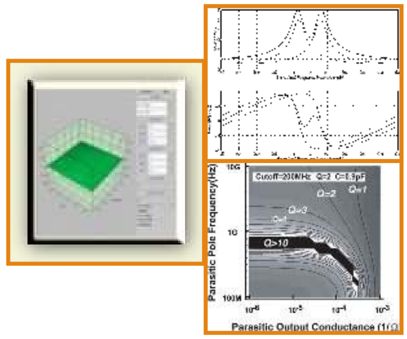
Mg.04: INVERSE LAPLACE TRANSFORM

Objectives:

- Rational Transforms
- Partial Fractions Expansion
- Distinct Roots
- Complex Poles
- Repeated Roots
- Numerator Has A Larger Order ($M \geq N$)
- Transforms With Exponentials



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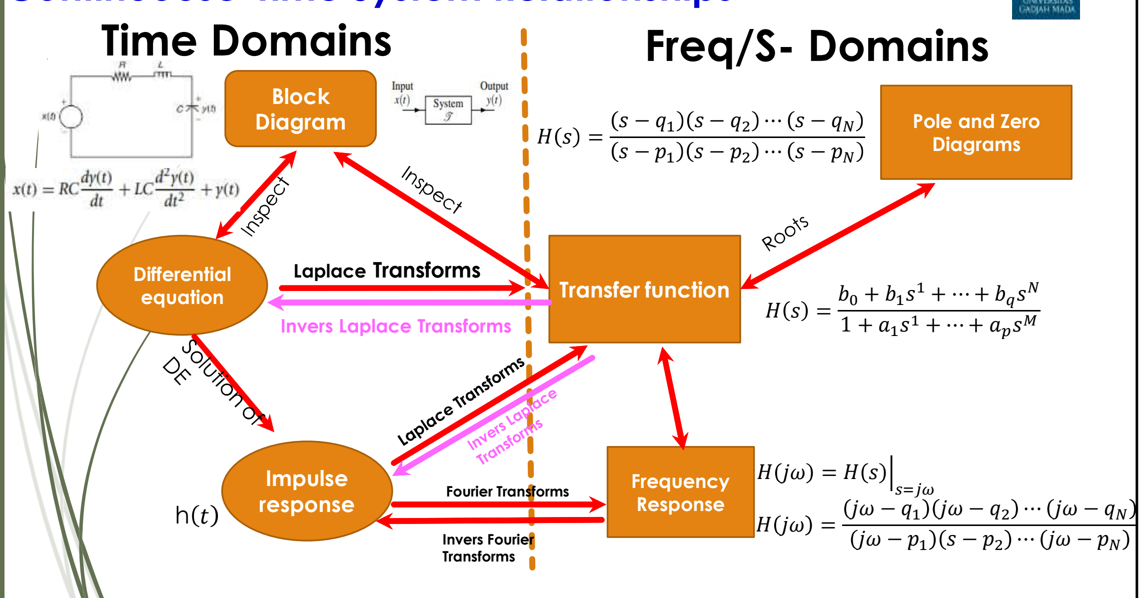
Continuouse-Time System Relationships

PSE - UGM



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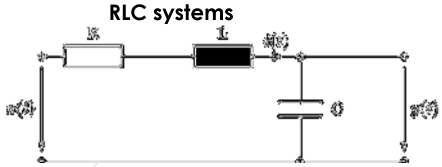
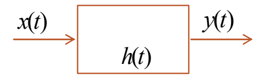
Why need Invers Laplace Transforms : RLC Filters ?

RLC systems



Block Diagram

CT LTI



$$y(t) = x(t) * h(t) \quad \longleftrightarrow \quad Y(s) = X(s).H(S)$$

$$CL\frac{d^2y(t)}{dt^2} + CR\frac{dy(t)}{dt} + y(t) = x(t) \xrightarrow{\mathcal{L}\{\dots\}} H(s) = \frac{1}{LCs^2 + RCs + 1}$$

$$x(t) = u(t) \xrightarrow{\mathcal{L}\{\dots\}} U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$Y(s) = X(s).H(S)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = ? \xleftarrow{\mathcal{L}^{-1}\{\dots\}} Y(s) = \frac{1}{LCs^2 + RCs + 1} \cdot \frac{1}{s}$$

$$C = 1\,F; \quad L: 1H; \quad R = 2\Omega$$

$$? \xleftarrow{\mathcal{L}^{-1}\{\dots\}} Y(s) = \frac{1}{s^2+2s+1} \cdot \frac{1}{s} = \frac{1}{(s+1)^2.s}$$

- Review your tables of transform properties (one-sided!) and common transform pairs:

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

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Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

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Rational Transforms

- Consider the problem of finding the **inverse Laplace transform** for:

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + b_2 s^2 \ldots + b_M s^M}{a_0 + a_1 s + a_2 s^2 \ldots + a_N s^N}$$

where **{a_i}** and **{b_i}** are **real numbers**, and ***M*** and ***N*** are **positive integers**.

