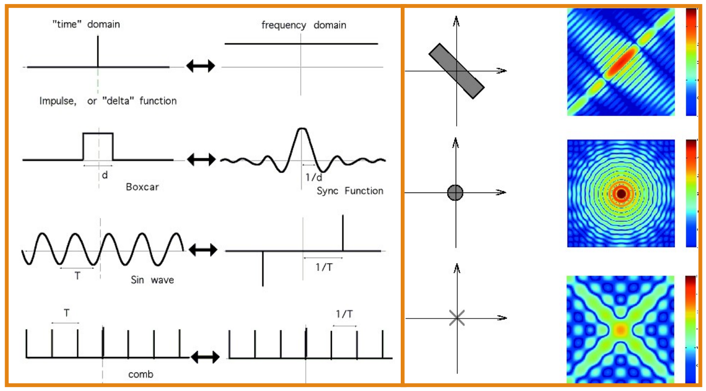
Weeks 06: THE FOURIER TRANSFORM

Objectives:

- Derivation
- Transform Pairs
- Response of LTI Systems
- Transforms of Periodic Signals
- Examples

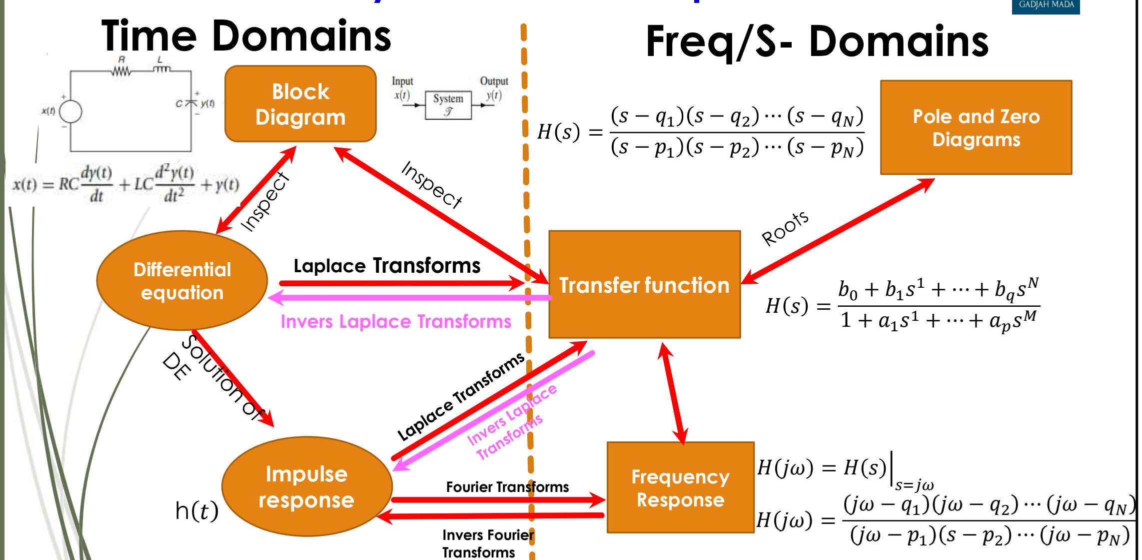
1



Continuouse-Time System Relationships

PSE - UGM

UNIVERSITAS GADJAH MADA



LOCALLY ROOTED, GLOBALLY RESPECTED

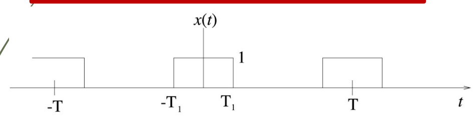
ugm.ac.id

Motivation

3

- We have introduced **a Fourier Series** for analyzing **periodic signals.** What about **aperiodic signals? (**e.g., a pulse instead of a pulse train**)**
- We can view **an aperiodic signal** as **the limit of a periodic signal as** $T \to \infty$**.**
- **The harmonic components** are **spaced** $\omega_0 = \dfrac{2\pi}{T}$ **apart.**
- **As** $T \to \infty$, $\omega_0 \to 0$, **then** $k\omega_0$ **becomes continuous.**
- **The Fourier Series** becomes **the Fourier Transform.**

