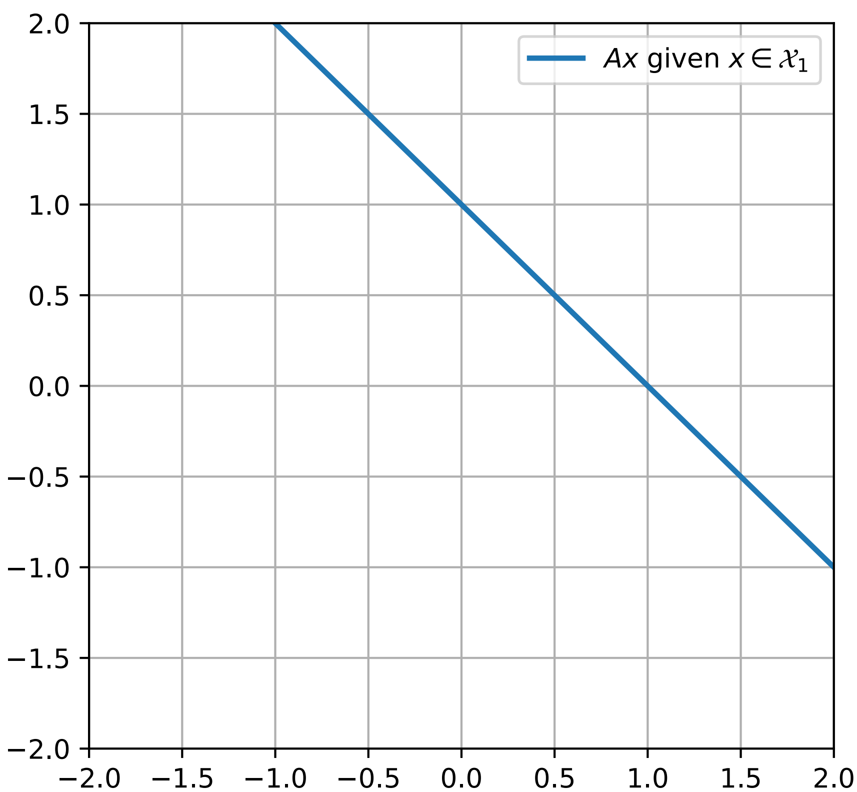
thus our second point is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In [5]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [0, 1]
x_2 = [1, 0]

x = np.linspace(-5, 5, num=100)
y = (x_2[1] - x_2[0])/(x_1[1] - x_1[0]) * x + (x_1[0]*x_2[1] - x_1[1]*x_2[0]) / (x_1[0]

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$Ax$ given $x \in \mathcal{X}_1$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



We can now define the set of points for when $x \in \mathcal{X}_2$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = -1$$

thus our first point is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = 0$$

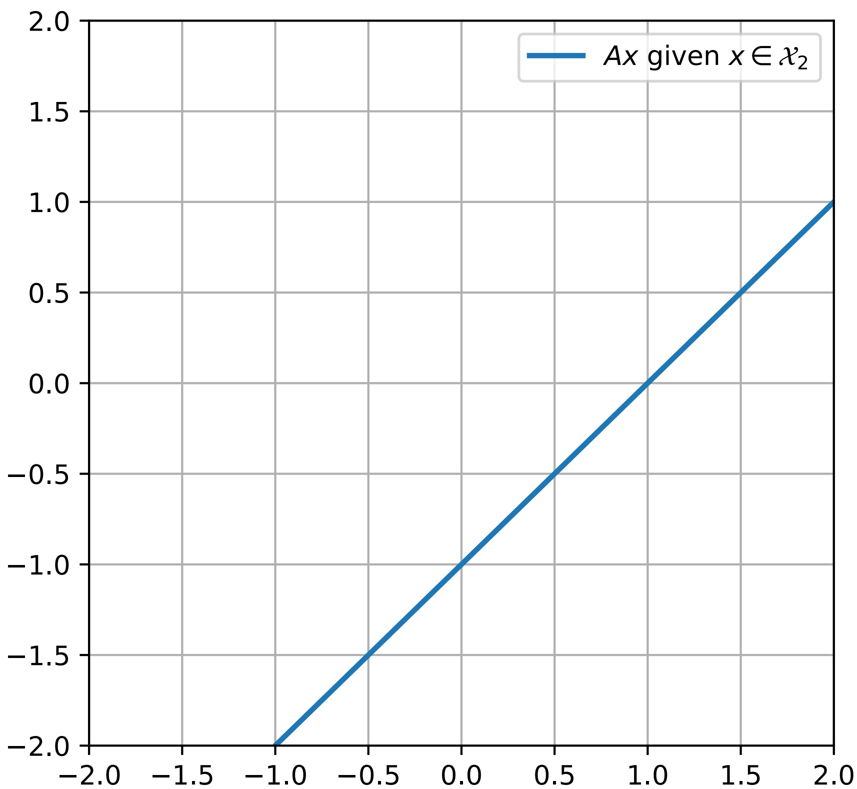
thus our second point is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In [6]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [0, 1]
x_2 = [-1, 0]

x = np.linspace(-5, 5, num=100)
y = (x_2[1] - x_2[0])/(x_1[1] - x_1[0]) * x + (x_1[0]*x_2[1] - x_1[1]*x_2[0]) / (x_1[0]

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$Ax$ given $x \in \mathcal{X}_2$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



For the condition where $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, we can solve for Ax such that

$$Ax = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_2 \end{bmatrix}$$

We can now draw the set of points by finding two points and then drawing the line through these two points.