- One straightforward approach is to **factor B(s)** into **a sum of simpler terms** whose **Laplace transforms** can be easily computed (or located in a table of transform pairs). The method of **Partial Fractions** is one approach:

$$\boxed{X(s) = \frac{b_0 + b_1 s + b_2 s^2 \ldots + b_M s^M}{a_N(s-p_1)(s-p_2)\ldots(s-p_N)} = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \ldots + \frac{c_N}{s-p_N}}$$

- The details of the expansion depends on the properties of ***A(s)***. For example, the presence of **repeated roots** slightly complicates things.

- Fortunately, MATLAB can be used to find the **roots of the polynomial**:

$$A(s) = s^3 + 4s^2 + 6s + 4$$

```
A = [1 4 6 4]
p = roots(A)
p =
    -2
    -1.0000 + 1.0000i
    -1.0000 – 1.0000i
```

$\rightarrow$ $A(s) = (s+2)(s+1-j)(s+1+j)$

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➢ 1. Distinct Roots: Method of Residues

- Suppose there are **distinct** or **non-repeated roots:** $p_i \neq p_j$ when $i \neq j$.

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \ldots + \frac{c_N}{s-p_N}$$
$$c_i = [(s-p_i)X(s)]_{s=p_i}, \quad i = 1, 2, \ldots, N$$

- The **constants**, c_i, are called **residues**, and this method of computation is called **the residue method**.
- The **constant** c_i is real if the corresponding pole is real. If two poles appear as complex conjugate pairs, the c_i must also appear in complex conjugate pairs.
- Once the **partial fractions expansion** is completed, the **inverse Laplace transform** can be easily found as:

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \ldots + c_N e^{p_N t} \qquad t \geq 0$$

- **Example:**

$$X(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{s+2}{s(s+1)(s+3)} = \frac{c_1}{s-0} + \frac{c_2}{s-(-1)} + \frac{c_3}{s-(-3)} = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+3}$$

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Distinct Roots (Cont.)

- **Example:**

$$X(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s+3}$$

$$c_1 = [sX(s)]_{s=0} = \left.\frac{s+2}{(s+1)(s+3)}\right|_{s=0} = \frac{2}{3}$$

$$c_2 = [(s+1)X(s)]_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$c_3 = [(s+3)X(s)]_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

- $$X(s) = \frac{(2/3)}{s} + \frac{(-1/2)}{s+1} + \frac{(-1/6)}{s+3} \quad \Rightarrow \quad x(t) = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}, \quad t \geq 0$$

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2. Distinct Complex Poles

- Recall if **all coefficients** of the **denominator** are **real**, then the polynomial must have a combination of real and/or complex conjugate poles.
- We can factor this polynomial into the following form:

$$\boxed{X(s) = \frac{c_1}{s-p_1} + \frac{\bar{c}_1}{s-\bar{p}_1} + \ldots + \frac{c_N}{s-p_N} \quad \text{where} \quad \begin{matrix} p_1 = \sigma + j\omega \\ \bar{p}_1 = \sigma - j\omega \end{matrix}}$$

- The **inverse transform** is given by:

$$\boxed{x(t) = c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + \ldots + c_N e^{p_N t} = 2|c_1| e^{\sigma t}\cos(\omega t + \angle c_1) + \ldots + c_N e^{p_N t}}$$

- How can this be visualized in **the *s*-plane**? (Hint: frequency and bandwidth)
- Example:

$$X(s) = \frac{s^2 - 2s + 1}{s^3 + 3s^2 + 4s + 2} = \frac{c_1}{s-(-1+j)} + \frac{\bar{c}_1}{s-(-1-j)} + \frac{c_3}{s-(-1)}$$

$$c_1 = [(s+1-j)X(s)]_{s=-1+j} = \left.\frac{s^2-2s+1}{(s+1+j)(s+1)}\right|_{s=-1+j} = -\frac{3}{2} + j2$$

$$c_3 = [(s+1)X(s)]_{s=-1} = \left.\frac{s^2-2s+1}{(s+1-j)(s+1+j)}\right|_{s=-1} = 4$$

$$|c_1| = 5/2 \quad \angle c_1 = 126.87° \quad \sigma = \mathrm{Re}\{p_1\} = -1$$

$$x(t) = 5e^{-t}\cos(t + 126.87°) + 4e^{-t}, \quad t \geq 0$$

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3. Repeated Poles

- If **the denominator** has **repeated poles**, the expansion is of the form:

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \ldots + \frac{c_r}{(s-p_1)^r} + \ldots + \frac{c_{r+1}}{s-p_{r+1}} + \ldots + \frac{c_N}{s-p_N}$$

- **The residues** for the **non-repeated poles** are calculated the same as before:

$$c_i = [(s-p_i)X(s)]_{s=p_i}, \quad i = r+1, r+2, \ldots, N$$

- **The residues** for the **repeated poles** are calculated by:

$$c_r = [(s-p_1)^r X(s)]_{s=p_1}$$

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{ds^i}[(s-p_1)^r X(s)]\right]_{s=p_1}, \quad i = 1, 2, \ldots, r-1$$

- If **the poles** are **all real**, **the inverse** can be compactly written using:

$$\boxed{\frac{t^{N-1}}{(N-1)!}e^{-at} \leftrightarrow \frac{1}{(s+a)^N}}$$

- **Repeated complex poles** and **complex conjugate poles** can also be dealt with by combining the **complex conjugate poles** into one **quadratic term,** and **solving for the residues** by **equating terms of a polynomial**:

$$X(s) = \frac{c_1}{s-p_1} + \frac{\bar{c}_1}{s-\bar{p}_1} + \ldots = \frac{cs+d}{(s-p_1)(s-\bar{p}_1)} + \ldots = \frac{cs+d}{s^2+|p_1|^2} + \ldots$$

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Repeated Poles (Cont.)

- **Example:**

$$X(s)=\frac{5s-1}{s^3-3s-2}=\frac{c_1}{s+1}+\frac{\bar{c}_1}{(s+1)^2}+\frac{c_3}{s-2}$$

$$c_1=\left[\frac{d}{ds}\left[(s+1)^2X(s)\right]\right]_{s=-1}=\left[\frac{d}{ds}\left[\frac{5s-1}{s-2}\right]\right]_{s=-1}$$

$$=\left[(5s-1)(\frac{-1}{(s-2)^2})+\left(\frac{1}{s-2}\right)(5)\right]_{s=-1}=\left[\frac{-9}{(s-2)^2}\right]_{s=-1}=-1$$

$$c_2=\left[(s+1)^2X(s)\right]_{s=-1}=\left[\frac{5s-1}{s-2}\right]_{s=-1}=2$$

$$c_3=\left[(s-2)^2X(s)\right]_{s=2}=1$$

Compute the **inverse** of each term individually and sum them:

$$x(t)=-e^{-t}+2te^{-t}+e^{2t},\quad t\geq 0$$

- This can be easily checked using MATLAB.

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➢ 4. Numerator Has A Larger Order ($M \geq N$)

- What **if the numerator** has **a degree larger than the denominator**?
- **The polynomials** can be **decomposed** using **long division**:

$$\boxed{X(s)=\frac{B(s)}{A(s)}=\frac{b_0+b_1s+b_2s^2...+b_Ms^M}{a_0+a_1s+a_2s^2...+a_Ns^N}=Q(s)+\frac{R(s)}{A(s)}}$$

- **The inverse transform** of Q(s) is computing using the **transform pair**:

$$\boxed{\frac{d^N}{dt^N}\delta(t)\leftrightarrow s^N,\quad N=1,2,...}$$

- **Example:**

$$X(s)=\frac{s^3+2s-4}{s2+4s-2}=s-4+\frac{20s-12}{s^2+4s-2}$$

$$x(t)=\frac{d}{dt}\delta(t)-4\delta(t)+...$$

The long division can be performed using the MATLAB command *deconv*:

```
num = [1 0 2 -4]
den = [1 4 -2]
[Q,R] = deconv(num, den)
```

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Pole Locations Determine the Behavior of a Signal

- We have seen that the structure of the poles in the **denominator polynomial** play a significant role in the determining the structure of the signal:

$$\boxed{\begin{array}{ccc} \text{distinct real pole} & \to & ce^{pt} \\ \text{repeated pole} & \to & c_1 e^{pt} + c_2 t e^{pt} \\ \text{complex conjugate poles} & \to & ce^{\sigma t}\cos(\omega t + \theta) \\ \text{repeated complex conjugate poles} & \to & c_1 e^{\sigma t}\cos(\omega t + \theta_1) + c_2 t e^{\sigma t}\cos(\omega t + \theta_2) \end{array}}$$

- As a result, **the signal** can be determined directly from **the poles**. Modifying **the numerator** does not change the signal significantly, it just changes the values of the **constants** associated with the above terms.
- We can observe that $x(t)$ converges to zero as $t \to \infty$ if and only if the poles of **the signal** all have **real parts strictly less than zero**:

$$\mathrm{Re}\{p_i\} < 0 \quad \text{for } i = 1, 2, \ldots, N$$

Hence, the **poles** play a **critical role** in **the stability** of a signal. The field of Control Systems deals with stability issues in signals and systems.