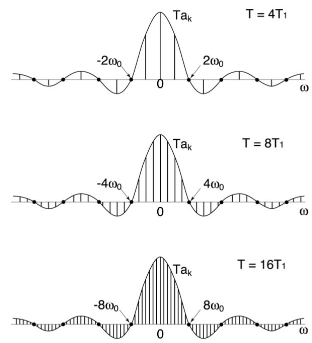
$$\boxed{\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt}$$



$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{2\sin(k2\pi(T_1/T))}{k\omega_0 T}$$

$$T_1 \ \text{fixed}, T \to \infty, k\omega_0 \to \omega$$

$$Tc_k = \frac{2\sin(\omega T_1)}{\omega}$$

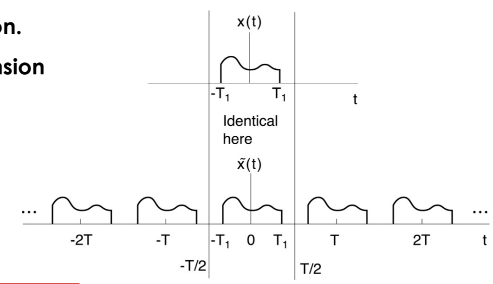


Derivation of Analysis Equation

4

- Assume x(t) **has a finite duration.**
- Define $\tilde{x}(t)$ **as a periodic extension of** x(t)**:**

$$\tilde{x}(t) = \begin{cases} x(t) & -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ periodic & |t| > \dfrac{T}{2} \end{cases}$$



- **As** $T \to \infty, \ \tilde{x}(t) \to x(t)$
- **Recall our Fourier series pair:**

$$\boxed{\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt}$$

- **Since** x(t) **and** $\tilde{x}(t)$ **are identical** over this interval**:**

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

- **As** $T \to \infty, \ k\omega_0 \to \omega$

$$\boxed{X(j\omega) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

5

Derivation of the Synthesis Equation

- **Recall:** $\tilde{x}(t) = x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad for \; -\frac{T}{2} \le t \le \frac{T}{2}$
- We can substitute **for** c_k the sampled value **of** $X(j\omega)$:

$$\tilde{x}(t) = x(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T} X(j\omega_0)\right) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(j\omega_0)\, e^{jk\omega_0 t}$$

- **As** $T \to \infty,\ \omega_0 \to 0,\ \sum_{k=-\infty}^{\infty} \omega_0 \to \int d\omega,\ k\omega_0 \to \omega$

and we arrive at our **Fourier Transform pair:**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \textbf{(synthesis)}$$

$$X(j\omega) = \frac{1}{T} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \textbf{(analysis)}$$

- Note the presence **of the eigenfunction:** $e^{st}\big|_{s=j\omega} = e^{j\omega t}$
- Also note **the symmetry** of these equations (e.g., integrals over time and frequency, change in the sign of the exponential, difference in scale factors).

6

Frequency Response of a CT LTI System

- Recall that **the impulse response** of **a CT system**, $h(t)$, defines the properties of that system.
- We apply **the Fourier Transform** to obtain **the system's frequency response:**

$$e^{j\omega t} \to \boxed{h(t) \Leftrightarrow H(j\omega)} \to H(j\omega)e^{j\omega t}$$

$$\begin{matrix} x(t) \\ X(j\omega) \end{matrix} \to \boxed{h(t) \Leftrightarrow H(j\omega)} \to \begin{matrix} y(t) \\ Y(j\omega) \end{matrix}$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega$$

$$H(j\omega) = \frac{1}{T} \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

except that now **this is valid for finite duration (energy) signals** as well as **periodic signals!**

