Week 05: FOURIER SERIES

Objectives:

- Eigenfunctions
- Fourier Series of CT Signals
- Trigonometric Fourier Series
- Dirichlet Conditions
- Gibbs Phenomena

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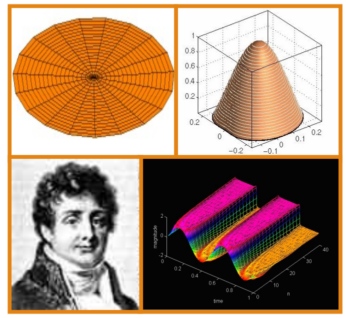

Representation of CT Signals (Again!)

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- What is an example of a function when applied to an LTI system produces an output that is a scaled version of itself?

$$\phi_k(t) \longrightarrow \boxed{h(t)} \longrightarrow \lambda_k \phi_k(t)$$

- The scale factor, λ_k, is referred to as an eigenvalue. The function, ϕ_k, is referred to as an eigenfunction.
- Using the superposition property of LTI systems:

$$x(t) = \sum_k a_k \phi_k(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_k \lambda_k a_k \phi_k(t)$$

- This reduces the problem of finding the response of any LTI system to any signal to the problem of finding the $\{\lambda_k\}$.
- What types of things influence the values $\{\lambda_k\}$? We will soon see that the frequency response of this LTI system is one thing that will influence the shape of the output.
- We will later generalize this concept of eigenvalues and eigenfunctions to many types of engineering systems and analyses.
- The Fourier series is one of many ways to decompose a signal.

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Eigenfunctions and LTI Systems

Ex. #1: Identity system

$$x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t) * \delta(t) = x(t)$$

Any function is an eigenfunction for this LTI system.

Ex. #2: A delay

$$x(t) \longrightarrow \boxed{\delta(t-T)} \longrightarrow x(t-T)$$

Any periodic function $x(t) = x(t+T)$ is an eigenfunction.

Ex. #3: $h(t)$ even

$$\cos \omega t \longrightarrow \boxed{h(t) = h(-t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cos[\omega(t-\tau)]d\tau = \int_{-\infty}^{\infty} h(\tau)[\cos\omega t \cdot \cos\omega\tau + \sin\omega t \cdot \sin\omega\tau)]d\tau$$

$$= \cos\omega t \underbrace{\int_{-\infty}^{\infty} h(\tau)\cos\omega\tau d\tau}_{H(j\omega)} \quad \text{— } \cos\omega t \text{ is an eigenfunction}$$

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Response of a CT LTI System

- **Complex exponentials are always eigenfunctions for an LTI system and extremely useful because the output is a scaled version of the input:**

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st}$$

$$= \underbrace{H(s)}_{\text{eigenvalue}}\underbrace{e^{st}}_{\text{eigenfunction}}$$

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} h[m]z^{n-m}$$

$$= \left[\sum_{m=-\infty}^{\infty} h[m]z^{-m}\right]z^n$$

$$= \underbrace{H(z)}_{\text{eigenvalue}}\underbrace{z^n}_{\text{eigenfunction}}$$

(Complex) Fourier Series Representations

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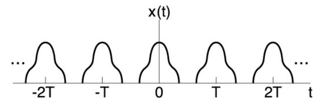
- What types of signals can be represented as sums of complex exponentials?
  - CT: $s = j\omega$ (Fourier Transform) ➢ signals of the form $e^{j\omega t}$
  - DT: $z = e^{j\omega}$ (z-Transform) ➢ signals of the form $e^{j\omega n}$
- These representations form the basis of the Fourier Series and Transform.
- Consider a periodic signal:

$$x(t) = x(t+T) \quad \textit{for all } T$$

smallest such T is the fundamental period

$$\omega_0 = \frac{2\pi}{T} \textit{ is the fundamental frequency (radians)}$$



- Consider representing a signal as a sum of these exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t} = x(t+T)$$

Notes:

- Periodic with period T
- $\{c_k\}$ are the (complex) Fourier series coefficients
- $k = 0$ corresponds to the DC value; $k = 1$ is the first harmonic; …

Alternate Fourier Series Representations

- For real, periodic signals:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \qquad \text{"trigonometric Fourier series"}$$

or,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \gamma_k \cos(k\omega_0 t + \phi_k) \qquad \text{"cosine wth phase"}$$

or,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{"complex"}$$

- These are essentially interchangeable representations. For example, note:

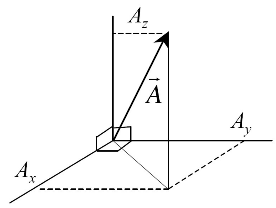
$$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

- The complex Fourier series is used for most engineering analyses.
- How do we compute the Fourier series coefficients?
  - There are several ways to arrive at the equations for estimating the coefficients. Most are based on concepts of orthogonal functions and vector space projections.