- If $X(s)$ has a **single pole** at $s = 0$, the limiting value of $x(t)$ is equal to **the residue** corresponding to that pole:

$$\lim_{t\to\infty} x(t) = [sX(s)]_{s=0}$$

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5. Transforms Containing Exponentials

- One last case we want to consider:

$$X(s) = \frac{B_0(s)}{A_0(s)} + \frac{B_1(s)}{A_1(s)}e^{-h_1 s} + \ldots + \frac{B_q(s)}{A_q(s)}e^{-h_q s}$$

- The function is **not rational** in s. Such functions are referred to as ***transcendental functions*** of s.
- Our approach is to use the methods previously described to compute the **inverse of the rational transforms**, and then to apply the **time shift property**:

$$\boxed{x(t) = x0(t) + \sum_{i=1}^{q} x_i(t-h_i)u(t-h_i), \quad t \geq 0}$$

- Such functions arise when **the Laplace transform** is applied to a piecewise-continuous function, such as **a pulse**:

$$u(t) - u(t-c) \quad \leftrightarrow \quad \frac{1}{s} - \frac{1}{s}e^{-cs}$$

- **Example:**

$$X(s) = \frac{s+1}{s^2+1} - \frac{1}{s+1}e^{-s} + \frac{s+2}{s^2+1}e^{-3s/2}$$

$$\text{Using}: \quad (\cos t + \sin t)u(t) \leftrightarrow \frac{s+1}{s^2+1} \qquad (\cos t + 2\sin t) \leftrightarrow \frac{s+2}{s^2+1}$$

$$\text{gives}: \quad x(t) = \cos t + \sin t - e^{-(t-1)}u(t-1) + [\cos(t-1.5) + 2\sin(t-1.5)]u(t-1.5), \quad t \geq 0$$

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Specific Cases of Inverse Laplace Transforms

- **We will restrict ourselves to two special case:**

1) Rational transforms: use **partial fractions expansion**

$$X(s) = \frac{N(s)}{D(s)} = \frac{A}{s+a} + \frac{B}{s+b} + \ldots$$

2) Exponentials: use **the shift property:**

$$X(s) = e^{-sT_1} X_1(s) + e^{-sT_2} X_2(s) + \ldots \Leftrightarrow x(t) = x_1(t - T_1) + x_2(t - T_2) + \ldots$$

- These **two building blocks** will allow us to construct **the inverse transforms** for many common **signals** and **systems**, including those used in **circuit analysis**.
- Therefore, the **unilateral Laplace transform** can be applied to finding both **the transient** and **steady-state responses** (as well as the frequency response) of **a circuit**. This is one of its **principal uses** in **electrical engineering**.
- The use of **partial fractions**, however, requires being able to factor a polynomial into its roots. You have previously used this in calculus, and have good MATLAB support for this as well.

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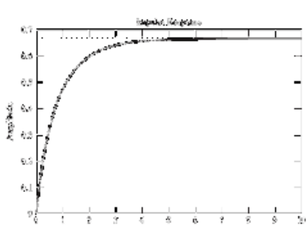
Summary

- Introduced a method for funding **the inverse Laplace transform** using **partial fractions expansion**:
  1) Factor the denominator.
  2) Assess the complexity of the poles (e.g., distinct vs. repeated, real vs. complex conjugate pairs).
  3) Compute the coefficients of the expansion using the **method of residues**.
  4) Write the **inverse Laplace transform** by inspection.
- Discussed the influence poles have on the resulting signal. We will study this more carefully later in the course.
- Noted that the MATLAB Symbolic Toolbox can be used to find the inverse transforms:

```
syms X s x
X = (s+2)/(s^3+4*s^2+3*s);
x = ilaplace(X);
x =
    -1/6*exp(-3*t)-1/2*exp(-t)+2/3
ezplot(x,[0,10])
```