- How does this relate to what you have learned in **circuit theory?**

7

Existence of the Fourier Transform

- **Under what conditions does this transform exist?**

$x(t)$ **can be infinite duration but must satisfy these conditions:**

a) Finite energy $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |e(t)|^2 \, dt = 0$$

b) Dirichlet conditions (including $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$)

(i) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2}[x(t-0) + x(t+0)) = \begin{cases} x(t) & \text{at points of continuity} \\ \text{midpoint at discontinuity} \end{cases}$$

(ii) Gibb' s phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

8

Example: Impulse Function

Example #1

(a) $$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega t} d\omega$$ — Synthesis equation for $\delta(t)$

(b) $$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt$$

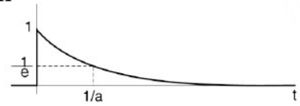
$$= e^{-j\omega t_o}$$ — Linear phase shift in ω

9

Example: Exponential Function

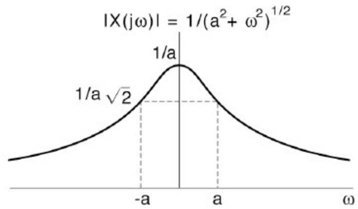
Example #2: Exponential function

$$x(t) = e^{-at}u(t)\ , \quad a > 0$$

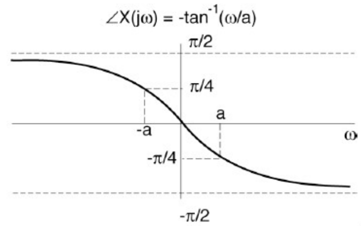


$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{0}^{+\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}}dt$$

$$= -\left(\frac{1}{a+j\omega}\right)e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$



Even



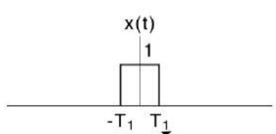
Odd

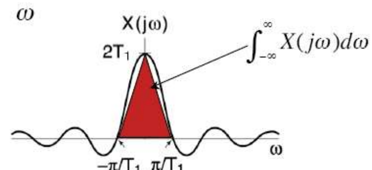
10

Example: Square Pulse

Example #3: A square pulse in the time-domain

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{2\sin\omega T_1}{\omega}$$





Note the inverse relation between the two widths ⇒ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{+\infty} x(t)dt \qquad \text{Example above: } \int_{-\infty}^{\infty} x(t)dt = 2T_1 = X(0)$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)d\omega \qquad \text{Ex. above: } x(0) = 1 = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} X(j\omega)d\omega}_{2\pi} = \frac{1}{2\pi}\cdot\frac{1}{2}\left(2T_1 \times \frac{2\pi}{T_1}\right)$$

$$= \frac{1}{2\pi} \times (\text{Area of the triangle})$$

11

Example: Gaussian Pulse

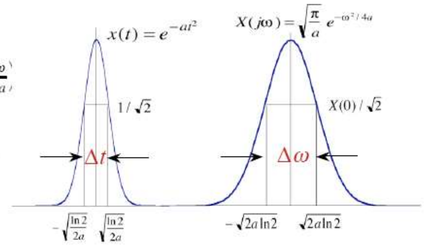
Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2}$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$



(Pulse width in t)•(Pulse width in ω)
$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a)^{1/2} \cdot (a)^{1/2} = 1$

Also a Gaussian!

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.

12

CT Fourier Transforms of Periodic Signals

CT Fourier Transforms of Periodic Signals

Suppose $X(j\omega) = \delta(\omega - \omega_0)$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is $\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)}$ — All the energy is concentrated in one frequency — ω_o

More generally, if $x(t) = x(t+T)$, then

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ Discrete spectra

13

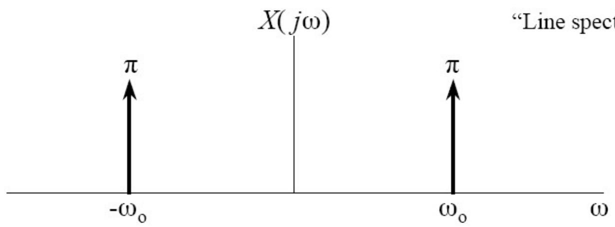
Example: Cosine Function

Example #4:

$$x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\Updownarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