Vector Space Projections

- How can we compute the Fourier series coefficients?
  - One approach is to use the concept of a vector projection:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \quad where \quad \hat{x}, \hat{y}, \hat{z} \text{ are unit vectors}$$



- How do we find the components:

$$A_x = \vec{A} \bullet \hat{x},\ A_y = \vec{A} \bullet \hat{y},\ A_z = \vec{A} \bullet \hat{z}$$

We project the vector onto the corresponding axis using a dot product.

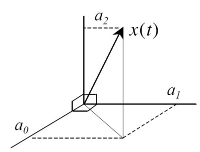
- Note that this works because the three axes are orthogonal:

$$\hat{x} \bullet \hat{y} = \hat{y} \bullet \hat{z} = \hat{x} \bullet \hat{z} = 0$$

- We can apply the same concept to signals using the notion of a Hilbert space in which each axis represents an eigenfunction (e.g., cos() and sin()):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$x(t) \leftrightarrow \vec{A}$$
$$e^{jk\omega_0 t} \leftrightarrow \hat{x}, \hat{y}, \hat{z}$$
$$c_k \leftrightarrow A_x, A_y, A_z$$



Computation of the Coefficients

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- We can make use of the principle of orthogonality in the Hilbert space:

$$\frac{1}{T}\int_T e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} = \delta(k-n)$$

(This can be thought of as an "inner product.")

- We can apply this to our Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

- Using the inner product:

$$\frac{1}{T}\int_T x(t) \cdot e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt = \frac{1}{T}\sum_{k=-\infty}^{\infty} c_k \int_T e^{j(k-n)\omega_0 t}\,dt = c_n$$

- This gives us our Fourier series pair ($\omega_0 = 2\pi/T$):

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{(synthesis)}$$

$$c_k = \frac{1}{T}\int_T x(t) \cdot e^{-jk\omega_0 t}\,dt \qquad \text{(analysis)}$$

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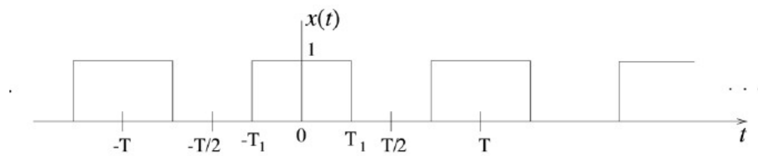
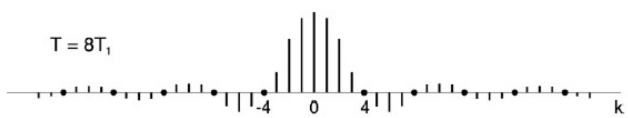
Example: Periodic Pulse Train



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1}}{-jk\omega_0} - \frac{e^{-jk\omega_0 (-T_1)}}{-jk\omega_0}\right]$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2}{k(2\pi/T)T}\left[\sin(k\omega_0 T_1)\right]$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi}$$



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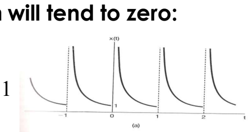
➢ Convergence of the Fourier Series

- **How can a series composed of continuous functions (e.g., sines and cosines) approximate a discontinuous function, such as a square wave?**
- **Conditions for which the error in this approximation will tend to zero:**

  1) X(t) is absolutely integrable over one period:

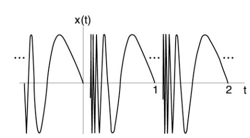
  $$\int_T |x(t)|\,dt < \infty$$

  $$x(t) = \frac{1}{t} \quad 0 < t \le 1$$



  2) In a finite time interval, $x(t)$ **has a finite number of maxima and minima.**

  $$x(t) = \sin(2\pi/t) \quad 0 < t \le 1$$



  3) In a finite time interval, $x(t)$ **has a finite number of discontinuities.**

- **These are known as the Dirichlet conditions. They will be satisfied for most signals we encounter in the real world. This implies:**