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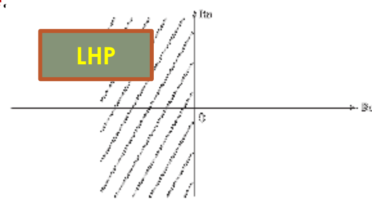
➢ Stability of CT Systems in the s-Plane

- **Recall our stability condition for the Laplace transform of the impulse response of a CT linear time-invariant system:**

$$H(s)=\frac{b_M s^M + b_{M-1}s^{M-1}+\ldots+b_0}{s^N + a_{N-1}s^{N-1}+\ldots+s_0} \quad \Leftrightarrow \quad \mathrm{Re}(p_i)<0 \quad \text{for} \quad i=1,2,\ldots,N$$

- **This implies the poles are in the left-half plane.**
  This also implies:



$$|h(t)|\to 0 \text{ as } t\to\infty \quad \text{and} \quad \int_{-\infty}^{\infty}|h(t)|\,dt<\infty$$

- **A system is said to be marginally stable if its impulse response is bounded:**

$$|h(t)|\le c \quad \forall t$$

In this case, at least one pole of the system lies on the $j\omega$-axis.

- **Recall periodic signals also have poles on the $j\omega$-axis because they are marginally stable.**
- **Also recall that the left-half plane maps to the inside of the unit circle in the z-plane for discrete-time (sampled) signals.**
- **We can show that circuits built from passive components (RLC) are always stable if there is some resistance in the circuit.**

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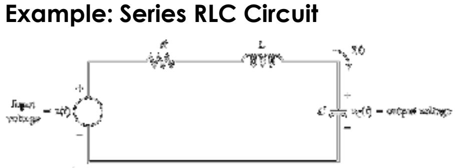
Stability of CT Systems in the s-Plane

- **Example: Series RLC Circuit**



$$H(s)=\frac{1/LC}{s^2+(R/L)s+1/LC}$$

Using the quadratic formula:

$$p_1,p_2=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

Case 1 :

$$\left(\frac{R}{2L}\right)^2-\frac{1}{LC}\ge 0 \;\Rightarrow \text{two real poles}$$

$$-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}<0 \quad\Rightarrow\quad \text{pole in LHP}$$

$\Rightarrow$ *always stable*

$$-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}<0$$

quadratic term must be $<\dfrac{R}{2L}$

$\Rightarrow$ *always stable*

Case 2 :

$$\left(\frac{R}{2L}\right)^2-\frac{1}{LC}<0 \;\Rightarrow \text{two complex poles,}$$

$$\mathrm{Re}\{p_1\}=-\frac{R}{2L} \quad\Rightarrow \text{always stable}$$

The RLC circuit is always stable. Why?

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The Routh-Hurwitz Stability Test

- **The procedures for determining stability do not require finding the roots of the denominator polynomial, which can be a daunting task for a high-order system (e.g., 32 poles).**
- **The Routh-Hurwitz stability test is a method of determining stability using simple algebraic operations on the polynomial coefficients. It is best demonstrated through an example.**
- **Consider:** $A(s) = a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0$
- **Construct the Routh array:**

$$
\begin{array}{c|cccc}
s^N & a_N & a_{N-2} & a_{N-4} & \cdots \\
s^{N-1} & a_{N-1} & a_{N-3} & a_{N-5} & \cdots \\
s^{N-2} & b_{N-2} & b_{N-4} & b_{N-6} & \cdots \\
s^{N-3} & c_{N-3} & c_{N-5} & c_{N-7} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s^2 & d_2 & d_0 & 0 & \cdots \\
s^1 & e_1 & 0 & 0 & \cdots \\
s^0 & f_0 & 0 & 0 & \cdots
\end{array}
$$

N even : $(N/2)+1$ columns

N odd : $(N+1)/2$ columns

$$b_{N-2} = \frac{a_{N-1}a_{N-2} - a_N a_{N-3}}{a_{N-1}} = a_{N-2} - \frac{a_N a_{N-3}}{a_{N-1}}$$

$$b_{N-4} = \frac{a_{N-1}a_{N-4} - a_N a_{N-5}}{a_{N-1}} = a_{N-4} - \frac{a_N a_{N-5}}{a_{N-1}}$$

Number of sign changes in 1st column = number of poles in the RHPRLC circuit is always stable