We can now define the set of points for when $x \in \mathcal{X}_1$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 1$$

thus our first point is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 0$$

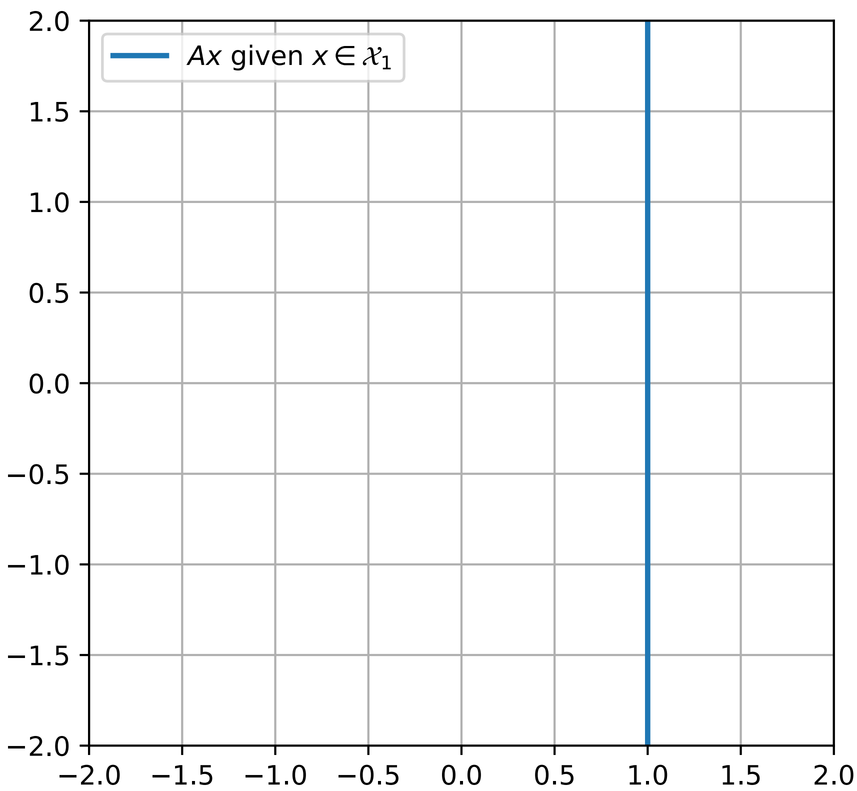
thus our second point is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In [27]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [1, 1]
x_2 = [-1, 0]

# By inspection, we see from the coordinates that Ax is a vertical line @ x_1 = 1, thus

fig, ax = plt.subplots(figsize=(5, 5))
ax.axvline(1, label='$Ax$ given $x \in \mathcal{X}_1$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



We can now define the set of points for when $x \in \mathcal{X}_2$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = -1$$

thus our first point is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = 0$$

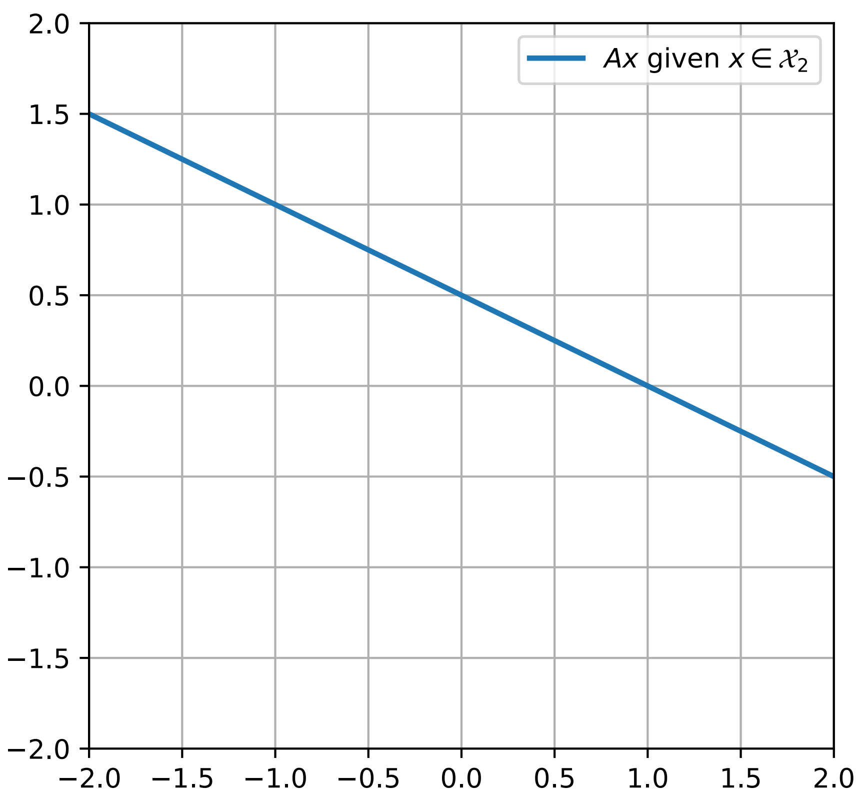
thus our second point is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In [28]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [-1, 1]
x_2 = [1, 0]

x = np.linspace(-5, 5, num=100)
y = (x_2[1] - x_2[0])/(x_1[1] - x_1[0]) * x + (x_1[0]*x_2[1] - x_1[1]*x_2[0]) / (x_1[0]

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$Ax$ given $x \in \mathcal{X}_2$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



For the condition where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, we can solve for Ax such that

$$Ax = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

We can now draw the set of points by finding two points and then drawing the line through these two points.