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2}\left[x(t \to 0^+) + x(t \to 0^-)\right]$$

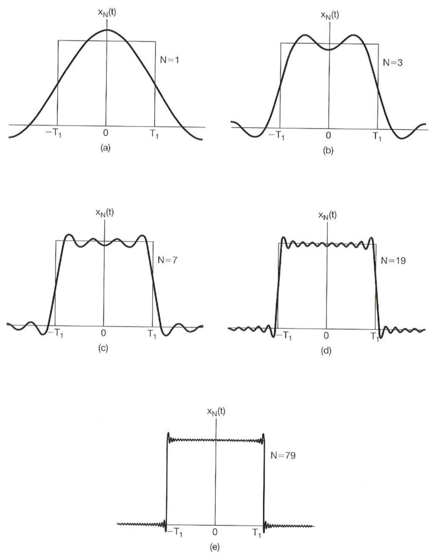
➢ Gibbs Phenomena

- Convergence in error can have some interesting characteristics:

$$x(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}$$

- This is known as Gibbs phenomena and was first observed by Albert Michelson in 1898.
- The Fourier series of a square wave is plotted as a function of N, the number of terms in the finite series.
- The limit as $N \to \infty$ is the average value of x(t) at the discontinuity.
- The squared error does converge:

$$\lim_{N \to \infty} \int_T \left(x(t) - \sum_{k=-N}^{N} c_k e^{jk\omega_0 t} \right)^2 dt = 0$$



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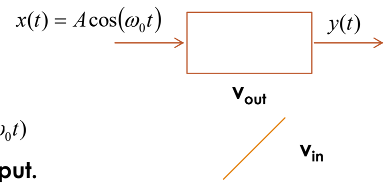
Summary

- Introduced the concept of eigenvalues and eigenfunctions.
- Developed the concept of a Fourier series.
- Discussed three representations of the Fourier series.
- Derived an expression for the estimation of the coefficients.
- Derived the Fourier series of a periodic pulse train.
- Introduced the Dirichlet conditions and Gibbs phenomena.
- Discussed convergence properties.

Appendix: Nonlinear Amplifier

Assume we apply a sinewave input to a voltage amplifier under two conditions:



- **Linear:** $v_{out} = a \cdot v_{in}$

  In this case, the output is: $v_{out} = aA\cos(\omega_0 t)$

  The output is a scaled version of the input.

- **Nonlinear:** $v_{out} = a \cdot (v_{in})^2$

  $$v_{out} = a \cdot (v_{in})^2 = a \cdot (A\cos(\omega_0 t))^2 = (aA)^2 \cos^2(\omega_0 t) = \frac{(aA)^2}{2}(1 + \cos(2\omega_0 t))$$

  Hence, the output signal is at a frequency twice that of the input, which clearly makes this a nonlinear system.

In practice, amplifiers, such as audio power amplifiers, are characterized using a power series:

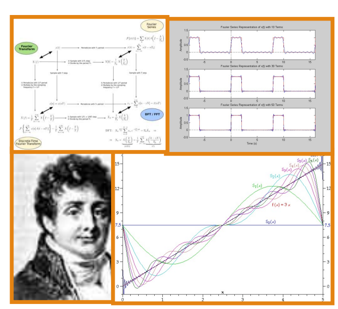
$$v_{out} = a_0 + a_1 v_{in} + a_2 v_{in}^2 + a_3 v_{in}^3 + \ldots$$

A single sinewave input generates many frequencies, all harmonically related to the input frequency. This distortion is characterized by figures of merit such as total harmonic distortion and intermodulation. See audio system measurements for more information.

THE TRIGONOMETRIC FOURIER SERIES

Objectives:

- The Trigonometric Fourier Series
- Pulse Train Example
- Symmetry (Even and Odd Functions)
- Line Spectra
- Power Spectra
- More Properties
- More Examples



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➢ The Trigonometric Fourier Series Representations

- **For real, periodic signals:**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \qquad \text{"trigonometric Fourier series"}$$

$or,$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \text{"complex"}$$

- **The analysis equations for a_k and b_k are:**

$$a_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt, \quad k = 1, 2, \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt, \quad k = 1, 2, \ldots$$

- **Note that a_0 represents the average, or DC, value of the signal.**
- **We can convert the trigonometric form to the complex form:**

$$c_0 = a_0 \quad c_k = \frac{1}{2}(a_k - jb_k) \quad c_{-k} = \frac{1}{2}(a_k + jb_k) \quad k = 1, 2, \ldots$$