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Routh-Hurwitz Examples

- **Example:** $A(s) = s^2 + a_1 s + a_0$

$$
\begin{array}{ccc}
s^2 & 1 & a_0 \\
s^1 & a_1 & 0 \\
s^0 & \dfrac{a_1 a_0 - (1)(0)}{a_1} = a_0 & 0
\end{array}
$$

if $a_1 > 0$ and $a_0 < 0 \Rightarrow$ one sign change $\Rightarrow$ 1 RHP pole $\Rightarrow$ unstable

if $a_1 < 0$ and $a_0 < 0 \Rightarrow$ one sign change $\Rightarrow$ 1 RHP pole $\Rightarrow$ unstable

if $a_1 < 0$ and $a_0 > 0 \Rightarrow$ two sign changes $\Rightarrow$ 2 RHP poles $\Rightarrow$ unstable

- **Example:** $A(s) = s^3 + a_2 s^2 + a_1 s + a_0$

$$
\begin{array}{ccc}
s^3 & 1 & a_1 \\
s^2 & a_2 & a_0 \\
s^1 & \dfrac{a_2 a_1 - (1)a_0}{a_2} = a_1 - \dfrac{a_0}{a_2} & 0 \\
s^0 & a_0 & 0
\end{array}
$$

if $a_2 < 0$ and $a_0 > 0 \Rightarrow$ two sign changes $\Rightarrow$ 2 RHP poles $\Rightarrow$ unstable

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Analysis of the Step Response For A 1st-Order System

- Recall the transfer function for a 1st-order differential equation:

$$H(s) = \frac{k}{s-p}$$

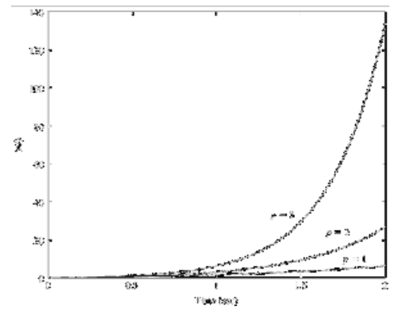
$$X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

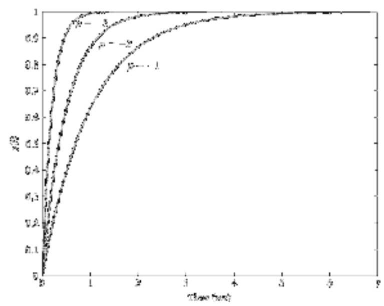
$$Y(s) = H(s)X(s) = \frac{k}{s(s-p)}$$

$$Y(s) = \frac{-k/p}{s} + \frac{k/p}{s-p}$$

$$y(t) = -\frac{k}{p}\left(1-e^{pt}\right)u(t)$$

```
num = 1; den = [1 –
    p];
t = 0:0.05:10;
y = step(num, den, t);
```





- Define a time constant as the time it takes for the response to reach $1/e$ (37%) of its value.
- The time constant in this case is equal to $-1/p$. Hence, the real part of the pole, which is the distance of the pole from the $j\omega$-axis, and is the bandwidth of the pole, is directly related to the time constant.

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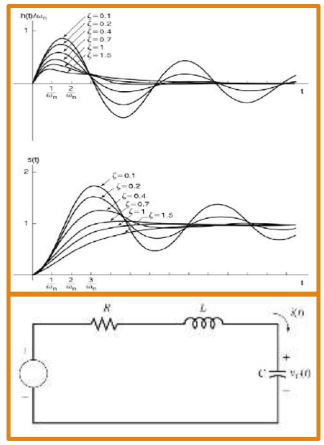
Summary

- Reviewed stability of CT systems in terms of the location of the poles in the s-plane.
- Demonstrated that an RLC circuit is unconditionally stable.
- Introduced the Routh-Hurwitz technique for determining the stability of a system: does not require finding the roots of a polynomial.
- Analyzed the properties of the impulse response of a first-order differential equation.
- Next: analyze the properties of a second-order differential equation.

SECOND-ORDER SYSTEMS

Objectives:
- Second-Order Transfer Function
- Real Poles
- Complex Poles
- Effect of Damping
- Circuit Example



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Second-Order Transfer Function

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- Recall our expression for a simple, 2nd-order differential equation:

$$\frac{d^2y}{dt}+a_1\frac{dy}{dt}+a_0y(t)=b_0x(t) \;\Rightarrow\; H(s)=\frac{b_0}{s^2+a_1s+a_0}$$

- Write this in terms of two parameters, ζ and ω_n, related to the poles:

$$H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (\zeta \text{ is "zeta"})$$

- From the quadratic equation:

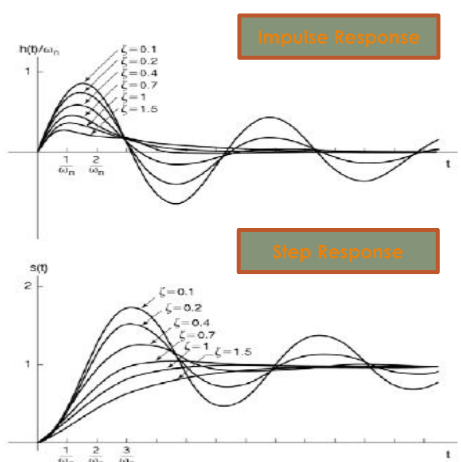
$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

- There are three types of interesting behavior of this system:

$0<\zeta<1$: complex poles (underdamped)

$\zeta=1$: double pole at $s=-\omega_n$ (critically damped)

$\zeta>1$: two poles (negative real axis) (overdamped)



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Step Response For Two Real Poles

- When $\zeta > 1$, both poles are real and distinct:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{(s - p_1)(s - p_2)}$$

$$X(s) = \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{(s - p_1)(s - p_2)}\left(\frac{1}{s}\right)$$

$$y(t) = \frac{\omega_n^2}{p_1 p_2}\left(1 + k_1 e^{p_1 t} + k_1 e^{p_2 t}\right), \quad t \geq 0$$

- There are two components to this response:

$$y_{tr}(t) = \frac{\omega_n^2}{p_1 p_2}\left(k_1 e^{p_1 t} + k_1 e^{p_2 t}\right), \quad t \geq 0$$

$$y_{ss}(t) = \frac{\omega_n^2}{p_1 p_2} \quad \text{(due to step input)}$$

- When $\zeta = 1$, both poles are real ($s = \omega_n$) and repeated:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{(s - p_1)^2}$$

$$X(s) = \frac{1}{s}$$

$$H(s) = \frac{\omega_n^2}{(s - p_1)^2}\left(\frac{1}{s}\right)$$

$$y(t) = 1 - \left(1 + \omega_n t e^{-\omega_n t}\right), \quad t \geq 0$$

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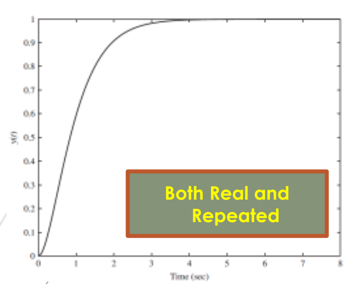
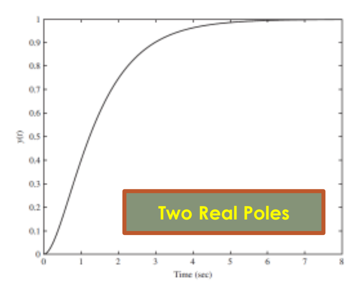
Step Response For Two Real Poles (Cont.)





- ζ is referred to as the damping ratio because it controls the time constant of the impulse response (and the time to reach steady state);
- ω_n is the natural frequency and controls the frequency of oscillation (which we will see next for the case of two complex poles).
- $\zeta > 1$: The system is considered overdamped because it does not achieve oscillation and simply directly approaches its steady-state value.
- $\zeta = 1$: The system is considered critically damped because it is on the verge of oscillation.

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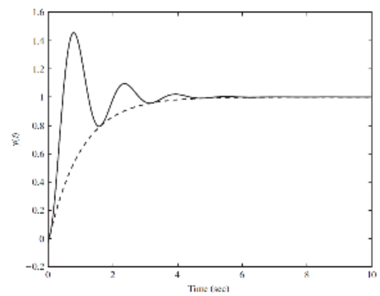
Step Response For Two Complex Poles

- When $0 < \zeta < 1$, we have two complex conjugate poles:

$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$p_1, p_2 = -\zeta\omega_n \pm j\omega_d$$



- The transfer function can be rewritten as:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$= \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2\right) + \left(\omega_n^2 - \zeta^2\omega_n^2\right)}$$
$$= \frac{\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2}$$

- The step response, after some simplification, can be written as:

$$y(t) = 1 - \frac{\omega_n}{\omega_n} e^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right), \quad t \geq 0 \quad where\ \phi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$$

- Hence, the response of this system eventually settles to a steady-state value of 1. However, the response can overshoot the steady-state value and will oscillate around it, eventually settling in to its final value.

