Homework 4

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In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.patches as mpatches
import matplotlib.lines as mlines
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
```

In [3]:
```
import warnings
warnings.simplefilter('ignore')
```

1. Constraint Reformulations

Given $f(x) = \frac{1}{2}x^TQx + c^Tx$ and minimizing $f(x)$ such that $x \in \mathcal{X}$ where $\mathcal{X}$ is expressed in two forms.

(a)

Given our optimality conditions $\frac{\partial f}{\partial x} = v^TA$ and $Ax = b$.

To solve for v^* and x^*, we first start by multiplying our 1st optimality condition by the value of AQ^{-1} such that

$$AQ^{-1}(Qx + c) = AQ^{-1}(-A^Tv)$$

$$AQ^{-1}Qx + AQ^{-1}c = -AQ^{-1}A^Tv$$

$$Ax + AQ^{-1}c = -AQ^{-1}A^Tv$$

Substituting our second optimality condition $Ax = b$,

$$b + AQ^{-1}c = -AQ^{-1}A^Tv$$

$$v^* = (-AQ^{-1}A^T)^{-1}(b + AQ^{-1}c)$$

To solve for x^* in terms of b, we can substitute our relationship for v^* into our first optimality condition such that

$$Qx^* + c = -A^Tv^*$$

$$x^* = -A^T(-AQ^{-1}A^T)^{-1}(b + AQ^{-1}c) - c$$

(b)

Given our optimality conditions $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{bmatrix} = \tau^T \begin{bmatrix} I & -N \end{bmatrix}$ and $x = Nz + x^0$.

To solve for x^*, z^*, and τ, we can solve for the partial derivatives to generate two equations as follows

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{bmatrix} = \tau^T \begin{bmatrix} I & -N \end{bmatrix}$$

$$\begin{bmatrix} x^T Q + c^T & 0 \end{bmatrix} = \tau^T \begin{bmatrix} I & -N \end{bmatrix}$$

From this, we find two equations

$$x^T Q + c^T = \tau^T$$

$$0 = -\tau^T N$$

We can use the second equation to find our value for τ. With the second equation, we can see that τ must be orthogonal to the columns of N and we know that $AN = 0$ given that N forms a basis for $\mathcal{A}$. Thus, we determine that $\tau = A^T v$ for a given v.

We can solve for z^* from our first equation and the relationship that $x^T = {x^0}^T + z^T N^T$.