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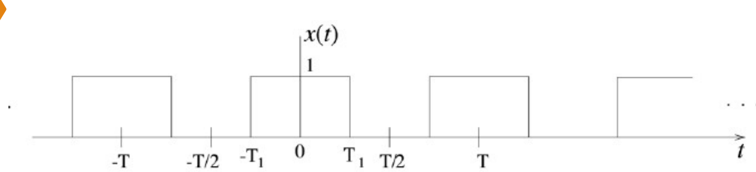
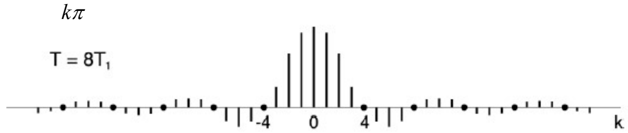
Example: Periodic Pulse Train (Complex Fourier Series)



$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1}}{-jk\omega_0} - \frac{e^{-jk\omega_0(-T_1)}}{-jk\omega_0}\right]$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2}{k(2\pi/T)T}\left[\sin(k\omega_0 T_1)\right]$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi}$$



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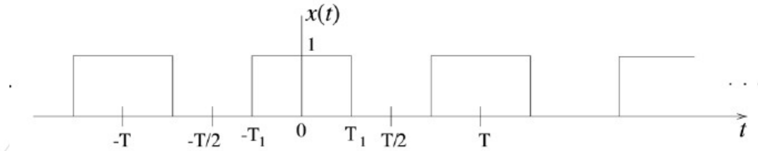
Example: Periodic Pulse Train (Tria Fourier Series)



$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1}(1)dt = \frac{1}{T}\left[t\right]_{-T_1}^{T_1} = \frac{1}{T}(T_1-(-T_1)) = \frac{2T_1}{T}$$

- **This is not surprising because a_0 is the average value ($2T_1/T$).**
- **Also,**

$$a_0 = c_0$$

$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \Rightarrow c_0 = \left.\frac{\sin(k\omega_0 T_1)}{k\pi}\right|_{k=0}$$

Use L'Hopital's Rule :

$$= \left.\frac{\partial/\partial k(\sin(k\omega_0 T_1))}{\partial/\partial k(k\pi)}\right|_{k=0} = \left.\frac{\omega_0 T_1\cos(k\omega_0 T_1)}{\pi}\right|_{k=0} = \frac{\omega_0 T_1}{\pi} = \frac{(2\pi/T)T_1}{\pi} = \frac{2T_1}{T}$$

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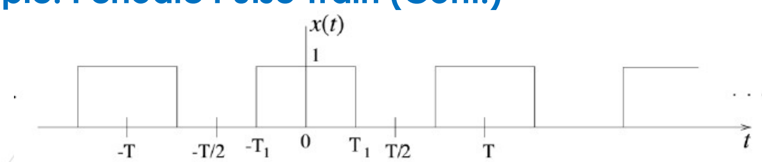
Example: Periodic Pulse Train (Cont.)



$$\begin{aligned} a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt \\ &= \frac{2}{T}\int_{-T_1}^{T_1} x(t)\cos(k\omega_0 t)dt = \frac{2}{T}\left[\frac{\sin(k\omega_0 t)}{k\omega_0}\right]_{-T_1}^{T_1} \\ &= \frac{2}{T}\left[\frac{\sin(k\omega_0 T_1)}{k\omega_0} - \frac{\sin(k\omega_0(-T_1))}{k\omega_0}\right] \\ &= \frac{4\sin(k\omega_0 T_1)}{k\omega_0 T} \end{aligned}$$

$$\begin{aligned} b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt \\ &= \frac{2}{T}\int_{-T_1}^{T_1} x(t)\sin(k\omega_0 t)dt = \frac{2}{T}\left[\frac{-\cos(k\omega_0 t)}{k\omega_0}\right]_{-T_1}^{T_1} \\ &= 0 \end{aligned}$$

- **Check this with our result for the complex Fourier series ($k > 0$):**

$$c_k = \frac{1}{2}(a_k - jb_k) = \frac{1}{2}\left(\frac{4\sin(k\omega_0 T_1)}{k\omega_0 T} - 0\right) = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Even and Odd Functions