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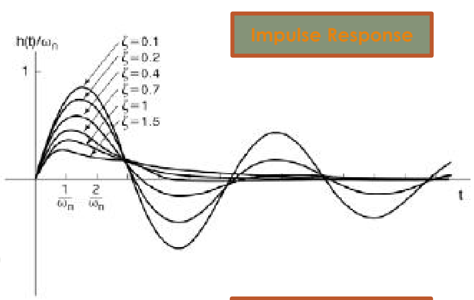
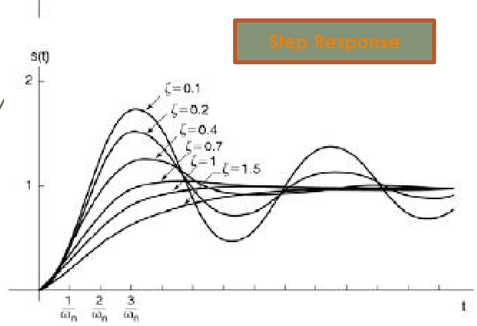
Analysis of the Step Response For Two Complex Poles





- $\zeta > 1$: the overdamped system experiences an exponential rise and decay. Its asymptotic behavior is a decaying exponential.
- $\zeta = 1$: the critically damped system has a fast rise time, and converges to the steady-state value in an exponetial fashion.
- $0 < \zeta > 1$: the underdamped system oscillates about the steady-state behavior at a frequency of ω_d.
- Note that you cannot control the rise time and the oscillation behavior independently!
- What can we conclude about the frequency response of this system?

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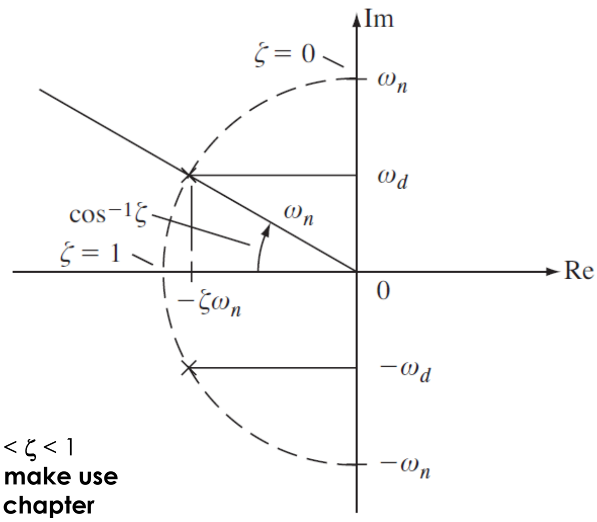
Implications in the s-Plane

Several important observations:

- The pole locations are:

$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\zeta\omega_n \pm j\omega_d$$

- Since the frequency response is computed along the $j\omega$-axis, we can see that the pole is located at $\pm\omega_d$.
- The bandwidth of the pole is proportional to the distance from the $j\omega$-axis, and is given by $\zeta\omega_n$.
- For a fixed ω_n, the range $0 < \zeta < 1$ describes a circle. We will make use of this concept in the next chapter when we discuss control systems.
- What happens if ζ is negative?



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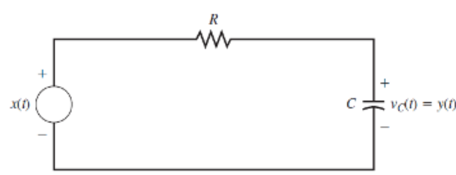
RC Circuit

- Example: Find the response to a sinewave:.

$$x(t) = C\cos(\omega_0 t)u(t) \quad \Leftrightarrow \quad X(s) = \frac{Cs}{s^2 + \omega_0^2}$$



- Solution:

$$H(s) = \frac{1/RC}{s + 1/RC} = \frac{k}{s - p}$$

$$Y(s) = \frac{kCs}{(s-p)(s^2 + \omega_0^2)}$$

$$y(t) = \frac{kCp}{\omega_0^2 + p^2}e^{pt} + \frac{Ck}{\sqrt{\omega_0^2 + p^2}}\cos\left(\omega_0 t - \tan^{-1}\left(-\frac{\omega_0}{p}\right)\right), \quad t \geq 0$$

- Again we see the solution is the superposition of a transient and steady-state response.
- The steady-state response could have been found by simply evaluating the Fourier transform at ω_0 and applying the magnitude scaling and phase shift to the input signal. Why?
- The Fourier transform is given by: $H(e^{j\omega}) = H(s)\big|_{s=e^{j\omega}} = \dfrac{1/RC}{j\omega + 1/RC}$

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Summary

- Analyzed the behavior of stable 2nd-order systems.
- Characterized these systems in terms of three possible behaviors: overdamped, critically-damped, or overdamped.
- Discussed the implications of this in the time and frequency domains.
- Analyzed the response of an RC circuit to a sinewave.
- Next: Frequency response and Bode plots.