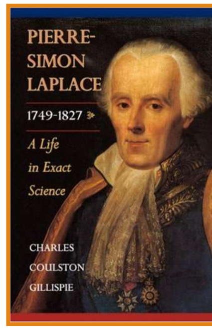
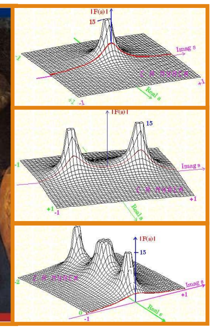
Mg.03: THE LAPLACE TRANSFORM

Materials:

- Motivation
- The Bilateral Transform
- Region of Convergence (ROC)
- Properties of the ROC
- Rational Transforms
- Properties Of The Laplace Transform





Continuouse-Time System Relationships

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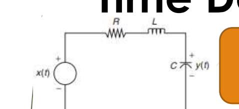
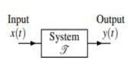
Time Domains

Freq/S- Domains



$$x(t) = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t)$$

Block Diagram



$$H(s) = \frac{(s-q_1)(s-q_2)\cdots(s-q_N)}{(s-p_1)(s-p_2)\cdots(s-p_N)}$$

Pole and Zero Diagrams

Inspect

Inspect

Roots

Differential equation

Laplace Transforms

Invers Laplace Transforms

Transfer function

$$H(s) = \frac{b_0 + b_1 s^1 + \cdots + b_q s^N}{1 + a_1 s^1 + \cdots + a_p s^M}$$

Solution of DE

Laplace Transforms

Invers Laplace Transforms

Impulse response

h(*t*)

Fourier Transforms

Invers Fourier Transforms

Frequency Response

$$H(j\omega) = H(s)\big|_{s=j\omega}$$

$$H(j\omega) = \frac{(j\omega-q_1)(j\omega-q_2)\cdots(j\omega-q_N)}{(j\omega-p_1)(s-p_2)\cdots(j\omega-p_N)}$$

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➢ Motivation for the Laplace Transform

- **The CT Fourier transform enables us to:**
  - Solve linear constant coefficient differential equations (LCCDE);
  - Analyze the frequency response of LTI systems;
  - Analyze and understand the implications of sampling;

$$x(t) = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t)$$

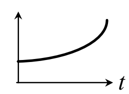
- **Why do we need another transform?**
  - The Fourier transform cannot handle **unstable signals**:

    (Recall this is related to the fact that the **eigenfunction**, $e^{j\omega t}$, has unit amplitude, $|e^{j\omega t}| = 1$.
  - There are many problems in which we desire to analyze and control unstable systems (e.g., the space shuttle, oscillators, lasers).
- **Consider the simple unstable system:** $\int_{-\infty}^{\infty} |x(t)|dt = \infty$
  - This is an **unstable causal system**.
  - We cannot analyze its behavior using:

    $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
  - **Note however we can use time domain techniques such as convolution and differential equations.**

$x(t)$ → CT LTI $h(t)$ → $y(t)$

$h(t) = e^t u(t)$



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➢ The Bilateral (Two-sided) Laplace Transform

- Recall **the eigenfunction property** of an **LTI system**:
  - e^{st} is an **eigenfunction** of any **LTI system.**
  - s can be **complex:** $s = \sigma + j\omega$

$x(t) = e^{st}$ → CT LTI $h(t)$ → $y(t) = H(s)e^{st}$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \quad \text{(assuming convergence)}$$

- We can define **the bilateral,** or **two-sided, Laplace transform:**

$$\boxed{x(t) \leftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}}$$

- **Several important observations are:**
  - Can be viewed as **a generalization of the Fourier transform**:

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t}\right)e^{-j\omega t}dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

  - $X(s)$ generally exists for a certain range of values of s. We refer to this as **the region of convergence (ROC).** Note that this only depends **on** σ **and not** ω.

$$ROC \in \left\{\sigma + j\omega \ni \int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right|dt < \infty\right\}$$

  - **If** $s = j\omega$ is in **the ROC (i.e.,** $\sigma = 0$**), then:**

    $$\mathcal{L}\{x(t)\}\big|_{s=j\omega} = X(s)\big|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

    and there is a clear relationship between **the Laplace** and **Fourier transforms.**

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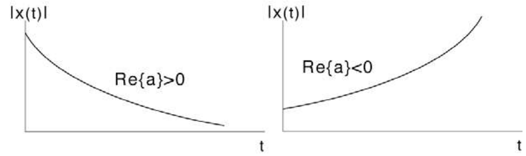
Example: A Right-Sided Signal

- **Example**: $x_1(t) = e^{-at}u(t)$ where a is an arbitrary real or complex number.