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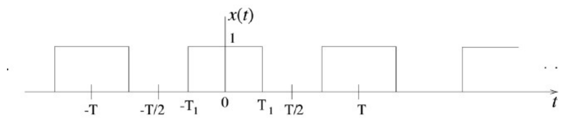

$$a_k = \frac{4\sin(k\omega_0 T_1)}{k\omega_0 T}, \quad b_k = 0$$

- **Was this result surprising? Note:** x(*t*) **is an even function:** x(*t*) = x(-*t*)

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt = 2\left[\frac{2}{T}\int_{0}^{T/2} x(t)\cos(k\omega_0 t)dt\right]$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt = 0$$

- **If** x(*t*) **is an odd function:** x(*t*) = –x(-*t*)

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt = 0$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt = 2[\frac{2}{T}\int_{0}^{T/2} x(t)\sin(k\omega_0 t)dt]$$

➢ Line Spectra

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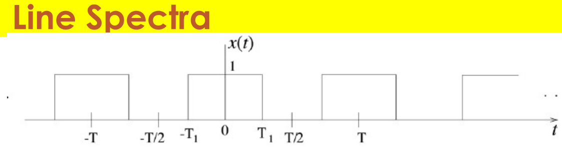
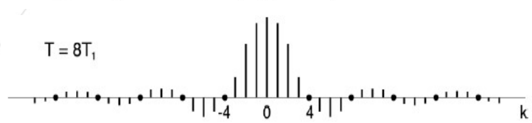


$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$$



$$k \Rightarrow k\omega_0 = k(\frac{2\pi}{T}) = k(2\pi)f_0$$

- **Recall:** $c_0 = a_0 \quad c_k = \frac{1}{2}(a_k - jb_k) \quad c_{-k} = \frac{1}{2}(a_k + jb_k) \quad k = 1, 2, ...$
- **From this we can show:**

$$|c_k| = \left|\frac{1}{2}(a_k - jb_k)\right| = \frac{1}{2}\sqrt{a_k^2 + b_k^2} = |c_{-k}|$$

$$\angle c_k = \left\{\begin{array}{ll} \tan^{-1}\left(-\frac{b_k}{a_k}\right) & a_k \geq 0 \\ \pi + \tan^{-1}\left(-\frac{b_k}{a_k}\right) & a_k < 0 \end{array}\right\} \Rightarrow \angle c_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right) = -\angle c_{-k}$$

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➢Energy and Power Spectra

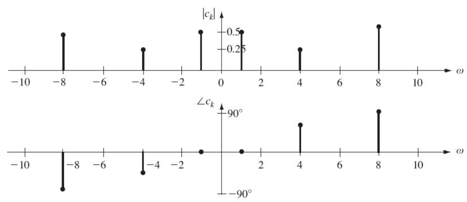
- The energy of a CT signal is: $E = \int_{-\infty}^{\infty} x^2(t)dt$
- The power of a signal is defined as: $P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt$

  Think of this as the power of a voltage across a 1-ohm resistor.
- Recall our expression for the signal: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$
- We can derive an expression for the power in terms of the Fourier series coefficients:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)^2 dt = \sum_{k=-\infty}^{\infty}|c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=0}^{\infty} a_k^2 + b_k^2$$

- Hence we can also think of the line spectrum as a power spectral density:

$$|c_k|^2 = \left|\frac{\sin(k\omega_0 T_1)}{k\pi}\right|^2$$



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➢ Properties of the Fourier Series

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$, $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

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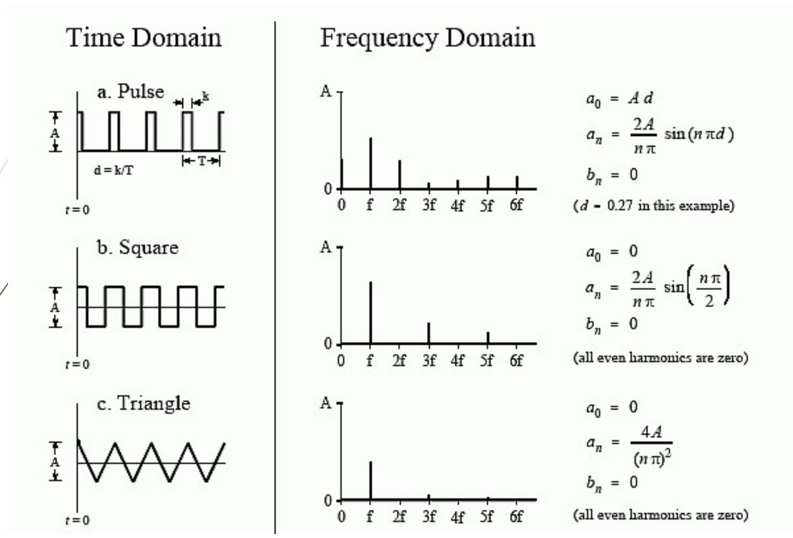
Properties of the Fourier Series