"Line spectra"

14

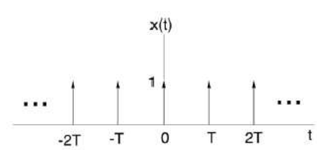
Example: Periodic Pulse Train

Example #5:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad \text{— Old friend, sampling function}$$

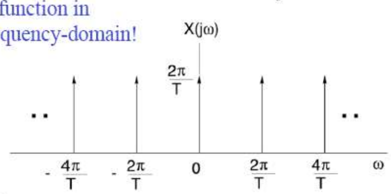
$$x(t) \longleftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}\,dt = \frac{1}{T}$$

$$\Downarrow\ x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$



$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

15

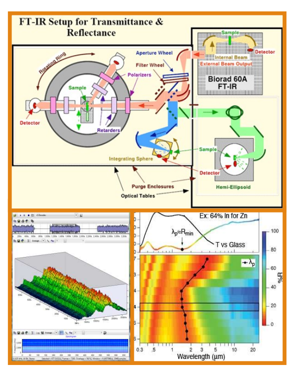
Summary

- Motivated the derivation of the CT Fourier Transform by starting with the Fourier Series and increasing the period: $T \rightarrow \infty$
- Derived the analysis and synthesis equations (Fourier Transform pairs).
- Applied the Fourier Transform to CT LTI systems and showed that we can obtain the frequency response of an LTI system by taking the Fourier Transform of its impulse response.
- Discussed the conditions under which the Fourier Transform exists. Demonstrated that it can be applied to periodic signals and infinite duration signals as well as finite duration signals.
- Worked several examples of important finite duration signals.
- Introduced the Fourier Transform of a periodic signal.
- Applied this to a cosinewave and a pulse train.

FOURIER TRANSFORM PROPERTIES

Objectives:

- Linearity
- Time Shift and Time Reversal
- Multiplication
- Integration
- Convolution
- Parseval's Theorem
- Duality



16

17

Linearity

- Recall our expressions for the Fourier Transform and its inverse:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad \textbf{(synthesis)}$$

$$X(j\omega) \equiv X(\omega) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \textbf{(analysis)}$$

- The property of linearity:

$$\boxed{\mathcal{F}\{ax(t)+by(t)\} = aX(j\omega)+bY(j\omega) \quad \Rightarrow \quad ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)}$$

Proof:

$$\mathcal{F}\{ax(t)+by(t)\} = \frac{1}{T}\int_{-\infty}^{\infty} \{ax(t)+by(t)\}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} ax(t)\, e^{-j\omega t}\, dt + \frac{1}{T}\int_{-\infty}^{\infty} by(t)\, e^{-j\omega t}\, dt$$

$$= a\left\{\frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right\} + b\left\{\frac{1}{T}\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt\right\}$$

$$= aX(j\omega)+bY(j\omega)$$

18

Time Shift

- Time Shift:

$$\boxed{x(t-t_0) \leftrightarrow X(j\omega)e^{-j\omega t_0}}$$

Proof:

$$\mathcal{F}\{x(t-t_0)\} = \frac{1}{T}\int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

make a change of variables : $\lambda = t\text{-}t_0$, which implies $t = \lambda + t_0$

$$\mathcal{F}\{x(t-t_0)\} = \frac{1}{T}\int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\lambda+t_0)}\, d\lambda$$

$$= \frac{1}{T}\left\{\int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda\right\} e^{-j\omega t_0}$$

$$= X(j\omega)e^{-j\omega t_0}$$

- Note that this means time delay is equivalent to a linear phase shift in the frequency domain (the phase shift is proportional to frequency).
- We refer to a system as an all-pass filter if:

$$|X(j\omega)| = 1 \quad \angle X(j\omega) \neq 0$$

- Phase shift is an important concept in the development of surround sound.