We can now define the set of points for when $x \in \mathcal{X}_1$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 1$$

thus our first point is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_1$,

$$x_2 = 1 - x_1 = 0$$

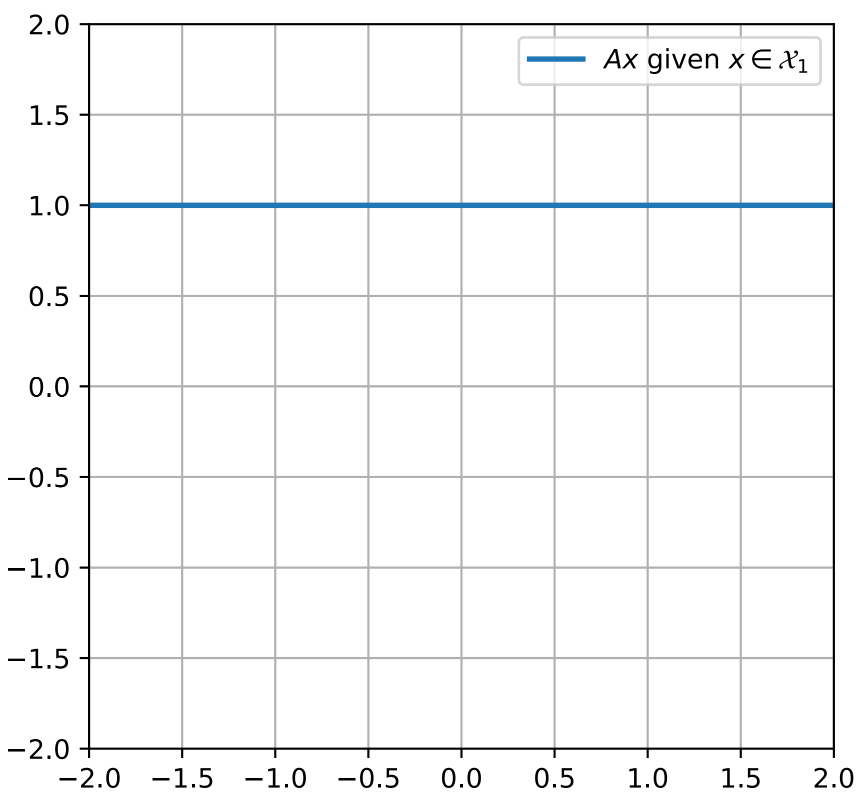
thus our second point is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

In [29]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [-1, 1]
x_2 = [1, 1]

x = np.linspace(-5, 5, num=100)
y = (x_2[1] - x_2[0])/(x_1[1] - x_1[0]) * x + (x_1[0]*x_2[1] - x_1[1]*x_2[0]) / (x_1[0]

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$Ax$ given $x \in \mathcal{X}_1$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



We can now define the set of points for when $x \in \mathcal{X}_2$. When $x_1 = 0$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = -1$$

thus our first point is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

When $x_1 = 1$, we find that given $x \in \mathcal{X}_2$,

$$x_2 = x_1 - 1 = 0$$

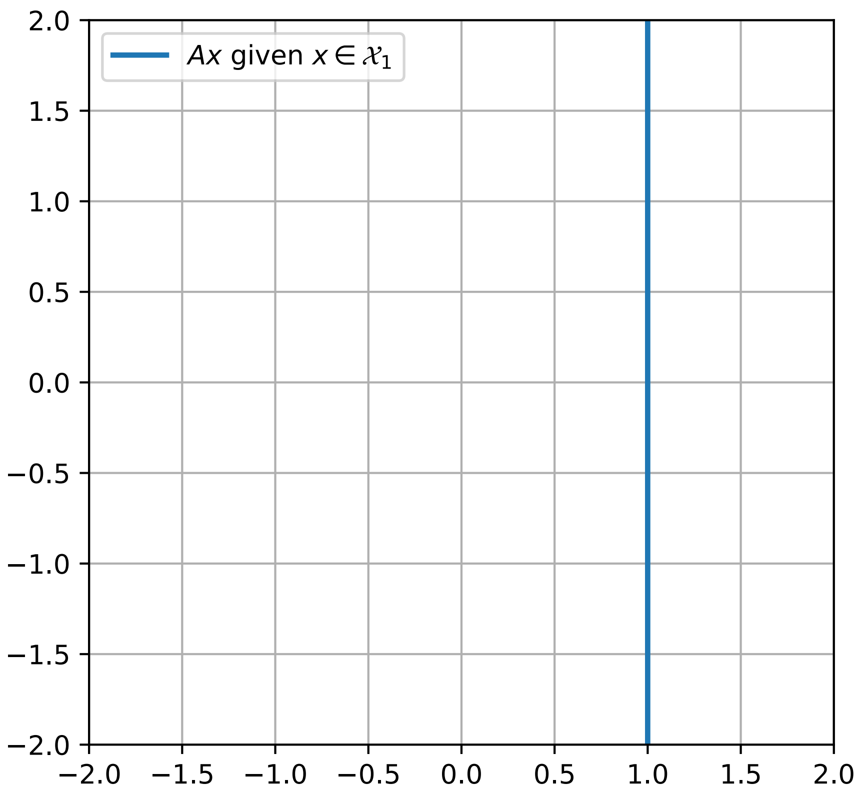
thus our second point is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

In [31]:

```
# Given coordinates defined above, define a line y that is the set of points Ax.
x_1 = [1, 1]
x_2 = [-1, 1]

# By inspection, we see from the coordinates that Ax is a vertical line @ x_1 = 1, thus

fig, ax = plt.subplots(figsize=(5, 5))
ax.axvline(1, label='$Ax$ given $x \in \mathcal{X}_1$', linewidth=2)
ax.set_xlim([-2, 2])
ax.set_ylim([-2, 2])
ax.grid()
ax.legend()
plt.show()
```