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Properties of the Fourier Series

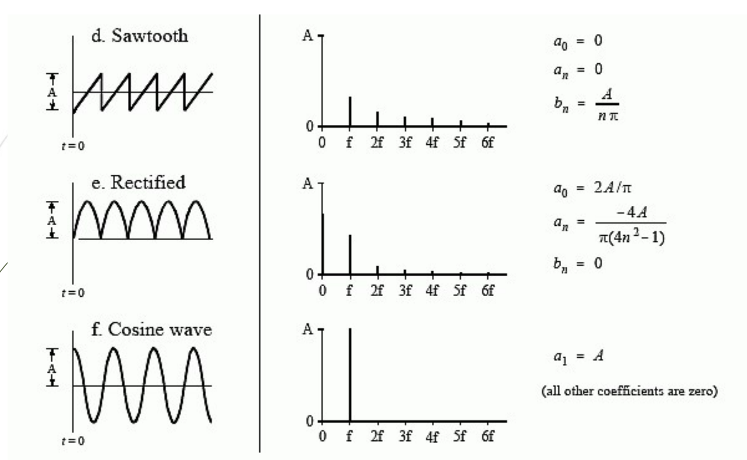

FIGURE 13-10
Examples of the Fourier series. Six common time domain waveforms are shown, along with the equations to calculate their "a" and "b" coefficients.

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Summary

- Reviewed the Trigonometric Fourier Series.
- Worked an example for a periodic pulse train.
- Analyzed the impact of symmetry on the Fourier series.
- Introduced the concept of a line spectrum.
- Discussed the relationship of the Fourier series coefficients to power.
- Introduced our first table of transform properties.
- Next: what do we do about non-periodic signals?

Continuouse-Time System Relationships

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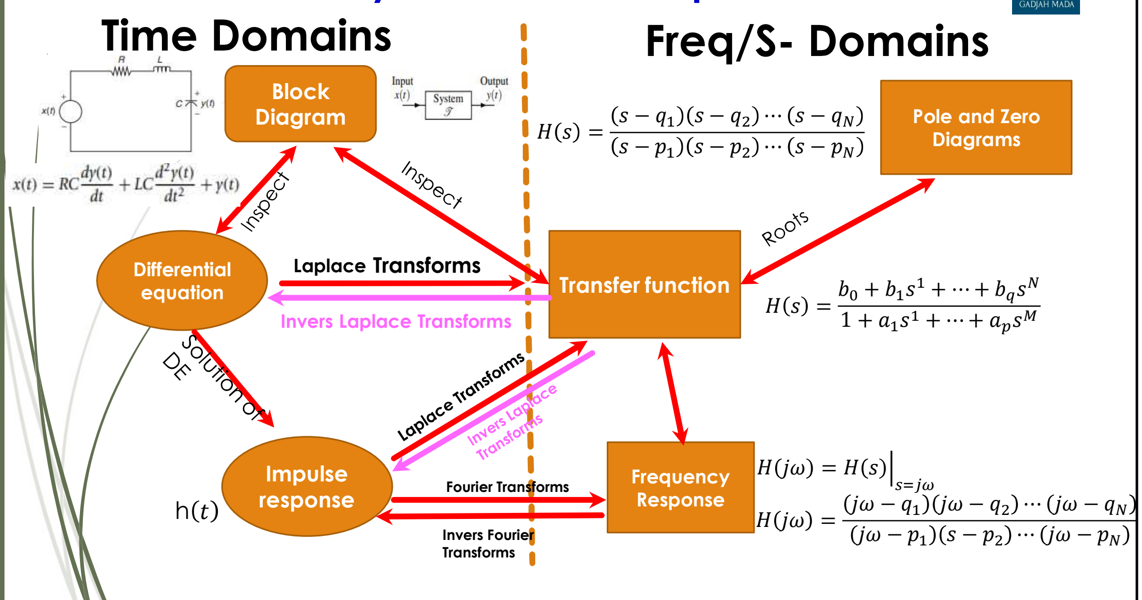

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