Summary: Apply Chebyshev's to the sample mean.

We start with Chebyshev's inequality:

$$\text{①} \quad \Pr(|X - E[X]| > a) \leq \frac{var(X)}{a^2} \quad \forall a > 0.$$

Let $S_n = X_1 + X_2 + \ldots + X_n$, where each X_i is i.i.d. with mean μ and variance σ^2. We compute the expectation and variance of S_n:

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\mu.$$

(linearity of exp).

$$var[S_n] = var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} var(X_i) = n\sigma^2.$$

(independence)

Now consider the sample average $\frac{S_n}{n}$, and find its mean and variance:

$$E\left[\frac{S_n}{n}\right] = \frac{1}{n} E[S_n] = \frac{1}{n}(n\mu) = \mu.$$

$$var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} var[S_n] = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}.$$

Now we apply ① to $\frac{S_n}{n}$:

$$\Pr\left(\left|\frac{S_n}{n} - E\left[\frac{S_n}{n}\right]\right| > a\right) \le \frac{\operatorname{var}\left(\frac{S_n}{n}\right)}{a}$$

$$= \Pr\left(\left|\frac{S_n}{n} - \mu\right| > a\right) \le \frac{\sigma^2}{na}$$

Trivially, this probability is lower bounded by zero:

$$0 \le \Pr\left(\left|\frac{S_n}{n} - \mu\right| > a\right) \le \frac{\sigma^2}{na}.$$

It is also upper bounded by zero in the limit as $n \to \infty$:

$$\lim_{n \to \infty} \frac{\sigma^2}{na} = 0$$

$$\Rightarrow \lim_{n \to \infty} \Pr\left(\left|\frac{S_n}{n} - \mu\right| > a\right) = 0 \quad \forall a > 0.$$

In words, the sample average converges in probability to the mean as the number of samples approaches infinity.

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