4. Affine and Half Spaces

(a)

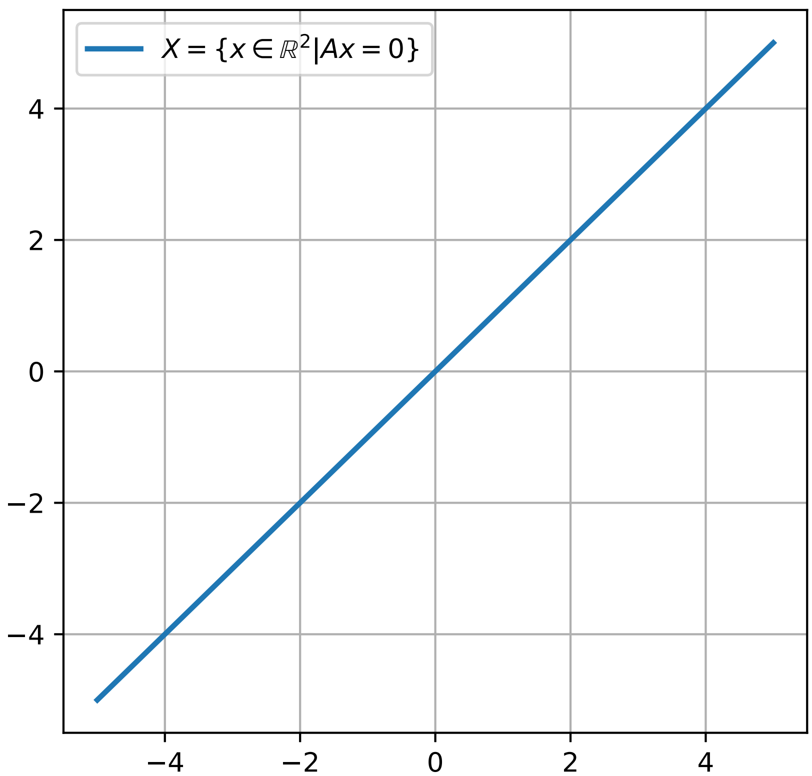
For $a^T = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $X = \{x \in \mathbb{R}^2 | a^T x = 0\}$, the set is defined as:

$$a^T x = 0$$
$$\begin{bmatrix} 1 & -1 \end{bmatrix} x = 0$$
$$x_1 - x_2 = 0$$
$$x_1 = x_2$$

In [15]:

```
x = np.linspace(-5, 5, num=100)
y = x

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$X = \{ x \in \mathbb{R}^{2} | Ax = 0 \}$', linewidth=2)
ax.legend()
ax.grid()
plt.show()
```



For $a^T = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $X = \{x \in \mathbb{R}^2 | a^T x = 1\}$, the set is defined as:

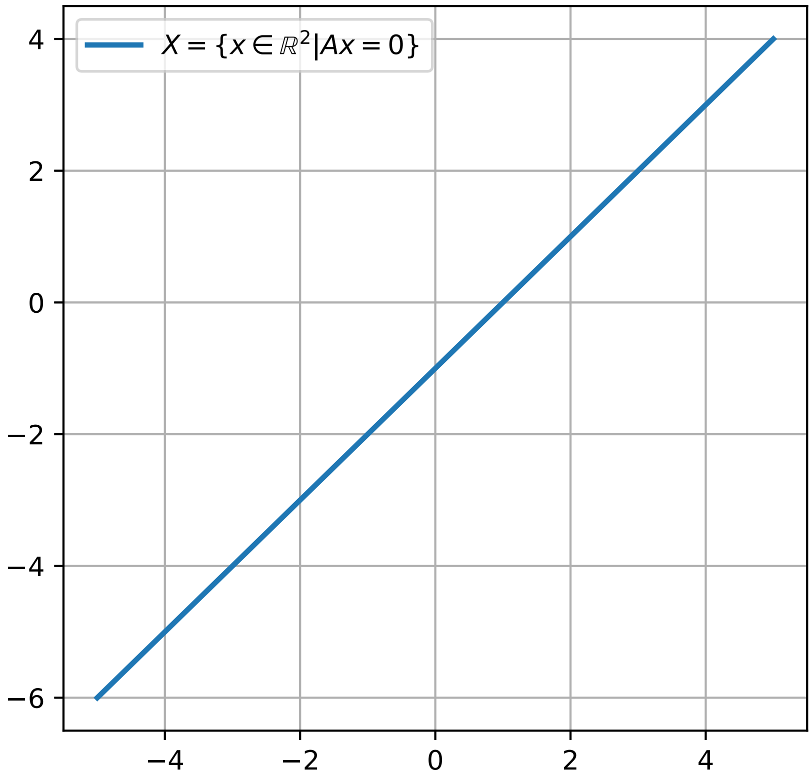
$$a^T x = 1$$
$$\begin{bmatrix} 1 & -1 \end{bmatrix} x = 1$$
$$x_1 - x_2 = 1$$
$$x_2 = x_1 - 1$$

In [17]:

```
x = np.linspace(-5, 5, num=100)
y = x - 1

fig, ax = plt.subplots(figsize=(5, 5))
```

```
ax.plot(x, y, label='$X = \{ x \in \mathbb{R}^{2} | Ax = 0 \}$', linewidth=2)
ax.legend()
ax.grid()
plt.show()
```