$$x^T Q + c^T = \tau^T$$

$$({x^0}^T + z^T N^T) Q + c^T = \tau^T$$

We can now right-hand multiply both sides of our equation by N such that

$$[({x^0}^T + z^T N^T) Q + c^T] N = \tau^T N$$

$${x^0}^T N + z^T N^T Q N + c^T N = \tau^T N$$

We know from our second equation that right-hand side of our equation is equal to zero ($0 = -\tau^T N$), thus we can now simplify the left-hand side and solve for z^*.

$${x^0}^T N + z^T N^T Q N + c^T N = 0$$

$$z^T = (-{x^0}^T Q N - c^T N)(N^T Q N)^{-1}$$

We know that x is a function of z given that $x = Nz + x^0$, thus we can see that

$$x = N[(-{x^0}^T Q N - c^T N)(N^T Q N)^{-1}]^T + x^0$$

$$x = -N(N^T Q N)^{-1}(N Q x^0 + N^T c) + x^0$$

(c)

If we turn our optimization problem into an unconstrained optimization problem using Form 2 of $\mathcal{X}$, we can solve for z. We are given that $f(x) = \frac{1}{2} x^T Q x + c^T x$ and $x = Nz + d$, thus we can substitue our relationship for x into our $f(x)$ such that,

$$f(z) = \frac{1}{2}(x^{0^T} + z^T N^T)Q(Nz + x^0) + c^T(Nz + x^0)$$

$$f(z) = \frac{1}{2}x^{0^T}Qx^0 + x^{0^T}QNz + \frac{1}{2}z^T N^T QNz + c^T Nz + c^T x^0$$

We can now take the partial derivative of $f(z)$ with respect to z and equate the resulting derivative to zero such that

$$\frac{\partial f}{\partial z} = z^T N^T QN + x^{0^T}QN + c^T N = 0$$

Solving for z given this relationship we find that

$$z^T = (-x^{0^T}QN - c^T N)(N^T QN)^{-1}$$

(d)

2. Inequality Constraint Reformulations

Given $Ax = b$ and $Cx \geq d$ where $A = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $b = 1$, $C = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, and $d = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(a)

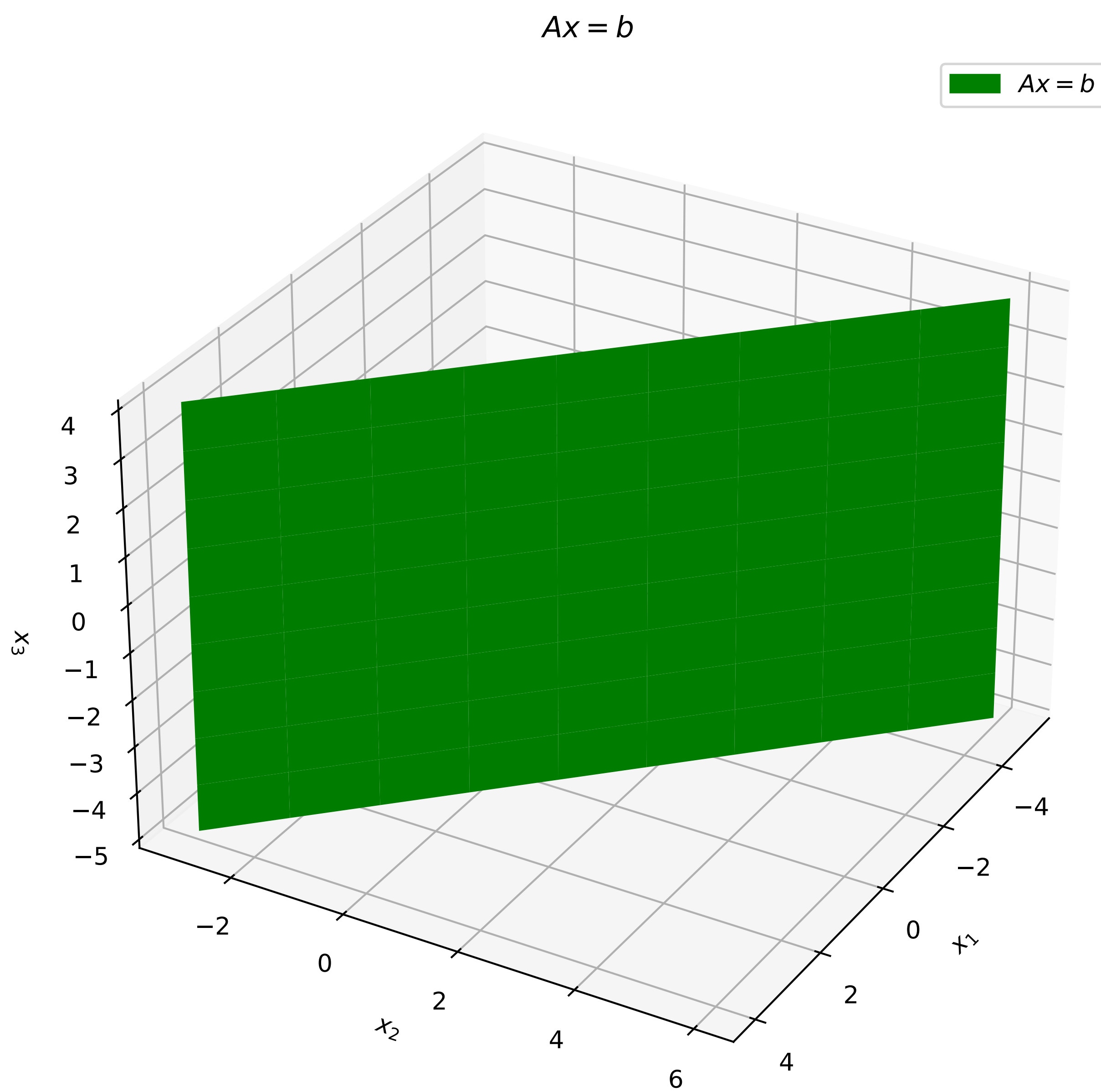
Plotting the set of $x's$ that satisfy the equations as defined above.

Two plots are given due to constraints of plotting multiple surfaces on a single plot. The first plot represents $Ax = b$; the second plot includes $Cx \geq d$ and the resulting space with combining both equations.

In [32]:

```
fig = plt.figure(figsize=(8, 8))
ax = plt.axes(projection='3d')

# Make data and plot Ax = b
x_1 = np.arange(-5, 5, 1)
x_3 = np.arange(-5, 5, 1)
x_1, x_3 = np.meshgrid(x_1, x_3)
x_2 = 1 - x_1
col1_patch = mpatches.Patch(color='green', label='$Ax=b$') # Info for legend

ax.plot_surface(x_1, x_2, x_3, color='green')

# Plot details
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
```

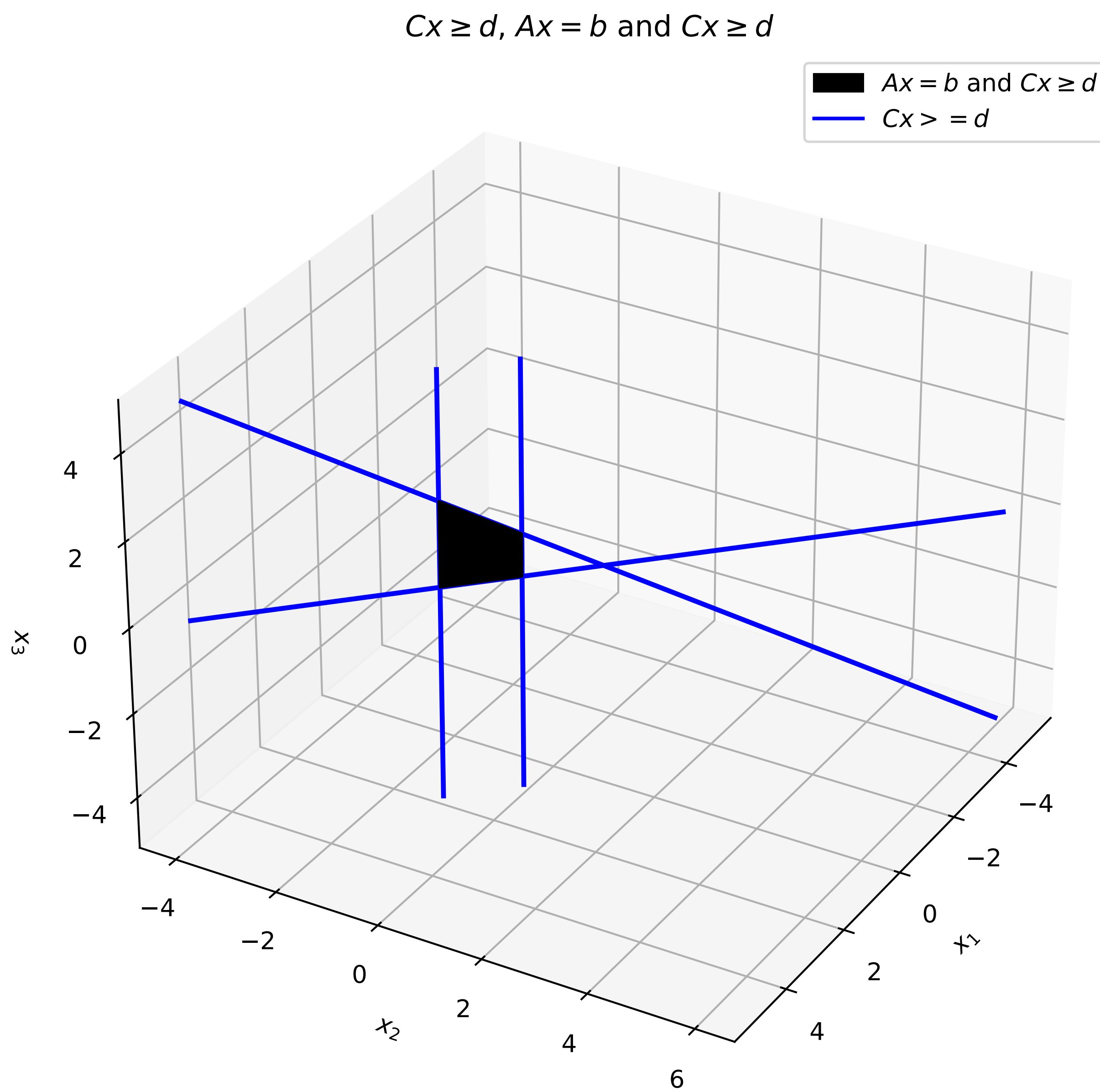
```python
ax.set_zlabel('$x_3$')
ax.legend(handles=[col1_patch])
ax.set_title('$Ax=b$')
ax.view_init(30, 30)
fig.show()
```



In [31]:

```python
fig = plt.figure(figsize=(8, 8))
ax = plt.axes(projection='3d')