- **Solution**:

$$X_1(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-at}e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt$$

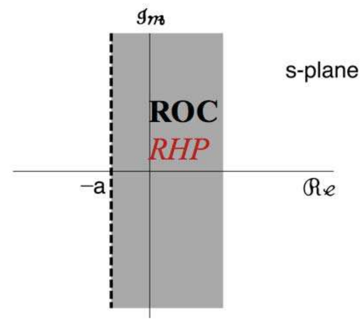
$$= \frac{-1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{-1}{s+a}\left[e^{-(s+a)\infty} - 1\right] = ???$$

This **converges** only if:

$$\mathrm{Re}\{s+a\} > 0 \Rightarrow \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$

and we can write:

$$X_1(s) = \frac{1}{s+a} \quad \text{when} \quad \mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$$



ROC

- The **ROC** can be visualized using **s-plane** plot shown above. **The shaded region** defines the values of s for which the **Laplace transform** exists. The **ROC** is a very importance property of **a two-sided Laplace transform**.

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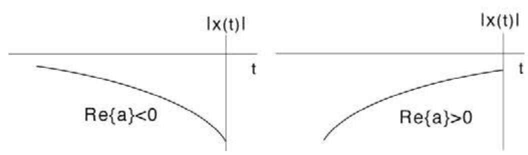
Example: A Left-Sided Signal

- **Example:** $x_2(t) = -e^{-at}u(-t)$
- **Solution:**



$$X_2(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

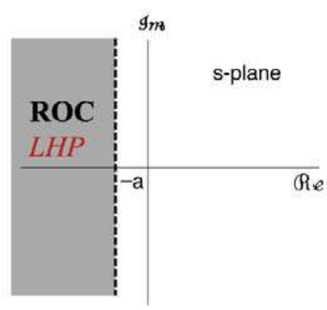
$$= \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a}\left[1 - e^{(s+a)\infty}\right] = ???$$

This **converges** only if:

$$\mathrm{Re}\{s+a\} < 0 \Rightarrow \mathrm{Re}\{s\} < -\mathrm{Re}\{a\}$$

and we can write**:**

$$X(s) = \frac{1}{s+a} \quad \text{when} \quad \mathrm{Re}\{s\} < -\mathrm{Re}\{a\}$$



ROC

- The transform is the same but the **ROC** is different. This is a major difference from the Fourier transform – we need both the transform and the ROC to uniquely specify the signal. The **FT does not have an ROC issue.**

Rational Transforms

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- Many transforms of interest to us will be **ratios of polynomials** in *s*, which we refer to as a **rational transform**:

$$\boxed{H(s)=\frac{N(s)}{D(s)}\quad \text{where}\quad N(s)=b_0+b_1s+b_2s^2+\ldots\quad D(s)=1+a_1s+a_2s^2+\ldots}$$

- The zeroes of the polynomial, *N*(*s*), are called **zeroes** of *H*(*s*).
- The zeroes of the polynomial, *D*(*s*), are called **poles** of *H*(*s*).
- Any signal that is **a sum of (complex) one-sided exponentials** can be expressed as **a rational transform**. Examples include **circuit analysis**.
- Example: $x(t)=3e^{2t}u(t)-2e^{-t}u(t)$