For $a^T = [1 - 1]$ and $X = \{x \in \mathbb{R}^2 | a^T x \leq 1\}$, the set is defined as:

$$a^T x \leq 1$$

$$[1 - 1]x \leq 1$$

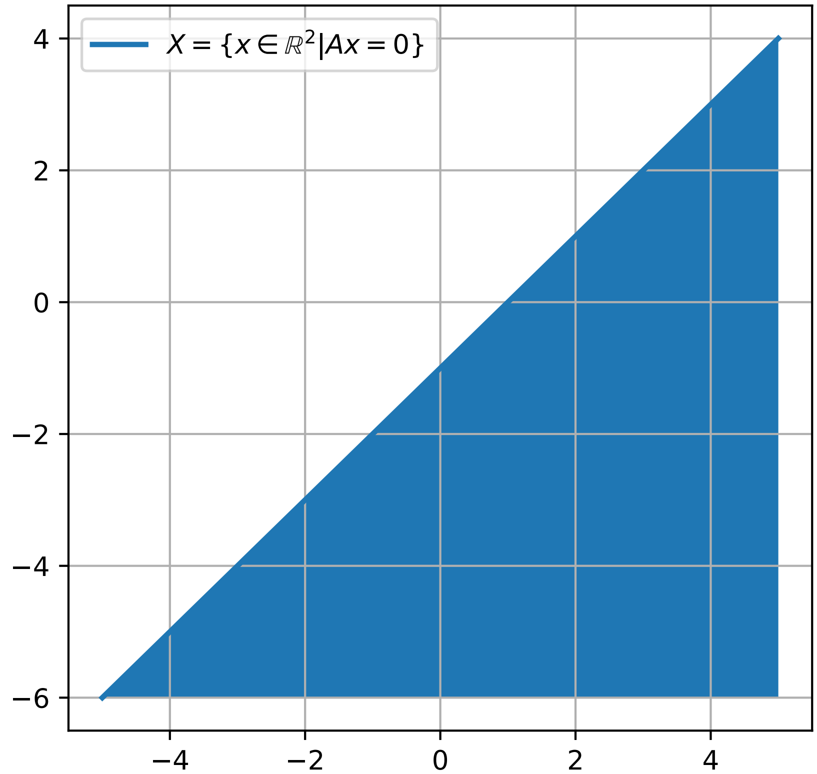
$$x_1 - x_2 \leq 1$$

$$x_2 \geq x_1 - 1$$

In [29]:

```
x = np.linspace(-5, 5, num=100)
y = x - 1
y2 = -6 + x*0

fig, ax = plt.subplots(figsize=(5, 5))
ax.plot(x, y, label='$X = \{ x \in \mathbb{R}^{2} | Ax = 0 \}$', linewidth=2)
ax.fill_between(x, y, y2)
ax.legend()
ax.grid()
plt.show()
```



(b)

For $a^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $X = \{x \in \mathbb{R}^2 | a^T x = 0\}$, the set is defined as:

$$a^T x = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x = 0$$

$$x_1 + x_2 + x_3 = 0$$

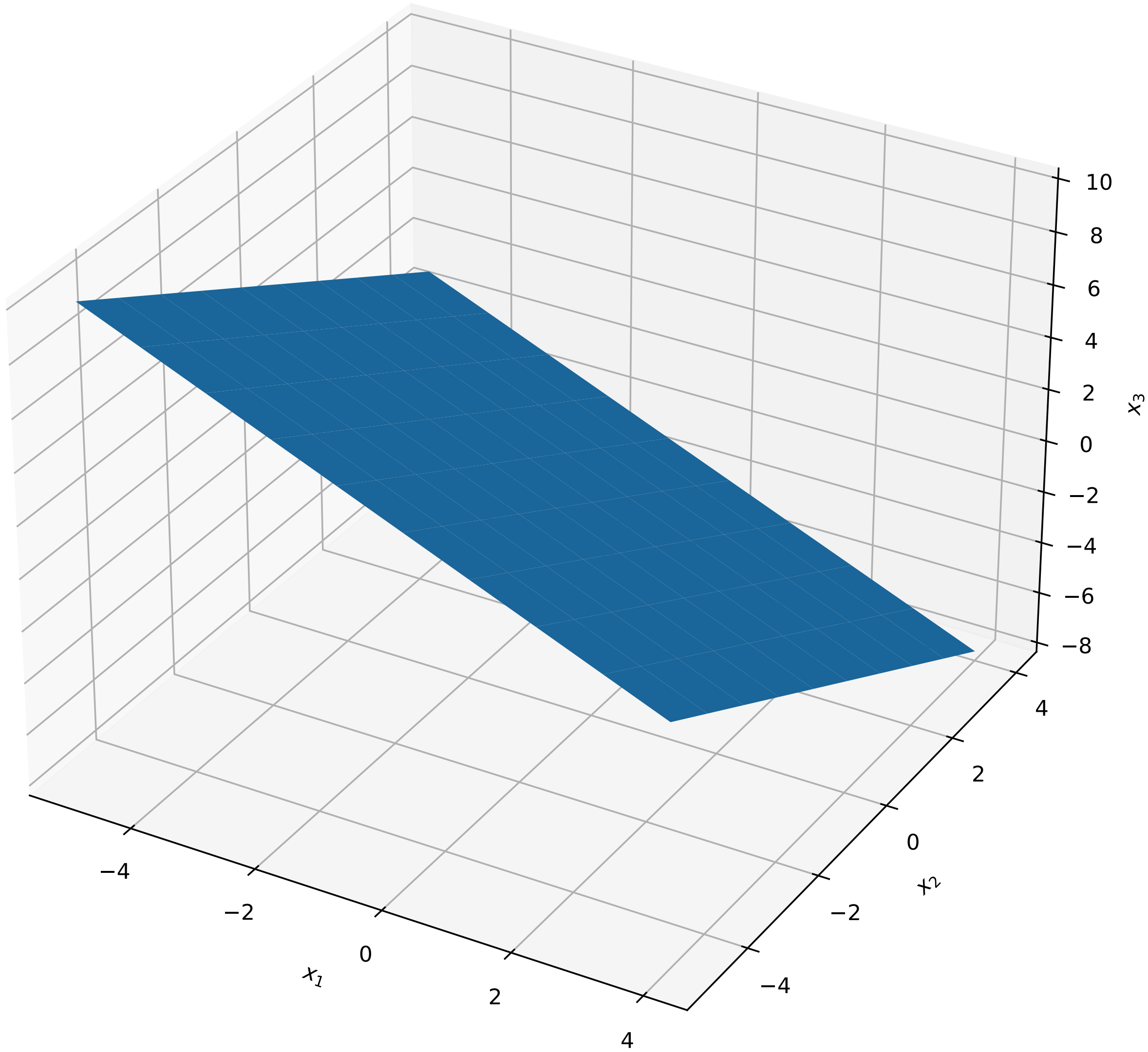
$$x_3 = -x_1 - x_2$$

In [41]:

```
fig = plt.figure(figsize=(10, 10))
ax = plt.axes(projection='3d')

# Make data.
x_1 = np.arange(-5, 5, 1)
x_2 = np.arange(-5, 5, 1)
x_2, x_1 = np.meshgrid(x_1, x_2)
x_3 = -x_1 - x_2

# Plot the surface.
surf = ax.plot_surface(x_1, x_2, x_3)
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$x_3$')
plt.show()
```



For $a^T = [1 \quad 1 \quad 1]$ and $X = \{x \in \mathbb{R}^2 | a^T x = 1\}$, the set is defined as:

$$a^T x = 0$$

$$[1 \quad 1 \quad 1]\, x = 1$$

$$x_1 + x_2 + x_3 = 1$$

$$x_3 = 1 - x_1 - x_2$$

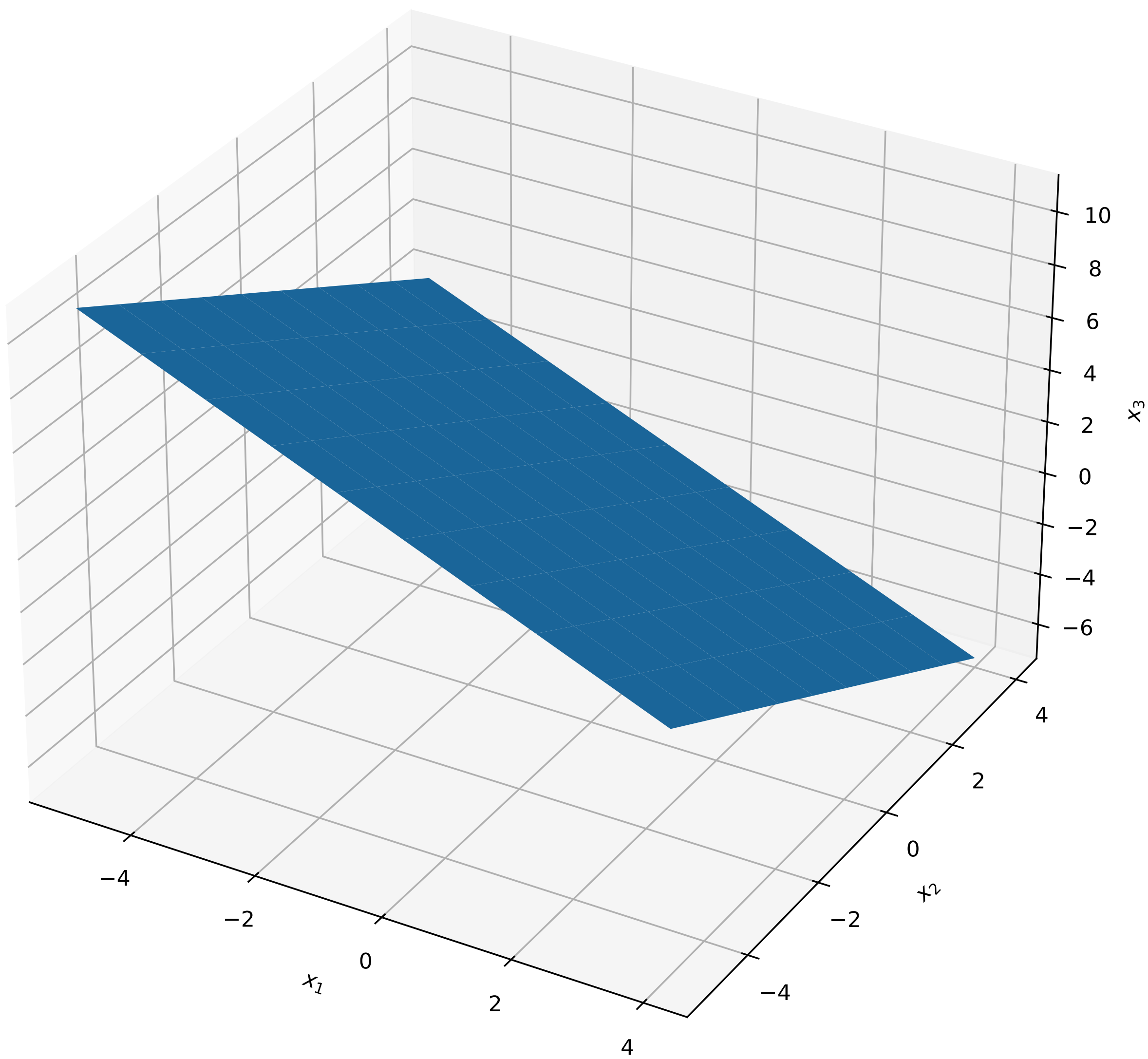
In [42]:

```
fig = plt.figure(figsize=(10, 10))
ax = plt.axes(projection='3d')

# Make data.
x_1 = np.arange(-5, 5, 1)
x_2 = np.arange(-5, 5, 1)
x_2, x_1 = np.meshgrid(x_1, x_2)
x_3 = 1 - x_1 - x_2

# Plot the surface.
```

```
surf = ax.plot_surface(x_1, x_2, x_3)
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$x_3$')
plt.show()
```