# Make data and plot Cx >= d
blue_line = mlines.Line2D([], [], color='blue', markersize=15, label='$Cx>=d$') # Info .

ax.plot(xs=[2, 2], ys=[-1, -1], zs=[-5, 5], color='blue', linewidth='2')

ax.plot(xs=[1, 1], ys=[0, 0], zs=[-5, 5], color='blue', linewidth='2')

ax.plot(xs=[5, -5], ys=[-4, 6], zs=[0, 0], color='blue', linewidth='2')

ax.plot(xs=[5, -5], ys=[-4, 6], zs=[5, -5], color='blue', linewidth='2')
```

```python
# Plot combination of Ax=b and Cx >= d
x_1 = [1, 1, 2, 2]
x_2 = [0, 0, -1, -1]
x_3 = [0, 1, 2, 0]
verts = [list(zip(x_1,x_2,x_3))]

col1_patch = mpatches.Patch(color='black', label='$Ax=b$ and $Cx \geq d$') # Info for l

ax.add_collection3d(Poly3DCollection(verts, edgecolor='black', facecolor='black'))

# Plot details
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$x_3$')
ax.legend(handles=[col1_patch, blue_line])
ax.set_title('$Cx \geq d$, $Ax=b$ and $Cx \geq d$')
ax.view_init(30, 30)
fig.show()
```



(b)

Given A and b, we can represent this set in the form $x = Nz + x^0$ such that the columns of N form a basis for the nullspace of A.

We can define an N such that it spans the nullspace of A as

$$N = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can solve for x^0 by substituing $x = Nz + x^0$ into $Ax = b$ such that

$$A(Nz + x^0) = b$$

We know that the first term (ANz) is equal to zero, thus the equation is $Ax^0 = b$.

$$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x^0 = 1$$

$$x_1^0 + x_2^0 = 1$$

We select $x_3^0 = 0$ and $x_2^0 = 2$; thus we can solve for $x_1^0 = -1$.

Thus, we find an $N = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $x^0 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ for our given A and b.

Given this N and x^0, we can find our inequality constraints on z by substituing $x = Nz + x^0$ into $Cx \geq d$ such that

$$C(Nz + x^0) \geq d$$

$$CNz + Cx^0 \geq d$$

Solving this equation such that it relates to $Ez \geq h$ we find that

$$CNz \geq d - Cx^0$$

which implies that $E = CN$ and $d - Cx^0 = h$. Thus, we find that E is equal to

$$E = CN = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

And solving for h

$$h = d - Cx^0 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$h = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$h = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

If we chose a different N or x^0, E and h would be equivalent (or at least scaled factors of the same values we derived above).

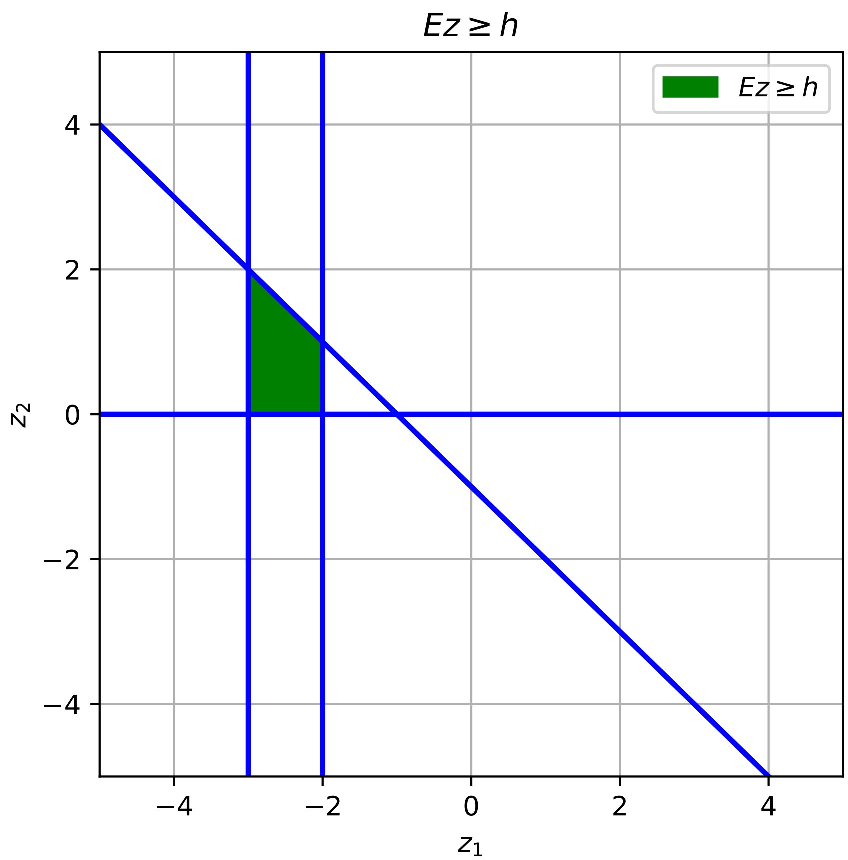
(c)

Given $E = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $h = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ as calculated in part (b), we can plot $z \in \mathbb{R}^2$ such that $Ez \geq h$.

In [30]:

```
# Plot the various constraints of Ez >= h
fig, ax = plt.subplots(figsize=(5, 5))

x = np.linspace(-5, 5, num=100)

# 1st row
ax.axvline(-3, linewidth=2, color='blue')

# 2nd row
ax.axvline(-2, linewidth=2, color='blue')

# 3rd row
y = 0 * x
ax.plot(x, y, linewidth=2, color='blue')

# 4th row
y = -1 - x
ax.plot(x, y, linewidth=2, color='blue')

# Plot the inequality as a patch
xy = [[-2, 0], [-3, 0], [-3, 2], [-2, 1]]
ax.add_patch(mpatches.Polygon(xy, color='green', label='$Ez \geq h$'))

ax.legend()

ax.set_xlim([-5, 5])
ax.set_ylim([-5, 5])
ax.set_xlabel('$z_1$')
```

```
ax.set_ylabel('$z_2$')
ax.set_title('$Ez \geq h$')
ax.legend()
ax.grid()
fig.show()
```



3. Inequality Constraints and Positive Matrices

4. Complementary Slackness

5. Lagrangians

(a)

Given $f(x) = \frac{1}{2}x^T x$ and $Ax = b$, we can write the Lagrangian for the optimization problem with a dual variable v as given below,

$$\mathcal{L}(x, v) = f(x) + v^T g(x)$$

We are given $f(x)$ and know that $g(x)$ can be derived from our $Ax = b$ relationship such that $g(x) = Ax - b = 0$. From this, we can now write our Lagrangian as