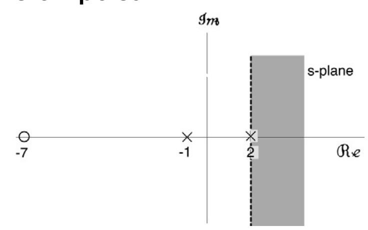
zero poles

$$X(s)=\int_0^{\infty}(3e^{2t}-2e^{-t})e^{-st}\,dt$$

$$=3\int_0^{\infty}e^{-(s-2)t}\,dt-2\int_0^{\infty}e^{-(s+1)t}\,dt$$

$$=\left.\frac{3}{s-2}\right|_0^{\infty}-\left.\frac{2}{s+1}\right|_0^{\infty}\qquad \begin{array}{l}1^{\text{st}}\ \text{term}:\mathrm{Re}\{s\}>2\\ 2^{\text{nd}}\ \text{term}:\mathrm{Re}\{s\}>-1\end{array}$$

$$=\frac{3}{s-2}-\frac{2}{s+1}=\frac{s+7}{(s-2)(s+1)}\qquad \mathrm{Re}\{s\}>2$$



- **Does this signal have a Fourier transform?**

Properties of the ROC

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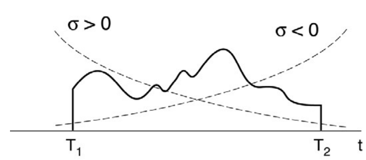
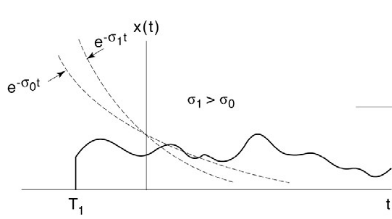
- There are some signals, particularly **two-sided signals** such as $x(t)=e^{t}$ and $x(t)=e^{j\omega_0 t}$ that **do not have Laplace transforms**.
- The **ROC** typically assumes **a few simple shapes**. It is usually the **intersection of lines parallel to the imaginary axis**. Why?
- For **rational transforms**, the **ROC does not include any poles**. Why?
- If $x(t)$ is of **finite duration** and absolutely **integrable**, its **ROC** is the **complete *s*-plane**.

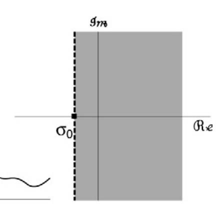


$$X_2(s)=\int_{-\infty}^{\infty}x(t)u(t)e^{-st}\,dt$$

$$=\int_{T_1}^{T_2}x(t)e^{-st}\,dt<\infty\quad \text{if}\ \int_{T_1}^{T_2}|x(t)|\,dt<\infty$$

- If $x(t)$ is **right-sided**, and if σ_0 is in **the ROC**, **all points to the right of σ_0 are in the ROC.**
- If $x(t)$ is **left-sided**, **points to the left of σ_0 are in the ROC.**





More Properties of the ROC

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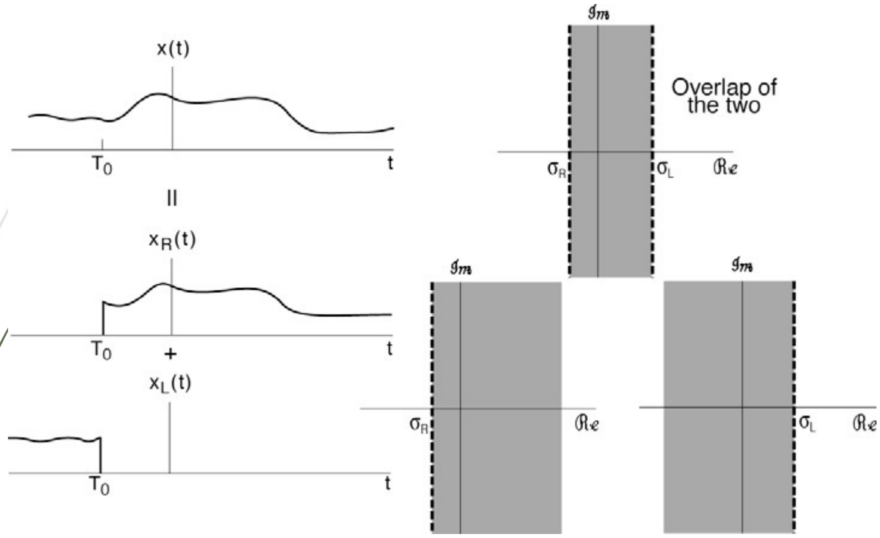
- If $x(t)$ is **two-sided**, then the **ROC** consists of the **intersection of a left-sided and right-sided** version of $x(t)$, which is **a strip in the *s*-plane**:



Example of a Two-sided Signal

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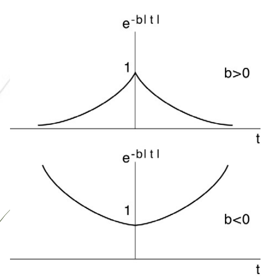
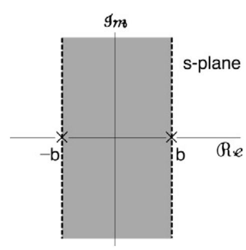
$$x(t) = e^{b|t|} = e^{bt}u(-t) + e^{-bt}u(t)$$



$$X(s) = -\frac{1}{s-b} + \frac{1}{s+b} \quad \begin{matrix} 1^{st}\ term: \mathrm{Re}\{s\} < b \\ 2^{nd}\ term: \mathrm{Re}\{s\} > -b \end{matrix}$$

$$= \frac{-2b}{s^2 - b^2} \quad (b > 0) \text{ with ROC below}$$



- **If b < 0, the Laplace transform does not exist.**
- **Hence, the ROC plays an integral role in the Laplace transform.**

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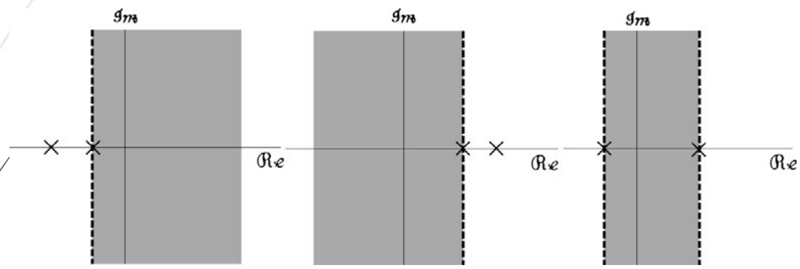
➢ROC for Rational Transforms

- Since **the ROC cannot include poles, the ROC** is **bounded by the** poles for a rational transform.
- If x(*t*) **is right-sided,** the ROC begins to the right of the rightmost pole**.** If x(*t*) **is left-sided,** the ROC begins to the left of the leftmost pole**.** If x(*t*) **is double-sided**, the ROC will be the intersection of these two regions**.**



- **If the ROC includes the** $j\omega$**-axis,** then **the Fourier transform of** x(*t*) **exists.** Hence, the **Fourier transform** can be considered to be the evaluation of the **Laplace transform** along the $j\omega$-axis**.**

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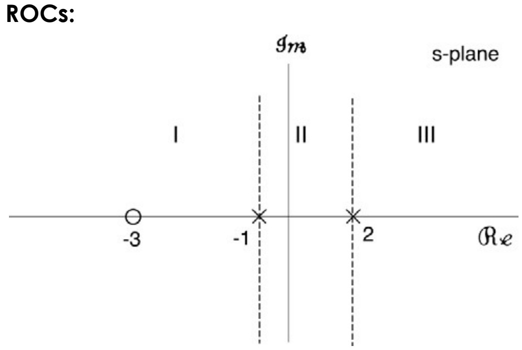
Example:

- Consider **the Laplace transform:** $X(s) = \frac{(s+3)}{(s+1)(s-2)}$
- Can we **uniquely determine the original signal,** x(*t*)**?**
- There are **three possible ROCs:**



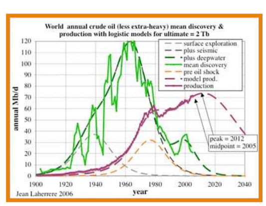
- **ROC III: only if** x(*t*) **is right-sided.**
- **ROC I: only if** x(*t*) **is left-sided.**
- **ROC II: only if** x(*t*) **has a Fourier transform.**

- **Review your tables of transform properties (one-sided!) and common transform pairs:**

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t+\theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t+\theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

PROPERTIES OF THE LAPLACE TRANSFORM



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➢ Familiar Properties of Linear Transforms

- To introduce **the inverse Laplace transform** and some important applications of the transform (e.g., circuits), we will need to introduce some **familiar properties of the transform** (e.g., **linearity**).
- There are some **properties unique** to **the Laplace transform** (e.g., **the Initial Value Theorem**).
- There are **some properties** of the **Fourier Transform** that do not have an equivalent for the **Laplace transform** (e.g., **duality, Parseval's theorem**).