19

➢ Time Scaling

- **Time Scaling:**

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a})}$$

Proof:

$$\mathcal{F}\{x(at)\} = \frac{1}{T}\int_{-\infty}^{\infty} x(at)\, e^{-j\omega t} dt$$

assume $a > 0$, make a change of variables : $\lambda = at$, which implies $t = \lambda / a$, and $dt = (1/a)d\lambda$

$$\mathcal{F}\{ax(t)\} = \frac{1}{T}\int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\frac{\lambda}{a})} (\frac{1}{a}) d\lambda$$

$$= (\frac{1}{a})\left\{\frac{1}{T}\int_{-\infty}^{\infty} x(\lambda)\, e^{-j(\omega/a)\lambda} d\lambda\right\}$$

$$= (\frac{1}{a}) X(\frac{j\omega}{a})$$

- **Generalization for** $a < 0$ **, the negative value is offset by the change in the limits of integration.**
- **What is the implication of** $a < 1$ **on the time-domain waveform? On the frequency response? What about** $a > 1$**?**
- **Any real-world applications of this property? Hint: sampled signals.78**

20

➢ Time Reversal

- **Time Reversal:**

$$\boxed{x(-t) \leftrightarrow X(-j\omega)}$$

Proof:

$$\mathcal{F}\{x(-t)\} = \frac{1}{|a|} X(\frac{j\omega}{a})\bigg|_{a=-1} = X(-j\omega)$$

We can also note that for real-valued signals:

$$X(-j\omega) = |X(-j\omega)|\angle X(-j\omega)$$
$$= |X(j\omega)|\angle X(-j\omega) = X^*(j\omega) \quad \text{(complex conjugate)}$$

- **Time reversal is equivalent to conjugation in the frequency domain.**
- **Can we time reverse a signal? If not, why is this property useful?**

21

➢ Multiplication by a Power of t

- **Multiplication by a power of** t**:**

$$\boxed{t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(j\omega)}$$

Proof: $$X(j\omega) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

differentiate with respect to ω:

$$\frac{dX(j\omega)}{d\omega} = \frac{1}{T}\int_{-\infty}^{\infty} (-jt)\, x(t)\, e^{-j\omega t}\, dt$$

multiply by j:

$$j\frac{dX(j\omega)}{d\omega} = (j)\frac{1}{T}\int_{-\infty}^{\infty} (-jt)\, x(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} (t)\, x(t)\, e^{-j\omega t}\, dt = \mathcal{F}\{t\, x(t)\}$$

- **We can repeat the process for higher powers of** t**.**

22

➢ Multiplication by a Complex Exponential (Modulation)

- **Multiplication by a complex exponential:**

$$\boxed{x(t)e^{j\omega t} \leftrightarrow X(j(\omega-\omega_0)) \quad \text{for any real number } \omega_0}$$

Proof:

$$\mathcal{F}\{x(t)e^{j\omega_0 t}\} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}\; e^{-j\omega t}\, dt$$

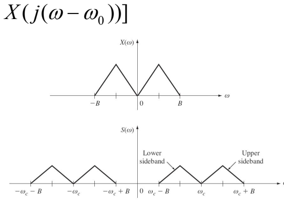
$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\, dt$$

$$= X(j(\omega-\omega_0))$$

- **Why is this property useful?**
- **First, another property:**

$$x(t)\cos(\omega_0 t) = x(t)\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] \leftrightarrow \frac{1}{2}[X(j(\omega+\omega_0)) + X(j(\omega-\omega_0))]$$

- **This produces a translation in the frequency domain. How might this be useful in a communication system?**