For $a^T = [1 \quad 1 \quad 1]$ and $X = \{x \in \mathbb{R}^2 | a^T x \leq 1\}$, the set is defined as:

$$a^T x \leq 0$$

$$[1 \quad 1 \quad 1]\, x \leq 1$$

$$x_1 + x_2 + x_3 \leq 1$$

$$x_3 \leq 1 - x_1 - x_2$$

```
In [ ]:
```

5. Coordinates

(a)

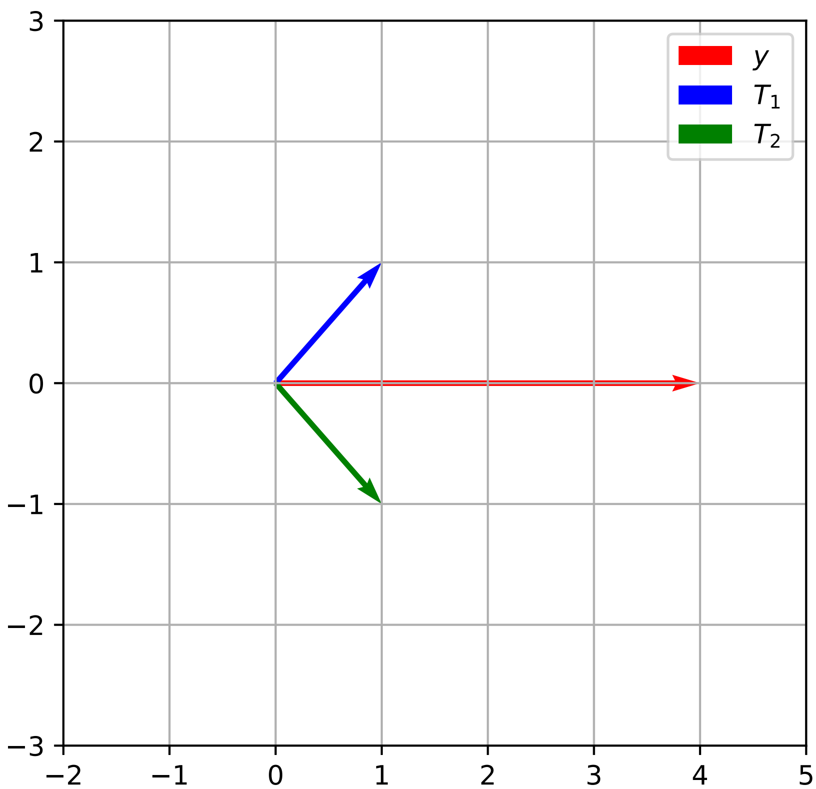
Given $y = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we can plot the columns of the matrix T and y to compute the coordinates of the vector y with respect to new basis.

In [11]:

```
y = np.array([[4], [0]])
T = np.array([[1, 1], [1, -1]])

origin = np.array([[0], [0]])

fig, ax = plt.subplots(figsize=(5, 5))
origin = np.array([[0, 0, 0], [0, 0, 0]])
ax.quiver([0], [0], y[0], y[1], angles='xy', color='r', scale_units='xy', scale=1, labe
ax.quiver([0], [0], T[0,0], T[1,0], angles='xy', color='b', scale_units='xy', scale=1,
ax.quiver([0], [0], T[0,1], T[1,1], angles='xy', color='g', scale_units='xy', scale=1,
ax.set_xlim([-2, 5])
ax.set_ylim([-3, 3])
ax.grid()
ax.legend()
plt.show()
```



By solving for x where $y = Tx$, we find that $x = T^{-1}y$. Solving for x, we find that $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

In [6]:

```
x = np.linalg.inv(T).dot(y)

print('Coordinates of y with respect to new basis:\n', x)
```

```
Coordinates of y with respect to new basis:
 [[2.]
 [2.]]
```

(b)

