Weak Duality & the Lagrangian

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Given an optimization problem:

$$\min f_0(x)$$
$$\text{s.t. } f_i(x) \leq 0 \quad i=1,\dots,m$$
$$h_i(x) = 0 \quad i=1,\dots,p$$

The Lagrangian is defined as:

$$\mathcal{L}(x, \lambda, \gamma) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \gamma_i h_i(x).$$

$\lambda \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^p$, $x \in \mathbb{R}^n$.

$\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$.

and the (Lagrangian) dual function is defined as:

$$g(\lambda, \gamma) = \inf_x \mathcal{L}(x, \lambda, \gamma)$$
$$= \inf_x f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \gamma_i h_i(x)$$

For any feasible $\tilde{x}$, $\lambda \geq 0$, and any γ,

$$f_0(\tilde{x}) \geq f_0(\tilde{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i f_i(\tilde{x})}_{\leq 0} + \underbrace{\sum_{i=1}^{p} \gamma_i h_i(\tilde{x})}_{=0}$$

$$= \mathcal{L}(\tilde{x}, \lambda, \gamma)$$

$$\geq \inf_x \mathcal{L}(x, \lambda, \gamma)$$

$$= g(\lambda, \gamma).$$

Since x^* is a feasible pt, this implies

$$f_0(x^*) = p^* \geq g(\lambda, \gamma).$$

The Lagrangian dual function gives us a lower bound on the optimal primal objective value for $\lambda \geq 0$.