$$\mathcal{L}(x, v) = \frac{1}{2}x^T x + v^T(Ax - b)$$

(b)

Given $A = \begin{bmatrix} - & \bar{A}_1^T & - \\ - & \bar{A}_2^T & - \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$ and $b = \begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix}$, we can generate a plot for the following $h(x)$ in the $\begin{bmatrix} x_1 & x_2 & h(x) \end{bmatrix}^T$ space.

$$h_1(x) = f(x)$$

$$h_2(x) = v_1(\bar{A}_1^T x - b_1)$$

$$h_3(x) = v_2(\bar{A}_2^T x - b_2)$$

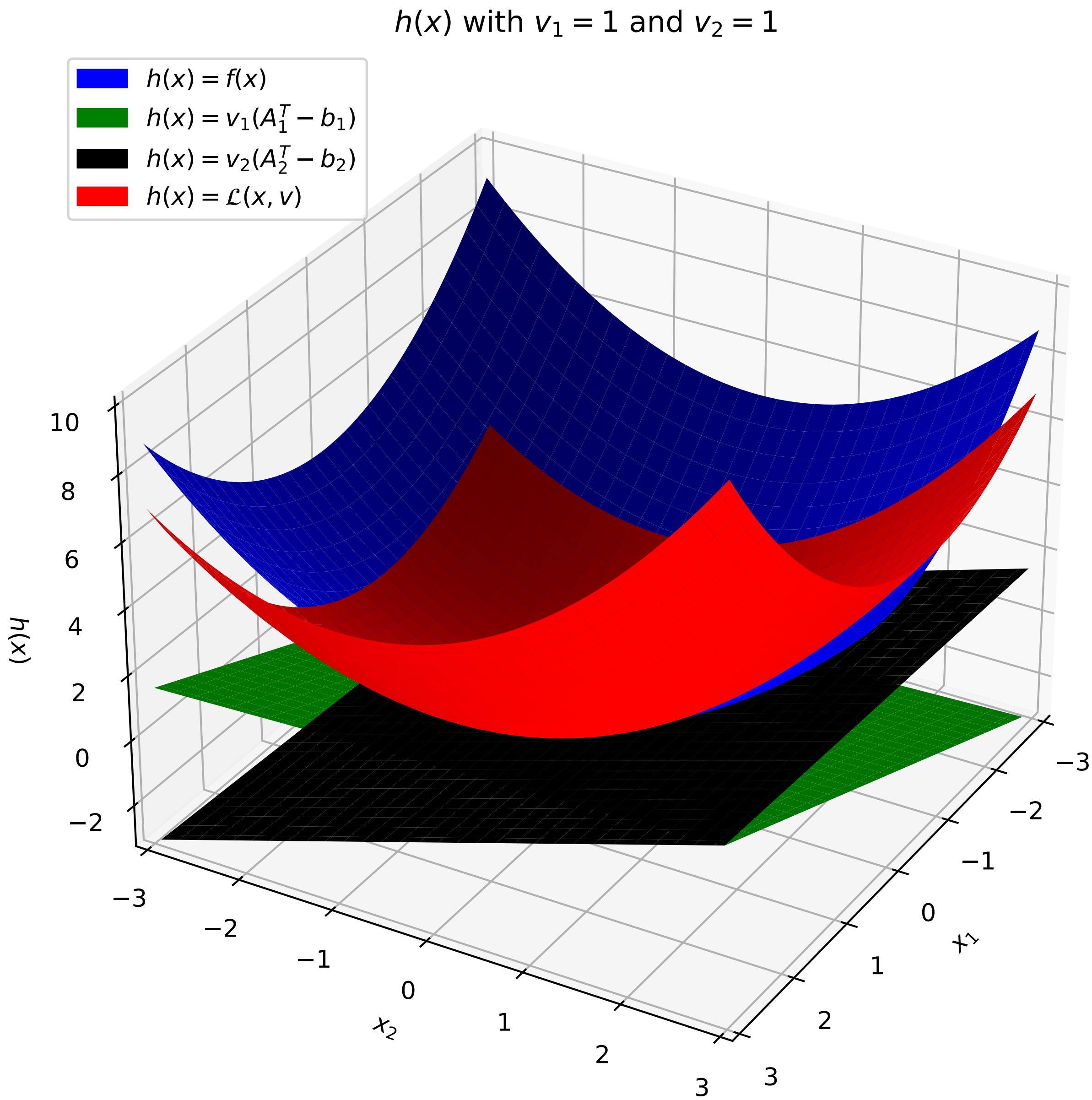
$$h_4(x) = \mathcal{L}(x, v)$$

Note: By inspection, we can see that $h_4(x) = h_1(x) + h_2(x) + h_3(x)$.

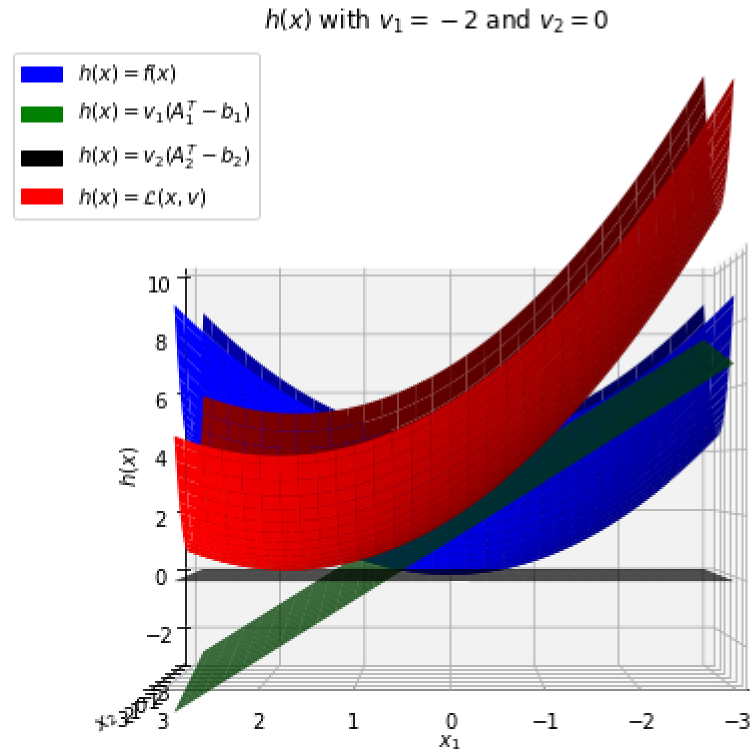
In [45]:

```
fig = plt.figure(figsize=(8, 8))

#3D
ax = fig.add_subplot(1, 1, 1, projection='3d')

# Plot h_1(x)
x_1 = np.arange(-3, 3, 0.1)
x_2 = np.arange(-3, 3, 0.1)
x_1, x_2 = np.meshgrid(x_1, x_2)
h_1 = 1/2 * (x_1**2 + x_2**2)

ax.plot_surface(x_1, x_2, h_1, color='blue')
h1_label = mpatches.Patch(color='blue', label='$h(x) = f(x)$') # Info for legend