23

➢ Differentiation / Integration

- **Differentiation in the Time Domain:**

$$\boxed{\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(j\omega)}$$

- **Integration in the Time Domain:**

$$\boxed{\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)}$$

- **What are the implications of time-domain differentiation in the frequency domain?**
- **Why might this be a problem? Hint: additive noise.**
- **How can we apply these properties? Hint: unit impulse, unit step, …**

24

➢ Convolution in the Time Domain

- **Convolution in the Time Domain:**

$$\boxed{x(t) * h(t) \leftrightarrow X(j\omega)H(j\omega)}$$

- **Proof:**

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

$$\mathcal{F}\{x(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} h(t-\lambda)e^{-j\omega t}dt\right]d\lambda$$

$$\text{change of variables}: \gamma = t - \lambda \Rightarrow d\gamma = dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} h(\gamma)e^{-j\omega(\gamma+\lambda)}d\gamma\right]d\lambda$$

$$= \left[\int_{-\infty}^{\infty} x(\lambda)e^{-j\omega\lambda}d\lambda\right]\left[\int_{-\infty}^{\infty} h(\gamma)e^{-j\omega\gamma}d\gamma\right]$$

$$= X(j\omega)H(j\omega)$$

25

➢Other Important Properties

- **Multiplication in the Time Domain:**

$$\boxed{x(t)\bullet y(t) \leftrightarrow \frac{1}{2\pi}[X(j\omega)*Y(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda}$$

- **Parseval's Theorem:**

$$\boxed{\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}$$

- **Duality:**

$$\boxed{X(t) \leftrightarrow 2\pi\, x(-\omega)}$$

- **Note:** please read the textbook carefully for the derivations and interpretation of these results.

26

Summary

TABLE 3.1 Properties of the Fourier Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)$ $a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega)$ $n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$ ω_0 real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega+\omega_0) - X(\omega-\omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)$ $n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t)*v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega)*V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X(\omega)}V(\omega)\,d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2\,d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

27

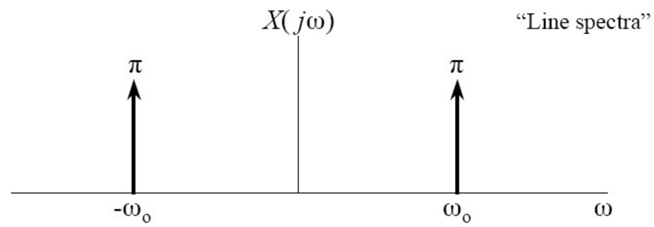
Example: Cosine Function

Example #4:

$$x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\updownarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



28

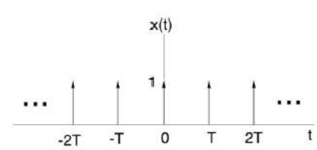
Example: Periodic Pulse Train

Example #5:

$$x(t) = \sum_{n=-\infty}^{+\infty}\delta(t - nT)$$ — Old friend, sampling function

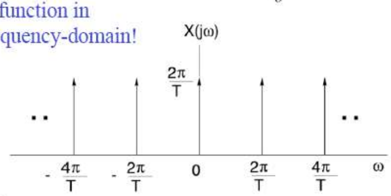
$$x(t) \longleftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T}$$

$$\Downarrow\ x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{jk\omega_o t}$$



$$X(j\omega) = \sum_{k=-\infty}^{+\infty}\frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

29

Example: Gaussian Pulse

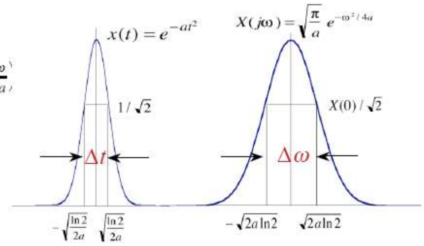
Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2}$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi / a}} \cdot e^{-\frac{\omega^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$



(Pulse width in t)•(Pulse width in ω)
$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a)^{1/2} \cdot (a)^{1/2} = 1$

Also a Gaussian!

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.