1. Linearity:

$$\boxed{ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)}$$

Note that the ROC for the sum is at least the intersection of the ROCs for each component (it must include regions for which both transforms converge).

- In some special cases, **the ROC** can **be large**r (e.g., when the zeroes of one component cancel the poles of the other component). For example:

$a = -b$ and $x_1(t) = x_2(t) \;\Rightarrow\; x(t) = 0 \;\Rightarrow\;$ ROC is the entire s-plane

Example:

$$x(t) = u(t) + e^{-t}u(t) \leftrightarrow X(s) = \frac{1}{s} + \frac{1}{s+1} = \frac{(s+1)}{(s)(s+1)} + \frac{s}{(s)(s+1)} = \frac{2s+1}{(s)(s+1)}$$

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Properties of the Laplace Transform

2. Time-Shift:

$$\boxed{x(t-T) \leftrightarrow e^{-sT}X(s) \quad \text{same ROC as } X(s)}$$

Example:

$$e^{-2t}u(t) \leftrightarrow X(s) = \frac{1}{s+2} \qquad \text{ROC}: \mathrm{Re}\{X(s)\} > -2$$

$$e^{-2(t-3)}u(t-3) \leftrightarrow X(s) = \frac{e^{-3s}}{s+2} \qquad \text{ROC}: \mathrm{Re}\{X(s)\} > -2$$

Note that **the ROC doesn't change** because it is defined by the **pole location.**

Example:

$$x(t) = u(t) - u(t-T) \leftrightarrow X(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1-e^{-Ts}}{s}$$

What is the **ROC**? Hint: This is **a time-limited signal**.

3. Time Scaling:

$$\boxed{x(at) \leftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right)}$$

Example:

$$u(at) \leftrightarrow \frac{1}{a}\left(\frac{1}{s/a}\right) = \frac{1}{s}$$

Is this result expected?

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More Properties

4. Multiplication by a Power of t:

$$\boxed{t^n x(t) \leftrightarrow (-1)^n \frac{d^n}{ds^n} X(s) \quad \text{same ROC as } X(s)}$$

Compare to **Fourier transform**:

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{ds^n} X(e^{j\omega})$$

Example:

$$r(t) = tu(t) \leftrightarrow R(s) = (-1)^1 \frac{d^1}{ds^1}(\frac{1}{s}) = \frac{1}{s^2}$$

5. Time-Domain Differentiation (Bilateral):

$$\boxed{\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s)} \quad \text{ROC can be bigger than the ROC for } x(t)$$

Example:

$$x(t) = \delta(t) = \frac{du(t)}{dt} \leftrightarrow X(s) = (s)(\frac{1}{s}) = 1 \quad \text{ROC for } \delta(t) \text{ is larger than the ROC for } u(t)$$

6. Integration:

$$\boxed{\int_0^t x(\lambda)d\lambda \leftrightarrow \frac{1}{s} X(s)} \quad \text{ROC must not include } s = 0$$

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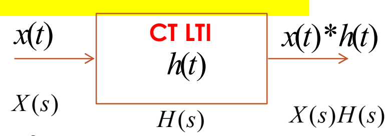
➢ Convolution

7. The convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(\lambda)h(t-\lambda)d\lambda$$



- **Laplace transform** is analogous **to the Fourier transform:**

$$\boxed{Y(s) = X(s)H(s)}$$

- But, because the **Laplace transform** and **its inverse** (to be discussed in a moment) are **not symmetric, the dual of this is not true:**

$$x(t)h(t) \not\leftrightarrow X(s) * H(s)$$

This is one reason the Fourier transform is more popular for applications involving communications systems and modulation.