# Plot h_2(x)
v_1 = 1 # Assume a fixed v_1 for now
h_2 = v_1 * (0.9*x_1 + 0.1*x_2 - 0.9)

ax.plot_surface(x_1, x_2, h_2, color='green')
h2_label = mpatches.Patch(color='green', label='$h(x) = v_1(A_1^T-b_1)$') # Info for le

# Plot h_3(x)
v_2 = 1 # Assume a fixed v_2 for now
h_3 = v_2 * (0.1*x_1 + 0.9*x_2 - 0.9)

ax.plot_surface(x_1, x_2, h_3, color='black')
h3_label = mpatches.Patch(color='black', label='$h(x) = v_2(A_2^T-b_2)$') # Info for le

# Plot h_4(x)
h_4 = h_1 + h_2 + h_3

ax.plot_surface(x_1, x_2, h_4, color='red')
h4_label = mpatches.Patch(color='red', label='$h(x) = \mathcal{L}(x,v)$') # Info for le

ax.set_xlim3d([-3, 3])
ax.set_ylim3d([-3, 3])
ax.set_zlim3d([-3, 10])
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$h(x)$')
ax.set_title('$h(x)$ with $v_1 = 1$ and $v_2 = 1$')
ax.legend(handles=[h1_label, h2_label, h3_label, h4_label], loc='upper left')
ax.view_init(30, 30)
```

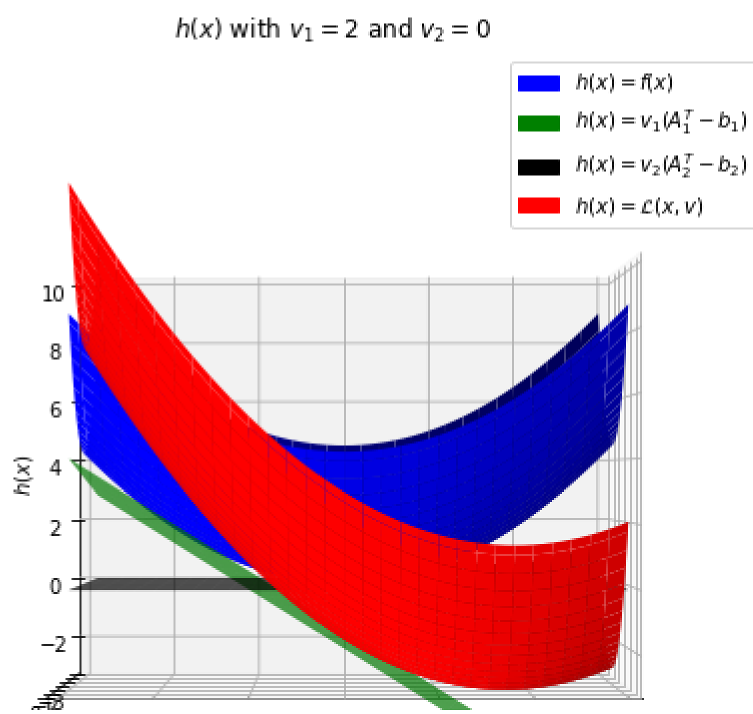
```
#ax.view_init(0, 90) # view of h(x), x_1 (x_2 into page) --> good for varying v_1
#ax.view_init(0, 0) # view of h(x), x_2 (x_1 into page) --> good for varying v_2
fig.show()