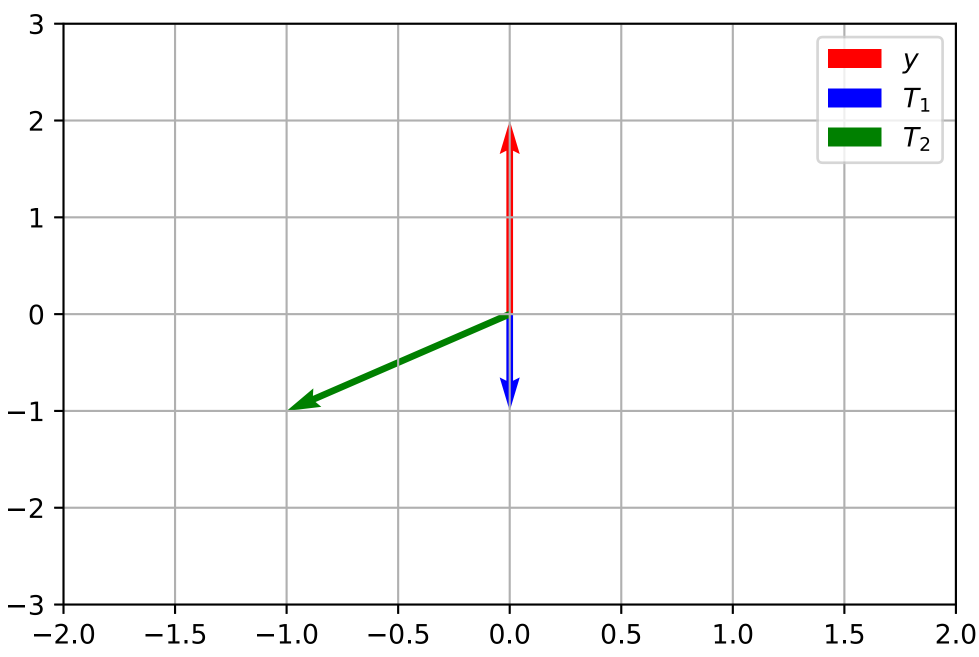
Given $y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$, we can plot the columns of the matrix T and y to compute the coordinates of the vector y with respect to new basis.

In [7]:

```
y = np.array([[0], [2]])
T = np.array([[0, -1], [-1, -1]])

origin = np.array([[0], [0]])

fig, ax = plt.subplots(figsize=(5, 5))
origin = np.array([[0, 0, 0], [0, 0, 0]])
ax.quiver([0], [0], y[0], y[1], angles='xy', color='r', scale_units='xy', scale=1, labe
ax.quiver([0], [0], T[0, 0], T[1, 0], angles='xy', color='b', scale_units='xy', scale=1
ax.quiver([0], [0], T[0,1], T[1,1], angles='xy', color='g', scale_units='xy', scale=1,
ax.set_xlim([-2, 2])
ax.set_ylim([-3, 3])
ax.grid()
ax.legend()
plt.show()
```



By solving for x where $y = Tx$, we find that $x = T^{-1}y$. Solving for x, we find that $x = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

In [8]:

```
x = np.linalg.inv(T).dot(y)

print('Coordinates of y with respect to new basis:\n', x)
```

```
Coordinates of y with respect to new basis:
 [[-2.]
 [ 0.]]
```

(c)

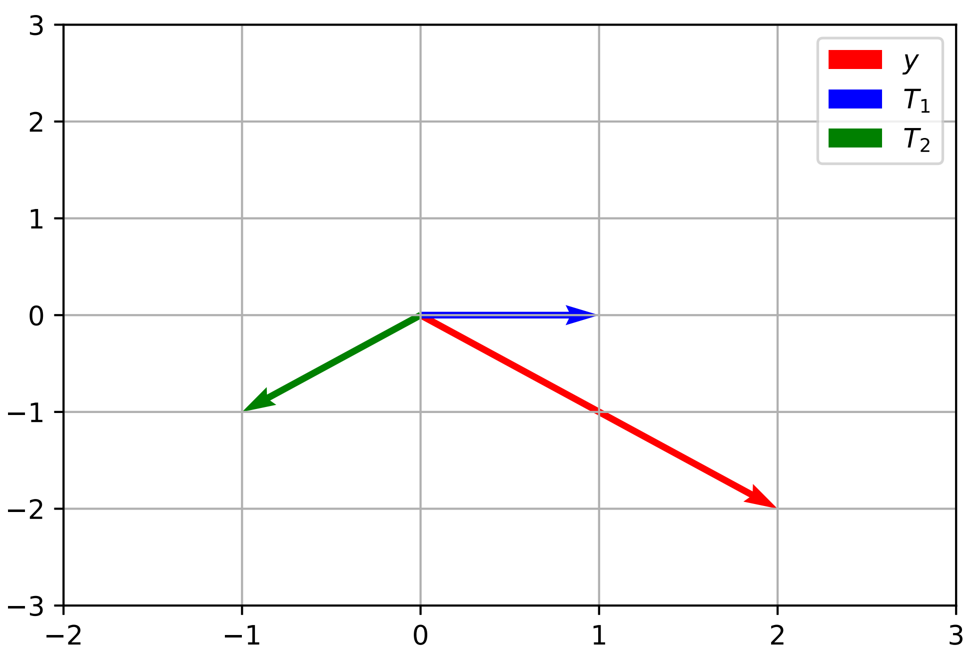
Given $y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, we can plot the columns of the matrix T and y to compute the coordinates of the vector y with respect to new basis.

In [9]:

```
y = np.array([[2], [-2]])
T = np.array([[1, -1], [0, -1]])

origin = np.array([[0], [0]])

fig, ax = plt.subplots(figsize=(5, 5))
origin = np.array([[0, 0, 0], [0, 0, 0]])
ax.quiver([0], [0], y[0], y[1], angles='xy', color='r', scale_units='xy', scale=1, labe
ax.quiver([0], [0], T[0, 0], T[1, 0], angles='xy', color='b', scale_units='xy', scale=1
ax.quiver([0], [0], T[0, 1], T[1, 1], angles='xy', color='g', scale_units='xy', scale=1
ax.set_xlim([-2, 3])
ax.set_ylim([-3, 3])
ax.grid()
ax.legend()
plt.show()
```



By solving for x where $y = Tx$, we find that $x = T^{-1}y$. Solving for x, we find that $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

In [10]:

```
x = np.linalg.inv(T).dot(y)

print('Coordinates of y with respect to new basis:\n', x)
```

```
Coordinates of y with respect to new basis:
 [[4.]
 [2.]]
```