- The ROC is at least the overlap of the ROCs for each signal (again, it can be larger than the ROC for either signal).
- **Example:**

$$x(t) = u(t) - u(t-1) \leftrightarrow X(s) = \left(\frac{1-e^{-s}}{s}\right)$$

$$y(t) = x(t) * x(t) \leftrightarrow X^2(s) = \left(\frac{1-e^{-s}}{s}\right)^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\} - 2\mathcal{L}^{-1}\{\frac{e^{-s}}{s^2}\} + \mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\} = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

The Unilateral (One-sided) Laplace Transform

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- Define a special case of **the Laplace transform** for **right-sided signals**: $$\boxed{X(s)=\int_0^\infty x(t)e^{-st}\,dt}$$
- For **right-sided signals (** $x(t)=0,\quad t<0$**)**, and for **causal systems (**$h(t)=0,\quad t<0$ **)**, **the one-sided** and **two-sided transforms are equal.**
- Several properties change slightly, such as **differentiation:**

$$\boxed{\frac{dx(t)}{dt}\leftrightarrow sX(s)-x(0^-)} \qquad \boxed{\frac{d^2x(t)}{dt^2}\leftrightarrow s^2X(s)-sx(0^-)-\left.\frac{dx}{dt}\right|_{t=0^-}}$$

Proof:

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\}=\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt=x(t)e^{-st}\Big|_{0^-}^{\infty}+s\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt\quad\text{(using integration by parts)}$$

$$=-x(0^-)+sX(s)$$

- Other properties, such as **convolution**, hold as is, as long as the system is **causal** and the input starts at *t* = *0***.**

Initial and Final Value Theorems

- **Theorem:**

$$\boxed{\begin{aligned} x(0)&=\lim_{s\to\infty}sX(s)\quad\text{(Initial Value Theorem)}\\ x(\infty)&=\lim_{s\to 0}sX(s)\quad\text{(Final Value Theorem)}\end{aligned}}$$

- **Proof:**

Applying the **differentiation** property:

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\}=sX(s)-x(0)$$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\}=\int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt=\begin{cases}\int_{0^-}^{\infty}\frac{dx(t)}{dt}(0)dt=0 & s\to\infty\\ \int_{0^-}^{\infty}\frac{dx(t)}{dt}(1)dt=x(\infty)-x(0) & s\to 0\end{cases}$$

Combining the two:

$$sX(s)-x(0)=0 \qquad s\to\infty \quad\Rightarrow\quad x(0)=\lim_{s\to\infty}sX(s)$$

$$x(\infty)-x(0)=sX(s)-x(0)\qquad s\to 0\quad\Rightarrow\quad x(\infty)=\lim_{s\to 0}sX(s)$$

- **The initial value theorem** can be extended **to higher-order derivatives:**

$$\left.\frac{dx(t)}{dt}\right|_{t=0}=\lim_{s\to\infty}[s^2X(s)-sx(0)]$$

- Allow **initial** and **final conditions** to be computed directly from **the transform.**

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Application of the Initial and Final Value Theorems

- **Consider a rational transform:**

$$X(s)=\frac{N(s)}{D(s)} \quad \text{where} \quad \begin{matrix} n: \text{order of N(s)} \\ d: \text{order of D(s)} \end{matrix}$$

- **Initial value:**

$$\boxed{x(0)=\lim_{s\to\infty} sX(s)=\lim_{s\to\infty}\frac{s^{n+1}}{s^d}=\begin{cases} 0 & d>n+1 \\ finite\neq 0 & d=n+1 \\ \infty & d<n+1 \end{cases}}$$

For example:

$$X(s)=\frac{1}{s+1} \quad \Rightarrow x(0)=\lim_{s\to\infty}\frac{s^1}{s^1}=1$$

$$X(s)=\frac{1}{(s+1)^2} \quad \Rightarrow x(0)=\lim_{s\to\infty}\frac{s^1}{s^2}=0$$