# Save figure (if desired)
#plt.savefig('v1_neg2.png')
```



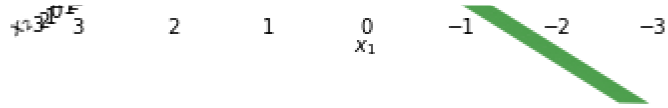
We can see the effect of varying v_1 and v_2 in the plots as shown below. When we oscillate v_1 between $[-2, 2]$ causes "see-sawing" about the x_2 axis (as shown by the 1st and 2nd image in which we see the rotation of the green plane about the x_2 axis). When we oscillate v_2 between $[-2, 2]$ causes "see-sawing" about the x_1 axis (as shown by the 3rd and 4th image in which we see the rotation of the black plane about the x_1 axis).



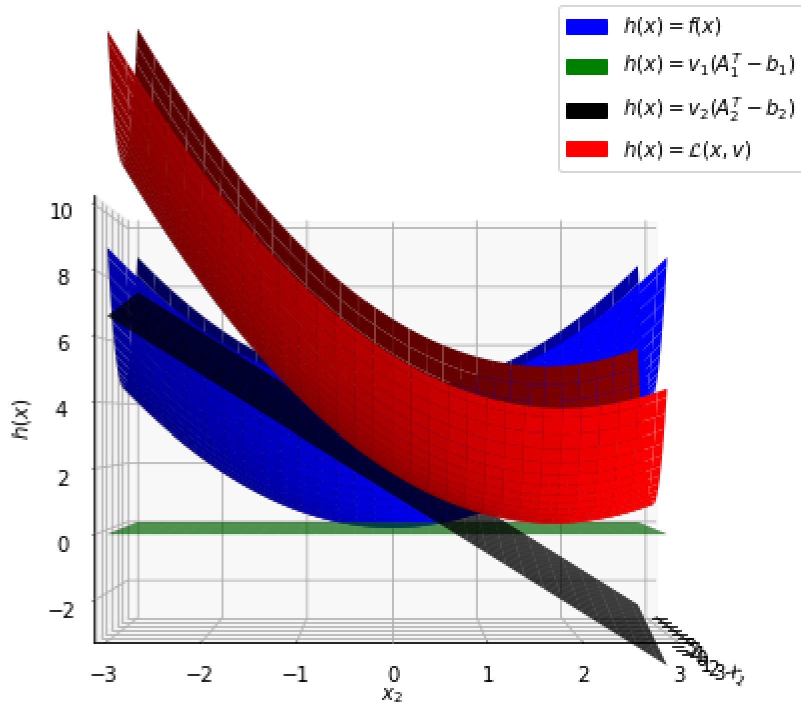
$h(x)$ with $v_1 = -2$ and $v_2 = 0$
$h(x) = f(x)$
$h(x) = v_1(A_1^T - b_1)$
$h(x) = v_2(A_2^T - b_2)$
$h(x) = \mathcal{L}(x, v)$
10
8
6
4
2
0
−2
$h(x)$
3
2
1
0
−1
−2
−3
x_1



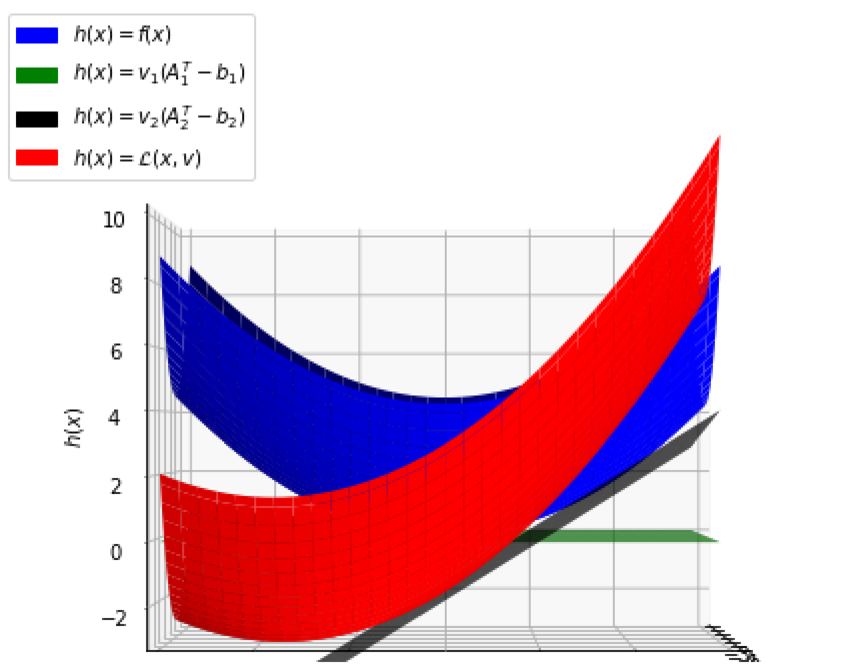
$h(x)$ with $v_1 = 2$ and $v_2 = 0$
$h(x) = f(x)$
$h(x) = v_1(A_1^T - b_1)$
$h(x) = v_2(A_2^T - b_2)$
$h(x) = \mathcal{L}(x, v)$
10
8
6
4
2
0
−2
$h(x)$


3
2
1
0
−1
−2
−3
x_1

$h(x)$ with $v_1 = 0$ and $v_2 = -2$



$h(x)$ with $v_1 = 0$ and $v_2 = 2$



(c)

When we replace $Ax = b$ with $Ax \geq b$, the Lagrangian does change to the function below,

$$\mathcal{L}(x, v) = f(x) - v^T g(x)$$

$$\mathcal{L}(x, v) = \frac{1}{2} x^T x - v^T (Ax - b)$$

With the new form of Lagrangian, we also must impose the the following constraints on v_1 and v_2: $v_1 \geq 0$ and $v_2 \geq 0$.

```
In [ ]:
```