- **Final Value:**

$$\boxed{\mathrm{x}(\infty)=\lim_{s\to 0} sX(s)=0 \quad \Rightarrow \quad \lim_{s\to 0} X(s)<\infty \quad \Rightarrow \quad \text{no poles at s} = 0}$$

$$X(s)=\frac{1}{s+1} \quad \Rightarrow x(\infty)=\lim_{s\to 0}\frac{s}{s+1}=1$$

$$X(s)=\frac{1}{(s+1)^2} \quad \Rightarrow x(\infty)=\lim_{s\to 0}\frac{s}{(s+1)^2}=0$$

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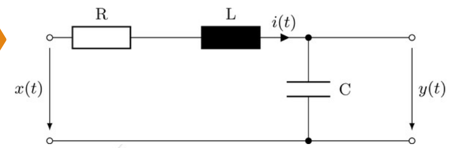
Table 3.2 Basic Properties of One-Sided Laplace Transforms

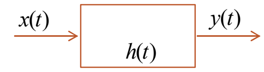
Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t)+\beta g(t)$	$\alpha F(s)+\beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s)-f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0-)-f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha\neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+)=\lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	

➢ Case Example: RLC Filters



Block Diagram

CT LTI



Differential Equation (Second Order):

$$\boxed{CL\frac{d^2y(t)}{dt^2} + CR\frac{dy(t)}{dt} + y(t) = x(t)} \Rightarrow \boxed{H(s) = \frac{1}{LCs^2 + RCs + 1}}$$

Impulse Response :

$C: \frac{2}{5},\quad L: \frac{1}{2},\quad R: 1$

$$h(t) = C_1 e^{\frac{t\left(-R-\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}} + C_2 e^{\frac{t\left(-R+\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}} - \frac{e^{\frac{t\left(-R-\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}}\theta(t)}{\sqrt{C(CR^2-4L)}} + \frac{e^{\frac{t\left(-R+\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}}\theta(t)}{\sqrt{C(CR^2-4L)}}$$

The integration constants C_1 and C_2 have to be determined from the initial conditions $y(t)=0$ and $dy(t)/dt=0$ for $t<0$

$C_1: 0,\quad C_2: 0$

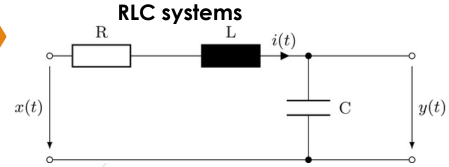
$$\boxed{h(t) = -\frac{e^{\frac{t\left(-R-\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}}\theta(t)}{\sqrt{C(CR^2-4L)}} + \frac{e^{\frac{t\left(-R+\frac{\sqrt{C(CR^2-4L)}}{C}\right)}{2L}}\theta(t)}{\sqrt{C(CR^2-4L)}}}$$

Function Transfers :

$$Y(s) = \frac{1}{LCs^2 + RCs + 1}\cdot X(s) \Rightarrow \boxed{H(s) = \frac{1}{LCs^2 + RCs + 1}}$$

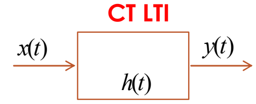
➢ Case Example: RLC Filters

RLC systems



Block Diagram

CT LTI



$$y(t) = x(t) * h(t) \Leftrightarrow Y(s) = X(s).H(S)$$

$$CL\frac{d^2y(t)}{dt^2} + CR\frac{dy(t)}{dt} + y(t) = x(t) \xrightarrow{\mathcal{L}\{\dots.\}} H(s) = \frac{1}{LCs^2 + RCs + 1}$$

$$x(t) = u(t) \xrightarrow{\mathcal{L}\{\dots.\}} U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = ? \xleftarrow{\mathcal{L}^{-1}\{\dots\}} Y(s) = X(s).H(S)$$

$$Y(s) = \frac{1}{LCs^2 + RCs + 1}\cdot\frac{1}{s}$$

$$C = 1\,F;\quad L: 1H;\quad R = 2\Omega$$

$$\xleftarrow{\mathcal{L}^{-1}\{\dots\}} Y(s) = \frac{1}{s^2+2s+1}\cdot\frac{1}{s} = \frac{1}{(s+1)^2.s}$$